Financial Mathematics - Assessed Problem Sheet 2

Dom Hutchinson

March 4, 2021

Answer 1. (a) i.

Consider time $t = 0$ and define $p_1 := \mathbb{Q}(S_1 = 16)$, then

$$\begin{array}{rcl} 10 & = & 16p_1 + 8(1-p_1) \\ \implies p_1 & = & 1/4 \end{array}$$

Thus $\mathbb{Q}(S_1 = 16) = 1/4$ and $\mathbb{Q}(S_1 = 8) = 1 - 1/4 = 3/4$. Now, consider time $t = 1$ and that event ω has occurred with $\omega \in \{\omega_1, \omega_2\}$. Define $p_2 := \mathbb{Q}(S_2 = 18|S_1 = 16)$, then

$$\begin{array}{rcl} 16 & = & 18p_2 + 12(1-p_2) \\ \implies p_2 & = & 2/3 \end{array}$$

Thus $\mathbb{Q}(S_2 = 18|S_1 = 16) = 2/3$ and $\mathbb{Q}(S_2 = 12|S_1 = 16) = 1/3$. Now, consider time $t = 1$ and that event ω has occurred with $\omega \in \{\omega_3, \omega_4\}$. Define $p_3 := \mathbb{Q}(S_2 = 12|S_1 = 8)$, then

$$\begin{array}{rcl} 8 & = & 12p_3 + 6(1-p_3) \\ \implies p_3 & = & 1/3 \end{array}$$

Thus $\mathbb{Q}(S_2 = 12|S_1 = 8) = 1/3$ and $\mathbb{Q}(S_2 = 6|S_1 = 8) = 2/3$.

We can use these conditional probabilities to work out the probability of each event $\{\omega_1, \omega_2, \omega_3, \omega_4\}$ under $\mathbb{Q}$.

$$\begin{array}{rcl} \mathbb{Q}(\{\omega_1\}) & = & \mathbb{Q}(S_2 = 18|S_1 = 16)\mathbb{Q}(S_1 = 16) \\ & = & (2/3) \times (1/4) \\ & = & 1/6 \\ \mathbb{Q}(\{\omega_2\}) & = & \mathbb{Q}(S_2 = 12|S_1 = 16)\mathbb{Q}(S_1 = 16) \\ & = & (1/3) \times (1/4) \\ & = & 1/12 \\ \mathbb{Q}(\{\omega_3\}) & = & \mathbb{Q}(S_2 = 12|S_1 = 8)\mathbb{Q}(S_1 = 8) \\ & = & (1/3) \times (3/4) \\ & = & 1/4 \\ \mathbb{Q}(\{\omega_4\}) & = & \mathbb{Q}(S_2 = 6|S_1 = 8)\mathbb{Q}(S_1 = 8) \\ & = & (2/3) \times (3/4) \\ & = & 1/2 \end{array}$$

To confirm that this produces a probability measure, note that

$$(1/6) + (1/12) + (1/4) + (1/2) = 1$$

Answer 1. (a) ii.

$$\begin{aligned}
X(\omega_1) &= \left\{\frac{1}{3}(10+16+18)-14\right\}_+ \\
&= \left\{\frac{44}{3}-14\right\}_+ \\
&= 2/3 \\
X(\omega_2) &= \left\{\frac{1}{3}(10+16+12)-14\right\}_+ \\
&= \left\{\frac{38}{3}-14\right\}_+ \\
&= 0 \\
X(\omega_3) &= \left\{\frac{1}{3}(10+8+12)-14\right\}_+ \\
&= \left\{\frac{30}{3}-14\right\}_+ \\
&= 0 \\
X(\omega_4) &= \left\{\frac{1}{3}(10+8+6)-14\right\}_+ \\
&= \left\{\frac{24}{3}-14\right\}_+ \\
&= 0
\end{aligned}$$

$$\begin{aligned}
Y(\omega_1) &= \{\max(10,16,18)-14\}_+ \\
&= \{18-14\}_+ \\
&= 4 \\
Y(\omega_2) &= \{\max(10,16,12)-14\}_+ \\
&= \{16-14\}_+ \\
&= 2 \\
Y(\omega_3) &= \{\max(10,8,12)-14\}_+ \\
&= \{12-14\}_+ \\
&= 0 \\
Y(\omega_4) &= \{\max(10,8,6)-14\}_+ \\
&= \{10-14\}_+ \\
&= 0
\end{aligned}$$

Answer 1. (a) iii.

The *Risk-Neutral Valuation Principle* states that, for all attainable contingent claims X, the following holds

$$V_t^* = \mathbb{E}_Q[X/B_t|\mathcal{F}_t] \text{ for } t=0,\dots,T \text{ and all } \mathbb{Q}$$

Thus, at time $t=0$

$$V_0 = \mathbb{E}_{\mathbb{Q}}[X|\mathcal{F}_0] = \mathbb{E}_{\mathbb{Q}}[X]$$

We can use this to derive the risk-neutral prices for the two options at time $t=0$

$$\begin{aligned}
V_0^X &= \mathbb{E}_{\mathbb{Q}}[X] \\
&= (2/3)\cdot(1/6) + 0\cdot(1/12) + 0\cdot(1/4) + 0\cdot(1/2) \\
&= 1/9
\end{aligned}$$

$$\begin{aligned}
V_0^Y &= \mathbb{E}_{\mathbb{Q}}[Y] \\
&= 4\cdot(1/6) + 2\cdot(1/12) + 0\cdot(1/4) + 0\cdot(1/2) \\
&= 5/6
\end{aligned}$$

Thus the risk-neutral price for the Asian option at time $t = 0$ is 1/9 and for the look-back option it is 5/6

Answer 1. (a) iv.
Note that since $r = 0$, $B_t = 1$ for $i = 0, 1, 2$.
Consider the Asian option and let $H^X(t) := \{H_0^Y(t), H_1^Y(t)\}$ denote a portfolio which only has access to a bank account & Asian options for each time point $t = 0, 1, 2$.

consider the value of the Asian option at time $t = 1$ and $t = 2$

$$\begin{aligned}
V_1^X(\omega) &= \begin{cases} \mathbb{E}[X|S_1 = 16] & \text{if } \omega \in \{\omega_1, \omega_2\} \\ \mathbb{E}[X|S_1 = 8] & \text{if } \omega \in \{\omega_3, \omega_4\} \end{cases} \\
&= \begin{cases} (2/3) \cdot p_2 + 0 \cdot (1 - p_2) & \text{if } \omega \in \{\omega_1, \omega_2\} \\ 0 & \text{if } \omega \in \{\omega_3, \omega_4\} \end{cases} \\
&= \begin{cases} (2/3) \cdot (2/3) & \text{if } \omega \in \{\omega_1, \omega_2\} \\ 0 & \text{if } \omega \in \{\omega_3, \omega_4\} \end{cases} \\
&= \begin{cases} 4/9 & \text{if } \omega \in \{\omega_1, \omega_2\} \\ 0 & \text{if } \omega \in \{\omega_3, \omega_4\} \end{cases} \\
V_2^X(\omega) &= X(\omega)\ \forall\ \omega
\end{aligned}$$

To find a replicating portfolio we start at time $t = 2$ and find that if $\omega \in \{\omega_1, \omega_2\}$

$$\begin{aligned}
V_2^X(\omega_1) = 2/3 &= H_0^X(2)(\omega_1, \omega_2) + 18 \cdot H_1^X(2)(\omega_1, \omega_2) \\
V_2^X(\omega_2) = 0 &= H_0^X(2)(\omega_1, \omega_2) + 12 \cdot H_1^X(2)(\omega_1, \omega_2)
\end{aligned}$$

$$\begin{aligned}
\implies \quad 2/3 &= 6H_1^X(2)(\omega_1, \omega_2) \\
\implies \quad H_1^X(2)(\omega_1, \omega_2) &= 1/9 \\
\implies \quad 2 &= H_0^X(2)(\omega_1, \omega_2) + 12 \cdot (1/9) \\
\implies \quad H_0^X(2)(\omega_1, \omega_2) &= -4/3
\end{aligned}$$

And if $\omega \in \{\omega_3, \omega_4\}$

$$\begin{aligned}
V_2^X(\omega_3) = 0 &= H_0^X(2)(\omega_3, \omega_4) + 12 \cdot H_1^X(2)(\omega_3, \omega_4) \\
V_2^X(\omega_4) = 0 &= H_0^X(2)(\omega_3, \omega_4) + 6 \cdot H_1^X(2)(\omega_3, \omega_4)
\end{aligned}$$

$$\begin{aligned}
\implies \quad H_0^X(2)(\omega_3, \omega_4) &= 0 \\
\implies \quad H_1^X(2)(\omega_3, \omega_4) &= 0
\end{aligned}$$

Let $H_0^X(1) := H_0^X(1)(\omega_1, \omega_2, \omega_3, \omega_4)$ and $H_1^X(1) := H_1(1)(\omega_1, \omega_2, \omega_3, \omega_4)$, then we have

$$\begin{aligned}
V_1^X(\omega) = 4/9 &= H_0^X(1) + 16 \cdot H_1^X(1) \quad \text{if } \omega \in \{\omega_1, \omega_2\} \\
V_1^X(\omega) = 0 &= H_0^X(1) + 8 \cdot H_1^X(1) \quad \text{if } \omega \in \{\omega_3, \omega_4\}
\end{aligned}$$

$$\begin{aligned}
\implies \quad 4/9 &= 8H_1^X(1) \\
\implies \quad H_1^X(1) &= 1/18 \\
\implies \quad 0 &= H_0^X(1) + 8 \cdot (1/18) \\
\implies \quad H_0^X(1) &= -4/9
\end{aligned}$$

Thus, a self-financing trading strategy with access to a bank account and the look-back option is as follows

$$\begin{aligned}
H^X(1)(\omega) &= (-4/9, 1/18) \quad \forall\ \omega \\
H^X(2)(\omega) &= (-4/3, 1/9) \quad \text{if } \omega \in \{\omega_1, \omega_2\} \\
H^X(2)(\omega) &= (0, 0) \quad \text{if } \omega \in \{\omega_3, \omega_4\}
\end{aligned}$$

To confirm this trading strategy is self-financing it is required that the cost of each portfolio is equal to the value process from the previous time-step

$$\begin{array}{l|rcccl} H^Y(1)(\omega) & -(4/9)+10\cdot(1/18) & = & 1/9 & = & V_0^X \\ H^Y(2)(\omega) & -(4/3)+16\cdot(1/9) & = & 4/9 & = & V_1^Y(\omega) \text{ if } \omega\in\{\omega_1,\omega_2\} \\ H^Y(2)(\omega) & 0+8\cdot 0 & = & 0 & = & V_1^Y(\omega) \text{ if } \omega\in\{\omega_3,\omega_4\} \end{array}$$

This requirement holds in all cases.

Now, consider the look-back option and let $H^Y(t) := \{H_0^Y(t), H_1^Y(t)\}$ denote a portfolio which only has access to a bank account & look-back options for each time point $t = 0, 1, 2$.

Consider the value of the look-back option at time $t = 1$ and $t = 2$

$$\begin{array}{rcl} V_1^Y(\omega) & = & \begin{cases} \mathbb{E}[Y|S_1=16] & \text{if } \omega\in\{\omega_1,\omega_2\} \\ \mathbb{E}[Y|S_1=8] & \text{if } \omega\in\{\omega_3,\omega_4\} \end{cases} \\ & = & \begin{cases} 4p_2+2(1-p_2) & \text{if } \omega\in\{\omega_1,\omega_2\} \\ 0 & \text{if } \omega\in\{\omega_3,\omega_4\} \end{cases} \\ & = & \begin{cases} 4\cdot(2/3)+2(1/3) & \text{if } \omega\in\{\omega_1,\omega_2\} \\ 0 & \text{if } \omega\in\{\omega_3,\omega_4\} \end{cases} \\ & = & \begin{cases} 10/3 & \text{if } \omega\in\{\omega_1,\omega_2\} \\ 0 & \text{if } \omega\in\{\omega_3,\omega_4\} \end{cases} \\ V_2^Y(\omega) & = & Y(\omega)\ \forall\ \omega \end{array}$$

To find a replicating portfolio we start at time $t = 2$ and find that if $\omega \in \{\omega_1, \omega_2\}$

$$\begin{array}{rcl} V_2^Y(\omega_1)=4 & = & H_0^Y(2)(\omega_1,\omega_2)+18\cdot H_1^Y(2)(\omega_1,\omega_2) \\ V_2^Y(\omega_2)=2 & = & H_0^Y(2)(\omega_1,\omega_2)+12\cdot H_1^Y(2)(\omega_1,\omega_2) \end{array}$$

$$\begin{array}{lrcl} \implies & 2 & = & 6H_1^Y(2)(\omega_1,\omega_2) \\ \implies & H_1^Y(2)(\omega_1,\omega_2) & = & 1/3 \\ \\ \implies & 2 & = & H_0^Y(2)(\omega_1,\omega_2)+12\cdot(1/3) \\ \implies & H_0^Y(2)(\omega_1,\omega_2) & = & -2/3 \end{array}$$

And if $\omega \in \{\omega_3, \omega_4\}$

$$\begin{array}{rcl} V_2^Y(\omega_3)=0 & = & H_0^Y(2)(\omega_3,\omega_4)+12\cdot H_1^Y(2)(\omega_3,\omega_4) \\ V_2^Y(\omega_4)=0 & = & H_0^Y(2)(\omega_3,\omega_4)+6\cdot H_1^Y(2)(\omega_3,\omega_4) \end{array}$$

$$\begin{array}{lrcl} \implies & H_0^Y(2)(\omega_3,\omega_4) & = & 0 \\ \implies & H_1^Y(2)(\omega_3,\omega_4) & = & 0 \end{array}$$

Let $H_0^Y(1) := H_0(1)(\omega_1, \omega_2, \omega_3, \omega_4)$ and $H_1^Y(1) := H_1(1)(\omega_1, \omega_2, \omega_3, \omega_4)$, then we have

$$\begin{array}{rcll} V_1^Y(\omega)=10/3 & = & H_0^Y(1)+16\cdot H_1^Y(1) & \text{if } \omega\in\{\omega_1,\omega_2\} \\ V_1^Y(\omega)=0 & = & H_0^Y(1)+8\cdot H_1^Y(1) & \text{if } \omega\in\{\omega_3,\omega_4\} \end{array}$$

$$\begin{array}{lrcl} \implies & 10/3 & = & 8H_1^Y(1) \\ \implies & H_1^Y(1) & = & 5/12 \\ \\ \implies & 0 & = & H_0^Y(1)+8\cdot(5/12) \\ \implies & H_0^Y(1) & = & -40/12 \end{array}$$

Thus, a self-financing trading strategy with access to a bank account and the look-back option is as follows

$$\begin{array}{rcll} H^Y(1)(\omega) & = & (-40/12, 5/12) & \forall\, \omega \\ H^Y(2)(\omega) & = & (-2/3, 1/3) & \text{if } \omega \in \{\omega_1, \omega_2\} \\ H^Y(2)(\omega) & = & (0, 0) & \text{if } \omega \in \{\omega_3, \omega_4\} \end{array}$$

To confirm this trading strategy is self-financing it is required that the cost of each portfolio is equal to the value process from the previous time-step

$$\begin{array}{r|rcccl} H^Y(1)(\omega) & -(40/12) + 10 \cdot (5/12) & = & 5/6 & = & V_0^Y \\ H^Y(2)(\omega) & -(2/3) + 16 \cdot (1/3) & = & 10/3 & = & V_1^Y(\omega) \text{ if } \omega \in \{\omega_1, \omega_2\} \\ H^Y(2)(\omega) & 0 + 8 \cdot 0 & = & 0 & = & V_1^Y(\omega) \text{ if } \omega \in \{\omega_3, \omega_4\} \end{array}$$

This requirement holds in all cases.

Answer 1. (b)
TODO