Financial Mathematics - Problem Sheet 3

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Answer 1. a)

$$\begin{array}{rcl} \mathcal{P}_0 & = & \{\Omega\} \\ \mathcal{P}_1 & = & \{\{\omega_1,\omega_2,\omega_3\},\{\omega_4,\omega_5\},\{\omega_6,\omega_7,\omega_8\}\} \\ \mathcal{P}_2 & = & \{\{\omega_1\},\{\omega_2\},\{\omega_3\},\{\omega_4\},\{\omega_5\},\{\omega_6\},\{\omega_7\},\{\omega_8\}\} \end{array}$$

Answer 1. b)

$$\begin{array}{rcl} \mathcal{F}_0 & = & \{\emptyset,\Omega\} \\ \mathcal{F}_1 & = & \{\emptyset,\Omega,\{\omega_1,\omega_2,\omega_3\},\{\omega_4,\omega_5,\omega_6,\omega_7,\omega_8\}, \\ & & \{\omega_4,\omega_5\},\{\omega_1,\omega_2,\omega_3,\omega_6,\omega_7,\omega_8\},\{\omega_6,\omega_7,\omega_8\}, \\ & & \{\omega_1,\omega_2,\omega_3,\omega_4,\omega_5\}\} \\ \mathcal{F}_2 & = & 2^\Omega \end{array}$$

Answer 2. a)
The value process $V_t(\omega)$ for this model is as follows

$\omega \setminus t$	0	1
ω_1	$H_0+15H_1+30H_2$	$(10/9)H_0+20H_1+40H_2$
ω_2	$H_0+15H_1+30H_2$	$(10/9)H_0+20H_1+(80/3)H_2$
ω_3	$H_0+15H_1+30H_2$	$(10/9)H_0+(40/3)H_1+(80/3)H_2$
ω_4	$H_0+15H_1+30H_2$	$(10/9)H_0+(20/3)H_1+(20/3)H_2$

and the discounted value process $V_t^*(\omega)=V_t(\omega)/B_t$ for this model is

$\omega \setminus t$	0	1
ω_1	$H_0+15H_1+30H_2$	$H_0+18H_1+36H_2$
ω_2	$H_0+15H_1+30H_2$	$H_0+18H_1+24H_2$
ω_3	$H_0+15H_1+30H_2$	$H_0+12H_1+24H_2$
ω_4	$H_0+15H_1+30H_2$	$H_0+6H_1+6H_2$

The gains process $G(\omega)$ for this model is as follows

ω	$G(\omega)$
ω_1	$(1/9)H_0+5H_1+10H_2$
ω_2	$(1/9)H_0+5H_1-(10/3)H_2$
ω_3	$(1/9)H_0-(5/3)H_1-(10/3)H_2$
ω_4	$(1/9)H_0-(25/3)H_1-(70/3)H_2$

and the discounted gains process $G^*(\omega)=\sum_{i=1}^{2} H_1\Delta S_i^*(\omega)$ for this model is as follows

ω	$G^*(\omega)$
ω_1	$3H_1+6H_2$
ω_2	$3H_1-6H_2$
ω_3	$-3H_1-6H_2$
ω_4	$-9H_1-24H_2$

Answer 2. b)
Consider $X\in\mathbb{R}^4$ st $X=G^*$. Then

$$\begin{pmatrix}X_1\\X_2\\X_3\\X_4\end{pmatrix}=\begin{pmatrix}3H_1+6H_2\\3H_1-6H_2\\-3H_1-6H_2\\-9H_1-24H_2\end{pmatrix}$$

Thus

$$\mathbb{W}=\text{span}\left\{\begin{pmatrix}3\\3\\-3\\-9\end{pmatrix},\begin{pmatrix}6\\-6\\-6\\-24\end{pmatrix}\right\}=\text{span}\left\{\begin{pmatrix}1\\1\\-1\\-3\end{pmatrix},\begin{pmatrix}1\\-1\\-1\\-4\end{pmatrix}\right\}$$

Consider the set $\mathbb{A}:=\{X\in\mathbb{R}^4:x\geq 0\;\forall\,x\in X\}$ and the elements of $\mathbb{W}\cap\mathbb{A}$. This elements form the set of possible gains which would produce a risk-free profit (i.e. an arbitrage opportunity).

$$\begin{array}{rcl}\mathbb{W}\cap\mathbb{A}&=&\left\{X\in\mathbb{R}^4:X_1=x_3=0;X_2,X_4>0\right\}\\X\in(\mathbb{W}\cap\mathbb{A})&\implies&X=\begin{pmatrix}0&\lambda&0&\mu\end{pmatrix}\text{ with }\lambda,\mu>0\end{array}$$

How consider the Xs where $X=G^*$ and $X\in(\mathbb{W}\cap\mathbb{A})$.

$$\begin{array}{rrcl}&X_1=3H_1+6H_2&=&0\\\implies&H_1&=&-2H_2\\\\&X_2=3H_1-6H_2&=&\lambda\\\implies&-12H_2&=&\lambda\\\implies&H_2&=&-\frac{1}{12}\lambda\\\implies&H_1&=&\frac{1}{6}\lambda\\\\&X_4=-9H_1-24H_2&=&\mu\\\implies&-6H_2&=&\mu\\\implies&H_2&=&-\frac{1}{6}\mu\\\implies&H_1&=&\frac{1}{3}\mu\\\implies&\mu&=&\frac{1}{2}\lambda\end{array}$$

Thus the trading strategies which exploit arbitrage opportunities in this model as those in the set

$$\left\{H\in\mathbb{R}^3:H=\begin{pmatrix}0&\frac{1}{6}\lambda&-\frac{1}{12}\lambda\end{pmatrix}:\lambda>0\right\}$$

Answer 2. c)
The "No-Arbitrage Theorem" states that risk-neutral probability measures only exists for a model <u>iff</u> no arbitrage opportunities exist. In this model arbitrage opportunities do exist, thus no risk-neutral probability measures can.

Consider matrix A which summarises this model

$$A = \begin{pmatrix} B_1(\omega_1) & S_1(1)(\omega_1) & S_2(1)(\omega_1) \\ B_1(\omega_2) & S_1(1)(\omega_2) & S_2(1)(\omega_2) \\ B_1(\omega_3) & S_1(1)(\omega_3) & S_2(1)(\omega_3) \\ B_1(\omega_4) & S_1(1)(\omega_4) & S_2(1)(\omega_4) \end{pmatrix} == \begin{pmatrix} 10/9 & 20 & 40 \\ 10/9 & 20 & 80/3 \\ 10/9 & 40/3 & 80/3 \\ 10/9 & 20/3 & 20/3 \end{pmatrix}$$

Matrix A only has three linearly-independent columns, thus AH can only span $\mathbb{R}^3$ for any trading strategy H. As AH cannot span $\mathbb{R}^4$, the market is <u>not</u> complete.

Answer 3. a)

$\mathcal{F}_1$ is a σ-algebra.

$\mathcal{F}_2$ is <u>not</u> a σ-algebra as (among other reasons) $\{\omega_1\}, \{\omega_2\} \in \mathcal{F}_2$ but $\{\omega_1, \omega_2\} \notin \mathcal{F}_2$.

Answer 3. b)

Consider the σ-algebra $\mathcal{G} := \{\emptyset, \Omega\}$ and the function $X(\omega_i) = i$. X is <u>not</u> measurable wrt $\mathcal{G}$ as $\{\omega_1\}, \{\omega_2\}, \{\omega_3\}, \{\omega_4\} \notin \mathcal{G}$.

Answer 3. c)

The powerset of Ω contains all possible subsets of Ω and thus all functions $\omega \to X(\omega)$ are measurable wrt to it.

$$\mathcal{H} = 2^\Omega$$