Financial Mathematics - Assessed Problem Sheet 2

Dom Hutchinson

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Answer 1. (a) i.
Consider time $t = 0$ and define $p_1 := \mathbb{Q}(S_1 = 16)$, then

$$\begin{array}{rcl} 10 & = & 16p_1 + 8(1 - p_1) \\ \implies p_1 & = & 1/4 \end{array}$$

Thus $\mathbb{Q}(S_1 = 16) = 1/4$ and $\mathbb{Q}(S_1 = 8) = 1 - 1/4 = 3/4$. Now, consider time $t = 1$ and that event ω has occurred with $\omega \in \{\omega_1, \omega_2\}$. Define $p_2 := \mathbb{Q}(S_2 = 18|S_1 = 16)$, then

$$\begin{array}{rcl} 16 & = & 18p_2 + 12(1 - p_2) \\ \implies p_2 & = & 2/3 \end{array}$$

Thus $\mathbb{Q}(S_2 = 18|S_1 = 16) = 2/3$ and $\mathbb{Q}(S_2 = 12|S_1 = 16) = 1/3$. Now, consider time $t = 1$ and that event ω has occurred with $\omega \in \{\omega_3, \omega_4\}$. Define $p_3 := \mathbb{Q}(S_2 = 12|S_1 = 8)$, then

$$\begin{array}{rcl} 8 & = & 12p_3 + 6(1 - p_3) \\ \implies p_3 & = & 1/3 \end{array}$$

Thus $\mathbb{Q}(S_2 = 12|S_1 = 8) = 1/3$ and $\mathbb{Q}(S_2 = 6|S_1 = 8) = 2/3$.

We can use these conditional probabilities to work out the probability of each event $\{\omega_1, \omega_2, \omega_3, \omega_4\}$ under $\mathbb{Q}$.

$$\begin{array}{rcl} \mathbb{Q}(\{\omega_1\}) & = & \mathbb{Q}(S_2 = 18|S_1 = 16)\mathbb{Q}(S_1 = 16) \\ & = & (2/3) \times (1/4) \\ & = & 1/6 \\ \mathbb{Q}(\{\omega_2\}) & = & \mathbb{Q}(S_2 = 12|S_1 = 16)\mathbb{Q}(S_1 = 16) \\ & = & (1/3) \times (1/4) \\ & = & 1/12 \\ \mathbb{Q}(\{\omega_3\}) & = & \mathbb{Q}(S_2 = 12|S_1 = 8)\mathbb{Q}(S_1 = 8) \\ & = & (1/3) \times (3/4) \\ & = & 1/4 \\ \mathbb{Q}(\{\omega_4\}) & = & \mathbb{Q}(S_2 = 6|S_1 = 8)\mathbb{Q}(S_1 = 8) \\ & = & (2/3) \times (3/4) \\ & = & 1/2 \end{array}$$

To confirm that this produces a probability measure, note that

$$(1/6) + (1/12) + (1/4) + (1/2) = 1$$

Answer 1. (a) ii.

$$\begin{aligned}
X(\omega_1) &= \left\{\frac{1}{3}(10+16+18)-14\right\}_+ \\
&= \left\{\frac{44}{3}-14\right\}_+ \\
&= 2/3 \\
X(\omega_2) &= \left\{\frac{1}{3}(10+16+12)-14\right\}_+ \\
&= \left\{\frac{38}{3}-14\right\}_+ \\
&= 0 \\
X(\omega_3) &= \left\{\frac{1}{3}(10+8+12)-14\right\}_+ \\
&= \left\{\frac{30}{3}-14\right\}_+ \\
&= 0 \\
X(\omega_4) &= \left\{\frac{1}{3}(10+8+6)-14\right\}_+ \\
&= \left\{\frac{24}{3}-14\right\}_+ \\
&= 0
\end{aligned}$$

$$\begin{aligned}
Y(\omega_1) &= \{\max(10,16,18)-14\}_+ \\
&= \{18-14\}_+ \\
&= 4 \\
Y(\omega_2) &= \{\max(10,16,12)-14\}_+ \\
&= \{16-14\}_+ \\
&= 2 \\
Y(\omega_3) &= \{\max(10,8,12)-14\}_+ \\
&= \{12-14\}_+ \\
&= 0 \\
Y(\omega_4) &= \{\max(10,8,6)-14\}_+ \\
&= \{10-14\}_+ \\
&= 0
\end{aligned}$$

Answer 1. (a) iii.

The *Risk-Neutral Valuation Principle* states that, for all attainable contingent claims X, the following holds

$$V_t^* = \mathbb{E}_Q[X/B_t|\mathcal{F}_t] \text{ for } t=0,\dots,T \text{ and all } \mathbb{Q}$$

Thus, at time $t=0$

$$V_0 = \mathbb{E}_{\mathbb{Q}}[X|\mathcal{F}_0] = \mathbb{E}_{\mathbb{Q}}[X]$$

We can use this to derive the risk-neutral prices for the two options at time $t=0$

$$\begin{aligned}
V_0^X &= \mathbb{E}_{\mathbb{Q}}[X] \\
&= (2/3)\cdot(1/6)+0\cdot(1/12)+0\cdot(1/4)+0\cdot(1/2) \\
&= 1/9
\end{aligned}$$

$$\begin{aligned}
V_0^Y &= \mathbb{E}_{\mathbb{Q}}[Y] \\
&= 4\cdot(1/6)+2\cdot(1/12)+0\cdot(1/4)+0\cdot(1/2) \\
&= 5/6
\end{aligned}$$

Thus the risk-neutral price for the Asian option at time $t = 0$ is $1/9$ and for the look-back option it is $5/6$

Answer 1. (a) iv.
Note that since $r = 0$, $B_t = 1$ for $i = 0, 1, 2$.
Consider the Asian option and let $H^X(t) := \{H_0^Y(t), H_1^Y(t)\}$ denote a portfolio which only has access to a bank account & Asian options for each time point $t = 0, 1, 2$.

consider the value of the Asian option at time $t = 1$ and $t = 2$

$$\begin{array}{rcl} V_1^X(\omega) &=& \begin{cases} \mathbb{E}[X|S_1 = 16] & \text{if } \omega \in \{\omega_1, \omega_2\} \\ \mathbb{E}[X|S_1 = 8] & \text{if } \omega \in \{\omega_3, \omega_4\} \end{cases} \\ &=& \begin{cases} (2/3) \cdot p_2 + 0 \cdot (1 - p_2) & \text{if } \omega \in \{\omega_1, \omega_2\} \\ 0 & \text{if } \omega \in \{\omega_3, \omega_4\} \end{cases} \\ &=& \begin{cases} (2/3) \cdot (2/3) & \text{if } \omega \in \{\omega_1, \omega_2\} \\ 0 & \text{if } \omega \in \{\omega_3, \omega_4\} \end{cases} \\ &=& \begin{cases} 4/9 & \text{if } \omega \in \{\omega_1, \omega_2\} \\ 0 & \text{if } \omega \in \{\omega_3, \omega_4\} \end{cases} \\ V_2^X(\omega) &=& X(\omega) \ \forall \ \omega \end{array}$$

To find a self-financing portfolio we start at time $t = 2$ and find that if $\omega \in \{\omega_1, \omega_2\}$

$$\begin{array}{rcl} V_2^X(\omega_1) = 2/3 &=& H_0^X(2)(\omega_1, \omega_2) + 18 \cdot H_1^X(2)(\omega_1, \omega_2) \\ V_2^X(\omega_2) = 0 &=& H_0^X(2)(\omega_1, \omega_2) + 12 \cdot H_1^X(2)(\omega_1, \omega_2) \end{array}$$

$$\begin{array}{rrcl} \implies & 2/3 &=& 6H_1^X(2)(\omega_1, \omega_2) \\ \implies & H_1^X(2)(\omega_1, \omega_2) &=& 1/9 \\ \\ \implies & 2 &=& H_0^X(2)(\omega_1, \omega_2) + 12 \cdot (1/9) \\ \implies & H_0^X(2)(\omega_1, \omega_2) &=& -4/3 \end{array}$$

And if $\omega \in \{\omega_3, \omega_4\}$

$$\begin{array}{rcl} V_2^X(\omega_3) = 0 &=& H_0^X(2)(\omega_3, \omega_4) + 12 \cdot H_1^X(2)(\omega_3, \omega_4) \\ V_2^X(\omega_4) = 0 &=& H_0^X(2)(\omega_3, \omega_4) + 6 \cdot H_1^X(2)(\omega_3, \omega_4) \end{array}$$

$$\begin{array}{rrcl} \implies & H_0^X(2)(\omega_3, \omega_4) &=& 0 \\ \implies & H_1^X(2)(\omega_3, \omega_4) &=& 0 \end{array}$$

Let $H_0^X(1) := H_0^X(1)(\omega_1, \omega_2, \omega_3, \omega_4)$ and $H_1^X(1) := H_1(1)(\omega_1, \omega_2, \omega_3, \omega_4)$, then we have

$$\begin{array}{rcll} V_1^X(\omega) = 4/9 &=& H_0^X(1) + 16 \cdot H_1^X(1) & \text{if } \omega \in \{\omega_1, \omega_2\} \\ V_1^X(\omega) = 0 &=& H_0^X(1) + 8 \cdot H_1^X(1) & \text{if } \omega \in \{\omega_3, \omega_4\} \end{array}$$

$$\begin{array}{rrcl} \implies & 4/9 &=& 8H_1^X(1) \\ \implies & H_1^X(1) &=& 1/18 \\ \\ \implies & 0 &=& H_0^X(1) + 8 \cdot (1/18) \\ \implies & H_0^X(1) &=& -4/9 \end{array}$$

Thus, a self-financing trading strategy with access to a bank account and the look-back option is as follows

$$\begin{array}{rcll} H^X(1)(\omega) &=& (-4/9, 1/18) & \forall \ \omega \\ H^X(2)(\omega) &=& (-4/3, 1/9) & \text{if } \omega \in \{\omega_1, \omega_2\} \\ H^X(2)(\omega) &=& (0, 0) & \text{if } \omega \in \{\omega_3, \omega_4\} \end{array}$$

To confirm this trading strategy is self-financing I show that $H_0(t+1)B_t+H_1(t+1)S_1(t) = V_t\ \forall\ t$.

$$\begin{array}{r|rcll} H^Y(1)(\omega) & B_0\cdot H_0^X(1)(\omega)+S_1(0)\cdot H_1^Y(1)(\omega) & = & -(4/9)+10\cdot(1/18) & \\ & & = & 1/9 = V_0^X & \\ H^Y(2)(\omega) & B_1\cdot H_0^X(2)(\omega)+S_1(1)\cdot H_1^Y(2)(\omega) & = & -(4/3)+16\cdot(1/9) & \text{if } \omega\in\{\omega_1,\omega_2\} \\ & & = & 4/9 = V_1^Y(\omega) & \\ H^Y(2)(\omega) & B_1\cdot H_0^X(2)(\omega)+S_1(1)\cdot H_1^Y(2)(\omega) & = & 0+8\cdot 0 & \text{if } \omega\in\{\omega_3,\omega_4\} \\ & & = & 0 = V_1^Y(\omega) & \end{array}$$

This requirement holds in all cases.

Now, consider the look-back option and let $H^Y(t) := \{H_0^Y(t), H_1^Y(t)\}$ denote a portfolio which only has access to a bank account & look-back options for each time point $t=0,1,2$.

Consider the value of the look-back option at time $t=1$ and $t=2$

$$\begin{array}{rcl} V_1^Y(\omega) & = & \begin{cases} \mathbb{E}[Y|S_1=16] & \text{if } \omega\in\{\omega_1,\omega_2\} \\ \mathbb{E}[Y|S_1=8] & \text{if } \omega\in\{\omega_3,\omega_4\} \end{cases} \\ & = & \begin{cases} 4p_2+2(1-p_2) & \text{if } \omega\in\{\omega_1,\omega_2\} \\ 0 & \text{if } \omega\in\{\omega_3,\omega_4\} \end{cases} \\ & = & \begin{cases} 4\cdot(2/3)+2(1/3) & \text{if } \omega\in\{\omega_1,\omega_2\} \\ 0 & \text{if } \omega\in\{\omega_3,\omega_4\} \end{cases} \\ & = & \begin{cases} 10/3 & \text{if } \omega\in\{\omega_1,\omega_2\} \\ 0 & \text{if } \omega\in\{\omega_3,\omega_4\} \end{cases} \\ V_2^Y(\omega) & = & Y(\omega)\ \forall\ \omega \end{array}$$

To find a self-financing portfolio we start at time $t=2$ and find that if $\omega\in\{\omega_1,\omega_2\}$

$$\begin{array}{rrcl} & V_2^Y(\omega_1)=4 & = & H_0^Y(2)(\omega_1,\omega_2)+18\cdot H_1^Y(2)(\omega_1,\omega_2) \\ & V_2^Y(\omega_2)=2 & = & H_0^Y(2)(\omega_1,\omega_2)+12\cdot H_1^Y(2)(\omega_1,\omega_2) \\ \\ \implies & 2 & = & 6H_1^Y(2)(\omega_1,\omega_2) \\ \implies & H_1^Y(2)(\omega_1,\omega_2) & = & 1/3 \\ \\ \implies & 2 & = & H_0^Y(2)(\omega_1,\omega_2)+12\cdot(1/3) \\ \implies & H_0^Y(2)(\omega_1,\omega_2) & = & -2/3 \end{array}$$

And if $\omega\in\{\omega_3,\omega_4\}$

$$\begin{array}{rrcl} & V_2^Y(\omega_3)=0 & = & H_0^Y(2)(\omega_3,\omega_4)+12\cdot H_1^Y(2)(\omega_3,\omega_4) \\ & V_2^Y(\omega_4)=0 & = & H_0^Y(2)(\omega_3,\omega_4)+6\cdot H_1^Y(2)(\omega_3,\omega_4) \\ \\ \implies & H_0^Y(2)(\omega_3,\omega_4) & = & 0 \\ \implies & H_1^Y(2)(\omega_3,\omega_4) & = & 0 \end{array}$$

Let $H_0^Y(1) := H_0(1)(\omega_1,\omega_2,\omega_3,\omega_4)$ and $H_1^Y(1) := H_1(1)(\omega_1,\omega_2,\omega_3,\omega_4)$, then we have

$$\begin{array}{rrcll} & V_1^Y(\omega)=10/3 & = & H_0^Y(1)+16\cdot H_1^Y(1) & \text{if } \omega\in\{\omega_1,\omega_2\} \\ & V_1^Y(\omega)=0 & = & H_0^Y(1)+8\cdot H_1^Y(1) & \text{if } \omega\in\{\omega_3,\omega_4\} \\ \\ \implies & 10/3 & = & 8H_1^Y(1) & \\ \implies & H_1^Y(1) & = & 5/12 & \\ \\ \implies & 0 & = & H_0^Y(1)+8\cdot(5/12) & \\ \implies & H_0^Y(1) & = & -40/12 & \end{array}$$

Thus, a self-financing trading strategy with access to a bank account and the look-back option is as follows

$$\begin{array}{rcll} H^Y(1)(\omega) &=& (-40/12, 5/12) & \forall\,\omega \\ H^Y(2)(\omega) &=& (-2/3, 1/3) & \text{if } \omega \in \{\omega_1, \omega_2\} \\ H^Y(2)(\omega) &=& (0, 0) & \text{if } \omega \in \{\omega_3, \omega_4\} \end{array}$$

To confirm this trading strategy is self-financing I show that $H_0(t+1)B_t + H_1(t+1)S_1(t) = V_t \;\forall\, t$.

$$\begin{array}{r|lcll} H^Y(1)(\omega) & B_0 \cdot H_0^Y(1)(\omega) + S_1(0) \cdot H_1^Y(1)(\omega) &=& -(40/12) + 10 \cdot (5/12) & \\ & &=& 5/6 = V_0^Y & \\ H^Y(2)(\omega) & B_1 \cdot H_0^Y(2)(\omega) + S_1(1) \cdot H_1^Y(2)(\omega) &=& -(2/3) + 16 \cdot (1/3) & \text{if } \omega \in \{\omega_1, \omega_2\} \\ & &=& 10/3 = V_1^Y(\omega) & \\ H^Y(2)(\omega) & B_1 \cdot H_0^Y(2)(\omega) + S_1(1) \cdot H_1^Y(2)(\omega) &=& 0 + 8 \cdot 0 & \text{if } \omega \in \{\omega_3, \omega_4\} \\ & &=& 0 = V_1^Y(\omega) & \end{array}$$

This requirement holds in all cases.

Answer 1. (b)

Assume that ϵ is known, T is large and note that $B_t = 1 \;\forall\, t \in [1, T]$ since the interest rate is $r = 0$.

Consider the following trading strategy

> If, at time-point $t \in (1, T)$ it is the case that $S(t) \leq \epsilon$ then:
>
> - In time-point t borrow £$S(t)$ from the bank-account and buy a unit of the security for £$S(t)$.
> - In time-point $t+1$ sell your security for £$S(t+1)$ and place all of this revenue into the bank account (reimbursing the bank at the same time).
>
> Otherwise, do nothing.

This strategy can be expressed mathematically as

$$\begin{array}{rcl} H(0) &=& (0, 0) \\ H(t+1) &=& \begin{cases} \big(H_0(t) - S(t), 1\big) & \text{if } S(t) \leq \epsilon \\ \big(H_0(t) + S(t), 0\big) & \text{if } S(t-1) \leq \epsilon \\ \big(H_0(t), 0\big) & \text{otherwise} \end{cases} \end{array}$$

For this specific strategy we need to further assume that $\epsilon > e^{1-T}$ as otherwise the main action could never be triggered.

I will how justify that this trading strategy is an arbitrage opportunity.

Consider the value of this portfolio at time-point $t = 0$

$$V_0 = H_0(0) \cdot B_0 + H_1(0) \cdot S(1) = 0 \cdot 1 + 0 \cdot 1 = 0$$

Thus the requirement that $V_0 = 0$ is fulfilled.

Suppose that $S(t) \leq \epsilon$ but $S(t-1) > \epsilon$. This means that the price process must have decreased in value between these time steps and thus $S(t) = S(t-1)e^{-1} \implies S(t)e > \epsilon$.

Now consider $S(t+1)$, we know that $S(t+1) = S(t)\exp\{X(t) + 2\}$ by the definition of the price process and since $S(t) \leq \epsilon$. Since $X(t) \in \{-1, 1\}$ then $X(t)+2 \geq 1$, meaning $S(t+1) \geq S(t)e > \epsilon$. This shows that the price process only ever goes below threshold ϵ for a single time-period at a time.

Now, suppose the action is triggered at time-point t and consider the price process over this and the next time-point

$$\begin{array}{rrcl} & S(t) & \leq & \epsilon \\ \implies & S(t+1) & = & S(t)\exp\{X(t)+2\} \\ \implies & S(t+1) & > & S(t) \text{ since } X(t)+2 \geq 1 \end{array}$$

This shows that when we sell the security in time-point $t+1$ it is always worth more than the price we bought it for, meaning we can fully reimburse the bank-account and have a surplus left over in our bank-account. Moreover, this shows that we are guaranteed to make a profit whenever the main action of this strategy is triggered. And, since this strategy takes no other actions, it cannot lose money between time-periods.

Since we assumed that T is large and since $\{X_t\}$ is a symmetric random walk it is expected that the action will occur at least once. This means the criteria that $V_T = H_0(T) \geq 0$ and $\mathbb{E}[V_T] > 0$ are fulfilled.

To show that this trading strategy H is self-financing I need to show that $V_t(\omega) = H_0(t+1) + H_1(t+1)S(t)$ for all t, ω. There are three cases to consider

i). Case $S(t) \leq \epsilon \implies S(t-1) > \epsilon$ (Shown above).

No action was taken in the last time-period. Thus $V_t = H_0(t)$.

$$\begin{array}{rl} & H_0(t+1) + H_1(t+1)S(t) \\ = & \big(H_0(t) - S(t)\big) + S(t) \\ = & H_0(t) \\ = & V_t \end{array}$$

ii). Case $S(t-1) \leq \epsilon \implies S(t) > \epsilon$ (Shown above).

This means an option was bought in the last-time period for $S(t-1)$ and that option is now worth $S(t)$, thus $V_t = H_0(t) - S(t-1) + S(t)$.

$$\begin{array}{rl} & H_0(t+1) + H_1(t+1)S(t) \\ = & \big(H_0(t) + S(t)\big) + 0 \\ = & H_0(t) + S(t) \\ = & (H_0(t-1) - S(t-1)) + S(t) \text{ as } S(t-1) \leq \epsilon \\ = & V_t \end{array}$$

iii). Otherwise (ie $S(t), S(t-1) > \epsilon$)

No actions are taken so $V_t = H_0(t)$.

$$\begin{array}{rl} & H_0(t+1) + H_1(t+1)S(t) \\ = & H_0(t) + 0 \\ = & V_t \end{array}$$

This shows that H is self-financing, and thus fulfils all criteria to be an arbitrage opportunity.