

Finance Mathematics - Problem Sheet 7

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Answer 1.

Define the probability q st

$$q = \frac{1 + r - d}{u - d} = 1 + 0.01 - 0.91 \cdot 1 - 0.9 = \frac{11}{20}$$

For this European call option, we have that the payoff function is

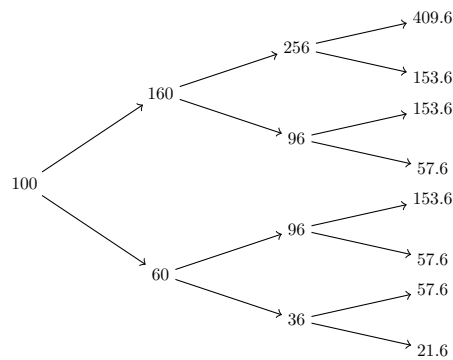
$$g(x) = \{x - e\}_+ = \{x - 1000\}_+$$

Under the Cox-Ross-Rubinstein model we have the time $t = 0$ price of this option is

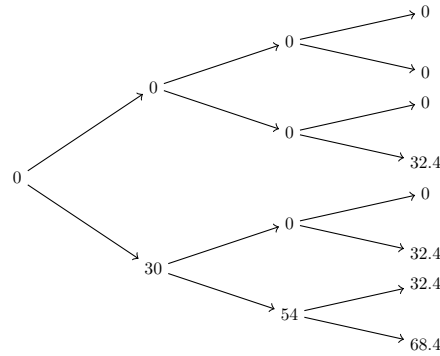
$$\begin{aligned} \Pi_e(0) &:= \frac{1}{(1+r)^2} \left\{ \sum_{n=0}^T \binom{T}{n} q^n (1-q)^{T-n} g(S_0 u^n d^{T-n}) \right\} \\ &= \frac{1}{1.01^2} \left\{ \sum_{n=0}^2 \binom{2}{n} \left(\frac{11}{20}\right)^n \left(\frac{9}{20}\right)^{2-n} \left(S_0 \left(\frac{11}{10}\right)^n \left(\frac{9}{20}\right)^{2-n} - 1000 \right) \right\} \\ &= \frac{1}{1.01^2} \left\{ \binom{2}{0} \left(\frac{11}{20}\right)^2 \left(S_0 \left(\frac{11}{10}\right)^2 - 1000 \right) + \binom{2}{1} \frac{11}{20} \cdot \frac{9}{20} \left(S_0 \frac{11}{10} \cdot \frac{9}{10} - 1000 \right) \right. \\ &\quad \left. + \binom{2}{2} \left(\frac{9}{20}\right)^2 \left(S_0 \left(\frac{9}{20}\right)^2 - 1000 \right) \right\} \\ &= \left(\frac{1}{1.01^2} \right) \left(\frac{1}{400} \right) \cdot \left\{ 121 \left(S_0 \frac{121}{100} - 1000 \right) + 198 \left(S_0 \frac{99}{100} - 1000 \right) + 81 \left(S_0 \frac{81}{100} - 1000 \right) \right\} \end{aligned}$$

Answer 2.

Consider the tree below which shows the possible evolutions of the price process S_t for each time-point and event.



For this American put option we have that the pay-out is $Y_t(\omega) = \{90 - S_t(\omega)\}_+$ for each $t = 0, 1, 2, 3$ and $\omega \in \Omega$. I summarise the pay-outs in the tree below



We can calculate the probability value q where

$$q := \frac{1 + r - d}{u - d} = \frac{1 + 0.1 - 0.6}{1.6 - 0.6} = \frac{1}{2}$$

This means the probability of the price increasing and the probability price decreasing during each time period are both $1/2$.

I will now construct a Snell Envelope $\{Z_t\}$ for $\{Y_t\}$.

By definition $Z_3(\omega) = Y_3(\omega) \forall \omega \in \Omega$.

Let ω_{ud} denote the events where the price increased once and decreased once (in some order) during the first two time-steps and ω_{dd} denote the events where the price decreased twice during the first two time-steps. Note these are disjoint and that no other events can lead to a profitable payout.

Suppose $t = 2$ and ω_{ud} occurred. Then

$$\begin{aligned} \mathbb{E}[Z_3|\mathcal{F}_2] &= q \cdot 0 + (1 - q) \cdot 32.4 \\ &= \frac{1}{2}(0 + 32.4) \\ &= 16.2 \\ \implies Z_2(\omega_{ud}) &= \max\{Y_2(\omega_{ud}), \mathbb{E}[Z_3, \mathcal{F}_2]\} \\ &= \max\{0, 16.2\} \\ &= 16.2 \end{aligned}$$

Suppose $t = 2$ and ω_{dd} occurred. Then

$$\begin{aligned} \mathbb{E}[Z_3|\mathcal{F}_2] &= \frac{1}{2}(32.4 + 68.4) = 50.4 \\ \implies Z_2(\omega_{dd}) &= \max\{Y_2(\omega_{dd}), \mathbb{E}[Z_3, \mathcal{F}_2]\} \\ &= \max\{54, 50.4\} \\ &= 54 \end{aligned}$$

Let ω_u denote the events where the price increased during the first time-step and ω_d denote the events where the price decreases during the first time-step.

Suppose $t = 1$ and ω_u occurred. Then

$$\begin{aligned} \mathbb{E}[Z_2|\mathcal{F}_1] &= \frac{1}{2}(0 + 16.2) = 8.1 \\ \implies Z_1(\omega_u) &= \max\{Y_1(\omega_u), \mathbb{E}[Z_2, \mathcal{F}_1]\} \\ &= \max\{0, 8.1\} \\ &= 8.1 \end{aligned}$$

Suppose $t = 1$ and ω_d occurred. Then

$$\begin{aligned} \mathbb{E}[Z_2|\mathcal{F}_1] &= \frac{1}{2}(16.2 + 54) = 35.1 \\ \Rightarrow Z_1(\omega_d) &= \max\{Y_1(\omega_d), \mathbb{E}[Z_2, \mathcal{F}_1]\} \\ &= \max\{30, 35.1\} \\ &= 35.1 \end{aligned}$$

Suppose $t = 0$. Then

$$\begin{aligned} \mathbb{E}[Z_1|\mathcal{F}_0] &= \mathbb{E}[Z_1] \\ &= \frac{1}{2}(8.1 + 35.1) \\ &= 21.6 \\ \Rightarrow Z_0 &= \max\{Y_0, \mathbb{E}[Z_1, \mathcal{F}_0]\} \\ &= \max\{0, 21.6\} \\ &= 21.6 \end{aligned}$$

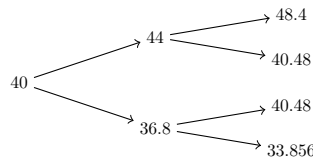
The time $t = 0$ fair-price of this American put is 21.6.

By inspecting the calculations of each Z_t above, the only time-point $t < 3$ where $Z_t(\omega) = Y_t(\omega)$ is when $t = 2$ and $\omega = \omega_{dd}$.

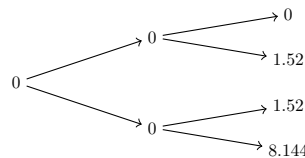
Thus, the only scenario where exercising early is optimal is when the price falls twice in the first two time-periods.

Answer 3. a)

Note that the interest rate for a three month period is $r = 0.03$. Consider the tree below which shows the possible evolutions of the price process S_t at each time point and event.



For this European put option we have that the pay-out is $Y_T(\omega) = g(S_T(\omega)) = \{42 - S_T(\omega)\}_+$ for each $\omega \in \Omega$ and 0 for all other t . I summarise the pay-outs in the tree below

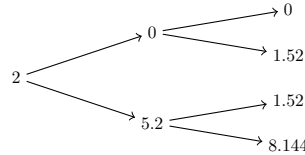


The time $t = 0$ value for this European put option is

$$\begin{aligned} \Pi(0) &= \frac{1}{(1+r)^T} \sum_{n=0}^T \binom{T}{n} q^n (1-q)^{T-n} g(S_0 u^n d^{T-n}) \\ &= \frac{1}{1.03^2} \sum_{n=0}^2 \binom{2}{n} \frac{11^n 7^{2-n}}{18^2} \{42 - S_0 u^n d^{T-n}\}_+ \\ &= \frac{1}{1.03^2} \sum_{n=0}^1 \binom{2}{n} \frac{11^n 7^{2-n}}{18^2} \{42 - 30 \cdot (1.1)^n \cdot (0.92^{2-n})\}_+ \\ &= \frac{1}{1.03^2} \left(1 \cdot \frac{7^2}{18^2} \cdot 8.144 + 2 \cdot \frac{11 \cdot 7}{18^2} \cdot 1.52 \right) \\ &\simeq 1.84194 \dots \end{aligned}$$

Answer 3. b)

The price process S_t is the same as described in 3. a). For this American put option we have that the pay-out is $Y_t(\omega) = \{42 - S_t(\omega)\}_+$ for each $t = 0, 1, 2, 3$ and $\omega \in \Omega$. I summarise the pay-outs in the tree below



I will now construct a Snell Envelope $\{Z_t\}$ for the payoff process $\{Y_t\}$.

By definition, $Z_2(\omega) = Y_2(\omega) \forall \omega \in \Omega$.

Let ω_u, ω_d represents the events the price of underlying asset increases/decreases during the first time-step, respectively.

Consider time-period $t = 1$ and that the event ω_u has occurred. Then

$$\mathbb{E}[Z_2|\mathcal{F}_1] = q \cdot 0 + (1 - q) \cdot 1.52 = \frac{11}{18} \cdot 0 + \frac{7}{18} \cdot 1.52 = 0.5911$$

Thus, given ω_u occurred, we can deduce Z_1

$$\begin{aligned} Z_1 &:= \max\{Y_1, \mathbb{E}[Z_2|\mathcal{F}_1]\} \\ &= \max\{0, 0.5911\} \\ &= 0.5911 \end{aligned}$$

Consider time-period $t = 1$ and that the event ω_d has occurred. Then

$$\mathbb{E}[Z_2|\mathcal{F}_1] = \frac{11}{18} \cdot 1.52 + \frac{7}{18} \cdot 8.144 = 4.096$$

Thus, given ω_d occurred, we can deduce Z_1

$$\begin{aligned} Z_1 &:= \max\{Y_1, \mathbb{E}[Z_2|\mathcal{F}_1]\} \\ &= \max\{5.2, 4.096\} \\ &= 5.2 \end{aligned}$$

Consider time-period $t = 0$. The

$$\mathbb{E}[Z_1|\mathcal{F}_0] = \mathbb{E}[Z_1] = \frac{11}{18} \cdot 0.5911 + \frac{7}{18} \cdot 5.2 = 2.3834$$

We can deduce Z_0

$$\begin{aligned} Z_0 &:= \max\{Y_0, \mathbb{E}[Z_1|\mathcal{F}_0]\} \\ &= \max\{2, 2.3834\} \\ &= 2.3834 \end{aligned}$$

The time $t = 0$ fair price of this American Option is 2.3834.

Answer 4.

In this problem we have that

$$S_0 = 69 \quad K = 70 \quad r = 0.06 \quad \sigma^2 = 0.35 \quad U = 1/2$$

U is defined to equal $1/2$ as the interest rate is defined over a year but the option matures in six months.

By the Black-Scholes formula we have that the time $t = 0$ fair price for this European call option is

$$\begin{aligned}\Pi^{BS}(0) &:= S_0 \Phi(d_1(S_0, U)) - Ke^{-rU} \Phi(d_2(S_0, U)) \\ &= 69 \Phi(d_1(69, 1/2)) - 70e^{-0.03} \Phi(d_2(69, 1/2))\end{aligned}$$

where

$$\begin{aligned}d_1(s, u) &:= \frac{\ln(s/K) + (r + \sigma^2/2)U}{\sqrt{\sigma^2 U}} \\ \Rightarrow d_1(69, 1/2) &= \frac{\ln(69/70) + (0.06 + 0.35/2) \cdot (1/2)}{\sqrt{0.35 \cdot (1/2)}} \\ &= 0.2468 \dots\end{aligned}$$

and

$$\begin{aligned}d_2(s, u) &:= \frac{\ln(s/K) + (r - \sigma^2/2)U}{\sqrt{\sigma^2 U}} \\ \Rightarrow d_2(69, 1/2) &= \frac{\ln(69/70) - (0.06 + 0.35/2) \cdot (1/2)}{\sqrt{0.35 \cdot (1/2)}} \\ &= -0.1718 \dots\end{aligned}$$

Note that

$$\Phi(0.2468) = 0.597 \quad \text{and} \quad \Phi(-0.1718) = 0.432$$

Thus

$$\Pi^{BS}(0) = 69 \cdot 0.597 - 70 \cdot 0.432 = 10.432$$

The time $t = 0$ fair-price for this European call option is 10.432.

I will now determine the time $t = 0$ price for the equivalent European put option.

The Put-Call parity states that

$$S_0 + P_0 - C_0 = Ke^{-rT}$$

Thus, in this problem we have

$$\begin{aligned}69 + P_0 - 10.432 &= 70e^{-0.06/2} \\ \Rightarrow P_0 &= 9.363\end{aligned}$$

The time $t = 0$ fair-price for the equivalent put option is 9.363.