Finance Mathematics - Problem Sheet 5

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Answer 2.
Consider time $t = 0$ and $p_1 := \mathbb{Q}(S_1 = 5)$, then

$$\begin{array}{rcl} 6 &=& 5p_1 + 7(1-p_1) \\ &=& 7 - 2p_1 \\ \implies p_1 &=& 1/2 \end{array}$$

Thus $\mathbb{Q}(S_1 = 5) = 1/2$ and $\mathbb{Q}(S_1 = 7) = 1/2$.
Now consider time $t = 1$, events $\{\omega_4, \omega_5\}$ and $p_2 := \mathbb{Q}(S_2 = 6|S_1 = 7)$, then

$$\begin{array}{rcl} 7 &=& 6p_2 + 8(1-p_2) \\ &=& 8 - 2p_2 \\ \implies p_2 &=& 1/2 \end{array}$$

Thus $\mathbb{Q}(S_2 = 6|S_1 = 7) = 1/2$ and $\mathbb{Q}(S_2 = 8|S_1 = 7) = 1/2$.
Now consider time $t = 1$, events $\{\omega_1, \omega_2, \omega_3\}$, $p_3 := \mathbb{Q}(S_2 = 3|S_1 = 5)$ and $p_4 := \mathbb{Q}(S_2 = 4|S_1 = 5)$, then

$$\begin{array}{rcl} 6 &=& 3p_3 + 4p_4 + 8(p_3 - p_4) \\ &=& 11p_3 - 4p_4 \\ \implies p_4 &=& \frac{1}{4}(11p_3 - 6) \end{array}$$

From this we can deduce that $\mathbb{Q}(S_2 = 3|S_1 = 5) = x$, $\mathbb{Q}(S_2 = 4|S_1 = 5) = \frac{1}{4}(11x - 6)$ and $\mathbb{Q}(S_2 = 8|S_1 = 5) = 1 - x - \frac{1}{4}(11x - 6) = \frac{1}{4}(10 - 15x)$.
It is clear that these values sum to 1 but we need to restrict the values of x so that each of these probabilities is in $(0, 1)$.

$$\begin{array}{rcl} \frac{1}{4}(11x-6) &\in& (0,1) \\ \implies 11x - 6 &\in& (0,4) \\ \implies x &\in& \left(\frac{6}{11}, \frac{10}{11}\right) \end{array} \qquad \begin{array}{rcl} \frac{1}{4}(10-15x) &\in& (0,1) \\ \implies 10 - 15x &\in& (0,4) \\ \implies x &\in& \left(\frac{6}{15}, \frac{10}{15}\right) \end{array}$$

Thus we need to restrict x st

$$\begin{array}{rcl} x &\in& \left\{(0,1) \cap \left(\frac{6}{11}, \frac{10}{11}\right) \cap \left(\frac{2}{5}, \frac{2}{3}\right)\right\} \\ &=& \left(\frac{6}{11}, \frac{2}{3}\right) \end{array}$$

We can use these conditional probabilities to determine the probability of each event $\{\omega_1,\dots,\omega_5\}$ under $\mathbb{Q}$

$$\begin{array}{rcl}\mathbb{Q}(\{\omega_1\}) &=& \mathbb{Q}(S_2=3|S_1=5)\mathbb{Q}(S_1=5)\\ &=& \dfrac{x}{2}\\ \mathbb{Q}(\{\omega_2\}) &=& \mathbb{Q}(S_2=4|S_1=5)\mathbb{Q}(S_1=5)\\ &=& \dfrac{1}{8}(11x-6)\\ \mathbb{Q}(\{\omega_3\}) &=& \mathbb{Q}(S_2=8|S_1=5)\mathbb{Q}(S_1=5)\\ &=& \dfrac{1}{8}(10-15x)\\ \mathbb{Q}(\{\omega_4\}) &=& \mathbb{Q}(S_2=6|S_1=7)\mathbb{Q}(S_1=7)\\ &=& 1/2\\ \mathbb{Q}(\{\omega_5\}) &=& \mathbb{Q}(S_2=8|S_1=7)\mathbb{Q}(S_1=7)\\ &=& 1/2\end{array}$$

for $x\in\left(\dfrac{6}{11},\dfrac{2}{3}\right)$.

Answer 3. (a)

For $\mathbb{Q}$ to be a martingale measure it needs to fulfil $\mathbb{Q}(\{\omega\})>0\ \forall\ \omega\in\Omega$ (which it does by definition) and that

$$\mathbb{E}[S_n^*(t+s)|\mathcal{F}_t]=S_n(t)\ \forall\ s,t,n$$

As the interest rate in this model is $r=0$, we have that $S_n^*(t)=S_n(t)\ \forall\ t,n$.

I now test whether this $\mathbb{Q}$ fulfils this criteria for all t,s,n. It is trivial that this holds for cases where $t=2,s=0$.

$$\begin{array}{r|rcl}(t=0,s=1,n=1) & \mathbb{E}_\mathbb{Q}[S_1(1)|\mathcal{F}_0] &=& \mathbb{E}[S_1(1)]\\ & &=& \dfrac{3}{9}\cdot 8+\dfrac{4}{12}\cdot 7+\dfrac{4}{12}6\\ & &=& 7=S_1(0)\\ & & & \\ (t=0,s=2,n=1) & \mathbb{E}_\mathbb{Q}[S_1(2)|\mathcal{F}_0] &=& \mathbb{E}[S_1(2)]\\ & &=& \dfrac{1}{9}(7+8+9)+\dfrac{1}{6}(6+6)+\dfrac{1}{12}(6+10+3+9)\\ & &=& 7=S_1(0)\\ & & & \\ (t=1,s=1,n=1) & \mathbb{E}_\mathbb{Q}[S_1(2)|\omega] &=& \dfrac{\frac{1}{9}(7+8+9)}{3\cdot\frac{1}{9}}\text{ if }\omega\in\{\omega_1,\omega_2,\omega_3\}\\ & &=& 8\\ & &=& S_1(1)\text{ if }\omega\in\{\omega_1,\omega_2,\omega_3\}\\ & & & \\ (t=1,s=1,n=1) & \mathbb{E}_\mathbb{Q}[S_1(2)|\omega] &=& \dfrac{\frac{1}{6}\cdot 6+\frac{1}{12}(6+10)}{\frac{1}{6}+2\cdot\frac{1}{12}}\text{ if }\omega\in\{\omega_4,\omega_5,\omega_6\}\\ & &=& 7\\ & &=& S_1(1)\text{ if }\omega\in\{\omega_4,\omega_5,\omega_6\}\\ & & & \\ (t=1,s=1,n=1) & \mathbb{E}_\mathbb{Q}[S_1(2)|\omega] &=& \dfrac{\frac{1}{6}\cdot 6+\frac{1}{12}(3+9)}{\frac{1}{6}+2\cdot\frac{1}{12}}\text{ if }\omega\in\{\omega_7,\omega_8,\omega_9\}\\ & &=& 6\\ & &=& S_1(1)\text{ if }\omega\in\{\omega_7,\omega_8,\omega_9\}\end{array}$$

$$\begin{array}{l|lcl}
(t=0,s=1,n=2) & \mathbb{E}_{\mathbb{Q}}[S_2(1)|\mathcal{F}_0] & = & \mathbb{E}[S_2(1)] \\
& & = & \dfrac{3}{9}\cdot 5+\dfrac{4}{12}\cdot 8+\dfrac{4}{12}5 \\
& & = & 6=S_2(0) \\
\\
(t=0,s=2,n=2) & \mathbb{E}_{\mathbb{Q}}[S_2(2)|\mathcal{F}_0] & = & \mathbb{E}[S_2(2)] \\
& & = & \dfrac{1}{9}(7+5+3)+\dfrac{1}{6}(8+3)+\dfrac{1}{12}(9+7+8+6) \\
& & = & 6=S_1(0) \\
\\
(t=1,s=1,n=2) & \mathbb{E}_{\mathbb{Q}}[S_2(2)|\omega] & = & \dfrac{\frac{1}{9}(7+5+3)}{3\cdot\frac{1}{9}} \text{ if } \omega\in\{\omega_1,\omega_2,\omega_3\} \\
& & = & 5 \\
& & = & S_2(1) \text{ if } \omega\in\{\omega_1,\omega_2,\omega_3\} \\
\\
(t=1,s=1,n=2) & \mathbb{E}_{\mathbb{Q}}[S_2(2)|\omega] & = & \dfrac{\frac{1}{6}\cdot 8+\frac{1}{12}(9+7)}{\frac{1}{6}+2\cdot\frac{1}{12}} \text{ if } \omega\in\{\omega_4,\omega_5,\omega_6\} \\
& & = & 8 \\
& & = & S_2(1) \text{ if } \omega\in\{\omega_4,\omega_5,\omega_6\} \\
\\
(t=1,s=1,n=2) & \mathbb{E}_{\mathbb{Q}}[S_2(2)|\omega] & = & \dfrac{\frac{1}{6}\cdot 3+\frac{1}{12}(8+6)}{\frac{1}{6}+2\cdot\frac{1}{12}} \text{ if } \omega\in\{\omega_7,\omega_8,\omega_9\} \\
& & = & 5 \\
& & = & S_2(1) \text{ if } \omega\in\{\omega_7,\omega_8,\omega_9\}
\end{array}$$

First, consider the payout from each event

	ω_1	ω_2	ω_3	ω_4	ω_5	ω_6	ω_7	ω_8	ω_9
$X(\omega)$	1	1	0	1	2	4	0	0	2

The fair-price for this option at time $t=0$ is

$$\begin{array}{rcl}
V_0 & = & \mathbb{E}[X] \\
& = & \frac{1}{9}(1+1+0)+\frac{1}{6}(1+0)+\frac{1}{12}(2+4+2) \\
& = & \frac{2}{9}+\frac{1}{6}+\frac{8}{12} \\
& = & 19/18
\end{array}$$

The fair-price for this option at time $t=1$ depends on which state the model is in

$$\begin{array}{rcl}
V_1(\omega) & = & \begin{cases} \mathbb{E}[X|S_1=8,S_2=5] & \text{if } \omega\in\{\omega_1,\omega_2,\omega_3\} \\ \mathbb{E}[X|S_1=7,S_2=8] & \text{if } \omega\in\{\omega_4,\omega_5,\omega_6\} \\ \mathbb{E}[X|S_1=6,S_2=5] & \text{if } \omega\in\{\omega_7,\omega_8,\omega_9\} \end{cases} \\
& = & \begin{cases} \dfrac{\frac{1}{9}(1+1+0)}{\frac{1}{9}\cdot 3} & \text{if } \omega\in\{\omega_1,\omega_2,\omega_3\} \\ \dfrac{16(1)+\frac{1}{12}(2+4)}{\frac{1}{6}+\frac{1}{12}\cdot 2} & \text{if } \omega\in\{\omega_4,\omega_5,\omega_6\} \\ \dfrac{\frac{1}{6})0+\frac{1}{12}(0+2)}{\frac{1}{6}+\frac{1}{12}\cdot 2} & \text{if } \omega\in\{\omega_7,\omega_8,\omega_9\} \end{cases} \\
& = & \begin{cases} 2/3 & \text{if } \omega\in\{\omega_1,\omega_2,\omega_3\} \\ 3/2 & \text{if } \omega\in\{\omega_4,\omega_5,\omega_6\} \\ 1/2 & \text{if } \omega\in\{\omega_7,\omega_8,\omega_9\} \end{cases}
\end{array}$$