Financial Mathematics - Reviewed Notes

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Contents

1 General

Definition 1.1 - *Modelling*
TODO

Definition 1.2 - *Risk-Free*
An activity is said to be "*Risk-Free*" if the potential profits & losses are completely known.[1]

2 Probability

2.1 General Probability

Definition 2.1 - *Sample Space* Ω
The *Sample Space* Ω is the set consisting of all elementary outcomes from a (series of) event(s).

Definition 2.2 - *Random Variable* X
A *Random Variable* X is a function from the *Sample Space* Ω to the real numbers $\mathbb{R}$.

$$X : \Omega \to \mathbb{R}$$

2.2 Stochastic Processes

Definition 2.3 - *Stochastic Process*
A *Stochastic Process* S is a real-valued function $S(t)(\omega)$

$$S : [0,T] \times \Omega \to \mathbb{R}$$

Proposition 2.1 - *Fixing components of a Stochastic Process*
Let a *Stochastic Process* $S : [0,T] \times \Omega \to \mathbb{R}$ and consider fixing different variables

- If we fix $t \in [0,T]$ then $S(t)(\cdot) : \Omega \to \mathbb{R}$[2] is a *Random Variable*.
- If we fix $\omega \in \Omega$ then $S(\cdot)(\omega) : [0,T] \to \mathbb{R}$[3] is called a *Sample Path*.

Definition 2.4 - *Predictable Stochastic Process*
A *Stochastic Process* $\{X_t\}_t$ is "*predictable*" if, for each t, X_t is $\mathcal{F}_{t-1}$-*Measurable* wrt some filtration $\{\mathcal{F}_t\}_t/$

Definition 2.5 - *Simple Stochastic Process*
A *Stochastic Process* $\{X_t\}_{t\in[0,T]}$ is "*simple*" if there exists

- A partition $\{t_0, t_1, \dots, t_n\}$ with $0 = t_0 < t_1 < \cdots < t_n = T$.

[1] It does not refer to whether there no chance of making a loss.
[2] The event ω is the only variable
[3] The time-point t is the only variable

- a set of random variables $\{\xi_k\}_{k\in[0,n]}$ which are $\mathcal{F}_{t_k}$-*Measurable* and have finite expected values[4].

such that $X_t(\omega)$ can be written as the stepped-function (1).

$$X_t(\omega) = \xi_0(\omega)\mathbb{1}\{t=0\} + \sum_{i=0}^{n-1}\xi_i(\omega)\mathbb{1}\{t\in[t_i,t_{i+1}]\} \tag{1}$$

Definition 2.6 - *Adapted Stochastic Process*
Let $S:[0,T]\times\Omega\to\mathbb{R}$ be a *Stochastic Process* and $\{\mathcal{F}_t\}_{t\in[0,T]}$ be a *Filtration.*

S is *Adapted* to *Filtration* $\{\mathcal{F}_t\}_{t\in[0,T]}$ if the *Random Variable* $S(t)$ is *Measurable* wrt σ-algebra $\mathcal{F}_t$, for all $t\in[0,T]$.[5]

Definition 2.7 - *Natural Filtration*
Let $S:[0,T]\times\Omega\to\mathbb{R}$ be a *Stochastic Process.*

We generate the *Natural Filtration* $\{\mathcal{F}_t\}_{t\in[0,T]}$ for S by doing the following for each $t=0,1,\dots,T$

i). Define $\mathcal{P}_t$ to be a partition of the *Sample Space* Ω st $S(t)(\cdot)$ takes the same value for each element in each subset of $\mathcal{P}_t$.

$$\mathcal{P}_t := \left\{A_1,\dots,A_m : S(t)(a) = S(t)(a')\ \forall\ a,a'\in A\right\} \text{ and } A_1,\dots,A_m \text{ form a partition.}$$

ii). Define $\mathcal{F}_t$ to be the σ-*Algebra* generated by[6] partition $\mathcal{P}_t$.

Definition 2.8 - *Bernoulli Process*
A stochastic process $\{X_t\}_{t\in\mathbb{N}}$ is a *Bernoulli Process*, with parameter p, if $X_1,X_2,\dots$ are independent RVS taking only values $\{0,1\}$ and with $\mathbb{P}(X_t=1)=p\ \forall\ t$.

Definition 2.9 - *Random Walk Process*
A stochastic process $\{N_t\}_{t\in\mathbb{N}}$ is a *Random Walk Process* when $N_t := X_1+\dots+X_t$ for some *Bernoulli Process* $\{X_t\}_{t\in\mathbb{N}}$.

Theorem 2.1 - *Binomial Distribution and Standard Normal*
Let $Y_n\sim\text{Binomial}(n,\pi_n)$. Then

$$\tilde{Y}_n := \frac{Y_n - n\pi_n}{\sqrt{n\pi_n(1-\pi_n)}}$$

converges in distribution to the standard normal distribution as $n\to\infty$.

2.3 Information Structures

Definition 2.10 - *Partition* $\mathcal{P}$
Let $\mathcal{P} := \{A_1,\dots,A_N\}$ be a set (of sets) and Ω be a *Sample Space.*

$\mathcal{P}$ is a *Partition* of Ω if it has the following properties

[4] $\mathbb{E}[|\xi_k|]<\infty$

[5] It is often easier to define a stochastic process first and then find a filtration for it (e.g. the *Natural Filtration*).

[6] See `Proposition 2.3`.

i). All elements in $\mathcal{P}$ are mutually disjoint

$$A_i \cap A_j = \emptyset \ \forall \ A_i, A_j \in \mathcal{P}$$

ii). The union of the elements form the *Sample Space* Ω.

$$\bigcup_{i=1}^{N} A_i = \Omega$$

Remark 2.1 - *Flow of Information*
At time $t = 0$ every state $\omega \in \Omega$ is a possible outcome at time $t = T$. And, at time $t = T$ we know for certain which outcome has occurred.

At each time in-between $t \in (0, T)$ our information about the world increases[7] meaning the set of possible outcomes at time $t = T$ may decrease. Let A_t denote the possible set of outcomes given we are at time t, then

$$\begin{array}{rcl} A_0 & = & \Omega \\ A_T & = & \{\omega\} \\ A_0 & \supseteq & A_1 \supseteq \cdots \supseteq A_T \end{array}$$

Flipping a coin 3 times is a motivating example. Before we start flipping ($t = 0$) it is possible that we will flip three tails, but if the first flip ($t = 1$) is heads then this is no longer possible.

Definition 2.11 - *Information Sequence* $\{\mathcal{P}_0, \dots, \mathcal{P}_T\}$
An *Information Sequence* is a sequence of *Partitions* $\{\mathcal{P}_0, \dots, \mathcal{P}_T\}$ of the *Sample Space* Ω, which fulfil the following criteria

i). $\mathcal{P}_0 = \big\{\Omega\big\}$.

ii). For $t \in [1, T)$ each $A \in \mathcal{P}_t$ is equal to the union of a subset of elements in $\mathcal{P}_{t+1}$.

iii). $\mathcal{P}_T = \big\{\{\omega_1\}, \dots, \{\omega_N\}\big\}$.

Information Sequences show the set of possible events, at each time point t, which could still occur.[8]

Remark 2.2 - *Visualising Information Structures*
TODO

Definition 2.12 - *σ-Algebra* $\mathcal{F}$
A *σ-Algebra* $\mathcal{F}$ is a set of subsets of the *Sample Space* Ω which satisfy the following conditions

i). $\Omega \in \mathcal{F}$.

ii). $\forall \ A \in \mathcal{F}, \ A^c \in \mathcal{F}$.

iii). $\forall \ A, B \in \mathcal{F}, \ (A \cup B) \in \mathcal{F}$.

Definition 2.13 - *Filtration* $\{\mathcal{F}_0, \dots, \mathcal{F}_T\}$
A *Filtration* is a sequence of *σ-Algebras* $\{\mathcal{F}_t : t = 0, 1, \dots, T\}$ where

i). $\mathcal{F}_0 = \{\emptyset, \Omega\}$.

[7]or, at least, does not decrease.
[8]An *Information Sequence* is a sequence of *σ-Algebras*.

ii). $\forall\ n < T,\ \mathcal{F}_n \subset \mathcal{F}_{n+1}$ (Meaning each subset of $\mathcal{F}_n$ must be an element of $\mathcal{F}_{n+1}$).

iii). $\mathcal{F}_T = 2^\Omega$.[9]

Each *σ-Algebra* $\mathcal{F}_t$ represents all the information generated up to time-point t by the random stock processes[10].

Definition 2.14 - *Measurable Function*
Consider a random variable $X : \Omega \to \mathbb{R}$ and a *σ-Algebra* $\mathcal{F}$.

X is *Measurable* wrt $\mathcal{F}$ if

$$\forall\ x \in \mathbb{R}, X^{-1}(x) \subset \mathcal{F} \text{ where } X^{-1}(x) := \{\omega \in \Omega : X(\omega) = x\}$$

This can be interpreted to mean that, if we known which set of $\mathcal{F}$ ω is in, then we know the values of $X(\omega)$.

Proposition 2.2 - *Measurability and Filtrations*
Consider a *Filtration* $\{\mathcal{F}_1, \dots, \mathcal{F}_T\}$ and a random variable $X : \Omega \to \mathbb{R}$.

If X is *Measurable* wrt $\mathcal{F}_t$ then it is *Measurable* wrt $\mathcal{F}_{t+1}$ since $\mathcal{F}_t \subseteq \mathcal{F}_{t+1}$.

Proposition 2.3 - *How to generate σ-Algebras*
Let $\mathcal{P}$ be a *Partition* of the *Sample Space* Ω.

We can generate a *σ-Algebra* $\mathcal{F}$ from $\mathcal{P}$ be defining $\mathcal{F}$ to be the set of all possible unions from elements in $\mathcal{P}$ as well as the compliments of all these unions.

2.4 Conditional Expectation

Definition 2.15 - *Conditional Expectation* $\mathbb{E}[\cdot|\cdot]$
Let Ω be a finite sample space, X be a discrete random variable and $A \subseteq \Omega$.

The *Conditional Expectation* of X given A has occurred is defined as

$$\mathbb{E}[X|A] = \sum_x x\mathbb{P}(X = x|A)$$

Remark 2.3 - *Alternative Definitions of Conditional Expectation*
Here are two restatements of the definition of *Conditional Expectation*, both are consequences of *Bayes Rule.*

$$\begin{aligned}\mathbb{E}[X|A] &= \sum_x \frac{\mathbb{P}(X = x, A)}{\mathbb{P}(A)} \\ \mathbb{E}[X|A] &= \sum_{\omega \in A} X(\omega)\frac{\mathbb{P}(\omega)}{\mathbb{P}(A)}\end{aligned}$$

Definition 2.16 - *Conditional Expectation \w σ-Algebra* $\mathbb{E}[\cdot|\mathcal{F}]$
Let $\mathcal{F}$ be a *σ-algebra*, $\mathcal{P}$ be the corresponding *Partition* of the sample space Ω and X be a discrete random variable.

[9]The set of all subsets of the sample space Ω.
[10]So we know how the stock has developed up to time t.

The *Conditional Expectation* of X given $\mathcal{F}$ is defined as

$$\mathbb{E}[X|\mathcal{F}] := \sum_{A\in\mathcal{P}} \mathbb{E}[X|A]\mathbb{1}\{A\}^{[11]}$$

[12] Note - This is a random variable as its value depends on which random event A occurs. Moreover, it is *Measurable* wrt $\mathcal{F}$ and for a given $A \in \mathcal{P}$

$$\forall\ \omega \in A,\ \mathbb{E}[X|\mathcal{F}](\omega) = \mathbb{E}[X|A]^{[13]}$$

Theorem 2.2 - *Tower Law*
Let X be a discrete random variable and $\mathcal{F}_1, \mathcal{F}_2$ be σ-*Algebras* with $\mathcal{F}_1 \subset \mathcal{F}_2$.

The *Tower Law* states that

$$\mathbb{E}\big[\mathbb{E}[X|\mathcal{F}]\big] = \mathbb{E}[X]$$

The *Generalised Tower Law* states that

$$\mathbb{E}\big[\mathbb{E}[X|\mathcal{F}_2]\big|\mathcal{F}_1\big] = \mathbb{E}[X|\mathcal{F}_1] = \mathbb{E}\big[\mathbb{E}[X|\mathcal{F}_1]\big|\mathcal{F}_2\big]$$

Proof 2.1 - `Theorem 2.2` - *Tower Law*
Let X be a discrete random variable, $\mathcal{F}$ be a σ-*Algebra* and $\mathcal{P}$ be the partition of the sample space Ω associated with $\mathcal{F}$.

$$\begin{array}{rcl}
\mathbb{E}\big[\mathbb{E}[X|\mathcal{F}]\big] & = & \mathbb{E}\left[\sum_{A\in\mathcal{P}} \mathbb{E}[X|A]\mathbb{1}\{A\}\right] \text{ by def.} \\
& = & \sum_{A\in\mathcal{P}} \mathbb{E}\big[\mathbb{E}[X|A]\mathbb{1}\{A\}\big] \text{ by linearity of expectation} \\
& = & \sum_{A\in\mathcal{P}} \mathbb{E}[X|A]\mathbb{E}[\mathbb{1}\{A\}] \\
& = & \sum_{A\in\mathcal{P}} \mathbb{P}(A)\cdot\left(\sum_{\omega\in A}\frac{X(\omega)\mathbb{P}(\omega)}{\mathbb{P}(A)}\right) \text{ by alt def.} \\
& = & \sum_{A\in\mathcal{P}}\sum_{\omega\in A} X(\omega)\mathbb{P}(\omega) \text{ as } \sum \mathbb{P}(A) = 1 \\
& = & \mathbb{E}[X] \text{ by def.}
\end{array}$$

$\square$

Proof 2.2 - `Theorem 2.2` - *Generalised Tower Law*
Let X be a discrete random variable, $\mathcal{F}_1, \mathcal{F}_2$ be σ-*Algebras* with $\mathcal{F}_1 \subset \mathcal{F}_2$ and $\mathcal{P}_1, \mathcal{P}_2$ be the

[12] This is not really a summation as there is only one event A st $\mathbb{1}\{A\} = 1$.
[13] This is intuitive from the definition of $\mathbb{E}[\cdot|\mathcal{F}]$.

partitions associated to $\mathcal{F}_1, \mathcal{F}_2$.

$$\begin{array}{rcl}\mathbb{E}\big[\mathbb{E}[X|\mathcal{F}_2]\big|\mathcal{F}_1\big] &=& \mathbb{E}\left[\sum_{B\in\mathcal{P}_2}\mathbb{E}[X|B]\mathbb{1}\{B\}\middle|\mathcal{F}_1\right]\\ &=& \sum_{B\in\mathcal{P}_2}\mathbb{E}[X|B]\mathbb{E}[\mathbb{1}\{B\}|\mathcal{F}_1]\\ &=& \sum_{B\in\mathcal{P}_\in}\mathbb{E}[X|B]\left(\sum_{A\in\mathcal{P}_1}\mathbb{E}[\mathbb{1}\{B\}|A]\mathbb{1}\{A\}\right)\\ &=& \sum_{A\in\mathcal{P}_1}\sum_{B\in\mathcal{P}_\in}\mathbb{E}[X|B]\mathbb{E}[\mathbb{1}\{B\}|A]\mathbb{1}\{A\}\\ &=& \sum_{A\in\mathcal{P}_1}\sum_{B\in\mathcal{P}_\in}\mathbb{E}[X|B]\cdot\frac{\mathbb{P}(A\cap B)}{\mathbb{P}(A)}\cdot\mathbb{1}\{A\}\end{array}$$

Since $\mathcal{P}_2$ is more refined than $\mathcal{P}_1$, either $B\subset A$ or $B\cap A=\emptyset$. Thus

$$\begin{array}{rcl}\mathbb{E}\big[\mathbb{E}[X|\mathcal{F}_2]\big|\mathcal{F}_1\big] &=& \sum_{A\in\mathcal{P}_1}\sum_{B\in\mathcal{P}_\in,B\subset A}\mathbb{E}[X|B]\cdot\frac{\mathbb{P}(B)}{\mathbb{P}(A)}\cdot\mathbb{1}\{A\}\\ &=& \sum_{A\in\mathcal{P}_1}\sum_{B\in\mathcal{P}_\in,B\subset A}\left(\sum_{\omega\in B}X(\omega)\frac{\mathbb{P}(\omega)}{\mathbb{P}(B)}\right)\cdot\frac{\mathbb{P}(B)}{\mathbb{P}(A)}\cdot\mathbb{1}\{A\}\\ &=& \sum_{A\in\mathcal{P}_1}\sum_{\omega\in B}X(\omega)\frac{\mathbb{P}(\omega)}{\mathbb{P}(A)}\cdot\mathbb{1}\{A\}\\ &=& \sum_{A\in\mathcal{P}_1}\mathbb{E}[X|A]\mathbb{1}\{A\}\\ &=& \mathbb{E}[X|\mathcal{F}_1]\end{array}$$

□

Theorem 2.3 - *Conditional Expectation & Measurable Random Variables*
Let $\mathcal{F}$ be a σ-*Algebra* and X, Y be discrete random variables with X being *Measurable* wrt $\mathcal{F}$. Then

$$\begin{array}{rcl}\mathbb{E}[X|\mathcal{F}] &=& X\\ \mathbb{E}[XY|\mathcal{F}] &=& X\mathbb{E}[Y|\mathcal{F}]\end{array}$$

Proof 2.3 - *Theorem 2.3*
Let $\mathcal{F}$ be a σ-*Algebra*, $\mathcal{P}$ be the partition associated with $\mathcal{F}$ and X, Y be discrete random variables with Y being *Measurable* wrt $\mathcal{F}$.

Since Y is *Measurable* it is constant on sets of $\mathcal{P}$ we write X as

$$Y=\sum_{A\in\mathcal{P}}Y_A\mathbb{1}\{A\}\text{ with }Y_A\in\mathbb{R}$$

Thus

$$\begin{aligned}
\mathbb{E}[XY|\mathcal{F}] &= \sum_{A\in\mathcal{P}} \mathbb{E}[XY|A]\mathbb{1}\{A\} \\
&= \sum_{A\in\mathcal{P}} \mathbb{E}[XY_A|A]\mathbb{1}\{A\} \\
&= \sum_{A\in\mathcal{P}} Y_A\mathbb{E}[X|A]\mathbb{1}\{A\} \text{ as } Y_A \text{ is a scalar} \\
&= \sum_{A\in\mathcal{P}} Y\mathbb{E}[X|A]\mathbb{1}\{A\}^{[14]} \\
&= Y\sum_{A\in\mathcal{P}} \mathbb{E}[X|A]\mathbb{1}\{A\} \\
&= Y\mathbb{E}[X|\mathcal{F}]
\end{aligned}$$

□

Theorem 2.4 - *Conditional Expectation & Independent Random Variables*
Let $\mathcal{F}$ be a σ-*Algebra* and X be a discrete random variable which is independent of $\mathcal{F}$.

Then

$$\mathbb{E}[X|\mathcal{F}] = \mathbb{E}[X]$$

Proof 2.4 - *Theorem 2.4*
Let $\mathcal{F}$ be a σ-*Algebra* and X be a discrete random variable which is independent of $\mathcal{F}$.

$$\begin{aligned}
\mathbb{E}[X|A] &= \textstyle\sum_x \mathbb{P}(X=x|A) \\
&= \textstyle\sum_x \mathbb{P}(X=x) \text{ by independence} \\
&= \mathbb{E}[X]
\end{aligned}$$

□

Theorem 2.5 - *General Conditional Expectation*
Let $\mathcal{F}$ be a σ-*Algebra* of a general sample space[15] Ω and X be a discrete random variable.

Then, the *Conditional Expectation* $\mathbb{E}[X|\mathcal{F}]$ is a unique random variable with the following properties

i). $\mathbb{E}[X|\mathcal{F}]$ is *Measurable* wrt $\mathcal{F}$.

ii). $\forall\ A\in\mathcal{F},\ \mathbb{E}\big[\mathbb{E}[X|\mathcal{F}]\mathbb{1}\{A\}\big] = \mathbb{E}[X\mathbb{1}\{A\}]$

Proof 2.5 - *Theorem 2.5*
Let $\mathcal{F}$ be a σ-*Algebra* of a general sample space Ω, $\mathcal{P}$ be the partition associated with $\mathcal{F}$ and X be a discrete random variable.

i). Let Y be a random variable which is *Measurable* wrt $\mathcal{F}$ and satisfies

$$\mathbb{E}[Y\mathbb{1}\{A\}] = \mathbb{E}[X\mathbb{1}\{A\}]\ \forall\ A\in\mathcal{F}$$

[14]As there is only one event A where $\mathbb{1}\{A\}=1$.
[15]i.e. Not necessarily finite

Consider the expression $\mathbb{E}[X\mathbb{1}\{A\}]$

$$\begin{aligned}\mathbb{E}[X\mathbb{1}\{A\}] &= \sum_{\omega\in A} X(\omega)\mathbb{P}(\omega)\\ &= \frac{\mathbb{P}(A)}{\mathbb{P}(A)}\sum_{\omega\in A} X(\omega)\mathbb{P}(\omega)\\ &= \mathbb{P}(A)\sum_{\omega\in A}\frac{X(\omega)\mathbb{P}(\omega)}{\mathbb{P}(A)}\\ &= \mathbb{P}(A)\mathbb{E}[X|A]\end{aligned}$$

Now, Note that $Y = \sum_{A\in\mathcal{P}} Y_A\mathbb{1}\{A\}$ (As in `Proof 2.3`).

It follows that

$$\forall\ A\in\mathcal{P},\ \mathbb{E}[Y\mathbb{1}\{A\}] = Y_A\mathbb{E}[\mathbb{1}\{A\}] = Y_A\mathbb{P}(A)$$

We now have that

$$\begin{aligned} & \mathbb{E}[X\mathbb{1}\{A\}] &= \mathbb{E}[Y\mathbb{1}\{A\}] \text{ by def. } Y\\ \implies & \mathbb{P}(A)\mathbb{E}[X|A] &= Y_A\mathbb{P}(A)\\ \implies & Y_A &= \mathbb{E}[X|A]\ \forall\ A\in\mathcal{P}\\ \implies & Y &= \mathbb{E}[X|\mathcal{F}]\end{aligned}$$

As we defined Y to be *Measurable* wrt $\mathcal{F}$, this means $\mathbb{E}[X|\mathcal{F}]$ is *Measurable* wrt $\mathcal{F}$.

ii). For any event $A\in\mathcal{F}$, the indicator function $\mathbb{1}\{A\}$ is $\mathcal{F}$-*Measurable*.

Thus, $\mathbb{E}[X\mathbb{1}\{A\}|\mathcal{F}] = \mathbb{1}\{A\}\cdot\mathbb{E}[X|\mathcal{F}]$ by `Theorem 2.3`.

Hence, by the Tower Law (`Theorem 2.2`).

$$\mathbb{E}\big[\mathbb{E}[X|\mathcal{F}]\cdot\mathbb{1}\{A\}\big] = \mathbb{E}\big[\mathbb{E}[X\mathbb{1}\{A\}|\mathcal{F}]\big] = \mathbb{E}[X\mathbb{1}\{A\}]$$

□

2.5 Random Walks

Remark 2.4 - *Random Walks*
Random walks are a class of discrete-time stochastic processes.

Definition 2.17 - *Random Walk*
Let $Y_0, Y_1, \dots$ be IID random variables with finite variance σ^2 and finite mean μ.

A *Random Walk* is the sequence $\{X_t\}_{t\geq 0}$ where $X_t := \sum_{i=1}^t Y_i$.

Definition 2.18 - *Simple Random Walk*
Let $\{X_t\}_{t\geq 0}$ be a *Random Walk* with $X_t := \sum_{i=1}^t Y_i$ where $Y_0, Y_1, \dots$ are IID RVs.

We say that $\{X_t\}_{t\geq 0}$ is a *Simple Random Walk* if

$$Y_t\in\{-1,1\}\quad \mathbb{P}(Y_t=1)=p\quad \mathbb{P}(Y_t=-1)=1-p$$

A *Simple Random Walk* can be thought of as a process where you only ever step forward or step backwards, with fixed probabilities.

Theorem 2.6 - *Distribution of a Simple Random Walk*
Let $\{X_t\}_{t\geq 0}$ be a *Simple Random Walk*. Then

$$\mathbb{P}(X_t = x) = \binom{t}{\frac{t+x}{2}} p^{(t+x)/2}(1-p)^{(t-x)/2} \quad \forall\ t \geq 0,\ x \in \{-t, -t+2, \dots, t\}$$

Note that the set of possible x values steps by 2.

Proof 2.6 - `Theorem 2.6`
Note that $x = \frac{1}{2}(2x + t - t) = (+1)\cdot\frac{1}{2}(t+x) + (-1)\cdot\frac{1}{2}(t-x)$.
For $X_t = x$ we require exact $\frac{1}{2}(t+x)$ of $Y_1, \dots, Y_t$ to take value 1, and then the remaining $\frac{1}{2}(t-x)$ will take value -1. There are $\binom{t}{\frac{t+x}{2}}$ different ways this can occur.
Note that each Y_i takes its value independently and takes value 1 with probability p and -1 with probability $1-p$. Thus

$$\mathbb{P}(X_t = x) = \binom{t}{\frac{t+x}{2}} p^{(t+x)/2}(1-p)^{(t-x)/2} \quad \forall\ t \geq 0,\ x \in \{-t, -t+2, \dots, t\}$$

2.6 Martingales

Definition 2.19 - *Discrete-Time Martingale* $\{Z_t\}_{t\in\mathbb{N}_0}$
Let $\{Z_t\}_{t\in[0,T]}$ be an *Adapted Stochastic Process* on a *Sample Space* Ω wrt a *Filtration* $\{\mathcal{F}_t\}_{t\in[0,T]}$.

- $\{Z_t\}_{t\in[0,T]}$ is a *Martingale* if

$$\forall\ t \geq 1,\ \mathbb{E}[Z_t|\mathcal{F}_{t-1}] = Z_{t-1}$$

  This can be interpreted to mean that, given all available information $\mathcal{F}_{t-1}$, our present state Z_{t-1} is the best indicator of the future state Z_t.

- $\{Z_t\}_{t\in[0,T]}$ is a Super-*Martingale* if

$$\forall\ t \geq 1,\ \mathbb{E}[Z_t|\mathcal{F}_{t-1}] \leq Z_{t-1}$$

  This can be interpreted to mean that, given all available information $\mathcal{F}_{t-1}$, our present state Z_{t-1} provides an upper-bound on the future state Z_t.

- $\{Z_t\}_{t\in[0,T]}$ is a Sub-*Martingale* if

$$\forall\ t \geq 1,\ \mathbb{E}[Z_t|\mathcal{F}_{t-1}] \geq Z_{t-1}$$

  This can be interpreted to mean that, given all available information $\mathcal{F}_{t-1}$, our present state Z_{t-1} provides a lower-bound on the future state Z_t.

Definition 2.20 - *Continuous Time Martingale*
Let $\{Z_t\}_{t\in\mathbb{R}^{\geq 0}}$ be an *Adapted Stochastic Process* on a *Sample Space* Ω wrt a *Filtration* $\{\mathcal{F}_t\}_{t\in\mathbb{R}^{\geq 0}}$.

- $\{Z_t\}_{t\in\mathbb{R}^{\geq 0}}$ is a *Martingale* if

$$\forall\ t \geq s \geq 0,\ \mathbb{E}\left[|X_t|\right] < \infty \text{ and } \mathbb{E}\left[X_t|\mathcal{F}_s\right] = X_s$$

- $\{Z_t\}_{t\in\mathbb{R}^{\geq 0}}$ is a Super-*Martingale* if
$$\forall\ t \geq s \geq 0,\ \mathbb{E}\left[|X_t|\right] < \infty \text{ and } \mathbb{E}\left[X_t|\mathcal{F}_s\right] \leq X_s$$

- $\{Z_t\}_{t\in\mathbb{R}^{\geq 0}}$ is a Sub-*Martingale* if
$$\forall\ t \geq s \geq 0,\ \mathbb{E}\left[|X_t|\right] < \infty \text{ and } \mathbb{E}\left[X_t|\mathcal{F}_s\right] \geq X_s$$

Proposition 2.4 - *Notable Martingales*
Let $\{X_t\}_{t\in\mathbb{N}_0}$ be a *Simple Random Walk* with parameter p and let $\mathcal{F}_t$ be the σ-*Algebra* generated by X_t. Then

i). If $p = 1/2$ $\{X_t\}_{t\in\mathbb{N}_0}$ is a *Martingale.*

ii). If $p \leq 1/2$ $\{X_t\}_{t\in\mathbb{N}_0}$ is a *Super-Martingale.*

iii). If $p \geq 1/2$ $\{X_t\}_{t\in\mathbb{N}_0}$ is a *Sub-Martingale.*

iv). If $p = 1/2$ then $\{Z_t\}_{t\in\mathbb{N}_0}$ where $Z_t := (X_t^2 - t)$ is a *Martingale.*

v). If $p \neq 1/2$ then $\{L_t\}_{t\in\mathbb{N}_0}$ where $L_0 := 1, L_t := \left(\frac{1-p}{p}\right)^{X_t}$ is a *Martingale Martingale.*

vi). If $p \neq 1/2$ then $\{M_t\}_{t\in\mathbb{N}_0}$ where $M_t := \big(X_t - t(2p-1)\big)$ is a *Martingale. Martingale.*

Proof 2.7 - `Proposition 2.4` *i)-iii)*
Let $\{X_t\}_{t\in\mathbb{N}_0}$ be a *Simple Random Walk* with parameter p and let $\mathcal{F}_t$ be the σ-*Algebra* generated by X_t.

Since $\{\mathcal{F}_t\}_{t\in\mathbb{N}}$ is the *Natural Filtration* of $\{X_t\}_{t\in\mathbb{N}}$, then $\{X_t\}_{t\in\mathbb{N}}$ is *Measurable* wrt $\mathcal{F}_t$ and Y_t[16] is independent of $\mathcal{F}_{t-1}$.

Then
$$\begin{array}{rcl}\mathbb{E}[X_t|\mathcal{F}_{t-1}] & = & \mathbb{E}[X_{t-1}+Y_t|\mathcal{F}_{t-1}] \text{ by def. } X_t \\ & = & \mathbb{E}[X_{t-1}|\mathcal{F}_{t-1}] + \mathbb{E}[Y_t|\mathcal{F}_{t-1}] \text{ by linearity of exp.} \\ & = & X_{t-1} + \mathbb{E}[Y_t] \text{ by } \texttt{Theorem 2.4}\end{array}$$

Thus

- If $p = 1/2$ then $\mathbb{E}[Y_t] = 0 \implies \mathbb{E}[X_t|\mathcal{F}_{t-1}] = X_{t-1}$. This is the definition of a *Martingale.*
- If $p \leq 1/2$ then $\mathbb{E}[Y_t] \leq 0 \implies \mathbb{E}[X_t|\mathcal{F}_{t-1}] \leq X_{t-1}$. This is the definition of a *Super-Martingale.*
- If $p \geq 1/2$ then $\mathbb{E}[Y_t] \geq 0 \implies \mathbb{E}[X_t|\mathcal{F}_{t-1}] \geq X_{t-1}$. This is the definition of a *Sub-Martingale.*

□

Proof 2.8 - `Proposition 2.4` *iv)*
Let $\{X_t\}_{t\in\mathbb{N}_0}$ be a *Simple Random Walk* with parameter p and let $\mathcal{F}_t$ be the σ-*Algebra* generated by X_t.

[16]The t^{th} step of the random walk.

As the definition of a *Martingale* depends on the conditional expectation of Z_t given $\mathcal{F}_{t-1}$ we consider its value

$$\begin{array}{rcl}\mathbb{E}[Z_t|\mathcal{F}_{t-1}] & = & \mathbb{E}[X_t^2 - t|\mathcal{F}_{t-1}] \text{ by def. } Z_t \\ & = & \mathbb{E}[(X_{t-1}+Y_t)^2 - t|\mathcal{F}_{t-1}] \text{ by def. } X_t \\ & = & \mathbb{E}[(X_{t-1}+Y_t)^2|\mathcal{F}_{t-1}] - t \\ & = & \mathbb{E}[X_{t-1}^2 + 2X_{t-1}Y_t + Y_t^2|\mathcal{F}_{t-1}] - t \\ & = & \mathbb{E}[X_{t-1}^2|\mathcal{F}_{t-1}] + 2\mathbb{E}[X_tY_t|\mathcal{F}_{t-1}] + \mathbb{E}[Y_t^2|\mathcal{F}_{t-1}] - t \\ & = & X_{t-1}^2 + 2X_{t-1}\mathbb{E}[Y_t|\mathcal{F}_{t-1}] + \mathbb{E}[Y_t^2] - t\end{array}$$

Since $p = 1/2 \implies \mathbb{E}[Y_t] = 0, \mathbb{E}[Y_t^2 = 1]$.

$$\begin{array}{rcl}\mathbb{E}[Z_t|\mathcal{F}_{t-1}] & = & X_{t-1}^2 + 2X_{t-1}\mathbb{E}[Y_t|\mathcal{F}_{t-1}] + \mathbb{E}[Y_t^2] - t \\ & = & X_{t-1}^2 + 0 + 1 - t \\ & = & X_{t-1}^2 - (t-1) \\ & = & Z_{t-1} \\ \implies \mathbb{E}[Z_t|\mathcal{F}_{t-1}] & = & Z_{t-1}\end{array}$$

This is the definition of a *Martingale*. □

Proof 2.9 - *`Proposition 2.4` v)*
TODO (Homework)

Proof 2.10 - *`Proposition 2.4` vi)*
TODO (Homework)

Theorem 2.7 - *Adapted Stochastic Processes as Martingales*
Let $\{Z_t\}_{t\in[0,T]}$ be an *Adapted Stochastic Process*.

$\{Z_t\}_{t\in[0,T]}$ is a *Martingale* <u>iff</u>

$$\forall\ t \geq s,\ \mathbb{E}[Z_t|\mathcal{F}_s] = Z_s$$ [17]

Proof 2.11 - *`Theorem 2.7`*
Let $\{Z_t\}_{t\in[0,T]}$ be an *Adapted Stochastic Process*.

I prove this statement in both directions[18]

$\implies$ Suppose $\{Z_t\}_{t\in[0,T]}$ is a *Martingale*.

Using `Theorem 2.5` we can deduce that

$$\begin{array}{rcll}\mathbb{E}[Z_t|\mathcal{F}_s] & = & \mathbb{E}\big[\mathbb{E}[Z_t|\mathcal{F}_{t-1}]\big|\mathcal{F}_s\big] & \text{by Theorem 2.5} \\ & = & \mathbb{E}[Z_{t-1}|\mathcal{F}_s] & \text{as } Z \text{ is a Martingale} \\ & = & \mathbb{E}[Z_s|\mathcal{F}_s] & \text{by recursion} \\ & = & \mathbb{E}[Z_s] & \text{by Theorem 2.3} \\ & = & Z_s & \end{array}$$

$\impliedby$ Suppose it holds that

$$\forall\ t \geq s,\ \mathbb{E}[Z_t|\mathcal{F}_s] = Z_s$$

[17] Equivalent results can be made for *Super-* and *Sup-Martingales* by replacing $=$ with $\leq, \geq$ respectively.
[18] The proofs for *Super-* and *Sup-Martingales* are very similar.

Consider the case where $s = t - 1$, it holds that

$$\mathbb{E}[Z_t|\mathcal{F}_{t-1}] = Z_{t-1}$$

This is the definition of a *Martingale*.

$\square$

Definition 2.21 - *Stopping Times τ*
Let $\{\mathcal{F}_t\}_{t\in\mathbb{N}_0}$ be a *Filtration* of *Sample Space* Ω and τ be a random variable which takes values in $(\mathbb{R} \geq 0 \cup \{\infty\})$[19].

τ is a *Stopping Time* if the event $\{\tau < t\}$ is an element of the σ*-Algebra* $\mathcal{F}_t$.

Stopping Times are used to determine whether an event has occurred, or not.[20]

Definition 2.22 - *Bounded Stopping Time τ*
Let τ be a *Stopping Time*.

A τ is a *Bounded Stopping Time* if

$$\exists\ t \in \mathbb{R}^{\geq 0},\ \mathbb{P}(\tau < t) = 1$$

Theorem 2.8 - *Stopping Times & σ Algebras*
Let τ be a random variable.

τ is a *Stopping Time* <u>iff</u> $\forall\ t \in \mathbb{N}_0$ the event $\{\tau = t\}$ is an element of the σ*-Algebra* $\mathcal{F}_t$.

Proof 2.12 - `Theorem 2.8`
Let τ be a random variable.

I prove this statement in both directions

$\implies$ Suppose τ is a *Stopping Time*.

Then the event $\{\tau \leq t\}$ is an element of the σ*-Algebra* $\mathcal{F}_t$ $\forall\ t \in \mathbb{N}_0$.

We can restate this event as

$$\{\tau \leq t\} = \bigcup_{k\leq t}\{\tau = k\}$$

As $\{\tau \leq t\} \in \mathcal{F}_t$, then each of $\{\tau = k\} \in \mathcal{F}_t$ due to the definition of a σ*-Algebra*.

$\impliedby$ Suppose the event $\{\tau = t\}$ is an element of the σ*-Algebra* $\mathcal{F}_t$ $\forall\ t \in \mathbb{N}_0$.

We can restate this event as

$$\{\tau \leq t\} = \big(\{\tau \leq t\} \setminus \{\tau \leq t - 1\}\big)$$

Since $\{\tau \leq t\}, \{\tau \leq t - 1\}$ are elements of $\mathcal{F}_t$, then $\{\tau \leq t\} \in \mathcal{F}_t$ due to the definition of a σ*-Algebra*.

$\square$

[19] ∞ is used for impossible events.

[20] Examples of *Stopping Times* are "RBS shares hit £1".

Theorem 2.9 - *Stopping Time for an Adapted Stochastic Process*
Let $\{X_t\}_{t\in\mathbb{N}_0}$ be an *Adapted Stochastic Process* and $c \in \mathbb{R}$.
The event $\tau_c := \inf\{t \geq 0 : X_t \geq c\}$[21] is a *Stopping Time.*

Proof 2.13 - *Theorem 2.9*
Let $\{X_t\}_{t\in\mathbb{N}_0}$ be an *Adapted Stochastic Process* and $c \in \mathbb{R}$.
Note that $\tau \leq t$ iff $\exists\ k \leq t$ st $X_k \geq c$ due to the definition of τ_c.
Therefore

$$\{\tau_c \leq t\} = \bigcup_{k\leq t}\{X_k \geq c\}$$

Since each $\{X_k \geq c\} \in \mathcal{F}_t$ then $\{\tau_c \leq t\} \in \mathcal{F}_t$ by the definition of σ*-Fields.*
Thus τ_c is a *Stopping Time.* □

Theorem 2.10 - *Optional Stopping Theorem*[22] *- Martingale*
Let τ be a *Bounded Stopping Time* and $\{X_t\}_{t\in\mathbb{N}_0}$ be a *Martingale.*
Then

$$\mathbb{E}[X_\tau] = \mathbb{E}[X_0] = X_0$$

Theorem 2.11 - *Optional Stopping Theorem - Super-Martingale*
Let τ be a *Bounded Stopping Time* and $\{X_t\}_{t\in\mathbb{N}_0}$ be a Super-*Martingale.*
Then

$$\mathbb{E}[X_\tau] \leq \mathbb{E}[X_0] = X_0$$

Remark 2.5 - *Weaker Conditions for Optional Stopping Theorem*
The following are weaker conditions[23] that suffice for the *Optional Stopping Theorem* to hold

i). $\mathbb{P}(\tau < \infty) = 1$ and X_τ is bounded.

ii). $\mathbb{E}[\tau] < \infty$ and $(X_t - X_{t-1})$ is bounded.

Proof 2.14 - *Theorem 2.10*
Let τ be a *Bounded Stopping Time* and $\{X_t\}_{t\in\mathbb{N}_0}$ be a *Martingale.*
Assume that $\tau \leq K$ (This is reasonable since τ is bounded). We can write

$$X_{\tau(\omega)}\omega = \sum_{t=0}^{K} X_t(\omega)\mathbb{1}\{\tau(\omega) = t\}$$

Note that this is not really a sum as there is only one event ω st $\mathbb{1}\{\tau(\omega) = t\} = 1$, the rest equal 0.

[21]The first time X_t reaches value c.
[22]AKA *Optional Sampling Theorem*
[23]Rather than τ being a bounded stopping time

Then

$$\begin{aligned}\mathbb{E}[X_\tau] &= \mathbb{E}\left[\sum_{t=0}^K X_t \mathbb{1}\{\tau=t\}\right]\\ &= \sum_{t=0}^K \mathbb{E}\left[X_t\mathbb{1}\{\tau=t\}\right] \text{ by linearity of exp.}\\ &= \sum_{t=0}^K \mathbb{E}\left[\mathbb{E}[X_K|\mathcal{F}_t]\mathbb{1}\{\tau=t\}\right] \text{ by } \texttt{Theorem 2.7}\end{aligned}$$

Since τ is a *Stopping Time* then $\{\tau=t\}$ is *Measurable* wrt $\mathcal{F}_t$.

Thus, by `Theorem 2.3`

$$\mathbb{E}[X_K|\mathcal{F}_t]\mathbb{1}\{\tau=t\}=\mathbb{E}[X_K\mathbb{1}\{\tau=t\}|\mathcal{F}_t]$$

We continue the analysis of $\mathbb{E}[X_\tau]$

$$\begin{aligned}\mathbb{E}[X_\tau] &= \sum_{t=0}^K \mathbb{E}\left[\mathbb{E}[X_K|\mathcal{F}_t]\mathbb{1}\{\tau=t\}\right]\\ &= \sum_{t=0}^K \mathbb{E}\left[\mathbb{E}[X_K\cdot\mathbb{1}\{\tau=t\}|\mathcal{F}_t]\right]\\ &= \sum_{t=0}^K \mathbb{E}[X_K\cdot\mathbb{1}\{\tau=t\}] \text{ by Tower Law}\\ &= \mathbb{E}\left[X_K\sum_{t=0}^K\mathbb{1}\{\tau=t\}\right]\\ &= \mathbb{E}\left[X_K\cdot 1\right]\\ &= \mathbb{E}\left[X_K\right]\\ &= \mathbb{E}[X_0] \text{ as } \{X_t\}_{t\in\mathbb{N}_0} \text{ is a Martingale}\\ &= X_0 \text{ as its value is known}\end{aligned}$$

Definition 2.23 - *Gambler's Ruin Problem*
The *Gambler's Ruin Problem* involves considering a gambler with an initial wealth of £C. The gambler is allowed to play a game until either they become bankrupt (i.e. have £0) or reach a target of £$(C+G)$ where $G>0$.

The simplest specification of the game is flipping a coin[24] and the gambler receives £1 if it lands heads, or loses £1 if it lands tails.

Proposition 2.5 - *Stopping Time in Gambler's Ruin Problem*
Consider the *Gambler's Ruin Problem* using the simple game described in `Definition 2.23`.

Let $\{X_t\}_{t\in\mathbb{N}_0}$ be a *Simple Random Walk* with parameter p[25] and $X_0=0$ and $C,G>0$.

Consider the *Stopping Time* $\tau:=\inf\{t:X_t=G \text{ or } X_t=-C\}$, the event the gambler stops playing[26]. Then

[24]potentially fair, potentially not.

[25]$\{X_t\}_{t\in\mathbb{N}_0}$ can be consider to model the net winnings of the gambler and p is the probability of the coin landing heads (i.e. the gambler wins money).

[26]Either due to reaching goal or going bankrupt.

- If $p = 1/2$ then

$$\begin{array}{rcl}\mathbb{P}(X_\tau = G)^{[27]} & = & \frac{C}{C+G} \\ \mathbb{P}(X_\tau = -C)^{[28]} & = & \frac{G}{C+G} \\ \mathbb{E}[\tau] & = & CG\end{array}$$

- If $p \neq 1/2$ then

$$\begin{array}{rcl}\mathbb{P}(X_\tau = G) & = & \dfrac{1-\left(\frac{p}{1-p}\right)^C}{\left(\frac{p}{1-p}\right)^G - \left(\frac{p}{1-p}\right)^C} \\ \mathbb{P}(X_\tau = -C) & = & 1 - \mathbb{P}(X_\tau = G) = \dfrac{\left(\frac{p}{1-p}\right)^G - 1}{\left(\frac{p}{1-p}\right)^G - \left(\frac{p}{1-p}\right)^C} \\ \mathbb{E}[\tau] & = & \dfrac{G\mathbb{P}(X_\tau = G) + (-C)\mathbb{P}(X_\tau = -C)}{2p-1}\end{array}$$

Proof 2.15 - ***Proposition 2.5***
Since τ is not *Bounded*, but X_τ is *Bounded* by $G, -C$, we can use a weaker condition from `Remark 2.5` to apply the *Optional Stopping Theorem* (`Theorem 2.10`) provided we can show that $\mathbb{P}(\tau < \infty) = 1$.

Note that whenever there is a run of $k \geq C + G$ successive 1's in the process $\{Y_t\}_{t\in\mathbb{N}_0}$ which defines the random walk X, the process will stop and $\tau < \infty$. Thus, for all m, the following hold

$$\begin{array}{rcl}\mathbb{P}(\tau > km) & = & \mathbb{P}(\text{No run of } k \text{ 1's in } Y_1 \text{ to } Y_{mk}) \\ & = & \prod_{j=0}^{m-1} \mathbb{P}(\text{No run of } k \text{ 1's in } Y_{jk+1} \text{ to } Y_{(j+1)k}) \\ & = & (1-p^k)^m \\ \implies \mathbb{P}(\tau < \infty) & = & 1\end{array}$$

We can now consider the two cases for the value of p

- If $p = 1/2$. Then by the *Optional Stopping Theorem* we can deduce the following

$$\begin{array}{rrcl} & 0 & = & \mathbb{E}[X_\tau] \text{ as } p = 1/2 \\ & & = & G\mathbb{P}(X_\tau = G) + (-C)\mathbb{P}(X_\tau = -C) \\ & & = & G\mathbb{P}(X_\tau = G) + (-C)(1 - \mathbb{P}(X_\tau = G)) \\ \implies & C & = & (G+C)\mathbb{P}(X_\tau = G) \\ \implies & \mathbb{P}(X_\tau = G) & = & \frac{C}{G+C} \\ \text{and} & \mathbb{P}(X_\tau = -C) & = & 1 - \mathbb{P}(X_\tau = G) = \frac{G}{G+C}\end{array}$$

  To determine $\mathbb{E}[X_\tau]$ we apply the *Optional Stopping Theorem* to the process $\{Z_t\}_{\mathbb{N}_0}$ where $Z_t := X_t^2 - t$. It was shown in `Proposition 2.4` that $\{Z_t\}_{t\in\mathbb{N}_0}$ is a *Martingale*.

  As $\{Z_t\}_{t\in\mathbb{N}_0}$ is a *Martingale* it holds that

$$0 = \mathbb{E}[Z_0] = \mathbb{E}[Z_\tau] = \mathbb{E}[X_\tau^2 - \tau] = \mathbb{E}[X_\tau^2] - \mathbb{E}[\tau]$$

[28]Gambler reaches goal.
[28]Gambler goes bankrupt.

By rearranging we obtain that

$$\begin{array}{rcl}\mathbb{E}[\tau] & = & \mathbb{E}[\tau] \\ & = & G^2\mathbb{P}(X_\tau = G) + C^2\mathbb{P}(X_\tau = -C) \\ & = & G^2 \cdot \dfrac{C}{C+G} + C^2\dfrac{G}{C+G} \\ & = & CG\end{array}$$

- Consider the case $p \neq 1/2$ and the process $\{L_t\}_{t\in\mathbb{N}_0}$ where $L_t := \left(\frac{1-p}{p}\right)^{X_t}$. It was shown in `Proposition 2.4` that $\{L_t\}_{t\in\mathbb{N}_0}$ is a *Martingale*.

  By the *Optional Stopping Theorem*

$$\begin{array}{rcl}1 & = & \mathbb{E}[L_0] \\ & = & \left(\dfrac{1-p}{p}\right)^G \mathbb{P}(X_\tau = G) + \left(\dfrac{1-p}{p}\right)^C \mathbb{P}(X_\tau = -C)\end{array}$$

  Remembering that $\mathbb{P}(X_\tau = G) + \mathbb{P}(X_\tau = -C) = 1$, we can derive the probabilities of each end event occurring

$$\begin{array}{rcl}1 & = & \left(\dfrac{1-p}{p}\right)^G \mathbb{P}(X_\tau = G) + \left(\dfrac{1-p}{p}\right)^C (1 - \mathbb{P}(X_\tau = G)) \\ & = & \left(\dfrac{1-p}{p}\right)^C + \left[\left(\dfrac{1-p}{p}\right)^G - \left(\dfrac{1-p}{p}\right)^C\right]\mathbb{P}(X_\tau = G) \\ \implies \mathbb{P}(X_\tau = G) & = & \dfrac{1-\left(\frac{1-p}{p}\right)^C}{\left(\frac{1-p}{p}\right)^G - \left(\frac{1-p}{p}\right)^C}\end{array}$$

  Consider the process $\{M_t\}_{t\in\mathbb{N}_0}$ where $M_t := X_t - t(2p-1)$. It was shown in `Proposition 2.4` that $\{M_t\}_{t\in\mathbb{N}_0}$ is a *Martingale*.

  We determine $\mathbb{E}[X_\tau]$ by applying the *Optional Stopping Theorem* to $\{M_t\}_{t\in\mathbb{N}_0}$.

$$\begin{array}{rcl}0 & = & \mathbb{E}[M_\tau] \\ & = & G\mathbb{P}(X_\tau = G) + (-C)\mathbb{P}(X_\tau = -C) - \mathbb{E}[\tau](2p-1)\end{array}$$

  By rearranging we obtain that

$$\begin{array}{rrcl} & 0 & = & G\mathbb{P}(X_\tau = G) + (-C)\mathbb{P}(X_\tau = -C) - \mathbb{E}[\tau](2p-1) \\ \implies & \mathbb{E}[\tau](2p-1) & = & G\mathbb{P}(X_\tau = G) + (-C)\mathbb{P}(X_\tau = -C) \\ \implies & \mathbb{E}[\tau] & = & \dfrac{G\mathbb{P}(X_\tau = G) + (-C)\mathbb{P}(X_\tau = -C)}{2p-1}\end{array}$$

$\square$

2.7 Brownian Motion

Definition 2.24 - *Continuous Random Walk* $\{S_t^n\}$

Let $\{S_t\}_{t\in\mathbb{N}_0}$ be the discrete-time random walk where $S_t := \sum_{i=1}^t Y_i$ where $Y_i \sim \text{Normal}(0,1)$.

Define $\{S_t^n\}_{t\in[0,1]}$ to be the continuous time-random walk defined by (2) and using linear interpolation.

$$S_t^n := \frac{1}{\sqrt{n}} S_{(t\cdot n)} = \frac{1}{\sqrt{n}}\sum_{i=1}^{(t\cdot n)} Y_i \text{ with } Y_i \overset{iid}{\sim} N(0,1) \tag{2}$$

Theorem 2.12 - *Properties of* $\{S_t^n\}$
Here are some propoerties of $\{S_t^n\}_{t\in[0,1]}$ defined in **Definition 2.24**.

i). $S_0^n = S_0/\sqrt{n} = 0$.

ii). Taking $t = j/n$ and $u = k/n$ gives

$$S_{t+u}^n - S_t^n = \frac{1}{\sqrt{n}}(S_{j+k} - S_j) = \frac{1}{\sqrt{n}}(Y_{j+1} + \dots + Y_{j+k})$$

As the Xs are independent, this shows that the change in value of a given period is independent of both the start and end points.

iii). Taking $t = j/n$ and $u = k/n$, for large n. Consider this expression for the change in value over a time period

$$S_{t+u}^n - S_t^n = \frac{X_{j+1} + \dots + X_{j+k}}{\sqrt{n}} = \frac{\sqrt{k}}{\sqrt{n}}\left(\sum_{i=j+1}^{j+k} \frac{X_i}{\sqrt{k}}\right)$$

By the *Central Limit Theorem* this tends to $\text{Normal}\,(0, k/n) = \text{Normal}\,(0, u)$.

iv). $S_t^n(\omega)$ is continuous as a function of t, for all n and all $\omega \in \Omega$.

v). As $n \to \infty$, S_t^n tends to a process W_t known as *Brownian Motion*.

Definition 2.25 - *Brownian Motion* $\{W_t\}$
Let $W := \{W_t\}_{t\geq 0}$ be an *Adapted Stochastic Process* wrt *Filtration* $\{\mathcal{F}_t\}_{t\geq 0}$.
W is a standard one-dimensional *Brownian Motion* if

i). $W_0 = 0$. (Almost surely)

ii). W has independent increments.

$$\forall\ u, t \geq 0,\ W_{t+u} - W_t \text{ is independent of } \mathcal{F}_t$$

iii). W has stationary Gaussian increments.

$$\forall\ u, t \geq 0,\ (W_{t+u} - W_t) \sim \text{Normal}(0, u)$$

iv). W has continuous paths. ($W_t(\omega)$ is a continuous function of t for all $\omega \in \Omega$.)

Remark 2.6 - *Differentiating Brownian Motion*
Brownian Motion is not differentiable.

Proposition 2.6 - *General Properties of Brownian Motion*
Let W be standard Brownian Motion, as defined in **Defintion 2.25**. Then

i). $\mathbb{E}[W_t] = 0,\ \text{Var}(W_t) = t\ \forall\ t$.

ii). $\forall\ s, t\ \text{Cov}(W_s, W_t) = \min\{s, t\}$.

iii). $-W_t$ is a *Standard Brownian Motion*.

iv). For fixed t, $X_s := W_{s+t} - W_t$ is a *Standard Brownian Motion.*

v). For any α, $y_s := \frac{1}{\sqrt{\alpha}} W_{\alpha s}$ is a *Standard Brownian Motion.*

Proof 2.16 - *Proposition 2.6 i)-v)*

i). Follows from properties i) & iii) of `Definition 2.25`.

ii). By the previous result we can conclude that

$$\begin{aligned}
\mathrm{Cov}(W_s, W_t) &= \mathbb{E}[W_s \cdot W_t] \\
&= \mathbb{E}[W_s \cdot (W_t - W_s)] + \mathbb{E}[W_s^2] \\
&= \mathbb{E}[W_s \cdot (W_t - W_s)] + \underbrace{\mathrm{Var}(W_s)}_{=s} \\
&= \underbrace{\mathbb{E}[W_s]}_{=0}\mathbb{E}[W_t - W_s] + s \\
&= s
\end{aligned}$$

iii). $\mathbb{E}[-W_t] = -\mathbb{E}[W_t] = 0$ and increments occur in the t-direction, not the X-direction, so the distribution of $W_{t+u} - W_t$ is unaffected.

iv). In the case $t = 0$ we find that $X_0 = W_t - W_t = 0$. The other properties in `Defintion 2.25` are shift-invariant and therefore follow as well.

v). The main part to check here is propert iii) of `Defintion 2.25`. Indeed

$$\begin{aligned}
Y_{t+s} - Y_t &= \underbrace{(W_{\alpha(t+s)} - W_{\alpha t})}_{\sim \mathrm{Nor}(0, \alpha s)} / \sqrt{\alpha} \\
&\sim \mathrm{Normal}(0, s)
\end{aligned}$$

□

Definition 2.26 - *Geometric Brownian Motion*
Let $\{W_t\}_{t \geq 0}$ be *Standard Brownian Motion.*
Geometric Brownian Motion $\{\tilde{Z}\}_{t \geq 0}$ with volatility $\sigma > 0$ and drift $a \in \mathbb{R}$ is defined as

$$\tilde{Z}_t = \exp\{\sigma W_t + at\}$$

Proposition 2.7 - *Martingales and Brownian Motion*
Let $\{W_t\}_{t \geq 0}$ be *Standard Brownian Motion* wrt *Filtration* $\{\mathcal{F}_t\}_{t \geq 0}$. Then

i). $\{W_t\}_{t \geq 0}$ is a *Martingale.*

ii). $\{W_t^2 - t\}_{t \geq 0}$ is a *Martingale.*

iii). The *Geometric Brownian Motion* with volatility $\sigma > 0$ and drift $a \in \mathbb{R}$ is a *Martingale* <u>iff</u> $a = -\frac{1}{2}\sigma^2$.

Proof 2.17 - *Proposition 2.7 i)-iii)* i). We can write $W_t = (W_t - W_s) + W_s$. As $(W_t - W_s)$ is independent of filtration $\mathcal{F}_s$ and has zero mean, we can conclude that

$$\mathbb{E}[W_t|\mathcal{F}_s] = \mathbb{E}[W_t - W_s|\mathcal{F}_s] + \mathbb{E}[W_s|\mathcal{F}_s] = \underbrace{\mathbb{E}[W_t - W_s]}_{=0} + W_s = W_s$$

ii). Similarly, we can write

$$W_t^2 = (W_t - W_s)^2 + 2W_s(W_t - W_s) + W_s^2$$

Then

$$\begin{array}{rcl}\mathbb{E}[W_t^2 - t|\mathcal{F}_s] & = & \left\{\mathbb{E}[(W_t - W_s)^2|\mathcal{F}_s] + 2W_s\mathbb{E}[W_t - W_s|\mathcal{F}_s] + W_s^2\right\} - t \\ & = & \text{Var}(W_t - W_s) + 0 + W_s^2 - t \\ & = & t - s + W_s^2 - t \\ & = & W_s^2 - s\end{array}$$

iii). We find that

$$\begin{array}{rcl}\mathbb{E}[\tilde{Z}_t|\mathcal{F}_s] & = & \mathbb{E}[\exp\{\sigma W_t + at\}|\mathcal{F}_s] \\ & = & \exp\{at\}\mathbb{E}[\exp\{\sigma(W_t - W_s + W_s)\}|\mathcal{F}_s] \\ & = & \exp\{\sigma W_s + at\} \cdot \mathbb{E}[\exp\{\sigma \overbrace{(W_t - W_s)}^{:=N}\}|\mathcal{F}_s] \\ & = & \exp\{\sigma W_s + at\}\mathbb{E}[\exp\{\sigma N\}|\mathcal{F}_s] \\ & = & \exp\{\sigma W_s + at\}\mathbb{E}[\exp\{\sigma N\}]\end{array}$$

where $N \sim \text{Normal}(0, t - s)$. The MGF of N is $\mathbb{E}[e^{\sigma N}] = e^{(t-s)\sigma^2/2}$, therefore

$$\begin{array}{rcl}\mathbb{E}[\tilde{Z}_t|\mathcal{F}_s] & = & \exp\{\sigma W_s + at\}\mathbb{E}[\exp\{\sigma N\}] \\ & = & \tilde{Z}_s e^{a(t-s)} e^{(t-s)\sigma^2/2} \\ & = & \tilde{Z}_s \exp\left\{(t - s)\left(a + \frac{\sigma^2}{2}\right)\right\}\end{array}$$

We conclude that $\tilde{Z}_t$ is a *Martingale* <u>iff</u> $a = -\sigma^2$[29].

$\square$

3 Financial Terminology

Definition 3.1 - *Underlying Asset*
The *Underlying Asset* is a real financial asset or security which a contract can be based on. (e.g. Oil, interest rate, shares).

Definition 3.2 - *Dividend*
A *Dividend* is a one-off payment provided to the holder of an *Underlying Asset* at a certain time. Whether an *Underlying Asset* pays a *Dividend*, and the value of the *Dividend*, will affect the value of the *Underlying Asset*.

A *Dividend* is generally used by companies to distribute yearly profits to its shareholders.

[29]As we require $\mathbb{E}[\tilde{Z}_t|\mathcal{F}_s] = \tilde{Z}_s$ which only occurs <u>iff</u> $a + \frac{\sigma^2}{2} = 0$.

Definition 3.3 - *Long Selling*
Long Selling is the practice of buying an asset (or security) and then selling it at some point in the future.

In *Long Selling* your profit/loss is $P_{\text{sell}} - P_{\text{buy}}$, thus you hope the price of the asset increases in the period between you buying and selling it.

Definition 3.4 - *Short Selling*
Short Selling is the practice of borrowing an asset (or security), immediately selling it[30] and at some point in the future buying an equivalent asset in order to reimburse your lender.

In *Short Selling* your profit/loss if $P_{\text{sell}} - P_{\text{buy}}$, thus you hope the price of the asset decreases in the period between you selling and having to reimburse your lender.

Remark 3.1 - *Short Selling & Dividends*
If the asset you borrowed in *Short Selling* pays a *Dividend* during the time you have borrowed the asset, then you must pay this *Dividend* to the lender.[31]

Definition 3.5 - *Arbitrage Opportunity*
An *Arbitrage Opportunity* occurs when it is possible to make a profit without being exposed to the risk of incurring a loss.[32]

Generally *Arbitrage Opportunities* occur by being able to buy and sell the same asset in different markets, as each market may have a different price.

Theorem 3.1 - *No-Arbitrage Principle*

> "*Arbitrage Opportunities* do not exist (for long) in real life markets."

As when the opportunities arise, the market activity cause by agents exploiting the opportunity would raise the cost of buying and thus remove the opportunity due to the *Law of Supply-and-Demand*.

Remark 3.2 - *Value of Money*
IRL the value of money is not constant due to inflation, interest rates & exchange rates. We generally want to normalise the returns of our portfolio wrt the change in value of money in order to determine the "real returns".

Definition 3.6 - *Portfolio*
TODO

3.1 Derivatives

Definition 3.7 - *Derivative Securities*
A *Deriviative Security* is a contract which has an expiry date T and pays out different amounts depending upon the value of some *Underlying Asset* in the time-period $[0, T]$.

[30]Receiving payment at this point.
[31]As you have already sold the asset, then this expense will come out of your own pocket.
[32]Someone who loos for *Arbitrage Opportunities* is called an *Arbitrageur*.

Remark 3.3 - *Valuing Contracts*
When valuing contracts we assume that arbitrage does not exist. This means we can derive a single price[33] for a contract, as any other price would create an *Arbitrage Opportunity*.

Theorem 3.2 - *Equivalent Contract Valuations over Time*
If two combinations of financial derivatives both have the same value $V_T = W_T$ at time $t = T$. Then their prices will be the same at all $t < T$

$$\text{if } V_T = W_t \text{ then } V_t = W_t \ \forall\ t < T$$

Proof 3.1 - `Theorem 3.2`
We assume the "No-Arbitrage Principle" holds throughout this proof.

Let V_t, W_t represent the fair price for two different combinations of financial derivatives at time t and that $V_T = W_T$. Suppose there is a risk-free profit of r.

Assume WLOG that $V_t > W_t$. Then an arbitrage opportunity exists and can be exploited by doing the following:

- At $t = 0$
  i). Sell/short the first combination, receiving £V_t.
  ii). Buy the second combination, costing £W_t.
  iii). Invest the difference (£$V_t - W_t > 0$).
- At $t = T$
  i). Sell the first combination, receiving £$V_T = W_T$
  ii). Buy the second combination, costing £$W_T = V_T$.

Following this will result in a "riskless" profit of $(V_t - W_t)e^{r(T-t)} > 0$.

Definition 3.8 - *Forward Contract*
A *Forward Contract* is a type of *Derivative Security*. In a *Forward Contract* two parties agree to an exchange on a predetermined future date for a predetermined amount, and are both obliged to fulfil this exchange.

All *Forward Contracts* have the following components

- *Delivery Date* T.
- *Delivery Price* K.

Remark 3.4 - *Positions in a Forward Contract*
In a *Forward Contract* agents can take two positions

- *Long Position* - Agree to buy the underlying asset for £K on date T. Makes a profit if the market-value of the underlying asset is greater than K in time-period T.
- *Short Position* - Agree to sell the underlying asset for £K on date T. Makes a profit if the market-value of the underlying asset is less than K in time-period T.

[33]Known as the *Fair Price*.

Remark 3.5 - *Utility of Forwards Contracts*
Forward Contracts allow you to agree terms of a purchase/sale some time in advance of actually transacting. This means business have greater certainty about their future cash-flows.[34]

Theorem 3.3 - *Fair Delivery Price of a Forward Contract*
Consider a *Forward Contract* with delivery date T, where the underlying asset has value S_0 at time $t=0$ and pays a dividend D at time $t_0 \in (0,T)$. Suppose there is a risk-free bank account with a constant interest rate r during the interval $[0,T]$.[35]

Then

- If $D=0$ (ie no dividend is payed) then the fair *Delivery Price* for this contract is
$$K = S_0 e^{rT}$$
- If $D>0$ then the fair *Delivery Price* for this contract is
$$K = (S_0 - I)e^{rT} \text{ where } I := De^{-rt_0}$$

Proof 3.2 - *Theorem 3.3*
We use the "No-Arbitrage Principle" to prove that these Ks are the fair prices under each scenario.

Case 1 - Suppose, for the sake-of-contradiction, that $K > (S_0 - I)e^{rT}$ with $I := De^{-rt_0}$. Then an arbitrage opportunity exists and can be exploited by doing the following:

- At $t=0$
  - i). Borrow £S_0 from the bank, at an interest rate of r.
  - ii). Buy the underlying asset.
  - iii). Taking a short position in the forward contract (receiving $K > (S_0 - I)e^{rT}$).
- At $t=t_0$
  - i). We will receive a dividend payment £D which we shall use to partially repay our loan. This leaves an outstanding balance of $S_0 e^{rt_0} - D$.
- At $t=T$
  - i). Sell the asset for K using the forward contract.
  - ii). Repay the outstanding balance on the loan ($(S_0 e^{rt_0} - D)e^{r(T-t_0)}$).

Doing all this will lead to a "riskless" profit of
$$K - (S_0 e^{rt_0} - D)e^{r(T-t_0)} = K - (S_0 - I)e^{rT} > 0 \text{ by def. } K$$

This means that this definition of K cannot be the fair-price, thus $K \leq (S_0 - I)e^{rT}$.

Case 2 - Suppose, for the sake-of-contradiction, that $K < (S_0 - I)e^{rT}$ with $I := De^{-rt_0}$. Then an arbitrage opportunity exists and can be exploited by doing the following:

[34] e.g. Farmers may agree to a price for their whole harvest a year in advance. Thus their next years income is completely known.
[35] This means $B_t = e^{rt}$.

- At $t = 0$
  i). Short sell the underlying asset. (Receiving £S_0).
  ii). Invest this revenue, receiving an interest rate of r.
  iii). Take a long position on the forward contract.
- At $t = t_0$
  i). Pay the dividend £D to our lender, from our bank account.
- At $t = T$
  i). Buy the asset for K using the forward contract.

Doing all this will lead to a "riskless" profit of

$$(S_0 e^{rt_0} - D)e^{r(T-t_0)} - K = (S_0 - I)e^{rT} - K > 0 \text{ by def. } K$$

This means that this definition of K cannot be the fair-price, thus $K \geq (S_0 - I)e^{rT}$.

Thus, by combining these two inequalities, the fair price for this *Forward Contract* is

$$K = (S_0 - I)e^{rT}$$

Definition 3.9 - *Options Contract*
An *Options Contract* is a type of *Derivative Security.* In an *Options Contract* two parties agree to an exchange on (or before) a predetermined future date for a predetermined amount, but the holder is not obliged to fulfil this exchange.

All *Options Contracts* have the following components

- *Delivery Date T.*
- *Strike Price K.*

There are two classes of *Options Contract*

- *Call Option* - The holder has the right to buy.
- *Put Option* - The holder has the right to sell.

Definition 3.10 - *European & American Options*
There are two categories of *Options Contract* which determine when the contract can be exercised

- *European Option* - The holder can only execute on the delivery date $t = T$.
- *American Option* - The holder can execute on any date before the delivery date T.

Definition 3.11 - *Positions in an Options Contract*
In *Options Contracts* agents can take one of two positions. The position they take determines their rights & potential cash-flows.

- *Holder* - Decides whether to execute the contract of not. Will pay the *Writer* a fee for creating the contract.

  The *Holder's* only expense is the fee they pay the *Writer* and they may make an income if they execute the contract.

- *Write* - Must complete the transaction if the *Holder* wishes to. Receives a fee from the *Holder*.

  The *Writer's* only income is the fee they receive from the *Holder* and they may incur a loss if the contract is executed.

Remark 3.6 - *When are options executed?*
Whether the holder should execute their option depends on the market price S_T at time T, the strike price K and the class of contract. Assuming (justifiably) that the holder will only execute the option if it will make them money, the holder should do the following

- For a *Call Option* the holder should execute if $S_T > K$. As they can immediately sell their newly bought asset for a profit of $S_T - K - F$ where F is the fee payed to the writer.
- For a *Put Option* the holder should execute if $S_T < K$. As they can buy the asset from the market and sell it to the writer of their option for a profit of $S_T - K - F$ where F is the fee payed to the writer.

Theorem 3.4 - *Put-Call Parity*[36]
Consider a *European Put Option* and a *European Call Option* where both have the same: underlying asset, strike price K and expiry date T. Let S_t be the value of the underlying asset at time point t and assume there is a "risk-free" interest rate of r available.

Then, if <u>no</u> *Arbitrage Opportunities* exist then the following hold

$$S_t + P_t - C_t = Ke^{-r(T-t)} \ \forall\ t \in [0,T]$$

where P_t, C_t are the prices of the put & call options at time t respectively, and $Ke^{-r(T-t)}$ is the discounted value of our bank account.

Theorem 3.5 - *Lower-Bound for a European Call Option*
Let S_t be the value of an underlying asset at time t.

For a *European Call Option* with strike price K and delivery date T we can determine the following lower bound on its price C_t

$$C_t \quad \geq \quad \{S_t - Ke^{-r(T-t)}\}_+$$

Proof 3.3 - *`Theorem 3.5`*
By *Put-Call Parity* (`Theorem 3.4`) we have that

$$\begin{array}{rrcl} & S_t + P_t - C_t & = & Ke^{-r(T-t)} \\ \implies & C_t & = & S_t + P_t - Ke^{-r(T-t)} \end{array}$$

Since *Put Options* cannot have a negative price, $P_t \geq 0$, we have that

$$C_t \geq S_t + P_t - Ke^{-r(T-t)}$$

Further, since *Call Options* cannot have a negative price, $C_t \geq 0$, we have that

$$C_t \geq \left\{S_t + P_t - Ke^{-r(T-t)}\right\}_+$$

[36]This is an application of `Theorem 3.2` to European Put & Call Options.

Theorem 3.6 - *Value of American Call Options w/o Dividends*
Consider an *American* & a *European Call Option*, for the same underlying asset, with the same strike price and expiry date.

Then, if the underlying asset does not pay a *Dividend*

$$C_A = C_E^{[37]}$$

where C_A, C_E are the price of the American & European call options, respectively.

Proof 3.4 - *`Theorem 3.6`*
If the *American Call Option* is executed early at time-point $t < T$ then it generates an income of $S_t - K$.

However, `Theorem 3.5` shows that selling a *Call Option* generates

$$\{S_t - Ke^{-r(T_t)}\}_+ \geq S_t - Ke^{-r(T-t)} > S_t - K$$

This shows that it is sub-optimal to exercise the call at any time $t < T$.

4 Discrete-Time

4.1 Processes of Financial Models

Remark 4.1 - *Time-Span T*
Below I generically define several processes which are commonly defined for different financial models. Different models use different time-spans T at which trades can occur:

- Single-Period Model - $T = \{0, 1\}$.
- Multi-Period Model - $T = \{0, 1, \dots, T\}$.
- Continuous - $T = [0, T]$.

Definition 4.1 - *Bank Account Process, B*
A *Bank Account Process B* models how much an initial deposit of one unit, at time $t = 0$, into a "risk-free" bank account, with interest rate r, would be worth at each time-point t.

$$\begin{array}{rrcl} & B &=& \{B_t : t \in T\} \\ \text{where} & B_0 &=& 1 \\ \text{and} & B_t(\omega) &\geq& 0 \ \forall\ \omega \in \Omega \end{array}$$

It is generally assumed that you can borrow money from these accounts, paying the same interest rate r.

Proposition 4.1 - *Value of Bank Account Process B*
Suppose our "Risk-Free" Bank Account pays a constant interest rate of r in each time-period, then after t time-periods our initial deposit is worth

[37]This shows that if an underlying asset does not pay a dividend then it is suboptimal to exercise an *American Call Option* early.

- *Single-Period Model* $B_1 = B_0(1+r)$.[38]
- *Multi-Period Model* $B_t = B_0(1+r)^t$.
- *Continuous Time Model* $B_t = B_0 e^{rt}$.

Definition 4.2 - *Price-Process S*
A *Price Process S* models the price of each security at each time-point

$$S := \{S(t) : t \in T\} \text{ where } S(t) = (S_1(t), \dots, S_N(t))$$

where $S_n(t)$ is the price of the n^{th} stock at time t and there are N different stock available.

The values of $S(t)$ only become known in time-period t.

Definition 4.3 - *Discounted Price-Process S^**
A *Discounted Price-Process S^** is the price of each security at each time-point t, but normalised by the *Bank Process B_t*.

$$S^* := \{S^*(t) : t \in T\} \text{ where } S^*(t) = (S^*_1(t), \dots, S^*_N(t)) \text{ and } S^*_n(t) := \frac{S_n(t)}{B_t}$$

Definition 4.4 - *Trading Strategy H*
A *Trading Strategy H* describes the changes in an investors portfolio over given time-periods.

$$H(t) := \big(H_0(t), H_1(t), \dots, H_N(t)\big) \text{ for } t \in T$$

where $H_0(t), \dots, H_N(t)$ are predictable stochastic processes with $H_n(t)$ denoting the number of units of stock n the investor carries from period $t-1$ to period t. Stock $n = 0$ is the bank account.

Definition 4.5 - *Self-Financing Trading Strategy*
A *Trading Strategy H* is to be *Self-Financing* if no money is introduced, or removed, between time-periods.

$$\forall\, t \in T, \quad V_t = H_0(t+1)B_t + \sum_{n=1}^N H_n(t+1)S_n(t)$$

Theorem 4.1 - *Self-Financing and Value Process*
A *Trading Strategy H* is self-financing iff $\forall\, t \in (T \setminus \{0\}),\ V^*_t = V^*_0 + G^*_t$

Proof 4.1 - *Theorem 4.1*
For all $t = 1, \dots, T$ it holds that

$$G^*_t = G^*_{t-1} + \sum_{n=1}^N H_n(t)\Delta S^*_n(t)$$

For convenience we define $G^*_0 = 0$.

I prove the statement in both directions

[38]Must be that $t = 1$ in a *Single-Period Model*.

$\implies$ Assume that H is *Self-Financing*.

By the definitions of *Self-Financing*, *Discounted Processes* and the above result, we can show the following for all $t = 1, \dots, T$

$$\begin{array}{rcl} V_t^* - G_t^* & = & H_0(t) + \left(\sum_{n=1}^N H_n(t)S_n^*(t)\right) - \left(\sum_{n=1}^N H_n(t)\Delta S_n^*(t)\right) - G_{t-1}^* \\ & = & H_0(t) + \left(\sum_{n=1}^N H_n(t)(S_n^*(t) - \Delta S_n^*(t))\right) - G_{t-1}^* \\ & = & H_0(t) + \left(\sum_{n=1}^N H_n(t)S_n^*(t-1)\right) - G_{t-1}^* \\ & = & V_{t-1}^* - G_{t-1}^* \end{array}$$

By recursion we find that $V_t^* - G_t^* = V_0^*$.

$\impliedby$ Assume that $V_t^* = V_0^* + G_t^*$ for all $t = 1, \dots, T$.

Then, for all $t = 1, \dots, T_1$ we have the following

$$\begin{array}{rcl} V_t^* - V_{t+1}^* & = & V_0^* + G_t^* - (V_0^* + G_{t+1}^*) \\ & = & G_t^* - G_{t-1}^* \end{array}$$

Therefore, by the definitions of discounted process and the result at the start of this proof

$$\begin{array}{rcl} V_t^* & = & V_{t+1}^* - (G_{t+1}^* - G_t^*) \\ & = & H_0(t+1) + \sum_{n=1}^N H_n(t+1)S_n^*(t+1) - \sum_{n=1}^N H_n(t+1)\Delta S_n^*(t+1) \\ & = & H_0(t+1) + \sum_{n=1}^N H_n(t+1) \end{array}$$

Thus H is *Self-Financing*.

$\square$

Definition 4.6 - *Value-Process V*

A *Value Process V* models the total value of a *Trading Strategy H* at each time-point t

$$V := \{V_t : t \in T\} \text{ where } V_t := H_0(t)B_t + \sum_{n=1}^N H_n(t)S_n(t)$$

Definition 4.7 - *Discounted Value-Process V^**

A *Discounted Value-Process V^** models the total value of a *Trading Strategy H* at each time-point t <u>but</u> normalised by the *Bank Process B*.

$$V^* := \{V_t^* : t \in T\} \text{ where } V_t^* := \frac{V_t}{B_t} = H_0 + \sum_{n=1}^N H_n \underbrace{\frac{S_n(t)}{B_t}}_{=S_n^*(t)}$$

Definition 4.8 - *Gains-Process G*

A *Gains Process G* models the total profit/loss made by a *Trading Strategy H* up to time-period t.

$$\begin{array}{rrcll} & G & := & \{G_t : t \in T \setminus \{0\}\} & \\ \text{where} & G_t & := & \left(\sum_{u=1}^t H_0(u)B_t\right) + \sum_{n=1}^N \sum_{u=1}^t H_n(u)\Delta S_n^*(u) & \text{Discrete} \\ \text{or} & G_t & := & \int_0^t H_0(u)dB_u + \sum_{n=1}^N \int_0^t H_n(u)dS_n(u) & \text{Continuous} \end{array}$$

Definition 4.9 - *Discounted Gains-Process G^**
A *Discounted Gains Process* G^* models the total discounted profit/loss made by a *Trading Strategy* H up to time-period t.

$$\begin{array}{rrcll} & G^* & := & \{G^*_t : t \in T \setminus \{0\}\} & \\ \text{where} & G^*_t & := & \dfrac{G_t}{B_t} = \displaystyle\sum_{n=1}^N \sum_{u=1}^t H_n(u)\Delta S^*_n(u) & \text{Discrete} \\ \text{and} & \Delta S_n(u) & := & S_n(t) - S_n(t-1) & \\ \text{or} & G^*_t & := & \displaystyle\sum_{n=1}^N \int_0^t H_n(u) dS^*_n(u) & \text{Continuous} \end{array}$$

4.2 Single-Period Model

Definition 4.10 - *Single-Period Model*
The *Single-Period Model* is a model for a financial market where actions can only occur on two dates. It has the following components

- *Initial Date* $t = 0$.
- *Terminal Date* $t = 1$.
- Trading is only allowed to occur on the *Initial & Terminal Dates*.
- A finite *Sample Space* $\Omega := \{\omega_1, \dots, \omega_K\}$ with $K < \infty$.
  Each event $\omega_1, \dots, \omega_K$ corresponds to some state of the world.
- A *Probability Measure* $\mathbb{P}$ on the *Sample Space* Ω with $\mathbb{P}(\{\omega_i\}) > 0\ \forall\ i \in [1, K]$.

Definition 4.11 - *Arbitrage Opportunity - Single-Period Model*
Consider a *Trading Strategy* $H = (H_0, H_1)$ for the *Single-Period Model*.
H exploits an *Arbitrage Opportunity* if it has the following three properties

i). $V_0 = 0$.

ii). $V_1(\omega) \geq 0\ \forall\ \omega \in \Omega$.

iii). $\mathbb{P}(V_1(\omega) \geq 0) > 0\ \forall\ \omega \in \Omega$.[39]

Theorem 4.2 - *Arbitrage Opportunities & Gains Process*
There exists an *Arbitrage Opportunity* in a market iff there exists a *Trading Strategy* H st[40]

$$G^* \geq 0 \quad \text{and} \quad \mathbb{E}[G^*] > 0$$

Proof 4.2 - *Theorem 4.2*

[39]Equivalently $\mathbb{E}[V_1] > 0$
[40]This means H never loses money, and it is expected to make money.

$\Rightarrow$ Let H be a *Trading Strategy* which exploits an *Arbitrage Opportunity.*

By the definition of an *Arbitrage Opportunity* $G^* = V_1^* - V_0^*$ and $B_t > 0 \ \forall \ t, \omega$, this means that $G^* \geq 0$ and thus

$$\mathbb{E}[G^*] = \mathbb{E}[V_1^*] > 0$$

$\Leftarrow$ Let H be a *Trading Strategy* which satisfies $G^* \geq 0$ and $\mathbb{E}[G^*] > 0$.

Define $\hat{H} := (\hat{H}_0, H_1, \dots, H_N)$ where $\hat{H}_0 := -\sum_{i=1}^N H_i S_i^*(0)$[41].

Under $\hat{H}_0$ we have that $V_0^* = 0$ and $V_1^* = V_0^* + G^* = G^*$.

Hence, $V_1^* \geq 0$ and $\mathbb{E}[V_1^*] = \mathbb{E}[G^*] > 0$, meaning $\hat{H}$ exploits an *Arbitrage Opportunity.*

As the result holds in both directions, we can say it holds iff.

4.2.1 Risk-Neutral Probability Measures $\mathbb{Q}$

Remark 4.2 - *Risk-Neutral Probability Measure vs Martingale Measure*
A risk-neutral probability measure (**Definition 4.12**) is the single-period version of a martingale measure (**Definition 4.15**).

Definition 4.12 - *Risk-Neutral Probability Measure* $\mathbb{Q}$
A *Probabiltiy Measure* $\mathbb{Q}$ on *Sample Space* Ω is said to be a *Risk-Neutral Probability Measure* if the following hold

i). $\mathbb{Q}(\{\omega\}) > 0 \ \forall \ \omega \in \Omega$.

ii). $\mathbb{E}_{\mathbb{Q}}[S_i * (1)] = S_i^*(0) \ \forall \ i \in [1, N]$

Theorem 4.3 - *Separating Hyperplane Theorem*[42]
Let $\mathbb{W}$ be a linear subspace of $\mathbb{R}^K$ and $\mathbb{K}$ be a compact convex subset in $\mathbb{R}^K$ which is disjoint from $\mathbb{W}$.

We can separate $\mathbb{W}$ and $\mathbb{K}$ strictly by using a hyperplane containing $\mathbb{W}$[43] st

$$u^T v > 0 \ \forall \ u \in \mathbb{K}$$

Theorem 4.4 - *No-Arbitrage Principle*
No *Arbitrage Opportunities* exist in a single-period model iff there exists a *Risk-Neural Probability Measure* $\mathbb{Q}$.

Proof 4.3 - *Theorem 4.4*
Consider the three following sets

i). $\mathbb{W} = \{X \in \mathbb{R}^K : X = G^* \text{ for some Trading Strategy } H\}$.

This is the set of possible *Gains* in our market for *Trading Strategies* which have zero initial investment. $\mathbb{W}$ is a linear subspace of $\mathbb{R}^K$[45].

[41]This ensures V_0, a requirement for H to exploit an *Arbitrage Opportunity.*
[42]This theorem is used to prove `Theorem 4.4`. The proof of this theorem is beyond the scope of this course.
[43]ie $\exists \ v \in \mathbb{R}^K$ which is *Orthogonal* to $\mathbb{W}$[44] $u^T v = 0 \ \forall \ u \in \mathbb{W}$.
[45]Proved by showing it is complete under: addition, and scalar multiplication.

ii). $\mathbb{A}=\left\{X\in\mathbb{R}^K:X\geq 0,X\neq 0\right\}$[46].

There exists an arbitrage opportunity iff $\mathbb{W}\cap\mathbb{A}\neq\emptyset$.

iii). $\mathbb{A}^+=\left\{X\in\mathbb{R}^N:X\geq 0,X\neq 0,\sum_{i=1}^K X_i=1\right\}$.

$\mathbb{A}^+$ is a convex and compact subset of $\mathbb{R}^K$.

$\Rightarrow$ Assume that there are no *Arbitrage Opportunities*, then $\mathbb{W}\cap\mathbb{A}\neq\emptyset$.

By the *Separating Hyperplane Theorem* (`Theorem 4.3`) $\exists\ Y\in\mathbb{R}^K$ which is *orthogonal* to $\mathbb{W}$ st

$$X^TY>0\ \forall\ X\in\mathbb{A}^+$$

For each $k\in\{1,\dots,K\}$ the k^{th} unit vector e_k is an element of $\mathbb{A}^+$. Therefore,

$$Y_k:=e_k^TY>0\ \forall\ k\in\{1,\dots,K\}$$

meaning all entries of Y are strictly positive.

Define a probability measure $\mathbb{Q}$ by setting

$$\mathbb{Q}(\{\omega_k\})=\frac{Y(\omega_k)}{Y(\omega_1)+\dots+Y(\omega_k)}$$

Furthermore, $\Delta S_n^*\in\mathbb{W}\ \forall\ n$ because $\Delta S_n^*:=S_n^*(1)-S_n^*(0)$ is the discounted wealth for the portfolio $H:=e_n$ which consists of one unit of the n^{th} asset only.

Since Y is orthogonal to $\mathbb{W}$ we can conclude that

$$\mathbb{E}_{\mathbb{Q}}[\Delta S_n^*]=\sum_{k=1}^K\Delta S_n^*(\omega_k)\mathbb{Q}(\{\omega_k\})=0\ \forall\ n$$

In other words

$$\mathbb{E}_{\mathbb{Q}}[S_n^*(1)]=S_n^*(0)\ \forall\ n$$

Thus $\mathbb{Q}$ is a *Risk-Neutral Probability Measure.*

$\Leftarrow$ Let $\mathbb{Q}$ be a *Risk-Neutral Probability Measure.*

Then for an arbitrary *Trading Strategy* H we have that

$$\mathbb{E}_{\mathbb{Q}}[G^*]=\mathbb{E}_{\mathbb{Q}}\left[\sum_{n=1}^N H_n\Delta S_n^*\right]=\sum_{n=1}^N H_n\mathbb{E}_{\mathbb{Q}}[\Delta S_n^*]=0$$

and, in particular

$$\sum_{k=1}^K G^*(\omega_k)\mathbb{Q}(\{\omega_k\})=0$$

which shows that either $G^*(\omega_k<0)$ for some k or $G^*=0$, but then $\mathbb{E}_{\mathbb{Q}}[G^*]=0$.

Hence, by `Theorem 4.2`, there cannot be any arbitrage opportunities.

The result holds in both directions. $\square$

[46] $\mathbb{A}$ is not compact, so can not be used for $\mathbb{K}$ in *Separating Hyperplane Theorem*

4.3 Multi-Period Model

Definition 4.13 - *Multi-Period Model*
The *Single-Period Model* is a model for a financial market where actions can only occur on multiple dates. This provides a more realistic model than the *Single-Period Mdeol.* It has the following components

- *Initial Date* $t = 0$.
- *Terminal Date* $t = T \in \mathbb{N}$.
- Trading can occur at any times $t \in \{0, 1, \dots, T\}$
- A finite *Sample Space* $\Omega = \{\omega_1, \dots, \omega_K\}$ with $K < \infty$. Each event $\omega_1, \dots, \omega_K$ corresponds to a state of the world.
- A *Probability Space* $\mathbb{P}$ on Ω with $\mathbb{P}(\omega) > 0 \ \forall \ \omega \in \Omega$.

Definition 4.14 - *Arbitrage Opportunity - Multi-Period Model*
An *Arbitrage Opportunity* exists in a multi-period model if there exists a *Trading Strategy* H with the following properties

i). $V_0 = 0$.

ii). $V_T \geq 0$.

iii). $\mathbb{E}[V_T] > 0$.

iv). H is *Self-Financing.*

Proposition 4.2 - *Arbitrage Opportunities for Single & Multi-Period Models*
If a multi-period model has no arbitrage opportunities, then no arbitrage opportunities exist for any of the underlying single-period models.

Proof 4.4 - *Proposition 4.2*
For each $t < T$ and for each $A \in \mathcal{P}_t$ there is one underlying single-period model where

- *Initial Time Discounted Price* is $S_n^*(t, \omega)$ for an arbitrary $\omega \in A$ since $S_n^*(t, \omega)$ are constant on A.
- *Sample Space* contains one state for each cell $A' \in \mathcal{P}_{t+1}$ st $A' \subset A$.
- *Terminal Time Discounted Price* is $S_n^*(t + 1, \omega)$ for each $n = 1, \dots, N$ for some $\omega \in A$.

If any underlying single-period model has an arbitrage opportunity in the single-period sense, then the multi-period model must have an arbitrage opportunity in the multi-period sense.

To see this, suppose there exists an *Arbitrage Opportunity* $\hat{H}$ for the single period model corresponding to some $A \in \mathcal{P}_t$ for $t < T$. This means that the discounted gain (3) is *non-negative* and not identical to zero on the event A.

$$\hat{H}_1 \Delta S_n^*(t + 1) + \cdots + \hat{H}_N \Delta S_N^*(t + 1) \tag{3}$$

We now construct a multi-period *Trading Strategy* H which is an *Arbitrage Opportunity.*

$$H_n(s,\omega)=\begin{cases}0 & \text{if } s\leq t \text{ or } \omega\notin A\\ \hat{H}_n & \text{if } s=t+1,\ \omega\in A, \text{ and } n=1,\dots,N\\ -\sum_{i=1}^N \hat{H}_iS_i^*(t) & \text{if } s=t+1,\ \omega\in A \text{ and } n=0\\ \sum_{i=1}^N \hat{H}_1\Delta S_1^*(t+1) & \text{if } s>t+1,\ \omega\in A \text{ and } n=0\\ 0 & \text{if } s>t+1,\ \ \text{and } n=0\end{cases}$$

This strategy starts with zero money and does nothing unless the event A occurs at time t, in which case at time t the position $\hat{H}_n$ is taken in the n^{th} risky security while the position in the bank account is used to self-finance.

Subsequently, no position is taken in any of the risky securities, and non-zero value of the portfolio is reflected by a position in the bank account.

THis *Trading Strategy* H is an *Arbitrage Opportunity* in the multi-period model. □

Theorem 4.5 - *Arbitrage Opportunity & Gains Process*
A *Self-Financing Trading Strategy* H is an *Arbitrage Opportunity* iff the following three properties hold

i). $G_T^*\geq 0$.

ii). $\mathbb{E}[G_T^*]>0$.

iii). $V_0=0$.

Proof 4.5 - *Theorem 4.5*

4.3.1 Martingale Measure $\mathbb{Q}$

Remark 4.3 - *Risk-Neutral Probability Measure vs Martingale Measure*
A martingale measure (**Definition 4.15**) is the multi-period version of the risk-neutral probability measure (**Definition 4.12**).

Definition 4.15 - *Martingale Measure*
A *Martingale Measure* $\mathbb{Q}$ is a probability measure with the following properties:

i). $\forall\ \omega\in\Omega,\ \mathbb{Q}(\{\omega\})>0$.

ii). The *Discounted Price Process* S^* is a *Martingale* under $\mathbb{Q}$

$$\forall\ t,s\geq 0,\ \mathbb{E}_\mathbb{Q}\left[\frac{S_n(t+s)}{B_{t+s}}\middle|F_t\right]=S_n(t)$$

Theorem 4.6 - *No-Arbitrage Principle*
No *Arbitrage Opportunities* exist in a multi-period model iff there exists a *Martingale Measure* $\mathbb{Q}$.

Proof 4.6 - *Theorem 4.6*

$\Longleftarrow$ We first show that there can be no *Arbitrage Opportunities* provided the existence of a *Martingale Measure* $\mathbb{Q}$.

Suppose H is any *Self-Finacing Trading Strategy* with

$$V_T^* \geq 0 \text{ and } \mathbb{E}[V_T^*] > 0$$

This implies

$$\mathbb{E}_{\mathbb{Q}}[V_T^*] > 0$$

Since V^* is a *Martingale* under $\mathbb{Q}$ by `Theorem 4.7`, it follows that

$$V_0^* = \mathbb{E}_{\mathbb{Q}}[V_T^*] > 0$$

Hence H cannot be an *Arbitrage Opportunity*, nor can any other trading strategies be *Arbitrage Opportunities* due to H being chosen arbitrarily.

$\Longrightarrow$[47] Using `Proposition 4.2`, we have that for each $t < T$ and each $A \in \mathcal{P}_t$ there is a *risk-neutral probability measure* $\mathbb{Q}(t, A)$ for the underlying single-period model.

This probability measure gives positive mass to each cell $A' \in \mathcal{P}_{t+1}$ st it sums to 1 over all such cells and it satisfies

$$\mathbb{E}_{\mathbb{Q}(t,A)}[\Delta S_n^*(t+1)] = 0 \text{ for } n = 1, \dots, N$$

Notice that $\mathbb{Q}(t, A)$ puts probability on each branch in the information tree which emerges from the node corresponding to (t, A).

We can calculate a *Martingale Measure* $\mathbb{Q}$ for the multi-period model from these probabilities by setting $\mathbb{Q}(\{\omega\})$ equal to the product of the conditional probabilities along the path from the node at $t = 0$ to the node at (T, ω).

Then

- $\sum_{\omega \in \Omega} \mathbb{Q}(\{\omega\}) = 1$.
- $\mathbb{Q}(\{\omega\}) > 0$ for every $\omega \in \Omega$ because all the conditional risk neutral probabilities are strictly positive.
- And, $\mathbb{E}_{\mathbb{Q}}[S_n^*(t+1)|\mathcal{F}_t] = S_n^*(t)$ for all t and n.

Thus $\mathbb{Q}$ is indeed a *Martingale Measure*. $\square$

Theorem 4.7 - *Finance Processes which are Martingales*
Consider a self-financing trading strategy H and a martingale measure $\mathbb{Q}$.

The following are martingales under $\mathbb{Q}$:

i). Discounted value process V^*.

ii). Discounted gains process G^*.

Proof 4.7 - *`Theorem 4.7`*
We have to show that $\mathbb{E}_{\mathbb{Q}}\left[V_{t+1}^*|\mathcal{F}_t\right] = V_t^* \ \forall \ t > 0$.

This is equivalent to showing that $\mathbb{E}_{\mathbb{Q}}[G_{t+1}^*|\mathcal{F}_t] = G_t^*$ using `Theorem 4.1`.

Using the expression for $G_{t+1}^* - G_t^*$ derived in `Proof 4.1` we can conclude that

$$\begin{aligned}
\mathbb{E}_{\mathbb{Q}}\left[G_{t+1}^* - G_t^*|\mathcal{F}_t\right] &= \mathbb{E}_{\mathbb{Q}}\left[\sum_{n=1}^N H_n(t+1)\Delta S_n^*(t+1)\middle|\mathcal{F}_t\right] \\
&= \sum_{n=1}^N \mathbb{E}_{\mathbb{Q}}\big[\underbrace{H_n(t+1)}_{\in \mathcal{F}_t{}^{[48]}}\Delta S_n^*(t+1)|\mathcal{F}_t\big] \\
&= \sum_{n=1}^N H_n(t+1)\mathbb{E}_{\mathbb{Q}}\left[\Delta S_n^*(t+1)|\mathcal{F}_t\right]
\end{aligned}$$

The last result is due to H_n being a *Trading Strategy* and thus *Predictable.*

Furthermore, $\mathbb{E}_{\mathbb{Q}}\left[\Delta S_n^*(t+1)|\mathcal{F}_t\right] =$ for all $t > 0$ since S_n^* is a under $\mathbb{Q}$.

Hence G_t^*, V_t^* are *Martingales.* □

4.4 American Claims

Definition 4.16 - *American Claim* Y_τ
Let $\{Y_t\}_{t\in T}$ be a payoff process and τ be a stopping time representing the exercise date of some "exercise strategy".

Y_τ is called an *American Claim* wrt $\{Y_t\}$ and τ.

Definition 4.17 - *Attainable American Claim*
An *American Claim* Y_τ is "*attainable*" if

$$\exists \text{ self-financing trading strategy } H \text{ st } V_\tau = Y_\tau \text{ when using } H.$$

Theorem 4.8 - *Complete Markets and American Claims*
If a financial market is *Complete* then every *American Claim* is *Attainable.*

Proof 4.8 - *Theorem 4.8*
Let $\{Y_t\}_{t\in T}$ be a payoff-process and τ be an exercise strategy.

We have to find a self-financing trading strategy H st $V_\tau = Y_\tau$.

Consider the *European Claim* $X = Y_\tau(B_T/B_\tau)$ which corresponds to someone exercising the *American Claim* Y at time-point $t = \tau$ and then earning interest from a bank-account until time-point T. Since the model is complete, there must be a replicating trading strategy H st $V_T = X = Y_\tau(B_T/B_\tau)$.

This portfolio which starts at time τ with the amount of Y_τ all of which is put into and kept in the bank account until time T, has the same value at time T as H.

We conclude that $V_\tau = Y_\tau$. □

Definition 4.18 - *Snell Envelope*
Let $\{X_t\}_{t\in T}$ be a stochastic process adapted to some filtration $\mathcal{F}_t$.

The process $\{Z_t\}_{t\in T}$, defined below, is called the *Snell Envelope* of X.

$$Z_t = \begin{cases} X_T & \text{if } t = T \\ \max\{X_t, \mathbb{E}[Z_{t+1}|\mathcal{F}_t]\} & \text{if } t < T \end{cases}$$

Theorem 4.9 - *Snell Envelope is the Smallest Super-Martingale*
The *Snell Envelope* $\{Z_t\}$ of X is the smallest *Super-Martingale* which dominates X.

Proof 4.9 - *Theorem 4.9*
First, $Z_t \geq \mathbb{E}[Z_{t+1}|\mathcal{F}_t]$ and $Z_t \geq X_t$, so $\{Z_t\}$ is a super-martingale and dominates X.

[48]This is due to $H_n(t+1)$ being the strategy we are building for time-step $t+1$ and thus we use all the information available in time-step t.

Next, let $\{U_t\}_{t\in T}$ be any other super-martingale which dominates X. Since, by definition, $Z_T = X_T$ and U dominates X we must have $U_T \geq Z_T$. Assume inductively that $U_t \geq Z_t$. Then

$$\begin{array}{rcl} U_{t-1} & \geq & \mathbb{E}[U_t|\mathcal{F}_{t-1}] \text{ since } U_t \text{ is a supermartingale} \\ & \geq & \mathbb{E}[Z_t|\mathcal{F}_{t-1}] \end{array}$$

and U dominates X

$$U_{t-1} \geq X_{t-1}$$

Combining

$$U_{t-1} \geq \max\{X_{t-1}\mathbb{E}[Z_t|\mathcal{F}_{t-1}]$$

By repeating this argument we get $U_t \geq Z_t \ \forall\ t$. □

Theorem 4.10 - *Optimal Stopping Theorem*
This is NOT the "Optional Stopping Theorem" (`Theorem 4.10`).

Let $\{X_t\}_{t\in T}$ be a stochastic process adapted to some *Filtration* $\mathcal{F}_t$ and Z_t be the *Snell Envelope* of $\{X_t\}_{t\in T}$.

For any $t = 0, \dots, T$ we define a stopping time by $\tau(t) = \min_{s\geq t}\{Z_s = X_s\}$, then the optimal stopping rule is

$$Z_t = \mathbb{E}[X_{\tau(t)}|\mathcal{F}_t] = \max_t \{\mathbb{E}[X_\tau|\mathcal{F}_t] : \text{ all stopping times } t \leq \tau \leq T\} \text{ for all } t = 0, \dots, T \tag{4}$$

In particular

$$Z_0 = \mathbb{E}[X_{\tau(0)}] = \max_t \left\{\mathbb{E}[X_{\tau)}|\mathcal{F}_t] : \text{ all stopping times } \tau \leq T\right\}$$

Proof 4.10 - `Theorem 4.10`
This proof is a backwards induction through time.

Base Case

Note first that `Eq. 4` is clearly true for $t = T$ because, by definition of the *Snell Envelope*, $Z_T = X_T$ and therefore the stopping time $\tau(T)$ stops at T.

Inducitve Step

Now assume that `Eq. 4` is satisfied for some t, we now need to show it holds for $t-1$. Let τ be an arbitrary stopping time between $t-1$ and T.

Define another stopping time $\tau' = \max\{\tau, t\}$[49]. Then, since $\tau \geq t \implies \tau' = \tau$

$$\begin{array}{rcl} \mathbb{E}[X_\tau|\mathcal{F}_{t-1}] & = & \mathbb{E}[\mathbb{1}\{\tau = t-1\}X_{t-1} + \mathbb{1}\{\tau > t-1\}X_\tau|\mathcal{F}_{t-1}] \\ & = & \mathbb{E}[\mathbb{1}\{\tau = t-1\}X_{t-1} + \mathbb{1}\{\tau > t-1\}X_{\tau'}|\mathcal{F}_{t-1}] \\ & = & \mathbb{1}\{\tau = t-1\}X_{t-1} + \mathbb{1}\{\tau > t-1\}\mathbb{E}[X_{\tau'}|\mathcal{F}_{t-1}] \\ & = & \mathbb{1}\{\tau = t-1\}X_{t-1} + \mathbb{1}\{\tau > t-1\}\mathbb{E}[\mathbb{E}[X_{\tau'}|\mathcal{F}_t]|\mathcal{F}_{t-1}] \text{ by Tower Law} \end{array}$$

Since τ' is a stopping time st $\tau' \in [t, T]$, we find that $\mathbb{E}[X_{\tau'}|\mathcal{F}_t] \leq Z_t$ because we have assumed `Eq. 4` holds for t.

Using the definition of Z_{t-1} we see that

$$\begin{array}{rcl} \mathbb{E}[X_\tau|\mathcal{F}_{t-1}] & \leq & \mathbb{1}\{\tau = t-1\}X_{t-1} + \mathbb{1}\{\tau > t-1\}\mathbb{E}[Z_t|\mathcal{F}_{t-1}] \\ & \leq & Z_{t-1} \text{by def. Snell Envelope} \end{array}$$

[49]This is similar to τ but can never take the value $t-1$.

In the special case where $\tau = \tau(t-1)$ we find that $\tau = (t-1)$ stops in $t-1$ iff $Z_{t-1} = X_{t-1}$, otherwise $Z_{t-1} > X_{t-1}$[50] and $\tau(t-1) = \tau(t)$. Hence

$$\begin{array}{rcl}\mathbb{E}[X_{\tau(t-1)}|\mathcal{F}_{t-1}] & = & \mathbb{1}\{Z_{t-1} = X_{t-1}\}X_{t-1} + \mathbb{1}\{Z_{t-1} > X_{t-1}\}\mathbb{E}[\mathbb{E}[X_{\tau(t)}|\mathcal{F}_t]|\mathcal{F}_{t-1}] \\ & = & \mathbb{1}\{Z_{t-1} = X_{t-1}\}X_{t-1} + \mathbb{1}\{Z_{t-1} > X_{t-1}\}\mathbb{E}[Z_t|\mathcal{F}_{t-1}] \\ & = & Z_{t-1} \text{ by def. Snell Envelope}\end{array}$$

Proposition 4.3 - *Snell Envelope as Discounted Value Process*
Consider a financial market with a *Martingale Measure* $\mathbb{Q}$ and an attainable *American Payoff Process* $\{Y_t\}$.

Then the *Snell Envelope* $\{Z_t\}_t$ of the discounted payoff process $\{Y_t/B_t\}_t$ is the discounted value process for Y.

Proof 4.11 - *`Proposition 4.3`*
This proof uses the Optimal Stopping Theorem (`Theorem 4.10`).

Let p denote the time t price of the American claim wrt the payoff process (Y_t/B_t).

Suppose, first, that $p < Z_t$ then:

- We buy the option for p.
- If $\tau(t) = t$[51] then $Z_t = Y_t/B_t$ and so we can exercise the option immediately for $(Y_t/B_t) > p$ to make a riskless profit.
- If $\tau(t) > t$ we undertake the negative of the trading strategy that replicates $Y_{\tau(t)}/B_{\tau(t)}$, as we want to short sell. The price of the replicating strategy is $\mathbb{E}_Q[Y_{\tau(t)}/B_{\tau(t)}|\mathcal{F}_t]$ by the *Risk-Netural Valuation Principle* (`Theorem 4.12`), but by the *Optimal Stopping Theorem* (`Theorem 4.10`) this is equal to Z_t and so we can invest the different $Z_t - p$ in the bank account.
- Later at time $\tau(t)$ we exercise the option and liquidate the replicating portfolio at the same time. The amount we collect from the option seller is equal to our liability on the portfolio[52]. Meanwhile, we have $(Z_t - p)\cdot(B_{\tau(t)}/B_t) > 0$ in the bank account.

This shows that we make a riskless profit in any case.

Now consider the case where $p > Z_t$ then

- We sell the option for p.

Consider in detail the case where $\tau(t) > t$. Then

- We undertake the trading strategy that replicates $Y_{\tau(t)}/B_{\tau(t)}$. Again we find by using *Risk-Netural Valuation Principle* (`Theorem 4.12`) and the *Optimal Stopping Theorem* (`Theorem 4.10`) that the price of building up the portfolio is

$$Z_t = \mathbb{E}_{\mathbb{Q}}[Y_{\tau(t)/B_{\tau(t)}}|\mathcal{F}_t]$$

- Therefore there is a profit of $p - Z_t$ which we put in the bank account.

[50]As $\{Z_t\}$ dominates $\{X_t\}$.
[51]This is unlikely as the arbitrage opportunity is obvious.
[52]Meaning the cash flow at this time-period is net 0

How we proceed will depend on when the buyer exercises the option.

i). If the buyer exercises the option at time-point $s \leq \tau(t)$ then

- We pay the buyer the payoff Y_s/B_s.
- We liquidate our portfolio and using risk-neutral valuation the value of our portfolio at time-point s is
$$\mathbb{E}_{\mathbb{Q}}[Y_{\tau(t)}/B_{\tau(t)}|\mathcal{F}_s]$$
Since $s \in [t, \tau(t)]$ we see from the definition of $\tau(t)$ that $\tau(t) = \tau(s)$.
Using this and the *Optimal Stopping Theorem* (`Theorem 4.10`) that the value of our portfolio at time s is
$$\mathbb{E}_{\mathbb{Q}}[Y_{\tau(t)}/B_{\tau(t)}|\mathcal{F}_s] = \mathbb{E}_{\mathbb{Q}}[Y_{\tau(s)}/B_{\tau(s)}|\mathcal{F}_s] = Z_s \geq (Y_s/B_s)$$
- Therefore all transaction at time-point s will only add to our portfolio and our total profit is strictly positive.

ii). If the buyer does not exercise by time $s = \tau(t)$ where $\tau(t) < T$ then:

- We repeat the process, undertaking the trading strategy that replicates $Y_{\tau(s+1)}/B_{\tau(s+1)}$.
- The value of the portfolio to be built up is equal to
$$\mathbb{E}_{\mathbb{Q}}[Y_{\tau(s+1)}/B_{\tau(s+1)}|\mathcal{F}_s] \leq \mathbb{E}_{\mathbb{Q}}[Y_{\tau(s)}/B_{\tau(s)}|\mathcal{F}_s] = Z_s$$
Therefore the change of the portfolio will only pay us some money which we put in the bank account.
- As before, if the option buyer exercises at some time $u \leq \tau(s+1)$ then the value of the portfolio will be enough to cover the payoff Y_u.

iii). If the buyer has not exercised by time $\tau(s+1)$ then we repeat this process again, and so forth. There will always be enough money in the portfolio to cover the payoff. Our overall profit will be at least $p - Z_t > 0/$

Finally, we consider the case where $\tau(t)$. Then the optimal strategy would be to exercise the option immediately.

- If the buyer indeed exercises at time t, then we pay them $(Y_t/B_t) = Z_t < p$ and make a riskless-profit.
- If note, then we proceed as in the previous case where the buyer does not exercise by the optimal stopping time and undertake the trading strategy which replicates $(Y_{\tau(t+1)}/B_{\tau(t+1)})$ and so forth making again profit of at least $p - Z_t$.

$\square$

4.5 Contingent Claims X

Definition 4.19 - *Contingent Claim X*
A *Contingent Claim* $X \in \mathbb{R}^N$ is the final payoff of a model with $N-1$ risky securities.

Definition 4.20 - *Attainable Contingent Claim X*
A *Contingent Claim* X is said to be "*attainable*" if

$\exists$ (self-financing, admissible[53]) trading strategy H st $\forall\ \omega \in \Omega,\ V_T(\omega) = X(\omega)$ when using H.

If such a strategy H exists, it is called a *Replicating Portfolio* and is said to "*generate*" X.

Remark 4.4 - *Computing a Replicating Portfolio H*
Our approach depends on what information we are provided.

- If we known the contingent claim X and the *Value Process* V for the replicating portfolio, we need to solve for the trading strategy H using the linear equations in the definition of the value process (keeping in mind that H is *Predictable*).

$$V_t(\omega_i) = H_0(t)B_t + \sum_{n=1}^N H_n(t)S_n(t)(\omega_i)\ \forall\ \omega_i \in \Omega$$

- If we only know the *Contingent Claim* X we need to work backwards in time, deriving V and H simultaneously.

  Since $V_T = X$ we must first solve the following for $H(T)$.

$$X(\omega_i) = H_0(T)B)T + \sum_{n=1}^N H_n(T)S_n(T)(\omega_i)$$

  Since H is self-financing, we can calculate V_{T-1}.

$$V_{T-1} = H_0(T)B_{T-1} + \sum_{n=1}^N H_n(T)S_n(T-1)(\omega_i)$$

  Therefore, our next step is to solve the following for $H(T_1)$.

$$V_{T-1} = H_0(T_1)B_{T-1} + \sum_{n=1}^N H_n(T-1)S_n(T-1)(\omega_i)$$

  We can now continue by calculating V_{T-2} etc. until we end up with V_0.

Remark 4.5 - *Determining whether a Contingent Claim X is Attainable*
Consider a *Single-Period Model* with $K-1$ securities, which can be described the following matrix A, and a *Contingent Claim* $X \in \mathbb{R}^K$. Then

X is *Attainable* iff $\exists$ a trading strategy $H \in \mathbb{R}^K$ st $AH = X$ where

$$A := \begin{pmatrix} B_1(\omega_1) & S_1(1)(\omega_1) & \dots & S_N(1)(\omega_1) \\ B_1(\omega_2) & S_1(1)(\omega_2) & \dots & S_N(1)(\omega_2) \\ \vdots & \vdots & \ddots & \vdots \\ B_1(\omega_K) & S_1(1)(\omega_K) & \dots & S_N(1)(\omega_K) \end{pmatrix}$$

Theorem 4.11 - *Fair Price of a Contingent Claim*
Let X be an *Attainable Contingent Claim* and H be a *Replicating Portfolio* which generates X.

The value of portfolio H at time $t = 0$ (V_0) is the "fair-price" of the contingent claim X.

[53]Depends on model

Proof 4.12 - *Theorem 4.11*
Let X be an *Attainable Contingent Claim* and H be a *Replicating Portfolio* which generates X.

Let p be the fair price for X and assume, for the sake of contradiction, that p does not equal the value of H at time $t = 0$. This means we assuming that $p \neq V_0$.

We have two cases

Case 1 - $p > V_0$.
In this case, an *arbitrage opportunity* exists and can be exploited by doing the following

- At $t = 0$ - Short the *Contingent Claim* for p; buy portfolio H for V_0; and invest the difference $p - V_0 > 0$.
- At $t = 1$ - Our portfolio has the same value as X so we sell H to fulfil our short position on the contingent claim.

Our profit in this scenario is $(p - V_0)B_1 = (p - V_0)(1 + r) > 0$.

Case 2 - $p < V_0$.
In this case, an *arbitrage opportunity* exists and can be exploited by doing the following

- At $t = 0$ - Buy the *Contingent Claim* for p; buy portfolio $-H$[54] for $-V_0$; and invest the difference $V_0 - p > 0$.
- At $t = 1$ - Our portfolio has value $-X$ so we sell our *Contingent Claim* for X to cover the portfolio, fulfilling any short positions.

Our profit in this scenario is $(V_0 - p)B_1 = (V_0 - p)(1 + r) > 0$.

Hence, in all scenarios where $p \neq V_0$ an arbitrage opportunity exists. This means $p \neq V_0$ cannot be the fair price for X and thus $p = V_0$ is the fair price. □

Theorem 4.12 - *Risk-Neutral Valuation Principle*
The *Risk-Neutral Valuation Principle* gives the fair-price of an attainable contingent claim X at each time-period for models where no arbitrage opportunities exist. The principle is different under different models

Single-Period The fair-price is $p = \mathbb{E}_{\mathbb{Q}}[X/B_1]$ where $\mathbb{Q}$ is a risk-neutral probability measure.

Multi-Period The fair-price at time t is the time t value V_t of the portfolio which replicates X. Moreover,

$$V_t^* = \mathbb{E}_{\mathbb{Q}}[X/B_T|\mathcal{F}_t]$$

where $\mathbb{Q}$ is a martingale measure.

Continuous The fair-price at time t is the time t value V_t of the portfolio which replicates X. Moreover,

$$V_t^* = \mathbb{E}_{\mathbb{Q}}[X/B_T|\mathcal{F}_t]$$

where $\mathbb{Q}$ is an equivalent-martingale measure. At time $t = 0$, the value is

$$\pi_0 = V_0 = \mathbb{E}_{\mathbb{Q}}[X/B_T]$$

[54]Note this is equivalent to shorting portfolio H

Proof 4.13 - `Theorem 4.12` - *Single-Period Models*
Consider a *Single-Period Model* with no arbitrage opportunities and let X be an *Attainable Contingent Claim* under this model.

Here we derive the fair-price for X and show that time price is unique.

Suppose there exists two trading strategies $H, \hat{H}$ st $V_1 = \hat{V}_1 = X$ but $\hat{V}_0 \neq V_0$.

Let $\mathbb{Q}$ be a *Risk-Neutral Probability Measure* under this model. Then, by the *No-Arbitrage Principle* (`Theorem 3.4`), we have that for any trading strategy H $\mathbb{E}_{\mathbb{Q}}[G^*] = 0$. Thus we can deduce that

$$\begin{array}{rcl} V_0 & = & V_0^* \\ & = & \mathbb{E}_{\mathbb{Q}}[V_0^*] \\ & = & \mathbb{E}_{\mathbb{Q}}[V_1^* - G^*] \\ & = & \mathbb{E}_{\mathbb{Q}}[V_1^*] - \mathbb{E}_{\mathbb{Q}}[G^*] \\ & = & \mathbb{E}_{\mathbb{Q}}[V_1^*] - 0 \\ & = & \mathbb{E}_{\mathbb{Q}}[V_1/B_1] \end{array}$$

This shows that any trading strategy H with $V_1 = X$ (ie is worth X at time $t = 1$), has the following value at time $t = 0$

$$V_0 = \mathbb{E}_{\mathbb{Q}}[V_1/B_1] = \mathbb{E}_{\mathbb{Q}}[X/B_1]$$

This holds for all *Risk-Neutral Probability Measures* $\mathbb{Q}$, so the fair-price for X at time $t = 0$ is constant between different *Risk-Neutral Probability Measures*. Further, all trading strategies with the same value at time $t = 1$ have the same value at time $t = 0$ (and we have a formula for this value). □

Proof 4.14 - `Theorem 4.12` - *Multi-Period Models*
Let P_t denote the actual price of the *Contingent Claim* at time point t.

By `Theorem 4.2`, $P_t = V_t$ is the only possibility to avoid arbitrage opportunities.

Now, let $\mathbb{Q}$ be an arbitrary *Martingale Measure* then for every $t < T$ we have that

$$V_t^* := \mathbb{E}_{\mathbb{Q}}[V_t^* | \mathcal{F}_t]$$

as V_t^* is a *Martingale* by `Theorem 4.7`.

Moreover, since V_T^* is the discounted value of the portfolio which replicated X we have that

$$\mathbb{E}_{\mathbb{Q}}[V_T^* | \mathcal{F}_t] = \mathbb{E}_{\mathbb{Q}}[X/B_T | \mathcal{F}_t]$$

Thus $V_t^* = \mathbb{E}_{\mathbb{Q}}[X/B_T | \mathcal{F}_t]$ independent of the choice of *Martingale Measure* $\mathbb{Q}$. □

4.6 Complete Markets

Definition 4.21 - *Complete & Incomplete Markets*
A model of a market is said to be *Complete* if each *Contingent Claim* X there exists a *Trading Strategy* H which generates X.

Otherwise, the model is said to be in*complete*.

Remark 4.6 - *Checking if a Market is Complete*

We can check whether the model of a market is *Complete* by defining the following matrix A, and if A spans the same space as *Contigent Claims*[55] then the market is *Complete*.

$$A=\begin{pmatrix} B_1(\omega_1) & S_1(1)(\omega_1) & \dots & S_N(1)(\omega_1)\\ B_1(\omega_2) & S_1(1)(\omega_2) & \dots & S_N(1)(\omega_2)\\ \vdots & \vdots & \ddots & \vdots\\ B_1(\omega_K) & S_1(1)(\omega_K) & \dots & S_N(1)(\omega_K)\end{pmatrix}$$

Theorem 4.13 - *Complete Markets and* $\mathbb{Q}$
Consider a model with no *Arbitrage Opportunities*, then

> The model is *Complete* iff $\exists$ a unique *Risk-Neutral Probability Measure* (or *Martingale Measure*) $\mathbb{Q}$.

Proof 4.15 - `Theorem 4.13` - *Single-Period Model*
Consider a *Single-Period Model* with no *Arbitrage Opportunities* and let $\mathbb{M}$ denote the set of all *Risk-Neutral Probability Measures* for this model.

Since there are no arbitrage opportunities then $\mathbb{M}\neq\emptyset$.

As this theorem is "iff" I shall prove it in both directions separately

$\implies$ Assume, for the sake of contraction, that the model is *Complete* but $\mathbb{M}=\{\mathbb{Q},\hat{\mathbb{Q}}\}$ (ie contains two distinct elements).

Then $\exists\ \omega_k\in\Omega$ st $\mathbb{Q}(\omega_k)\neq\hat{\mathbb{Q}}(\omega_k)$. Consider the following *Contingent Claim* X

$$\begin{array}{rcl} X(\omega) &=& \begin{cases} B_1(\omega) & \text{if } \omega=\omega_k\\ 0 & \text{otherwise}\end{cases}\\ &=& B_1\mathbb{1}\{\omega=\omega_k\}\end{array}$$

Then

$$\begin{array}{rcl} \mathbb{E}_{\mathbb{Q}}[V_0]=\mathbb{E}_{\mathbb{Q}}[X/B_1] &=& \mathbb{E}_{\mathbb{Q}}[\mathbb{1}\{\omega=\omega_k\}]\\ &=& \mathbb{Q}(\{\omega_k\}))\\ &\neq& \mathbb{Q}(\{\omega_k\}) \text{ by def. } X\\ &=& \mathbb{E}_{\hat{\mathbb{Q}}}[\mathbb{1}\{\omega=\omega_k\}]\\ &=& \mathbb{E}_{\hat{\mathbb{Q}}}[X/B_1]=\mathbb{E}_{\hat{\mathbb{Q}}}[V_0]\\ \implies\ \mathbb{E}_{\mathbb{Q}}[V_0] &\neq& \mathbb{E}_{\hat{\mathbb{Q}}}[V_0]\end{array}$$

This contradicts `Proof 4.13` when we showed that if X is attainable then $\mathbb{E}_{\mathbb{Q}}[V_0]$ is the same for all $\mathbb{Q}\in\mathbb{M}$.

Thus, if the model is *Complete* then it has a unique *Risk-Neutral Probability Measure*.

$\impliedby$ Assume, for the sake of contradiction, that the model has a unique *Risk-Neutral Probability Measure* $\hat{\mathbb{Q}}$ but there exists a *Contingent Claim* X which is not *Attainable*.

Then, there does not exist a trading strategy H which solves $AH=X$.

By the *Separating Hyperplane Theorem* (`Theorem 4.3`) it follows that

$$\exists\ \pi\in\mathbb{R}^K \text{ st } \pi^TA=0^{[56]} \text{ and } \pi^TX>0$$

[55]This is done by determining whether $\text{rank}(A)=\dim(X)$.

Let $\lambda > 0$ be small enough that

$$\mathbb{Q}(\{\omega_j\}) := \hat{\mathbb{Q}}(\{\omega_j\}) + \lambda\pi_j \cdot B_1(\omega_j) > 0 \quad \forall\, j \in [1, K]$$

As A is defined st all the terms in its first column are B_1 and $\pi^T A = 0$, the $\mathbb{Q}$ defined above is a probability measure.

Moreover, for any *Discounted Price Process* $s^* = (S_1^*, \dots, S_N^*)$ and any $n \in [1, N]$ we have

$$\begin{array}{rcl}\mathbb{E}_{\mathbb{Q}}[S_n^*(1)] &=& \displaystyle\sum_{j=1}^K \frac{S_n(1)(\omega_j)}{B_1(\omega_j)} \cdot \mathbb{Q}(\{\omega_j\}) \\ &=& \displaystyle\sum_{j=1}^K \frac{S_n(1)(\omega_j)}{B_1(\omega_j)} \cdot \left(\hat{\mathbb{Q}}(\{\omega_j\}) + \lambda\pi_j B_1(\omega_j)\right) \\ &=& \displaystyle\sum_{j=1}^K \frac{S_n(1)(\omega_j)}{B_1(\omega_j)} \cdot \hat{\mathbb{Q}}(\{\omega_j\}) + \underbrace{\lambda\sum_{j=1}^K \pi_j S_n(1)(\omega_j)}_{=0} \\ &=& \displaystyle\sum_{j=1}^K \frac{S_n(1)(\omega_j)}{B_1(\omega_j)} \cdot \hat{\mathbb{Q}}(\{\omega_j\}) \\ &=& \displaystyle\sum_{j=1}^K S_n^*(1)(\omega_j)\hat{\mathbb{Q}}(\{\omega_j\}) \\ &=& \mathbb{E}_{\hat{\mathbb{Q}}}[S_n^*(1)] \\ &=& S_n^*[0]^{[57]}\end{array}$$

This shows that $\mathbb{Q}$ is a *Risk-Neutral Probability Measure* and so $\mathbb{Q} \in \mathbb{M}$, a contradiction to the uniqueness of $\hat{\mathbb{Q}}$.

If there is a unique *Risk-Neutral Probability Measure* for a model, then all *Contingent Claims* are attainable under the model.

This has proved the theorem in both directions. □

Proof 4.16 - `Theorem 4.13` *- Multi-Period Model*

- If the multi-period model is complete, for any claim X we can work backwards in time to compute the trading strategy that generates X. Hence, for each underlying single-period model the matrix A must have independent columns and the model is complete.
- Conversely, if every underlying single-period model is complete then the computational procedure for the multi-period model succeeds.

Therefore, completeness of the multi-period model is equivalent to completeness of all underlying single-period models.

In particular the multi-period model is complete iff each underlying model has a unique risk-neutral probability measure.

On the other hand, uniqueness of the martingale measure $\mathbb{Q}$ is equivalent to uniqueness of the risk-neutral probability measures of the underlying single-period models.

Obviously, the existence of several risk-neutral probability measures for a single-period model leads to several multi-period martingale measures.

[56]ie π is orthogonal to A.

[57]As $\hat{\mathbb{Q}}$ is a *Risk-Neutral Probability Measure*.

However, assume there are two multi-period martingale measures. Then the conditional probability must be different for at least one specific single-period model. □

4.7 Cox-Ross-Rubinstein Model

Definition 4.22 - *Cox-Ross-Rubinstein Model*
The *Cox-Ross-Rubinstein Model* is the special case of a multi-period model, defined by the following properties.

i). Risk-free constant interest rate r.

ii). A single risk security.

iii). Only two events can occur:

  (a) The price increases by a factor of u with probability p. ($S_{t+1} = uS_t$ with $u > 1$).
  (b) The price decreases by a factor of d with probability $1 - p$. ($S_{t+1} = dS_t$ with $d < 1$).

iv). Price process $S_t := S_0 u^{N_t} d^{t-N_t}$ where S_0 is the initial price and $\{N_t\}_{t\in\mathbb{N}}$ is a random walk process with parameter p

Note $S_0, p, u, d \in \mathbb{R}^{\geq 0}$ and $0 < d < 1 < u$.

Theorem 4.14 - *Arbitrage in Cox-Ross-Rubinstein Model*
The *Cox-Ross-Rubinstein Model* has no arbitrage opportunities iff $d < 1 + r < u$.

Remark 4.7 - *Cox-Ross-Rubinstein Model without Arbitrage Opportunities*
If the Cox-Ross-Rubinstein Model has no arbitrage opportunities then the model is complete and the unique *Martingale Measure* $\mathbb{Q}$ is defined by

$$\mathbb{Q}(\{\omega\}) = q^n(1-q)^{T-n} \text{ where } q = \frac{1+r-d}{u-d}$$

where $\omega \in \Omega$ is a state which corresponds to n up-steps and $(T - n)$ down-steps.

In particular, adding all contributions from states with exactly n up-steps and $(T-n)$ down-steps we get

$$\mathbb{Q}\left(S_t = S_0 u^n d^{T-n}\right) = \binom{t}{n} q^n (1-q)^{t-n} \text{ for } n = 0, \dots, t$$

Moreover, if X is a *Contingent Claim* of the form $X = g(S_T)$ for some real-valued function g, then the time $t = 0$ value of X is given by

$$V_0 = \frac{1}{(1+r)^T} \sum_{n=0}^{T} \binom{T}{n} q^n (1-q)^{T-n} g(S_0 u^n d^{T-n})$$

More generally, the value of contingent claim X at time-point t is

$$\Pi_t = \frac{1}{(1+r)^{T-t}} \sum_{n=0}^{T-t} \binom{T-t}{n} q^n (1-q)^{T-t-n} g(S_t U^n d^{T-t-n})$$

Proof 4.17 - *Theorem 4.14*

The *No-Arbitrage Theorem* (`Theorem 4.4`) shows the *Cox-Ross-Rubinstein Model* is free of arbitrage iff there is a *Martingale Measure* $\mathbb{Q}$ with

$$S_t = \frac{1}{1+r}\mathbb{E}_{\mathbb{Q}}[S_{t+1}|\mathcal{F}_t]\ \forall\ t < T$$

Let $q := \mathbb{Q}(X_{t+1} = 1) = \mathbb{Q}(S_{t+1}/S_t = u)$ which means that $\mathbb{Q}(S_{t+1}/S_t = d) = 1 - q$.

S_t is known at time-point t, so we can put it inside the conditional expectation wrt $\mathcal{F}_t$ in the expression for S_t above. This means we can divide by S_t and get

$$1 = \frac{1}{1+r}\mathbb{E}_{\mathbb{Q}}\left[S_{t+1}/S_t|\mathcal{F}_t\right] = \frac{1}{1+r}\big(uq + d(1-q)\big)$$

By rearranging, we see the *No-Arbitrage Principle* is satisfied iff $qu + (1-q)d = 1 + r$. Thus

$$q = \frac{1+r-d}{u-d}$$

By its definition, $\mathbb{Q}$ must be positive everywhere meaning $q \in (0,1)$. Equivalently $d < 1 + r < u$ must be satisfied.

We can now state the form of the *Martingale Measure* $\mathbb{Q}$ by multiplying the conditional probabilities along the paths that lead to each state ω_i. This is made easier since the *Cox-Ross-Rubinstein Model* assumes that the probability of each movement is constant at all time-period.

By the uniqueness of q we can determine that the *Cox-Ross-Rubinstein Model* is complete.

The expression for $\mathbb{Q}\left(S_t = S_0 u^n d^{T-n}\right)$ is straightforward.

The time $t = 0$ value of the contingent claim X is calculated as the expectation wrt $\mathbb{Q}$.

We think of $\Pi(t)$ as a time-shifted version of the formula for the time $t = 0$ price[58], replacing T by $(T - t)$ and S_0 by S_t. $\square$

Proposition 4.4 - *Value of European Call Option in Cox-Ross-Rubinstein Model*
Consider a *Cox-Ross-Rubinstein Model* with T periods, starting price S_0, interest rate r and parameters d, u.

Then the time t price of a *European Call Option* with exercise price K is

$$\Pi_K(t) = \frac{1}{(1+r)^{T-t}}\sum_{n=0}^{T-t}\binom{(T-t)}{n}q^n(1-q)^{T-t-n}\left\{S_0u^nd^{T-t-n} - K\right\}_+$$

where $q = \frac{1+r-d}{u-d}$ as usual.

4.7.1 Black-Scholes Formula

Theorem 4.15 - *Black-Scholes Formula*
Consider a *European Call Option* with exercise price K and matures at time U.

Let $\Pi_K^{(t)}(0)$ denote its fair-price in a *Cox-Ross-Rubinstein Model* with $T+1$ time-points $\left\{0, \frac{U}{T}, \dots, U\right\}$, constant interest rate $r_T = e^{-\frac{rU}{T}} - 1$ and $u_T = e^{\sigma\sqrt{U/T}} = \frac{1}{d_T}$. Then

$$\lim_{T\to\infty}\Pi_K^{(T)}(0) = \Pi_K^{BS}(0)$$

[58] Effectively assume that the model starts at time t and runs for $T - t$ steps.

where

$$\begin{aligned}\Pi_K^{BS}(0) &= S_0\Phi\big(d_1(S_0,U)\big) - Ke^{-rU}\big(d_2(S_0,U)\big)\\ \text{with}\quad d_1(s,u) &= \frac{\ln(s/K)+\big(r+(\sigma^2/2)\big)U}{\sigma\sqrt{U}}\\ d_2(s,u) &= \frac{\ln(s/K)+\big(r-(\sigma^2/2)\big)U}{\sigma\sqrt{U}}\end{aligned}$$

Φ is the CDF of the standard Normal distribution.

Proof 4.18 - *`Theorem 4.15`*

Let $\alpha_T := \min\left\{n : S_0u_T^nd_T^{T-n} > l\right\}$. This allows us to consider only terms in `Proposition 4.4` which are positive.

We rewrite the time $t=0$ price $\Pi_K^{(T)}(0)$ in the T^{th} *Cox–Ross-Rubinstein Model* as

$$\begin{aligned}\Pi_K^{(T)}(0) &= (1+r_T)^{-T}\sum_{n=\alpha_T}^T\binom{T}{n}q_T^n(1-q_T)^{T-n}(S_0u_T^nd_T^{T-n}-K)\\ &= S_0\left(\sum_{n=\alpha_T}^T\binom{T}{n}\left(\frac{q_Tu_T}{1+r_T}\right)^n\left(\frac{(1-q_T)d_T}{1+r_T}\right)^{T-n}\right)\\ &- (1+r_T)^{-T}K\left(\sum_{n=\alpha_T}^T\binom{T}{n}q_T^n(1-q_T)^{T-n}\right) \qquad [1]\end{aligned}$$

We can identify terms involved in the second sum as the density of a $\text{Bin}(T,q_T)$ distribution. For notational ease we define

$$\hat{q}_T = \frac{q_Tu_T}{1+r_T}$$

This $\hat{q}_T$ is a probability[59] because

$$\begin{aligned}0<\hat{q}_T & \\ &= \frac{q_Tu_T}{1+r_T}\\ &= \frac{\left(\frac{1+r_T-d_T}{u_T-d_T}\right)\cdot u_T}{q+r_T}\\ &= \frac{u_T-\frac{d_Tu_T}{1+r_T}}{u_T-d_T}\\ &< \frac{u_T-\frac{d_Tu_T}{u_T}}{u_T-d_T} \text{ since } u_T>1+r_T\\ &= 1\end{aligned}$$

Moreover, we see from the definition of q_T that

$$\begin{aligned}1-\hat{q}_T &= \frac{1+r_T-q_Tu_T}{1+r_T}\\ &= \frac{1+r_T-(1+r_T-d_T)-q_Td_T}{1+r_T}\\ &= \frac{(1-q_T)d_T}{1+r_T}\end{aligned}$$

Thus we identify the first sum in [1] as the density of $Bin(T,\hat{q}_T)$ distribution.

[59]ie $\hat{q}_T \in (0,1)$

Now, let $Y_T \sim \text{Bin}(T, q_T)$ and $\hat{Y}_T \sim \text{Bin}(T, \hat{q}_T)$. We can rewrite [1] as

$$\Pi_K^{(T)}(0) = S_0\mathbb{P}\left(\hat{Y} > d_T - 1\right) - K(1+r_T)^{-T}\mathbb{P}(Y_T > d_T - 1)$$

Thus we need to show that

i). $\lim_{T\to\infty} \mathbb{P}\left(\hat{Y}_T > \alpha_T - 1\right) = \Phi\left(d_1(S_0, U)\right)$. **Proof 4.19 -** *This result is not proved in detail*

ii). $\lim_{T\to\infty} \mathbb{P}\left(Y_T > \alpha_T - 1\right) = \Phi\left(d_2(S_0, U)\right)$.

Consider ii) first

$$\mathbb{P}(Y_T > \alpha_T - 1) = \mathbb{P}\left(\frac{Y_T - Tq_t}{\sqrt{Tq_T(1-q_T)}} > \frac{d_T - 1 - Tq_T}{\sqrt{Tq_T(1-q_T)}}\right)$$

We want to use `Theorem 2.1` and therefore determine the convergence of the term $\dfrac{\alpha_T - 1 - Tq_T}{\sqrt{Tq_T(1-q_T)}}$.
Note that we defined

$$u_T := \exp\{\sigma\sqrt{U/T}\} = 1/d_T \quad \text{and} \quad r_T = \exp\{rU/T\} - 1$$

Using a *Taylor Decomposition* we see that

$$\begin{aligned}
q_T &= \frac{1 + r_T - d_T}{u_T - d_T} \\
&= \frac{\exp\{rU/T\} - e^{-\sqrt{U/T}}}{e^{\sigma\sqrt{U/T}} - e^{-\sigma\sqrt{U/T}}} \\
&= \frac{\{1 + (rU/T) + o(1/T)\} + \{-1\sigma\sqrt{U/T} - \frac{1}{2}\sigma^2(U/T) - o(1/T)\}}{\{1 + (\sigma\sqrt{U/T}) + \frac{1}{2}\sigma^2(U/T) + o(1/T)\} + \{-1 + \sigma\sqrt{U/T} - \frac{1}{2}\sigma^2(U/T) - o(1/T)\}} \\
&= \frac{(rU/T) + \sigma\sqrt{U/T} - \frac{1}{2}\sigma^2(U/T) + o(1/T)}{2\sigma\sqrt{U/T} + o(1/T)} \\
&= \frac{1}{2}\left\{\frac{r(U/T)}{\sigma\sqrt{U/T}} + \frac{\sigma\sqrt{U/T}}{\sigma\sqrt{U/T}} - \frac{(1/2)\sigma^2(U/T)}{\sigma\sqrt{U/T}}\right\} + o(1/\sqrt{T}) \\
&= \frac{1}{2}\left(\left(\frac{r}{\sigma} - \frac{\sigma}{2}\right)\sqrt{U/T} + 1\right) + o(1/\sqrt{T})
\end{aligned}$$

Thus we obtain the limiting relations

$$\lim_{T\to\infty} q_T = \frac{1}{2} \text{ and } \lim_{T\to\infty}(1 - 2q_T)\sqrt{UT} = -U\left(\frac{r}{\sigma} - \frac{\sigma}{2}\right)$$

Finally, we see from the definition of α_T that for some $|\gamma_T| < 1$

$$\begin{aligned}
\alpha_T &= \frac{\ln(K/S_0 d_T^T)}{\ln(u_T/d_T)} + \gamma_T \\
&= \frac{\ln(k/S_0) - T\ln(d_T)}{\ln(u_T^2)} + \gamma_T \\
&= \frac{\ln(K/S_0) + T\sigma\sqrt{U/T}}{2\sigma\sqrt{U/T}} + \gamma_T
\end{aligned}$$

Hence

$$\begin{aligned}
&\lim_{T\to\infty}\frac{\alpha_T-1-Tq_T}{\sqrt{Tq_T(1-q_T)}}\\
=&\lim_{T\to\infty}\frac{\left(\frac{\ln(K/S_0)+T\sigma\sqrt{U/T}}{2\sigma\sqrt{U/T}}+\gamma_T\right)-1-Tq_T}{\sqrt{Tq_T(1-q_T)}}\\
=&\lim_{T\to\infty}\frac{\ln(K/S_0)+\sigma\sqrt{U_T}(1-2q_T)}{2\sigma\sqrt{Uq_T(1-q_T)}}\text{[60]}\\
=&\frac{\ln(K/S_0)-(r-\sigma^2/2)U}{\sigma\sqrt{U}}\\
=&-d_2(S_0,U)
\end{aligned}$$

By `Theorem 2.1` we conclude that

$$\lim_{T\to\infty}\mathbb{P}(Y_T>\alpha_T-1)=1-\Phi(-d_2(S_0,U))=\Phi(d_2(S_0,u))$$

This proves result ii).

To prove result i) a very similar argument is used.

Using the limiting relations

$$\lim_{T\to\infty}\hat{q}_T=\frac{1}{2}\text{ and }\lim_{T\to\infty}(1-2\hat{q}_T)\sqrt{UT}=-U\left(\frac{r}{\sigma}+\frac{\sigma}{2}\right)$$

we can conlude that

$$\lim_{T\to\infty}\frac{\alpha_T-1-T\hat{q}_T}{\sqrt{T\hat{q}_T(1-\hat{q}_T)}}=\frac{\ln(k/S_0)0(r+\sigma^2/2)U}{\sigma\sqrt{U}=-d_1(S_0,U)}$$

Hence

$$\lim_{T\to\infty}\mathbb{P}(\hat{Y}_T>\alpha_T-1)=1-\Phi(-d_1(S_0,U))=\Phi(d_1(S_0,U))$$

□

5 Continuous-Time

5.1 Stochastic Integration

Definition 5.1 - *Stochastic Integral* $\{I_t\}_t$

Let $\{X_t\}_{t\in[0,T]}$ be a simple stochastic process (**Definition 2.5**) and $\{W_t\}_{t\in[0,T]}$ be standard Brownian motion.

The *Stochastic Integral* I_t wrt $\{X_t\}_t$ is defined as

$$\begin{aligned}
I_t(X) &:= \int_0^t X_t dW\\
&= \sum_{k=0}^{n-1}\xi_k\cdot\left(W_{\min(t,t_{k+1})}-W_{\min(t,t_k)}\right)
\end{aligned}$$

Remark 5.1 - *Stochastic Integrals are Stochastic Processes*

Remark 5.2 - *Expanding a Stochastic Integral*

[60] γ_T disappears due to taking the limit.

If $t \in [t_k, t_{k+1}]$ for some $k \in [0, n-1]$ then

$$I_t(X) = \left\{\sum_{i=0}^{k-1} \xi_i \cdot (W_{t_{i+1}} - W_{t_i})\right\} + \xi_k \cdot (W_t - W_{t-k})$$

Example 5.1 - *Stochastic Integrals*
Consider the simple stochastic process $\{X_t\}_{t\in[0,T]}$

i). Suppose each $X_t = c$. This is arguably the simplest random process possible and means $n = 1$ in the partition. Then
$$I_t(X) = cW_t$$

ii). Suppose each $X_t = Y$ where Y is some random variable. This means $n = 1$ in the partition and
$$I_t(X) = YW_t$$

iii). Suppose $X_t = c$ for $t \leq 1/2$ and $X_t = d$ for $t > 1/2$. Then

$$I_t(X) = \begin{cases} cW_t & \text{if } t < 1/2 \\ cW_{1/2} + d(W_t - W_{1/2}) & \text{if } t \geq 1/2 \end{cases}$$

Theorem 5.1 - *Properties of Simple Stochastic Processes*
Let $X := \{X_t\}_{t\in[0,T]}$ be a *Simple Stochastic Process*. Then

i). $\mathbb{E}[I_t(X)] = 0$.

ii). The *Stochastic Process* satisfies Itô's isometry.

$$\mathbb{E}\left[(I_t(X))^2\right] = \int_0^t \mathbb{E}[X_s^2 ds]$$

iii). $I_t(X)$ is a continuous *Martingale* wrt the *Natural Brownian Motion Filtration* $\mathcal{F}_t$ for all $0 \leq s \leq t \leq T$.
$$\mathbb{E}\left[I_t(X)|\mathcal{F}_s\right] = I_s(X)$$

iv). *Linearity* - $I_t(aX + bY) = aI_t(X) + bI_t(Y)$ where Y is another *Simple Stochastic Process* and $a, b \in \mathbb{R}$.

v). The stochastic process $I_t(X)$ has continuous sample paths.

Proof 5.1 - *Theorem 5.1*

i). Note that since ξ_i is $\mathcal{F}_{t_i}$ measurable, then ξ_i & $(W_{t_{i+1}} - W_{t_i})$ are independent for all i. Also $\mathbb{E}[W_{t_{i+1}} - W_{t_i}] = 0$, thus

$$\mathbb{E}[\xi_i(W_{t_{i+1}} - W_{t_i})] = \mathbb{E}[\xi_i]\mathbb{E}[W_{t_{i+1}} - W_{t_i}] = 0$$

Meaning
$$\mathbb{E}[I_t(X)] = 0$$

ii). Consider a partition of $[0,t]$ st $0=t_0<\dots<t_k=t$. Then

$$\mathbb{E}[(I_t(X))^2]=\sum_{i=0}^{k-1}\sum_{j=0}^{k-1}\mathbb{E}\left[\xi_i(W_{t_{i+1}}-W_{t_i})\xi_j(W_{t_{j+1}}-W_{t_j})\right]$$

If $i>j$ then $(W_{t_{i+1}}-W_{t_i})$ is independent of all the other factors and has expectation 0

$$\begin{array}{rl}\mathbb{E}\left[\xi_i(W_{t_{i+1}}-W_{t_i})\xi_j(W_{t_{j+1}}-W_{t_j})\right] & \\ = & \underbrace{\mathbb{E}[\underbrace{W_{t_{i+1}}-W_{t_i}}_{\sim N(0,t_{i+1}-t_i)}]}_{=0}\mathbb{E}[\xi_i\xi_j(W_{t_{j+1}}-W_{t_j})]\end{array}$$

So all terms with $i>j$ and, similarly $i<j$, disappear and we conclude that

$$\begin{array}{rcl}\mathbb{E}[I_t(X)^2] & = & \sum_{i=0}^{k-1}\mathbb{E}[\xi_i(W_{t_{i+1}}-W_{t_i})\xi_i(W_{t_{i+1}}-W_{t_i})]\\ & = & \sum_{i=0}^{k-1}\mathbb{E}[\xi_i^2]\mathbb{E}[(W_{t_{i+1}}-W_{t_i})^2]\\ & = & \sum_{i=0}^{k-1}\mathbb{E}[\xi_i^2](t_{i+1}-t_i)\end{array}$$

The last step comes from considering the variance of $(W_{t_{i+1}}-W_{t_i})$.

The RHS is the usual Riemann Integral $\displaystyle\int_0^T f(s)ds$ of the step function $f(x)=\mathbb{E}[X_s^2]$ which coincides with $\mathbb{E}[\xi^2]$ for $s\in[t_i,t_{i+1}]$.

iii). Adaptedness of $I(X)$ follows since at time t all ξ_i and $(W_{t_{i+1}}-W_{t_i})$ contributing to $I_t(X)$ are functions of *Brownian Motion* up to time t.

The condition $\mathbb{E}\left[|I_t(X)|\right]$ follows from the isometry property ii).

It remains to show that $\mathbb{E}\left[I_t(X)|\mathcal{F}_s\right]$ for all $s<t$.

First, assume that $s<t$ and $s,t\in[t_k,t_{k+1}]$. Notice that

$$\begin{array}{rcl}I_t(X) & = & I_{t_k}(X)+\xi_k(W_t-W_{t_k})\\ I_s(X) & = & I_{t_k}(X)+\xi_k(W_s-W_{t_k})\end{array}$$

Hence $I_t(X)=I_s(X)+\xi_k(W_t-W_s)$ where $I_s(X)$ and ξ_k are known at time s, and (W_t-W_s) is independent of $\mathcal{F}_s$ and has mean zero. Hence

$$\mathbb{E}\left[I_t(X)|\mathcal{F}_s\right]=I_s(X)+\xi_k\underbrace{\mathbb{E}\left[W_t-W_s\right]}_{=0}=I_s(X)$$

The case where $s<t_k<t$ can be handled analogously.

iv). Assume that the simple process X is defined using the partition

$$0=t_0<t_1<\dots<t_n=T$$

and Y uses the partition

$$0=s_0<s_1<\dots<s_m=T$$

Note that these partitions can be different lengths and take different values.

Consider the joint-partition

$$0 = u_0 < u_1 < \cdots < u_l = T$$

which combines all the points from the other partitions, ensuring the correct ordering of values. The values of $I_t(X)$ and $I_t(Y)$ wrt the finer partition $\{u_0, \dots, u_l\}$ remain the same.

Linearity follows from the linearity of the underlying sums.

v). This follows from the definition of $I_t(X)$ since

$$I_t(X) = I_{t_{k-1}}(X) + \xi_k(W_t - W_{t_{k-1}}) \; \forall \; t \in [t_{k-1}, t_k]$$

The only unfixed term in this expression is the *Brownian Motion* W_t. We know that *Brownian Motion* has continuous sample paths, thus $I_t(X)$ has continuous sample paths.

Definition 5.2 - *Itô Stochastic Integral* $I_t(X)$
Let $\{X_t\}_{t\in[0,T]}$ be a *Stochastic Process* which is adapted to *Brownian Motion*[61].

Itô's Stochastic Integral. The *Itô Integral* $\{I_t(X)\}_t$ of X wrt W is denoted as

$$I_t(X) = \int_0^t X_s dW_s$$

Remark 5.3 - *Rule of Thumb for Itô Stochastic Integral*
The *Itô Stochastic Integral* $\{I_t(X)\}_{t\in[0,T]}$ constitute a *Stochastic Process*. For a given partition $0 = t_0 < t_1 < \cdots < t_n = T$ and $t \in [t_k, t_{k+1}]$ the random variable $I_t(X)$ is approximately

$$I_t(X) \approx \sum_{i=0}^{k-1} \left\{ X_{t_i}(W_{t_{i-1}} - W_{t_i}) \right\} + X_{t_k}(W_t - W_{t_k})$$

This approximation is closer to the value of $I_t(X)$ the denser the partition is in $[0, T]$.

Theorem 5.2 - *Properties of Itô Stochastic Integral*[62]
Proofs are out of scope of this course! Let $\{X_t\}_{t\in[0,T]}$ be a *Stochastic Process* which is adapted to *Brownian Motion*. Then

i). $\mathbb{E}[I_t(X)] = 0$.

ii). $I_t(X)$ satisfies the Itô Isometry Property (`Theorem 5.1`).

iii). $I_t(X)$ is a *Martingale* wrt the *Natural Brownian Filtration*.

iv). $I_t(X)$ is *Linear*.

v). $I_t(X)$ has continuous sample paths.

[61] X_t is a function of W_s for $s \leq t$ which satisfies that $\int_0^T \mathbb{E}[X_t^2] < \infty$
[62] The proof for these properties is not covered in this course.

5.2 Itô's Lemma

Definition 5.3 - *Itô Process*
Let $\{X_t\}_{t\in[0,T]}$ be a stochastic process. $\{X_t\}$ is an *Itô Process* when X_t takes the form (5).

$$X_t = X_0 + \int_0^t b_u du + \int_0^t \sigma_u dW_u \tag{5}$$

where both b, σ are functions which are adapted to Brownian motion.
Process $\{X_t\}$ has a stochastic differential

$$dX_t = b_t d_t + \sigma_t dW_t$$

Theorem 5.3 - *Itô's Lemma - Special Case*
This is a special case of Itô's Lemma.
Let $f(x)$ be a twice continuously differentiable function. Then for any $t > 0$

$$f(W_t) - f(W_0) = \underbrace{\int_0^t f'(W_u)dW_u}_{\text{Standard Integral}} + \underbrace{\frac{1}{2}\int_0^t f''(W_u)du}_{\text{Stochastic Integral}}$$

or in differential form

$$df(W_t) = f'(W_t)dW_t + \underbrace{\frac{1}{2}f''(W_t)dt}_{\text{Itô's Correction Term}}$$

Proof 5.2 - *Theorem 5.3*
Consider a partition $\{t_0, \dots, t_n\}$ of $[0, t]$ with $0 = t_0 < \dots < t_n = t$.
By Taylor's formula we obtain

$$\begin{array}{rcl} f(W_t) - f(W_0) & = & \sum_{i=0}^{n-1} f(W_{t_{i+1}}) - f(W_{t_i}) \\ & = & \sum_{i=0}^{n-1} f'(W_{t_i})(W_{t_{i+1}} - w_{t_i}) \\ & + & \frac{1}{2}\sum_{i=0}^{n-1} f''(W_{t_i} + \theta_i(W_{t_{i+1}} - W_{t_i}))(W_{t_{i+1}} - w_{t_i})^2 \text{ with } \theta_i \in (0,1) \end{array}$$

The first sum is an approximating sequence of a stochastic integral. Indeed, we find

$$\sum_{i=0}^{n-1} f'(W_{t_i})(W_{t_{i+1}} - W_{t_i}) \overset{n\to\infty}{\longrightarrow} \int_0^t f'(W_u)dW_u$$

We also know that

$$\lim_{n\to\infty}\sum_{i=0}^{n-1}(W_{t_{i+1}} - W_{t_i})^2 = t$$

and with a little more effort we can prove that[63]

$$\lim_{n\to\infty}\sum_{i=0}^{n-1} f''(W_{t_i} + \theta_i(W_{t_{i+1}} - W_{t_i}))(W_{t_{i+1}} - W_{t_i})^2 = \int_0^t f''(W_u)du$$

[63] Proof is beyond scope of course. Won't be asked to reproduce this step.

□

Theorem 5.4 - *Itô's Lemma - More General Case*
This is a more general case of Itô's Lemma than `Theorem 5.3`*, but not as general as* `Theorem 5.5`*.*

Let $f(t,x)$ be a function which is continuously differentiable once in its first argument (the time parameter t) and twice in its second argument x. Then

$$f(t,W_t)-f(0,W_0)=\int_0^t f_t(u,W_u)+\frac{1}{2}f_{xx}(u,W_u)du+\int_0^t f_x(u,W_i)dW_u$$

where $f_t:=\frac{\partial f}{\partial t},\ f_{xx}:=\frac{\partial^2 f}{\partial x^2}$ and $f_x:=\frac{\partial f}{\partial x}$.
Or, in differentiable form

$$df(t,W_t)=(f_t(t,W_t)+\frac{1}{2}f_{xx}(t,W_t))dt+f_x(t,W_t)dW_t$$

Proof 5.3 - `Theorem 5.4`
By the Taylor expansion of a smooth function of several variables we get for t close to t_0 that

$$\begin{array}{rcl} f(t,W_t) & = & f(t_0,W_{t_0})+(t-t_0)f_t(t_0,W_{t_0})+(W_t-W_{t_0})f_x(t_0,W_{t_0}) \\ & + & \frac{1}{2}(t-t_0)^2 f_{tt}(t_0,W_{t_0})+\frac{1}{2}(W_t-W_{t_0})^2 f_{xx}(t_0,W_{t_0}) \\ & + & (t-T_0)(W_t-W_{t_0})f_{tx}(t_0,W_{t_0})+\text{Higher order terms} \end{array}$$

This can be written symbolically as

$$df=f_t dt+fxdW+\frac{1}{2}f_{tt}(dt)^2+f_{tx}dtdW+\frac{1}{2}f_{xx}(dW)^2+\dots$$

Note that $(dt)^2=0$. Now using the formal multiplication rules

$$dt\cdot dt=0 \quad dt\cdot dW=0 \quad dW\cdot dW=dt$$

We get

$$df=f_t d_t+f_x dW+\frac{1}{2}f_{xx}dt=(f_t+\frac{1}{2}f_{xx})df+f_x dW$$

□

Theorem 5.5 - *Itô's Lemma - Most General*
Let X be an *Itô Process* and $f(t,x)$ be a function whose second order partial derivatives are continuous. Then for any $t>0$

$$f(t,X_t)-f(0,X_0)=\int_0^t f_t(u,X_u)+b_u f_x(u,X_u)+\frac{1}{2}\sigma_u^2 f_{xx}(u,X_u)du+\int_0^t \sigma_u f_x(u,X_u)dW_u$$

Or in differential form

$$df=\left(f_t+b_t f_X+\frac{1}{2}\sigma_t^2 f_{xx}\right)dt+\sigma_t f_x dW_t$$

Proof 5.4 - `Theorem 5.5`

We proceed as in the preceding version of Itô's formula and consider a Taylor expansion of $f(t, X_t)$ which is in differential notation

$$df = f_t dt + f_x dX + \frac{1}{2} f_{tt}(dt)^2 + f_{tx} dt dx + \frac{1}{2} f_{xx}(dX)^2 + \text{High order terms}$$

Now we substitute $dX = bdt + \sigma dW$ and obtain

$$df = f_t dt + f_x(bd_t + \sigma dW) + \tfrac{1}{2} f_{tt}(dt)^2 + f_{tx} dt(bdt + \sigma dW) + \tfrac{1}{2} f_{xx}(bd_t + \sigma dW) + \text{High order terms}$$

Again, neglecting all $(ft)^2)$ and $dtdW$ terms as well as the high order terms we obtain

$$\begin{aligned} df &= f_t d_t + f_x(bd_t + \sigma dW) + \tfrac{1}{2} f_{xx}(\sigma dW)^2 \\ &= (f_t + f_x b + \tfrac{1}{2}\sigma^2 f_{xx})dt + f_x \sigma dW \end{aligned}$$

$\square$

Theorem 5.6 - *Product Rule for Stochastic Calculus*
We can derive the produce rule for stochastic calculus from `Theorem 5.5`.

Suppose two process X_t, Y_t are adapated to the same Brownian motion

$$\begin{aligned} dX_t &= \sigma_t dW_t + \mu_t dt \\ dY_t &= \rho_t dW_t + \nu_t dt \end{aligned}$$

Then

$$d(X_t Y_t) = X_t dY_t + Y_t dX_t + \sigma_t \rho_t dt$$

Proof 5.5 - `Theorem 5.6`
Since $d(X_t + Y_t) = (\sigma_t + \rho_t)dW_t + (\mu_t + \sigma_t)dt$, using `Theorem 5.5` applied to $f(t, x) = x^2$ we obtain that

$$\begin{aligned} d(X_t^2) &= (2\mu_t X_t + \sigma_t^2)dt + 2\sigma_t X_t dW_t \\ d(Y_t^2) &= (2u\nu_t Y_t + \rho_t^2)dt + 2\rho_t Y_t dW_t \\ d((X_t + Y_t)^2) &= (2(\mu_t + \nu_t)(X_t + Y_t) + (\sigma_t + \rho_t)^2)dt \\ &+ 2(\sigma_t + \rho_t)(X_t + Y_t)dW_t \end{aligned}$$

Subtracting, the result follows since

$$\begin{aligned} 2d(X_t Y_t) = d((X_t + Y_t)^2 - X_t^2 - Y_t^2) \\ &= 2(\mu_t Y_t + \nu_t X_t)dt + 2\sigma_t \rho_t dt + 2(\sigma_t Y_t + \rho_t X_t)dW_t \\ &= 2Y_t dX_t + 2X_t dY_t + 2\sigma_t \rho_t dt \end{aligned}$$

$\square$

5.3 Continuous-Time Model

Definition 5.4 - *Continuous-Time Financial Model*
The *Continuous-Time Financial Model* has the following components

- Initial date $t = 0$.
- Terminal date $t = T \in \mathbb{N}$.
- Trading interval $[0, T]$ with trading allowed at any time-point $t \in [0, T]$.

- A probability space $(\Omega, \mathcal{F}, \mathbb{P})$.
- Price, Value, Gains and Bank Account Process from **Section 4.1** over continuous time-interval $[0, T]$.

Remark 5.4 - *Discounted Gains process in Multi-Period vs continuous Model*
In the multi-period model the discounted gains process is always a martingale. This is not the case for the continuous model. Rather, we restrict ourselves to the subclass of *Admissible Trading Strategies* (**Definition 5.6**).

Definition 5.5 - *Equivalent Martingale Measure*
An *Equivalent Martingale Measure* is a probability measure $\mathbb{Q}$ st

i). $\mathbb{Q}$ is equivalent to $\mathbb{P}$.
$$\mathbb{P}(A) = 0 \text{ \underline{iff} } \mathbb{Q}(A) = 0$$

ii). The discounted price process S_n^* is a martingale under $\mathbb{Q}$ for all $n = 1, \dots, N$.

Definition 5.6 - *Admissible Trading Strategy*
A self-financing trading strategy H is called *Admissible* wrt to an *Equivalent Martingale Measure* $\mathbb{Q}$ if the discounted gains process $G^*(t)$ is a martingale under $\mathbb{Q}$.

Theorem 5.7 - *Conditions for discount value process to be a Martingale*
For each *Admissible Trading Strategy* H and each equivalent martingale measure $\mathbb{Q}$, the discounted value process $V^*(t)$ is a Martingale wrt $\mathbb{Q}$.

Proof 5.6 - *Theorem 5.7*
Immediate from **Theorem 4.1**.

Definition 5.7 - *Arbitrage Opportunity - Continuous-Time Model*
Let H be a trading strategy for a continuous-time financial model. H exploits an *Arbitrage Opportunity* if it has the following properties

i). $V_0 = 0$.

ii). $V_T \geq 0$.

iii). $\mathbb{E}[V_T] > 0$ and

iv). H is self-financing.

Theorem 5.8 - *No-Admissible Arbitrage Theorem*
Assume there exists an equivalent Martingale Measure $\mathbb{Q}$ for the market model.

Then the market model contains no *Admissible Arbitrage Opportunities*.

Proof 5.7 - *Theorem 5.8*
Assume that H is an admissible arbitrage opportunity. Then

$$V_0^* = 0, \; V_T^* \geq 0 \text{ and } \mathbb{E}\left[V_T^*\right] > 0$$

Since $\mathbb{Q}$ is equivalent to $\mathbb{P}$, we also have $\mathbb{E}_{\mathbb{Q}}[V_T^*] > 0$.

On the other hand V_T^* is a martingale measure, using `Theorem ??.7`.

That is

$$\mathbb{E}_{\mathbb{Q}}[V_t^*|\mathcal{F}_u] = V_u^* \text{ for all } u \leq t \leq T$$

This implies

$$\mathbb{E}_{\mathbb{Q}}[V_T^*] = V_0^* = 0$$

This is a contradiction. Thus there cannot be an admissible arbitrage opportunity. □

Proof 5.8 - `Theorem 5.12` *- Continuous Time*
Let H be an arbitrary replicating portfolio for X and P_t be the time t price of claim X.

If P_t does not match the value of the strategy H at time t, then there exists arbitrage opportunities as described in the proof of the Risk-neutral valuation principle.

Moreover, by `Theorem 5.7` the discounted value process V_t^* is a martingale under $\mathbb{Q}$ and thus

$$V_t^* = \mathbb{E}_{\mathbb{Q}}[V_T^*|\mathcal{F}_t] = \mathbb{E}_{\mathbb{Q}}\left[V_T/B_T|\mathcal{F}_t\right] = \mathbb{E}_{\mathbb{Q}}\left[X/B_T|\mathcal{F}_t\right]$$

Note that the last expression is independent of the particular replicating strategy. □

5.4 The Black-Scholes Model

Definition 5.8 - *Black-Scholes Model*
The *Black-Scholes Model* is a continuous-time financial market model with two assets:

- A riskless bond with a constant interest rate r and corresponding bank account process

$$B_t = e^{rt}$$

- A risky asset with price process $\{S_t\}_{t\in[0<T]}$ modelled as *Geometric Brownian Motion*

$$S_t = S_0 e^{\sigma W_t + t\cdot\left(\mu - \frac{\sigma^2}{2}\right)}$$

  where W_t is Brownian motion, μ is drift and σ is volatility.

Brownian motion has the characteristic randomness of share prices and Geometric Brownian motion ensures prices do not become negative.

Remark 5.5 - *Assumptions of Black-Scholes Framework*
In the *Black-Scholes Framework* the following assumptions are made

i). No arbitrage opportunities exist. This is a standard assumption as when an arbitrage opportunity exists then it will disappear quickly due to arbitrageurs.

ii). No transactions costs. (Relaxable).

iii). Unlimited short-selling is available, at not cost.

iv). Constant risk-free interest rate. (Relaxable).

v). The underlying price follows a geometric Brownian motion. In reality, prices can jump which does not fit with geometric Brownian motion. Some extensions of the Black-Scholes model account have jump components.

vi). Constant volatility. In reality, volatility is not known nor constant.

Remark 5.6 - *Estimating Volatility*
Estimating the volatility σ is hard. There are two approaches:

i). *Statistical Approach.* Estimate volatility σ by looking at historical prices S and deduce the historical volatility. This is naturally a poor approach as there is no reason for the volatility over the past to be a good estimator of future volatility. Also, how far back do you want to consider.

ii). *Market-Based Approach.* Extract information of volatility from traded instruments. Volatility obtained in this way is the market view of the volatility, called "Implied Volatility". The Black-Scholes formula is numerically inverted to find the implied volatility of a quoted price.

Moreover, the Black-Scholes model assumes volatility to be constant but in practice there is a "Volatility Smile"[64].

Proposition 5.1 - *Differential Equations of Black-Scholes Model*
In the *Black-Scholes Model* the following differential equations hold

- Of the bond price $dB_t = rB_t dt$.
- Of the price process $dS_t = \mu S_t dt + \sigma S_t dW_t$ (A stochastic differential equation).

Proof 5.9 - *5.1*
(First part proved in **Example 4.5** of long notes).

The function $f(t, W_t) = S_0 \cdot \exp\left\{t \cdot \left(\mu - \frac{\sigma^2}{2}\right) + \sigma W_t\right\}$ has the following partial derivatives

$$f_t(t, W_t) = (\mu - \frac{\sigma^2}{2})S_t \quad f_X(t, W_t) = \sigma S_t \quad f_{XX}(t, W_t) = \sigma^2 S_t$$

We can now apply Itô's Lemma to obtain

$$\begin{array}{rcl} dS_t & = & (f_t(t, W_t) + f_{XX}(t, W_t)/2)dt + f_X(t, W_t)dW_t \\ & = & \left((\mu - \sigma^2/2) + \sigma^2/2\right) S_t d_t + \sigma S_t dW_t \\ & = & \mu S_t d_t + \sigma S_t dW_t \end{array}$$

5.4.1 Equivalent Martingale Measures for Black-Scholes Model

Theorem 5.9 - *Girsanov's Theorem for Black-Scholes Model*[65]
Let $\{W_t\}_{t\geq 0}$ be standard Brownian motion, $\{X_t\}_{t\geq 0}$ be Brownian motion with drift b and volatility σ (ie $X_t = \sigma W_t + bt$), $a \in \mathbb{R}$ and $T > 0$.

Define the *Girsanov Density* L_T as

$$L_T = \exp\left\{\frac{a-b}{\sigma^2}X_T - \frac{a^2-b^2}{2\sigma^2}T\right\}$$

Setting

$$\mathbb{Q}(A) = \mathbb{E}[L_T \mathbb{1}\{A\}] \text{ for all } A \in \mathcal{F}_T$$

we can define a probability measure $\mathbb{Q}$ st

[64]Volatility actually depends on strike price, typically in the shape of a smile.
[65]This is a special case of Girsanov's theorem

- $\mathbb{Q}$ is equivalent to $\mathbb{P}$; and,
- $\{X_t\}_{t\geq 0}$ is Brownian motion with drift a and volatility σ under $\mathbb{Q}$.

Proof 5.10 - *Theorem 5.9*
This is just a sketch proof.
Write $X_T = \sigma W_T + bT$ and obtain

$$\begin{array}{rcl}\mathbb{Q}(A) & = & \mathbb{E}\left[\mathbb{1}\{A\}\cdot\exp\left\{\frac{a-b}{\sigma}W_T + \frac{a-b}{\sigma^2}bT - \frac{a^2-b^2}{2\sigma^2}T\right\}\right] \\ & = & \mathbb{E}\left[\mathbb{1}\{A\}\cdot\exp\left\{\frac{a-b}{\sigma}W_T + \frac{T}{2\sigma^2}(a-b)\left(2b-(a+b)\right)\right\}\right] \\ & = & \mathbb{E}\left[\mathbb{1}\{A\}\cdot\exp\left\{\frac{a-b}{\sigma}W_T + \frac{T}{2\sigma^2}(a-b)\left(-(a-b)\right)\right\}\right] \\ & = & \mathbb{E}\left[\mathbb{1}\{A\}\cdot\exp\left\{\theta W_T + T\theta^2/2\right\}\right]\end{array}$$

where $\theta := (a-b)/\sigma$.
Since W_T is distribution $\mathcal{N}(0,T)$ we obtain in particular for $A=\Omega$ that

$$\begin{array}{rcl}\mathbb{Q}(\Omega) & = & \displaystyle\int_{-\infty}^{\infty}\exp\left\{\theta z - T\theta^2/2\right\}\frac{1}{\sqrt{2\pi T}}\exp\left\{-\frac{z^2}{2T}\right\}dz \\ & = & \displaystyle\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi T}}\exp\left\{-\frac{z^2-2T\theta z+T^2\theta^2}{2T}\right\}dz \\ & = & \displaystyle\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi T}}\exp\left\{-\frac{(z-T\theta)^2}{2T}\right\}dz \\ & = & \displaystyle\int_{-\infty}^{\infty}g(z)dz\end{array}$$

We identify the intergrand $g(z)$ as the density of a $\mathcal{N}(T\theta,T)$ distribution.
Therefore

$$\mathbb{Q}(\Omega) = \int_{-\infty}^{\infty}g(z)dz = 1$$

Hence $\mathbb{Q}$ is a probability measure.
Since $L_T > 0$, $\mathbb{Q}$ is equivalent to $\mathbb{P}$.
Similarly we obtain for every $s\in\mathbb{R}$ that

$$\begin{array}{rcl}\mathbb{Q}(X_T \leq s) & = & \mathbb{Q}\left(W_T \leq \dfrac{s-bT}{\sigma}\right) \\ & = & \displaystyle\int_{-\infty}^{\frac{s-bT}{\sigma}} g(z)dz \\ & = & \displaystyle\int_{-\infty}^{s} g\left(\frac{u-bT}{\sigma}\right)du\end{array}$$

using the transformation $z=\frac{u-bT}{\sigma}$. We can rewrite the integrand as

$$\begin{array}{rcl}\dfrac{1}{\sigma}g\left(\dfrac{u-bT}{\sigma}\right) & = & \dfrac{1}{\sigma 2\pi T\sigma^2}\exp\left\{-\dfrac{\left(\frac{u-bT}{\sigma}-T\theta\right)^2}{2T}\right\} \\ & = & \dfrac{1}{\sigma 2\pi T\sigma^2}\exp\left\{-\dfrac{(u-T(b+\theta^2\sigma))^2}{2T\sigma^2}\right\}\end{array}$$

Note that $\theta := (a-b)/\sigma \implies b+\theta\sigma = a$. Therefore X_T is, under $\mathbb{Q}$, normally distributed with mean aT and variance $T\sigma^2$.

It remains to show that for $t<T$ the distribution of X_t is again a normal distribution with mean at and variance $t\sigma^2$ and that the increments of X_t are independent.[66] □

[66] Not covered in notes, may be out of scope of course.

Proposition 5.2 - *Equivalent Martingale Measure for Black-Scholes Model*
An equivalent martingale measure $\mathbb{Q}$ is defined by setting

$$\mathbb{Q}(A)=\mathbb{E}[L_T\mathbb{1}\{A\}]$$

where

$$L_T=\exp\left\{\frac{r-\mu}{\sigma}W_T-\frac{(r-\mu)^2}{2\sigma^2}T\right\}$$

Proof 5.11 - `Theorem 5.2`
The discounted price process in the *Black-Scholes model* is given by

$$S_t^*=e^{-rt}S_t=S_0e^{\sigma W_t+\left(\mu-r-\frac{\sigma^2}{2}\right)}=S_0e^{X_t}$$

where X_t is a Brownian motion with volatility σ and drift $\left(\mu-r-\frac{\sigma^2}{2}\right)$.
Writing

$$\begin{array}{rcl}L_T&=&\exp\left\{\frac{r-\mu}{\sigma^2}\left(X_T-T(\mu-r-\frac{\sigma^2}{2})\right)-\frac{(r-\mu)^2}{2\sigma^2}T\right\}\\&=&\exp\left\{\frac{r-\mu}{\sigma^2}X_T-\frac{(r-\mu)(\mu-r-\frac{\sigma^2}{2})T}{\sigma^2}-\frac{(r-\mu)^2}{2\sigma^2}T\right\}\\&=&\exp\left\{\frac{r-\mu}{\sigma^2}X_T-\frac{-2(r-\mu)^2-(r-\mu)\sigma^2}{2\sigma^2}T-\frac{(r-\mu)^2}{2\sigma^2}T\right\}\\&=&\exp\left\{\frac{r-\mu}{\sigma^2}X_T-\frac{-(r-\mu)^2-(r-\mu)\sigma^2}{2\sigma^2}T\right\}\end{array}$$

and applying *Girsanov's Theorem* (`Theorem 5.9`) with $a=-\sigma^2/2$ and $b=\mu-r-\frac{\sigma^2}{2}$, we see that X_T is, under $\mathbb{Q}$, a Brownian motion with volatility σ and drift $-\sigma^2/2$
Using `Proposition 2.7` we see that S_t^* is a martingale under $\mathbb{Q}$. □

Remark 5.7 - *Price Process under* $\mathbb{P}$ *and* $\mathbb{Q}$
Under $\mathbb{P}$, the stock price process is given by

$$S_t=S_0\exp\left\{\sigma W_t+(\mu-\frac{\sigma^2}{2})t\right\}$$

Under $\mathbb{Q}$, the stock price process is given by

$$S_t=S_0\exp\left\{\sigma\bar{W}_t+(r-\frac{\sigma^2}{2})t\right\}$$

where

$$\bar{W}_t=W_t-\frac{r-\mu}{\sigma}t$$

Note that $\bar{W}$ is a standard Brownian motion wrt $\mathbb{Q}$. This show that computations using the risk-neutral measure can be made by replacing the trend μ with the constant interest rate r.

5.4.2 Pricing a Black-Scholes Model

Proposition 5.3 - *Price of Black-Scholes Model*

Consider the Black-Scholes model and a contingent claim X. The price of X at time t is given by
$$P_t = e^{-r(T-t)}\mathbb{E}_{\mathbb{Q}}[X|\mathcal{F}_t]$$
where $\mathbb{E}_{\mathbb{Q}}$ is given via the Girsanov density (*Theorem* **??***.9*)

Theorem 5.10 - *Technical Lemma before Black-Scholes Formula*
For $Z \sim \mathcal{N}(a,\gamma^2)$ and any $b,c>0$
$$\mathbb{E}[\{be^Z - c\}_+] = be^{a+\frac{\gamma^2}{2}}\Phi\left(\frac{1}{\gamma}\left(\ln(b/c)+a+\gamma^2\right)\right) - c\Phi\left(\frac{1}{\gamma}\left(\ln(b/c)+a\right)\right)$$

Proof 5.12 - *Theorem 5.10*
Recall that for a random variable X with a standard normal distribution
$$\mathbb{P}(X>x) = 1-\Phi(x) = \Phi(-x)\ \forall\ x\in\mathbb{R}$$
Therefore we can calculate $\mathbb{E}\left[\{be^Z-c\}_+\right]$ as follows
$$\begin{array}{rcl}
\mathbb{E}\left[\{be^Z-c\}_+\right] &=& \displaystyle\int_{-\infty}^{\infty}\{be^x-c\}_+\frac{1}{\sqrt{2\pi\gamma^2}}e^{-\frac{(x-a)^2}{2\gamma^2}}dx\\
&=& \displaystyle\int_{\ln(c/b)}^{\infty}(be^x-c)\frac{1}{\sqrt{2\pi\gamma^2}}e^{-\frac{(x-a)^2}{2\gamma^2}}dx\\
&=& \displaystyle b\int_{\ln(c/b)}^{\infty}\frac{1}{\sqrt{2\pi\gamma^2}}e^{-\frac{2x\gamma^2+(x-a)^2}{2\gamma^2}}dx - c\underbrace{\int_{\ln(c/b)}^{\infty}\frac{1}{\sqrt{2\pi\gamma^2}}e^{-\frac{(x-a)^2}{2\gamma^2}}dx}_{=\mathbb{P}(Z>\ln(c/b))}\\
&=& \displaystyle be^{\frac{2a\gamma^2+\gamma^4}{2\gamma^2}}\underbrace{\int_{\ln(c/b)}^{\infty}\frac{1}{\sqrt{2\pi\gamma^2}}e^{-\frac{(x-a-\gamma^2)^2}{2\gamma^2}}dx}_{=\mathbb{P}(Z+\gamma^2>\ln(c/b))} - c\mathbb{P}(Z>\ln(c/b))\\
&=& be^{a+\frac{\gamma^2}{2}}\mathbb{P}(Z+\gamma^2>\ln(c/b)) - c\mathbb{P}(Z>\ln(c/b))\\
&=& \displaystyle be^{a+\frac{\gamma^2}{2}}\mathbb{P}\left(\frac{Z-a}{\gamma}>\frac{1}{\gamma}\left(\ln(c/b)-\gamma^2-a\right)\right) - c\mathbb{P}\left(\frac{Z-a}{\gamma}>\frac{1}{\gamma}\left(\ln(c/b)-a\right)\right)\\
&=& \displaystyle be^{a+\frac{\gamma^2}{2}}\Phi\left(\frac{1}{\gamma}\left(\ln(c/b)-\gamma^2-a\right)\right) - c\Phi\left(\frac{1}{\gamma}\left(\ln(c/b)-a\right)\right)
\end{array}$$
□

Proposition 5.4 - *Black-Scholes Formula - Continuous Time*
This is the same as the Black-Scholes Formula in Theorem 4.15.

Consider a European call option with exercise price K maturing at time T which corresponds to the claim $X=\{S_T-K\}_+$. Then the Black-Scholes price process of that European class is given by
$$\Pi_t = S_t\Phi(d_1(S_t,T_t)) - Ke^{-r(T-t)}\Phi(d_2(S_t,T-t))$$
The functions $d_1(u,v)$ and $d_2(u,v)$ are given by
$$\begin{array}{rcl}
d_1(u,v) &=& \dfrac{\ln(u/K)+\left(r+\frac{\sigma^2}{2}\right)v}{\sigma\sqrt{v}}\\
d_2(u,v) &=& \dfrac{\ln(u/K)+\left(r-\frac{\sigma^2}{2}\right)v}{\sigma\sqrt{v}}
\end{array}$$

Proof 5.13 - *Proposition 5.4.*
Let $\bar{W}_t$ be a standard Brownian motion under $\mathbb{Q}$. The price Π_t is given by

$$\begin{array}{rcl}\Pi_t & = & e^{-r(T-t)}\mathbb{E}_{\mathbb{Q}}\left[\{S_T-K\}_+|\mathcal{F}_t\right] \\ & = & e^{-r(T-t)}\mathbb{E}_{\mathbb{Q}}\left[\left\{S_te^{\sigma(\bar{W}_T-\bar{W}_t)+\left(r-\frac{\sigma^2}{2}\right)(T-t)}-K\right\}_+\middle|\mathcal{F}_t\right] \\ & = & e^{-r(T-t)}\mathbb{E}_{\mathbb{Q}}\left[|\{S_te^Z-K\}_+\mathcal{F}_t\right]\end{array}$$

where $Z\sim\mathcal{N}(a,\gamma^2)$ under $\mathbb{Q}$ with mean $a=\left(r-\frac{\sigma^2}{2}\right)(T-t)$ and standard deviation $\gamma=\sigma\sqrt{T-t}$ and is independent of $\mathcal{F}_t$.

We obtain the Black-Scholes formula by applying `Theorem 5.10` with $b=S_t,c=K$

$$\begin{array}{rcl}\Pi_t & = & e^{-r(T-t)}S_te^{a+\frac{1}{2}(\sigma^2(T-t))}\Phi\left(\frac{1}{\sigma\sqrt{T-t}}\left(\ln(S_t/K)+a+\sigma^2(T-t)\right)\right) \\ - & & e^{-r(T-t)}K\Phi\left(\frac{1}{\sigma\sqrt{T-t}}(\ln(S_t/K)+a)\right)\end{array}$$

$\square$

5.4.3 Black-Scholes-Merton Equation

Remark 5.8 - *Motivation*
The risk-neutral pricing principle (**Theorem 4.12**) does not provide an explicitly expression for the replicating portfolio.

Theorem 5.11 - *Black-Scholes-Mertons Equation*
Let $X=g(S_T)$ be an attainable claim in the Black-Scholes model with drift μ and volatility σ.

Assume that the price Π_t at time t satisfies $\Pi_t=f(t,S_t)$ for some function f with two continuous derivatives in x and one in t.

Then $f(t,x)$ satisfies the *Black-Scholes-Mertons Equation*

$$f_t+rS_tf_x+\frac{1}{2}\sigma^2S_t^2f_{xx}-rf=0,\quad f(T,S_T)=g(S_T)$$

Further, the replicating strategy is given by

$$\begin{array}{rcl}H_0(t) & = & \frac{1}{B_t}\left(f(t,S_t)-f_x(t,S_t)S_t\right) \\ & = & \frac{1}{rB_t}\left(f_t(t,S_t)+\frac{1}{2}f_{xx}(t,S_t)S_t^2\sigma^2\right) \\ H_1(t) & = & f_x(t,S_t)\end{array}$$

Proof 5.14 - *Theorem 5.11*
The price process S_t of the stock under $\mathbb{Q}$ is

$$S_t=S_0\exp\left\{\sigma\bar{W}_t+t\cdot\left(r-\frac{\sigma^2}{2}\right)\right\}$$

An application of Itô's lemma shows that it has dynamics

$$dS_t=S_t(rdt+\sigma d\bar{W}_t) \tag{6}$$

Similarly, the discounted price process $S_t^* = S_0 \exp\left\{\sigma \bar{W}_t - t(\sigma^2/2)\right\}$ has dynamics

$$dS_t^* = S_t^* \sigma d\bar{W}_t \tag{7}$$

Let now $h(t, S_t) = f(t, S_t)/B_t = \exp^{-rt} f(t, S_t)$ be the discounted fair price of the claim X at time t.

Since, (6) means that S_t is an Itô process with $b_t = rS_t$ and $\sigma_t = \sigma S_t$ using Itô's lemma we find that

$$dh = \left(h_t + rS_t h_x + \frac{1}{2}\sigma^2 S_t^2 h_{xx}\right) dt + \sigma S_t h_x d\bar{W}_t \tag{8}$$

We know from the risk-neutral valuation principle that

$$h_t(t, S_t) = \mathbb{E}_{\mathbb{Q}}\left[X/B_T \middle| \mathcal{F}_t\right] \tag{9}$$

and in particular that h is a martingale with respect to $\mathbb{Q}$.

Therefore, there is no drift meaning that the dt term in (8) is zero

$$h_t(t, S_t) + rS_t h_x(t, S_t) + \frac{1}{2}\sigma^2 S_t^2 h_{xx}(t, S_t) = 0$$

Moreover, the dynamics of (8) reduce to

$$dh(t, S_t) = \sigma S_t h_x(t, S_t) d\bar{W}_t \tag{10}$$

Black-Scholes-Merton Equation

The ratio between h and f is a function of t, not x, and therefore $h_x = f_x/B_t$ and $g_{xx} = f_{xx}/B_t$.

We also know that $h_t = (f_t/B_t) - (rf/B_t)$. Therefore, (9) can be rewritten as

$$\begin{array}{rcl} 0 & = & h_t + rS_t h_x + \frac{1}{2}\sigma^2 S_t^2 h_{xx} \\ & = & \frac{1}{B_t}\left(f_t - rf + rS_t f_x + \frac{1}{2}\sigma^2 S_t^2 f_{xx}\right) \end{array}$$

Multiplying by B_t we see that f satisfies the *Black-Scholes-Merten Equation.* $\square$

Replicating Portfolio.

For all t the value of the replicating portfolio $(H_0(t), H_1(t))$ equals the price process $f(t, S_t)$ of the claim X and the replicating strategy must be self-financing.

Therefore, (7) implies

$$\begin{array}{rcl} dh(t, S_t) & = & H_1(t) dS_t^* \\ & = & H_1(t) S_t^* \sigma d\bar{W}_t \\ & = & H_1(t) \frac{S_t}{B_t} \sigma d\bar{W}_t \end{array}$$

Equating this result to (10) we get

$$\begin{array}{rrcl} & H_1(t) S_t^* \sigma & = & \sigma S_t h_x(t, S_t) \\ \implies & H_1(t)\left(\frac{S_t}{B_t}\right) & = & S_t h_x(t, S_t) \\ \implies & H_1(t) & = & B_t h_x(t, S_t) \\ & & = & f_x(t, S_t) \end{array}$$

Finally, the replicating portfolio has to satisfy

$$\begin{array}{rcl} f(t, S_t) & = & H_0(t) B_t + H_1(t) S_t \\ & = & H_0(t) B_t + f_x(t, S_t) S_t \end{array}$$

From this we obtain that

$$H_0(t) = \frac{f(t, S_t) - f_x(t, S_t)S_t}{B_t}$$

Remark 5.9 - *Black-Scholes-Merton Equation and Contingency Claims*
The *Black-Scholes-Merton Equation* is an alternative way of calculating the price of a contingent claim.

The boundary condition has to be chosen according to the contingent claim we want to analyse. In particular, for $f(T, S_T) = \{S_T - k\}^+$ we obtain yet another way of deriving the Black-Scholes formula (See **Example 5.4** in long notes).

0 Reference

0.1 Notation

0.2 Definitions