Financial Mathematics - Assessed Problem Sheet 1

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Question 1. a)
The current price for AKOCOM shares is £37.50 per share. A put option on 1 share of AKOCOM maturing in exactly half a year, with a strike price of £30, is traded at £11.

The price of the corresponding call option, with the same strike price and maturity date, is currently £18.50.

Assume that the interest rate for lending money as well as positing money on a bank account is 5% per annum.

Question 1. a) i.
Use the put-call parity to show that there exists arbitrage opportunities in the model above.

Answer 1. a) i.
From the question we know the strike price $K = \pounds 30$, the annual interest rate is $r = 0.05$, initial value of AKOCOM $S_0 = \pounds 37.5$, initial price of the put option $P_0 = \pounds 11$ and initial price of the call option $C_0 = \pounds 18.5$. The strike date is $T = \frac{1}{2}$, as the length of each option is 6 months, but we are given the annual interest rate.

The put-call parity states that if $S_t + P_t - C_t = Ke^{r(T-t)}$ for all t then no arbitrage opportunities exist.

Consider each side of this expression at time $t = 0$

$$\begin{array}{rcl} S_0 + P_0 - C_0 & = & \pounds 37.50 + \pounds 11 - \pounds 18.50 \\ & = & \pounds 30 \\ Ke^{-r(T-0)} & = & 30e^{-(0.05)(0.5-0)} \\ & = & 30e^{-0.025} \\ & = & \pounds 29.26 \end{array}$$

This means that the put-call parity does not hold at time $t = 0$, as $S_0 + P_0 - C_0 \neq Ke^{-rT}$, and thus an arbitrage opportunity exists.

Question 1. a) ii.
Describe explicitly what actions an arbitrageur would carry out today and in six months time to attain riskless profit. You can assume that the arbitrageur does not have to pay any fees and that short selling is possible.

Answer 1. a) ii.
Today, $t = 0$, the arbitrageur should do the following

- Short sell a share of AKOCOM, receiving £37.50.
- Take a long position on a call option, costing £18.50.

- Take a short position on a put option, receiving £11.
- Invest the net amount receiving from these transactions, $B_0 = 37.5 - 18.5 + 11 = £30$.

In six months time, $t = T$, our arbitrageur's bank balance will be $B_1 = B_0 e^{T/20}$.

Let S_T be the price of AKOCOM shares at this time. The arbitrageur should do the following

- If $S_T \geq £30$ then exercise their call option. This costs £30 and fulfils the arbitrageur's short position on AKOCOM. The gains in this scenario is

  $$B_1 - 30 = 30e^{T/20} - 30 > 0$$

  The holder of long position in our put option will not exercise the option in this scenario as they would loose money.

- If $S_T < £30$ then tear-up our call option. The holder of the long position in our put option will exercise their option in this scenario as they will make a profit. This means we have to buy a share of AKOCOM from them for £30, this fulfils our short position on AKOCOM. Our profit in this scenario is again

  $$B_1 - 30 = 30e^{T/20} - 30 > 0$$

These two scenarios cover all outcomes in six months time, and show that we make a risk-free profit under both scenarios.

Question 1. b)

Let S_t be the time t price of some stock and C_t^1 and C_t^2 the time t prices of European call options on that stock, with maturity T and strike prices K_1 and K_2, with $K_1 < K_2$. We also consider a *capped* call option X on the same stock with same maturity T and value

$$X_T = \begin{cases} K_2 - K_1 & \text{if } K_2 \leq S_T \\ S_T - K_1 & \text{if } S_T \in [K_1, K_2] \\ 0 & \text{if } S_T \leq K_1 \end{cases}$$

Find the time t price X_t of the capped call option in terms of C_t^1 and C_t^2.

Answer 1. b)

Consider the payouts from the two European call options at time $t = T$

$$\begin{array}{rcl} C_T^1 & = & \{S_T - K_1\}_+ \\ C_T^2 & = & \{S_T - K_2\}_+ \end{array}$$

where $\{x\}_+ = \max\{0, x\}$. Note that since $K_1 < K_2$ then $C_T^2 \leq C_T^1 \ \forall \ S_T \in \mathbb{R}^{\geq 0}$.

We can restate X_T in terms of C_T^1 and C_T^2 as

$$\begin{array}{rcl} X_T & = & \begin{cases} C_T^1 - C_T^2 & \text{if } S_T \geq K_2 \\ C_T^1 & \text{if } S_T \in [K_1, K_2] \\ 0 & \text{if } S_T \leq K_1 \end{cases} \\ & = & C_T^1 - C_T^2 \end{array}$$

Using the "No-Arbitrage Principle" it can be shown that if two, or more, financial derivatives have the same value at time T, then their prices will coincide at all times $t < T$. This means the fair price at time t for this capped call is

$$X_t = C_t^1 - C_t^2$$

Question 1. c)
Consider a single-period model with one stock that has a current price of $S_0 = 1$ and a future price of either $S_1(\omega_1) = 1.3$ or $S_1(\omega_2) = 1.1$. The interest rate of the risk-free asset is r.

Question 1. c) i.
Find all values for r such that there exists a risk neutral probability measure, and determine that measure in terms of r.

Answer 1. c) i.
A probability measure $\mathbb{Q}$ is a *Risk-Neutral Probability Measure* if the following all hold

i). $\mathbb{Q}(\{\omega\}) > 0\ \forall\ \omega \in \Omega$; and,

ii). $\mathbb{E}_{\mathbb{Q}}[S_1^*(1)] = S_1^*(0)$

Additionally, as $\mathbb{Q}$ is a probability measure we have $\sum_{\omega\in\Omega} \mathbb{Q}(\{\omega\}) = 1$.

From the question we have that $S_1(0) = 1,\ S_1(\omega_1) = 1.3$ and $S_2(\omega_2) = 1.1$. Let r denote the risk-free interest rate, $q_1 := \mathbb{Q}(\{\omega_1\}),\ q_2 := \mathbb{Q}(\{\omega_2\})$. Note that the Bank process at time $t = 1$ has value $B_1 = 1 + r$.

Under the conditions of this question, for $\mathbb{Q}$ to be a *Risk-Neutral Probability Measure*, the following must hold

$$\begin{array}{lrcll}
 & q_1 + q_2 & = & 1 & (1)\\
\text{and} & \mathbb{E}_{\mathbb{Q}}[S_1^*(1)] & = & S_1^*(0) & \\
\implies & q_1 S_1^*(\omega_1) + q_2 S_1^*(\omega_2) & = & \dfrac{S_1(0)}{B_0} & \\
\implies & q_1 \cdot \dfrac{S_1(\omega_1)}{B_1} + q_2 \cdot \dfrac{S_1(\omega_2)}{B_1} & = & \dfrac{S_1(0)}{B_0} & \\
\implies & q_1 \cdot \dfrac{1.1}{1+r} + q_2 \cdot \dfrac{1.3}{1+r} & = & 1 & (2)
\end{array}$$

These equations are only well defined when $r \neq 1$.

From equations $(1), (2)$ we can deduce values for q_1, q_2 in terms of r.

$$\begin{array}{lrcl}
 & q_1 \cdot \dfrac{1.1}{1+r} + (1 - q_1) \cdot \dfrac{1.3}{1+r} & = & 1\\
\implies & \dfrac{13}{10(1+r)} - q_1 \cdot \dfrac{2}{10(1+r)} & = & 1\\
\implies & q_1 & = & \dfrac{13 - 10(1+r)}{2}\\
 & & = & \dfrac{3 - 10r}{2}\\
\implies & q_2 & = & 1 - \dfrac{3-10r}{2}\\
 & & = & \dfrac{10r - 1}{2}
\end{array}$$

Thus, probability measure $\mathbb{Q}$ can be stated as

$$\mathbb{Q}(\{\omega_1\}) = \frac{3 - 10r}{2} \quad \mathbb{Q}(\{\omega_2\}) = \frac{10r - 1}{2}$$

As both these quantities must take values in $[0,1]$ we can deduce the range of interest rates r where a *Risk-Neutral Probability Measure* exists.

$$\begin{array}{rrcl} & \mathbb{Q}(\{\omega_1\}) & \in & [0,1] \\ \implies & \dfrac{3-10r}{2} & \in & [0,1] \\ \implies & 3-10r & \in & [0,2] \\ \implies & 10r & \in & [1,3] \\ \implies & r & \in & [0.1,0.3] \end{array}$$

Thus, there exists a *Risk-Neutral Probability Measure* if $r \in [0.1, 0.3]$.

Question 1. c) ii.
Assume that $r = 0.4$ and that stock holders are getting paid a dividend of 0.1 at time point 1. Describe explicitly what an arbitrageur would have to do at time point 0 and at time point 1 to gain a riskless profit, and calculate the profit in each state ω_1, ω_2. Again, assume that the arbitrageur does not have to pay any fees and that short-selling is possible.

Answer 1. c) ii.
At time-point $t = 0$ the agent should do the following

- Short a share.
- Invest the $S_0 = £1$, received from shorting, in the risk-free asset.

Then, at time-point $t = 1$ the agent should do the following.

- Pay the dividend $D = 0.1$ to the agent who leant us the share used for our short position.
- Buy a share, at whatever the current price is, in order to fulfil our short position.

Our investment in the risk-free asset is now worth $B_1 = 1 + r = 1.4$.

If our arbitrageur follows this strategy and event ω_1 occurs, then they share a make risk-free profit of

$$B_1 - S_1(\omega_1) - D = 1.4 - 1.3 - 0.1 = 0$$

If our arbitrageur follows this strategy and event ω_2 occurs, then they share a make risk-free profit of

$$B_1 - S_1(\omega_2) - D = 1.4 - 1.1 - 0.1 = 0.2$$

Thus, under either event our arbitrageur is guaranteed not to loose money.