Financial Mathematics - Problem Sheet 2

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Question 2.
Consider a single-period model with one risky asset that has a current price of $S_0 = 15$ and a future price of either $S_1(\omega_1) = 20,\ S_1(\omega_2) = 40/3$ or $S_1(\omega_3) = 10$. There is also a bank account with a constant interest rate $r = 1/9$.

Question 2. a)
Specify the value and gains processes V, G for this model as well as their discounted versions V^*, G^*.

Answer 2. a)
When using a trading strategy $H := (H_0, H_1)$, this model has value process V and gains process G, defined as

$$\begin{array}{rrcl} & V & := & \{V_t : t = 0, 1\} \\ \text{where} & V_t & := & H_0(1+rt) + H_1 \cdot S_1(t) \\ & & = & H_0\left(1 + \dfrac{t}{9}\right) + H_1 \cdot S_1(t) \\ \\ & G & = & H_0 r + H_1 \Delta S_1(t) \\ & & = & \dfrac{1}{9}H_0 + (S_1(t) - 15)H_1 \end{array}$$

Note that in this model $B_t = 1 + rt$ with $r = 1/9$. Thus the discounted value process V^* and discounted gains process G^* for this model are defined as

$$\begin{array}{rrcl} & V^* & := & \{V_t^* : t = 0, 1\} \\ \text{where} & V_t^* & := & \dfrac{V_t}{B_t} \\ & & = & \dfrac{V_t}{1+rt} \\ \\ & G^* & := & \dfrac{G}{B_1} \\ & & = & \dfrac{G}{1+r} \end{array}$$

Question 2. b)
By describing the discounted price process S^*, and using the condition that $\mathbb{E}_{\mathbb{Q}}[S^*(1)] = S^*(0)$, determine all risk neutral probability measures $\mathbb{Q}$.

Answer 2. b)
For this model, the discounted price process S^* is defined as

$$\begin{array}{rrcl} & S^* & := & \{S_1^*(t) : t = 0, 1\} \\ \text{where} & S_1^*(t) & := & \dfrac{S_1(t)}{B_t} \\ & & = & \dfrac{S_1(t)}{1+rt} \end{array}$$

Define a probability measure $\mathbb{Q}$ and let $q_i := \mathbb{Q}(\{\omega_i\})$ for $i \in \{1, 2, 3\}$. Consider both sides of the expression $\mathbb{E}_{\mathbb{Q}}[S_1^*(1)] = S_1^*(0)$.

$$\begin{array}{rcl} S_1^*(0) & = & \dfrac{S_1(0)}{B_0} \\ & = & 15 \\ \mathbb{E}_{\mathbb{Q}}[S_1^*(1)] & = & \displaystyle\sum_{i=1}^{3} q_i S^*(1)(\omega_i) \\ & = & \displaystyle\sum_{i=1}^{3} q_i \frac{S(1)(\omega_i)}{B_1} \\ & = & \dfrac{1}{1+r}(20q_1 + (40/3)q_2 + 10q_3) \end{array}$$

For $\mathbb{Q}$ to be a risk-neutral probability measure, the following must all hold

i). $\mathbb{Q}(\{\omega_i\}) > 0 \ \forall \ i \in \{1, 2, 3\}$.

ii). $\displaystyle\sum_{i=1}^{3} \mathbb{Q}(\{\omega_i\}) = 1$.

iii). $\mathbb{E}_{\mathbb{Q}}[S_1^*(1)] = S_1^*(0) \implies \dfrac{1}{1+r}(20q_1 + (40/3)q_2 + 10q_3) = 15$.

We can state, and find solutions to, the last two conditions as the following matrix

$$\begin{array}{rrcl} & & & \left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ \dfrac{20}{1+r} & \dfrac{40}{3(1+r)} & \dfrac{10}{1+r} & 15 \end{array}\right) \\ & & = & \left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 18 & 12 & 9 & 15 \end{array}\right) \\ & & = & \left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 6 & 4 & 3 & 5 \end{array}\right) \\ & & = & \left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 3 & 1 & 0 & 2 \end{array}\right) \\ \implies & q_3 & = & 1 - q_1 - q_2 \\ \text{and} & q_2 & = & 2 - 3q_1 \\ \implies & q_3 & = & 1 - q_1 - (2 - 3q_1) \\ & & = & 2q_1 - 1 \\ \implies & \mathbf{q} & = & \begin{pmatrix} q_1 & 2 - 3q_1 & 2q_1 - 1 \end{pmatrix} \end{array}$$

In order for the first condition to be fulfilled we require

$$\begin{array}{rrcl} & q_1 & \in & [0, 1] \\ & q_2 = 2 - 3q_1 & \in & [0, 1] \\ \implies & q_1 & \in & [1/3, 2/3] \\ & q_3 = 2q_1 - 1 & \in & [0, 1] \\ \implies & q_1 & \in & [0, 1] \end{array}$$

Thus, for the first condition to be true we must restrict q_1 to values in $[1/3, 2/3]$.

To summarise, $\mathbb{Q}$ is a risk-neutral probability measure for this model if

$$\mathbb{Q}(\{\omega_1\}) = q_1 \quad \mathbb{Q}(\{\omega_2\}) = 2 - 3q_1 \quad \mathbb{Q}(\{\omega_3\}) = 2q_1 - 1 \quad \text{for any } q_1 \in [1/3, 2/3]$$

Question 2. c)
Is the market complete? Describe the set of attainable contingent claims $X = (X_1, X_2, X_3)$ for this model.

Answer 2. c)
Consider the following matrix A which summarises this market

$$A = \begin{pmatrix} B_1(\omega_1) & S_1(1)(\omega_1) \\ B_1(\omega_2) & S_1(1)(\omega_2) \\ B_1(\omega_3) & S_1(1)(\omega_3) \end{pmatrix} = \begin{pmatrix} 10/9 & 20 \\ 10/9 & 40/3 \\ 10/9 & 10 \end{pmatrix}$$

Matrix A only has two linearly-independent columns (ie $\text{rank}(A) = 2$), this means AH can only span R^2 for any H. Thus, this market is <u>not</u> complete.

Under this model, a contingent claim X can have any value for two of its three dimensions, but the value of the third dimension has to fulfil a specific relationship with the other two in order for X to be attainable. This dependency is defined in the requirement that $\exists\ H \in \mathbb{R}^2$ st $AH = X$ for X to be attainable

Question 3.
Consider a single-period model with two risky securities S_1 and S_2. Assume that there is a bank account with interest rate $r = 1/9$ and that the price process is given in the following table

n	$S_n(0)$	$S_n(1)(\omega_1)$	$S_n(1)(\omega_2)$	$S_n(1)(\omega_3)$
1	15	20	20	40/3
2	30	40	80/3	80/3

Question 3. a)
Specify the value and gains processes V, G for this model as well as their discounted versions V^*, G^*.

Answer 3. a)
When using a trading strategy $H := (H_0, H_1, H_2)$, this model has value process V and gains process G, defined as

$$\begin{array}{rrcl} & V & := & \{V_t : t = 0, 1\} \\ \text{where} & V_t & := & H_0(1 + rt) + H_1 \cdot S_1(t) \\ & & = & H_0\left(1 + \frac{t}{9}\right) + H_1 \cdot S_1(t) + H_2 \cdot S_2(t) \\ \\ & G & = & H_0 r + H_1 \Delta S_1(t) + H_2 \Delta S_2(t) \\ & & = & \frac{1}{9}H_0 + (S_1(t) - 15)H_1 + (S_2(t) - 30)H_2 \end{array}$$

Note that in this model $B_t = 1 + rt$ with $r = 1/9$. Thus the discounted value process V^* and discounted gains process G^* for this model are defined as

$$\begin{array}{rrcl} & V^* & := & \{V_t^* : t = 0, 1\} \\ \text{where} & V_t^* & := & \dfrac{V_t}{B_t} \\ & & = & \dfrac{V_t}{1+rt} \\ & G^* & := & \dfrac{G}{B_1} \\ & & = & \dfrac{G}{1+r} \end{array}$$

Question 3. b)
By considering the two discounted stock processes S_1^*, S_2^*, show that there is no risk neutral probability measure.

Answer 3. b)
For this model, the discounted price processes S_1^* and S_2^* are defined as

$$\begin{array}{rcl} S_1^*(t) & := & \dfrac{S_1(t)}{B_t} = \dfrac{S_1(t)}{1+rt} \\ S_2^*(t) & := & \dfrac{S_2(t)}{B_t} = \dfrac{S_2(t)}{1+rt} \end{array}$$

Let $\mathbb{Q}$ be a probability measure and defined $q_i := \mathbb{Q}(\{\omega_i\})$ for $i \in \{1, 2, 3\}$. For $\mathbb{Q}$ to be a risk-neutral probability measure, the following must all hold

i). $q_i > 0 \ \forall \ i \in \{1, 2, 3\}$.

ii). $q_1 + q_2 + q_3 = 1$.

iii). $\mathbb{E}_\mathbb{Q}[S_1^*(1)] = S_1^*(0) \implies \dfrac{1}{1+r}(20q_1 + 20q_2 + (40/3)q_3) = 15$.

iv). $\mathbb{E}_\mathbb{Q}[S_2^*(1)] = S_2^*(0) \implies \dfrac{1}{1+r}(40q_1 + (80/3)q_2 + (80/3)q_3) = 30$.

We can express the last three equations in the following matrix

$$\begin{array}{rl} & \left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ \dfrac{20}{1+r} & \dfrac{20}{1+r} & \dfrac{40}{3(1+r)} & 15 \\ \dfrac{40}{1+r} & \dfrac{80}{3(1+r)} & \dfrac{80}{3(1+r)} & 30 \end{array}\right) \\ = & \left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 18 & 18 & 12 & 15 \\ 36 & 24 & 24 & 30 \end{array}\right) \\ = & \left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 0 & -6 & -3 \\ 12 & 0 & 0 & 6 \end{array}\right) \\ = & \left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1/2 \\ 1 & 0 & 0 & 1/2 \end{array}\right) \\ = & \left(\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1/2 \\ 1 & 0 & 0 & 1/2 \end{array}\right) \end{array}$$

This shows there is a unique solution to this system of equations $(q_1, q_2, q_3) = (0, 1/2, 1/2)$, but this violates the first condition for $\mathbb{Q}$ to be a risk-neutral probability measure. Thus, there are no risk-neutral probability measures for this model.

Question 3. c)
Determine expressions for $\mathbb{W}$ and $\mathbb{W} \cap \mathbb{A}$ where

$$\begin{array}{rcl} \mathbb{W} & := & \left\{X \in \mathbb{R}^K : X = G^* \text{ for some strat. } H\right\} \\ \mathbb{A} & := & \left\{X \in \mathbb{R}^K : X \geq 0, X \neq 0\right\} \end{array}$$

Determine all arbitrage opportunities by deriving for any $X \in \mathbb{W} \cap \mathbb{A}$ a trading strategy H which gives rise to the time $t = 1$ portfolio value (X_1, X_2, X_3).

Answer 3. c)
Consider generic $X, H \in \mathbb{R}^3$ and define matrix $A \in \mathbb{R}^{3\times 3}$ as

$$A = \begin{pmatrix} B_1(\omega_1) & S_1(1)(\omega_1) & S_2(1)(\omega_1) \\ B_1(\omega_2) & S_1(1)(\omega_2) & S_2(1)(\omega_2) \\ B_1(\omega_3) & S_1(1)(\omega_3) & S_2(1)(\omega_3) \end{pmatrix} = \begin{pmatrix} 1+r & 20 & 40 \\ 1+r & 20 & 80/3 \\ 1+r & 40/3 & 80/3 \end{pmatrix} = \begin{pmatrix} 10/9 & 20 & 40 \\ 10/9 & 20 & 80/3 \\ 10/9 & 40/3 & 80/3 \end{pmatrix}$$

The strategy H attains contingent claim X if $AH = X$. Thus

$$\begin{array}{rcl} H & = & A^{-1}X \\ \begin{pmatrix} H_1 \\ H_2 \\ H_3 \end{pmatrix} & = & \begin{pmatrix} -9/5 & 0 & 27/10 \\ 0 & 3/20 & -3/20 \\ 3/40 & -3/40 & 0 \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix} \\ & = & \begin{pmatrix} (9/10)(3X_3 - 2X_1) \\ (3/20)(X_2 - X_3) \\ (3/40)(X_1 - X_2) \end{pmatrix} \end{array}$$

This gives a formula for what trading strategy H to give rise to a portfolio of value (X_1, X_2, X_3) at time $t = 1$.