Financial Mathematics - Problem Sheet 8

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Answer 2. a)

$$\begin{array}{rcl} \text{Cov}(U_s, U_t) & = & \mathbb{E}[U_s U_t] - \mathbb{E}[U_s]\mathbb{E}[U_t] \\ & = & \mathbb{E}[U_s U_t] \\ & = & \mathbb{E}[(W_s - sW_1)(W_t - tW_1)] \\ & = & \mathbb{E}[W_s W_t] - \mathbb{E}[sW_1 W_t] - \mathbb{E}[tW_1 W_s] + \mathbb{E}[stW_1^2] \\ & = & \text{Cov}(W_s, W_t) - s\text{Cov}(W_1, W_t) - t\text{Cov}(W_1, W_s) + st\text{Var}(W_1) \\ & = & s - st - st + st \\ & = & s(1-t) \end{array}$$

Answer 2. b)

Y_t can be restated as

$$\begin{array}{rcl} Y_t & = & (1+t)U_{t/1+t} \\ & = & (1+t)\left\{W_{t/1+t} - \frac{t}{1+t}W_1\right\} \\ & = & (1+t)W_{t/1+t} - tW_1 \end{array}$$

For Y_t to be a standard Brownian motion, it must fulfil the following properties

i). $Y_0 = 0$.

$$\begin{array}{rcl} Y_0 & = & (1+0)W_{0/1+0} - 0W_1 \\ & = & W_0 = 0 \end{array}$$

ii). *Increments of Y_t are independent.*

Consider $(Y_t - Y_s)$ with $s \leq t$.

$$\begin{array}{rcl} (Y_t - Y_s) & = & \{(1+t)W_{t/1+t} - tW_1\} - \{(1+s)W_{s/1+s} - sW_1\} \\ & = & (1+s)\left(W_{t/1+t} - W_{s/1+s}\right) - (t-s)\left(W_1 - W_{t/1+t}\right) \end{array}$$

Since W_t is standard Brownian motion, the increments $\left(W_{t/1+t} - W_{s/1+s}\right), \left(W_1 - W_{t/1+t}\right)$ are independent of $\mathcal{F}_s$ and thus the increment $(Y_t - Y_s)$ is independent of $\mathcal{F}_s$ for all $s \leq t$.

iii). *Increments have stationary Gaussian Distributions.*

As W_t is standard Brownian motion, its increments have gaussian distributions. Thus, by linearity, the increments of $\{Y_t\}$ have gaussian distributions too.

Consider the mean and variance of increment $(Y_t - Y_s)$ with $s \leq t$.

$$\begin{aligned}\mathbb{E}[Y_t - Y_s] &= (1+s)\mathbb{E}\left[W_{t/1+t} - W_{s/1+s}\right] - (t-s)\mathbb{E}\left[W_1 - W_{t/1+t}\right] \\ &= 0 - 0 = 0\end{aligned}$$

$$\begin{aligned}\text{Var}[Y_t - Y_s] &= (1+s)^2\text{Var}\left[W_{t/1+t} - W_{s/1+s}\right] - (t-s)^2\text{Var}\left[W_1 - W_{t/1+t}\right] \\ &= (1+s)^2\left(\frac{t}{1+t} - \frac{s}{1+s}\right) - (t-s)^2\left(1 - \frac{t}{1+t}\right) \\ &= \frac{(1+s)(t-s)}{1+t} - \frac{(t-s)^2}{1+t} \\ &= \dots\end{aligned}$$

Answer 3. a)

I wrote the following python code

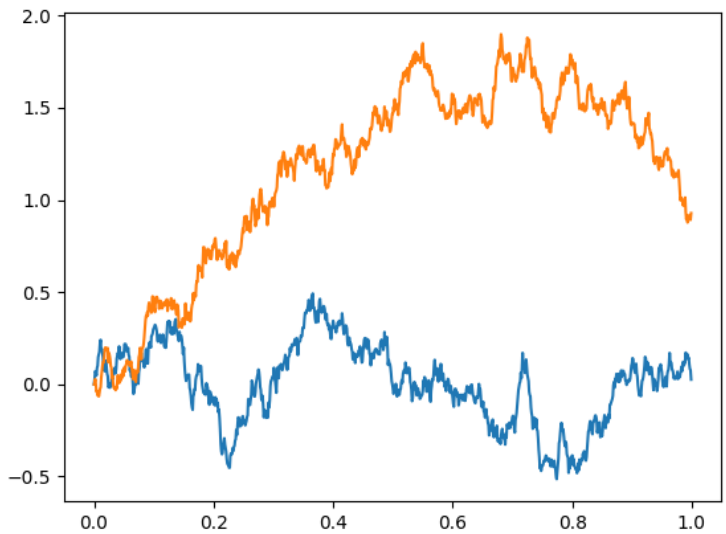
```
import numpy as np
from scipy import stats
import matplotlib.pyplot as plt

def sample_standard_norm():
    return stats.norm(0,1).rvs(1)[0]

def standard_brownian_motion(n:int):
    X_is=[sample_standard_norm() for _ in range(1,n)]
    S_ns=[0]+list(np.cumsum(X_is))
    W_ts=[x/np.sqrt(n) for x in S_ns]
    return W_ts

n=1000
for i in range(2):
    plt.plot([x/n for x in range(n)],standard_brownian_motion(n))
plt.show()
```

Which produced this plot



Answer 3. b)

I extended the code above

```
def drift_brownian_motion(n:int,mu:float):
    W_ts=standard_brownian_motion(n)
    W_ts_drift=[W_ts[t]+mu*t for t in range(n)]
    return W_ts_drift

n=1000

plt.plot([x/n for x in range(n)],drift_brownian_motion(n,-3))
plt.plot([x/n for x in range(n)],drift_brownian_motion(n,30))
plt.show()
```

Which produced this plot