Financial Mathematics - Problem Sheet 3

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Question 1.
The following table shows the price process of a certain stock at three successive time points

t	$S(t,\omega_1)$	$S(t,\omega_2)$	$S(t,\omega_3)$	$S(t,\omega_4)$	$S(t,\omega_5)$	$S(t,\omega_6)$	$S(t,\omega_7)$	$S(t,\omega_8)$
0	10	10	10	10	10	10	10	10
1	15	15	15	10	10	9	9	9
2	19	18	14	11	9	12	13	8

Question 1. a)
Identify for all three time points the corresponding partitions $\mathcal{P}_0, \mathcal{P}_1, \mathcal{P}_2$ of $\Omega = \{\omega_1, \dots, \omega_8\}$.

Answer 1. a)

$$\begin{array}{rcl}
\mathcal{P}_0 &=& \big\{\Omega\big\} \\
\mathcal{P}_1 &=& \big\{\{\omega_1,\omega_2,\omega_3\},\{\omega_4,\omega_5\},\{\omega_6,\omega_7,\omega_8\}\big\} \\
\mathcal{P}_2 &=& \big\{\{\omega_1\},\{\omega_2\},\{\omega_3\},\{\omega_4\},\{\omega_5\},\{\omega_6\},\{\omega_7\},\{\omega_8\}\big\}
\end{array}$$

Question 1. b)
Write down explicitly the corresponding σ-algebras $\mathcal{F}_0, \mathcal{F}_1, \mathcal{F}_2$.

Answer 1. b)

$$\begin{array}{rcl}
\mathcal{F}_0 &=& \big\{\emptyset,\Omega\big\} \\
\mathcal{F}_1 &=& \big\{\emptyset,\Omega,\{\omega_1,\omega_2,\omega_3\},\{\omega_4,\omega_5,\omega_6,\omega_7,\omega_8\}, \\
&& \{\omega_4,\omega_5\},\{\omega_1,\omega_2,\omega_3,\omega_6,\omega_7,\omega_8\},\{\omega_6,\omega_7,\omega_8\}, \\
&& \{\omega_1,\omega_2,\omega_3,\omega_4,\omega_5\}\big\} \\
\mathcal{F}_2 &=& 2^\Omega
\end{array}$$

Question 2.
Consider a single-period model with two risky securities S_1 and S_2. We assume that there is a bank account with a risk-free interest rate $r = 1/9$ and the price process is given by the following table

n	$S_n(0)$	$S_n(1)(\omega_1)$	$S_n(1)(\omega_2)$	$S_n(1)(\omega_3)$	$S_n(1)(\omega_4)$
1	15	20	20	40/3	20/3
2	30	40	80/3	80/3	20/3

Question 2. a)
Specify the value and gains processes V and G for this model as well as their discounted versions V^* and G^*.

Answer 2. a)
The value process $V_t(\omega)$ for this model is as follows

$\omega \setminus t$	0	1
ω_1	$H_0 + 15H_1 + 30H_2$	$(10/9)H_0 + 20H_1 + 40H_2$
ω_2	$H_0 + 15H_1 + 30H_2$	$(10/9)H_0 + 20H_1 + (80/3)H_2$
ω_3	$H_0 + 15H_1 + 30H_2$	$(10/9)H_0 + (40/3)H_1 + (80/3)H_2$
ω_4	$H_0 + 15H_1 + 30H_2$	$(10/9)H_0 + (20/3)H_1 + (20/3)H_2$

and the discounted value process $V_t^*(\omega) = V_t(\omega)/B_t$ for this model is

$\omega \setminus t$	0	1
ω_1	$H_0 + 15H_1 + 30H_2$	$H_0 + 18H_1 + 36H_2$
ω_2	$H_0 + 15H_1 + 30H_2$	$H_0 + 18H_1 + 24H_2$
ω_3	$H_0 + 15H_1 + 30H_2$	$H_0 + 12H_1 + 24H_2$
ω_4	$H_0 + 15H_1 + 30H_2$	$H_0 + 6H_1 + 6H_2$

The gains process $G(\omega)$ for this model is as follows

ω	$G(\omega)$
ω_1	$(1/9)H_0 + 5H_1 + 10H_2$
ω_2	$(1/9)H_0 + 5H_1 - (10/3)H_2$
ω_3	$(1/9)H_0 - (5/3)H_1 - (10/3)H_2$
ω_4	$(1/9)H_0 - (25/3)H_1 - (70/3)H_2$

and the discounted gains process $G^*(\omega) = \sum_{i=1}^{2} H_1 \Delta S_i^*(\omega)$ for this model is as follows

ω	$G^*(\omega)$
ω_1	$3H_1 + 6H_2$
ω_2	$3H_1 - 6H_2$
ω_3	$-3H_1 - 6H_2$
ω_4	$-9H_1 - 24H_2$

Question 2. b)
Specify $\mathbb{W}$ where

$$\mathbb{W} = \{X \in \mathbb{R}^K : X = G^* \text{ for some trading strategy } H\}$$

and using the fact that an arbitrage opportunity is a trading strategy which requires $G^*(\omega) \geq 0$, determine the arbitrage opportunities in this model.

Hint: Consider $G^*(\omega_1)$ and $G^*(\omega_3)$ first.

Answer 2. b)
Consider $X \in \mathbb{R}^4$ st $X = G^*$. Then

$$\begin{pmatrix} X_1 \\ X_2 \\ X_3 \\ X_4 \end{pmatrix} = \begin{pmatrix} 3H_1 + 6H_2 \\ 3H_1 - 6H_2 \\ -3H_1 - 6H_2 \\ -9H_1 - 24H_2 \end{pmatrix}$$

Thus

$$\mathbb{W} = \text{span}\left\{\begin{pmatrix} 3 \\ 3 \\ -3 \\ -9 \end{pmatrix}, \begin{pmatrix} 6 \\ -6 \\ -6 \\ -24 \end{pmatrix}\right\} = \text{span}\left\{\begin{pmatrix} 1 \\ 1 \\ -1 \\ -3 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ -1 \\ -4 \end{pmatrix}\right\}$$

Consider the set $\mathbb{A} := \{X \in \mathbb{R}^4 : x \geq 0 \; \forall \; x \in X\}$ and the elements of $\mathbb{W} \cap \mathbb{A}$. This elements form the set of possible gains which would produce a risk-free profit (i.e. an arbitrage opportunity).

$$\begin{array}{rcl}\mathbb{W} \cap \mathbb{A} & = & \big\{X \in \mathbb{R}^4 : X_1 = x_3 = 0; X_2, X_4 > 0\big\} \\ X \in (\mathbb{W} \cap \mathbb{A}) & \implies & X = \begin{pmatrix}0 & \lambda & 0 & \mu\end{pmatrix} \text{ with } \lambda, \mu > 0\end{array}$$

How consider the Xs where $X = G^*$ and $X \in (\mathbb{W} \cap \mathbb{A})$.

$$\begin{array}{rrcl} & X_1 = 3H_1 + 6H_2 & = & 0 \\ \implies & H_1 & = & -2H_2 \\ & & & \\ & X_2 = 3H_1 - 6H_2 & = & \lambda \\ \implies & -12H_2 & = & \lambda \\ \implies & H_2 & = & -\dfrac{1}{12}\lambda \\ \implies & H_1 & = & \dfrac{1}{6}\lambda \\ & & & \\ & X_4 = -9H_1 - 24H_2 & = & \mu \\ \implies & -6H_2 & = & \mu \\ \implies & H_2 & = & -\dfrac{1}{6}\mu \\ \implies & H_1 & = & \dfrac{1}{3}\mu \\ \implies & \mu & = & \dfrac{1}{2}\lambda \end{array}$$

Thus the trading strategies which exploit arbitrage opportunities in this model as those in the set

$$\left\{H \in \mathbb{R}^3 : H = \begin{pmatrix}0 & \frac{1}{6}\lambda & -\frac{1}{12}\lambda\end{pmatrix} : \lambda > 0\right\}$$

Question 2. c)
Briefly explain why there are no risk-neutral probability measures and why this model is not complete

Answer 2. c)
The "No-Arbitrage Theorem" states that risk-neutral probability measures only exists for a model <u>iff</u> no arbitrage opportunities exist. In this model arbitrage opportunities do exist, thus no risk-neutral probability measures can.

Consider matrix A which summarises this model

$$A = \begin{pmatrix} B_1(\omega_1) & S_1(1)(\omega_1) & S_2(1)(\omega_1) \\ B_1(\omega_2) & S_1(1)(\omega_2) & S_2(1)(\omega_2) \\ B_1(\omega_3) & S_1(1)(\omega_3) & S_2(1)(\omega_3) \\ B_1(\omega_4) & S_1(1)(\omega_4) & S_2(1)(\omega_4) \end{pmatrix} == \begin{pmatrix} 10/9 & 20 & 40 \\ 10/9 & 20 & 80/3 \\ 10/9 & 40/3 & 80/3 \\ 10/9 & 20/3 & 20/3 \end{pmatrix}$$

Matrix A only has three linearly-independent columns, thus AH can only span $\mathbb{R}^3$ for any trading strategy H. As AH cannot span $\mathbb{R}^4$, the market is <u>not</u> complete.

Question 3.
Consider a sample space $\Omega = \{\omega_1, \omega_2, \omega_3, \omega_4\}$.

Question 3. a)
Decide whether the following collections of subsets of Ω are σ-algebras or not:

$$\mathcal{F}_1 = \{\emptyset, \{\omega_1, \omega_2\}, \{\omega_3, \omega_4\}, \Omega\} \quad \mathcal{F}_2 = \{\emptyset, \{\omega_1\}, \{\omega_2\}, \{\omega_3\}, \{\omega_4\}\}$$

Answer 3. a)

$\mathcal{F}_1$ is a σ-algebra.

$\mathcal{F}_2$ is <u>not</u> a σ-algebra as (among other reasons) $\{\omega_1\}, \{\omega_2\} \in \mathcal{F}_2$ but $\{\omega_1, \omega_2\} \notin \mathcal{F}_2$.

Question 3. b)
Define a σ-algebra $\mathcal{F}$ and a function $\omega \to X(\omega)$ with values in the real numbers st X is not measurable wrt $\mathcal{G}$.

Answer 3. b)
Consider the σ-algebra $\mathcal{G} := \{\emptyset, \Omega\}$ and the function $X(\omega_i) = i$. X is <u>not</u> measurable wrt $\mathcal{G}$ as $\{\omega_1\}, \{\omega_2\}, \{\omega_3\}, \{\omega_4\} \notin \mathcal{G}$.

Question 3. c)
Find a σ-algebra $\mathcal{H}$ st <u>all</u> function $\omega \to X(\omega)$ are measurable wrt $\mathcal{H}$.

Answer 3. c)
The powerset of Ω contains all possible subsets of Ω and thus all functions $\omega \to X(\omega)$ are measurable wrt it to.

$$\mathcal{H} = 2^{\Omega}$$