Combinatorics - Reviewed Notes

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April 30, 2019

Contents

1 Counting Techniques

Theorem 1.1 - *Bijection Rule*
A set X has $n \in \mathbb{N}$ elements iff $\exists$ a bijection $f : X \to [n]$.

Theorem 1.2 - *Addition Rule*
Let $A_1, \dots, A_n$ be finite pairwise disjoint sets. Then

$$\left|\bigcup_{i=1}^{n} A_i\right| = \sum_{i=1}^{n} |A_i|$$

Theorem 1.3 - *Multiplication Rule*
If a counting problem can be split into n independent stages, each of which involves choosing an option out of a set A_i. Then the total number of possible outcomes is

$$\text{Possible Outcomes} = \prod_{i=1}^{n} |A_i|$$

Theorem 1.4 - *Inclusion-Exclusion Principle*
Let $A_1, \dots, A_n$ be finite sets. Then

$$\left|\bigcup_{i=1}^{n} A_i\right| = \sum_{i=1}^{n} |A_i| - \sum_{i_1<i_2} |A_{i_1} \cap A_{i_2}| + \sum_{i_1<i_2<i_3} |A_{i_1} \cap A_{i_2} \cap A_{i_3}| - \dots + (-1)^{n+1}|A_1 \cap \dots \cap A_n|$$

Theorem 1.5 - *Binomial Coefficient Identities*

$$\binom{n}{k} = \binom{n}{n-k}, \qquad \binom{n}{0} = 1 = \binom{n}{n}$$
$$\binom{n}{k} = 0 \ \forall \ k > n, \quad \binom{n}{k} = \frac{(n)_k}{k!} = \frac{n!}{k!(n-k)!}$$

Definition 1.1 - *Ordered Selection*
In an *Ordered Selection* problem we are selecting k elements from a set of n, and care about the order in which we pick them, so $\{x_1, x_2\} \not\equiv \{x_2, x_1\}$.

Definition 1.2 - *Unordered Selection*
In an *Unordered Selection* problem we are selecting k elements from a set of n, and do not care about the order we pick them so $\{x_1, x_2\} \equiv \{x_2, x_1\}$.

Proposition 1.1 - *Number of Possible Selections*
The table below summarises the number of possible ways to choose k elements from a set of n, for each scenario of selection problems.

	Ordered Selection	Unordered Selection
With Repetition	n^k	$\binom{n+k-1}{k}$
Without Repetition	$\frac{n!}{(n-k)!}$	$\binom{n}{k}$

Theorem 1.6 - *Pascal's Identity*

$$\binom{n}{i} = \binom{n-1}{i} + \binom{n-1}{i-1} \ \forall \ i \in \mathbb{N}^{\leq n}$$

Theorem 1.7 - *Binomial Theorem*

$$(a+b)^n = \sum_{i=0}^{n} \binom{n}{i} a^i b^{n-i} \ \forall \ a, b \in \mathbb{R} \ n \in \mathbb{N}$$

Theorem 1.8 - *Sum of Binomial Coefficients*

$$\sum_{i=0}^{n} \binom{n}{i} = 2^n$$

Theorem 1.9 - *Identity*

$$(j+1)\binom{n}{j+1} = n\binom{n-1}{j}$$

Theorem 1.10 - *Pigeonhole Principle*
If there are more pigeons than pigeonholes, then at least 2 pigeons must occupy the same hole.

Theorem 1.11 - *Generalised Pigeonhole Principle*
If m objects are distributed into n boxes and $m > nk$ for $k \in \mathbb{N}$, then at least one box contains at least $k+1$ objects.

2 Generating Functions

Definition 2.1 - *Generating Functions*
Given a sequence $(a_n)_{n\geq 0}$ with $a_i \in \mathbb{R}$ we associate the power series $f(x) = \sum_{i=0}^{\infty} a_n x^n$.
$f(x)$ is the *Generating Function* of the sequence $(a_n)_{n\geq 0}$.

Theorem 2.1 - *Scaling Rule*

$$\text{If } f(x) \leftrightarrows (a_0, a_1, a_2, \dots) \implies cf(x) \leftrightarrows (ca_0, ca_1, ca_2, \dots)$$

Theorem 2.2 - *Addition Rule*

$$\text{If } f(x) \leftrightarrows (a_0, a_1, a_2, \dots) \;\&\; g(x) \leftrightarrows (b_0, b_1, b_2, \dots) \implies f(x) + g(x) \leftrightarrows (a_0 + b_0, a_1 + b_1, \dots)$$

Theorem 2.3 - *Right Shift Rule*

$$\text{If } f(x) \leftrightarrows (a_0, a_1, a_2, \dots) \implies x^k f(x) \leftrightarrows (\underbrace{0, \dots, 0}_{k \text{ times}}, a_0, a_1, a_2, \dots)$$

Theorem 2.4 - *Differentiation Rule*

$$\text{If } f(x) = \sum_{i=0}^{\infty} a_i x^i \leftrightarrows (a_0, a_1, a_2, \dots) \implies f'(x) = \sum_{i=0}^{\infty} i a_{i+1} x^i \leftrightarrows (a_1, 2a_2, 3a_3, \dots)$$

Theorem 2.5 - *Convolution Rule*

$$\text{If } f(x) \leftrightarrows (a_0, a_1, a_2, \dots) \;\&\; g(x) \leftrightarrows (b_0, b_1, b_2, \dots) \implies f(x) \cdot g(x) \leftrightarrows (c_0, c_1, c_2, \dots) \text{ where } c_i := \sum_{j=0}^{i} a_j b_{i-j}$$

Definition 2.2 - *Recurrence Relation*
A sequence, $(a_n)_{n\geq 0}$ is a *Recurrence Relation* if, for large enough n, a_n is defined in terms of its previous terms. Generating functions can be used to express $(a_n)_{n\geq 0}$ in terms of n.

Theorem 2.6 - *Generating Function Identities*

$$\begin{aligned}\frac{1}{1-x} &\leftrightarrows (1,1,1,\dots)\\ (1+x)^n &\leftrightarrows \left(\binom{n}{k}\right)_{k\geq 0}\\ (1+x)^{-n} &\leftrightarrows \sum_{i=0}^{\infty}(-1)^k\binom{n+k-1}{k}x^k\end{aligned}$$

Theorem 2.7 - *Sum Identities*

$$\begin{aligned}\left(\sum_{i=0}^{\infty}x^i\right)^n &= \sum_{i=0}^{\infty}\binom{n+k-1}{k}x^k\\ \frac{1}{(1-x)^n} &= \sum_{i=0}^{\infty}\binom{n+k-1}{k}x^k\end{aligned}$$

3 Combinatorial Design

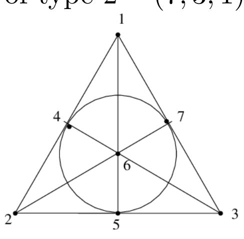
Definition 3.1 - *Set System*
$(V,\mathcal{B})$ is a *Set System* where V is a finite set & $\mathcal{B}$ is a set of subsets, which are not necessarily disjoint), of V. V is called the *Ground Set* & $\mathcal{B}$ is called *Blocks*.

Definition 3.2 - *k-Uniform*
A *Set System*, $(V,\mathcal{B})$, is k-uniform iff $\forall\ B\in\mathcal{B},\ |B|=k$.

Definition 3.3 - *Block Design*
Let $v,k,t,\lambda\in\mathbb{N}$ with $v>k\geq k\geq t\geq 1$.
A *Block Design* of type $t-(v,k,\lambda)$ is a set system $(V,\mathcal{B})$ with

- $|V|=v$;
- $\forall\ B\in\mathcal{B},\ |B|=k$; and,
- $\forall\ T\subseteq V$ with $|T|=t\ \exists\ \lambda$ blocks, $B\in\mathcal{B}$, with $T\subseteq B$.

Definition 3.4 - *Fano Plane*
The *Fano Plane* is a set-system with $V=\{1,2,3,4,5,6,7\}$
& $\mathcal{B}=\{\{1,2,3\},\{1,4,7\},\{3,6,7\},\{1,5,6\},\{3,4,5\},\{2,5,7\},\{2,4,6\}\}$. This is a *Block Design* of type $2-(7,3,1)$.



N.B. - This is the smallest example of a *Finite Projective Plane.*

Definition 3.5 - *Replication Number*
The *Replication Number* is the number of blocks each element appears in in a *Block Design.*

Theorem 3.1 - *Number of Blocks in Block Design*
The number of blocks in a block design of type $t-(v,k,\lambda)$ is found by the following formula

$$b := \frac{\lambda\binom{v}{t}}{\binom{k}{t}}$$

Theorem 3.2 - *Replication Number*
In a block design of type $2-(v,k,\lambda)$ every element lies in r blocks where

$$r := \frac{(k-1)}{\lambda(v-1)} = \frac{bk}{v}$$

Theorem 3.3 - *Fisher's Inequality*
Let $(V,\mathcal{B})$ be a block design of type $2-(v,k,\lambda)$ with $v>k$. Then

$$|\mathcal{B}| \geq |V|$$

Definition 3.6 - *Incidence Matrix*
The *Incidence Matrix* of a *Set System* $(V,\mathcal{B})$ is a $|V|\times|\mathcal{B}|$ matrix where

$$a_{i,j} = \begin{cases} 1 & \text{if } v_i \in B_j \\ 0 & \text{otherwise} \end{cases}$$

4 Graph Theory

4.1 Introduction

Definition 4.1 - *Graph*
A *Graph*, G, is an ordered pair (V,E) where V is a set of vertices & E is a set of edges between these vertices. Edges are represented by a 2-element subset of V.

Definition 4.2 - *Sub-Graph*
Consider a graph $G=(V,E)$. $G'=(V',E')$ is a *Sub-Graph* of G if

$$V' \subseteq V \ \&\ E' = \{\{u,v\} \in E : u,v \in V'\}$$

Definition 4.3 - *Induced Sub-Graph*
A *Sub-Graph* $G'=(V',E')$ of $G=(V,E)$ is an *Induced Sub-Graph* if E' contains all the edges in E that run between vertices in V'.

Definition 4.4 - *Order of a Graph*
The *Order* of a *Graph* $G=(V,E)$ is the number of vertices, $|V|$.

Definition 4.5 - *Simple Graph*
A *Simple Graph* is an unweighted, undirected with no edges which start & end on the same vertex, nor duplicate edges.

Proposition 4.1 - *Graphs are* 2*-Uniform Set Systems*

Definition 4.6 - *Adjacency*
For a graph $G=(V,E)$, $u,v \in V$ are *Adjacent* if $\{u,v\} \in E$.

Definition 4.7 - *Neighbourhood*
Consider a graph $G = (V, E)$ & $v \in V$. The *Neighbourhood* of v in G is the vertices adjacent to v in G.

$$N_G(v) := \{u : u \in V \ \& \ \{u, v\} \in E\}$$

Definition 4.8 - *Neighbourhood of a Set of Vertices*
Let $G = (V, E)$ be a graph, then for $S \subseteq V$

$$N_G(S) := \bigcup_{s \in S} N_G(s)$$

Definition 4.9 - *Degree*
Consider a graph $G = (V, E)$ & $v \in V$. The *Degree* of v is the size of its *Neighbourhood* of v.

$$deg_G(v) := |N_G(v)|$$

Theorem 4.1 - *Handshaking Lemma*
Consider a graph $G = (V, E)$. Then

$$|E| = \frac{1}{2} \sum_{v \in V} deg_G(v)$$

Theorem 4.2 - *Maximum Number of Edges*
A graph has at most $\binom{n}{2} = \frac{1}{2}n(n-1)$ edges.
Definition 4.10 - *Graph Isomorphism*
Two graphs $G_1 = (V_1, E_1)$ & $G_2 = (V_2, E_2)$ are *Isomorphic* if $\exists$ a bijection $\phi : V_1 \to V_2$ st

$$\forall \ \{u, v\} \in E_1 \implies \{\phi(u), \phi(v)\} \in E_2$$

Definition 4.11 - *Degree Sequence*
Consider a graph $G = (V, E)$. Order the vertices $\{v_1, \dots, v_n\}$ with $deg(v_i) \leq deg(v_j) \ \forall \ i < j$. The *Degree Sequence* of G is

$$\big(deg(v_1), \dots, deg(v_n)\big)$$

Theorem 4.3 - *Isomorphic Graphs*
If two graphs are isomorphic then the following properties hold

i) They have the same number of edges;

ii) They have the same *Degree Sequence*.

4.2 Types of Graph

Definition 4.12 - *Walk*
Consider a graph $G = (V, E)$ and $u, v \in V$. A *Walk* from u to v is a sequence of not-necessarily-distinct vertices $V_c = (u = x_1, x_2, \dots, x_n = v)$ with $E_c = \big\{\{x_i, x_{i+1}\} : i \in [1, n-1]\big\} \subseteq E$. Neither the edges, nor vertices, are necessarily unique.

Definition 4.13 - *Circuit*
A *Circuit* is a closed *Walk*.
N.B. - A *Circuit* is generally not a graph since they often have repeated edges.

Definition 4.14 - *Trail*
A *Trail* is a *Walk* with no repeated edges. It can still have repeated vertices.

Definition 4.15 - *Path*
A *Path* of length n has vertex set $V_P = \{v_1, \dots, v_n : v_i \neq v_j \ \forall \ i \neq j\}$ & edge set $E_p = \{\{v_i, v_{i+1}\} : i \in [1, n-1]\} \subseteq E$. All vertices, and thus edges, in a path are distinct.

Definition 4.16 - *Connected*
Two vertices are *Connected* if there exists a path between them.
A *Graph* is *Connected* if there exists a path between all pairs of vertices in it.

Definition 4.17 - *Disconnected*
A *Graph* $G = (V, E)$ is *Disconnected* if there $\exists \ u, v \in V \ st \ \nexists$ a path in G between u & v. The *Maximally Connected Sub-Graphs* of G are called *Components* of G.

Definition 4.18 - *Cycle*
A *Cycle* of length n has vertex set $V_c = \{v_1, \dots, v_n : v_i \neq v_j \ \forall \ i \neq j\}$ & edge set $E_c = \{v_n, v_1\} \cup \big\{\{v_i, v_j\} : i \in [1, n-1]\big\}$. This is a path that starts and ends on the same vertex.

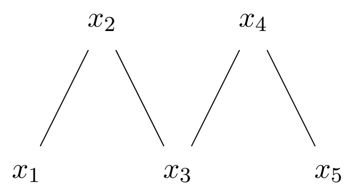
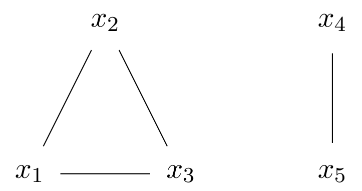
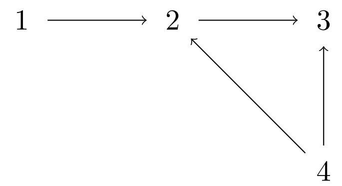
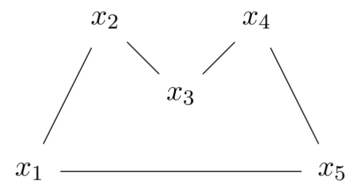
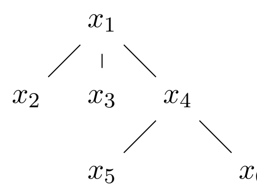
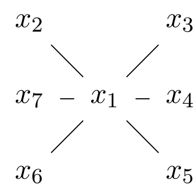
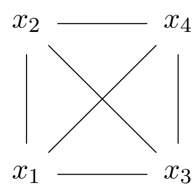
Theorem 4.4 - *If a graph admits an odd circuit, then it admits an odd cycle*

Definition 4.19 - *Tree*
A *Tree* is a *Graph* with no cycles.

Definition 4.20 - *Star*
A *Star* is a *Graph* $G = (V, E)$ with $V = \{v_1, \dots, v_n\}$ & $E = \{\{v_1, v_i\} : i \in [2, n]\}$.

Definition 4.21 - *Complete Graph*
A *Complete Graph* is a *Graph* $G = (V, E)$ with $V = \{v_1, \dots, v_n\}$ & $E = \{\{v_i, v_j\} : i \neq j\}$. There exists an edge between each pair of vertices.

Example 4.1 - *Common Graphs*

Path, P_4		Disconnected	
Wal & Trail		Cycle, C_5	
Tree		Star	
Complete, K_4			

Definition 4.22 - *Eulerian Circuit*
An *Eulerian Circuit* is a *Circuit* that traverses every edge exactly once. It traverses every edge exactly once, with no repeated vertices.
N.B. - A *Eulerian Graph* admits an *Eulerian Circuit.*

Theorem 4.5 - *Every vertex of a connected graph has even degree* $\Leftrightarrow$ *Eulerian Graph*

Theorem 4.6 - *Decomposing into Cycles*
If every vertex of a graph has even degree then its edge set can be partitioned into disjoint subsets st each subset is a cycle.

Definition 4.23 - *Hamiltonian Path*
A *Hamiltonian Path* is a path that visits every vertex.

Definition 4.24 - *Hamiltonian Cycle*
A *Hamiltonian Cycle* is a cycle that visits every vertex in a graph.
N.B. - A *Hamiltonian Graph* admits a *Hamiltonian Cycle.*

Theorem 4.7 - *Dirac's Theorem*
Let G be a graph of order $n \geq 3$. If $\delta(G) \geq \frac{n}{2}$ then G is *Hamiltonian.*

Definition 4.25 - *k-Partite Graph*
Consider a graph $G = (V, E)$, G is a *k-Partite Graph* if V can be partitioned into k disjoint vertex classes, $\{V_1, \dots, V_k\}$, st $E \subseteq \{\{x, y\} : x \in V_i,\ y \in V_j,\ i \neq j\}$.

Definition 4.26 - *Complete k-Partite Graph*
A k-Partite Graph, $G = (V, E)$ with vertex classes $\{V_1, \dots, V_k\}$, is a *Complete k-Partite Graph*

if $E = \{\{x, y\} : x \in V_i,\ y \in V_j,\ i \neq j\}$.

Definition 4.27 - *Bipartite Graph*
A graph $G = (V, E)$ is a *Bipartite Graph* iff V can be partitioned into two sets V_1 & V_2 st $V_1 \cup V_2 = V$, $V_1 \cap V_2 = \emptyset$ & $\forall\ u \in V_1,\ v \in V_2\ \{u, v\} \notin E$.
N.B. - - This graph is denoted $G = (V_1 \cup V_2, E)$.

Theorem 4.8 - *Characterisation of Bipartite Graph*
A graph is bipartite iff it contains no odd cycles.

Theorem 4.9 - *Handshaking Lemma for Bipartite Graph*
Let $G = (V_1 \cup V_2, E)$ be a bipartite graph. Then

$$\sum_{v \in V_1} deg_G(v) = \sum_{u \in V_2} deg_G(u)$$

Definition 4.28 - *Matching*
Consider a bipartite graph $G = (X \cup Y, E)$ a *Matching* from X to Y is a set of edges $\{\{x, y\} : x \in X, y \in Y\}$ where no two edges have a vertex in common.

Theorem 4.10 - *Hall's Theorem*
Consider a bipartite graph $G = (X \cup Y, E)$. Then

$$\exists \text{ matching from } X \text{ to } Y \Leftrightarrow\ \forall\ s \subseteq X, |N_G(S)| \geq |S|$$

Theorem 4.11 - *Degree Constrained Version of Hall's Theorem*
Consider a bipartite graph $G = (X \cup Y, E)$. Then

$$\delta(X) \geq \Delta(Y) \Leftrightarrow \exists \text{ matching from } X \text{ to } Y$$

4.3 Trees & Forests

Definition 4.29 - *Acyclic*
A graph is *Acyclic* if it contains no cycles.
N.B. - An *Acyclic Graph* is called a *Forest.*

Proposition 4.2 - *A tree is an acyclic connected graph*

Definition 4.30 - *Leaf*
A vertex of degree one is called a *Leaf.*

Theorem 4.12 - *Every tree on at least two vertices has a leaf*

Theorem 4.13 - *Characterisation of Trees*
The following are different ways to characterise a tree $G = (V, E)$

i) G is maximally acyclic (*i.e.* G is acyclic & any additional edge creates a cycle).

ii) G is minimally collected (*i.e.* G is connected & the removal of any edge disconnects G).

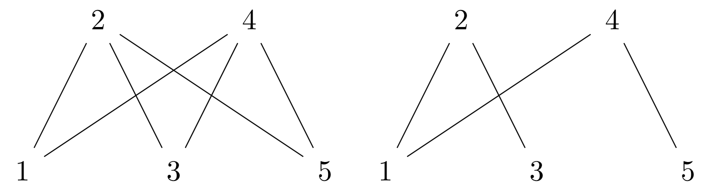
iii) G is connected and $|E| = |V| - 1$.

iv) G is acyclic and $|E| = |V| - 1$.

v) Any two vertices in G has a unique path.

Definition 4.31 - *Spanning Tree*
Consider a graph $G = (V, E)$, any tree on all vertices in V is a *Spanning Tree.*
N.B. - $T = (V, E')$ with $E' \subseteq E$.

Example 4.2 - *Spanning Tree*
Below is a graph and then a spanning tree of that graph



Theorem 4.14 - *Every connected graph contains a spanning tree*

Definition 4.32 - *Weight Function*
For $G = (V, E)$ we define a *Weight Function* $W : V \to \mathbb{R}$.

Definition 4.33 - *Minimum Spanning Tree*
Consider a graph $G = (V, E)$ with a weight function $W : V \to \mathbb{R}$. Let T be a *Spanning Tree* of G. $T = (V, E')$ is a *Minimum Spanning Tree* if $W(E') := \sum_{e \in E'} W(e)$ is minimised relative to the other *Spanning Trees* of G.

Theorem 4.15 - *Algorithm for Finding Spanning Tree*
Let $G = (V, E)$ be a graph with n vertices and m edges.
Order the edges of G arbitrarily into a sequence $e_1, \dots, e_m$.
The algorithm constructs sets of edges $E_0, E_1, \dots, \subseteq E$ in stances.
Set $E_0 = \emptyset$.
At state i the algorithm has already defined E_{i-1}. Then

$$E_i = \begin{cases} E_{i-1} \bigcup \{e_i\} & \text{If graph } (V, E_{i-1} \bigcup \{e_i\} \text{ contains no cycles} \\ E_{i-1} & \text{otherwise} \end{cases}$$

The algorithm stops if after stage i we have that$|E_i| = n - 1$.
This condition means that (V, E_i) is a tree.

Theorem 4.16 - *Kruskal's Algorithm*
Let $G = (V, E)$ be a connected weighted graph equipped with weight function $W : E \to \mathbb{R}$.
Label the edges of G with $e_1, \dots, e_m$, with $m = |E|$, in such a way that

$$W(e_1) \leq \dots \leq W(e_m)$$

Set $E_0 = \emptyset$.
At state i the algorithm has already defined E_{i-1}. Then

$$E_i = \begin{cases} E_{i-1} \bigcup \{e_i\} & \text{If graph } (V, E_{i-1} \bigcup \{e_i\} \text{ contains no cycles} \\ E_{i-1} & \text{otherwise} \end{cases}$$

The algorithm stops if after stage i we have that$|E_i| = n - 1$.
N.B. - This is the algorithm from **Definition 6.6**.

4.4 Cliques

Definition 4.34 - *k-Clique*
A *k-Clique* is a sub-graph which is a *Complete Graph* on k vertices.

Theorem 4.17 - *Mantel's Theorem*
Consider a graph $G = (V, E)$, with $n := |V|$, containing no 3-cycle sub-graphs. Then

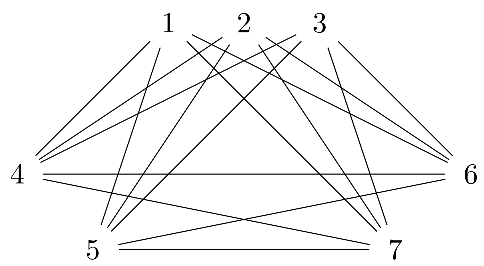
i) $|E| \leq \lfloor \frac{n^2}{4} \rfloor$; &

ii) $\exists$ a graph where $|E| = \lfloor \frac{n^2}{4} \rfloor$.

N.B. - The graph described *ii)* is isomorphic to $K_{\lfloor n/2 \rfloor, n - \lfloor n/2 \rfloor}$.

Definition 4.35 - *Turán Graph*
A *Turán Graph* is a graph $T_k(n)$ that is a *Complete k-Partite Graph* on n vertices with vertex classes $\{V_1, \dots, V_k\}$ where the difference in size of any pair of vertices is no more than 1 (*i.e.* $\big||V_i| - |V_j|\big| \leq 1 \ \forall \ i, j \in [1, k]$).

Example 4.3 - *Turan Graph, $T_3(7)$*



Theorem 4.18 - *Degree of Vertex Classes in Turán Graph*
In the *Turán Graph* $T_k(n)$ each vertex class has size: $\lfloor \frac{n}{k} \rfloor$; or $\lceil \frac{n}{k} \rceil$.

Theorem 4.19 - *Degree of Vertices in Turán Graph*
Consider a *Turán Graph* $T_k(n)$ with vertex classes $\{V_1, \dots, V_k\}$. Then

$$\forall \ V_i \ \forall \ x \in V_i, \ deg_G(x) = |V| - |V_i|$$

Theorem 4.20 - *Number of Edges in a Turán Graph*
Consider the *Turán Graph* $T_k(n) = (V, E)$ then

$$\max |E| = \left\lfloor \frac{(k-1)n^2}{2k} \right\rfloor$$

Theorem 4.21 - *Turán's Theorem*
Consider a graph $G = (V, E)$ with $n := |V|$ with no *k-Cliques*. Then

$$|E| \leq |E(T_{k-1}(n))|$$

Theorem 4.22 - *Distance in Bounded Diameter Sets*
Let $\{x_1, \dots, x_n\}$ be a finite set of points in a circular plane of diameter ≤ 1. Then the maximum number of pairs of points whose distance exceeds $\frac{1}{\sqrt{2}}$ is $\lfloor \frac{n^2}{3} \rfloor$.

4.5 Planar Graphs

Definition 4.36 - *Arc*
An *Arc* is a line in the $2D$ plane which does not intersect itself. An *Arc* can be described by an injective, continuous map $\gamma : [0, 1] \to \mathbb{R}^2$. $\gamma(0)$ & $\gamma(1)$ are endpoints of the arc.

Definition 4.37 - *Drawing of a Graph*
A *Drawing* of a graph $G = (V, E)$ is an assignment of the following form

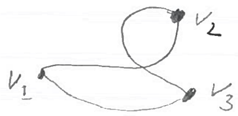
i) $\forall\ v \in V$ assign a unique point, p_v, in the plane, in such a way that the map $v \mapsto p_v$ is injective.

ii) $\forall\ e =: \{u, v\} \in E$ assign an arc γ_e in the plane whose endpoints as p_u & p_v and does not pass through any other points p_u with $u \in V$.

Definition 4.38 - *Planar Drawing*
A *Planar Drawing* of a graph $G = (V, E)$ is a drawing of G where any two arcs, corresponding to edges, either have no intersections or only share an endpoint. A *Planar Graph* is a graph which can be drawn in the $2D$ plane, with no edges intersecting.

Definition 4.39 - *Planar Graph*
A *Planar Graph* is a *Graph* which admits at least one *Planar Drawing.*

Example 4.4 - *Non-Planar Drawing of K_3*
Below is a *Drawing* of K_3 on the vertices $\{v_1, v_2, v_3\}$ which is not planar.



Definition 4.40 - *Jordan Curve*
A *Jordan Curve* is a closed arc. This is a non-intersecting, closed curve.

Theorem 4.23 - *Jordan Curve Theorem*
Any *Jordan Curve*, C, divides the plane into precisely two connected parts. These parts are called the interior & exterior. C is called the boundary of both regions.

Proposition 4.3 - *If G contains a non-planar sub-graph, then G is non-planar*

Definition 4.41 - *Sub-Division*
Consider a graph G, G' is a *Sub-Division* of G if the additional vertices it includes partition the edges of E. G' contains the partitions of the edges, but not the edges themselves.

Theorem 4.24 - *Kuratowski's Theorem*
A graph is *Planar* $\Leftrightarrow$ Every sub-division of G is planar.

Definition 4.42 - *Face of a Planar Graph*
Consider the set of all points which do not lie on any arc of a *Planar Drawing.* This set consists of a finite number of connected regions. Each of these regions is called a *Face* of the planar Graph.

Theorem 4.25 - *Euler's Theorem*
Consider a connected graph $G = (V, E)$ with faces F being the set of faces of a *Planar Drawing* of G. Then

$$|V| - |E| + |F| = 2$$

N.B. - The number of faces does not depend upon how the *Planar Drawing* is drawn.

Theorem 4.26 - *Planar Graph's have Few Edges*
Consider a connected planar graph $G = (V, E)$ with $|V| \geq 3$. Then

$$|E| \leq 3|V| - 6$$

4.6 Graph Colouring

Definition 4.43 - *k-Colouring*
A *k-Colouring* of a graph G is an assignment of one of k colours to each vertex st no adjacent vertices share a colour.

Definition 4.44 - *Chromatic Number*
The *Chromatic Number* of a graph G is the lowest $k \in \mathbb{N}$ for which there is a valid *k-Colouring* of G.

Theorem 4.27 - *Chromatic Number & Max Degree*
For any graph $G = (V, E)$ we have

$$\chi(G) \leq \Delta(G) + 1$$

N.B. - $\forall\ k \in \mathbb{N}$ with $k \geq \Delta(G) \implies \chi(G) \leq k + 1 \implies G$ is $k+1$-Colourable.

Theorem 4.28 - *Five-Colour Theorem*
At most five colours are required to colour a map st no two adjacent territories share a colour.

Proposition 4.4 - *Assumptions for Five-Colour Theorem*
We make the following assumptions about the *maps* referred to in the *Five-Colour Theorem*

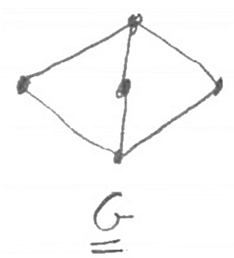
i) Each state is a single connected region;

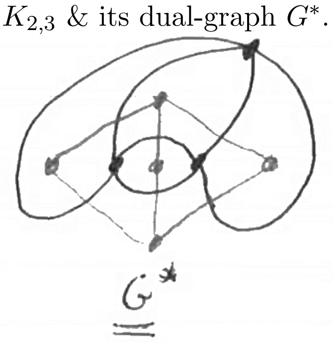
ii) Two states are neighbours iff they share a continuous interval of a border (*i.e.* If two states just meet at a point they are not neighbours & no pair of states share two borders).

Proposition 4.5 - *Map as a Planar Drawing*
We can consider a map to be a planar drawing of a graph where the faces correspond to states & edges correspond to borders between them.

Definition 4.45 - *Dual Graph*
Consider a planar graph $G = (V, E)$ along with its planar drawing. The dual graph $G^* = (V^*, E^*)$ of the planar drawing of G is obtained by drawing a vertex inside each face of the planar drawing & an edge between the vertices which correspond to adjacent faces.
N.B. - We can ignore duplicate edges in a *Dual Graph*.

Example 4.5 - *Dual-Graph*
Below is a planar drawing of $G = K_{2,3}$ & its dual-graph G^*.





Proposition 4.6 - *Using Dual-Graphs*
Consider a map G & its dual-graph G^*. A valid colouring of the faces of G corresponds to a valid colouring of the vertices of G^*. Thus there is a k-colouring of G iff there is a k-colouring of G^*.
N.B. - Thus proving the *Five-Colour Theorem* is equivalent to showing that every planar graph s 5-colourable.

4.7 Ramsey Numbers

Definition 4.46 - *Ramsey Number*
The *Ramsey Number* for $s \in \mathbb{N}^{\geq 2}$, $r(s)$, is the least $n \in \mathbb{N}$ st $\forall$ 2-colourings of the edges of K_n $\exists$ a *Monochromatic* K_s subgraph.
N.B. - $\forall\ s \in \mathbb{N}^{\geq 2},\ r(s)$ exists.

Definition 4.47 - *Off-Diagonal Ramsey Number*
The *Off-Diagonal Ramsey Number* for $s, t \in \mathbb{N}$, $r(s,t)$ is the least $n \in \mathbb{N}$ st $\forall$ 2-Colourings of the edges of K_n $\exists$ a *Monochromatic* K_s **or** K_t subgraph.

Theorem 4.29 - *Off-Diagonal Ramsey Number Identities*

$$\begin{array}{rcll} r(s,s) & = & r(s) & \forall\ s \in \mathbb{N} \\ r(s,t) & = & r(t,s) & \forall\ s,t \in \mathbb{N} \\ r(2,t) & = & t & \forall\ t \in \mathbb{N} \end{array}$$

Theorem 4.30 - *Ramsey's Theorem*
The *Off-Diagonal Ramsey Number* $r(s,t)$ exists $\forall\ s,t \in \mathbb{N}^{\geq 2}$. Moreover,

$$r(s,t) \leq r(s-1,t) + r(s,t-1)\ \forall\ s,t \in \mathbb{N}^{\geq 3}$$

Theorem 4.31 - *Upper Bound on Ramsey Number*
$\forall\ s,t \geq 2$ we have

$$r(s,t) \leq 2^{s+t}$$

Equivalently, $r(s) < 4^s$.
Further

$$r(s) \leq r(s-1,s) + r(s,s-1) = 2r(s-1,s)\ \forall\ s \in \mathbb{N}^{\geq 3}$$

If $r(s-1,t)$ & $r(s,t-1)$ are both even then

$$r(s,t) < r(s-1,t) + r(s,t-1)$$

N.B. - This is a strict inequality.

0 Reference

0.1 Definition

Definition 0.1 - *Linearly Independent Vertices*
A vector v is linearly independent of vertices $u_1, \dots, u_n$ iff

$$\nexists\ a_1, \dots, a_n \in \mathbb{R}\ st\ v = \sum_{i=1}^{n} a_i u_i$$

Definition 0.2 - *Rank of Matrix*
The *Rank* of a *Matrix* is the number of linearly independent columns in the matrix.

0.2 Notation

Notation 0.1 - *Binomial Coefficient*
Let $n, k \in \mathbb{N}$. We denote the number of k-element subsets of an n-element set as

$$\binom{n}{k}$$

Notation 0.2 - *Chromatic Number, $\chi(G)$*
$\chi(G)$ denotes the *Chromatic Number* of G.

Notation 0.3 - *Complete Graph, K_n*
We denote the *Complete Graph* on n vertices as K_n.

Notation 0.4 - *Complete Bipartite Graph, $K_{m,n}$*
We denote the *Complete Bipartite Graph* with vertex classes of sizes n & m as $K_{m,n}$.

Notation 0.5 - *Consecutive Numbers, $[n]$*
For $n \in \mathbb{N}$ we have $[n] := \{1, 2, \dots, n\}$.

Notation 0.6 - *Generating Function*
Let $f(x)$ be the generating function of sequence $(a_1, a_2, \dots)$ we denote this as

$$f(x) \leftrightarrows (a_1, a_2, \dots)$$

Notation 0.7 - *Minimum & Maximum Degree*

$$\delta(G) := \min\{deg_G(x) : x \in V\} \quad \& \quad \Delta(G) := \max\{deg_G(x) : x \in V\}$$

Notation 0.8 - $(n)_k$
$(n)_k$ denote the product of k natural numbers less than n, including n.

$$(n)_k := n \times (n-1) \times \cdots \times (n-k+1) = \frac{n!}{(n-k)!}$$