Financial Mathematics - Problem Sheet 6

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Answer 1.
First I find a risk-neutral probability measure $\mathbb{Q}$ for this security.
Let $p_1 := \mathbb{Q}(S_1 = 8)$. Then

$$\begin{array}{rcl} 5 & = & 8p_1 + 4(1-p_1) \\ & = & 4 + 4p_1 \\ \implies p_1 & = & 1/4 \end{array}$$

Thus $\mathbb{Q}(S_1 = 8) = 1/4$ and $\mathbb{Q}(S_1 = 4) = 1 - p_1 = 3/4$.
Let $p_2 := \mathbb{Q}(S_2 = 9|S_1 = 8)$. Then

$$\begin{array}{rcl} 8 & = & 9p_2 + 6(1-p_2) \\ & = & 6 + 3p_2 \\ \implies p_2 & = & 2/3 \end{array}$$

Thus $\mathbb{Q}(S_2 = 9|S_1 = 8) = 2/3$ and $\mathbb{Q}(S_2 = 6|S_1 = 8) = 1 - p_2 = 1/3$.
Let $p_3 := \mathbb{Q}(S_2 = 6|S_1 = 4)$. Then

$$\begin{array}{rcl} 4 & = & 6p_2 + 3(1-p_3) \\ & = & 3 + 3p_3 \\ \implies p_3 & = & 1/3 \end{array}$$

Thus $\mathbb{Q}(S_2 = 6|S_1 = 4) = 1/3$ and $\mathbb{Q}(S_2 = 3|S_1 = 4) = 1 - p_3 = 2/3$.
Using these conditional probabilities we can deduce the risk-neutral probability measure $\mathbb{Q}$

$$\begin{array}{rcl} \mathbb{Q}(\omega_1) & = & \mathbb{Q}(S_2 = 9|S_1 = 8)\mathbb{Q}(S_1 = 8) = (2/3) \cdot (1/4) = 1/6 \\ \mathbb{Q}(\omega_2) & = & \mathbb{Q}(S_2 = 7|S_1 = 8)\mathbb{Q}(S_1 = 8) = (1/3) \cdot (1/4) = 1/12 \\ \mathbb{Q}(\omega_3) & = & \mathbb{Q}(S_2 = 6|S_1 = 4)\mathbb{Q}(S_1 = 8) = (1/3) \cdot (3/4) = 1/4 \\ \mathbb{Q}(\omega_4) & = & \mathbb{Q}(S_2 = 6|S_1 = 3)\mathbb{Q}(S_1 = 8) = (2/3) \cdot (3/4) = 1/2 \\ \implies \mathbb{Q} & = & (1/6, 1/12, 1/4, 1/2) \end{array}$$

Since I am considering an American put option, the payout is $Y_t(\omega) = \{e - S_t(\omega)\}^+$. These values are summarised in the following table

$$Y_t(\omega) = \begin{array}{c|ccc} t \setminus \omega & 0 & 1 & 2 \\ \hline \omega_1 & 1 & 0 & 0 \\ \omega_2 & 1 & 0 & 0 \\ \omega_3 & 1 & 2 & 0 \\ \omega_4 & 1 & 2 & 3 \end{array}$$

Now I calculate a Snell Envelope $\{Z_t\}$ for this payout process $\{Y_t\}$.

In states ω_1, ω_2 and time $t = 1$ we have

$$\begin{aligned} Z_1 &= \max\{Y_1, \mathbb{E}_{\mathbb{Q}}[Z_2|\mathcal{F}_1]\} \\ &= \max\{0, 0 \cdot (2/3) + 0 \cdot (1/3)\} \\ &= \max\{0, 0\} \\ &= 0 \end{aligned}$$

In states ω_3, ω_4 and time $t = 1$ we have

$$\begin{aligned} Z_1 &= \max\{Y_1, \mathbb{E}_{\mathbb{Q}}[Z_2|\mathcal{F}_1]\} \\ &= \max\{2, 0 \cdot (1/3) + 3 \cdot (2/3)\} \\ &= \max\{2, 2\} \\ &= 2 \end{aligned}$$

Thus, at time $t = 0$ we have

$$\begin{aligned} Z_0 &= \max\{Y_0, \mathbb{E}_{\mathbb{Q}}[Z_1|\mathcal{F}_0]\} \\ &= \max\{1, 0 \cdot (1/4) + 2 \cdot (3/4)\} \\ &= \max\{1, 3/2\} \\ &= 3/2 \end{aligned}$$

The time $t = 0$ fair price for this American put option is $3/2$.

By the optimal stopping theorem, the optimal exercise strategies $\tau(t)(\omega) = \min\{s \geq t : Z_s(\omega) = Y_s(\omega)$. For this problem these are

$$\begin{aligned} \tau(0)(\omega) &= \begin{cases} 1 & \text{if } \omega \in \{\omega_1, \omega_2\} \\ 2 & \text{if } \omega \in \{\omega_1, \omega_2\} \end{cases} \\ \tau(1)(\omega) &= \begin{cases} 1 & \text{if } \omega \in \{\omega_1, \omega_2\} \\ 2 & \text{if } \omega \in \{\omega_1, \omega_2\} \end{cases} \\ \tau(2)(\omega) &= 2\ \forall\ \omega \in \Omega \end{aligned}$$

This means it is optimal to exercise early whenever it becomes apparent you are in states ω_1, ω_2 (ie if $S_1 = 8$).

I now find a replicating portfolio for $\tau(0)$. In the second time-period the following equations must be satisfied if $\omega \in \{\omega_3, \omega_4\}$

$$\begin{aligned} & H_0(2) + 3H_1(2) &= Y_2(\omega_3) = 0 \\ \& \quad & H_0(2) + 6H_1(2) &= Y_2(\omega_4) = 3 \\ \\ \implies & 3H_1(2) &= 3 \\ \implies & H_1(2) &= 1 \\ \implies & H_0(2) + 3 \cdot 1 &= 0 \\ \implies & H_0(2) &= -3 \end{aligned}$$

Thus $H(2)(\omega) = (-3, 1)$ if $\omega \in \{\omega_3, \omega_4\}$. We do not consider the states ω_1, ω_2 in the second time-period as the option would already have been exercised when using the optimal stopping strategy.

In the first time-period the following equations must be satisfied

$$\begin{aligned} & H_0(1) + 8H_1(1) &= Z_1(\omega_1) = Z_2(\omega_2) = 0 \\ \& \quad & H_0(1) + 4H_1(1) &= Z_1(\omega_3) = Z_2(\omega_4) = 2 \\ \\ \implies & 4H_1(1) &= -2 \\ \implies & H_1(1) &= -1/2 \\ \implies & H_0(1) + 8 \cdot (-1/2) &= 0 \\ \implies & H_0(1) &= 4 \end{aligned}$$

Thus $H(1)(\omega) = (4, -1/2)$. This has value $4 + 5 \cdot (-1/2) = 3/2$ at time $t = 0$ which is the same as the fair price deduced above for the American put option.

Answer 2.
Let $\{X_t\}_t$ be the scores achieved each spin.
Note that each spin is independent so $\mathbb{E}[X_t|\mathcal{F}_s] = \mathbb{E}[X_t] = 51/2 \ \forall \ t, s$.

Now, consider constructing a *Snell Envelope* $\{Z_t\}_t$ for this game.

$$\begin{aligned} Z_3 &= \max\{X_3, \mathbb{E}[Z_4|\mathcal{F}_3]\} = \max\{X_3, \mathbb{E}[X_4|\mathcal{F}_3]\} = \max\{X_3, \mathbb{E}[Z_4]\} \\ &= \max\{X_3, 51/2\} \end{aligned}$$

This means, if your third spin is 26, or greater, then you should take the money. Otherwise, proceed to the next spin.

Consider the expected value of Z_3

$$\begin{aligned} \mathbb{E}[Z_3] &= \mathbb{E}[\max\{X_3, 51/2\}] \\ &= \frac{1}{50}\left(25 \cdot \frac{51}{2} + \sum_{i=26}^{50} i\right) \\ &= \frac{1}{50}\left(25 \cdot \frac{51}{2} + 25 \cdot \frac{1}{2}(50 + 26)\right) \\ &= 127/4 \end{aligned}$$

Now we can deduce Z_2

$$\begin{aligned} Z_2 &= \max\{X_2, \mathbb{E}[Z_3|\mathcal{F}_2]\} = \max\{X_2, \mathbb{E}[Z_3]\} \\ &= \max\{X_3, 127/4\} \end{aligned}$$

This means, if your second spin is 32, or greater, then you should take the money. Otherwise, proceed to the next spin.

Consider the expected value of Z_2

$$\begin{aligned} \mathbb{E}[Z_2] &= \mathbb{E}[\max\{X_2, 127/4\}] \\ &= \frac{1}{50}\left(31 \cdot \frac{127}{4} + \sum_{i=32}^{50} i\right) \\ &= \frac{1}{50}\left(31 \cdot \frac{127}{4} + 19 \cdot \frac{1}{2}(50 + 32)\right) \\ &= 7053/200 \end{aligned}$$

Now we can deduce Z_1

$$\begin{aligned} Z_1 &= \max\{X_1, \mathbb{E}[Z_2|\mathcal{F}_1]\} = \max\{X_1, \mathbb{E}[Z_2]\} \\ &= \max\{X_3, 7053/200\} \end{aligned}$$

This means, if your first spin is 36, or greater, then you should take the money. Otherwise, proceed to the next spin.

Answer 3. a)
Consider the following three cases

i). *Case 1 -* $C_t \geq P_t$.

The time T payoff is

$$\{S_T - e\}^+ = \{S_T - e\}^+ + (e - S_T)\mathbb{1}\{C_t < P_t\}$$

the equality is due to $\mathbb{1}\{C_t < P_t\} = 0$.

ii). *Case 2* - $C_t < P_t$ and $e \geq S_T$.

The time T payoff is

$$\begin{array}{rcl} e - S_T & = & (e - S_T)\mathbb{1}\{C_t < P_t\} \\ & = & \{S_T - e\}^+ + (e - S_T)\mathbb{1}\{C_t < P_t\} \text{ since } S_T - e \leq 0 \end{array}$$

iii). *Case 3* - $C_t < P_t$ and $e < S_T$.

The time T payoff is

$$\begin{array}{rcl} 0 & = & (S_T - e) + (e - S_T) \\ & = & \{S_T - e\}^+ + (e - S_T)\mathbb{1}\{C_t < P_t\} \end{array}$$

The result holds in call cases.

Answer 3. b)

I struggled with this question and have not managed to produce a proof. To be honest, I don't think much of what I have down below would help reach a proof.

Note that the discounted price process S_T^* is a Martingale. Thus

$$\begin{array}{rrcl} & \mathbb{E}\left[S_T^* | \mathcal{F}_0\right] & = & S_0^* \text{ by def. Martingale} \\ \implies & \mathbb{E}\left[S_T^* | \mathcal{F}_0\right] & = & S_0 \\ \implies & \mathbb{E}\left[S_T(1+r)^{-T}\right] & = & \\ \implies & (1+r)^{-T}\mathbb{E}[S_T] & = & \\ \implies & \mathbb{E}[S_T] & = & S_0(1+r)^T \end{array}$$

The payoff for the chooser option at time-period $t = T$ is

$$Y_T = \{S_T - e\}_+ + (e - S_T)\mathbb{1}\{C_t < P_t\}$$

Thus

$$\begin{array}{rcl} \mathbb{E}[Y_T] & = & \{\mathbb{E}[S_T] - e\}_+ + (e - \mathbb{E}[S_T])\mathbb{P}(C_t < P_t) \\ & = & \left\{S_0(1+r)^T - e\right\}_+ + (e - S_0(1+r)^T)\mathbb{P}(C_t < P_t) \end{array}$$