Data Structures & Algorithms - Problem Sheet 2

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My Solution 1

$$10^{100},\ 3\log_2 n,\ \sqrt{n},\ n\log_2 n,\ n^2,\ 3(\log_2 n)^5,\ n^4-10n^2,\ n2^n+n,\ n3^n,\ n!-14$$

My Solution 2
TODO

$$\begin{aligned} f(0) &= 0 \\ f(1) &= 1000 \\ f(2) &= 4000 \\ f(n) &= 3[f(n-1)-f(n-2)]+2[f(n-2)-f(n-3)]+1000 \\ &= 3f(n-1)-f(n-2)-2f(n-3)+1000 \end{aligned}$$

My Solution 3

$$\begin{aligned} T_1(n) &= 10T_1(\lceil \tfrac{n}{10} \rceil)+100n \\ T_2(n) &= T_2(\lfloor \tfrac{9n}{10} \rfloor)+T_2(\lceil \tfrac{n}{10} \rceil)+\tfrac{n^{3/2}}{100} \end{aligned}$$

Analysing T_1 with the *Master Theorem.*

$$a=10,\ b=10,\ f(n)\in O(n) \implies c=1$$

Let $p=log_b a \implies p=log_{10}10=1=c \implies T_1\in\Theta(n\log_2 n)$.
Analysing T_2 with the *Akra-Bazzi Formula.*

$$k=2,\ a_1=1,\ a_2=1,\ d_1=\frac{9}{10},\ d_2=\frac{1}{100},\ \alpha=100,\ f(n)\in O(n^{3/2}) \implies c=\frac{3}{2}$$

Set

$$\sum_{i=1}^{k} a_i d_i^p = 1 \implies \left(\frac{9}{10}\right)^p+\left(\frac{1}{10}\right)^p=1 \implies p=1<c$$

So $T_2(n)\in\Theta(n^{3/2})$.
Since $T_1\in\Theta(n\log_2 n)$ & $T_2(n)\in\Theta(n^{3/2})$ then T_1 scales better than T_2.
I would recommend using T_1.

My Solution 4.1
Analysing with *Master Theorem*

$$a=27,\ b=3,\ b\in O(n) \implies c=1$$

Let $p=\log_b a=log_3 27=3 \implies c<p$.
Thus $T(n)\in\Theta(n^3)$.

My Solution 4.2
Analysing with *Akra-Bazzi Formula*

$$k=2,\ a_1=2,\ a_2=1,\ d_1=\frac{1}{4},\ d_2=\frac{1}{4},\ \alpha=15,\ c=2$$

This obeys all conditions of *Akra-Bazzi Formula.*
Set

$$\begin{array}{rcrcl} \sum_{i=1}^{2} a_i d_i^p = 1 & \implies & 2(\frac{1}{4})^p + (\frac{1}{4}^p) & = & 1 \\ & \implies & (\frac{1}{4})^p & = & \frac{1}{3} \\ & \implies & p < 1 < 2 & = & 2 \\ & \implies & p & < & c \end{array}$$

Thus $T(n) \in \Theta(n^2)$.

My Solution 4.3
Analysing with *Akra-Bazzi Formula*

$$k=3,\ a_1=\frac{1}{2},\ a_2=\frac{1}{3},\ a_3=\frac{1}{6},\ d_1=\frac{1}{3},\ d_2=\frac{1}{3},\ d_3=\frac{1}{6},\ \alpha=6,\ c=0$$

This obeys all conditions of *Akra-Bazzi Formula.*
Set

$$\begin{array}{rcrcl} \sum_{i=1}^{2} a_i d_i^p = 1 & \implies & \frac{1}{2}(\frac{1}{4})^p + \frac{1}{3}(\frac{1}{3})^p + \frac{1}{6}(\frac{1}{6})^p & = & 1 \\ & \implies & p & = & 0 \\ & \implies & p & = & c \end{array}$$

Thus $T(n) \in \Theta(n^0 \log_2 n) \equiv T(n) \in \Theta(\log_2 n)$.

My Solution 4.4
Analysing with *Akra-Bazzi Formula*

$$k=2,\ a_1=3,\ a_2=1,\ d_1=\frac{1}{2},\ d_2=\frac{1}{5},\ \alpha=1,\ c=2$$

This obeys all conditions of *Akra-Bazzi Formula.*
Set

$$\begin{array}{rcrcl} \sum_{i=1}^{2} a_i d_i^p = 1 & \implies & 3(\frac{1}{2})^p + (\frac{1}{5})^p = 1 & & \\ \text{Suppose } p=2 & \implies & 3(\frac{1}{4}) + \frac{1}{25} = \frac{79}{100} & < & 1 \\ \text{Suppose } p=0 & \implies & 3+1=4 & & \\ & \implies & 0 < p < 2 & = & c \\ & \implies & p & < & c \end{array}$$

Thus $T(n) \in \Theta(n^2)$.

My Solution 5.1
Let $n = 6m+1$.
There are $2m$ full groups & one group with 1 element.
If x is in the solo group then it is the median of its group. Then there are m groups with medians less than x, each of which can have only 1 element greater than x.
And there are m groups with medians greater than x, each of these can have 3 elements greater than x.
This means there are at most $1(m)+3(m)=4m$ elements greater than x.

My Solution 5.2

My Solution 6

Since $f(n) \in O(g_1(n))\ \exists\ n_0 \in \mathbb{N},\ c \in \mathbb{R}^{>0}\ st\ 0 \leq f(n) \leq cg_1(n)\ \forall\ b \geq n_0$.
And, since $g_1, g_2 \in \Theta(j(n))\ \exists\ n_1 \in \mathbb{N},\ d_1, d_2 \in \mathbb{R}^{>0}\ st\ 0 < d_1 j(n) \leq g_1(n), g_2(n) \leq d_2 j(n)$.
Then

$$\begin{array}{rcccl} & d_1 j(n) & \leq & g_2(n) + f(n) & \leq & cg_1(n) + d_2 j(n) \\ & & & & \leq & d_2(1+c)j(n) \\ d_2(1+c) > 0 & \implies & f(n) + g_2(n) & \in & \Theta(j(n)) \end{array}$$

My Solution 7.1

There are 3 elements in x's group which are less than x.
There are $\frac{1}{2}\lceil\frac{n}{7}\rceil - 2$ full groups with medians greater than x.
Each of these has at least 4 elemetns greater than x.
So there are at least $4(\frac{1}{2}\lceil\frac{n}{7}\rceil - 2)$ elements greater than x.
So there are at most $n - 4(\frac{1}{2}\lceil\frac{n}{7}\rceil - 2)$ elements less than x.

My Solution 7.2

There are 3 elements in x's group which are greater than x.
There are $\frac{1}{2}\lceil\frac{n}{7}\rceil - 2$ full groups with medians less than x.
Each of these has at least 4 elements less than x.
So at least $4(\frac{1}{2}\lceil\frac{n}{7}\rceil - 2)$ elements are less than x.
So there are at most $n - 4(\frac{1}{2}\lceil\frac{n}{7}\rceil - 2)$ greater than x.

My Solution 7.3

$$\begin{array}{rcl} n - 2(\lceil\frac{n}{7}\rceil - 2) & \leq & n - 2((\frac{n}{7}) - 2) \\ & = & \frac{5n}{7} + 4\ \forall\ n \in \mathbb{N} \end{array}$$

$n - 2(\lceil\frac{n}{7}\rceil - 2)$ is bounded above by $\frac{5n}{7} + 4$.

$$\lfloor\frac{11n}{14}\rfloor \quad \geq \quad \frac{11x}{14} - 1$$

$\lfloor\frac{11n}{14}\rfloor$is bounded below by $\frac{11n}{14} - 1$.

$$\begin{array}{rrcl} \text{Set} & \frac{11n}{14} - 1 & \geq & \frac{5n}{7} + 4 \\ \implies & \frac{n}{14} & \geq & 5 \\ \implies & n & \geq & 70 \end{array}$$

$n - 2(\lceil\frac{n}{7}\rceil - 2)$ is bounded above by $\frac{11n}{14} - 1$. Thus $\exists\ n_0 \in \mathbb{N}$ st $n - 2(\lceil\frac{n}{7}\rceil - 2)$ is bounded above & $\lfloor\frac{11n}{14}\rfloor$ is bounded below by $\frac{11n}{14} - 1$, thus $n - 2(\lceil\frac{n}{7}\rceil - 2) \leq \lfloor\frac{11n}{14}\rfloor\ \forall\ n > n_0$.