

# Financial Mathematics - Assessed Problem Sheet 2

Dom Hutchinson

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## Answer 1. (a) i.

Consider time  $t = 0$  and define  $p_1 := \mathbb{Q}(S_1 = 16)$ , then

$$\begin{aligned} 10 &= 16p_1 + 8(1 - p_1) \\ \implies p_1 &= 1/4 \end{aligned}$$

Thus  $\mathbb{Q}(S_1 = 16) = 1/4$  and  $\mathbb{Q}(S_1 = 8) = 1 - 1/4 = 3/4$ . Now, consider time  $t = 1$  and that event  $\omega$  has occurred with  $\omega \in \{\omega_1, \omega_2\}$ . Define  $p_2 := \mathbb{Q}(S_2 = 18|S_1 = 16)$ , then

$$\begin{aligned} 16 &= 18p_2 + 12(1 - p_2) \\ \implies p_2 &= 2/3 \end{aligned}$$

Thus  $\mathbb{Q}(S_2 = 18|S_1 = 16) = 2/3$  and  $\mathbb{Q}(S_2 = 12|S_1 = 16) = 1/3$ . Now, consider time  $t = 1$  and that event  $\omega$  has occurred with  $\omega \in \{\omega_3, \omega_4\}$ . Define  $p_3 := \mathbb{Q}(S_2 = 12|S_1 = 8)$ , then

$$\begin{aligned} 8 &= 12p_3 + 6(1 - p_3) \\ \implies p_3 &= 1/3 \end{aligned}$$

Thus  $\mathbb{Q}(S_2 = 12|S_1 = 8) = 1/3$  and  $\mathbb{Q}(S_2 = 6|S_1 = 8) = 2/3$ .

We can use these conditional probabilities to work out the probability of each event  $\{\omega_1, \omega_2, \omega_3, \omega_4\}$  under  $\mathbb{Q}$ .

$$\begin{aligned} \mathbb{Q}(\{\omega_1\}) &= \mathbb{Q}(S_2 = 18|S_1 = 16)\mathbb{Q}(S_1 = 16) \\ &= (2/3) \times (1/4) \\ &= 1/6 \\ \mathbb{Q}(\{\omega_2\}) &= \mathbb{Q}(S_2 = 12|S_1 = 16)\mathbb{Q}(S_1 = 16) \\ &= (1/3) \times (1/4) \\ &= 1/12 \\ \mathbb{Q}(\{\omega_3\}) &= \mathbb{Q}(S_2 = 12|S_1 = 8)\mathbb{Q}(S_1 = 8) \\ &= (1/3) \times (3/4) \\ &= 1/4 \\ \mathbb{Q}(\{\omega_4\}) &= \mathbb{Q}(S_2 = 6|S_1 = 8)\mathbb{Q}(S_1 = 8) \\ &= (2/3) \times (3/4) \\ &= 1/2 \end{aligned}$$

To confirm that this produces a probability measure, note that

$$(1/6) + (1/12) + (1/4) + (1/2) = 1$$

## Answer 1. (a) ii.

$$\begin{aligned}
X(\omega_1) &= \left\{ \frac{1}{3}(10 + 16 + 18) - 14 \right\}_+ \\
&= \left\{ \frac{44}{3} - 14 \right\}_+ \\
&= \frac{2}{3} \\
X(\omega_2) &= \left\{ \frac{1}{3}(10 + 16 + 12) - 14 \right\}_+ \\
&= \left\{ \frac{38}{3} - 14 \right\}_+ \\
&= 0 \\
X(\omega_3) &= \left\{ \frac{1}{3}(10 + 8 + 12) - 14 \right\}_+ \\
&= \left\{ \frac{30}{3} - 14 \right\}_+ \\
&= 0 \\
X(\omega_4) &= \left\{ \frac{1}{3}(10 + 8 + 6) - 14 \right\}_+ \\
&= \left\{ \frac{24}{3} - 14 \right\}_+ \\
&= 0
\end{aligned}$$

$$\begin{aligned}
Y(\omega_1) &= \{\max(10, 16, 18) - 14\}_+ \\
&= \{18 - 14\}_+ \\
&= 4 \\
Y(\omega_2) &= \{\max(10, 16, 12) - 14\}_+ \\
&= \{16 - 14\}_+ \\
&= 2 \\
Y(\omega_3) &= \{\max(10, 8, 12) - 14\}_+ \\
&= \{12 - 14\}_+ \\
&= 0 \\
Y(\omega_4) &= \{\max(10, 8, 6) - 14\}_+ \\
&= \{10 - 14\}_+ \\
&= 0
\end{aligned}$$

**Answer 1. (a) iii.**

The *Risk-Neutral Valuation Principle* states that, for all attainable contingent claims  $X$ , the following holds

$$V_t^* = \mathbb{E}_Q[X/B_t | \mathcal{F}_t] \text{ for } t = 0, \dots, T \text{ and all } \mathbb{Q}$$

Thus, at time  $t = 0$

$$V_0 = \mathbb{E}_Q[X | \mathcal{F}_0] = \mathbb{E}_Q[X]$$

We can use this to derive the risk-neutral prices for the two options at time  $t = 0$

$$\begin{aligned}
V_0^X &= \mathbb{E}_Q[X] \\
&= (2/3) \cdot (1/6) + 0 \cdot (1/12) + 0 \cdot (1/4) + 0 \cdot (1/2) \\
&= 1/9
\end{aligned}$$

$$\begin{aligned}
V_0^Y &= \mathbb{E}_Q[Y] \\
&= 4 \cdot (1/6) + 2 \cdot (1/12) + 0 \cdot (1/4) + 0 \cdot (1/2) \\
&= 5/6
\end{aligned}$$

Thus the risk-neutral price for the Asian option at time  $t = 0$  is  $1/9$  and for the look-back option it is  $5/6$

**Answer 1. (a) iv.**

Note that since  $r = 0$ ,  $B_t = 1$  for  $i = 0, 1, 2$ .

Consider the Asian option and let  $H^X(t) := \{H_0^X(t), H_1^X(t)\}$  denote a portfolio which only has access to a bank account & Asian options for each time point  $t = 0, 1, 2$ .

consider the value of the Asian option at time  $t = 1$  and  $t = 2$

$$\begin{aligned}
 V_1^X(\omega) &= \begin{cases} \mathbb{E}[X|S_1 = 16] & \text{if } \omega \in \{\omega_1, \omega_2\} \\ \mathbb{E}[X|S_1 = 8] & \text{if } \omega \in \{\omega_3, \omega_4\} \end{cases} \\
 &= \begin{cases} (2/3) \cdot p_2 + 0 \cdot (1 - p_2) & \text{if } \omega \in \{\omega_1, \omega_2\} \\ 0 & \text{if } \omega \in \{\omega_3, \omega_4\} \end{cases} \\
 &= \begin{cases} (2/3) \cdot (2/3) & \text{if } \omega \in \{\omega_1, \omega_2\} \\ 0 & \text{if } \omega \in \{\omega_3, \omega_4\} \end{cases} \\
 &= \begin{cases} 4/9 & \text{if } \omega \in \{\omega_1, \omega_2\} \\ 0 & \text{if } \omega \in \{\omega_3, \omega_4\} \end{cases} \\
 V_2^X(\omega) &= X(\omega) \quad \forall \omega
 \end{aligned}$$

To find a self-financing portfolio we start at time  $t = 2$  and find that if  $\omega \in \{\omega_1, \omega_2\}$

$$\begin{aligned}
 V_2^X(\omega_1) = 2/3 &= H_0^X(2)(\omega_1, \omega_2) + 18 \cdot H_1^X(2)(\omega_1, \omega_2) \\
 V_2^X(\omega_2) = 0 &= H_0^X(2)(\omega_1, \omega_2) + 12 \cdot H_1^X(2)(\omega_1, \omega_2) \\
 \implies 2/3 &= 6H_1^X(2)(\omega_1, \omega_2) \\
 \implies H_1^X(2)(\omega_1, \omega_2) &= 1/9 \\
 \implies 2 &= H_0^X(2)(\omega_1, \omega_2) + 12 \cdot (1/9) \\
 \implies H_0^X(2)(\omega_1, \omega_2) &= -4/3
 \end{aligned}$$

And if  $\omega \in \{\omega_3, \omega_4\}$

$$\begin{aligned}
 V_2^X(\omega_3) = 0 &= H_0^X(2)(\omega_3, \omega_4) + 12 \cdot H_1^X(2)(\omega_3, \omega_4) \\
 V_2^X(\omega_4) = 0 &= H_0^X(2)(\omega_3, \omega_4) + 6 \cdot H_1^X(2)(\omega_3, \omega_4) \\
 \implies H_0^X(2)(\omega_3, \omega_4) &= 0 \\
 \implies H_1^X(2)(\omega_3, \omega_4) &= 0
 \end{aligned}$$

Let  $H_0^X(1) := H_0^X(1)(\omega_1, \omega_2, \omega_3, \omega_4)$  and  $H_1^X(1) := H_1^X(1)(\omega_1, \omega_2, \omega_3, \omega_4)$ , then we have

$$\begin{aligned}
 V_1^X(\omega) = 4/9 &= H_0^X(1) + 16 \cdot H_1^X(1) \quad \text{if } \omega \in \{\omega_1, \omega_2\} \\
 V_1^X(\omega) = 0 &= H_0^X(1) + 8 \cdot H_1^X(1) \quad \text{if } \omega \in \{\omega_3, \omega_4\} \\
 \implies 4/9 &= 8H_1^X(1) \\
 \implies H_1^X(1) &= 1/18 \\
 \implies 0 &= H_0^X(1) + 8 \cdot (1/18) \\
 \implies H_0^X(1) &= -4/9
 \end{aligned}$$

Thus, a self-financing trading strategy with access to a bank account and the look-back option is as follows

$$\begin{aligned}
 H^X(1)(\omega) &= (-4/9, 1/18) \quad \forall \omega \\
 H^X(2)(\omega) &= (-4/3, 1/9) \quad \text{if } \omega \in \{\omega_1, \omega_2\} \\
 H^X(2)(\omega) &= (0, 0) \quad \text{if } \omega \in \{\omega_3, \omega_4\}
 \end{aligned}$$

To confirm this trading strategy is self-financing I show that  $H_0(t+1)B_t + H_1(t+1)S_1(t) = V_t \forall t$ .

$$\begin{array}{l|l}
 H^Y(1)(\omega) & B_0 \cdot H_0^X(1)(\omega) + S_1(0) \cdot H_1^Y(1)(\omega) = -(4/9) + 10 \cdot (1/18) \\
 & = 1/9 = V_0^X \\
 H^Y(2)(\omega) & B_1 \cdot H_0^X(2)(\omega) + S_1(1) \cdot H_1^Y(2)(\omega) = -(4/3) + 16 \cdot (1/9) \quad \text{if } \omega \in \{\omega_1, \omega_2\} \\
 & = 4/9 = V_1^Y(\omega) \\
 H^Y(2)(\omega) & B_1 \cdot H_0^X(2)(\omega) + S_1(1) \cdot H_1^Y(2)(\omega) = 0 + 8 \cdot 0 \quad \text{if } \omega \in \{\omega_3, \omega_4\} \\
 & = 0 = V_1^Y(\omega)
 \end{array}$$

This requirement holds in all cases.

Now, consider the look-back option and let  $H^Y(t) := \{H_0^Y(t), H_1^Y(t)\}$  denote a portfolio which only has access to a bank account & look-back options for each time point  $t = 0, 1, 2$ .

Consider the value of the look-back option at time  $t = 1$  and  $t = 2$

$$\begin{aligned}
 V_1^Y(\omega) &= \begin{cases} \mathbb{E}[Y|S_1 = 16] & \text{if } \omega \in \{\omega_1, \omega_2\} \\ \mathbb{E}[Y|S_1 = 8] & \text{if } \omega \in \{\omega_3, \omega_4\} \end{cases} \\
 &= \begin{cases} 4p_2 + 2(1 - p_2) & \text{if } \omega \in \{\omega_1, \omega_2\} \\ 0 & \text{if } \omega \in \{\omega_3, \omega_4\} \end{cases} \\
 &= \begin{cases} 4 \cdot (2/3) + 2(1/3) & \text{if } \omega \in \{\omega_1, \omega_2\} \\ 0 & \text{if } \omega \in \{\omega_3, \omega_4\} \end{cases} \\
 &= \begin{cases} 10/3 & \text{if } \omega \in \{\omega_1, \omega_2\} \\ 0 & \text{if } \omega \in \{\omega_3, \omega_4\} \end{cases} \\
 V_2^Y(\omega) &= Y(\omega) \forall \omega
 \end{aligned}$$

To find a self-financing portfolio we start at time  $t = 2$  and find that if  $\omega \in \{\omega_1, \omega_2\}$

$$\begin{aligned}
 V_2^Y(\omega_1) = 4 &= H_0^Y(2)(\omega_1, \omega_2) + 18 \cdot H_1^Y(2)(\omega_1, \omega_2) \\
 V_2^Y(\omega_2) = 2 &= H_0^Y(2)(\omega_1, \omega_2) + 12 \cdot H_1^Y(2)(\omega_1, \omega_2) \\
 \implies 2 &= 6H_1^Y(2)(\omega_1, \omega_2) \\
 \implies H_1^Y(2)(\omega_1, \omega_2) &= 1/3 \\
 \implies 2 &= H_0^Y(2)(\omega_1, \omega_2) + 12 \cdot (1/3) \\
 \implies H_0^Y(2)(\omega_1, \omega_2) &= -2/3
 \end{aligned}$$

And if  $\omega \in \{\omega_3, \omega_4\}$

$$\begin{aligned}
 V_2^Y(\omega_3) = 0 &= H_0^Y(2)(\omega_3, \omega_4) + 12 \cdot H_1^Y(2)(\omega_3, \omega_4) \\
 V_2^Y(\omega_4) = 0 &= H_0^Y(2)(\omega_3, \omega_4) + 6 \cdot H_1^Y(2)(\omega_3, \omega_4) \\
 \implies H_0^Y(2)(\omega_3, \omega_4) &= 0 \\
 \implies H_1^Y(2)(\omega_3, \omega_4) &= 0
 \end{aligned}$$

Let  $H_0^Y(1) := H_0(1)(\omega_1, \omega_2, \omega_3, \omega_4)$  and  $H_1^Y(1) := H_1(1)(\omega_1, \omega_2, \omega_3, \omega_4)$ , then we have

$$\begin{aligned}
 V_1^Y(\omega) = 10/3 &= H_0^Y(1) + 16 \cdot H_1^Y(1) \quad \text{if } \omega \in \{\omega_1, \omega_2\} \\
 V_1^Y(\omega) = 0 &= H_0^Y(1) + 8 \cdot H_1^Y(1) \quad \text{if } \omega \in \{\omega_3, \omega_4\} \\
 \implies 10/3 &= 8H_1^Y(1) \\
 \implies H_1^Y(1) &= 5/12 \\
 \implies 0 &= H_0^Y(1) + 8 \cdot (5/12) \\
 \implies H_0^Y(1) &= -40/12
 \end{aligned}$$

Thus, a self-financing trading strategy with access to a bank account and the look-back option is as follows

$$\begin{aligned} H^Y(1)(\omega) &= (-40/12, 5/12) \quad \forall \omega \\ H^Y(2)(\omega) &= (-2/3, 1/3) \quad \text{if } \omega \in \{\omega_1, \omega_2\} \\ H^Y(2)(\omega) &= (0, 0) \quad \text{if } \omega \in \{\omega_3, \omega_4\} \end{aligned}$$

To confirm this trading strategy is self-financing I show that  $H_0(t+1)B_t + H_1(t+1)S_1(t) = V_t \forall t$ .

$$\begin{array}{l|l} H^Y(1)(\omega) & \begin{aligned} B_0 \cdot H_0^Y(1)(\omega) + S_1(0) \cdot H_1^Y(1)(\omega) &= -(40/12) + 10 \cdot (5/12) \\ &= 5/6 = V_0^Y \end{aligned} \\ H^Y(2)(\omega) & \begin{aligned} B_1 \cdot H_0^Y(2)(\omega) + S_1(1) \cdot H_1^Y(2)(\omega) &= -(2/3) + 16 \cdot (1/3) \quad \text{if } \omega \in \{\omega_1, \omega_2\} \\ &= 10/3 = V_1^Y(\omega) \\ B_1 \cdot H_0^Y(2)(\omega) + S_1(1) \cdot H_1^Y(2)(\omega) &= 0 + 8 \cdot 0 \quad \text{if } \omega \in \{\omega_3, \omega_4\} \\ &= 0 = V_1^Y(\omega) \end{aligned} \end{array}$$

This requirement holds in all cases.

### Answer 1. (b)

Assume that  $\epsilon$  is known,  $T$  is large and note that  $B_t = 1 \forall t \in [1, T]$  since the interest rate is  $r = 0$ .

Consider the following trading strategy

If, at time-point  $t \in (1, T)$  it is the case that  $S(t) \leq \epsilon$  then:

- In time-point  $t$  borrow  $\mathcal{L}S(t)$  from the bank-account and buy a unit of the security for  $\mathcal{L}S(t)$ .
- In time-point  $t+1$  sell your security for  $\mathcal{L}S(t+1)$  and place all of this revenue into the bank account (reimbursing the bank at the same time).

Otherwise, do nothing.

This strategy can be expressed mathematically as

$$\begin{aligned} H(0) &= (0, 0) \\ H(t+1) &= \begin{cases} (H_0(t) - S(t), 1) & \text{if } S(t) \leq \epsilon \\ (H_0(t) + S(t), 0) & \text{if } S(t-1) \leq \epsilon \\ (H_0(t), 0) & \text{otherwise} \end{cases} \end{aligned}$$

For this specific strategy we need to further assume that  $\epsilon > e^{1-T}$  as otherwise the main action could never be triggered.

I will now justify that this trading strategy is an arbitrage opportunity.

Consider the value of this portfolio at time-point  $t = 0$

$$V_0 = H_0(0) \cdot B_0 + H_1(0) \cdot S(1) = 0 \cdot 1 + 0 \cdot 1 = 0$$

Thus the requirement that  $V_0 = 0$  is fulfilled.

Suppose that  $S(t) \leq \epsilon$  but  $S(t-1) > \epsilon$ . This means that the price process must have decreased in value between these time steps and thus  $S(t) = S(t-1)e^{-1} \implies S(t)e > \epsilon$ .

Now consider  $S(t+1)$ , we know that  $S(t+1) = S(t) \exp\{X(t) + 2\}$  by the definition of the price process and since  $S(t) \leq \epsilon$ . Since  $X(t) \in \{-1, 1\}$  then  $X(t) + 2 \geq 1$ , meaning  $S(t+1) \geq S(t)e > \epsilon$ . This shows that the price process only ever goes below threshold  $\epsilon$  for a single time-period at a time.

Now, suppose the action is triggered at time-point  $t$  and consider the price process over this and the next time-point

$$\begin{aligned} S(t) &\leq \epsilon \\ \implies S(t+1) &= S(t) \exp\{X(t) + 2\} \\ \implies S(t+1) &> S(t) \text{ since } X(t) + 2 \geq 1 \end{aligned}$$

This shows that when we sell the security in time-point  $t+1$  it is always worth more than the price we bought it for, meaning we can fully reimburse the bank-account and have a surplus left over in our bank-account. Moreover, this shows that we are guaranteed to make a profit whenever the main action of this strategy is triggered. And, since this strategy takes no other actions, it cannot lose money between time-periods.

Since we assumed that  $T$  is large and since  $\{X_t\}$  is a symmetric random walk it is expected that the action will occur at least once. This means the criteria that  $V_T = H_0(T) \geq 0$  and  $\mathbb{E}[V_T] > 0$  are fulfilled.

To show that this trading strategy  $H$  is self-financing I need to show that  $V_t(\omega) = H_0(t+1) + H_1(t+1)S(t)$  for all  $t, \omega$ . There are three cases to consider

- i). Case  $S(t) \leq \epsilon \implies S(t-1) > \epsilon$  (Shown above).

No action was taken in the last time-period. Thus  $V_t = H_0(t)$ .

$$\begin{aligned} &H_0(t+1) + H_1(t+1)S(t) \\ &= (H_0(t) - S(t)) + S(t) \\ &= H_0(t) \\ &= V_t \end{aligned}$$

- ii). Case  $S(t-1) \leq \epsilon \implies S(t) > \epsilon$  (Shown above).

This means an option was bought in the last-time period for  $S(t-1)$  and that option is now worth  $S(t)$ , thus  $V_t = H_0(t) - S(t-1) + S(t)$ .

$$\begin{aligned} &H_0(t+1) + H_1(t+1)S(t) \\ &= (H_0(t) + S(t)) + 0 \\ &= H_0(t) + S(t) \\ &= (H_0(t-1) - S(t-1)) + S(t) \text{ as } S(t-1) \leq \epsilon \\ &= V_t \end{aligned}$$

- iii). Otherwise (ie  $S(t), S(t-1) > \epsilon$ )

No actions are taken so  $V_t = H_0(t)$ .

$$\begin{aligned} &H_0(t+1) + H_1(t+1)S(t) \\ &= H_0(t) + 0 \\ &= V_t \end{aligned}$$

This shows that  $H$  is self-financing, and thus fulfils all criteria to be an arbitrage opportunity.