Financial Mathematics - Assessed Problem Sheet 3

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Question 1. a)
A "Down-and-Out Call" with knockout price b and rebate R is an ordinary American call with strike price K as long as the stock price does not fall below the knockout price. If the stock price falls below the knockout price at a random time point $t < T$, then the option expires and the owner of the option receives at time point τ the rebate R.

Question 1. a) i.
By defining an appropriate stopping time, give a mathematical description of the payoff of the down-and-out call. Find also a European claim that is equivalent to this claim.

Answer 1. a) i.
Define stopping time $\tau = \inf\{t : S_t \leq b\}$, this represents the first time the stock price falls below the knockout price b.

The payoff process $\{Y_t\}$ for the Down-and-Out call can be expressed as

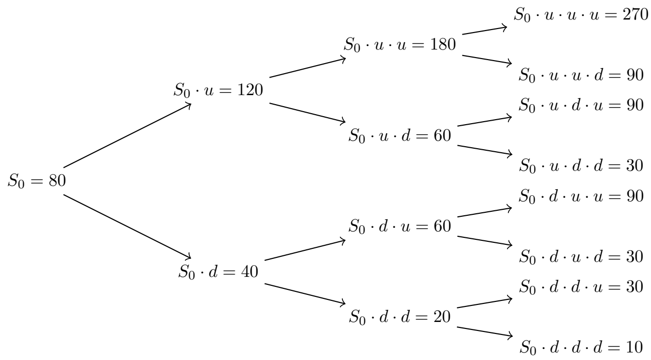
$$Y_t = \begin{cases} \{S_t - K\}_+ & \text{if } t < \tau \\ R & \text{if } t \geq \tau \end{cases}$$

Consider the European Claim $X = Y_\tau \dfrac{B_T}{B_\tau}$ which corresponds to exercising the Down-and-Out call at time τ and then accumulating interest from the bank account until the expiry date of the claim at time T.

Question 1. a) ii.
Consider now a Cox-Ross-Rubinstein model with $T = 3$ periods and stock price process $\{S_t\}$ where $S_0 = 80, u = 3/2, d = 1/2$ and interest rate $r = 0.1$.

Calculate the risk-neutral probability and the stock prices at each node in the tree.

Answer 1. a) ii.
Consider the tree below which shows the possible evolutions of the price process S_t for each time-point and event.



The risk-neutral probability measure for a Cox-Ross-Rubinstein model satisfies

$$\mathbb{Q}(S_t = S_0 u^n d^{t-n}) = \binom{t}{n} q^n (1-q)^{t-n} \text{ where } q = \frac{1+r-d}{u-d} \text{ for } n = 0, \dots, t$$

where n is the number of up steps taken in the first t time-periods.

Under this specification of the Cox-Ross-Rubinstein model

$$q = \frac{1 + 0.1 - 0.5}{1.5 - 0.5} = \frac{3}{5}$$

Thus

$$\mathbb{Q}(S_t = S_0 u^n d^{t-n}) = \binom{t}{n} \frac{3^n 2^{t-n}}{5^t} \text{ for } n = 0, \dots, t$$

By inspecting the tree of stock prices above we can determine the possible prices at each time-point, and thus the risk-neutral probability of each node.

At time $t = 0$

$$\mathbb{Q}(S_0 = 80) \quad = \quad \mathbb{Q}(S_0 = S_0) = 1$$

At time $t = 1$

$$\begin{aligned} \mathbb{Q}(S_1 = 120) &= \mathbb{Q}(S_1 = S_0 u) = \binom{1}{1} \cdot \frac{3}{5} = \frac{3}{5} \\ \mathbb{Q}(S_1 = 40) &= \mathbb{Q}(S_1 = S_0 d) = \binom{1}{0} \cdot \frac{2}{5} = \frac{2}{5} \end{aligned}$$

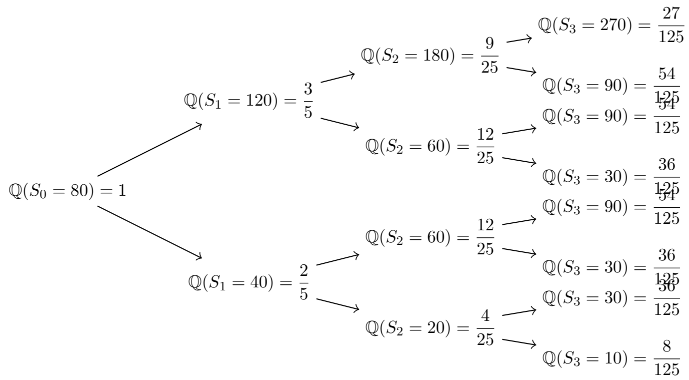
At time $t = 2$

$$\begin{aligned} \mathbb{Q}(S_2 = 180) &= \mathbb{Q}(S_2 = S_0 u^2) = \binom{2}{2} \cdot \frac{3^2}{5^2} = \frac{9}{25} \\ \mathbb{Q}(S_2 = 60) &= \mathbb{Q}(S_2 = S_0 ud) = \binom{2}{1} \cdot \frac{3 \cdot 2}{5^2} = 2 \cdot \frac{6}{25} = \frac{12}{25} \\ \mathbb{Q}(S_2 = 40) &= \mathbb{Q}(S_2 = S_0 d^2) = \binom{2}{0} \cdot \frac{2^2}{5^2} = \frac{4}{25} \end{aligned}$$

At time $t = 3$

$$\begin{aligned} \mathbb{Q}(S_3 = 270) &= \mathbb{Q}(S_3 = S_0 u^3) &= \binom{3}{3} \cdot \frac{3^3}{5^3} = \frac{27}{125} \\ \mathbb{Q}(S_3 = 90) &= \mathbb{Q}(S_3 = S_0 u^2 d) &= \binom{3}{2} \cdot \frac{3^2 \cdot 2}{5^3} = 3 \cdot \frac{18}{125} = \frac{54}{125} \\ \mathbb{Q}(S_3 = 30) &= \mathbb{Q}(S_3 = S_0 u d^2) &= \binom{3}{1} \cdot \frac{3 \cdot 2^2}{5^3} = 3 \cdot \frac{12}{125} = \frac{36}{125} \\ \mathbb{Q}(S_3 = 10) &= \mathbb{Q}(S_3 = S_0 3^3) &= \binom{3}{0} \cdot \frac{2^3}{5^3} = \frac{8}{125} \end{aligned}$$

I summarise these values in the tree below



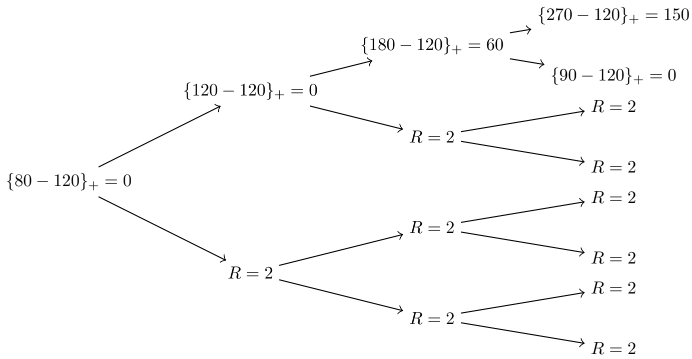
Question 1. a) iii.

Considering the same Cox-Ross-Rubinstein model, calculate the value of the down-and-out call with knockout price $b = 70$, rebate $R = 2$ and exercise price $K = 120$ at every time point t and for every state ω.

Then calculate a replicating strategy.

Answer 1. a) iii.

The tree below specifies the pay-out process $\{Y_t\}$ of the down-and-out call option is exercised at each possible time-point and sequence of events



I construct a Snell Envelope $\{Z_t\}$ to determine the the value of the down-and-out option at each time-point t and state ω. At time-point $t = 3$

$$Z_3(\omega) = Y_3(\omega) \ \forall \ \omega$$

At time-point $t = 2$ and state ω_{uu} (ie Two up steps have occurred)

$$\mathbb{E}[Z_3|\mathcal{F}_2] = 150q + 0 \cdot (1-q) = 90$$

At time-point $t = 2$ and state ω_{ud} (ie One up step and one down step have occurred)

$$\mathbb{E}[Z_3|\mathcal{F}_2] = 2q + 2 \cdot (1-q) = 2$$

At time-point $t = 2$ and state ω_{dd} (ie Two down steps have occurred)

$$\mathbb{E}[Z_3|\mathcal{F}_2] = 2q + 2 \cdot (1-q) = 2$$

Thus

$$\begin{aligned} Z_2(\omega) &= \max(\mathbb{E}[Z_3|\mathcal{F}_2], Y_2(\omega)) \\ &= \begin{cases} \max(60, 90) & \text{if } \omega = \omega_{uu} \\ \max(2, 2) & \text{otherwise} \end{cases} \\ &= \begin{cases} 90 & \text{if } \omega = \omega_{uu} \\ 2 & \text{otherwise} \end{cases} \end{aligned}$$

At time-point $t = 1$ and state ω_u (ie One up steps has occurred)

$$\mathbb{E}[Z_2|\mathcal{F}_1] = 90q + 2 \cdot (1-q) = 54.8$$

At time-point $t = 1$ and state ω_d (ie One down step has occurred)

$$\mathbb{E}[Z_2|\mathcal{F}_1] = 2q + 2 \cdot (1-q) = 2$$

Thus

$$\begin{array}{rcl} Z_1(\omega) & = & \max(\mathbb{E}[Z_2|\mathcal{F}_1], Y_1(\omega)) \\ & = & \begin{cases} \max(54.8, 0) & \text{if } \omega = \omega_u \\ \max(2, 2) & \text{if } \omega = \omega_d \end{cases} \\ & = & \begin{cases} 54.8 & \text{if } \omega = \omega_u \\ 2 & \text{if } \omega = \omega_d \end{cases} \end{array}$$

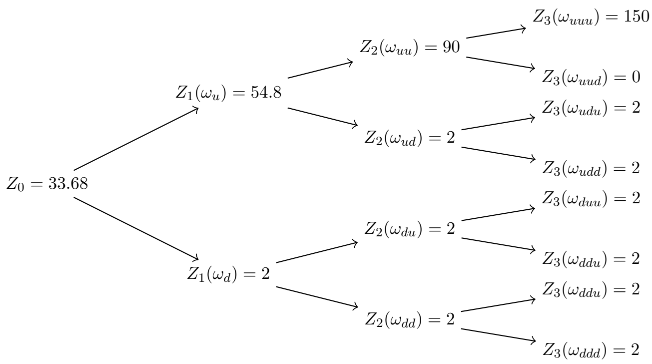
At time-point $t = 0$

$$\begin{array}{rcl} \mathbb{E}[Z_1|\mathcal{F}_0] & = & \mathbb{E}[Z_1] \\ & = & (54.8)q + 2(1-q) \\ & = & 33.68 \end{array}$$

Thus

$$Z_0 = \max(\mathbb{E}[Z_1|\mathcal{F}_0], Y_0(\omega)) = \max(33.68, 0) = 33.68$$

I summarise the value of this call option at each time-point and event in the tree below



Using the optimal stopping theorem, the optimal stopping strategy $\{\tau(t)\}_t$ is

$$\begin{array}{rcl} \tau(3)(\omega) & = & 3 \ \forall\ \omega \\ \tau(2)(\omega) & = & \begin{cases} 3 & \text{if } \omega \in \{\omega_{uuu}, \omega_{uud}\} \\ 2 & \text{otherwise} \end{cases} \\ \tau(1)(\omega) & = & \begin{cases} 3 & \text{if } \omega \in \{\omega_{uuu}, \omega_{uud}\} \\ 2 & \text{if } \omega \in \{\omega_{udu}, \omega_{udd}\} \\ 1 & \text{otherwise} \end{cases} \\ \tau(0)(\omega) & = & \begin{cases} 3 & \text{if } \omega \in \{\omega_{uuu}, \omega_{uud}\} \\ 2 & \text{if } \omega \in \{\omega_{udu}, \omega_{udd}\} \\ 1 & \text{otherwise} \end{cases} \end{array}$$

I now find a replicating strategy $\{H(t)\}$ for $\tau(0)$. In the third-time period the following equations must be satisfied

$$\begin{array}{lrcl} & H_0(3) + 270H_1(3) & = & Z_3(\omega_{uuu}) = 150 \\ & H_0(3) + 90H_1(3) & = & Z_3(\omega_{uud}) = 0 \\ \implies & H_1(3) & = & \dfrac{150}{180} = \dfrac{5}{6} \\ \implies & H_0(3) & = & 0 - 90 \cdot \dfrac{5}{6} = -75 \end{array}$$

Thus $H(3)(\omega) = (-75, 5/6)$ if $\omega \in \{\omega_{uuu}, \omega_u ud\}$. We do not consider any other states as the option would have already been exercised by time-point $t = 3$.

In the second-time period the following equations must be satisfied

$$\begin{array}{rrcl} & H_0(2) + 180H_1(2) & = & Z_2(\omega_{uu}) = 90 \\ & H_0(2) + 60H_1(2) & = & Z_2(\omega_{ud}) = 2 \\ \implies & H_1(2) & = & \frac{88}{120} = \frac{11}{15} \\ \implies & H_1(2) & = & 2 - 60 \cdot \frac{11}{15} = -42 \end{array}$$

Thus $H(2)(\omega) = (-43, 11/15)$ if $\omega \in \{\omega_{uu}, \omega_u d\}$. We do not consider any other states as the option would have already been exercised by time-point $t = 2$.

In the first-time period the following equations must be satisfied

$$\begin{array}{rrcl} & H_0(1) + 120H_1(1) & = & Z_1(\omega_u) = 54.8 \\ & H_0(1) + 40H_1(1) & = & Z_1(\omega_d) = 2 \\ \implies & H_1(1) & = & \frac{52.8}{80} = \frac{33}{50} \\ \implies & H_1(1) & = & 2 - 40 \cdot \frac{33}{50} = -\frac{122}{5} \end{array}$$

Thus $H(1)(\omega) = (-122/5, 33/50)$ for all ω.

Question 1. a) iv.
Consider the same Cox-Ross-Rubinstein model and a usual European call option which matures at time $T = 3$ with exercise price $K = 120$. Calculate the time $t = 0$ price and compare with the price of the down-and-out call.

Answer 1. a) iv.
The time $t = 0$ of a European call option in a Cox-Ross-Rubinstein model is

$$\Pi(0) = \frac{1}{(1+r)^T} \sum_{n=0}^{T} \binom{T}{n} q^n (1-q)^{T-n} \{S_0 u^n d^{T-n} - K\}_+$$

More specifically, for the model in this question, a European call option with exercise price $K = 120$ and exercise date $T =$

$$\begin{array}{rcl} \Pi(0) & = & \frac{1}{1.1^3} \sum_{n=0}^{3} \binom{3}{n} \frac{3^n 2^{3-n}}{5^3} \left\{ 80 \cdot \frac{3^n \cdot 1^{3-n}}{2^3} - 120 \right\}_+ \\ & = & \frac{1}{1.1^3} \left\{ \binom{3}{0} \cdot \frac{2^3}{5^3} \{10 - 120\}_+ + \binom{3}{1} \cdot \frac{3 \cdot 2^2}{5^3} \{30 - 120\}_+ \right. \\ & & \left. + \binom{3}{2} \cdot \frac{3^2 \cdot 2}{5^3} \{90 - 120\}_+ + \binom{3}{3} \cdot \frac{3^3}{5^3} \{270 - 120\}_+ \right\} \\ & = & \frac{1}{1.1^3} \cdot \frac{3^3}{5^3} \cdot 150 \\ & = & 24.34259\dots \end{array}$$

Question 1. b) i.
Find the probability that standard Brownian Motion lies between some values a, b, with $a \leq b$, at time T, in terms of the normal distribution function Φ.

Find the expected value of $\exp\{cW_T\}$ $\forall$ $c \in \mathbb{R}$.

Answer 1. b) i.
Let $a \leq b$ and $\{W_t\}_t$ be standard Brownian motion.
Note that $W_T \sim \text{Normal}(0, T)$, thus $\frac{1}{\sqrt{T}} W_T \sim \Phi$. Thus

$$\begin{aligned}
\mathbb{P}(W_T \in [a,b]) &= \mathbb{P}(W_T \leq b) - \mathbb{P}(W_T \leq a) \\
&= \mathbb{P}\left(\frac{1}{\sqrt{T}} W_T \leq \frac{b}{\sqrt{T}}\right) - \mathbb{P}\left(\frac{1}{\sqrt{T}} W_T \leq \frac{a}{\sqrt{T}}\right) \\
&= \Phi(b/\sqrt{T}) - \Phi(a/\sqrt{T})
\end{aligned}$$

Consider the expected value of e^{cW_T} for $c \in \mathbb{R}$

$$\begin{aligned}
\mathbb{E}\left[e^{cW_T}\right] &= \int e^{cx} f_{W_T}(x) dx \\
&= \int e^{cx} \cdot \frac{1}{\sqrt{2\pi T}} e^{-\frac{1}{2} \cdot \frac{x^2}{T}} dx \\
&= \int \frac{1}{\sqrt{2\pi T}} \exp\left\{-\frac{1}{2} \cdot \frac{(x-c)^2 + c^2}{T}\right\} dx \\
&= e^{c^2/T} \int \frac{1}{\sqrt{2\pi T}} \exp\left\{-\frac{1}{2} \cdot \frac{(x-c)^2}{T}\right\} dx \\
&= e^{c^2/T} \cdot \mathbb{E}\left[\text{Normal}(c, T)\right] \\
&= ce^{c^2/T}
\end{aligned}$$

Question 1. b) ii.
Consider two independent standard Brownian motions $W_t^{(1)}$ and $W_t^{(2)}$ and write the linear combination

$$X_t = \gamma W_t^{(1)} + \sqrt{1-\gamma^2} W_t^{(2)}$$

where $\gamma \in [-1, 1]$ is a constant. Show that X_t is a standard Brownian Motion.

Answer 1. b) ii.
Let $\{W_t^{(1)}\}_t \{W_t^{(2)}\}$ be independent standard Brownian motions and define stochastic process $\{X_t\}_t$ as

$$X_t = \gamma W_t^{(1)} + W_t^{(2)} \sqrt{1-\gamma^2} \text{ with } \gamma \in [-1, 1]$$

There are four properties I need to show for X_t to be a standard Brownian motion

i). *That $X_0 = 0$.*

$$\begin{aligned}
X_0 &= \gamma W_0^{(1)} + W_t^{(2)} \sqrt{1-\gamma^2} \\
&= \gamma \cdot 0 + 0 \cdot \sqrt{1-\gamma^2} \\
&= 0
\end{aligned}$$

X_t has this property.

ii). *Increments of X_t are independent.*

Consider the increment $(X_{t+u} - X_t)$ for $t, u \geq 0$

$$\begin{aligned}
(X_{t+u} - X_t) &= \left(\gamma W_{t+u}^{(1)} + W_{t+u}^{(2)} \sqrt{1-\gamma^2}\right) - \left(\gamma W_t^{(1)} + W_t^{(2)} \sqrt{1-\gamma^2}\right) \\
&= \gamma(W_{t+u}^{(1)} - W_t^{(1)}) + \sqrt{1-\gamma^2}(W_{t+u}^{(1)} - W_t^{(1)})
\end{aligned}$$

Since $W_t^{(1)}, W_t^{(2)}$ are standard Brownian motions then their increments $(W_{t+u}^{(1)} - W_t^{(1)}), (W_{t+u}^{(2)} - W_t^{(2)})$ are both independent of $\mathcal{F}_t$. Thus, by linearity the increment of X_t is independent of $\mathcal{F}_t$.

iii). *X_t has stationary Gaussian increments.*

Consider the increment $(X_{t+u} - X_t)$ for $t, u \geq 0$

$$(X_{t+u} - X_t) = \gamma(W_{t+u}^{(1)} - W_t^{(1)}) + \sqrt{1-\gamma^2}(W_{t+u}^{(1)} - W_t^{(1)})$$

As the increments of $W_t^{(1)}, W_t^{(2)}$ have gaussian distributions, the increments of X_t have gaussian distributions to. Now consider the mean and variance of these increments

$$\begin{array}{rcl}\mathbb{E}[X_{t+u} - X_t] & = & \gamma\mathbb{E}[W_{t+u}^{(1)} - W_t^{(1)}] + \sqrt{1-\gamma^2}\mathbb{E}[W_{t+u}^{(2)} - W_t^{(2)}] \\ & = & \gamma \cdot 0 + \sqrt{1-\gamma^2} \cdot 0 \\ & = & 0\end{array}$$

$$\begin{array}{rcl}\mathrm{Var}[X_{t+u} - X_t] & = & \gamma^2\mathrm{Var}[W_{t+u}^{(1)} - W_t^{(1)}] + (\sqrt{1-\gamma^2})^2\mathrm{Var}[W_{t+u}^{(2)} - W_t^{(2)}] \\ & = & \gamma^2 \cdot u + (1-\gamma^2)u \\ & = & u\end{array}$$

Thus

$$(X_{t+u} - X_t) \sim \mathrm{Normal}(0, u) \text{ for all } t, u \geq 0$$

iv). *X_t has continuous paths.*

We know that $W_t^{(1)}(\omega), W_t^{(2)}(\omega)$ are continuous functions of t for all ω.

Thus, by linearity, $X_t(\omega)$ is a continuous function of t for all ω.

Since X_t has all four properties, it is a standard Brownian motion.