Financial Mathematics - Problem Sheet 5

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Answer 2.
Consider time $t = 0$ and $p_1 := \mathbb{Q}(S_1 = 5)$, then

$$\begin{array}{rcl} 6 & = & 5p_1 + 7(1 - p_1) \\ & = & 7 - 2p_1 \\ \implies p_1 & = & 1/2 \end{array}$$

Thus $\mathbb{Q}(S_1 = 5) = 1/2$ and $\mathbb{Q}(S_1 = 7) = 1/2$.
Now consider time $t = 1$, events $\{\omega_4, \omega_5\}$ and $p_2 := \mathbb{Q}(S_2 = 6|S_1 = 7)$, then

$$\begin{array}{rcl} 7 & = & 6p_2 + 8(1 - p_2) \\ & = & 8 - 2p_2 \\ \implies p_2 & = & 1/2 \end{array}$$

Thus $\mathbb{Q}(S_2 = 6|S_1 = 7) = 1/2$ and $\mathbb{Q}(S_2 = 8|S_1 = 7) = 1/2$.
Now consider time $t = 1$, events $\{\omega_1, \omega_2, \omega_3\}$, $p_3 := \mathbb{Q}(S_2 = 3|S_1 = 5)$ and $p_4 := \mathbb{Q}(S_2 = 4|S_1 = 5)$, then

$$\begin{array}{rcl} 6 & = & 3p_3 + 4p_4 + 8(p_3 - p_4) \\ & = & 11p_3 - 4p_4 \\ \implies p_4 & = & \frac{1}{4}(11p_3 - 6) \end{array}$$

From this we can deduce that $\mathbb{Q}(S_2 = 3|S_1 = 5) = x$, $\mathbb{Q}(S_2 = 4|S_1 = 5) = \frac{1}{4}(11x - 6)$ and $\mathbb{Q}(S_2 = 8|S_1 = 5) = 1 - x - \frac{1}{4}(11x - 6) = \frac{1}{4}(10 - 15x)$.
It is clear that these values sum to 1 but we need to restrict the values of x so that each of these probabilities is in $(0, 1)$.

$$\begin{array}{rrcl} & \frac{1}{4}(11x - 6) & \in & (0, 1) \\ \implies & 11x - 6 & \in & (0, 4) \\ \implies & x & \in & \left(\frac{6}{11}, \frac{10}{11}\right) \end{array} \qquad \begin{array}{rrcl} & \frac{1}{4}(10 - 15x) & \in & (0, 1) \\ \implies & 10 - 15x & \in & (0, 4) \\ \implies & x & \in & \left(\frac{6}{15}, \frac{10}{15}\right) \end{array}$$

Thus we need to restrict x st

$$\begin{array}{rcl} x & \in & \left\{(0, 1) \cap \left(\frac{6}{11}, \frac{10}{11}\right) \cap \left(\frac{2}{5}, \frac{2}{3}\right)\right\} \\ & = & \left(\frac{6}{11}, \frac{2}{3}\right) \end{array}$$

We can use these conditional probabilities to determine the probability of each event $\{\omega_1, \dots, \omega_5\}$ under $\mathbb{Q}$

$$\begin{array}{rcl}
\mathbb{Q}(\{\omega_1\}) & = & \mathbb{Q}(S_2 = 3 | S_1 = 5)\mathbb{Q}(S_1 = 5) \\
& = & \frac{x}{2} \\
\mathbb{Q}(\{\omega_2\}) & = & \mathbb{Q}(S_2 = 4 | S_1 = 5)\mathbb{Q}(S_1 = 5) \\
& = & \frac{1}{8}(11x - 6) \\
\mathbb{Q}(\{\omega_3\}) & = & \mathbb{Q}(S_2 = 8 | S_1 = 5)\mathbb{Q}(S_1 = 5) \\
& = & \frac{1}{8}(10 - 15x) \\
\mathbb{Q}(\{\omega_4\}) & = & \mathbb{Q}(S_2 = 6 | S_1 = 7)\mathbb{Q}(S_1 = 7) \\
& = & 1/2 \\
\mathbb{Q}(\{\omega_5\}) & = & \mathbb{Q}(S_2 = 8 | S_1 = 7)\mathbb{Q}(S_1 = 7) \\
& = & 1/2
\end{array}$$

for $x \in \left(\frac{6}{11}, \frac{2}{3}\right)$.

Answer 3. (a)

For $\mathbb{Q}$ to be a martingale measure it needs to fulfil $\mathbb{Q}(\{\omega\}) > 0 \ \forall \ \omega \in \Omega$ (which it does by definition) and that

$$\mathbb{E}[S_n^*(t+s)|\mathcal{F}_t] = S_n(t) \ \forall \ s, t, n$$

As the interest rate in this model is $r = 0$, we have that $S_n^*(t) = S_n(t) \ \forall \ t, n$.

I now test whether this $\mathbb{Q}$ fulfils this criteria for all t, s, n. It is trivial that this holds for cases where $t = 2, s = 0$.

$$\begin{array}{r|rcl}
(t=0, s=1, n=1) & \mathbb{E}_{\mathbb{Q}}[S_1(1)|\mathcal{F}_0] & = & \mathbb{E}[S_1(1)] \\
& & = & \frac{3}{9}\cdot 8 + \frac{4}{12}\cdot 7 + \frac{4}{12}6 \\
& & = & 7 = S_1(0) \\
& & & \\
(t=0, s=2, n=1) & \mathbb{E}_{\mathbb{Q}}[S_1(2)|\mathcal{F}_0] & = & \mathbb{E}[S_1(2)] \\
& & = & \frac{1}{9}(7+8+9) + \frac{1}{6}(6+6) + \frac{1}{12}(6+10+3+9) \\
& & = & 7 = S_1(0) \\
& & & \\
(t=1, s=1, n=1) & \mathbb{E}_{\mathbb{Q}}[S_1(2)|\omega] & = & \frac{\frac{1}{9}(7+8+9)}{3\cdot\frac{1}{9}} \text{ if } \omega \in \{\omega_1, \omega_2, \omega_3\} \\
& & = & 8 \\
& & = & S_1(1) \text{ if } \omega \in \{\omega_1, \omega_2, \omega_3\} \\
& & & \\
(t=1, s=1, n=1) & \mathbb{E}_{\mathbb{Q}}[S_1(2)|\omega] & = & \frac{\frac{1}{6}\cdot 6 + \frac{1}{12}(6+10)}{\frac{1}{6} + 2\cdot\frac{1}{12}} \text{ if } \omega \in \{\omega_4, \omega_5, \omega_6\} \\
& & = & 7 \\
& & = & S_1(1) \text{ if } \omega \in \{\omega_4, \omega_5, \omega_6\} \\
& & & \\
(t=1, s=1, n=1) & \mathbb{E}_{\mathbb{Q}}[S_1(2)|\omega] & = & \frac{\frac{1}{6}\cdot 6 + \frac{1}{12}(3+9)}{\frac{1}{6} + 2\cdot\frac{1}{12}} \text{ if } \omega \in \{\omega_7, \omega_8, \omega_9\} \\
& & = & 6 \\
& & = & S_1(1) \text{ if } \omega \in \{\omega_7, \omega_8, \omega_9\}
\end{array}$$

$$\begin{array}{r|rcl}
(t=0,s=1,n=2) & \mathbb{E}_{\mathbb{Q}}[S_2(1)|\mathcal{F}_0] & = & \mathbb{E}[S_2(1)] \\
& & = & \frac{3}{9}\cdot 5+\frac{4}{12}\cdot 8+\frac{4}{12}5 \\
& & = & 6 = S_2(0) \\
& & & \\
(t=0,s=2,n=2) & \mathbb{E}_{\mathbb{Q}}[S_2(2)|\mathcal{F}_0] & = & \mathbb{E}[S_2(2)] \\
& & = & \frac{1}{9}(7+5+3)+\frac{1}{6}(8+3)+\frac{1}{12}(9+7+8+6) \\
& & = & 6 = S_1(0) \\
& & & \\
(t=1,s=1,n=2) & \mathbb{E}_{\mathbb{Q}}[S_2(2)|\omega] & = & \frac{\frac{1}{9}(7+5+3)}{3\cdot\frac{1}{9}} \text{ if } \omega\in\{\omega_1,\omega_2,\omega_3\} \\
& & = & 5 \\
& & = & S_2(1) \text{ if } \omega\in\{\omega_1,\omega_2,\omega_3\} \\
& & & \\
(t=1,s=1,n=2) & \mathbb{E}_{\mathbb{Q}}[S_2(2)|\omega] & = & \frac{\frac{1}{6}\cdot 8+\frac{1}{12}(9+7)}{\frac{1}{6}+2\cdot\frac{1}{12}} \text{ if } \omega\in\{\omega_4,\omega_5,\omega_6\} \\
& & = & 8 \\
& & = & S_2(1) \text{ if } \omega\in\{\omega_4,\omega_5,\omega_6\} \\
& & & \\
(t=1,s=1,n=2) & \mathbb{E}_{\mathbb{Q}}[S_2(2)|\omega] & = & \frac{\frac{1}{6}\cdot 3+\frac{1}{12}(8+6)}{\frac{1}{6}+2\cdot\frac{1}{12}} \text{ if } \omega\in\{\omega_7,\omega_8,\omega_9\} \\
& & = & 5 \\
& & = & S_2(1) \text{ if } \omega\in\{\omega_7,\omega_8,\omega_9\}
\end{array}$$

Answer 3. (b)
First, consider the payout from each event

	ω_1	ω_2	ω_3	ω_4	ω_5	ω_6	ω_7	ω_8	ω_9
$X(\omega)$	1	1	0	1	2	4	0	0	2

The fair-price for this option at time $t=0$ is

$$\begin{array}{rcl}
V_0 & = & \mathbb{E}[X] \\
& = & \frac{1}{9}(1+1+0)+\frac{1}{6}(1+0)+\frac{1}{12}(2+4+2) \\
& = & \frac{2}{9}+\frac{1}{6}+\frac{8}{12} \\
& = & 19/18
\end{array}$$

The fair-price for this option at time $t = 1$ depends on which state the model is in

$$
\begin{aligned}
V_1(\omega) &= \begin{cases} \mathbb{E}[X|S_1 = 8, S_2 = 5] & \text{if } \omega \in \{\omega_1, \omega_2, \omega_3\} \\ \mathbb{E}[X|S_1 = 7, S_2 = 8] & \text{if } \omega \in \{\omega_4, \omega_5, \omega_6\} \\ \mathbb{E}[X|S_1 = 6, S_2 = 5] & \text{if } \omega \in \{\omega_7, \omega_8, \omega_9\} \end{cases} \\
&= \begin{cases} \dfrac{\frac{1}{9}(1+1+0)}{\frac{1}{9}\cdot 3} & \text{if } \omega \in \{\omega_1, \omega_2, \omega_3\} \\ \dfrac{16(1) + \frac{1}{12}(2+4)}{\frac{1}{6} + \frac{1}{12}\cdot 2} & \text{if } \omega \in \{\omega_4, \omega_5, \omega_6\} \\ \dfrac{\frac{1}{6})0_{+}\frac{1}{12}(0+2)}{\frac{1}{6} + \frac{1}{12}\cdot 2} & \text{if } \omega \in \{\omega_7, \omega_8, \omega_9\} \end{cases} \\
&= \begin{cases} 2/3 & \text{if } \omega \in \{\omega_1, \omega_2, \omega_3\} \\ 3/2 & \text{if } \omega \in \{\omega_4, \omega_5, \omega_6\} \\ 1/2 & \text{if } \omega \in \{\omega_7, \omega_8, \omega_9\} \end{cases}
\end{aligned}
$$