Data Structures & Algorithms - Problem Sheet 2

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My Solution 1

$$10^{100},\ 3\log_2 n,\ \sqrt{n},\ n\log_2 n,\ n^2,\ 3(\log_2 n)^5,\ n^4-10n^2,\ n2^n+n,\ n3^n,\ n!-14$$

My Solution 2

$$\begin{array}{rrcl}
 & f(0) & = & 0 \\
 & f(1) & = & 1000 \\
 & f(n) & = & 2f(n-1)+3f(n-2)+1000 \\
\text{Set} & \lambda^2 & = & 2\lambda+3 \\
\implies & \lambda^2-2\lambda-3 & = & 0 \\
 & (\lambda-3)(\lambda+1) & = & 0 \\
\implies & \lambda_1=-1 & \& & \lambda_2=3 \\
\text{Set} & f(n) & = & \alpha(-1)^n+\beta 3^n+\gamma \\
 & & & \\
\text{Since} & f(n) & = & 2f(n-1)+3f(n-2)+1000 \\
 & & = & 2[\alpha(-1)^{n-1}+\beta 3^{n-1}+\gamma]+3[\alpha(-1)^{n-2}+\beta 3^{n-2}+\gamma]+1000 \\
 & & = & 2\alpha(-1)^{n-1}+2\beta 3^{n-1}+2\gamma-3\alpha(-1)^{n-1}+\beta 3^{n-1}+3\gamma+1000 \\
 & & = & -\alpha(-1)^{n-1}+\beta 3^n+5\gamma+1000 \\
 & & = & \alpha(-1)^n+\beta 3^n+5\gamma+1000 \\
 & & = & f(n)+4\gamma+1000 \\
\implies & \gamma & = & \underline{-250} \\
 & & & \\
 & \text{From conditions} & & \\
 & f(0) & = & 0 \\
\implies & \alpha+\beta-250 & = & 0 \\
\implies & \beta & = & 250-\alpha \\
 & f(1) & = & 1000 \\
\implies & -\alpha+3\beta-250 & = & 1000 \\
\implies & -\alpha+3(250-\alpha) & = & 1250 \\
\implies & -4\alpha & = & 500 \\
\implies & \alpha & = & \underline{-125} \\
\implies & \beta & = & 250-(-125) \\
 & & = & \underline{375} \\
 & & & \\
\text{So} & f(n) & = & 125(-1)^{n+1}+375(3^n)-250
\end{array}$$

When $n=9$, $f(9)$ is greater than the population of Albanian.

My Solution 3

$$\begin{array}{rcl} T_1(n) & = & 10T_1(\lceil \frac{n}{10} \rceil) + 100n \\ T_2(n) & = & T_2(\lfloor \frac{9n}{10} \rfloor) + T_2(\lceil \frac{n}{10} \rceil) + \frac{n^{3/2}}{100} \end{array}$$

Analysing T_1 with the *Master Theorem.*

$$a = 10,\ b = 10,\ f(n) \in O(n) \implies c = 1$$

Let $p = log_b a \implies p = log_{10} 10 = 1 = c \implies T_1 \in \Theta(n \log_2 n)$.
Analysing T_2 with the *Akra-Bazzi Formula.*

$$k = 2,\ a_1 = 1,\ a_2 = 1,\ d_1 = \frac{9}{10},\ d_2 = \frac{1}{100},\ \alpha = 100,\ f(n) \in O(n^{3/2}) \implies c = \frac{3}{2}$$

Set

$$\sum_{i=1}^{k} a_i d_i^p = 1 \implies \left(\frac{9}{10}\right)^p + \left(\frac{1}{10}\right)^p = 1 \implies p = 1 < c$$

So $T_2(n) \in \Theta(n^{3/2})$.
Since $T_1 \in \Theta(n \log_2 n)$ & $T_2(n) \in \Theta(n^{3/2})$ then T_1 scales better than T_2.
I would recommend using T_1.

My Solution 4.1
Analysing with *Master Theorem*

$$a = 27,\ b = 3,\ b \in O(n) \implies c = 1$$

Let $p = \log_b a = log_3 27 = 3 \implies c < p$.
Thus $T(n) \in \Theta(n^3)$.

My Solution 4.2
Analysing with *Akra-Bazzi Formula*

$$k = 2,\ a_1 = 2,\ a_2 = 1,\ d_1 = \frac{1}{4},\ d_2 = \frac{1}{4},\ \alpha = 15,\ c = 2$$

This obeys all conditions of *Akra-Bazzi Formula.*
Set

$$\begin{array}{rcrcl} \sum_{i=1}^{2} a_i d_i^p = 1 & \implies & 2(\frac{1}{4})^p + (\frac{1}{4}^p) & = & 1 \\ & \implies & (\frac{1}{4})^p & = & \frac{1}{3} \\ & \implies & p < 1 < 2 & = & 2 \\ & \implies & p & < & c \end{array}$$

Thus $T(n) \in \Theta(n^2)$.

My Solution 4.3
Analysing with *Akra-Bazzi Formula*

$$k = 3,\ a_1 = \frac{1}{2},\ a_2 = \frac{1}{3},\ a_3 = \frac{1}{6},\ d_1 = \frac{1}{3},\ d_2 = \frac{1}{3},\ d_3 = \frac{1}{6},\ \alpha = 6,\ c = 0$$

This obeys all conditions of *Akra-Bazzi Formula.*
Set

$$\begin{array}{rcrcl} \sum_{i=1}^{2} a_i d_i^p = 1 & \implies & \frac{1}{2}(\frac{1}{4})^p + \frac{1}{3}(\frac{1}{3})^p + \frac{1}{6}(\frac{1}{6})^p & = & 1 \\ & \implies & p & = & 0 \\ & \implies & p & = & c \end{array}$$

Thus $T(n) \in \Theta(n^0 \log_2 n) \equiv T(n) \in \Theta(\log_2 n)$.

My Solution 4.4
Analysing with *Akra-Bazzi Formula*

$$k = 2,\ a_1 = 3,\ a_2 = 1,\ d_1 = \frac{1}{2},\ d_2 = \frac{1}{5},\ \alpha = 1,\ c = 2$$

This obeys all conditions of *Akra-Bazzi Formula*.
Set

$$\begin{array}{lcrcl} \sum_{i=1}^{2} a_i d_i^p = 1 & \implies & 3(\frac{1}{2})^p + (\frac{1}{5})^p = 1 & & \\ \text{Suppose } p = 2 & \implies & 3(\frac{1}{4}) + \frac{1}{25} = \frac{79}{100} & < & 1 \\ \text{Suppose } p = 0 & \implies & 3 + 1 = 4 & & \\ & \implies & 0 < p < 2 & = & c \\ & \implies & p & < & c \end{array}$$

Thus $T(n) \in \Theta(n^2)$.

My Solution 5.1
Let $n = 6m + 1$.
There are $2m$ full groups & one group with 1 element.
If x is in the solo group then it is the median of its group. Then there are m groups with medians less than x, each of which can have only 1 element greater than x.
And there are m groups with medians greater than x, each of these can have 3 elements greater than x.
This means there are at most $1(m) + 3(m) = 4m$ elements greater than x.

My Solution 5.2

My Solution 6
Since $f(n) \in O(g_1(n))\ \exists\ n_0 \in \mathbb{N},\ c \in \mathbb{R}^{>0}\ st\ 0 \leq f(n) \leq cg_1(n)\ \forall\ b \geq n_0$.
And, since $g_1, g_2 \in \Theta(j(n))\ \exists\ n_1 \in \mathbb{N},\ d_1, d_2 \in \mathbb{R}^{>0}\ st\ 0 < d_1 j(n) \leq g_1(n), g_2(n) \leq d_2 j(n)$.
Then

$$\begin{array}{rclcl} d_1 j(n) & \leq & g_2(n) + f(n) & \leq & cg_1(n) + d_2 j(n) \\ & & & \leq & d_2(1+c)j(n) \\ d_2(1+c) > 0 & \implies & f(n) + g_2(n) & \in & \Theta(j(n)) \end{array}$$

My Solution 7.1
There are 3 elements in x's group which are less than x.
There are $\frac{1}{2}\lceil\frac{n}{7}\rceil - 2$ full groups with medians greater than x.
Each of these has at least 4 elemetns greater than x.
So there are at least $4(\frac{1}{2}\lceil\frac{n}{7}\rceil - 2)$ elements greater than x.
So there are at most $n - 4(\frac{1}{2}\lceil\frac{n}{7}\rceil - 2)$ elements less than x.

My Solution 7.2
There are 3 elements in x's group which are greater than x.
There are $\frac{1}{2}\lceil\frac{n}{7}\rceil - 2$ full groups with medians less than x.
Each of these has at least 4 elements less than x.
So at least $4(\frac{1}{2}\lceil\frac{n}{7}\rceil - 2)$ elements are less than x.
So there are at most $n - 4(\frac{1}{2}\lceil\frac{n}{7}\rceil - 2)$ greater than x.

My Solution 7.3

$$\begin{aligned} n-2(\lceil \tfrac{n}{7} \rceil - 2) &\leq n-2((\tfrac{n}{7})-2) \\ &= \tfrac{5n}{7}+4 \ \forall \ n \in \mathbb{N} \end{aligned}$$

$n-2(\lceil \frac{n}{7} \rceil - 2)$ is bounded above by $\frac{5n}{7}+4$.

$$\lfloor \tfrac{11n}{14} \rfloor \geq \tfrac{11x}{14} - 1$$

$\lfloor \frac{11n}{14} \rfloor$is bounded below by $\frac{11n}{14}-1$.

$$\begin{aligned} \text{Set} \quad \tfrac{11n}{14}-1 &\geq \tfrac{5n}{7}+4 \\ \implies \quad \tfrac{n}{14} &\geq 5 \\ \implies \quad n &\geq 70 \end{aligned}$$

$n-2(\lceil \frac{n}{7} \rceil - 2)$ is bounded above by $\frac{11n}{14}-1$. Thus $\exists \ n_0 \in \mathbb{N}$ st $n-2(\lceil \frac{n}{7} \rceil - 2)$ is bounded above & $\lfloor \frac{11n}{14} \rfloor$ is bounded below by $\frac{11n}{14}-1$, thus $n-2(\lceil \frac{n}{7} \rceil - 2) \leq \lfloor \frac{11n}{14} \rfloor \ \forall \ n > n_0$.