

# Finance Mathematics - Problem Sheet 4

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## Answer 2.

Consider time  $t = 0$  and  $p_1 := \mathbb{Q}(S_1 = 5)$ , then

$$\begin{aligned} 6 &= 5p_1 + 7(1 - p_1) \\ &= 7 - 2p_1 \\ \implies p_1 &= 1/2 \end{aligned}$$

Thus  $\mathbb{Q}(S_1 = 5) = 1/2$  and  $\mathbb{Q}(S_1 = 7) = 1/2$ .

Now consider time  $t = 1$ , events  $\{\omega_4, \omega_5\}$  and  $p_2 := \mathbb{Q}(S_2 = 6|S_1 = 7)$ , then

$$\begin{aligned} 7 &= 6p_2 + 8(1 - p_2) \\ &= 8 - 2p_2 \\ \implies p_2 &= 1/2 \end{aligned}$$

Thus  $\mathbb{Q}(S_2 = 6|S_1 = 7) = 1/2$  and  $\mathbb{Q}(S_2 = 8|S_1 = 7) = 1/2$ .

Now consider time  $t = 1$ , events  $\{\omega_1, \omega_2, \omega_3\}$ ,  $p_3 := \mathbb{Q}(S_2 = 3|S_1 = 5)$  and  $p_4 := \mathbb{Q}(S_2 = 4|S_1 = 5)$ , then

$$\begin{aligned} 6 &= 3p_3 + 4p_4 + 8(p_3 - p_4) \\ &= 11p_3 - 4p_4 \\ \implies p_4 &= \frac{1}{4}(11p_3 - 6) \end{aligned}$$

From this we can deduce that  $\mathbb{Q}(S_2 = 3|S_1 = 5) = x$ ,  $\mathbb{Q}(S_2 = 4|S_1 = 5) = \frac{1}{4}(11x - 6)$  and  $\mathbb{Q}(S_2 = 8|S_1 = 5) = 1 - x - \frac{1}{4}(11x - 6) = \frac{1}{4}(10 - 15x)$ .

It is clear that these values sum to 1 but we need to restrict the values of  $x$  so that each of these probabilities is in  $(0, 1)$ .

$$\begin{aligned} \implies \frac{1}{4}(11x - 6) &\in (0, 1) & \implies \frac{1}{4}(10 - 15x) &\in (0, 1) \\ \implies 11x - 6 &\in (0, 4) & \implies 10 - 15x &\in (0, 4) \\ \implies x &\in \left(\frac{6}{11}, \frac{10}{11}\right) & \implies x &\in \left(\frac{6}{15}, \frac{10}{15}\right) \end{aligned}$$

Thus we need to restrict  $x$  st

$$\begin{aligned} x &\in \left\{ (0, 1) \cap \left(\frac{6}{11}, \frac{10}{11}\right) \cap \left(\frac{2}{5}, \frac{2}{3}\right) \right\} \\ &= \left(\frac{6}{11}, \frac{2}{3}\right) \end{aligned}$$

We can use these conditional probabilities to determine the probability of each event  $\{\omega_1, \dots, \omega_5\}$  under  $\mathbb{Q}$

$$\begin{aligned}
 \mathbb{Q}(\{\omega_1\}) &= \mathbb{Q}(S_2 = 3|S_1 = 5)\mathbb{Q}(S_1 = 5) \\
 &= \frac{x}{2} \\
 \mathbb{Q}(\{\omega_2\}) &= \mathbb{Q}(S_2 = 4|S_1 = 5)\mathbb{Q}(S_1 = 5) \\
 &= \frac{1}{8}(11x - 6) \\
 \mathbb{Q}(\{\omega_3\}) &= \mathbb{Q}(S_2 = 8|S_1 = 5)\mathbb{Q}(S_1 = 5) \\
 &= \frac{1}{8}(10 - 15x) \\
 \mathbb{Q}(\{\omega_4\}) &= \mathbb{Q}(S_2 = 6|S_1 = 7)\mathbb{Q}(S_1 = 7) \\
 &= 1/2 \\
 \mathbb{Q}(\{\omega_5\}) &= \mathbb{Q}(S_2 = 8|S_1 = 7)\mathbb{Q}(S_1 = 7) \\
 &= 1/2
 \end{aligned}$$

for  $x \in \left(\frac{6}{11}, \frac{2}{3}\right)$ .

### Answer 3. (a)

For  $\mathbb{Q}$  to be a martingale measure it needs to fulfil  $\mathbb{Q}(\{\omega\}) > 0 \forall \omega \in \Omega$  (which it does by definition) and that

$$\mathbb{E}[S_n^*(t+s)|\mathcal{F}_t] = S_n(t) \forall s, t, n$$

As the interest rate in this model is  $r = 0$ , we have that  $S_n^*(t) = S_n(t) \forall t, n$ .

I now test whether this  $\mathbb{Q}$  fulfils this criteria for all  $t, s, n$ . It is trivial that this holds for cases where  $t = 2, s = 0$ .

$(t = 0, s = 1, n = 1)$	$  \begin{aligned}  \mathbb{E}_{\mathbb{Q}}[S_1(1) \mathcal{F}_0] &= \mathbb{E}[S_1(1)] \\  &= \frac{3}{9} \cdot 8 + \frac{4}{12} \cdot 7 + \frac{4}{12} \cdot 6 \\  &= 7 = S_1(0)  \end{aligned}  $
$(t = 0, s = 2, n = 1)$	$  \begin{aligned}  \mathbb{E}_{\mathbb{Q}}[S_1(2) \mathcal{F}_0] &= \mathbb{E}[S_1(2)] \\  &= \frac{1}{9}(7 + 8 + 9) + \frac{1}{6}(6 + 6) + \frac{1}{12}(6 + 10 + 3 + 9) \\  &= 7 = S_1(0)  \end{aligned}  $
$(t = 1, s = 1, n = 1)$	$  \begin{aligned}  \mathbb{E}_{\mathbb{Q}}[S_1(2) \omega] &= \frac{\frac{1}{9}(7 + 8 + 9)}{3 \cdot \frac{1}{9}} \text{ if } \omega \in \{\omega_1, \omega_2, \omega_3\} \\  &= 8 \\  &= S_1(1) \text{ if } \omega \in \{\omega_1, \omega_2, \omega_3\}  \end{aligned}  $
$(t = 1, s = 1, n = 1)$	$  \begin{aligned}  \mathbb{E}_{\mathbb{Q}}[S_1(2) \omega] &= \frac{\frac{1}{6} \cdot 6 + \frac{1}{12}(6 + 10)}{\frac{1}{6} + 2 \cdot \frac{1}{12}} \text{ if } \omega \in \{\omega_4, \omega_5, \omega_6\} \\  &= 7 \\  &= S_1(1) \text{ if } \omega \in \{\omega_4, \omega_5, \omega_6\}  \end{aligned}  $
$(t = 1, s = 1, n = 1)$	$  \begin{aligned}  \mathbb{E}_{\mathbb{Q}}[S_1(2) \omega] &= \frac{\frac{1}{6} \cdot 6 + \frac{1}{12}(3 + 9)}{\frac{1}{6} + 2 \cdot \frac{1}{12}} \text{ if } \omega \in \{\omega_7, \omega_8, \omega_9\} \\  &= 6 \\  &= S_1(1) \text{ if } \omega \in \{\omega_7, \omega_8, \omega_9\}  \end{aligned}  $

$(t = 0, s = 1, n = 2)$	$\mathbb{E}_{\mathbb{Q}}[S_2(1) \mathcal{F}_0] = \mathbb{E}[S_2(1)]$ $= \frac{3}{9} \cdot 5 + \frac{4}{12} \cdot 8 + \frac{4}{12} \cdot 5$ $= 6 = S_2(0)$
$(t = 0, s = 2, n = 2)$	$\mathbb{E}_{\mathbb{Q}}[S_2(2) \mathcal{F}_0] = \mathbb{E}[S_2(2)]$ $= \frac{1}{9}(7 + 5 + 3) + \frac{1}{6}(8 + 3) + \frac{1}{12}(9 + 7 + 8 + 6)$ $= 6 = S_1(0)$
$(t = 1, s = 1, n = 2)$	$\mathbb{E}_{\mathbb{Q}}[S_2(2) \omega] = \frac{\frac{1}{9}(7 + 5 + 3)}{3 \cdot \frac{1}{9}} \text{ if } \omega \in \{\omega_1, \omega_2, \omega_3\}$ $= 5$ $= S_2(1) \text{ if } \omega \in \{\omega_1, \omega_2, \omega_3\}$
$(t = 1, s = 1, n = 2)$	$\mathbb{E}_{\mathbb{Q}}[S_2(2) \omega] = \frac{\frac{1}{6} \cdot 8 + \frac{1}{12}(9 + 7)}{\frac{1}{6} + 2 \cdot \frac{1}{12}} \text{ if } \omega \in \{\omega_4, \omega_5, \omega_6\}$ $= 8$ $= S_2(1) \text{ if } \omega \in \{\omega_4, \omega_5, \omega_6\}$
$(t = 1, s = 1, n = 2)$	$\mathbb{E}_{\mathbb{Q}}[S_2(2) \omega] = \frac{\frac{1}{6} \cdot 3 + \frac{1}{12}(8 + 6)}{\frac{1}{6} + 2 \cdot \frac{1}{12}} \text{ if } \omega \in \{\omega_7, \omega_8, \omega_9\}$ $= 5$ $= S_2(1) \text{ if } \omega \in \{\omega_7, \omega_8, \omega_9\}$

**Answer 3. (b)**

First, consider the payout from each event

	$\omega_1$	$\omega_2$	$\omega_3$	$\omega_4$	$\omega_5$	$\omega_6$	$\omega_7$	$\omega_8$	$\omega_9$
$X(\omega)$	1	1	0	1	2	4	0	0	2

The fair-price for this option at time  $t = 0$  is

$$\begin{aligned}
 V_0 &= \mathbb{E}[X] \\
 &= \frac{1}{9}(1 + 1 + 0) + \frac{1}{6}(1 + 0) + \frac{1}{12}(2 + 4 + 2) \\
 &= \frac{2}{9} + \frac{1}{6} + \frac{8}{12} \\
 &= 19/18
 \end{aligned}$$

The fair-price for this option at time  $t = 1$  depends on which state the model is in

$$\begin{aligned}
 V_1(\omega) &= \begin{cases} \mathbb{E}[X|S_1 = 8, S_2 = 5] & \text{if } \omega \in \{\omega_1, \omega_2, \omega_3\} \\ \mathbb{E}[X|S_1 = 7, S_2 = 8] & \text{if } \omega \in \{\omega_4, \omega_5, \omega_6\} \\ \mathbb{E}[X|S_1 = 6, S_2 = 5] & \text{if } \omega \in \{\omega_7, \omega_8, \omega_9\} \end{cases} \\
 &= \begin{cases} \frac{\frac{1}{9}(1+1+0)}{\frac{1}{9} \cdot 3} & \text{if } \omega \in \{\omega_1, \omega_2, \omega_3\} \\ \frac{16(1) + \frac{1}{12}(2+4)}{\frac{1}{6} + \frac{1}{12} \cdot 2} & \text{if } \omega \in \{\omega_4, \omega_5, \omega_6\} \\ \frac{\frac{1}{6}0 + \frac{1}{12}(0+2)}{\frac{1}{6} + \frac{1}{12} \cdot 2} & \text{if } \omega \in \{\omega_7, \omega_8, \omega_9\} \end{cases} \\
 &= \begin{cases} 2/3 & \text{if } \omega \in \{\omega_1, \omega_2, \omega_3\} \\ 3/2 & \text{if } \omega \in \{\omega_4, \omega_5, \omega_6\} \\ 1/2 & \text{if } \omega \in \{\omega_7, \omega_8, \omega_9\} \end{cases}
 \end{aligned}$$