Financial Mathematics - Problem Sheet 1

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Answer 1)
From the question we can define the following two equations, equating the values of the contract & portfolio on July 1st for each of the outcomes

$$\begin{array}{rrcl} \text{(Share Increases)} & \frac{11}{10}H_0 + 16H_1 & = & 2 \\ \text{(Share Decreases)} & \frac{11}{10}H_0 + 4H_1 & = & 1 \end{array}$$

This system of equations can be solved as follows

$$\begin{array}{rrcl} & \left(\frac{11}{10} - \frac{11}{10}\right) H_0 + (16-4)H_1 & = & (2-1) \\ \implies & 12H_1 & = & 1 \\ \implies & H_1 & = & \frac{1}{12} \\ \\ \implies & \frac{11}{10}H_0 + 16 \cdot \frac{1}{12} & = & 2 \\ \implies & H_0 & = & \frac{20}{33} \end{array}$$

On Jan 1st this portfolio is worth

$$H_0 + 8H_1 = \frac{20}{33} + 8 \cdot \frac{1}{12} = \frac{14}{11}$$

Thus, the contract is worth £14/11 $\simeq$£1.27 on Jan 1st.

Answer 2)
We have that $S^* = S/(1+r)$. By the question $r = .1$, so $S* = \frac{11}{10}S$.
Thus, the expected value of S^* on July 1st, given variable q, is

$$\begin{array}{rcl} \mathbb{E}[S^*;q] & = & \frac{10}{11}(16q + 4(1-q)) \\ & = & \frac{10}{11}(12q+4) \end{array}$$

We find the probability q^* which produces an expected value of 8 as follows.

$$\begin{array}{rrcl} & \frac{10}{11}(12q^*+4) & = & 8 \\ \implies & 12q^*+4 & = & \frac{88}{10} \\ \implies & 12q^* & = & \frac{48}{10} \\ \implies & q^* & = & \frac{4}{10} = \frac{2}{5} \end{array}$$

With this probability, the portfolio derived in `Q 1)` has the expected following value on July 1st

$$\begin{aligned}\mathbb{E}[\text{Portfolio Value}] &= \frac{11}{10}H_0 + \{(q^* \cdot H_1 \cdot 16 + (1-q^*) \cdot H_1 \cdot 4\} \\ &= \frac{11}{10} \cdot \frac{20}{33} + \left\{\frac{2}{5} \cdot \frac{1}{12} \cdot 16 + \frac{3}{5} \cdot \frac{1}{12} \cdot 4\right\} \\ &= \frac{7}{5}\end{aligned}$$

This is the same as the normalised gain from part 1 as $\frac{10}{11} \cdot \frac{7}{5} = \frac{14}{11}$

Answer 3a)
Consider the minimum proportion of our total stake needed to be placed on each horse in order to cover the costs of our losses if that horse wins.

	Horse #1	Horse #2	Horses #3-#6
Required Pct.	.5%	.3333	.0196%

This sums to a total of 0.9509%. This means that we do not require our total stake in order to cover any loses, regardless of outcome (ie An arbitrage opportunity exists).

Distributing our total stake between each horse in the ratio of the proportions given in the above table will produce an arbitrage. Here are the details

	Horse #1	Horse #2	Horses #3-#6
Stake	£50	£33.33	£1.96
Returns	£100	£99.99	£99.96
Losses	£45.09	£61.76	£93.13
Profit	£54.91	£38.23	£6.83

Returns & *Losses* are the flows of money if that horse wins the race. As you can see from the table, the returns are greater than the losses in all outcomes. Naturally, a risk-free profit is made for any scaling of this set of stakes.

Answer 3b)
Consider the scenario in `Q 3a)`, but with the number of horses whose odds are offered at 50:1 now being a variable $n \in \mathbb{N}$.

Arbitrage opportunities do not exist when the sum of minimum proportions of our total stake required for the winnings from each horse to cover the losses from the other horses exceeds 100%. We can find such an n as follows

$$\begin{aligned} & \frac{1}{2} + \frac{1}{3} + \frac{n}{51} &>& \ 1 \\ \implies & \frac{n}{51} &>& \ \frac{1}{6} \\ \implies & n &>& \ \frac{51}{6} \\ \implies & n &\geq& \ 9 \text{ as } n \in \mathbb{N}\end{aligned}$$

So, if one horse has odds 1:1, another has odds 2:1, and 9, or more, have odds 50:1 then no arbitrage opportunity exists.