Financial Mathematics - Reviewed Notes

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February 24, 2021

Contents

1 General

Definition 1.1 - *Modelling*
TODO

Definition 1.2 - *Risk-Free*
An activity is said to be "*Risk-Free*" if the potential profits & losses are completely known.[1]

2 Probability

2.1 General Probability

Definition 2.1 - *Sample Space* Ω
The *Sample Space* Ω is the set consisting of all elementary outcomes from a (series of) event(s).

Definition 2.2 - *Random Variable* X
A *Random Variable* X is a function from the *Sample Space* Ω to the real numbers $\mathbb{R}$.

$$X : \Omega \to \mathbb{R}$$

2.2 Information Structures

Definition 2.3 - *Partition* $\mathcal{P}$
Let $\mathcal{P} := \{A_1, \dots, A_N\}$ be a set (of sets) and Ω be a *Sample Space*.

$\mathcal{P}$ is a *Partition* of Ω if it has the following properties

i). All elements in $\mathcal{P}$ are mutually disjoint

$$A_i \cap A_j = \emptyset \ \forall \ A_i, A_j \in \mathcal{P}$$

ii). The union of the elements form the *Sample Space* Ω.

$$\bigcup_{i=1}^{N} A_i = \Omega$$

Remark 2.1 - *Flow of Information*
At time $t = 0$ every state $\omega \in \Omega$ is a possible outcome at time $t = T$. And, at time $t = T$ we know for certain which outcome has occurred.

At each time in-between $t \in (0, T)$ our information about the world increases[2] meaning the set of possible outcomes at time $t = T$ may decrease. Let A_t denote the possible set of outcomes given we are at time t, then

$$\begin{array}{rcl} A_0 & = & \Omega \\ A_T & = & \{\omega\} \\ A_0 & \supseteq & A_1 \supseteq \cdots \supseteq A_T \end{array}$$

[1]It does not refer to whether there no chance of making a loss.
[2]or, at least, does not decrease.

Flipping a coin 3 times is a motivating example. Before we start flipping ($t = 0$) it is possible that we will flip three tails, but if the first flip ($t = 1$) is heads then this is no longer possible.

Definition 2.4 - *Information Sequence* $\{\mathcal{P}_0, \dots, \mathcal{P}_T\}$
An *Information Sequence* is a set of *Partitions* $\{\mathcal{P}_0, \dots, \mathcal{P}_T\}$ of the *Sample Space* Ω, which fulfil the following criteria

i). $\mathcal{P}_0 = \big\{\Omega\big\}$.

ii). For $t \in [1, T)$ each $A \in \mathcal{P}_t$ is equal to the union of a subset of elements in $\mathcal{P}_{t+1}$.

iii). $\mathcal{P}_T = \big\{\{\omega_1\}, \dots, \{\omega_N\}\big\}$.

Information Sequences show the set of possible events, at each time point t, which could still occur.[3]

Remark 2.2 - *Visualising Information Structures*
TODO

Definition 2.5 - *σ-Algebra $\mathcal{F}$*
A *σ-Algebra* $\mathcal{F}$ is a set of subsets of the *Sample Space* Ω which satisfy the following conditions

i). $\Omega \in \mathcal{F}$.

ii). $\forall\ A \in \mathcal{F},\ A^c \in \mathcal{F}$.

iii). $\forall\ A, B \in \mathcal{F},\ (A \cup B) \in \mathcal{F}$.

Definition 2.6 - *Filtration* $\{\mathcal{F}_0, \dots, \mathcal{F}_T\}$
A *Filtration* is a sequence of *σ-Algebras* $\{\mathcal{F}_t : t = 0, 1, \dots, T\}$ where

i). $\mathcal{F}_0 = \{\emptyset, \Omega\}$.

ii). $\forall\ n < T,\ \mathcal{F}_n \subset \mathcal{F}_{n+1}$ (Meaning each subset of $\mathcal{F}_n$ must be an element of $\mathcal{F}_{n+1}$).

iii). $\mathcal{F}_T = 2^\Omega$.[4]

A *Filtration* represents our understanding of available information at each time point.

Definition 2.7 - *Measurable Function*
Consider a random variable $X : \Omega \to \mathbb{R}$ and a *σ-Algebra* $\mathcal{F}$.

X is *Measurable* wrt $\mathcal{F}$ if

$$\forall\ x \in \mathbb{R}, X^{-1}(x) \subset \mathcal{F} \text{ where } X^{-1}(x) := \{\omega \in \Omega : X(\omega) = x\}$$

This can be interpreted to mean that, if we known which set of $\mathcal{F}$ ω is in, then we know the values of $X(\omega)$.

Proposition 2.1 - *Measurability and Filtrations*
Consider a *Filtration* $\{\mathcal{F}_1, \dots, \mathcal{F}_T\}$ and a random variable $X : \Omega \to \mathbb{R}$.

[3] An *Information Sequence* is a sequence of *σ-Algebras*.
[4] The set of all subsets of the sample space Ω.

If X is *Measurable* wrt $\mathcal{F}_t$ then it is *Measurable* wrt $\mathcal{F}_{t+1}$ since $\mathcal{F}_t \subseteq \mathcal{F}_{t+1}$.

Proposition 2.2 - *How to generate σ-Algebras*
Let $\mathcal{P}$ be a *Partition* of the *Sample Space* Ω.

We can generate a *σ-Algebra* $\mathcal{F}$ from $\mathcal{P}$ be defining $\mathcal{F}$ to be the set of all possible unions from elements in $\mathcal{P}$ as well as the compliments of all these unions.

2.3 Conditional Expectation

Definition 2.8 - *Conditional Expectation* $\mathbb{E}[\cdot|\cdot]$
Let Ω be a finite sample space, X be a discrete random variable and $A \subseteq \Omega$.

The *Conditional Expectation* of X given A has occurred is defined as

$$\mathbb{E}[X|A] = \sum_x x\mathbb{P}(X = x|A)$$

Remark 2.3 - *Alternative Definitions of Conditional Expectation*
Here are two restatements of the definition of *Conditional Expectation*, both are consequences of *Bayes Rule*.

$$\begin{aligned}\mathbb{E}[X|A] &= \sum_x \frac{\mathbb{P}(X = x, A)}{\mathbb{P}(A)} \\ \mathbb{E}[X|A] &= \sum_{\omega\in A} X(\omega)\frac{\mathbb{P}(\omega)}{\mathbb{P}(A)}\end{aligned}$$

Definition 2.9 - *Conditional Expectation \w σ-Algebra* $\mathbb{E}[\cdot|\mathcal{F}]$
Let $\mathcal{F}$ be a *σ-algebra*, $\mathcal{P}$ be the corresponding *Partition* of the sample space Ω and X be a discrete random variable.

The *Conditional Expectation* of X given $\mathcal{F}$ is defined as

$$\mathbb{E}[X|\mathcal{F}] := \sum_{A\in\mathcal{P}} \mathbb{E}[X|A]\mathbb{1}\{A\}^{[5]}$$

[6] Note - This is a random variable as its value depends on which random event A occurs. Moreover, it is *Measurable* wrt $\mathcal{F}$ and for a given $A \in \mathcal{P}$

$$\forall\ \omega \in A,\ \mathbb{E}[X|\mathcal{F}](\omega) = \mathbb{E}[X|A]^{[7]}$$

Theorem 2.1 - *Tower Law*
Let X be a discrete random variable and $\mathcal{F}_1, \mathcal{F}_2$ be *σ-Algebras* with $\mathcal{F}_1 \subset \mathcal{F}_2$.

The *Tower Law* states that

$$\mathbb{E}\big[\mathbb{E}[X|\mathcal{F}]\big] = \mathbb{E}[X]$$

The *Generalised Tower Law* states that

$$\mathbb{E}\big[\mathbb{E}[X|\mathcal{F}_2]\big|\mathcal{F}_1\big] = \mathbb{E}[X|\mathcal{F}_1] = \mathbb{E}\big[\mathbb{E}[X|\mathcal{F}_1]\big|\mathcal{F}_2\big]$$

[6] This is not really a summation as there is only one event A st $\mathbb{1}\{A\} = 1$.
[7] This is intuitive from the definition of $\mathbb{E}[\cdot|\mathcal{F}]$.

Proof 2.1 - *Theorem 2.1 - Tower Law*
Let X be a discrete random variable, $\mathcal{F}$ be a σ-*Algebra* and $\mathcal{P}$ be the partition of the sample space Ω associated with $\mathcal{F}$.

$$\begin{aligned}
\mathbb{E}\big[\mathbb{E}[X|\mathcal{F}]\big] &= \mathbb{E}\left[\sum_{A\in\mathcal{P}}\mathbb{E}[X|A]\mathbb{1}\{A\}\right] \text{ by def.}\\
&= \sum_{A\in\mathcal{P}}\mathbb{E}\big[\mathbb{E}[X|A]\mathbb{1}\{A\}\big] \text{ by linearity of expectation}\\
&= \sum_{A\in\mathcal{P}}\mathbb{E}[X|A]\mathbb{E}[\mathbb{1}\{A\}]\\
&= \sum_{A\in\mathcal{P}}\mathbb{P}(A)\cdot\left(\sum_{\omega\in A}\frac{X(\omega)\mathbb{P}(\omega)}{\mathbb{P}(A)}\right) \text{ by alt def.}\\
&= \sum_{A\in\mathcal{P}}\sum_{\omega\in A}X(\omega)\mathbb{P}(\omega) \text{ as } \sum\mathbb{P}(A)=1\\
&= \mathbb{E}[X] \text{ by def.}
\end{aligned}$$

□

Proof 2.2 - *Theorem 2.1 - Generalised Tower Law*
Let X be a discrete random variable, $\mathcal{F}_1,\mathcal{F}_2$ be σ-*Algebras* with $\mathcal{F}_1\subset\mathcal{F}_2$ and $\mathcal{P}_1,\mathcal{P}_2$ be the partitions associated to $\mathcal{F}_1,\mathcal{F}_2$.

$$\begin{aligned}
\mathbb{E}\big[\mathbb{E}[X|\mathcal{F}_2]\big|\mathcal{F}_1\big] &= \mathbb{E}\left[\sum_{B\in\mathcal{P}_2}\mathbb{E}[X|B]\mathbb{1}\{B\}\middle|\mathcal{F}_1\right]\\
&= \sum_{B\in\mathcal{P}_2}\mathbb{E}[X|B]\mathbb{E}[\mathbb{1}\{B\}|\mathcal{F}_1]\\
&= \sum_{B\in\mathcal{P}_\in}\mathbb{E}[X|B]\left(\sum_{A\in\mathcal{P}_1}\mathbb{E}[\mathbb{1}\{B\}|A]\mathbb{1}\{A\}\right)\\
&= \sum_{A\in\mathcal{P}_1}\sum_{B\in\mathcal{P}_\in}\mathbb{E}[X|B]\mathbb{E}[\mathbb{1}\{B\}|A]\mathbb{1}\{A\}\\
&= \sum_{A\in\mathcal{P}_1}\sum_{B\in\mathcal{P}_\in}\mathbb{E}[X|B]\cdot\frac{\mathbb{P}(A\cap B)}{\mathbb{P}(A)}\cdot\mathbb{1}\{A\}
\end{aligned}$$

Since $\mathcal{P}_2$ is more refined than $\mathcal{P}_1$, either $B\subset A$ or $B\cap A=\emptyset$. Thus

$$\begin{aligned}
\mathbb{E}\big[\mathbb{E}[X|\mathcal{F}_2]\big|\mathcal{F}_1\big] &= \sum_{A\in\mathcal{P}_1}\sum_{B\in\mathcal{P}_\in,B\subset A}\mathbb{E}[X|B]\cdot\frac{\mathbb{P}(B)}{\mathbb{P}(A)}\cdot\mathbb{1}\{A\}\\
&= \sum_{A\in\mathcal{P}_1}\sum_{B\in\mathcal{P}_\in,B\subset A}\left(\sum_{\omega\in B}X(\omega)\frac{\mathbb{P}(\omega)}{\mathbb{P}(B)}\right)\cdot\frac{\mathbb{P}(B)}{\mathbb{P}(A)}\cdot\mathbb{1}\{A\}\\
&= \sum_{A\in\mathcal{P}_1}\sum_{\omega\in B}X(\omega)\frac{\mathbb{P}(\omega)}{\mathbb{P}(A)}\cdot\mathbb{1}\{A\}\\
&= \sum_{A\in\mathcal{P}_1}\mathbb{E}[X|A]\mathbb{1}\{A\}\\
&= \mathbb{E}[X|\mathcal{F}_1]
\end{aligned}$$

□

Theorem 2.2 - *Conditional Expectation & Measurable Random Variables*
Let $\mathcal{F}$ be a σ-*Algebra* and X, Y be discrete random variables with X being *Measurable* wrt $\mathcal{F}$.

Then
$$\begin{array}{rcl}\mathbb{E}[X|\mathcal{F}] & = & X \\ \mathbb{E}[XY|\mathcal{F}] & = & X\mathbb{E}[Y|\mathcal{F}]\end{array}$$

Proof 2.3 - *Theorem 2.2*
Let $\mathcal{F}$ be a σ-*Algebra*, $\mathcal{P}$ be the partition associated with $\mathcal{F}$ and X, Y be discrete random variables with Y being *Measurable* wrt $\mathcal{F}$.

Since Y is *Measurable* it is constant on sets of $\mathcal{P}$ we write X as
$$Y = \sum_{A \in \mathcal{P}} Y_A \mathbb{1}\{A\} \text{ with } Y_A \in \mathbb{R}$$

Thus
$$\begin{array}{rcl}\mathbb{E}[XY|\mathcal{F}] & = & \sum_{A\in\mathcal{P}} \mathbb{E}[XY|A]\mathbb{1}\{A\} \\ & = & \sum_{A\in\mathcal{P}} \mathbb{E}[XY_A|A]\mathbb{1}\{A\} \\ & = & \sum_{A\in\mathcal{P}} Y_A\mathbb{E}[X|A]\mathbb{1}\{A\} \text{ as } Y_A \text{ is a scalar} \\ & = & \sum_{A\in\mathcal{P}} Y\mathbb{E}[X|A]\mathbb{1}\{A\}^{[8]} \\ & = & Y\sum_{A\in\mathcal{P}} \mathbb{E}[X|A]\mathbb{1}\{A\} \\ & = & Y\mathbb{E}[X|\mathcal{F}]\end{array}$$

□

Theorem 2.3 - *Conditional Expectation & Independent Random Variables*
Let $\mathcal{F}$ be a σ-*Algebra* and X be a discrete random variable which is independent of $\mathcal{F}$.

Then
$$\mathbb{E}[X|\mathcal{F}] = \mathbb{E}[X]$$

Proof 2.4 - *Theorem 2.3*
Let $\mathcal{F}$ be a σ-*Algebra* and X be a discrete random variable which is independent of $\mathcal{F}$.
$$\begin{array}{rcl}\mathbb{E}[X|A] & = & \sum_x \mathbb{P}(X=x|A) \\ & = & \sum_x \mathbb{P}(X=x) \text{ by independence} \\ & = & \mathbb{E}[X]\end{array}$$

□

Theorem 2.4 - *General Conditional Expectation*
Let $\mathcal{F}$ be a σ-*Algebra* of a general sample space[9] Ω and X be a discrete random variable.

Then, the *Conditional Expectation* $\mathbb{E}[X|\mathcal{F}]$ is a unique random variable with the following properties

[8] As there is only one event A where $\mathbb{1}\{A\} = 1$.
[9] i.e. Not necessarily finite

i). $\mathbb{E}[X|\mathcal{F}]$ is *Measurable* wrt $\mathcal{F}$.

ii). $\forall\ A \in \mathcal{F},\ \mathbb{E}\big[\mathbb{E}[X|\mathcal{F}]\mathbb{1}\{A\}\big] = \mathbb{E}[X\mathbb{1}\{A\}]$

Proof 2.5 - *Theorem 2.4*
Let $\mathcal{F}$ be a σ-*Algebra* of a general sample space Ω, $\mathcal{P}$ be the partition associated with $\mathcal{F}$ and X be a discrete random variable.

i). Let Y be a random variable which is *Measurable* wrt $\mathcal{F}$ and satisfies

$$\mathbb{E}[Y\mathbb{1}\{A\}] = \mathbb{E}[X\mathbb{1}\{A\}]\ \forall\ A \in \mathcal{F}$$

Consider the expression $\mathbb{E}[X\mathbb{1}\{A\}]$

$$\begin{aligned}\mathbb{E}[X\mathbb{1}\{A\}] &= \sum_{\omega\in A} X(\omega)\mathbb{P}(\omega)\\ &= \frac{\mathbb{P}(A)}{\mathbb{P}(A)}\sum_{\omega\in A} X(\omega)\mathbb{P}(\omega)\\ &= \mathbb{P}(A)\sum_{\omega\in A}\frac{X(\omega)\mathbb{P}(\omega)}{\mathbb{P}(A)}\\ &= \mathbb{P}(A)\mathbb{E}[X|A]\end{aligned}$$

Now, Note that $Y = \sum_{A\in\mathcal{P}} Y_A\mathbb{1}\{A\}$ (As in `Proof 2.3`).

It follows that

$$\forall\ A \in \mathcal{P},\ \mathbb{E}[Y\mathbb{1}\{A\}] = Y_A\mathbb{E}[\mathbb{1}\{A\}] = Y_A\mathbb{P}(A)$$

We now have that

$$\begin{aligned} & \mathbb{E}[X\mathbb{1}\{A\}] &= \mathbb{E}[Y\mathbb{1}\{A\}] \text{ by def. } Y\\ \implies & \mathbb{P}(A)\mathbb{E}[X|A] &= Y_A\mathbb{P}(A)\\ \implies & Y_A &= \mathbb{E}[X|A]\ \forall\ A\in\mathcal{P}\\ \implies & Y &= \mathbb{E}[X|\mathcal{F}]\end{aligned}$$

As we defined Y to be *Measurable* wrt $\mathcal{F}$, this means $\mathbb{E}[X|\mathcal{F}]$ is *Measurable* wrt $\mathcal{F}$.

ii). For any event $A \in \mathcal{F}$, the indicator function $\mathbb{1}\{A\}$ is $\mathcal{F}$-*Measurable*.

Thus, $\mathbb{E}[X\mathbb{1}\{A\}|\mathcal{F}] = \mathbb{1}\{A\}\cdot\mathbb{E}[X|\mathcal{F}]$ by `Theorem 2.2`.

Hence, by the Tower Law (`Theorem 2.1`).

$$\mathbb{E}\big[\mathbb{E}[X|\mathcal{F}]\cdot\mathbb{1}\{A\}\big] = \mathbb{E}\big[\mathbb{E}[X\mathbb{1}\{A\}|\mathcal{F}]\big] = \mathbb{E}[X\mathbb{1}\{A\}]$$

□

2.4 Stochastic Processes in Discrete Time

Definition 2.10 - *Stochastic Process*
A *Stochastic Process* S is a real-valued function $S(t)(\omega)$

$$S : [0,T]\times\Omega \to \mathbb{R}$$

Proposition 2.3 - *Fixing components of a Stochastic Process*
Let a *Stochastic Process* $S : [0,T]\times\Omega\to\mathbb{R}$ and consider fixing different variables

- If we fix $t \in [0, T]$ then $S(t)(\cdot) : \Omega \to \mathbb{R}$[10] is a *Random Variable.*
- If we fix $\omega \in \Omega$ then $S(\cdot)(\omega) : [0, T] \to \mathbb{R}$[11] is called a *Sample Path.*

Definition 2.11 - *Adapted Stochastic Process*
Let $S : [0, T] \times \Omega \to \mathbb{R}$ be a *Stochastic Process* and $\{\mathcal{F}_t\}_{t\in[0,T]}$ be a *Filtration.*

S is *Adapted* to *Filtration* $\{\mathcal{F}_t\}_{t\in[0,T]}$ if the *Random Variable* $S(t)$ is *Measurable* wrt σ-algebra $\mathcal{F}_t$, for all $t \in [0, T]$.[12]

Definition 2.12 - *Natural Filtration*
Let $S : [0, T] \times \Omega \to \mathbb{R}$ be a *Stochastic Process.*

We generate the *Natural Filtration* $\{\mathcal{F}_t\}_{t\in[0,T]}$ for S by doing the following for each $t = 0, 1, \dots, T$

i). Define $\mathcal{P}_t$ to be a partition of the *Sample Space* Ω st $S(t)(\cdot)$ takes the same value for each element in each subset of $\mathcal{P}_t$.

$$\mathcal{P}_t := \big\{A_1, \dots, A_m : S(t)(a) = S(t)(a')\ \forall\ a, a' \in A\big\} \text{ and } A_1, \dots, A_m \text{ form a partition.}$$

ii). Define $\mathcal{F}_t$ to be the *σ-Algebra* generated by[13] partition $\mathcal{P}_t$.

Definition 2.13 - *Random Walk*
Let $Y_0, Y_1, \dots$ be IID random variables with finite variance σ^2 and finite mean μ.

A *Random Walk* is the sequence $\{X_t\}_{t\geq 0}$ where $X_t := \sum_{i=1}^t Y_i$.

Definition 2.14 - *Simple Random Walk*
Let $\{X_t\}_{t\geq 0}$ be a *Random Walk* with $X_t := \sum_{i=1}^t Y_i$ where $Y_0, Y_1, \dots$ are IID RVs.

We say that $\{X_t\}_{t\geq 0}$ is a *Simple Random Walk* if

$$Y_t \in \{-1, 1\} \quad \mathbb{P}(Y_t = 1) = p \quad \mathbb{P}(Y_t = -1) = 1 - p$$

A *Simple Random Walk* can be thought of as a process where you only ever step forward or step backwards, with fixed probabilities.

Theorem 2.5 - *Distribution of a Simple Random Walk*
Let $\{X_t\}_{t\geq 0}$ be a *Simple Random Walk.* Then

$$\mathbb{P}(X_t = x) = \binom{t}{\frac{t+x}{2}} p^{(t+x)/2}(1-p)^{(t-x)/2} \quad \forall\ t \geq 0,\ x \in \{-t, -t+2, \dots, t\}$$

Note that the set of possible x values steps by 2.

Proof 2.6 - `Theorem 2.5`
Note that $x = \frac{1}{2}(2x + t - t) = (+1) \cdot \frac{1}{2}(t + x) + (-1) \cdot \frac{1}{2}(t - x)$.

For $X_t = x$ we require exact $\frac{1}{2}(t + x)$ of $Y_1, \dots, Y_t$ to take value 1, and then the remaining $\frac{1}{2}(t - x)$ will take value -1. There are $\binom{t}{\frac{t+x}{2}}$ different ways this can occur.

[10]The event ω is the only variable
[11]The time-point t is the only variable
[12]It is often easier to define a stochastic process first and then find a filtration for it (e.g. the *Natural Filtration*).
[13]See `Proposition 2.2`.

Note that each Y_i takes its value independently and takes value 1 with probability p and -1 with probability $1-p$. Thus

$$\mathbb{P}(X_t = x) = \binom{t}{\frac{t+x}{2}} p^{(t+x)/2}(1-p)^{(t-x)/2} \quad \forall\ t \geq 0,\ x \in \{-t, -t+2, \dots, t\}$$

2.5 Martingales

Definition 2.15 - *Martingale* $\{Z_t\}_{t\in\mathbb{N}_0}$
Let $\{Z_t\}_{t\in[0,T]}$ be an *Adapted Stochastic Process* on a *Sample Space* Ω with a *Filtration* $\{\mathcal{F}_t\}_{t\in[0,T]}$.

- $\{Z_t\}_{t\in[0,T]}$ is a *Martingale* if

$$\forall\ t \geq 1,\ \mathbb{E}[Z_t|\mathcal{F}_{t-1}] = Z_{t-1}$$

  This can be interpreted to mean that, given all available information $\mathcal{F}_{t-1}$, our present state Z_{t-1} is the best indicator of the future state Z_t.

- $\{Z_t\}_{t\in[0,T]}$ is a Super-*Martingale* if

$$\forall\ t \geq 1,\ \mathbb{E}[Z_t|\mathcal{F}_{t-1}] \leq Z_{t-1}$$

  This can be interpreted to mean that, given all available information $\mathcal{F}_{t-1}$, our present state Z_{t-1} provides an upper-bound on the future state Z_t.

- $\{Z_t\}_{t\in[0,T]}$ is a Sub-*Martingale* if

$$\forall\ t \geq 1,\ \mathbb{E}[Z_t|\mathcal{F}_{t-1}] \geq Z_{t-1}$$

  This can be interpreted to mean that, given all available information $\mathcal{F}_{t-1}$, our present state Z_{t-1} provides a lower-bound on the future state Z_t.

Proposition 2.4 - *Notable Martingales*
Let $\{X_t\}_{t\in\mathbb{N}_0}$ be a *Simple Random Walk* with parameter p and let $\mathcal{F}_t$ be the σ-*Algebra* generated by X_t. Then

i). If $p = 1/2$ $\{X_t\}_{t\in\mathbb{N}_0}$ is a *Martingale.*

ii). If $p \leq 1/2$ $\{X_t\}_{t\in\mathbb{N}_0}$ is a *Super-Martingale.*

iii). If $p \geq 1/2$ $\{X_t\}_{t\in\mathbb{N}_0}$ is a *Sub-Martingale.*

iv). If $p = 1/2$ then $\{Z_t\}_{t\in\mathbb{N}_0}$ where $Z_t := (X_t^2 - t)$ is a *Martingale.*

v). If $p \neq 1/2$ then $\{L_t\}_{t\in\mathbb{N}_0}$ where $L_0 := 1, L_t := \left(\frac{1-p}{p}\right)^{X_t}$ is a *Martingale Martingale.*

vi). If $p \neq 1/2$ then $\{M_t\}_{t\in\mathbb{N}_0}$ where $M_t := \big(X_t - t(2p-1)\big)$ is a *Martingale. Martingale.*

Proof 2.7 - *Proposition 2.4 i)-iii)*
Let $\{X_t\}_{t\in\mathbb{N}_0}$ be a *Simple Random Walk* with parameter p and let $\mathcal{F}_t$ be the σ-*Algebra* generated by X_t.

Since $\{\mathcal{F}_t\}_{t\in\mathbb{N}}$ is the *Natural Filtration* of $\{X_t\}_{t\in\mathbb{N}}$, then $\{X_t\}_{t\in\mathbb{N}}$ is Measurable wrt $\mathcal{F}_t$ and Y_t[14] is independent of $\mathcal{F}_{t-1}$.

Then

$$\begin{array}{rcl}\mathbb{E}[X_t|\mathcal{F}_{t-1}] & = & \mathbb{E}[X_{t-1}+Y_t|\mathcal{F}_{t-1}] \text{ by def. } X_t\\ & = & \mathbb{E}[X_{t-1}|\mathcal{F}_{t-1}]+\mathbb{E}[Y_t|\mathcal{F}_{t-1}] \text{ by linearity of exp.}\\ & = & X_{t-1}+\mathbb{E}[Y_t] \text{ by } \texttt{Theorem 2.3}\end{array}$$

Thus

- If $p=1/2$ then $\mathbb{E}[Y_t]=0 \implies \mathbb{E}[X_t|\mathcal{F}_{t-1}]=X_{t-1}$. This is the definition of a *Martingale*.
- If $p\leq 1/2$ then $\mathbb{E}[Y_t]\leq 0 \implies \mathbb{E}[X_t|\mathcal{F}_{t-1}]\leq X_{t-1}$. This is the definition of a *Super-Martingale*.
- If $p\geq 1/2$ then $\mathbb{E}[Y_t]\geq 0 \implies \mathbb{E}[X_t|\mathcal{F}_{t-1}]\geq X_{t-1}$. This is the definition of a *Sub-Martingale*.

□

Proof 2.8 - *`Proposition 2.4` iv)*
Let $\{X_t\}_{t\in\mathbb{N}_0}$ be a *Simple Random Walk* with parameter p and let $\mathcal{F}_t$ be the *σ-Algebra* generated by X_t.

As the definition of a *Martingale* depends on the conditional expectation of Z_t given $\mathcal{F}_{t-1}$ we consider its value

$$\begin{array}{rcl}\mathbb{E}[Z_t|\mathcal{F}_{t-1}] & = & \mathbb{E}[X_t^2-t|\mathcal{F}_{t-1}] \text{ by def. } Z_t\\ & = & \mathbb{E}[(X_{t-1}+Y_t)^2-t|\mathcal{F}_{t-1}] \text{ by def. } X_t\\ & = & \mathbb{E}[(X_{t-1}+Y_t)^2|\mathcal{F}_{t-1}]-t\\ & = & \mathbb{E}[X_{t-1}^2+2X_{t-1}Y_t+Y_t^2|\mathcal{F}_{t-1}]-t\\ & = & \mathbb{E}[X_{t-1}^2|\mathcal{F}_{t-1}]+2\mathbb{E}[X_tY_t|\mathcal{F}_{t-1}]+\mathbb{E}[Y_t^2|\mathcal{F}_{t-1}]-t\\ & = & X_{t-1}^2+2X_{t-1}\mathbb{E}[Y_t|\mathcal{F}_{t-1}]+\mathbb{E}[Y_t^2]-t\end{array}$$

Since $p=1/2 \implies \mathbb{E}[Y_t]=0, \mathbb{E}[Y_t^2=1]$.

$$\begin{array}{rcl}\mathbb{E}[Z_t|\mathcal{F}_{t-1}] & = & X_{t-1}^2+2X_{t-1}\mathbb{E}[Y_t|\mathcal{F}_{t-1}]+\mathbb{E}[Y_t^2]-t\\ & = & X_{t-1}^2+0+1-t\\ & = & X_{t-1}^2-(t-1)\\ & = & Z_{t-1}\\ \implies \mathbb{E}[Z_t|\mathcal{F}_{t-1}] & = & Z_{t-1}\end{array}$$

This is the definition of a *Martingale*. □

Proof 2.9 - *`Proposition 2.4` v)*
TODO (Homework)

Proof 2.10 - *`Proposition 2.4` vi)*
TODO (Homework)

Theorem 2.6 - *Adapted Stochastic Processes as Martingales*
Let $\{Z_t\}_{t\in[0,T]}$ be an *Adapted Stochastic Process*.

[14]The t^{th} step of the random walk.

$\{Z_t\}_{t\in[0,T]}$ is a *Martingale* iff
$$\forall\ t \geq s,\ \mathbb{E}[Z_t|\mathcal{F}_s] = Z_s$$[15]

Proof 2.11 - *Theorem 2.6*
Let $\{Z_t\}_{t\in[0,T]}$ be an *Adapted Stochastic Process*.

I prove this statement in both directions[16]

$\implies$ Suppose $\{Z_t\}_{t\in[0,T]}$ is a *Martingale*.

Using `Theorem 2.4` we can deduce that

$$\begin{array}{rcll}\mathbb{E}[Z_t|\mathcal{F}_s] &=& \mathbb{E}\big[\mathbb{E}[Z_t|\mathcal{F}_{t-1}]\big|\mathcal{F}_s\big] & \text{by Theorem 2.4}\\ &=& \mathbb{E}[Z_{t-1}|\mathcal{F}_s] & \text{as } Z \text{ is a Martingale}\\ &=& \mathbb{E}[Z_s|\mathcal{F}_s] & \text{by recursion}\\ &=& \mathbb{E}[Z_s] & \text{by Theorem 2.2}\\ &=& Z_s & \end{array}$$

$\impliedby$ Suppose it holds that
$$\forall\ t \geq s,\ \mathbb{E}[Z_t|\mathcal{F}_s] = Z_s$$

Consider the case where $s = t-1$, it holds that
$$\mathbb{E}[Z_t|\mathcal{F}_{t-1}] = Z_{t-1}$$

This is the definition of a *Martingale*.

$\square$

Definition 2.16 - *Stopping Times* τ
Let $\{\mathcal{F}_t\}_{t\in\mathbb{N}_0}$ be a *Filtration* of *Sample Space* Ω and τ be a random variable which takes values in $(\mathbb{R}\geq 0 \cup \{\infty\})$[17].

τ is a *Stopping Time* if the event $\{\tau < t\}$ is an element of the σ-*Algebra* $\mathcal{F}_t$.

Stopping Times are used to determine whether an event has occurred, or not.[18]

Definition 2.17 - *Bounded Stopping Time* τ
Let τ be a *Stopping Time*.

A τ is a *Bounded Stopping Time* if
$$\exists\ t \in \mathbb{R}^{\geq 0},\ \mathbb{P}(\tau < t) = 1$$

Theorem 2.7 - *Stopping Times & σ Algebras*
Let τ be a random variable.

τ is a *Stopping Time* iff $\forall\ t \in \mathbb{N}_0$ the event $\{\tau = t\}$ is an element of the σ-*Algebra* $\mathcal{F}_t$.

[15]Equivalent results can be made for *Super*- and *Sup-Martingales* by replacing $=$ with $\leq, \geq$ respectively.
[16]The proofs for *Super*- and *Sup-Martingales* are very similar.
[17]∞ is used for impossible events.
[18]Examples of *Stopping Times* are "RBS shares hit £1".

Proof 2.12 - *Theorem 2.7*
Let τ be a random variable.

I prove this statement in both directions

$\implies$ Suppose τ is a *Stopping Time.*

Then the event $\{\tau \leq t\}$ is an element of the σ*-Algebra* $\mathcal{F}_t$ $\forall$ $t \in \mathbb{N}_0$.

We can restate this event as

$$\{\tau \leq t\} = \bigcup_{k \leq t} \{\tau = k\}$$

As $\{\tau \leq t\} \in \mathcal{F}_t$, then each of $\{\tau = k\} \in \mathcal{F}_t$ due to the definition of a σ*-Algebra.*

$\impliedby$ Suppose the event $\{\tau = t\}$ is an element of the σ*-Algebra* $\mathcal{F}_t$ $\forall$ $t \in \mathbb{N}_0$.

We can restate this event as

$$\{\tau \leq t\} = \big(\{\tau \leq t\} \setminus \{\tau \leq t-1\}\big)$$

Since $\{\tau \leq t\}, \{\tau \leq t-1\}$ are elements of $\mathcal{F}_t$, then $\{\tau \leq t\} \in \mathcal{F}_t$ due to the definition of a σ*-Algebra.*

$\square$

Theorem 2.8 - *Stopping Time for an Adapted Stochastic Process*
Let $\{X_t\}_{t \in \mathbb{N}_0}$ be an *Adapted Stochastic Process* and $c \in \mathbb{R}$.

The event $\tau_c := \inf\{t \geq 0 : X_t \geq c\}$[19] is a *Stopping Time.*

Proof 2.13 - *Theorem 2.8*
Let $\{X_t\}_{t \in \mathbb{N}_0}$ be an *Adapted Stochastic Process* and $c \in \mathbb{R}$.

Note that $\tau \leq t$ iff $\exists$ $k \leq t$ st $X_k \geq c$ due to the definition of τ_c.

Therefore

$$\{\tau_c \leq t\} = \bigcup_{k \leq t} \{X_k \geq c\}$$

Since each $\{X_k \geq c\} \in \mathcal{F}_t$ then $\{\tau_c \leq t\} \in \mathcal{F}_t$ by the definition of σ*-Fields.*

Thus τ_c is a *Stopping Time.* $\square$

Theorem 2.9 - *Optional Stopping Theorem*[20] *- Martingale*
Let τ be a *Bounded Stopping Time* and $\{X_t\}_{t \in \mathbb{N}_0}$ be a *Martingale.*

Then

$$\mathbb{E}[X_\tau] = \mathbb{E}[X_0] = X_0$$

Theorem 2.10 - *Optional Stopping Theorem - Super-Martingale*
Let τ be a *Bounded Stopping Time* and $\{X_t\}_{t \in \mathbb{N}_0}$ be a Super-*Martingale.*

Then

$$\mathbb{E}[X_\tau] \leq \mathbb{E}[X_0] = X_0$$

[19] The first time X_t reaches value c.
[20] AKA *Optional Sampling Theorem*

Remark 2.4 - *Weaker Conditions for Optional Stopping Theorem*
The following are weaker conditions[21] that suffice for the *Optional Stopping Theorem* to hold

i). $\mathbb{P}(\tau < \infty) = 1$ and X_τ is bounded.

ii). $\mathbb{E}[\tau] < \infty$ and $(X_t - X_{t-1})$ is bounded.

Proof 2.14 - `Theorem 2.9`
Let τ be a *Bounded Stopping Time* and $\{X_t\}_{t\in\mathbb{N}_0}$ be a *Martingale*.

Assume that $\tau \leq K$ (This is reasonable since τ is bounded). We can write

$$X_{\tau(\omega)}\omega = \sum_{t=0}^{K} X_t(\omega)\mathbb{1}\{\tau(\omega) = t\}$$

Note that this is not really a sum as there is only one event ω st $\mathbb{1}\{\tau(\omega) = t\} = 1$, the rest equal 0.

Then

$$\begin{array}{rcl}\mathbb{E}[X_\tau] &=& \mathbb{E}\left[\sum_{t=0}^{K} X_t\mathbb{1}\{\tau = t\}\right] \\ &=& \sum_{t=0}^{K}\mathbb{E}\left[X_t\mathbb{1}\{\tau = t\}\right] \text{ by linearity of exp.} \\ &=& \sum_{t=0}^{K}\mathbb{E}\left[\mathbb{E}[X_K|\mathcal{F}_t]\mathbb{1}\{\tau = t\}\right] \text{ by } \texttt{Theorem 2.6}\end{array}$$

Since τ is a *Stopping Time* then $\{\tau = t\}$ is Measurable wrt $\mathcal{F}_t$.

Thus, by `Theorem 2.2`

$$\mathbb{E}[X_K|\mathcal{F}_t]\mathbb{1}\{\tau = t\} = \mathbb{E}[X_K\mathbb{1}\{\tau = t\}|\mathcal{F}_t]$$

We continue the analysis of $\mathbb{E}[X_\tau]$

$$\begin{array}{rcl}\mathbb{E}[X_\tau] &=& \sum_{t=0}^{K}\mathbb{E}\left[\mathbb{E}[X_K|\mathcal{F}_t]\mathbb{1}\{\tau = t\}\right] \\ &=& \sum_{t=0}^{K}\mathbb{E}\left[\mathbb{E}[X_K\cdot\mathbb{1}\{\tau = t\}|\mathcal{F}_t]\right] \\ &=& \sum_{t=0}^{K}\mathbb{E}[X_K\cdot\mathbb{1}\{\tau = t\}] \text{ by Tower Law} \\ &=& \mathbb{E}\left[X_K\sum_{t=0}^{K}\mathbb{1}\{\tau = t\}\right] \\ &=& \mathbb{E}\left[X_K\cdot 1\right] \\ &=& \mathbb{E}\left[X_K\right] \\ &=& \mathbb{E}[X_0] \text{ as } \{X_t\}_{t\in\mathbb{N}_0} \text{ is a Martingale} \\ &=& X_0 \text{ as its value is known}\end{array}$$

Definition 2.18 - *Gambler's Ruin Problem*

[21]Rather than τ being a bounded stopping time

The *Gambler's Ruin Problem* involves considering a gambler with an initial wealth of £C. The gambler is allowed to play a game until either they become bankrupt (i.e. have £0) or reach a target of £$(C+G)$ where $G>0$.

The simplest specification of the game is flipping a coin[22] and the gambler receives £1 if it lands heads, or loses £1 if it lands tails.

Proposition 2.5 - *Stopping Time in Gambler's Ruin Problem*
Consider the *Gambler's Ruin Problem* using the simple game described in `Definition 2.18`.

Let $\{X_t\}_{t\in\mathbb{N}_0}$ be a *Simple Random Walk* with parameter p[23] and $X_0=0$ and $C,G>0$.

Consider the *Stopping Time* $\tau:=\inf\{t:X_t=G \text{ or } X_t=-C\}$, the event the gambler stops playing[24]. Then

- If $p=1/2$ then

$$\begin{array}{rcl}\mathbb{P}(X_\tau=G)^{[25]}&=&\dfrac{C}{C+G}\\ \mathbb{P}(X_\tau=-C)^{[26]}&=&\dfrac{G}{C+G}\\ \mathbb{E}[\tau]&=&CG\end{array}$$

- If $p\neq 1/2$ then

$$\begin{array}{rcl}\mathbb{P}(X_\tau=G)&=&\dfrac{1-\left(\frac{p}{1-p}\right)^C}{\left(\frac{p}{1-p}\right)^G-\left(\frac{p}{1-p}\right)^C}\\ \mathbb{P}(X_\tau=-C)&=&1-\mathbb{P}(X_\tau=G)=\dfrac{\left(\frac{p}{1-p}\right)^G-1}{\left(\frac{p}{1-p}\right)^G-\left(\frac{p}{1-p}\right)^C}\\ \mathbb{E}[\tau]&=&\dfrac{G\mathbb{P}(X_\tau=G)+(-C)\mathbb{P}(X_\tau=-C)}{2p-1}\end{array}$$

Proof 2.15 - ***`Proposition 2.5`***
Since τ is <u>not</u> *Bounded*, but X_τ is *Bounded* by $G,-C$, we can use a weaker condition from `Remark 2.4` to apply the *Optional Stopping Theorem* (`Theorem 2.9`) provided we can show that $\mathbb{P}(\tau<\infty)=1$.

Note that whenever there is a run of $k\geq C+G$ successive 1's in the process $\{Y_t\}_{t\in\mathbb{N}_0}$ which defines the random walk X, the process will stop and $\tau<\infty$. Thus, for all m, the following hold

$$\begin{array}{rcl}\mathbb{P}(\tau>km)&=&\mathbb{P}(\text{No run of } k \text{ 1's in } Y_1 \text{ to } Y_{mk})\\ &=&\displaystyle\prod_{j=0}^{m-1}\mathbb{P}(\text{No run of } k \text{ 1's in } Y_{jk+1} \text{ to } Y_{(j+1)k})\\ &=&(1-p^k)^m\\ \implies \mathbb{P}(\tau<\infty)&=&1\end{array}$$

We can now consider the two cases for the value of p

[22]potentially fair, potentially not.

[23]$\{X_t\}_{t\in\mathbb{N}_0}$ can be consider to model the net winnings of the gambler and p is the probability of the coin landing heads (i.e. the gambler wins money).

[24]Either due to reaching goal or going bankrupt.

[26]Gambler reaches goal.

[26]Gambler goes bankrupt.

- If $p=1/2$. Then by the *Optional Stopping Theorem* we can deduce the following

$$\begin{array}{rrcl} & 0 & = & \mathbb{E}[X_\tau] \text{ as } p=1/2 \\ & & = & G\mathbb{P}(X_\tau=G)+(-C)\mathbb{P}(X_\tau=-C) \\ & & = & G\mathbb{P}(X_\tau=G)+(-C)(1-\mathbb{P}(X_\tau=G)) \\ \implies & C & = & (G+C)\mathbb{P}(X_\tau=G) \\ \implies & \mathbb{P}(X_\tau=G) & = & \dfrac{C}{G+C} \\ \text{and} & \mathbb{P}(X_\tau=-C) & = & 1-\mathbb{P}(X_\tau=G)=\dfrac{G}{G+C} \end{array}$$

To determine $\mathbb{E}[X_\tau]$ we apply the *Optional Stopping Theorem* to the process $\{Z_t\}_{\mathbb{N}_0}$ where $Z_t:=X_t^2-t$. It was shown in `Proposition 2.4` that $\{Z_t\}_{t\in\mathbb{N}_0}$ is a *Martingale*.

As $\{Z_t\}_{t\in\mathbb{N}_0}$ is a *Martingale* it holds that

$$0=\mathbb{E}[Z_0]=\mathbb{E}[Z_\tau]=\mathbb{E}[X_\tau^2-\tau]=\mathbb{E}[X_\tau^2]-\mathbb{E}[\tau]$$

By rearranging we obtain that

$$\begin{array}{rcl} \mathbb{E}[\tau] & = & \mathbb{E}[\tau] \\ & = & G^2\mathbb{P}(X_\tau=G)+C^2\mathbb{P}(X_\tau=-C) \\ & = & G^2\cdot\dfrac{C}{C+G}+C^2\dfrac{G}{C+G} \\ & = & CG \end{array}$$

- Consider the case $p\neq 1/2$ and the process $\{L_t\}_{t\in\mathbb{N}_0}$ where $L_t:=\left(\frac{1-p}{p}\right)^{X_t}$. It was shown in `Proposition 2.4` that $\{L_t\}_{t\in\mathbb{N}_0}$ is a *Martingale*.

By the *Optional Stopping Theorem*

$$\begin{array}{rcl} 1 & = & \mathbb{E}[L_0] \\ & = & \left(\dfrac{1-p}{p}\right)^G\mathbb{P}(X_\tau=G)+\left(\dfrac{1-p}{p}\right)^C\mathbb{P}(X_\tau=-C) \end{array}$$

Remembering that $\mathbb{P}(X_\tau=G)+\mathbb{P}(X_\tau=-C)=1$, we can derive the probabilities of each end event occurring

$$\begin{array}{rrcl} & 1 & = & \left(\dfrac{1-p}{p}\right)^G\mathbb{P}(X_\tau=G)+\left(\dfrac{1-p}{p}\right)^C(1-\mathbb{P}(X_\tau=G)) \\ & & = & \left(\dfrac{1-p}{p}\right)^C+\left[\left(\dfrac{1-p}{p}\right)^G-\left(\dfrac{1-p}{p}\right)^C\right]\mathbb{P}(X_\tau=G) \\ \implies & \mathbb{P}(X_\tau=G) & = & \dfrac{1-\left(\frac{1-p}{p}\right)^C}{\left(\frac{1-p}{p}\right)^G-\left(\frac{1-p}{p}\right)^C} \end{array}$$

Consider the process $\{M_t\}_{t\in\mathbb{N}_0}$ where $M_t:=X_t-t(2p-1)$. It was shown in `Proposition 2.4` that $\{M_t\}_{t\in\mathbb{N}_0}$ is a *Martingale*.

We determine $\mathbb{E}[X_\tau]$ by applying the *Optional Stopping Theorem* to $\{M_t\}_{t\in\mathbb{N}_0}$.

$$\begin{array}{rcl} 0 & = & \mathbb{E}[M_\tau] \\ & = & G\mathbb{P}(X_\tau=G)+(-C)\mathbb{P}(X_\tau=-C)-\mathbb{E}[\tau](2p-1) \end{array}$$

By rearranging we obtain that

$$\begin{array}{rrcl} & 0 & = & G\mathbb{P}(X_\tau=G)+(-C)\mathbb{P}(X_\tau=-C)-\mathbb{E}[\tau](2p-1) \\ \implies & \mathbb{E}[\tau](2p-1) & = & G\mathbb{P}(X_\tau=G)+(-C)\mathbb{P}(X_\tau=-C) \\ \implies & \mathbb{E}[\tau] & = & \dfrac{G\mathbb{P}(X_\tau=G)+(-C)\mathbb{P}(X_\tau=-C)}{2p-1} \end{array}$$

□

3 Financial Terminology

Definition 3.1 - *Underlying Asset*
The *Underlying Asset* is a real financial asset or security which a contract can be based on. (e.g. Oil, interest rate, shares).

Definition 3.2 - *Dividend*
A *Dividend* is a one-off payment provided to the holder of an *Underlying Asset* at a certain time. Whether an *Underlying Asset* pays a *Dividend*, and the value of the *Dividend*, will affect the value of the *Underlying Asset*.

A *Dividend* is generally used by companies to distribute yearly profits to its shareholders.

Definition 3.3 - *Long Selling*
Long Selling is the practice of buying an asset (or security) and then selling it at some point in the future.

In *Long Selling* your profit/loss is $P_{\text{sell}} - P_{\text{buy}}$, thus you hope the price of the asset increases in the period between you buying and selling it.

Definition 3.4 - *Short Selling*
Short Selling is the practice of borrowing an asset (or security), immediately selling it[27] and at some point in the future buying an equivalent asset in order to reimburse your lender.

In *Short Selling* your profit/loss if $P_{\text{sell}} - P_{\text{buy}}$, thus you hope the price of the asset decreases in the period between you selling and having to reimburse your lender.

Remark 3.1 - *Short Selling & Dividends*
If the asset you borrowed in *Short Selling* pays a *Dividend* during the time you have borrowed the asset, then you must pay this *Dividend* to the lender.[28]

Definition 3.5 - *Arbitrage Opportunity*
An *Arbitrage Opportunity* occurs when it is possible to make a profit without being exposed to the risk of incurring a loss.[29]

Generally *Arbitrage Opportunities* occur by being able to buy and sell the same asset in different markets, as each market may have a different price.

Theorem 3.1 - *No-Arbitrage Principle*

> "*Arbitrage Opportunities* do not exist (for long) in real life markets."

As when the opportunities arise, the market activity cause by agents exploiting the opportunity would raise the cost of buying and thus remove the opportunity due to the *Law of Supply-and-Demand*.

[27]Receiving payment at this point.
[28]As you have already sold the asset, then this expense will come out of your own pocket.
[29]Someone who loos for *Arbitrage Opportunities* is called an *Arbitrageur*.

Remark 3.2 - *Value of Money*
IRL the value of money is not constant due to inflation, interest rates & exchange rates. We generally want to normalise the returns of our portfolio wrt the change in value of money in order to determine the "real returns".

Definition 3.6 - *Bank Account Process,* B_t
A *Bank Account Process*[30] B_t is how much an initial deposit of one unit at time $t = 0$ would be worth at time point t if the deposit was made into a "Risk-Free Bank Account", given some risk-free *Interest Rate* r. This is a measure of how the value of money changes over the t time-periods.

The *Bank Account Process* must fulfil the following criteria

$$B_0 = 1 \quad \text{and} \quad B_t(\omega) \geq 0 \ \forall \ \omega \in \Omega$$

It is generally assumed that you can borrow money from these accounts, paying the same interest rate r.

Proposition 3.1 - *Value of Bank Account Process* B_t
Suppose our "Risk-Free Bank Account" pays a constant interest rate of r in each time-period, then after t time-periods our initial deposit would be worth

- *Continuous Time Model* $B_t = B_0 e^{rt}$.
- *Single-Period Model* $B_1 = B_0(1 + r)$.[31]
- *Multi-Period Model* $B_t = B_0(1 + r)^t$.

Definition 3.7 - *Portfolio*

3.1 Derivatives

Definition 3.8 - *Derivative Securities*
A *Deriviative Security* is a contract which has an expiry date T and pays out different amounts depending upon the value of some *Underlying Asset* in the time-period $[0, T]$.

Remark 3.3 - *Valuing Contracts*
When valuing contracts we assume that arbitrage does not exist. This means we can derive a single price[32] for a contract, as any other price would create an *Arbitrage Opportunity.*

Theorem 3.2 - *Equivalent Contract Valuations over Time*
If two combinations of financial derivatives both have the same value $V_T = W_T$ at time $t = T$. Then their prices will be the same at all $t < T$

$$\text{if } V_T = W_t \text{ then } V_t = W_t \ \forall \ t < T$$

Proof 3.1 - `Theorem 3.2`

[30] AKA a *Bond* or a *Numeraire.*
[31] Must be that $t = 1$ in a *Single-Period Model.*
[32] Known as the *Fair Price.*

We assume the "No-Arbitrage Principle" holds throughout this proof.

Let V_t, W_t represent the fair price for two different combinations of financial derivatives at time t and that $V_T = W_T$. Suppose there is a risk-free profit of r.

Assume WLOG that $V_t > W_t$. Then an arbitrage opportunity exists and can be exploited by doing the following:

- At $t = 0$
  - i). Sell/short the first combination, receiving £V_t.
  - ii). Buy the second combination, costing £W_t.
  - iii). Invest the difference (£$V_t - W_t > 0$).
- At $t = T$
  - i). Sell the first combination, receiving £$V_T = W_T$
  - ii). Buy the second combination, costing £$W_T = V_T$.

Following this will result in a "riskless" profit of $(V_t - W_t)e^{r(T-t)} > 0$.

Definition 3.9 - *Forward Contract*
A *Forward Contract* is a type of *Derivative Security.* In a *Forward Contract* two parties agree to an exchange on a predetermined future date for a predetermined amount, and are both obliged to fulfil this exchange.

All *Forward Contracts* have the following components

- *Delivery Date T.*
- *Delivery Price K.*

Remark 3.4 - *Positions in a Forward Contract*
In a *Forward Contract* agents can take two positions

- *Long Position* - Agree to buy the underlying asset for *£K* on date T. Makes a profit if the market-value of the underlying asset is greater than K in time-period T.
- *Short Position* - Agree to sell the underlying asset for *£K* on date T. Makes a profit if the market-value of the underlying asset is less than K in time-period T.

Remark 3.5 - *Utility of Forwards Contracts*
Forward Contracts allow you to agree terms of a purchase/sale some time in advance of actually transacting. This means business have greater certainty about their future cash-flows.[33]

Theorem 3.3 - *Fair Delivery Price of a Forward Contract*
Consider a *Forward Contract* with delivery date T, where the underlying asset has value S_0 at time $t = 0$ and pays a dividend D at time $t_0 \in (0, T)$. Suppose there is a risk-free bank account with a constant interest rate r during the interval $[0, T]$.[34]

Then

[33]e.g. Farmers may agree to a price for their whole harvest a year in advance. Thus their next years income is completely known.
[34]This means $B_t = e^{rt}$.

- If $D = 0$ (ie no dividend is payed) then the fair *Delivery Price* for this contract is

$$K = S_0 e^{rT}$$

- If $D > 0$ then the fair *Delivery Price* for this contract is

$$K = (S_0 - I)e^{rT} \text{ where } I := De^{-rt_0}$$

Proof 3.2 - *Theorem 3.3*
We use the "No-Arbitrage Principle" to prove that these Ks are the fair prices under each scenario.
Case 1 - Suppose, for the sake-of-contradiction, that $K > (S_0 - I)e^{rT}$ with $I := De^{-rt_0}$. Then an arbitrage opportunity exists and can be exploited by doing the following:

- At $t = 0$
  - i). Borrow £S_0 from the bank, at an interest rate of r.
  - ii). Buy the underlying asset.
  - iii). Taking a short position in the forward contract (receiving $K > (S_0 - I)e^{rT}$).
- At $t = t_0$
  - i). We will receive a dividend payment £D which we shall use to partially repay our loan. This leaves an outstanding balance of $S_0 e^{rt_0} - D$.
- At $t = T$
  - i). Sell the asset for K using the forward contract.
  - ii). Repay the outstanding balance on the loan ($(S_0 e^{rt_0} - D)e^{r(T-t_0)}$).

Doing all this will lead to a "riskless" profit of

$$K - (S_0 e^{rt_0} - D)e^{r(T-t_0)} = K - (S_0 - I)e^{rT} > 0 \text{ by def. } K$$

This means that this definition of K cannot be the fair-price, thus $K \leq (S_0 - I)e^{rT}$.
Case 2 - Suppose, for the sake-of-contradiction, that $K < (S_0 - I)e^{rT}$ with $I := De^{-rt_0}$. Then an arbitrage opportunity exists and can be exploited by doing the following:

- At $t = 0$
  - i). Short sell the underlying asset. (Receiving £S_0).
  - ii). Invest this revenue, receiving an interest rate of r.
  - iii). Take a long position on the forward contract.
- At $t = t_0$
  - i). Pay the dividend £D to our lender, from our bank account.
- At $t = T$
  - i). Buy the asset for K using the forward contract.

Doing all this will lead to a "riskless" profit of

$$(S_0e^{rt_0} - D)e^{r(T-t_0)} - K = (S_0 - I)e^{rT} - K > 0 \text{ by def. } K$$

This means that this definition of K cannot be the fair-price, thus $K \geq (S_0 - I)e^{rT}$.

Thus, by combining these two inequalities, the fair price for this *Forward Contract* is

$$K = (S_0 - I)e^{rT}$$

Definition 3.10 - *Options Contract*
An *Options Contract* is a type of *Derivative Security*. In an *Options Contract* two parties agree to an exchange on (or before) a predetermined future date for a predetermined amount, but the holder is <u>not obliged</u> to fulfil this exchange.

All *Options Contracts* have the following components

- *Delivery Date* T.
- *Strike Price* K.

There are two classes of *Options Contract*

- *Call Option* - The holder has the right to buy.
- *Put Option* - The holder has the right to sell.

Definition 3.11 - *European & American Options*
There are two categories of *Options Contract* which determine when the contract can be exercised

- *European Option* - The holder can only execute on the delivery date $t = T$.
- *American Option* - The holder can execute on <u>any</u> date before the delivery date T.

Definition 3.12 - *Positions in an Options Contract*
In *Options Contracts* agents can take one of two positions. The position they take determines their rights & potential cash-flows.

- *Holder* - Decides whether to execute the contract of not. Will pay the *Writer* a fee for creating the contract.

  The *Holder's* only expense is the fee they pay the *Writer* and they may make an income if they execute the contract.
- *Write* - Must complete the transaction if the *Holder* wishes to. Receives a fee from the *Holder*.

  The *Writer's* only income is the fee they receive from the *Holder* and they may incur a loss if the contract is executed.

Remark 3.6 - *When are options executed?*
Whether the holder should execute their option depends on the market price S_T at time T, the strike price K and the class of contract. Assuming (justifiably) that the holder will only execute the option if it will make them money, the holder should do the following

- For a *Call Option* the holder should execute if $S_T > K$. As they can immediately sell their newly bought asset for a profit of $S_T - K - F$ where F is the fee payed to the writer.
- For a *Put Option* the holder should execute if $S_T < K$. As they can buy the asset from the market and sell it to the writer of their option for a profit of $S_T - K - F$ where F is the fee payed to the writer.

Theorem 3.4 - *Put-Call Parity*[35]
Consider a *European Put Option* and a *European Call Option* where both have the same: underlying asset, strike price K and expiry date T. Let S_t be the value of the underlying asset at time point t and assume there is a "risk-free" interest rate of r available.

Then, if no *Arbitrage Opportunities* exist then the following hold

$$S_t + P_t - C_t = Ke^{-r(T-t)} \ \forall \ t \in [0, T]$$

where P_t, C_t are the prices of the put & call options at time t respectively, and $Ke^{-r(T-t)}$ is the discounted value of our bank account.

Theorem 3.5 - *Lower-Bound for a European Call Option*
Let S_t be the value of an underlying asset at time t.

For a *European Call Option* with strike price K and delivery date T we can determine the following lower bound on its price C_t

$$C_t \quad \geq \quad \{S_t - Ke^{-r(T-t)}\}_+$$

Proof 3.3 - `Theorem 3.5`
By *Put-Call Parity* (`Theorem 3.4`) we have that

$$\begin{array}{rrcl} & S_t + P_t - C_t & = & Ke^{-r(T-t)} \\ \implies & C_t & = & S_t + P_t - Ke^{-r(T-t)} \end{array}$$

Since *Put Options* cannot have a negative price, $P_t \geq 0$, we have that

$$C_t \geq S_t + P_t - Ke^{-r(T-t)}$$

Further, since *Call Options* cannot have a negative price, $C_t \geq 0$, we have that

$$C_t \geq \left\{S_t + P_t - Ke^{-r(T-t)}\right\}_+$$

Theorem 3.6 - *Value of American Call Options w/o Dividends*
Consider an *American* & a *European Call Option*, for the same underlying asset, with the same strike price and expiry date.

Then, if the underlying asset does not pay a *Dividend*

$$C_A = C_E$$ [36]

where C_A, C_E are the price of the American & European call options, respectively.

[35]This is an application of `Theorem 3.2` to European Put & Call Options.
[36]This shows that if an underlying asset does not pay a dividend then it is suboptimal to exercise an *American Call Option* early.

Proof 3.4 - `Theorem 3.6`
If the *American Call Option* is executed early at time-point $t < T$ then it generates an income of $S_t - K$.

However, `Theorem 3.5` shows that selling a *Call Option* generates

$$\{S_t - Ke^{-r(T_t)}\}_+ \geq S_t - Ke^{-r(T-t)} > S_t - K$$

This shows that it is sub-optimal to exercise the call at any time $t < T$.

4 Discrete-Time

4.1 Single-Period Model

Definition 4.1 - *Single-Period Model*
The *Single-Period Model* is a model for a financial market where actions can only occur on two dates. It has the following components

- *Initial Date* $t = 0$.
- *Terminal Date* $t = 1$.
- Trading is only allowed to occur on the *Initial & Terminal Dates*.
- A finite *Sample Space* $\Omega := \{\omega_1, \dots, \omega_K\}$ with $K < \infty$.
  Each event $\omega_1, \dots, \omega_K$ corresponds to some state of the world.
- A *Probability Measure* $\mathbb{P}$ on the *Sample Space* Ω with $\mathbb{P}(\{\omega_i\}) > 0 \ \forall\ i \in [1, K]$.

Definition 4.2 - *Price-Process* S[37]
A *Price Process* S models the price of each security at each time-point

$$S := \{S(t) : t = 0, 1\} \text{ where } S(t) = (S_1(t), \dots, S_N(t))$$

where $S(t)$ is the collection of the prices of each available stock at time t, $S_i(t)$ is the price of the i^{th} stock at time t and there are N different stock available.

The values of $S(0)$ are known to the investors, but $S(1)$ are unknown non-negative random variables whose value only become known at time $t = 1$.

Definition 4.3 - *Discounted Price-Process* S^*
A *Discounted Price-Process* S^* models the price of each security at each time-point <u>but</u> normalised for the change in the value of money due to the *Bank Process* B.

$$S^* := \{S^*(t) : t = 0, 1\} \text{ where } S^*(t) = (S^*_1(t), \dots, S^*_N(t)) \text{ and } S^*_i(t) := \frac{S_i(t)}{B_t}$$

Definition 4.4 - *Trading Strategy* H
A *Trading Strategy* H describes the changes in an investors portfolio between the *Initial Date* $t = 0$ and *Terminal Date* $t = 1$.

$$H := (H_0, H_1, \dots, H_N)$$

[37]AKA *Stock Process*

where H_0 is the amount invested/borrowed in a risk-free bank account, H_i is the amount bought/shorted of the i^{th} stock for $i \in [1, N]$ and there are N stocks available in the market.

Definition 4.5 - *Value-Process V*
A *Value Process V* describes the total value of a *Trading Strategy H* at each time-point $t = 0, 1$

$$V := \{V_t : t = 0, 1\} \text{ where } V_t = H_0 B_t + \sum_{i=1}^{N} H_i S_i(t)$$

Definition 4.6 - *Discounted Value-Process V^**
A *Discounted Value-Process V^** describes the total value of a *Trading Strategy H* at each time-point but normalised for the change in the value of money due to the *Bank Process B*.

$$V^* := \{V_t^* : t = 0, 1\} \text{ where } V_t^* := \frac{V_t}{B_t} = H_0 + \sum_{i=1}^{N} H_i S_i^*(t)$$

Definition 4.7 - *Gains-Process G*
A *Gains Process G* describes the total profit/loss generated made by a *Trading Strategy H* between time-points $t = 0, 1$

$$G := H_0 r + \sum_{i=1}^{N} H_i \Delta_{S_i} \text{ where } \Delta_{S_i} = S_i(1) - S_i(0)^{[38]}$$

Definition 4.8 - *Discounted Gains-Process G^**
A *Discounted Gains-Process G^** describes the total profit/loss generated made by a *Trading Strategy H* between time-points $t = 0, 1$ but normalised for the change in the value of money due to the *Bank Process B*.

$$G^* := \frac{G}{B_t} = \sum_{i=1}^{N} H_i \Delta_{S_i^*} \text{ where } \Delta_{S_i^*} = S_i^*(1) - S_i^*(0)$$

Definition 4.9 - *Arbitrage Opportunity - Single-Period Model*
Consider a *Trading Strategy $H = (H_0, H_1)$* for the *Single-Period Model.*

H exploits an *Arbitrage Opportunity* if it has the following three properties

i). $V_0 = 0$.

ii). $V_1(\omega) \geq 0 \ \forall \ \omega \in \Omega$.

iii). $\mathbb{P}(V_1(\omega) \geq 0) > 0 \ \forall \ \omega \in \Omega$.[39]

Theorem 4.1 - *Arbitrage Opportunities & Gains Process*

[38] Δ_{S_i} The change in value of the i^{th} stock.
[39] Equivalently $\mathbb{E}[V_1] > 0$

There exists an *Arbitrage Opportunity* in a market iff there exists a *Trading Strategy* H st[40]

$$G^* \geq 0 \quad \text{and} \quad \mathbb{E}[G^*] > 0$$

Proof 4.1 - *Theorem 4.1*

$\Rightarrow$ Let H be a *Trading Strategy* which exploits an *Arbitrage Opportunity*.

By the definition of an *Arbitrage Opportunity* $G^* = V_1^* - V_0^*$ and $B_t > 0 \ \forall \ t, \omega$, this means that $G^* \geq 0$ and thus

$$\mathbb{E}[G^*] = \mathbb{E}[V_1^*] > 0$$

$\Leftarrow$ Let H be a *Trading Strategy* which satisfies $G^* \geq 0$ and $\mathbb{E}[G^*] > 0$.

Define $\hat{H} := (\hat{H}_0, H_1, \dots, H_N)$ where $\hat{H}_0 := -\sum_{i=1}^N H_i S_i^*(0)$[41].

Under $\hat{H}_0$ we have that $V_0^* = 0$ and $V_1^* = V_0^* + G^* = G^*$.

Hence, $V_1^* \geq 0$ and $\mathbb{E}[V_1^*] = \mathbb{E}[G^*] > 0$, meaning $\hat{H}$ exploits an *Arbitrage Opportunity.*

As the result holds in both directions, we can say it holds iff.

4.1.1 Risk-Neutral Probability Measures $\mathbb{Q}$

Definition 4.10 - *Risk-Neutral Probability Measure* $\mathbb{Q}$
A *Probabiltiy Measure* $\mathbb{Q}$ on *Sample Space* Ω is said to be a *Risk-Neutral Probability Measure* if the following hold

i). $\mathbb{Q}(\{\omega\}) > 0 \ \forall \ \omega \in \Omega$.

ii). $\mathbb{E}_{\mathbb{Q}}[S_i * (1)] = S_i^*(0) \ \forall \ i \in [1, N]$

Theorem 4.2 - *Separating Hyperplane Theorem*[42]
Let $\mathbb{W}$ be a linear subspace of $\mathbb{R}^K$ and $\mathbb{K}$ be a compact convex subset in $\mathbb{R}^K$ which is disjoint from $\mathbb{W}$.

We can separate $\mathbb{W}$ and $\mathbb{K}$ strictly by using a hyperplane containing $\mathbb{W}$[43] st

$$u^T v > 0 \ \forall \ u \in \mathbb{K}$$

Theorem 4.3 - *Arbitrage Opportunities & Risk-Neutral Probability Measures*
No *Arbitrage Opportunities* exists iff there exists a *Risk-Neural Probability Measure* $\mathbb{Q}$.

Proof 4.2 - *Theorem 4.3*
Consider the three following sets

[40] This means H never loses money, and it is expected to make money.
[41] This ensures V_0, a requirement for H to exploit an *Arbitrage Opportunity.*
[42] This theorem is used to prove `Theorem 4.3`. The proof of this theorem is beyond the scope of this course.
[43] ie $\exists \ v \in \mathbb{R}^K$ which is *Orthogonal* to $\mathbb{W}$[44] $u^T v = 0 \ \forall \ u \in \mathbb{W}$.

i). $\mathbb{W} = \big\{X \in \mathbb{R}^K : X = G^* \text{ for some Trading Strategy } H\big\}$.

This is the set of possible *Gains* in our market for *Trading Strategies* which have zero initial investment. $\mathbb{W}$ is a linear subspace of $\mathbb{R}^K$[45].

ii). $\mathbb{A} = \big\{X \in \mathbb{R}^K : X \geq 0, X \neq 0\big\}$[46].

There exists an arbitrage opportunity <u>iff</u> $\mathbb{W} \cap \mathbb{A} \neq \emptyset$.

iii). $\mathbb{A}^+ = \Big\{X \in \mathbb{R}^N : X \geq 0, X \neq 0, \sum_{i=1}^K X_i = 1\Big\}$.

$\mathbb{A}^+$ is a convex and compact subset of $\mathbb{R}^K$.

$\Rightarrow$ Assume that there are no *Arbitrage Opportunities*, then $\mathbb{W} \cap \mathbb{A} \neq \emptyset$.

By the *Separating Hyperplane Theorem* (`Theorem 4.2`) $\exists\ Y \in \mathbb{R}^K$ which is *orthogonal* to $\mathbb{W}$ st

$$X^T Y > 0\ \forall\ X \in \mathbb{A}^+$$

For each $k \in \{1, \dots, K\}$ the k^{th} unit vector e_k is an element of $\mathbb{A}^+$. Therefore,

$$Y_k := e_k^T Y > 0\ \forall\ k \in \{1, \dots, K\}$$

meaning all entries of Y are strictly positive.

Define a probability measure $\mathbb{Q}$ by setting

$$\mathbb{Q}(\{\omega_k\}) = \frac{Y(\omega_k)}{Y(\omega_1) + \dots + Y(\omega_k)}$$

Furthermore, $\Delta S_n^* \in \mathbb{W}\ \forall\ n$ because $\Delta S_n^* := S_n^*(1) - S_n^*(0)$ is the discounted wealth for the portfolio $H := e_n$ which consists of one unit of the n^{th} asset only.

Since Y is orthogonal to $\mathbb{W}$ we can conclude that

$$\mathbb{E}_{\mathbb{Q}}[\Delta S_n^*] = \sum_{k=1}^K \Delta S_n^*(\omega_k)\mathbb{Q}(\{\omega_k\}) = 0\ \forall\ n$$

In other words

$$\mathbb{E}_{\mathbb{Q}}[S_n^*(1)] = S_n^*(0)\ \forall\ n$$

Thus $\mathbb{Q}$ is a *Risk-Neutral Probability Measure.*

$\Leftarrow$ Let $\mathbb{Q}$ be a *Risk-Neutral Probability Measure.*

Then for an arbitrary *Trading Strategy* H we have that

$$\mathbb{E}_{\mathbb{Q}}[G^*] = \mathbb{E}_{\mathbb{Q}}\left[\sum_{n=1}^N H_n \Delta S_n^*\right] = \sum_{n=1}^N H_n \mathbb{E}_{\mathbb{Q}}[\Delta S_n^*] = 0$$

and, in particular

$$\sum_{k=1}^K G^*(\omega_k)\mathbb{Q}(\{\omega_k\}) = 0$$

which shows that either $G^*(\omega_k < 0)$ for some k or $G^* = 0$, but then $\mathbb{E}_{\mathbb{Q}}[G^*] = 0$.

Hence, by `Theorem 4.1`, there cannot be any arbitrage opportunities.

The result holds in both directions. $\square$

[45] Proved by showing it is complete under: addition, and scalar multiplication.

[46] $\mathbb{A}$ is not compact, so can not be used for $\mathbb{K}$ in *Separating Hyperplane Theorem*

4.1.2 Contingent Claims X

Definition 4.11 - *Contingent Claim X - Single-Period Model*
Consider a *Single-Period Model* with a risk-free bank process and $K-1$ risky securities on offer.

A *Contingent Claim* $X \in \mathbb{R}^K$ in a *Single-Period Model* is a random variable which represents a payoff at time $t = 1$ for each security[47].

Definition 4.12 - *Attainable Contingent Claim X*
A *Contingent Claim* X is said to be "*attainable*" if

$$\exists \text{ trading strategy } H \text{ st } V_1 = X \text{ when using } H^{[48]}.$$

Otherwise, X is said to be "un*attainable*".

Remark 4.1 - *Determining whether a Contingent Claim X is Attainable*
Consider a *Single-Period Model* with $K-1$ securities, which can be described the following matrix A, and a *Contingent Claim* $X \in \mathbb{R}^K$. Then

$$X \text{ is } \textit{Attainable} \text{ iff } \exists \text{ a trading strategy } H \in \mathbb{R}^K \text{ st } AH = X \text{ where}$$

$$A := \begin{pmatrix} B_1(\omega_1) & S_1(1)(\omega_1) & \dots & S_N(1)(\omega_1) \\ B_1(\omega_2) & S_1(1)(\omega_2) & \dots & S_N(1)(\omega_2) \\ \vdots & \vdots & \ddots & \vdots \\ B_1(\omega_K) & S_1(1)(\omega_K) & \dots & S_N(1)(\omega_K) \end{pmatrix}$$

Theorem 4.4 - *Fair Price of a Contingent Claim*
Let X be an *Attainable Contingent Claim* and H be a *Replicating Portfolio* which generates X.

The value of portfolio H at time $t = 0$ (V_0) is the "fair-price" of the contingent claim X.

Proof 4.3 - *Theorem 4.4*
Let X be an *Attainable Contingent Claim* and H be a *Replicating Portfolio* which generates X.

Let p be the fair price for X and assume, for the sake of contradiction, that p does not equal the value of H at time $t = 0$. This means we assuming that $p \neq V_0$.

We have two cases

Case 1 - $p > V_0$.

In this case, an *arbitrage opportunity* exists and can be exploited by doing the following

- At $t = 0$ - Short the *Contingent Claim* for p; buy portfolio H for V_0; and invest the difference $p - V_0 > 0$.
- At $t = 1$ - Our portfolio has the same value as X so we sell H to fulfil our short position on the contingent claim.

Our profit in this scenario is $(p - V_0)B_1 = (p - V_0)(1 + r) > 0$.

Case 2 - $p < V_0$.

In this case, an *arbitrage opportunity* exists and can be exploited by doing the following

[47]This payoff is not necessarily achievable

[48]This H called a *Replicating Portfolio* and is said to "*generate*" X

- At $t = 0$ - Buy the *Contingent Claim* for p; buy portfolio $-H$[49] for $-V_0$; and invest the difference $V_0 - p > 0$.
- At $t = 1$ - Our portfolio has value $-X$ so we sell our *Contingent Claim* for X to cover the portfolio, fulfilling any short positions.

Our profit in this scenario is $(V_0 - p)B_1 = (V_0 - p)(1 + r) > 0$.

Hence, in all scenarios where $p \neq V_0$ an arbitrage opportunity exists. This means $p \neq V_0$ cannot be the fair price for X and thus $p = V_0$ is the fair price. $\square$

Theorem 4.5 - *Risk-Neutral Valuation Principle*
Consider a *Single-Period Model* where no *Arbitrage Opportunities* exists, and let $\mathbb{Q}$ be a *Risk-Neutral Probability* for this model.

Then the "fair-price" of an *Attainable Contingent Claim* X at time $t = 0$

$$p = \mathbb{E}_{\mathbb{Q}}[X/B_1]$$

Proof 4.4 - *Theorem 4.5*
Consider a *Single-Period Model* with no arbitrage opportunities and let X be an *Attainable Contingent Claim* under this model.

Here we derive the fair-price for X and show that time price is unique.

Suppose there exists two trading strategies $H, \hat{H}$ st $V_1 = \hat{V}_1 = X$ but $\hat{V}_0 \neq V_0$.

Let $\mathbb{Q}$ be a *Risk-Neutral Probability Measure* under this model. Then, by the *No-Arbitrage Principle* (`Theorem 3.3`), we have that for any trading strategy H $\mathbb{E}_{\mathbb{Q}}[G^*] = 0$. Thus we can deduce that

$$\begin{array}{rcl} V_0 & = & V_0^* \\ & = & \mathbb{E}_{\mathbb{Q}}[V_0^*] \\ & = & \mathbb{E}_{\mathbb{Q}}[V_1^* - G^*] \\ & = & \mathbb{E}_{\mathbb{Q}}[V_1^*] - \mathbb{E}_{\mathbb{Q}}[G^*] \\ & = & \mathbb{E}_{\mathbb{Q}}[V_1^*] - 0 \\ & = & \mathbb{E}_{\mathbb{Q}}[V_1/B_1] \end{array}$$

This shows that any trading strategy H with $V_1 = X$ (ie is worth X at time $t = 1$), has the following value at time $t = 0$

$$V_0 = \mathbb{E}_{\mathbb{Q}}[V_1/B_1] = \mathbb{E}_{\mathbb{Q}}[X/B_1]$$

This holds for all *Risk-Neutral Probability Measures* $\mathbb{Q}$, so the fair-price for X at time $t = 0$ is constant between different *Risk-Neutral Probability Measures*. Further, all trading strategies with the same value at time $t = 1$ have the same value at time $t = 0$ (and we have a formula for this value). $\square$

4.1.3 Complete Markets

Definition 4.13 - *Complete & Incomplete Markets*
A model of a market is said to be *Complete* if each *Contingent Claim* X there exists a *Trading Strategy* H which generates X.

[49] Note this is equivalent to shorting portfolio H

Otherwise, the model is said to be <u>in</u>*complete*.

Remark 4.2 - *Checking if a Market is Complete*
We can check whether the model of a market is *Complete* by defining the following matrix A, and if A spans the same space as *Contigent Claims*[50] then the market <u>is</u> *Complete*.

$$A=\begin{pmatrix}B_1(\omega_1) & S_1(1)(\omega_1) & \dots & S_N(1)(\omega_1)\\ B_1(\omega_2) & S_1(1)(\omega_2) & \dots & S_N(1)(\omega_2)\\ \vdots & \vdots & \ddots & \vdots\\ B_1(\omega_K) & S_1(1)(\omega_K) & \dots & S_N(1)(\omega_K)\end{pmatrix}$$

Theorem 4.6 - *Complete Markets and Risk-Neutral Probability Measure*
Consider a model with <u>no</u> *Arbitrage Opportunities*, then

The model is *Complete* <u>iff</u> $\exists$ a <u>unique</u> *Risk-Neutral Probability Measure* $\mathbb{Q}$.

Proof 4.5 - `Theorem 4.6`
Consider a *Single-Period Model* with <u>no</u> *Arbitrage Opportunities* and let $\mathbb{M}$ denote the set of all *Risk-Neutral Probability Measures* for this model.

Since there are <u>no</u> arbitrage opportunities then $\mathbb{M}\neq\emptyset$.

As this theorem is "iff" I shall prove it in both directions separately

$\implies$ Assume, for the sake of contraction, that the model <u>is</u> *Complete* but $\mathbb{M}=\{\mathbb{Q},\hat{\mathbb{Q}}\}$ (ie contains two distinct elements).

Then $\exists\ \omega_k\in\Omega$ st $\mathbb{Q}(\omega_k)\neq\hat{\mathbb{Q}}(\omega_k)$. Consider the following *Contingent Claim* X

$$\begin{array}{rcl}X(\omega)&=&\begin{cases}B_1(\omega) & \text{if } \omega=\omega_k\\ 0 & \text{otherwise}\end{cases}\\ &=& B_1\mathbb{1}\{\omega=\omega_k\}\end{array}$$

Then

$$\begin{array}{rcl}\mathbb{E}_{\mathbb{Q}}[V_0]=\mathbb{E}_{\mathbb{Q}}[X/B_1]&=&\mathbb{E}_{\mathbb{Q}}[\mathbb{1}\{\omega=\omega_k\}]\\ &=&\mathbb{Q}(\{\omega_k\}))\\ &\neq&\mathbb{Q}(\{\omega_k\})\text{ by def. }X\\ &=&\mathbb{E}_{\hat{\mathbb{Q}}}[\mathbb{1}\{\omega=\omega_k\}]\\ &=&\mathbb{E}_{\hat{\mathbb{Q}}}[X/B_1]=\mathbb{E}_{\hat{\mathbb{Q}}}[V_0]\\ \implies\mathbb{E}_{\mathbb{Q}}[V_0]&\neq&\mathbb{E}_{\hat{\mathbb{Q}}}[V_0]\end{array}$$

This contradicts `Proof 4.4` when we showed that if X is attainable then $\mathbb{E}_{\mathbb{Q}}[V_0]$ is the same for all $\mathbb{Q}\in\mathbb{M}$.

Thus, if the model is *Complete* then it has a unique *Risk-Neutral Probability Measure*.

$\impliedby$ Assume, for the sake of contradiction, that the model has a unique *Risk-Neutral Probability Measure* $\hat{\mathbb{Q}}$ but there exists a *Contingent Claim* X which is <u>not</u> *Attainable*.

Then, there does not exist a trading strategy H which solves $AH=X$.

By the *Separating Hyperplane Theorem* (`Theorem 4.2`) it follows that

$$\exists\ \pi\in\mathbb{R}^K \text{ st } \pi^TA=0^{[51]} \text{ and } \pi^TX>0$$

[50]This is done by determining whether $\text{rank}(A)=\dim(X)$.

Let $\lambda > 0$ be small enough that

$$\mathbb{Q}(\{\omega_j\}) := \hat{\mathbb{Q}}(\{\omega_j\}) + \lambda\pi_j \cdot B_1(\omega_j) > 0 \quad \forall\ j \in [1, K]$$

As A is defined st all the terms in its first column are B_1 and $\pi^T A = 0$, the $\mathbb{Q}$ defined above is a probability measure.

Moreover, for any *Discounted Price Process* $s^* = (S_1^*, \dots, S_N^*)$ and any $n \in [1, N]$ we have

$$\begin{array}{rcl}\mathbb{E}_{\mathbb{Q}}[S_n^*(1)] & = & \displaystyle\sum_{j=1}^K \frac{S_n(1)(\omega_j)}{B_1(\omega_j)} \cdot \mathbb{Q}(\{\omega_j\}) \\ & = & \displaystyle\sum_{j=1}^K \frac{S_n(1)(\omega_j)}{B_1(\omega_j)} \cdot \Big(\hat{\mathbb{Q}}(\{\omega_j\}) + \lambda\pi_j B_1(\omega_j)\Big) \\ & = & \displaystyle\sum_{j=1}^K \frac{S_n(1)(\omega_j)}{B_1(\omega_j)} \cdot \hat{\mathbb{Q}}(\{\omega_j\}) + \underbrace{\lambda\sum_{j=1}^K \pi_j S_n(1)(\omega_j)}_{=0} \\ & = & \displaystyle\sum_{j=1}^K \frac{S_n(1)(\omega_j)}{B_1(\omega_j)} \cdot \hat{\mathbb{Q}}(\{\omega_j\}) \\ & = & \displaystyle\sum_{j=1}^K S_n^*(1)(\omega_j)\hat{\mathbb{Q}}(\{\omega_j\}) \\ & = & \mathbb{E}_{\hat{\mathbb{Q}}}[S_n^*(1)] \\ & = & S_n^*[0]^{[52]}\end{array}$$

This shows that $\mathbb{Q}$ is a *Risk-Neutral Probability Measure* and so $\mathbb{Q} \in \mathbb{M}$, a contradiction to the uniqueness of $\hat{\mathbb{Q}}$.

If there is a unique *Risk-Neutral Probability Measure* for a model, then all *Contingent Claims* are attainable under the model.

This has proved the theorem in both directions. □

4.2 Multi-Period Model

Definition 4.14 - *Multi-Period Model*
The *Single-Period Model* is a model for a financial market where actions can only occur on multiple dates. This provides a more realistic model than the *Single-Period Mdeol.* It has the following components

- *Initial Date* $t = 0$.
- *Terminal Date* $t = T \in \mathbb{N}$.
- Trading can occur at any times $t \in \{0, 1, \dots, T\}$
- A finite *Sample Space* $\Omega = \{\omega_1, \dots, \omega_K\}$ with $K < \infty$. Each event $\omega_1, \dots, \omega_K$ corresponds to a state of the world.
- A *Probability Space* $\mathbb{P}$ on Ω with $\mathbb{P}(\omega) > 0\ \forall\ \omega \in \Omega$.

[51]ie π is orthogonal to A.
[52]As $\hat{\mathbb{Q}}$ is a *Risk-Neutral Probability Measure.*

Remark 4.3 - *Processes*
When studying *Multi-Period Models* we use many of the same processes as are used to study *Single-Period Model*, but with their definitions extended to allow for more time periods. Namely

- Bank Account Process (`Definition 3.6`).
- Price-Process (`Definition 4.2`) & Discounted Price-Process (`Definition 4.3`).
- Value-Process (`Definition 4.5`) & Discounted Value-Process (`Definition 4.6`).
- Gains-Process (`Definition 4.7`) & Discounted Gains-Process (`Definition 4.8`).

5 Continuous-Time

0 Reference

0.1 Notation

0.2 Definitions