Financial Mathematics - Assessed Problem Sheet 4

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Answer 1. (a) i.
Let $f(t,x)=x^3-t^2$ and note that

$$\begin{array}{rcl} f_t &=& -2t \\ f_x &=& 3x^2 \\ f_{xx} &=& 6x \end{array}$$

Define $X_t=f(t,W_t)$. Thus, by Itô's Lemma

$$\begin{array}{rcl} dX_t &=& df(t,W_t) \\ &=& \left(-2t+\frac{1}{2}6W_t\right)dt+3W_t^2dW_t \\ &=& (3W_t-2t)\,dt+3W_t^2dW_t \end{array}$$

Answer 1. (a) ii.
Let $f(x)=\exp\{3x\}$ and note that

$$\begin{array}{rcl} f' &=& 3f(x) \\ f'' &=& 9f(x) \end{array}$$

Define $X_t=f(W_t)$. Thus, by Itô's Lemma

$$\begin{array}{rcl} dX_t &=& df(W_t) \\ &=& 3f(W_t)dW_t+\frac{1}{2}\left(9f(W_t)\right)dt \\ &=& 3X_tdW_t+\frac{9}{2}X_tdt \end{array}$$

Answer 1. (a) iii.
Let $f(t,x)=\exp\{\sigma x+at\}$ and note that

$$\begin{array}{rcl} f_t &=& af(t,x) \\ f_x &=& \sigma f(t,x) \\ f_{xx} &=& \sigma^2 f(t,x) \end{array}$$

Define $X_t=f(t,W_t)$. Thus, by Itô's Lemma

$$\begin{array}{rcl} dX_t &=& df(t,W_t) \\ &=& \left(af(t,W_t)+\frac{1}{2}\sigma^2 f(t,W_t)\right)dt+\sigma f(t,W_t)dW_t \\ &=& \left(a+\frac{1}{2}\sigma^2\right)X_tdt+\sigma X_tdW_t \end{array}$$

Since W_t is a martingale, then X_t is a martingale if $\left(a+\frac{1}{2}\sigma^2\right)X_t=0$. This means, either of the two must hold at time t

i). $X_t = 0$.

ii). $a + \frac{1}{2}\sigma^2 = 0$.

Answer 1. (b)
Let W_t be standard brownian motion and define $X_t = aW_t + b$ for some $a, b \in \mathbb{R}$.
Note that by the linearity of expectation the following holds for all $s, t \geq 0$

$$\begin{array}{rcl}\mathbb{E}[X_t|\mathcal{F}_s] & = & \mathbb{E}[aW_t + b|\mathcal{F}_s] \\ & = & a\mathbb{E}[W_t|\mathcal{F}_s] + b \\ & = & aW_s + b \\ & = & X_s\end{array}$$

This shows that all linear functions of standard brownian motion are martingales.