

Financial Mathematics - Assessed Problem Sheet 1

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March 29, 2021

Answer 1. a) i.

From the question we know the strike price $K = £30$, the annual interest rate is $r = 0.05$, initial value of AKOCOM $S_0 = £37.5$, the initial price of the put option $P_0 = £11$ and the initial price of the call option $C_0 = £18.50$. The strike date is $T = 1/2$, as the length of each option is 6 months, but we are given the annual interest rate.

The put-call parity states that if $S_t + P_t - C_t = Ke^{r(T-t)}$ for all $t \in [0, T]$ then no arbitrage opportunities exist.

Consider each side of this expression at time $t = 0$

$$\begin{aligned} S_0 + P_0 - C_0 &= £37.50 + £11 - £18.50 \\ &= £30 \\ Ke^{-r(T-0)} &= 30e^{-(0.05)(0.5-0)} \\ &= 30e^{-0.025} \\ &= £29.26 \\ \implies S_0 + P_0 - C_0 &\neq Ke^{-r(T-0)} \end{aligned}$$

This shows that the put-call parity does not hold at time $t = 0$ and thus an arbitrage opportunity exists.

Answer 1. a) ii.

Today, $t = 0$, the arbitrageur should do the following

- Short sell a share of AKOCOM, receiving £37.50.
- Take a long position on a call option, costing £18.50.
- Take a short position on a put option, receiving £11.
- Invest the net amount receiving from these transactions, $B_0 = 37.5 - 18.5 + 11 = £30$.

In six months time, $t = T = 1/2$, our arbitrageur's bank balance will be $B_1 = B_0e^{T/20}$.

Let S_T be the price of AKOCOM shares at time T . The arbitrageur should do the following

- If $S_T \geq £30$ then exercise their call option. This costs £30 and fulfils the arbitrageur's short position on AKOCOM. The gains in this scenario are

$$B_1 - 30 = 30e^{T/20} - 30 > 0$$

The holder of long position in our put option will not exercise their option in this scenario as they would lose money.

- If $S_T < £30$ then tear-up the call option. The holder of the long position in our put option will exercise their option in this scenario as they will make a profit. This means we have to buy a share of AKOCOM from them for £30, this fulfils our short position on AKOCOM. Our profit in this scenario is again

$$B_1 - 30 = 30e^{T/20} - 30 > 0$$

These two scenarios cover all outcomes in six months time, and show that our arbitrageur makes a risk-free profit under both scenarios.

Answer 1. b)

Consider the payouts from the two European call options at time $t = T$

$$\begin{aligned} C_T^1 &= \{S_T - K_1\}_+ \\ C_T^2 &= \{S_T - K_2\}_+ \end{aligned}$$

where $\{x\}_+ := \max\{0, x\}$. Note that since $K_1 < K_2$ then $C_T^2 \leq C_T^1 \forall S_T \in \mathbb{R}^{\geq 0}$.

We can restate X_T in terms of C_T^1 and C_T^2 as

$$\begin{aligned} X_T &= \begin{cases} C_T^1 - C_T^2 & \text{if } S_T \geq K_2 \\ C_T^1 & \text{if } S_T \in [K_1, K_2] \\ 0 & \text{if } S_T \leq K_1 \end{cases} \\ &= C_T^1 - C_T^2 \end{aligned}$$

Using the “No-Arbitrage Principle” it can be shown that if two, or more, financial derivatives have the same value at time T , then their prices will coincide at all times $t < T$. This means the fair price at time t for this capped call option is

$$X_t = C_t^1 - C_t^2$$

Answer 1. c) i.

A probability measure \mathbb{Q} is a *Risk-Neutral Probability Measure* if the following all hold

- $\mathbb{Q}(\{\omega\}) > 0 \forall \omega \in \Omega$; and,
- $\mathbb{E}_{\mathbb{Q}}[S_1^*(1)] = S_1^*(0)$

Additionally, as \mathbb{Q} is a probability measure we have $\sum_{\omega \in \Omega} \mathbb{Q}(\{\omega\}) = 1$.

From the question we have that $S_0 = 1$, $S_1(\omega_1) = 1.3$ and $S_2(\omega_2) = 1.1$. Let r denote the risk-free interest rate, $q_1 := \mathbb{Q}(\{\omega_1\})$ and $q_2 := \mathbb{Q}(\{\omega_2\})$. Note that the Bank process at time $t = 1$ has value $B_1 = 1 + r$.

Under the conditions of this question, we can derive the follow equations which must hold in order for \mathbb{Q} to be a *Risk-Neutral Probability Measure*

$$\begin{aligned} \text{and} \quad q_1 + q_2 &= 1 & (1) \\ \mathbb{E}_{\mathbb{Q}}[S_1^*(1)] &= S_1^*(0) \\ \implies q_1 S_1^*(\omega_1) + q_2 S_1^*(\omega_2) &= \frac{S_1(0)}{B_0} \\ \implies q_1 \cdot \frac{S_1(\omega_1)}{B_{1.1}} + q_2 \cdot \frac{S_1(\omega_2)}{B_{1.3}} &= \frac{S_1(0)}{B_0} \\ \implies q_1 \cdot \frac{1.1}{1+r} + q_2 \cdot \frac{1.3}{1+r} &= 1 & (2) \end{aligned}$$

From equations (1), (2) we can deduce values for q_1, q_2 in terms of r .

$$\begin{aligned}
 & q_2 = 1 - q_1 \\
 \Rightarrow & q_1 \cdot \frac{1.1}{1+r} + (1 - q_1) \cdot \frac{1.3}{1+r} = 1 \\
 \Rightarrow & \frac{1.1}{10(1+r)} - q_1 \cdot \frac{1.3}{10(1+r)} = 1 \\
 \Rightarrow & q_1 = \frac{13 - 10(1+r)}{3 - 10r} \\
 & = \frac{3 - 10r}{3 - 10r} \\
 \Rightarrow & q_2 = 1 - \frac{3 - 10r}{3 - 10r} \\
 & = \frac{10r - 1}{2}
 \end{aligned}$$

Thus, probability measure \mathbb{Q} can be stated as

$$\mathbb{Q}(\{\omega_1\}) = \frac{3 - 10r}{2} \quad \mathbb{Q}(\{\omega_2\}) = \frac{10r - 1}{2}$$

As both these quantities must take values in $[0, 1]$ we can deduce the range of interest rates r where a *Risk-Neutral Probability Measure* exists.

$$\begin{aligned}
 \Rightarrow & \frac{\mathbb{Q}(\{\omega_1\})}{\frac{3 - 10r}{2}} \in [0, 1] & \frac{\mathbb{Q}(\{\omega_2\})}{\frac{10r - 1}{2}} \in [0, 1] \\
 \Rightarrow & \frac{3 - 10r}{2} \in [0, 1] & \frac{10r - 1}{2} \in [0, 1] \\
 \Rightarrow & 3 - 10r \in [0, 2] & 10r - 1 \in [0, 2] \\
 \Rightarrow & 10r \in [1, 3] & 10r \in [1, 3] \\
 \Rightarrow & r \in [0.1, 0.3] & r \in [0.1, 0.3]
 \end{aligned}$$

Thus, there exists a *Risk-Neutral Probability Measure* if $r \in [0.1, 0.3]$.

Answer 1. c) ii.

At time-point $t = 0$ the agent should do the following

- Short a share, receiving $S_0 = £1$.
- Invest this £1 in the risk-free asset.

At time-point $t = 1$ this investment is worth $B_1 = 1 + r = 1.4$. Then, at time-point $t = 1$ the agent should do the following.

- Pay the dividend $D = 0.1$ to the agent who lent us the share used for our short position.
- Buy a share, at whatever the current price is, in order to fulfil our short position.

If our arbitrageur follows this strategy and event ω_1 occurs, then they make a risk-free profit of

$$B_1 - S_1(\omega_1) - D = 1.4 - 1.3 - 0.1 = 0$$

If our arbitrageur follows this strategy and event ω_2 occurs, then they make a risk-free profit of

$$B_1 - S_1(\omega_2) - D = 1.4 - 1.1 - 0.1 = 0.2$$

Thus, under either event our arbitrageur is guaranteed not to lose money.