Finance Mathematics - Problem Sheet 7

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Answer 1.
Define the probability q st

$$q = \frac{1+r-d}{u-d} = 1 + 0.01 - 0.91.1 - 0.9 = \frac{11}{20}$$

For this European call option, we have that the payoff function is

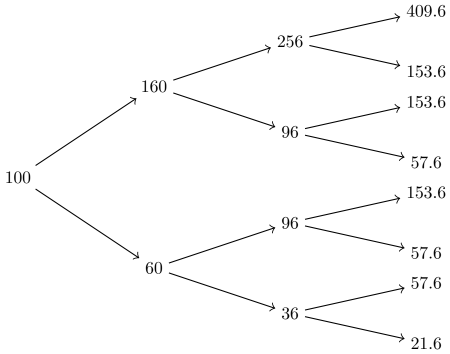
$$g(x) = \{x - e\}_+ = \{x - 1000\}_+$$

Under the Cox-Ross-Rubinstein model we have the time $t = 0$ price of this option is

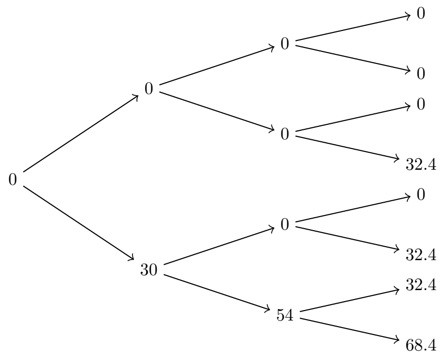
$$\begin{aligned}
\Pi_e(0) &:= \frac{1}{(1+r)^2}\left\{\sum_{n=0}^{T}\binom{T}{n}q^n(1-q)^{T-n}g(S_0u^nd^{T-n})\right\} \\
&= \frac{1}{1.01^2}\left\{\sum_{n=0}^{2}\binom{2}{n}\left(\frac{11}{20}\right)^n\left(\frac{8}{20}\right)^{T-n}\left(S_0\left(\frac{11}{10}\right)^n\left(\frac{9}{20}\right)^{T-n} - 1000\right)\right\} \\
&= \frac{1}{1.01^2}\left\{\binom{2}{2}\left(\frac{11}{20}\right)^2\left(S_0\left(\frac{11}{10}\right)^2 - 1000\right) + \binom{2}{1}\frac{11}{20}\cdot\frac{9}{20}\left(S_0\frac{11}{10}\cdot\frac{9}{10} - 1000\right)\right. \\
&+ \left.\binom{2}{0}\left(\frac{9}{20}\right)^2\left(S_0\left(\frac{9}{20}\right)^2 - 1000\right)\right\} \\
&= \left(\frac{1}{1.01^2}\right)\left(\frac{1}{400}\right)\cdot\left\{121\left(S_0\frac{121}{100} - 1000\right) + 198\left(S_0\frac{99}{100} - 1000\right) + 81\left(S_0\frac{81}{100} - 1000\right)\right\}
\end{aligned}$$

By inspection, each term of this function is clearly piecewise-linear wrt the initial price S_0.

Answer 2.
Consider the tree below which shows the possible evolutions of the price process S_t for each time-point and event.



For this American put option we have that the pay-out is $Y_t(\omega) = \{90 - S_t(\omega)\}_+$ for each $t = 0, 1, 2, 3$ and $\omega \in \Omega$. I summarise the pay-outs in the tree below



We can calculate the probability value q where

$$q := \frac{1+r-d}{u-d} = \frac{1+0.1-0.6}{1.6-0.6} = \frac{1}{2}$$

This means the probability of the price increasing and the probability price decreasing during each time period are both 1/2.

I will now construct a Snell Envelope $\{Z_t\}$ for $\{Y_t\}$.

By defintiion $Z_3(\omega) = Y_3(\omega)\ \forall\ \omega \in \Omega$.

Let ω_{ud} denote the events where the price increased once and decreased once (in some order) during the first two time-steps and ω_{dd} denote the events where the price decreased twice during the first two time-steps. Note these are disjoint and that no other events can lead to a profitable payout.

Suppose $t = 2$ and ω_{ud} occurred. Then

$$\begin{array}{rrcl} & \mathbb{E}[Z_3|\mathcal{F}_2] & = & q \cdot 0 + (1-q) \cdot 32.4 \\ & & = & \frac{1}{2}(0 + 32.4) \\ & & = & 16.2 \\ \implies & Z_2(\omega_{ud}) & = & \max\{Y_2(\omega_{ud}), \mathbb{E}[Z_3, \mathcal{F}_2]\} \\ & & = & \max\{0, 16.2\} \\ & & = & 16.2 \end{array}$$

Suppose $t = 2$ and ω_{dd} occurred. Then

$$\begin{array}{rrcl} & \mathbb{E}[Z_3|\mathcal{F}_2] & = & \frac{1}{2}(32.4 + 68.4) = 50.4 \\ \implies & Z_2(\omega_{dd}) & = & \max\{Y_2(\omega_{dd}), \mathbb{E}[Z_3, \mathcal{F}_2]\} \\ & & = & \max\{54, 50.4\} \\ & & = & 54 \end{array}$$

Let ω_u denote the events where the price increased during the first time-step and ω_d denote the events where the price decreases during the first time-step.

Suppose $t = 1$ and ω_u occurred. Then

$$\begin{array}{rrcl} & \mathbb{E}[Z_2|\mathcal{F}_1] & = & \frac{1}{2}(0 + 16.2) = 8.1 \\ \implies & Z_1(\omega_u) & = & \max\{Y_1(\omega_u), \mathbb{E}[Z_2, \mathcal{F}_1]\} \\ & & = & \max\{0, 8.1\} \\ & & = & 8.1 \end{array}$$

Suppose $t = 1$ and ω_d occurred. Then

$$\begin{array}{rrcl} & \mathbb{E}[Z_2|\mathcal{F}_1] & = & \frac{1}{2}(16.2 + 54) = 35.1 \\ \implies & Z_1(\omega_d) & = & \max\{Y_1(\omega_d), \mathbb{E}[Z_2, \mathcal{F}_1]\} \\ & & = & \max\{30, 35.1\} \\ & & = & 35.1 \end{array}$$

Suppose $t = 0$. Then

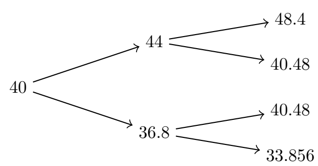
$$\begin{array}{rrcl} & \mathbb{E}[Z_1|\mathcal{F}_0] & = & \mathbb{E}[Z_1] \\ & & = & \frac{1}{2}(8.1 + 35.1) \\ & & = & 21.6 \\ \implies & Z_0 & = & \max\{Y_0, \mathbb{E}[Z_1, \mathcal{F}_0]\} \\ & & = & \max\{0, 21.6\} \\ & & = & 21.6 \end{array}$$

The time $t = 0$ fair-price of this American put is 21.6.

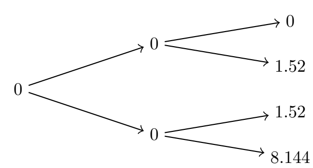
By inspecting the calculations of each Z_t above, the only time-point $t < 3$ where $Z_t(\omega) = Y_t(\omega)$ is when $t = 2$ and $\omega = \omega_{dd}$.

Thus, the only scenario where exercising early is optimal is when the price falls twice in the first two time-periods.

Answer 3. a)
Note that the interest rate for a three month period is $r = 0.03$. Consider the tree below which shows the possible evolutions of the price process S_t at each time point and event.



For this European put option we have that the pay-out is $Y_T(\omega) = g(S_T(\omega)) = \{42 - S_T(\omega)\}_+$ for each $\omega \in \Omega$ and 0 for all other t. I summarise the pay-outs in the tree below

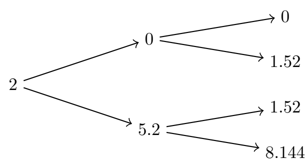


The time $t = 0$ value for this European put option is

$$\begin{array}{rcl} \Pi(0) & = & \frac{1}{(1+r)^T} \sum_{n=0}^{T} \binom{T}{n} q^n (1-q)^{T-n} g(S_0 u^n d^{T-n}) \\ & = & \frac{1}{1.03^2} \sum_{n=0}^{2} \binom{2}{n} \frac{11^n 7^{2-n}}{18^2} \left\{42 - S_0 u^n d^{T-n}\right\}_+ \\ & = & \frac{1}{1.03^2} \sum_{n=0}^{1} \binom{2}{n} \frac{11^n 7^{2-n}}{18^2} \left\{42 - 30 \cdot (1.1)^n \cdot (0.92^{2-n})\right\}_+ \\ & = & \frac{1}{1.03^2} \left(1 \cdot \frac{7^2}{18^2} \cdot 8.144 + 2 \cdot \frac{11 \cdot 7}{18^2} \cdot 1.52\right) \\ & \simeq & 1.84194\ldots \end{array}$$

Answer 3. b)

The price process S_t is the same as described in `3. a)`. For this American put option we have that the pay-out is $Y_t(\omega) = \{42 - S_t(\omega)\}_+$ for each $t = 0, 1, 2, 3$ and $\omega \in \Omega$. I summarise the pay-outs in the tree below



I will now construct a Snell Envelope $\{Z_t\}$ for the payoff process $\{Y_t\}$.

By definition, $Z_2(\omega) = Y_2(\omega)\ \forall\ \omega \in \Omega$.

Let ω_u, ω_d represents the events the price of underlying asset increases/decreases during the first time-step, respectively.

Consider time-period $t = 1$ and that the event ω_u has occurred. Then

$$\mathbb{E}[Z_2|\mathcal{F}_1] = q \cdot 0 + (1-q) \cdot 1.52 = \frac{11}{18} \cdot 0 + \frac{7}{18} \cdot 1.52 = 0.5911$$

Thus, given ω_u occurred, we can deduce Z_1

$$\begin{array}{rcl} Z_1 & := & \max\{Y_1, \mathbb{E}[Z_2|\mathcal{F}_1]\} \\ & = & \max\{0, 0.5911\} \\ & = & 0.5911 \end{array}$$

Consider time-period $t = 1$ and that the event ω_d has occurred. Then

$$\mathbb{E}[Z_2|\mathcal{F}_1] = \frac{11}{18} \cdot 1.52 + \frac{7}{18} \cdot 8.144 = 4.096$$

Thus, given ω_d occurred, we can deduce Z_1

$$\begin{array}{rcl} Z_1 & := & \max\{Y_1, \mathbb{E}[Z_2|\mathcal{F}_1]\} \\ & = & \max\{5.2, 4.096\} \\ & = & 5.2 \end{array}$$

Consider time-period $t = 0$. The

$$\mathbb{E}[Z_1|\mathcal{F}_0] = \mathbb{E}[Z_1] = \frac{11}{18} \cdot 0.5911 + \frac{7}{18} \cdot 5.2 = 2.3834$$

We can deduce Z_0

$$\begin{array}{rcl} Z_0 & := & \max\{Y_0, \mathbb{E}[Z_1|\mathcal{F}_0]\} \\ & = & \max\{2, 2.3834\} \\ & = & 2.3834 \end{array}$$

The time $t = 0$ fair price of this American Option is 2.3834.

Answer 4.

In this problem we have that

$$S_0 = 69 \quad K = 70 \quad r = 0.06 \quad \sigma^2 = 0.35 \quad U = 1/2$$

U is defined to equal $1/2$ as the interest rate is defined over a year but the option matures in six months.

By the Black-Scholes formula we have that the time $t=0$ fair price for this European call option is

$$\begin{array}{rcl}\Pi^{BS}(0) & := & S_0\Phi\left(d_1(S_0,U)\right) - Ke^{-rU}\Phi\left(d_2(S_0,U)\right) \\ & = & 69\Phi\left(d_1(69,1/2)\right) - 70e^{-0.03}\Phi\left(d_2(69,1/2)\right) \\ \text{where} & & \\ d_1(s,u) & := & \dfrac{\ln(s/K)+(r+\sigma^2/2)U}{\sqrt{\sigma^2 U}} \\ \implies\quad d_1(69,1/2) & = & \dfrac{\ln(69/70)+(0.06+0.35/2)\cdot(1/2)}{\sqrt{0.35\cdot(1/2)}} \\ & = & 0.2468\dots \\ \text{and} & & \\ d_2(s,u) & := & \dfrac{\ln(s/K)+(r-\sigma^2/2)U}{\sqrt{\sigma^2 U}} \\ \implies\quad d_2(69,1/2) & = & \dfrac{\ln(69/70)-(0.06+0.35/2)\cdot(1/2)}{\sqrt{0.35\cdot(1/2)}} \\ & = & -0.1718\dots\end{array}$$

Note that

$$\Phi(0.2468)=0.597 \quad \text{and} \quad \Phi(-0.1718)=0.432$$

Thus

$$\Pi^{BS}(0)=69\cdot 0.597-70\cdot 0.432=10.432$$

The time $t=0$ fair-price for this European call option is 10.432.

I will now determine the time $t=0$ price for the equivalent European put option.

The Put-Call parity states that

$$S_0+P_0-C_0=Ke^{-rT}$$

Thus, in this problem we have

$$\begin{array}{rrcl} & 69+P_0-10.432 & = & 70e^{-0.06/2} \\ \implies & P_0 & = & 9.363\end{array}$$

The time $t=0$ fair-price for the equivalent put option is 9.363.