Finance Mathematics - Problem Sheet 4

Dom Hutchinson

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Answer 1)
Let ω_t denote the event that a red ball was taken from the bag in time-period t, meaning $\mathbb{P}[\omega_t] = X_t$, and R_t denote the number of red balls in the urn at the start of time-period t.

Note that we can restate X_t as

$$X_t = \frac{R_t}{a+b+tc}$$

Now consider the conditional expectation of X_t wrt its natural filtration

$$\begin{aligned}
\mathbb{E}[X_t|\mathcal{F}_{t-1}] &= \mathbb{E}\left[\frac{R_t}{a+b+tc}\middle|\mathcal{F}_{t-1}\right] \\
&= \frac{1}{a+b+tc}\mathbb{E}[R_t|\mathcal{F}_{t-1}] \\
&= \frac{1}{a+b+tc}\mathbb{E}[R_{t-1}+c\mathbb{1}\{\omega_{t-1}\}|\mathcal{F}_{t-1}] \\
&= \frac{1}{a+b+tc}(R_{t-1}+cX_{t-1}) \\
&= \frac{1}{a+b+tc}\left(R_{t-1}+c\frac{cR_{t-1}}{a+b+(t-1)c}\right) \\
&= \frac{1}{a+b+tc}\cdot\frac{R_{t-1}(a+b+(t-1)c+c)}{a+b+(t-1)c} \\
&= \frac{1}{a+b+tc}\cdot\frac{R_{t-1}(a+b+tc)}{a+b+(t-1)c} \\
&= \frac{R_{t-1}}{a+b+(t-1)c} \\
&= X_{t-1} \\
\implies \mathbb{E}[X_t|\mathcal{F}_{t-1}] &= X_{t-1}
\end{aligned}$$

This is the definition of $\{X_t\}_{t\in\mathbb{N}_0}$ being a *Martingale*.

Answer 2) a)

$$
\begin{aligned}
\mathbb{E}[S_n|\mathcal{F}_{n-1}] &= \mathbb{E}[\exp\left\{\lambda\sum_{j=1}^n X_j - n\ln(\psi(\lambda))\right\}|\mathcal{F}_{n-1}] \\
&= \mathbb{E}\left[\frac{\exp\left\{\lambda\sum_{j=1}^n X_j\right\}}{\psi(\lambda)^n}\middle|\mathcal{F}_{n-1}\right] \\
&= \mathbb{E}\left[\frac{\exp\left\{\lambda\sum_{j=1}^{n-1} X_j\right\}\cdot\exp\{\lambda X\}}{\psi(\lambda)^{n-1}\psi(\lambda)}\middle|\mathcal{F}_{n-1}\right] \\
&= \frac{\exp\left\{\lambda\sum_{j=1}^{n-1} X_j\right\}}{\psi(\lambda)^{n-1}}\mathbb{E}\left[\frac{\exp\{\lambda X\}}{\psi(\lambda)}\middle|\mathcal{F}_{n-1}\right] \\
&= S_{n-1}\mathbb{E}\left[\frac{\exp\{\lambda X\}}{\mathbb{E}[\exp\{\lambda X\}]}\middle|\mathcal{F}_{n-1}\right] \\
&= S_{n-1}\cdot 1 \\
&= S_{n-1} \\
\implies \mathbb{E}[S_n|\mathcal{F}_{n-1}] &= S_{n-1}
\end{aligned}
$$

This is the definition of $\{S_n\}_{n\in\mathbb{N}_0}$ being a *Martingale*.

Answer 2) b)

$$
\begin{aligned}
\mathbb{E}[X_n|\mathcal{F}_{n-1}] &= \mathbb{E}\left[\left(\sum_{i=1}^n Y_i\right)^2 - n\sigma^2|\mathcal{F}_{n-1}\right] \\
&= \mathbb{E}\left[\left(\sum_{i=1}^{n-1} Y_i + Y_n\right)^2 |\mathcal{F}_{n-1}\right] - n\sigma^2 \\
&= \mathbb{E}\left[\left(\sum_{i=1}^{n-1} Y_i\right)^2 + 2Y_n\left(\sum_{i=1}^{n-1} Y_i + Y_n^2\right)|\mathcal{F}_{n-1}\right] - n\sigma^2 \\
&= \left(\sum_{i=1}^{n-1} Y_i\right)^2 + 2\mathbb{E}[Y_n|\mathcal{F}_{n-1}]\sum_{i=1}^{n-1} Y_i + \mathbb{E}[Y_n^2|\mathcal{F}_{n-1}] - n\sigma^2 \\
&= \left(\sum_{i=1}^{n-1} Y_i\right)^2 + 2\mathbb{E}[Y_n]\sum_{i=1}^{n-1} Y_i + \mathbb{E}[Y_n^2] - n\sigma^2 \\
&= \left(\sum_{i=1}^{n-1} Y_i\right)^2 + 2\cdot 0\cdot\sum_{i=1}^{n-1} Y_i + \left(\mathbb{E}[Y_n^2] - 0^2\right) - n\sigma^2 \\
&= \left(\sum_{i=1}^{n-1} Y_i\right)^2 + \left(\mathbb{E}[Y_n^2] - \mathbb{E}[Y_n]^2\right) - n\sigma^2 \\
&= \left(\sum_{i=1}^{n-1} Y_i\right)^2 + \sigma^2 - n\sigma^2 \\
&= \left(\sum_{i=1}^{n-1} Y_i\right)^2 - (n-1)\sigma^2 \\
&= X_{n-1}
\end{aligned}
$$

This is the definition of $\{X_n\}_{n\in\mathbb{N}_0}$ being a *Martingale* wrt the natural filtration of $\{Y_n\}_{n\in\mathbb{N}}$

Answer 3)

Consider the conditional expectation of L_t wrt the natural filtration $\mathcal{F}_t$ of $\{X_t\}_{t\geq 0}$.

$$
\begin{array}{rcl}
\mathbb{E}[L_t|\mathcal{F}_{t-1}] & = & \mathbb{E}\left[\left(\frac{1-p}{p}\right)^{X_t}\middle|\mathcal{F}_{t-1}\right] \\
& = & \mathbb{E}\left[\left(\frac{1-p}{p}\right)^{X_{t-1}+Y_t}\middle|\mathcal{F}_{t-1}\right] \\
& = & \mathbb{E}\left[\left(\frac{1-p}{p}\right)^{X_{t-1}}\cdot\left(\frac{1-p}{p}\right)^{Y_t}\middle|\mathcal{F}_{t-1}\right] \\
& = & \left(\frac{1-p}{p}\right)^{X_{t-1}}\mathbb{E}\left[\left(\frac{1-p}{p}\right)^{Y_t}\middle|\mathcal{F}_{t-1}\right] \\
& = & \left(\frac{1-p}{p}\right)^{X_{t-1}}\mathbb{E}\left[\left(\frac{1-p}{p}\right)^{Y_t}\right] \\
& = & \left(\frac{1-p}{p}\right)^{X_{t-1}}\cdot\left\{\left(\frac{1-p}{p}\right)^{1}p+\left(\frac{1-p}{p}\right)^{-1}(1-p)\right\} \\
& = & \left(\frac{1-p}{p}\right)^{X_{t-1}}\cdot\{1-p+p\} \\
& = & \left(\frac{1-p}{p}\right)^{X_{t-1}} \\
& = & L_{t-1}
\end{array}
$$

This is the definition of $\{L_t\}_{t\geq 0}$ being a *Martingale*.
Consider the conditional expectation of M_t wrt the natural filtration $\mathcal{F}_t$ of $\{X_t\}_{t\geq 0}$.

$$
\begin{array}{rcl}
\mathbb{E}[M_t|\mathcal{F}_{t-1}] & = & \mathbb{E}[X_t-t(2p-1)|\mathcal{F}_{t-1}] \\
& = & \mathbb{E}[X_{t-1}+Y_t-t(2p-1)|\mathcal{F}_{t-1}] \\
& = & X_{t-1}-t(2p-1)+\mathbb{E}[Y_t|\mathcal{F}_{t-1}] \\
& = & X_{t-1}-t(2p-1)+\mathbb{E}[Y_t] \\
& = & X_{t-1}-t(2p-1)+(2p-1) \\
& = & X_{t-1}-(t-1)(2p-1) \\
& = & M_{t-1}
\end{array}
$$

This is the definition of $\{M_t\}_{t\geq 0}$ being a *Martingale*.

Answer 4)
Consider the following cases for the conditional expectation of Y_n wrt its natural filtration $\mathcal{F}_n$.

i). Case $n<\nu(\omega)$.

$$
\begin{array}{rcll}
\mathbb{E}[Y_n(\omega)|\mathcal{F}_{n-1}] & = & \mathbb{E}\left[X_n^{(1)}(\omega)|\mathcal{F}_{n-1}\right] & \text{by def } Y_n(\omega) \\
& \leq & X_{n-1}^{(1)}(\omega) & \text{since } X_n^{(1)} \text{ is a supermartingale} \\
& = & Y_{n-1}(\omega) &
\end{array}
$$

ii). Case $n>\nu(\omega)$.

$$
\begin{array}{rcll}
\mathbb{E}[Y_n(\omega)|\mathcal{F}_{n-1}] & = & \mathbb{E}\left[X_n^{(2)}(\omega)|\mathcal{F}_{n-1}\right] & \text{by def } Y_n(\omega) \\
& \leq & X_{n-1}^{(2)}(\omega) & \text{since } X_n^{(2)} \text{ is a supermartingale} \\
& = & Y_{n-1}(\omega) &
\end{array}
$$

It is worth noting that as $n>\nu(\omega)$ then $(n-1)>\nu(\omega)$, so the last equality is sound.

iii). Case $n = \nu(\omega)$.

$$\begin{array}{rcll}\mathbb{E}[Y_\nu(\omega)|\mathcal{F}_{\nu-1}] & = & \mathbb{E}[X_\nu^{(2)}|\mathcal{F}_{\nu-1}] & \\ & \leq & X_{\nu-1}^{(2)} & \\ & \leq & X_{\nu-1}^{(1)} & \text{by def } \nu \\ & = & Y_{\nu-1}(\omega) & \end{array}$$

In all cases Y_n fulfils the condition for it to be a supermartingale.