Financial Mathematics - Reviewed Notes

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Contents

1   General

Definition 1.1 - *Modelling*
Definition 1.2 - *Risk-Free*
An activity is said to be "*Risk-Free*" if the potential profits & losses are completely known.[1]

1.1   Probability

Definition 1.3 - *Sample Space* Ω
The *Sample Space* Ω is the set consisting of all elementary outcomes from a (series of) event(s).

Definition 1.4 - *Random Variable X*
A *Random Variable X* is a function from the *Sample Space* Ω to the real numbers $\mathbb{R}$.

$$X : \Omega \to \mathbb{R}$$

Definition 1.5 - *Stochastic Process*
Definition 1.6 - σ-*Algebra*
Definition 1.7 - *Generated* σ-*Algebra*
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1.2   Information Structures

2   Financial Terminology

Definition 2.1 - *Underlying Asset*
The *Underlying Asset* is a real financial asset or security which a contract can be based on. (e.g. Oil, interest rate, shares).

Definition 2.2 - *Dividend*
A *Dividend* is a one-off payment provided to the holder of an *Underlying Asset* at a certain time. Whether an *Underlying Asset* pays a *Dividend*, and the value of the *Dividend*, will affect the value of the *Underlying Asset*.

A *Dividend* is generally used by companies to distribute yearly profits to its shareholders.

Definition 2.3 - *Long Selling*
Long Selling is the practice of buying an asset (or security) and then selling it at some point in the future.

In *Long Selling* your profit/loss is $P_{\text{sell}} - P_{\text{buy}}$, thus you hope the price of the asset <u>increases</u> in the period between you buying and selling it.

Definition 2.4 - *Short Selling*
Short Selling is the practice of borrowing an asset (or security), immediately selling it[2] and at some point in the future buying an equivalent asset in order to reimburse your lender.

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[1]It does not refer to whether there no chance of making a loss.
[2]Receiving payment at this point.

In *Short Selling* your profit/loss if $P_{\text{sell}} - P_{\text{buy}}$, thus you hope the price of the asset <u>decreases</u> in the period between you selling and having to reimburse your lender.

Remark 2.1 - *Short Selling & Dividends*
If the asset you borrowed in *Short Selling* pays a *Dividend* during the time you have borrowed the asset, then you must pay this *Dividend* to the lender.[3]

Definition 2.5 - *Arbitrage Opportunity*
An *Arbitrage Opportunity* occurs when it is possible to make a profit without being exposed to the risk of incurring a loss.[4]

Generally *Arbitrage Opportunities* occur by being able to buy and sell the same asset in different markets, as each market may have a different price.

Theorem 2.1 - *No-Arbitrage Principle*

"*Arbitrage Opportunities* do not exist (for long) in real life markets."

As when the opportunities arise, the market activity cause by agents exploiting the opportunity would raise the cost of buying and thus remove the opportunity due to the *Law of Supply-and-Demand*.

Remark 2.2 - *Value of Money*
IRL the value of money is not constant due to inflation, interest rates & exchange rates. We generally want to normalise the returns of our portfolio wrt the change in value of money in order to determine the "real returns".

Definition 2.6 - *Bank Account Process, B_t*
A *Bank Account Process*[5] B_t is how much an initial deposit of one unit at time $t = 0$ would be worth at time point t if the deposit was made into a "Risk-Free Bank Account", given some risk-free *Interest Rate* r. This is a measure of how the value of money changes over the t time-periods.

The *Bank Account Process* must fulfil the following criteria

$$B_0 = 1 \quad \text{and} \quad B_t(\omega) \geq 0 \ \forall \ \omega \in \Omega$$

It is generally assumed that you can borrow money from these accounts, paying the same interest rate r.

Proposition 2.1 - *Value of Bank Account Process B_t*
Suppose our "Risk-Free Bank Account" pays a constant interest rate of r in each time-period, then after t time-periods our initial deposit would be worth

- *Continuous Time Model $B_t = B_0 e^{rt}$.*

- *Single-Period Model $B_1 = B_0(1 + r)$.*[6]

- *Multi-Period Model $B_t = B_0(1 + r)^t$.*

Definition 2.7 - *Portfolio*

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[3]As you have already sold the asset, then this expense will come out of your own pocket.

[4]Someone who loos for *Arbitrage Opportunities* is called an *Arbitrageur*.

[5]AKA a *Bond* or a *Numeraire*.

[6]Must be that $t = 1$ in a *Single-Period Model*.

2.1   Derivatives

Definition 2.8 - *Derivative Securities*
A *Deriviative Security* is a contract which has an expiry date T and pays out different amounts depending upon the value of some *Underlying Asset* in the time-period $[0, T]$.

Remark 2.3 - *Valuing Contracts*
When valuing contracts we assume that arbitrage does not exist. This means we can derive a single price[7] for a contract, as any other price would create an *Arbitrage Opportunity*.

Theorem 2.2 - *Equivalent Contract Valuations over Time*
If two combinations of financial derivatives both have the same value $V_T = W_T$ at time $t = T$. Then their prices will be the same at all $t < T$

$$\text{if } V_T = W_t \text{ then } V_t = W_t \; \forall \; t < T$$

Proof 2.1 - *Theorem 2.2*
We assume the "No-Arbitrage Principle" holds throughout this proof.

Let V_t, W_t represent the fair price for two different combinations of financial derivatives at time t and that $V_T = W_T$. Suppose there is a risk-free profit of r.

Assume WLOG that $V_t > W_t$. Then an arbitrage opportunity exists and can be exploited by doing the following:

- At $t = 0$

    i). Sell/short the first combination, receiving £V_t.
    ii). Buy the second combination, costing £W_t.
    iii). Invest the difference (£$V_t - W_t > 0$).

- At $t = T$

    i). Sell the first combination, receiving £$V_T = W_T$
    ii). Buy the second combination, costing £$W_T = V_T$.

Following this will result in a "riskless" profit of $(V_t - W_t)e^{r(T-t)} > 0$.

Definition 2.9 - *Forward Contract*
A *Forward Contract* is a type of *Derivative Security*. In a *Forward Contract* two parties agree to an exchange on a predetermined future date for a predetermined amount, and are both <u>obliged</u> to fulfil this exchange.

All *Forward Contracts* have the following components

- *Delivery Date T.*

- *Delivery Price K.*

Remark 2.4 - *Positions in a Forward Contract*
In a *Forward Contract* agents can take two positions

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[7]Known as the *Fair Price.*

- *Long Position* - Agree to buy the underlying asset for £K on date T. Makes a profit if the market-value of the underlying asset is <u>greater</u> than K in time-period T.

- *Short Position* - Agree to sell the underlying asset for £K on date T. Makes a profit if the market-value of the underlying asset is <u>less</u> than K in time-period T.

Remark 2.5 - *Utility of Forwards Contracts*
Forward Contracts allow you to agree terms of a purchase/sale some time in advance of actually transacting. This means business have greater certainty about their future cash-flows.[8]

Theorem 2.3 - *Fair Delivery Price of a Forward Contract*
Consider a *Forward Contract* with delivery date T, where the underlying asset has value S_0 at time $t = 0$ and pays a dividend D at time $t_0 \in (0, T)$. Suppose there is a risk-free bank account with a constant interest rate r during the interval $[0, T]$.[9]

Then

- If $D = 0$ (ie no dividend is payed) then the fair *Delivery Price* for this contract is

$$K = S_0 e^{rT}$$

- If $D > 0$ then the fair *Delivery Price* for this contract is

$$K = (S_0 - I)e^{rT} \text{ where } I := De^{-rt_0}$$

Proof 2.2 - *Theorem 2.3*
We use the "No-Arbitrage Principle" to prove that these Ks are the fair prices under each scenario.

Case 1 - Suppose, for the sake-of-contradiction, that $K > (S_0 - I)e^{rT}$ with $I := De^{-rt_0}$. Then an arbitrage opportunity exists and can be exploited by doing the following:

- At $t = 0$

    i). Borrow £S_0 from the bank, at an interest rate of r.
    ii). Buy the underlying asset.
    iii). Taking a short position in the forward contract (receiving $K > (S_0 - I)e^{rT}$).

- At $t = t_0$

    i). We will receive a dividend payment £D which we shall use to partially repay our loan. This leaves an outstanding balance of $S_0 e^{rt_0} - D$.

- At $t = T$

    i). Sell the asset for K using the forward contract.
    ii). Repay the outstanding balance on the loan $((S_0 e^{rt_0} - D)e^{r(T - t_0)})$.

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[8]e.g. Farmers may agree to a price for their whole harvest a year in advance. Thus their next years income is completely known.
[9]This means $B_t = e^{rt}$.

Doing all this will lead to a "riskless" profit of

$$K - (S_0 e^{rt_0} - D)e^{r(T-t_0)} = K - (S_0 - I)e^{rT} > 0 \text{ by def. } K$$

This means that this definition of K cannot be the fair-price, thus $K \le (S_0 - I)e^{rT}$.

Case 2 - Suppose, for the sake-of-contradiction, that $K < (S_0 - I)e^{rT}$ with $I := De^{-rt_0}$. Then an arbitrage opportunity exists and can be exploited by doing the following:

- At $t = 0$

  i). Short sell the underlying asset. (Receiving £S_0).
  ii). Invest this revenue, receiving an interest rate of r.
  iii). Take a long position on the forward contract.

- At $t = t_0$

  i). Pay the dividend £D to our lender, from our bank account.

- At $t = T$

  i). Buy the asset for K using the forward contract.

Doing all this will lead to a "riskless" profit of

$$(S_0 e^{rt_0} - D)e^{r(T-t_0)} - K = (S_0 - I)e^{rT} - K > 0 \text{ by def. } K$$

This means that this definition of K cannot be the fair-price, thus $K \ge (S_0 - I)e^{rT}$.

Thus, by combining these two inequalities, the fair price for this *Forward Contract* is

$$K = (S_0 - I)e^{rT}$$

Definition 2.10 - *Options Contract*
An *Options Contract* is a type of *Derivative Security*. In an *Options Contract* two parties agree to an exchange on (or before) a predetermined future date for a predetermined amount, but the holder is <u>not obliged</u> to fulfil this exchange.

All *Options Contracts* have the following components

- *Delivery Date T.*
- *Strike Price K.*

There are two classes of *Options Contract*

- *Call Option* - The holder has the right to buy.
- *Put Option* - The holder has the right to sell.

Definition 2.11 - *European & American Options*
There are two categories of *Options Contract* which determine when the contract can be exercised

- *European Option* - The holder can only execute on the delivery date $t = T$.
- *American Option* - The holder can execute on <u>any</u> date before the delivery date T.

Definition 2.12 - *Positions in an Options Contract*
In *Options Contracts* agents can take one of two positions. The position they take determines their rights & potential cash-flows.

- *Holder* - Decides whether to execute the contract of not. Will pay the *Writer* a fee for creating the contract.

  The *Holder's* only expense is the fee they pay the *Writer* and they may make an income if they execute the contract.

- *Write* - Must complete the transaction if the *Holder* wishes to. Receives a fee from the *Holder*.

  The *Writer's* only income is the fee they receive from the *Holder* and they may incur a loss if the contract is executed.

Remark 2.6 - *When are options executed?*
Whether the holder should execute their option depends on the market price S_T at time T, the strike price K and the class of contract. Assuming (justifiably) that the holder will only execute the option if it will make them money, the holder should do the following

- For a *Call Option* the holder should execute if $S_T > K$. As they can immediately sell their newly bought asset for a profit of $S_T - K - F$ where F is the fee payed to the writer.

- For a *Put Option* the holder should execute if $S_T < K$. As they can buy the asset from the market and sell it to the writer of their option for a profit of $S_T - K - F$ where F is the fee payed to the writer.

Theorem 2.4 - *Put-Call Parity*[10]
Consider a *European Put Option* and a *European Call Option* where both have the same: underlying asset, strike price K and expiry date T. Let S_t be the value of the underlying asset at time point t and assume there is a "risk-free" interest rate of r available.

Then, if <u>no</u> *Arbitrage Opportunities* exist then the following hold

$$S_t + P_t - C_t = Ke^{-r(T-t)} \ \forall \ t \in [0, T]$$

where P_t, C_t are the prices of the put & call options at time t respectively, and $Ke^{-r(T-t)}$ is the discounted value of our bank account.

Theorem 2.5 - *Lower-Bound for a European Call Option*
Let S_t be the value of an underlying asset at time t.

For a *European Call Option* with strike price K and delivery date T we can determine the following lower bound on its price C_t

$$C_t \ \geq \ \{S_t - Ke^{-r(T-t)}\}_+$$

Proof 2.3 - `Theorem 2.5`
By *Put-Call Parity* (`Theorem 2.4`) we have that

$$\begin{aligned} S_t + P_t - C_t &= Ke^{-r(T-t)} \\ \implies \qquad C_t &= S_t + P_t - Ke^{-r(T-t)} \end{aligned}$$

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[10]This is an application of `Theorem 2.2` to European Put & Call Options.

Since *Put Options* cannot have a negative price, $P_t \geq 0$, we have that

$$C_t \geq S_t + P_t - Ke^{-r(T-t)}$$

Further, since *Call Options* cannot have a negative price, $C_t \geq 0$, we have that

$$C_t \geq \left\{ S_t + P_t - Ke^{-r(T-t)} \right\}_+$$

Theorem 2.6 - *Value of American Call Options w/o Dividends*
Consider an *American* & a *European Call Option*, for the same underlying asset, with the same strike price and expiry date.

Then, if the underlying asset does <u>not</u> pay a *Dividend*

$$C_A = C_E^{[11]}$$

where C_A, C_E are the price of the American & European call options, respectively.

Proof 2.4 - `Theorem 2.6`
If the *American Call Option* is executed early at time-point $t < T$ then it generates an income of $S_t - K$.

However, `Theorem 2.5` shows that selling a *Call Option* generates

$$\{S_t - Ke^{-r(T_t)}\}_+ \geq S_t - Ke^{-r(T-t)} > S_t - K$$

This shows that it is sub-optimal to exercise the call at any time $t < T$.

3   Discrete-Time

3.1   Single-Period Model

Definition 3.1 - *Single-Period Model*
The *Single-Period Model* is a model for a financial market where actions can only occur on two days. It has the following components.

- *Initial Date $t = 0$.*

- *Terminal Date $t = 1$.*

- Trading is only allowed to occur on the *Initial & Terminal Dates.*

- A finite *Sample Space* $\Omega := \{\omega_1, \ldots, \omega_K\}$ with $K < \infty$.

  Each event $\omega_1, \ldots, \omega_K$ corresponds to some state of the world.

- A *Probability Measure* $\mathbb{P}$ on the *Sample Space* Ω with $\mathbb{P}(\{\omega_i\}) > 0 \; \forall \; i \in [1, K]$.

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[11]This shows that if an underlying asset does not pay a dividend then it is suboptimal to exercise an *American Call Option* early.

Definition 3.2 - *Price-Process S*[12]
A *Price Process S* models the price of each security at each time-point

$$S := \{S(t) : t = 0, 1\} \text{ where } S(t) = (S_1(t), \dots, S_N(t))$$

where $S(t)$ is the collection of the prices of each available stock at time t, $S_i(t)$ is the price of the ith stock at time t and there are N different stock available.

The values of $S(0)$ are known to the investors, but $S(1)$ are unknown non-negative random variables whose value only become known at time $t = 1$.

Definition 3.3 - *Discounted Price-Process S^**
A *Discounted Price-Process S^** models the price of each security at each time-point <u>but</u> normalised for the change in the value of money due to the *Bank Process B*.

$$S^* := \{S^*(t) : t = 0, 1\} \text{ where } S^*(t) = (S_1^*(t), \dots, S_N^*(t)) \text{ and } S_i^*(t) := \frac{S_i(t)}{B_t}$$

Definition 3.4 - *Trading Strategy H*
A *Trading Strategy H* describes the changes in an investors portfolio between the *Initial Date* $t = 0$ and *Terminal Date* $t = 1$.
$$H := (H_0, H_1, \dots, H_N)$$

where H_0 is the amount invested/borrowed in a risk-free bank account, H_i is the amount bought/shorted of the ith stock for $i \in [1, N]$ and there are N stocks available in the market.

Definition 3.5 - *Value-Process V*
A *Value Process V* describes the total value of a *Trading Strategy H* at each time-point $t = 0, 1$

$$V := \{V_t : t = 0, 1\} \text{ where } V_t = H_0 B_t + \sum_{i=1}^{N} H_i S_i(t)$$

Definition 3.6 - *Discounted Value-Process V^**
A *Discounted Value-Process V^** describes the total value of a *Trading Strategy H* at each time-point <u>but</u> normalised for the change in the value of money due to the *Bank Process B*.

$$V^* := \{V_t^* : t = 0, 1\} \text{ where } V_t^* := \frac{V_t}{B_t} = H_0 + \sum_{i=1}^{N} H_i S_i^*(t)$$

Definition 3.7 - *Gains-Process G*
A *Gains Process G* describes the total profit/loss generated made by a *Trading Strategy H* between time-points $t = 0, 1$

$$G := H_0 r + \sum_{i=1}^{N} H_i \Delta_{S_i} \text{ where } \Delta_{S_i} = S_i(1) - S_i(0)^{[13]}$$

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[12]AKA *Stock Process*
[13]Δ_{S_i} The change in value of the ith stock.

Definition 3.8 - *Discounted Gains-Process G^**
A *Discounted Gains-Process G^** describes the total profit/loss generated made by a *Trading Strategy H* between time-points $t = 0, 1$ <u>but</u> normalised for the change in the value of money due to the *Bank Process B*.

$$G^* := \frac{G}{B_t} = \sum_{i=1}^{N} H_i \Delta_{S_i^*} \text{ where } \Delta_{S_i^*} = S_i^*(1) - S_i^*(0)$$

Definition 3.9 - *Arbitrage Opportunity - Single-Period Model*
Consider a *Trading Strategy $H = (H_0, H_1)$* for the *Single-Period Model.*

H exploits an *Arbitrage Opportunity* if it has the following three properties

   i). $V_0 = 0$.

   ii). $V_1(\omega) \geq 0 \; \forall \; \omega \in \Omega$.

   iii). $\mathbb{P}(V_1(\omega) \geq 0) > 0 \; \forall \; \omega \in \Omega$.[14]

Theorem 3.1 - *Arbitrage Opportunities & Gains Process*
There exists an *Arbitrage Opportunity* in a market <u>iff</u> there exists a *Trading Strategy H* st[15]

$$G^* \geq 0 \quad \text{and} \quad \mathbb{E}[G^*] > 0$$

Proof 3.1 - `Theorem 3.1`

   $\Rightarrow$ Let H be a *Trading Strategy* which exploits an *Arbitrage Opportunity.*

     By the definition of an *Arbitrage Opportunity* $G^* = V_1^* - V_0^*$ and $B_t > 0 \; \forall \; t, \omega$, this means that $G^* \geq 0$ and thus
$$\mathbb{E}[G^*] = \mathbb{E}[V_1^*] > 0$$

   $\Leftarrow$ Let H be a *Trading Strategy* which satisfies $G^* \geq 0$ and $\mathbb{E}[G^*] > 0$.

     Define $\hat{H} := (\hat{H}_0, H_1, \ldots, H_N)$ where $\hat{H}_0 := -\sum_{i=1}^{N} H_i S_i^*(0)$[16].

     Under $\hat{H}_0$ we have that $V_0^* = 0$ and $V_1^* = V_0^* + G^* = G^*$.

     Hence, $V_1^* \geq 0$ and $\mathbb{E}[V_1^*] = \mathbb{E}[G^*] > 0$, meaning $\hat{H}$ exploits an *Arbitrage Opportunity.*

As the result holds in both directions, we can say it holds <u>iff</u>.

3.1.1   Risk-Neutral Probability Measures $\mathbb{Q}$

Definition 3.10 - *Risk-Neutral Probability Measure $\mathbb{Q}$*
A *Probabiltiy Measure $\mathbb{Q}$* on *Sample Space Ω* is said to be a *Risk-Neutral Probability Measure* if the following hold

   i). $\mathbb{Q}(\{\omega\}) > 0 \; \forall \; \omega \in \Omega$.

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[14]Equivalently $\mathbb{E}[V_1] > 0$

[15]This means H never loses money, and it is expected to make money.

[16]This ensures V_0, a requirement for H to exploit an *Arbitrage Opportunity.*

ii). $\mathbb{E}_{\mathbb{Q}}[S_i * (1)] = S_i^*(0) \ \forall \ i \in [1, N]$

Theorem 3.2 - *Separating Hyperplane Theorem*[17]
Let $\mathbb{W}$ be a linear subspace of $\mathbb{R}^K$ and $\mathbb{K}$ be a compact convex subset in $\mathbb{R}^K$ which is disjoint from $\mathbb{W}$.

We can separate $\mathbb{W}$ and $\mathbb{K}$ strictly by using a hyperplane containing $\mathbb{W}$[18] st

$$u^T v > 0 \ \forall \ u \in \mathbb{K}$$

Theorem 3.3 - *Arbitrage Opportunities & Risk-Neutral Probability Measures*
No *Arbitrage Opportunities* exists <u>iff</u> there exists a *Risk-Neural Probability Measure* $\mathbb{Q}$.

Proof 3.2 - `Theorem 3.3`
Consider the three following sets

i). $\mathbb{W} = \left\{ X \in \mathbb{R}^K : X = G^* \text{ for some Trading Strategy } H \right\}$.

This is the set of possible *Gains* in our market for *Trading Strategies* which have zero initial investment. $\mathbb{W}$ is a linear subspace of $\mathbb{R}^{K}$[20].

ii). $\mathbb{A} = \left\{ X \in \mathbb{R}^K : X \geq 0, X \neq 0 \right\}$[21].

There exists an arbitrage opportunity <u>iff</u> $\mathbb{W} \cap \mathbb{A} \neq \emptyset$.

iii). $\mathbb{A}^+ = \left\{ X \in \mathbb{R}^N : X \geq 0, X \neq 0, \sum_{i=1}^K X_i = 1 \right\}$.

$\mathbb{A}^+$ is a convex and compact subset of $\mathbb{R}^K$.

$\Rightarrow$ Assume that there are no *Arbitrage Opportunities*, then $\mathbb{W} \cap \mathbb{A} \neq \emptyset$.

By the *Separating Hyperplane Theorem* (`Theorem 3.2`) $\exists \ Y \in \mathbb{R}^K$ which is *orthogonal* to $\mathbb{W}$ st
$$X^T Y > 0 \ \forall \ X \in \mathbb{A}^+$$

For each $k \in \{1, \dots, K\}$ the kth unit vector e_k is an element of $\mathbb{A}^+$. Therefore,

$$Y_k := e_k^T Y > 0 \ \forall \ k \in \{1, \dots, K\}$$

meaning all entries of Y are strictly positive.

Define a probability measure $\mathbb{Q}$ by setting

$$\mathbb{Q}(\{\omega_k\}) = \frac{Y(\omega_k)}{Y(\omega_1) + \cdots + Y(\omega_k)}$$

Furthermore, $\Delta S_n^* \in \mathbb{W} \ \forall \ n$ because $\Delta S_n^* := S_n^*(1) - S_n^*(0)$ is the discounted wealth for the portfolio $H := e_n$ which consists of one unit of the nth asset only.

Since Y is orthogonal to $\mathbb{W}$ we can conclude that

$$\mathbb{E}_{\mathbb{Q}}[\Delta S_n^*] = \sum_{k=1}^{K} \Delta S_n^*(\omega_k) \mathbb{Q}(\{\omega_k\}) = 0 \ \forall \ n$$

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[17]This theorem is used to prove `Theorem 3.3`. The proof of this theorem is beyond the scope of this course.
[18]ie $\exists \ v \in \mathbb{R}^K$ which is *Orthogonal* to $\mathbb{W}$ [19]$u^T v = 0 \ \forall \ u \in \mathbb{W}$.
[20]Proved by showing it is complete under: addition, and scalar multiplication.
[21]$\mathbb{A}$ is not compact, so can not be used for $\mathbb{K}$ in *Separating Hyperplane Theorem*

In other words

$$\mathbb{E}_{\mathbb{Q}}[S_n^*(1)] = S_n^*(0) \ \forall \ n$$

Thus $\mathbb{Q}$ is a *Risk-Neutral Probability Measure*.

$\Leftarrow$ Let $\mathbb{Q}$ be a *Risk-Neutral Probability Measure*.

Then for an arbitrary *Trading Strategy H* we have that

$$\mathbb{E}_{\mathbb{Q}}[G^*] = \mathbb{E}_{\mathbb{Q}}\left[\sum_{n=1}^N H_n \Delta S_n^*\right] = \sum_{n=1}^N H_n \mathbb{E}_{\mathbb{Q}}[\Delta S_n^*] = 0$$

and, in particular

$$\sum_{k=1}^K G^*(\omega_k)\mathbb{Q}(\{\omega_k\}) = 0$$

which shows that either $G^*(\omega_k < 0)$ for some k or $G^* = 0$, but then $\mathbb{E}_{\mathbb{Q}}[G^*] = 0$.

Hence, by `Theorem 3.1`, there cannot be any arbitrage opportunities.

The result holds in both directions. $\square$

3.1.2   Contingent Claims X

Definition 3.11 - *Contingent Claim X - Single-Period Model*
Consider a *Single-Period Model* with a risk-free bank process and $K-1$ risky securities on offer.

A *Contingent Claim $X \in \mathbb{R}^K$* in a *Single-Period Model* is a random variable which represents a payoff at time $t = 1$ for each security[22].

Definition 3.12 - *Attainable Contingent Claim X*
A *Contingent Claim X* is said to be "*attainable*" if

$$\exists \text{ trading strategy } H \text{ st } V_1 = X \text{ when using } H^{[23]}.$$

Otherwise, X is said to be "<u>un</u>*attainable*".

Remark 3.1 - *Determining whether a Contingent Claim X is Attainable*
Consider a *Single-Period Model* with $K-1$ securities, which can be described the following matrix A, and a *Contingent Claim $X \in \mathbb{R}^K$*. Then

X is *Attainable* iff $\exists$ a trading strategy $H \in \mathbb{R}^K$ st $AH = X$ where

$$A := \begin{pmatrix} B_1(\omega_1) & S_1(1)(\omega_1) & \dots & S_N(1)(\omega_1) \\ B_1(\omega_2) & S_1(1)(\omega_2) & \dots & S_N(1)(\omega_2) \\ \vdots & \vdots & \ddots & \vdots \\ B_1(\omega_K) & S_1(1)(\omega_K) & \dots & S_N(1)(\omega_K) \end{pmatrix}$$

Theorem 3.4 - *Fair Price of a Contingent Claim*
Let X be an *Attainable Contingent Claim* and H be a *Replicating Portfolio* which generates X.

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[22]This payoff is not necessarily achievable
[23]This H called a *Replicating Portfolio* and is said to "*generate*" X

The value of portfolio H at time $t = 0$ (V_0) is the "fair-price" of the contingent claim X.

Proof 3.3 - *Theorem 3.4*
Let X be an *Attainable Contingent Claim* and H be a *Replicating Portfolio* which generates X.

Let p be the fair price for X and assume, for the sake of contradiction, that p does <u>not</u> equal the value of H at time $t = 0$. This means we assuming that $p \neq V_0$.

We have two cases

Case 1 - $p > V_0$.

In this case, an *arbitrage opportunity* exists and can be exploited by doing the following

- At $t = 0$ - Short the *Contingent Claim* for p; buy portfolio H for V_0; and invest the difference $p - V_0 > 0$.
- At $t = 1$ - Our portfolio has the same value as X so we sell H to fulfil our short position on the contingent claim.

Our profit in this scenario is $(p - V_0)B_1 = (p - V_0)(1 + r) > 0$.

Case 2 - $p < V_0$.

In this case, an *arbitrage opportunity* exists and can be exploited by doing the following

- At $t = 0$ - Buy the *Contingent Claim* for p; buy portfolio $-H$[24] for $-V_0$; and invest the difference $V_0 - p > 0$.
- At $t = 1$ - Our portfolio has value $-X$ so we sell our *Contingent Claim* for X to cover the portfolio, fulfilling any short positions.

Our profit in this scenario is $(V_0 - p)B_1 = (V_0 - p)(1 + r) > 0$.

Hence, in all scenarios where $p \neq V_0$ an arbitrage opportunity exists. This means $p \neq V_0$ cannot be the fair price for X and thus $p = V_0$ is the fair price. $\qquad\square$

Theorem 3.5 - *Risk-Neutral Valuation Principle*
Consider a *Single-Period Model* where <u>no</u> *Arbitrage Opportunities* exists, and let $\mathbb{Q}$ be a *Risk-Neutral Probability* for this model.

Then the "fair-price" of an *Attainable Contingent Claim* X at time $t = 0$

$$p = \mathbb{E}_{\mathbb{Q}}[X/B_1]$$

Proof 3.4 - *Theorem 3.5*
Consider a *Single-Period Model* with <u>no</u> arbitrage opportunities and let X be an *Attainable Contingent Claim* under this model.

Here we derive the fair-price for X and show that time price is unique.

Suppose there exists two trading strategies $H, \hat{H}$ st $V_1 = \hat{V}_1 = X$ but $\hat{V}_0 \neq V_0$.

Let $\mathbb{Q}$ be a *Risk-Neutral Probability Measure* under this model. Then, by the *No-Arbitrage Principle* (Theorem 2.3), we have that for any trading strategy H $\mathbb{E}_{\mathbb{Q}}[G^*] = 0$. Thus we can

---

[24]Note this is equivalent to shorting portfolio H

deduce that

$$
\begin{aligned}
V_0 &= V_0^* \\
&= \mathbb{E}_{\mathbb{Q}}[V_0^*] \\
&= \mathbb{E}_{\mathbb{Q}}[V_1^* - G^*] \\
&= \mathbb{E}_{\mathbb{Q}}[V_1^*] - \mathbb{E}_{\mathbb{Q}}[G^*] \\
&= \mathbb{E}_{\mathbb{Q}}[V_1^*] - 0 \\
&= \mathbb{E}_{\mathbb{Q}}[V_1/B_1]
\end{aligned}
$$

This shows that any trading strategy H with $V_1 = X$ (ie is worth X at time $t = 1$), has the following value at time $t = 0$

$$
V_0 = \mathbb{E}_{\mathbb{Q}}[V_1/B_1] = \mathbb{E}_{\mathbb{Q}}[X/B_1]
$$

This holds for all *Risk-Neutral Probability Measures* $\mathbb{Q}$, so the fair-price for X at time $t = 0$ is constant between different *Risk-Neutral Probability Measures*. Further, all trading strategies with the same value at time $t = 1$ have the same value at time $t = 0$ (and we have a formula for this value). $\square$

3.1.3 Complete Markets

Definition 3.13 - *Complete & Incomplete Markets*
A model of a market is said to be *Complete* if each *Contingent Claim* X there exists a *Trading Strategy* H which generates X.

Otherwise, the model is said to be <u>in</u>*complete*.

Remark 3.2 - *Checking if a Market is Complete*
We can check whether the model of a market is *Complete* by defining the following matrix A, and if A spans the same space as *Contigent Claims*[25] then the market <u>is</u> *Complete*.

$$
A = \begin{pmatrix}
B_1(\omega_1) & S_1(1)(\omega_1) & \dots & S_N(1)(\omega_1) \\
B_1(\omega_2) & S_1(1)(\omega_2) & \dots & S_N(1)(\omega_2) \\
\vdots & \vdots & \ddots & \vdots \\
B_1(\omega_K) & S_1(1)(\omega_K) & \dots & S_N(1)(\omega_K)
\end{pmatrix}
$$

Theorem 3.6 - *Complete Markets and Risk-Neutral Probability Measure*
Consider a model with <u>no</u> *Arbitrage Opportunities*, then

> The model is *Complete* <u>iff</u> $\exists$ a <u>unique</u> *Risk-Neutral Probability Measure* $\mathbb{Q}$.

Proof 3.5 - *Theorem 3.6*
Consider a *Single-Period Model* with <u>no</u> *Arbitrage Opportunities* and let $\mathbb{M}$ denote the set of all *Risk-Neutral Probability Measures* for this model.

Since there are <u>no</u> arbitrage opportunities then $\mathbb{M} \neq \emptyset$.

As this theorem is "iff" I shall prove it in both directions separately

$\Longrightarrow$ Assume, for the sake of contraction, that the model <u>is</u> *Complete* but $\mathbb{M} = \{\mathbb{Q}, \hat{\mathbb{Q}}\}$ (ie contains two distinct elements).

---

[25]This is done by determining whether $\text{rank}(A) = \dim(X)$.

Then $\exists \, \omega_k \in \Omega$ st $\mathbb{Q}(\omega_k) \neq \hat{\mathbb{Q}}(\omega_k)$. Consider the following *Contingent Claim X*

$$
\begin{aligned}
X(\omega) &= \begin{cases} B_1(\omega) & \text{if } \omega = \omega_k \\ 0 & \text{otherwise} \end{cases} \\
&= B_1 \mathbb{1}\{\omega = \omega_k\}
\end{aligned}
$$

Then

$$
\begin{aligned}
\mathbb{E}_{\mathbb{Q}}[V_0] = \mathbb{E}_{\mathbb{Q}}[X/B_1] &= \mathbb{E}_{\mathbb{Q}}[\mathbb{1}\{\omega = \omega_k\}] \\
&= \mathbb{Q}(\{\omega_k\})) \\
&\neq \mathbb{Q}(\{\omega_k\}) \text{ by def. } X \\
&= \mathbb{E}_{\hat{\mathbb{Q}}}[\mathbb{1}\{\omega = \omega_k\}] \\
&= \mathbb{E}_{\hat{\mathbb{Q}}}[X/B_1] = \mathbb{E}_{\hat{\mathbb{Q}}}[V_0] \\
\implies \mathbb{E}_{\mathbb{Q}}[V_0] &\neq \mathbb{E}_{\hat{\mathbb{Q}}}[V_0]
\end{aligned}
$$

This contradicts `Proof 3.4` when we showed that if X is attainable then $\mathbb{E}_{\mathbb{Q}}[V_0]$ is the same for all $\mathbb{Q} \in \mathbb{M}$.

Thus, if the model is *Complete* then it has a unique *Risk-Neutral Probability Measure*.

$\impliedby$ Assume, for the sake of contradiction, that the model has a unique *Risk-Neutral Probability Measure* $\hat{\mathbb{Q}}$ but there exists a *Contingent Claim X* which is <u>not</u> *Attainable*.

Then, there does not exist a trading strategy H which solves $AH = X$.

By the *Separating Hyperplane Theorem* (`Theorem 3.2`) it follows that

$$\exists \, \pi \in \mathbb{R}^K \text{ st } \pi^T A = 0^{[26]} \text{ and } \pi^T X > 0$$

Let $\lambda > 0$ be small enough that

$$\mathbb{Q}(\{\omega_j\}) := \hat{\mathbb{Q}}(\{\omega_j\}) + \lambda \pi_j \cdot B_1(\omega_j) > 0 \quad \forall \, j \in [1, K]$$

As A is defined st all the terms in its first column are B_1 and $\pi^T A = 0$, the $\mathbb{Q}$ defined above is a probability measure.

Moreover, for any *Discounted Price Process* $s^* = (S_1^*, \ldots, S_N^*)$ and any $n \in [1, N]$ we have

$$
\begin{aligned}
\mathbb{E}_{\mathbb{Q}}[S_n^*(1)] &= \sum_{j=1}^{K} \frac{S_n(1)(\omega_j)}{B_1(\omega)} \cdot \mathbb{Q}(\{\omega_j\}) \\
&= \sum_{j=1}^{K} \frac{S_n(1)(\omega_j)}{B_1(\omega)} \cdot \left(\hat{\mathbb{Q}}(\{\omega_j\}) + \lambda \pi_j B_1(\omega_j) \right) \\
&= \sum_{j=1}^{K} \frac{S_n(1)(\omega_j)}{B_1(\omega)} \cdot \hat{\mathbb{Q}}(\{\omega_j\}) + \lambda \underbrace{\sum_{j=1}^{K} \pi_j S_n(1)(\omega_j)}_{=0} \\
&= \sum_{j=1}^{K} \frac{S_n(1)(\omega_j)}{B_1(\omega)} \cdot \hat{\mathbb{Q}}(\{\omega_j\}) \\
&= \sum_{j=1}^{K} S_n^*(1)(\omega_j) \hat{\mathbb{Q}}(\{\omega_j\}) \\
&= \mathbb{E}_{\hat{\mathbb{Q}}}[S_n^*(1)] \\
&= S_n^*[0]^{[27]}
\end{aligned}
$$

---

[26] ie π is orthogonal to A.

This shows that $\mathbb{Q}$ is a *Risk-Neutral Probability Measure* and so $\mathbb{Q} \in \mathbb{M}$, a contradiction to the uniqueness of $\hat{\mathbb{Q}}$.

If there is a unique *Risk-Neutral Probability Measure* for a model, then all *Contingent Claims* are attainable under the model.

This has proved the theorem in both directions. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

4 Continuous-Time

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[27]As $\hat{\mathbb{Q}}$ is a *Risk-Neutral Probability Measure*.

0   Reference

0.1   Notation

0.2   Definitions