Financial Mathematics - Assessed Problem Sheet 1

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Answer 1. a) i.
From the question we know the strike price $K = £30$, the annual interest rate is $r = 0.05$, initial value of AKOCOM $S_0 = £37.5$, the initial price of the put option $P_0 = £11$ and the initial price of the call option $C_0 = £18.50$. The strike date is $T = 1/2$, as the length of each option is 6 months, but we are given the annual interest rate.

The put-call parity states that if $S_t + P_t - C_t = Ke^{r(T-t)}$ for all $t \in [0, T]$ then no arbitrage opportunities exist.

Consider each side of this expression at time $t = 0$

$$\begin{array}{rcl} S_0 + P_0 - C_0 & = & £37.50 + £11 - £18.50 \\ & = & £30 \\ Ke^{-r(T-0)} & = & 30e^{-(0.05)(0.5-0)} \\ & = & 30e^{-0.025} \\ & = & £29.26 \\ \implies S_0 + P_0 - C_0 & \neq & Ke^{-r(T-0)} \end{array}$$

This shows that the put-call parity does not hold at time $t = 0$ and thus an arbitrage opportunity exists.

Answer 1. a) ii.
Today, $t = 0$, the arbitrageur should do the following

- Short sell a share of AKOCOM, receiving £37.50.
- Take a long position on a call option, costing £18.50.
- Take a short position on a put option, receiving £11.
- Invest the net amount receiving from these transactions, $B_0 = 37.5 - 18.5 + 11 = £30$.

In six months time, $t = T = 1/2$, our arbitrageur's bank balance will be $B_1 = B_0 e^{T/20}$.

Let S_T be the price of AKOCOM shares at time T. The arbitrageur should do the following

- If $S_T \geq £30$ then exercise their call option. This costs £30 and fulfils the arbitrageur's short position on AKOCOM. The gains in this scenario are

  $$B_1 - 30 = 30e^{T/20} - 30 > 0$$

  The holder of long position in our put option will not exercise their option in this scenario as they would loose money.

- If $S_T < £30$ then tear-up the call option. The holder of the long position in our put option will exercise their option in this scenario as they will make a profit. This means we have to buy a share of AKOCOM from them for £30, this fulfils our short position on AKOCOM. Our profit in this scenario is again

$$B_1 - 30 = 30e^{T/20} - 30 > 0$$

These two scenarios cover all outcomes in six months time, and show that our arbitrageur makes a risk-free profit under both scenarios.

Answer 1. b)
Consider the payouts from the two European call options at time $t = T$

$$\begin{array}{rcl} C_T^1 & = & \{S_T - K_1\}_+ \\ C_T^2 & = & \{S_T - K_2\}_+ \end{array}$$

where $\{x\}_+ := \max\{0, x\}$. Note that since $K_1 < K_2$ then $C_T^2 \leq C_T^1 \ \forall \ S_T \in \mathbb{R}^{\geq 0}$.
We can restate X_T in terms of C_T^1 and C_T^2 as

$$\begin{array}{rcl} X_T & = & \begin{cases} C_T^1 - C_T^2 & \text{if } S_T \geq K_2 \\ C_T^1 & \text{if } S_T \in [K_1, K_2] \\ 0 & \text{if } S_T \leq K_1 \end{cases} \\ & = & C_T^1 - C_T^2 \end{array}$$

Using the "No-Arbitrage Principle" it can be shown that if two, or more, financial derivatives have the same value at time T, then their prices will coincide at all times $t < T$. This means the fair price at time t for this capped call option is

$$X_t = C_t^1 - C_t^2$$

Answer 1. c) i.
A probability measure $\mathbb{Q}$ is a *Risk-Neutral Probability Measure* if the following all hold

i). $\mathbb{Q}(\{\omega\}) > 0 \ \forall \ \omega \in \Omega$; and,

ii). $\mathbb{E}_{\mathbb{Q}}[S_1^*(1)] = S_1^*(0)$

Additionally, as $\mathbb{Q}$ is a probability measure we have $\sum_{\omega \in \Omega} \mathbb{Q}(\{\omega\}) = 1$.

From the question we have that $S_0 = 1$, $S_1(\omega_1) = 1.3$ and $S_2(\omega_2) = 1.1$. Let r denote the risk-free interest rate, $q_1 := \mathbb{Q}(\{\omega_1\})$ and $q_2 := \mathbb{Q}(\{\omega_2\})$. Note that the Bank process at time $t = 1$ has value $B_1 = 1 + r$.

Under the conditions of this question, we can derive the follow equations which must hold in order for $\mathbb{Q}$ to be a *Risk-Neutral Probability Measure*

$$\begin{array}{lrcll} & q_1 + q_2 & = & 1 & (1) \\ \text{and} & \mathbb{E}_{\mathbb{Q}}[S_1^*(1)] & = & S_1^*(0) & \\ \implies & q_1 S_1^*(\omega_1) + q_2 S_1^*(\omega_2) & = & \frac{S_1(0)}{B_0} & \\ \implies & q_1 \cdot \frac{S_1(\omega_1)}{B_1} + q_2 \cdot \frac{S_1(\omega_2)}{B_1} & = & \frac{S_1(0)}{B_0} & \\ \implies & q_1 \cdot \frac{1.1}{1+r} + q_2 \cdot \frac{1.3}{1+r} & = & 1 & (2) \end{array}$$

From equations $(1), (2)$ we can deduce values for q_1, q_2 in terms of r.

$$\begin{array}{rrcl}
 & q_2 & = & 1 - q_1 \\
\implies & q_1 \cdot \dfrac{1.1}{1+r} + (1-q_1) \cdot \dfrac{1.3}{1+r} & = & 1 \\
\implies & \dfrac{13}{10(1+r)} - q_1 \cdot \dfrac{2}{10(1+r)} & = & 1 \\
\implies & q_1 & = & \dfrac{13-10(1+r)}{2} \\
 & & = & \dfrac{3-10r}{2} \\
\implies & q_2 & = & 1 - \dfrac{3-10r}{2} \\
 & & = & \dfrac{10r-1}{2}
\end{array}$$

Thus, probability measure $\mathbb{Q}$ can be stated as

$$\mathbb{Q}(\{\omega_1\}) = \frac{3-10r}{2} \quad \mathbb{Q}(\{\omega_2\}) = \frac{10r-1}{2}$$

As both these quantities must take values in $[0,1]$ we can deduce the range of interest rates r where a *Risk-Neutral Probability Measure* exists.

$$\begin{array}{rrcl rcl}
 & \mathbb{Q}(\{\omega_1\}) & \in & [0,1] & \mathbb{Q}(\{\omega_2\}) & \in & [0,1] \\
\implies & \dfrac{3-10r}{2} & \in & [0,1] & \dfrac{10r-1}{2} & \in & [0,1] \\
\implies & 3-10r & \in & [0,2] & 10r-1 & \in & [0,2] \\
\implies & 10r & \in & [1,3] & 10r & \in & [1,3] \\
\implies & r & \in & [0.1, 0.3] & r & \in & [0.1, 0.3]
\end{array}$$

Thus, there exists a *Risk-Neutral Probability Measure* if $r \in [0.1, 0.3]$.

Answer 1. c) ii.
At time-point $t = 0$ the agent should do the following

- Short a share, receiving $S_0 =$£1.
- Invest this £1 in the risk-free asset.

At time-point $t = 1$ this investment is work $B_1 = 1 + r = 1.4$. Then, at time-point $t = 1$ the agent should do the following.

- Pay the dividend $D = 0.1$ to the agent who leant us the share used for our short position.
- Buy a share, at whatever the current price is, in order to fulfil our short position.

If our arbitrageur follows this strategy and event ω_1 occurs, then they make a risk-free profit of

$$B_1 - S_1(\omega_1) - D = 1.4 - 1.3 - 0.1 = 0$$

If our arbitrageur follows this strategy and event ω_2 occurs, then they make a risk-free profit of

$$B_1 - S_1(\omega_2) - D = 1.4 - 1.1 - 0.1 = 0.2$$

Thus, under either event our arbitrageur is guaranteed not to loose money.