Financial Mathematics - Notes

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Contents

1 Financial Terminology & Single-Period Models

Proposition 1.1 - *Types of Financial Objects*
In this course we consider two types of financial objects

i). *Underlying Traded Assets* (e.g. Oil, interest rates & exchange rates).

ii). *Derivative Securities* (contracts based on *Underlying Traded Assets*)

Proposition 1.2 - *Derivative Securities*
Derivative Securities can act as insurance, by paying the holder when the value of the *Underlying Traded Asset* goes in the opposite direction.

e.g. "I will pay you £1mn for every dollar the price of oil is over £80 on 1st December 2025."

Our question is what such a contract is worth. To answer this we consider both *Arbitrage* and *Modelling*.

Definition 1.1 - *Modelling*
Modelling is the practice of forecasting the future value of an *Underlying Traded Asset*. Classical models for price processes involve *Brownian Motion*.

Definition 1.2 - *Arbitrage*
Arbitrage is the possibility of being able to make a profit on a transaction without being exposed to the risk of incurring a loss. Traders who try to lock in riskless profit are called *Arbitrageurs*.

Generally realised by buying and selling the same asset in different markets, with the asset having a different value in each market.

(2)

Example 1.1 - *Arbitrage*
Consider a stock that is traded on both the NYSE and the LSE. Suppose the sotck price is \$189 on the NYSE and £100 on the LSE, and the exchange rate is \$1.87=£1. The following *Arbitrage Opportunity* exists:

i). Buy 100 shares in London.

ii). Sell all these shares in NY.

iii). Exchange dollars to sterling.

This has a riskless profit of \$200=£106.95 (assuming negligible transaction costs).

Proposition 1.3 - *"No Arbitrage Principle"*
The *"No Arbitrage Principle"* is the principle that *Arbitrage* opportunity do not exist (for long) in real life markets. If they did then the market activity caused by agents exploiting the *Arbitrage* opportunity[1] would raise the cost of buying and thus remove the *Arbitrage* opportunity.

Remark 1.1 - *Arbitrage & Valuing Contracts*
When valuing contracts we assume there is no arbitrage (The *"No Arbitrage Principle"*). This means we can decide a single price for a contract, as at any other price there would be

[1]The forces of *Supply-and-Demand*.

arbitrage.

Definition 1.3 - *Short Selling*
Short Selling is the practice of borrowing an asset, selling it and then at some point in the future buying an equivalent asset to reimburse whoever lent you the original asset. If the value of the asset has fallen in this time then you make a profit.

Example 1.2 -
Consider a share which trades at £10 on 1st Jan. Suppose we know that on 1st July there is a 2/3 chance it will be worth £25 and 1/3 chance it will be worth £5. Suppose the following contract is offer on the 1st Jan:

i). "If the share goes up, I will pay you £4. If the share price goes down, I pay you nothing."

What is this contract worth? (Assuming your bank account pays no interest).

You could argue that the contract should be worth the expected payout of the contract , $\frac{2}{3} \times \pounds 4 + \frac{1}{3} \times \pounds 0 = \pounds 2.66$.

However, consider creating a replicating portfolio which buys $\frac{1}{5}$ of a unit of the stock and borrows £1 from the bank. On 1st Jan this portfolio is worth $(1/5) \times \pounds 10 - \pounds 1 = \pounds 1$.

- If the stock goes up, the portfolio is worth $\frac{1}{5} \times \pounds 25 - \pounds 1 = \pounds 4$.
- If the stock goes down, the portfolio is worth $\frac{1}{5} \times \pounds 5 - \pounds 1 = \pounds 0$.

These two outcomes show that this replicating portfolio has exactly the same payouts as the contract being offer. This means that whatever the portfolio costs, the contract must cost the same.

Suppose the portfolio costs £1 and the contract costs £1.50. Then you would sell the contract, buy the portfolio and make £0.50 profit independent of the price of the share on 1st July. (and visa-versa).

This valuation is independent of the probability of the asset's value increasing, but there is an interesting "phantom probability" $q = \frac{1}{4}$. The price of the contract is the expected payout of the contract, if the probability of the share price increasing was q, $\mathbb{E}[\text{payout}] = \frac{1}{4} \times \pounds 4 + \frac{3}{4} \times 0 = \pounds 1$. Also, using this probability the expected value of the stock on July 1st is $\frac{1}{4} \times \pounds 25 + \frac{3}{4} \times \pounds 5 = \pounds 10$, the same as it was on Jan 1st. (This is how we find q).

This q is known as the *Equivalent Martingale Measure*

Definition 1.4 - *Equivalent Martingale Measure*
TODO

Remark 1.2 - *Value of Money*
These problems are more interesting when we consider that the value of money is not constant due to inflation & interest rates.

Remark 1.3 - *Risk-Free*
When referring to an activity being "Risk-Free" we mean that the loses & profits from it are known, not that there are no loses.

Definition 1.5 - *Bank Process, B_t*

Bank Process, B_t, is a measure of how much the value of money has changed over t time-periods.

Assume the existence of a risk-free bank account with known interest rate r (assumed to be constant over interval $[0, T]$). This means the initial deposit of one unit becomes the following after t time-steps

inuous Time Model $B_t = e^{rt}$

Multi-Period Model $B_t = (1 + r)^t$

We must consider the performance of our portfolio relative to the *Bank Process*, by discounting our profits by factor B_t.

N.B. - The *Bank Process* is also known as a *Bond* or a *Numeraire*.

Definition 1.6 - *Derivative Security*
A *Derivative Security* is a contract whose value at expiration date T is a function of the values of the assets within the time interval $[0, T]$. Often, the value is just a function of the value at time T.

Definition 1.7 - *Forward Contract*
A *Forward Contract* is an agreement to buy or sell an asset S at future *Delivery Date*[2] T for *Delivery Price* K. Both parties are obliged to fulfil the contract.

The agent who agrees to buy the underlying asset is said to have a *Long Position*, the other agent has a *Short Position*.

The *Forward Price* $f(t; T)$ is the delivery price which would make the contract have zero value at time t.

Remark 1.4 - *Usefulness of Forward Contracts*
Forward Contracts allow you to agree terms of a future purchase/sale in advance of actually transacting. This means you know the price you will receive/pay and thus can plan accordingly. (e.g. Farmers may agree to price per tonne of corn well before the harvest).

Definition 1.8 - *Option Contract*
An *Option* is a financial instrument giving one the right, but not the obligation, to buy or sell an asset S at (or before) a specified date T for an agreed *Strike Price*[3] K. There are two classes of *Option Contract*

- *Call Option* - The holder has the right to buy.
- *Put Option* - The holder has the right to sell

N.B. Only one party (the holder) decides whether to exercise the contract or not, the other (the writer) has to do what the former wishes.

Definition 1.9 - *European & American Options*
European vs *American Options* determine when the *Option Contract* can be exercised.

- *European Option* can only be executed on the expiry date.

[2] AKA *Settlement Date*
[3] AKA *Exercise Price*

- *American Option* can be executed on any date up to and inc. the expiry date.

Example 1.3 - *Derivative*
Consider the following call option

> "The holder has the right to buy 1,000 litres of petrol for £1,000 next Jan 1st"

If the market price S_T on 1st Jan is greater than £1,000 it is profitable to exercise the option. You would make profit $S_T - 1000$. However, if S_T was less than £1,000 then it is better to buy petrol at the market price.

The value on Jan 1st of the option will be $\{S_T - 1000\}_+$, a function of the random price S_T.

N.B. - a put option would the opposite analysis with profit $1000 - S_T$.

Definition 1.10 - *Dividend*
A *Dividend* is a one-off payment provided made to the holder of an underlying asset at a certain time. Not all underlying assets provide a *Dividend*.

Proposition 1.4 - *Fair Price of a Forward Contract with a Dividend*
Consider a forward contract on an asset with current price S_0 which provides a known dividend D at time point $t_0 \in (0, T)$ and has delivery date T.

Assume the existence of a risk-free bank account with constant interest rate r during the interval $[0, T]$. This means an initial deposit of one unit grows to $B_t = e^{rt}$ up to time point t.

If $D = 0$ (ie no dividend is paid) then the fair delivery price for the forward contract is $K = S_0 e^{rT}$. Else, if $D > 0$ (ie a dividend is paid) then the fair delivery price for the forward contract is $K = (S_0 - I)e^{rT}$ where $I = De^{-rt_0}$.

Proof 1.1 - *Proposition 1.4*
We use the "no-arbitrage principle" to prove that this is the fair price.

First assume that the price of the contract is $K > (S_0 - I)e^{rT}$ where $I = De^{-rt_0}$. Then an arbitrageur will adopt a short position on this contract by doing the following

i). Borrow $\pounds S_0$ at an interest rate of r.

ii). Buy the underlying asset.

iii). Take a short position in the forward contract (i.e. agree to sell the asset for K at time T).

At time point T_0 we use the dividend to partially repay the loan. Finally, at time point T we sell the asset for K and repay the outstanding balance of the loan. The riskless profit is $K - (S_0 e^{rt_0} D)e^{(T-t_0)} = K - (S_0 - I)e^{rT}$, regardless of the price of the underlying asset at time point T.

Now assume the converse, the price of the contract is $K - (S_0 - I)e^{rT} > 0$ where $I = De^{-rt_0}$. Then an arbitrageur will adopt a long position on this contract by doing the following

i). Short sell the underlying asset (N.B. you are obliged to pay dividends to the lender).

ii). Invest the proceeds of S_0 at the risk-free interest rate of r.

iii). Take a long position in the forward contract (i.e. agree to buy the asset for K at time T).

At time t_0 we have to pay the dividend from our bank account. On the delivery date the arbitrageur buys the asset for K and makes a riskless profit of $(S_0 e^{rt_0} - D)e^{r(T-t_0)} - K = (S_0 - I)e^{rT} - K > 0$ where $I = De^{-rt_0}$.[4]

Theorem 1.1 - *Equivalent Contract Valuations over Time*
Consider two combinations of financial derivatives that both have the same value $V = W$ at time point T. Then their prices V_t and W_t at time point $t < T$ must also coincide.

Proof 1.2 - *Theorem 1.1*
We use the "no-arbitrage principle" and assume WLOG that $V_t > W_t$. Then, at time t, we would do the following

i). Sell or short the first combination.

ii). Buy the second combination.

iii). Invest the difference $V_t - W_t > 0$.

At time point T we would do the following

i). Buy the second combination for W.

ii). Sell the first combination for $V = W$.

The risk-free profit, assuming a risk-free interest rate of r, is $(V_t - W_t)e^{r(T-t)} > 0$

Proposition 1.5 - *Put-Call Parity*
We apply `Theorem 1.1` to *European Put & Call Options*.

Consider a *European Put Option* and a *European Call Option* for the same underlying asset, both with strike price K and expiry date T. Assume that S_T is the price of the underlying asset at time point T then the payoff of the *Call Option* is C and the *Put Option* is P at time T, where

$$\begin{array}{rcl} C &=& \{S_T - K\}_+ \\ P &=& \{K - S_T\}_+ \end{array}$$

For the first combination choose the *Underlying Asset* and *European Put Option*. The value of this combination at time point T is $V = S_T + P = \max\{S_T, K\}$.

For the second combination choose the *European Call Option* and a bond which matures at time point T with a price of K. The value of the second combination is $W = K + C = \max\{S_T, K\}$

Theorem 1.2 - *Put-Call Parity - Formal*
Let S_t be the price of the asset at time point t, $Ke^{-r(T-t)}$ the discounted value of the bond and C_t, P_t be the prices of the *European Call* and *Put Options* at time t, respectively. Then

$$S_t + P_t - C_t = Ke^{-r(T-t)} \quad \forall t \in [0, T]$$

If, at some time t, this relationship does not hold then an *Arbitrage Opportunity* exists at this time.

[4] D is subtracted as you have to pay the lender the dividend, but as you have already sold the asset you do not receive the dividend from the asset.

Theorem 1.3 - *Lower Bound for a European Call Option*
We can use *Put-Call Parity* to determine a lower bound for a *European Call Option*

$$\begin{array}{rcl} S_t + P_t - C_t & = & Ke^{-r(T-t)} \\ \implies C_t & = & S_t + P_t - Ke^{-r(T-t)} \\ \implies C_t \geq \{S_t - Ke^{-r(T-t)}\}_+ \text{ as } P_t \geq 0 & & \end{array}$$

Theorem 1.4 - *American Call Options*
Let C_A be the price of an *American Call Option* and C_E be the price of a *European Call Option* for the same underlying asset, with the same strike price and expiry date.

Then, for a non-dividend paying stock we have that

$$C_A = C_E$$

This means that, for non-dividend paying stock, it is suboptimal to exercise an American call optional early.

Proof 1.3 - *Theorem 1.4*
First, note that exercising the American call early at time $t < T$ generates an income of $S_t - K$. However, from the inequality above, we know that selling the call options yields a cash-flow of $\{S_t - Ke^{-r(T-t)}\}_+ \geq S_t - Ke^{-r(T-t)}$. Since $e^{-r(T-t)} < 1$, exercising the call at any $t < T$ (i.e. early) is suboptimal.

1.1 Single-Period Model

Definition 1.11 - *Sample Space, Ω*
The *Sample Space* Ω is the set consisting of all elementary outcomes.

Definition 1.12 - *Random Variable*
A *Random Variable* X is a function from the *Sample Space* to real numbers

$$X : \Omega \to \mathbb{R}$$

Definition 1.13 - *Single-Period Model*
The *Single-Period Model* is a model for a financial market with the following components

- Initial date $t = 0$ and terminal date $t = 1$, with trading and consumption only allowed on these two dates.
- A finite *Sample Space* Ω with $|\Omega| = K < \infty$.

  $$\Omega = \{\omega_1, \dots, \omega_k\}$$

  with each element corresponding to some state of the world.
- A *Probability Measure* $\mathbb{P}$ on Ω with $\mathbb{P}(\omega_i) > 0 \ \forall \ \omega_i \in \Omega$.

Definition 1.14 - *Bank Account Process B*
A *Bank Account Process* $B = \{B_t : t = 0, 1\}$ where $B_0 = 1$ and B_1 is a *Random Variable.*

The *Bank Account Process* is distinguished from other securities because its price $B_1(\omega)$ at time $t = 1$ is assumed to be strictly positive for all $\omega \in \Omega$. Usually, in fact, $B_1 \geq 1$ in which case B_1 should be though of as the value of the bank account at time $t = 1$, if 1 unit of currency was deposited at time $t = 0$. And, $r = B_t - 1 \geq 0$ should be thought of as the *Interest Rate.* In many applications r and B_1 are deterministic scalars.

Definition 1.15 - *Price Process*[5]
A *Price Process* $S = \{S(t) : t = 0, 1\}$ where $S(t) = (S_1(t), S_2(t), \dots, S_N(t)),\ N < \infty$ and $S_i(t)$ is the price of the i^{th} security at time t. In many applications these N risky securities are stocks.

The prices at time $t = 0$ are positive scalars that are known to the investors, whereas the prices at time $t = 1$ are non-negative random variables whose value only become known to investors at time $t = 1$.

Definition 1.16 - *Trading Strategy H*
A *Trading Strategy* $H = (H_0, \dots, H_N)$ describes an investor's portfolio as carried from time $t = 0$ to time $t = 1$. Specifically, H_0 is the number of units of the currency invested in the *Savings Account* and H_i with $i \geq 1$ is a scalar of the number of units invested in the i^{th} security.

Note that H_i can be positive or negative. Positive means you have bought/invested and negative means you are borrowing or short selling.

Example 1.4 - *Single-Period Model (Zero Interest)*
Consider a share which trades on 1st Jan for £10. Suppose we know with probability $p_1 = \frac{2}{3}$ it will be worth £25 on 1st July, and with probability $p_2 = \frac{1}{3}$ it will be worth £5 on 1st July. You also have access to a bank account which pays no interest ($r = 0$), which you can pay into or borrow from. Create a *Replicating Portfolio* as follows: buy $\frac{1}{5}$ unit of the stock, and borrow £1 from the bank.

We can rephrase this as a *Single-Period Model* for a single period ($T = 1$).

- *Bank Account Process* - $B_0 = B_1 = 1$. As no interest rate.
- *Sample Space* - $\Omega = \{\omega_1, \omega_2\}$ with ω_1 being the event the stock rises to £25 and ω_2 being the event the stock falls to £5.
- There are $N = 1$ stocks. So the *Price Process* is

$$\begin{aligned} S_1(0)(\omega_i) &= 10 \text{ for } i = 1, 2 \\ S_1(1)(\omega_i) &= \begin{cases} 25 & \text{if } i = 1 \\ 5 & \text{if } i = 2 \end{cases} \end{aligned}$$

- We can write the *Replicating Portfolio* as a vector (H_0, H_1).

Definition 1.17 - *Value Process, V*
A *Value Process* $V = \{V_t : t = 0, 1\}$ describes the total value of the portfolio at each point in time

$$V_t = H_0 B_t + \sum_{n=1}^{N} H_n S_n(t)$$

[5]AKA *Stock Process*

Definition 1.18 - *Gains Process, G*
The *Gains Process G* is a random variable that describes the total profit or loss generated by the portfolio between times 0 and 1.

$$G = H_0 r + \sum_{n=1}^{N} H_n \Delta S_n$$

where $\Delta S_n = S_n(1) - S_n(0)$ is the change in price of the n^{th} asset.

Remark 1.5 - *Normalising Prices*
It is convenient to normalize the prices, so that the bank account becomes constant by defining discounted versions of the processes defined earlier. See `Definitions 1.19, 1.20, 1.21`.

Definition 1.19 - *Discounted Price Process, S^**
A *Discounted Price Process* S^* is the *Price Process* S normalised by the *Bank Process* B_t

$$S^* := \{S_t^* : t = 0, 1\} \text{ with } S_n^*(t) := \frac{S_n(t)}{B_t} \text{ for } n \in \{1, \dots, N\},\ t = \{0, 1\}$$

Definition 1.20 - *Discounted Value Process, V^**
A *Discounted Value Process* V_t^* is the *Value Process* V_t normalised by the *Bank Process* B_t

$$V_t^* := \frac{V_t}{B_t} = H_0 + \sum_{n=1}^{N} H_n S_n^*(t) \text{ for } t = 0, 1$$

Definition 1.21 - *Discounted Gains Process, G^**
A *Discounted Gains Process* $G^*$is the *Gains Process* G normalised by the *Bank Process* B_t

$$G^* := \frac{G}{B_t} = \sum_{n=1}^{N} H_n \Delta S_n^* \text{ with } \Delta S_n^* := S_n^*(1) - S_n^*(0)$$

Example 1.5 - *Single-Period Model (Non-Zero Interest)*
Consider a share which trades on 1st Jan for £10. Suppose we know with probability $p_1 = \frac{2}{3}$ it will be worth £25 on 1st July, and with probability $p_2 = \frac{1}{3}$ it will be worth £5 on 1st July. You also have access to a bank account which pays interest $r \in \mathbb{R}$, which you can pay into or borrow from.

We can rephrase this as a *Single-Period Model* for a single period ($T = 1$).

- Bank Account Process - $B_0 = 1,\ B_1 = 1 + r$.
- Probability Space - $\Omega = \{\omega_1, \omega_2\}$ where ω_1 indicates the stock going up and ω_2 indicates the stock going down.

- There is $N=1$ units of the stock, so we have the following price process

$$\begin{array}{rcl} S_1(0)(\omega_i) & = & 10 \text{ for } i\in\{1,2\} \\ S_1(1)(\omega_1) & = & 25 \\ S_1(1)(\omega_2) & = & 5 \end{array}$$

  And discounted price process

$$\begin{array}{rcl} S_1^*(0)(\omega_i) & = & 10 \text{ for } i\in\{1,2\} \\ S_1^*(1)(\omega_1) & = & \frac{25}{1+r} \\ S_1^*(1)(\omega_2) & = & \frac{5}{1+r} \end{array}$$

- For an arbitrary strategy H, our portfolio has initial value $V_0=V_0^*=H_0+10H_1$. Thus, we can determine the gains and future values of this portfolio

$$\begin{array}{rcl} V_1(\omega_1) & = & (1+r)H_0+25H_1 \\ V_1^*(\omega_1) & = & H_0+\frac{25}{1+r}H_1 \\ G(\omega_1) & = & rH_0+15H_1 \\ G^*(\omega_1) & = & \left(\frac{25}{1+r}-10\right)H_1 \\ V_1(\omega_2) & = & (1+r)H_0+5H_1 \\ V_1^*(\omega_2) & = & H_0+\frac{5}{1+r}H_1 \\ G(\omega_2) & = & rH_0-5H_1 \\ G^*(\omega_2) & = & \left(\frac{5}{1+r}-10\right)H_1 \end{array}$$

Definition 1.22 - *Arbitrage Opportunity*
An *Arbitrage Opportunity* is a *Trading Strategy* H with the following three properties

i). $V_0=0$.

ii). $V_1(\omega)\geq 0\ \forall\ \omega\in\Omega$.

iii). $\mathbb{P}(V1(\omega)\geq 0)>0\ \forall\ \omega\in\Omega$.[6]

Proposition 1.6 - *Gains Process & Arbitrage Opportunity*
There exists an *Arbitrage Opportunity* iff there is some trading strategy H st $G^*\geq 0$ and $\mathbb{E}[G^*]>0$.

This means that *Gains* is always non-negative (so not losses), and as its expected value is strictly positive there must be at least one positive value of G^* which has a non-zero probability.

Proof 1.4 - *Proposition 1.6*
As this is an <u>iff</u> statement, I prove the statement in both directions. First consider the forwards direction. Let H be an *Arbitrage Opportunity*. Since $G^*=V_1^*-V_0^*$ and $B_t>0\ \forall\ t,\omega$, by the definition of an *Arbitrage Opportunity*, we find that $G^*\geq 0$ and thus $\mathbb{E}[G^*]=\mathbb{E}[V_1^*]>0$.

Now consider the other direction, suppose H satisfies $G^*\geq 0$ and $\mathbb{E}[G^*]>0$. Define $\hat{H}:=(\hat{H}_0,H_1\dots,H_N)$ where $\hat{H}_0:=-\sum_{n=1}^N H_nS_n^*(0)$[7]. Under $\hat{H}$ one has $V_0^*=0$ and $V_1^*=V_0^*+G^*=G^*$.

Hence $V_1^*\geq 0$ and $\mathbb{E}[V_1^*]=\mathbb{E}[G^*]>0$, meaning $\hat{H}$ is an arbitrage opportunity.

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[6] Equivalently, $\mathbb{E}[V_1]>0$.

[7] Any money used to buy a stock is borrowed from the bank, and any money made from short selling is deposited into the bank. This means the total value of the portfolio at $t=0$ is 0 (requirement of an *Arbitrage Opportunity*).

1.2 Risk-Neutral Probability Measures

Definition 1.23 - *Risk-Neutral Probability Measure*
A probability measure $\mathbb{Q}$ on Ω is said to be a *Risk-Neutral Probability Measure* if both the following hold

i). $\mathbb{Q}(\{\omega\}) > 0 \ \forall \ \omega \in \Omega$.

ii). $\mathbb{E}_{\mathbb{Q}}(S_n^*(1)) = S_n^*(0)$ for $n \in \{1, \dots, N\}$.

Theorem 1.5 - *No-Arbitrage Theorem*
There are no *Arbitrage Opportunities* iff there exists a *Risk-Neutral Probability Measure* $\mathbb{Q}$.

Theorem 1.6 - *Separating Hyperplane Theorem*[8]
Let $\mathbb{W}$ be a linear subspace of $\mathbb{R}^K$ and $\mathbb{K}$ be a compact convex subset in $\mathbb{R}^K$ which is disjoint from $\mathbb{W}$. Then we can separate $\mathbb{W}$ and $\mathbb{K}$ strictly by a hyperplane containing $\mathbb{W}$ (ie $\exists \ v \in \mathbb{R}^K$ which is *Orthogonal* to $\mathbb{W}$[9]) such that

$$u^T v > 0 \ \forall \ u \in \mathbb{K}$$

N.B. - Proof of this is beyond the scope of this course.

Proof 1.5 - *No-Arbitrage Theorem*
Consider the three following sets

i). $\mathbb{W} = \left\{X \in \mathbb{R}^K : X = G^* \text{ for some } \textit{Trading Strategy} H\right\}$.

This can be considered the set of random variables which each represent a possible $t = 1$ discounted wealth when the initial value of the investment is zero. Here, $\mathbb{W}$ is a linear subspace of $\mathbb{R}^K$[10].

ii). $\mathbb{A} = \left\{X \in \mathbb{R}^K : X \geq 0, X \neq 0\right\}$. ($\mathbb{A}$ is not compact, so can not be used for $\mathbb{K}$ in *Separating Hyperplane Theorem*).

Note there exists an arbitrage opportunity iff $\mathbb{W} \cap \mathbb{A} \neq \emptyset$.

iii). $\mathbb{A}^+ = \left\{X \in \mathbb{R}^N : X \geq 0, X \neq 0, \sum_{i=1}^K X_i = 1\right\}$.

$\mathbb{A}^+$ is a convex and compact subset of $\mathbb{R}^K$.

Assume that there is no *Arbitrage Opportunity*, then $\mathbb{W} \cap \mathbb{A} \neq \emptyset$ (They are disjoint). By the *Separating Hyperplane Theorem* $\exists \ Y \in \mathbb{R}^K$ which is *Orthogonal* to $\mathbb{W}$ ($X^T Y = 0 \ \forall \ X \in \mathbb{W}$) st

$$X^T Y > 0 \ \forall \ X \in \mathbb{A}^+$$

For each $k \in \{1, \dots, K\}$ the k^{th} unit vector e_k is an element of $\mathbb{A}^+$. Therefore, $\forall \ k \in \{1, \dots, K\}$ we have that

$$Y_K := e_K^T Y > 0$$

which means all entries of Y are strictly positive.

[8]This is a consequence of the *Hahn-Banach Theorem* from functional analysis.
[9]$u^T v = 0 \ \forall \ u \in \mathbb{W}$
[10]Proved by showing it is complete under: addition, and scalar multiplication.

Define a probability measure $\mathbb{Q}$ by setting

$$\mathbb{Q}(\{\omega_k\}) = \frac{Y(\omega_k)}{Y(\omega_1) + \cdots + Y(\omega_k)}$$

Furthermore, $\Delta S_n^* \in \mathbb{W}$ for all n because $\Delta S_n^* := S_n^*(1) - S_n^*(0)$ is the discounted wealth for the portfolio $H = e_n$ which consists of one unit of the n^{th} asset only. Since Y is orthogonal to $\mathbb{W}$ we can conclude that

$$\mathbb{E}_{\mathbb{Q}}[\Delta S_n^*] = \sum_{k=1}^{K} \Delta S_n^*(\omega_k)\mathbb{Q}(\{\omega_k\}) = 0 \ \forall\ n$$

In other words

$$\mathbb{E}_{\mathbb{Q}}[S_n^*(1)] = S_n^*(0)\ \forall n$$

Thus $\mathbb{Q}$ is a *Risk-Neutral Probability Measure.*

For the converse, assume that $\mathbb{Q}$ is a *Risk-Neutral Probability Measure.* Then for an arbitrary *Trading Strategy* H we have

$$\mathbb{E}_{\mathbb{Q}}[G^*] = \mathbb{E}_{\mathbb{Q}}\left[\sum_{n=1}^{N} H_n \Delta S_n^*\right] = \sum_{n=1}^{N} H_n \mathbb{E}_{\mathbb{Q}}[\Delta S_n^*] = 0$$

and, in particular

$$\sum_{k=1}^{K} G^*(\omega_k)\mathbb{Q}(\{\omega_k\}) = 0$$

which shows that either $G^*(\omega_k < 0)$ for some k or $G^* = 0$, but then $\mathbb{E}_{\mathbb{P}}[G^*] = 0$. Hence, by `Proposition 1.6` , there cannot be any arbitrage opportunities. $\square$

1.3 Valuation of Contingent Claims

Definition 1.24 - *Contingent Claim*
A *Contingent Claim* in the *Single-Period Model* is a random variable X which represents the payoff at time $t = 1$. A contingent claim X is said to be "attainable" (or "marketable") if there exists a trading strategy H st $V_1 = X$. In this case one says that H generates X and H is called the *Replicating Portfolio.*

Proposition 1.7 - *Fair Price of a Contingent Claim?*
Consider a *Contingent Claim* which is "attainable", and suppose that H is its *Replicating Portfolio.* The value V_0 of H at time $t = 0$ is the fair price of the *Contingent Claim.*

Proof 1.6 - *Proposition 1.7*
Suppose the fair price of the *Contingent Claim* p is not equal to the value V_0 of H at time $t = 0$ (ie $p \neq V_0$). Then we have two cases

$p > V_0$ - An *Arbitrage Opportunity* arises because an arbitrageur would sell the *Contingent Claim* for p at time 0, follow the trading strategy H at a time $t = 0$ cost of V_0 and pocket the difference $p - V_0$.

At time $t = 1$ the value V_1 of the portfolio matches the obligation X of the *Contingent Claim.* We make profit $(p - V_0)e^{rT}$.

$p < V_0$ - An arbitrageur would follow the trading strategy H and purchase the *Contingent Claim*, again making a riskless profit.

Theorem 1.7 - *Risk-Neural Valuation Principle*
If the *Single Period Model* is free of *Arbitrage Opportunities*, then the time $t = 0$ value of an *Attainable Contingent Claim* X is $\mathbb{E}_\mathbb{Q}[X/B_1]$ where $\mathbb{Q}$ is any *Risk-Neutral Probability Measure*[11].

Proof 1.7 - *Risk-Neutral Valuation Principle*
Suppose there exists a second trading strategy $\hat{H}$ st $\hat{V}_1 = X$, but $\hat{V}_0 \neq V_0$.

Let $\mathbb{Q}$ be a *Risk-Neutral Probability Measure*, then for any trading strategy H we have seen that $\mathbb{E}_\mathbb{Q}[G^*] = 0$. Consequently

$$V_0 = V_0^* = \mathbb{E}_\mathbb{Q}[V_0^*] = \mathbb{E}_\mathbb{Q}[V_1^* - G^*] = \mathbb{E}_\mathbb{Q}[V_1^*] - \mathbb{E}_\mathbb{Q}[G^*] = \mathbb{E}_\mathbb{Q}[V_1^*] = \mathbb{E}_\mathbb{Q}\left[\frac{V_1}{B_1}\right]$$

In particular, any replicating strategy with $V_1 = X$ has time $t = 0$ value

$$V_0 = \mathbb{E}_\mathbb{Q}\left[\frac{X}{B_1}\right]$$

As this holds for all $\mathbb{Q}$, this calculation does not depend on the choice of $\mathbb{Q}$ and therefore $\mathbb{E}_\mathbb{Q}[V_1^*]$ is constant even if there are two or more *Risk-Neutral Probability Measures*.

This means all *Replicating Porfolios* for the same *Attainable Contingent Claim* have the same values in time $t = 0$ and $t = 1$.

Example 1.6 - *Valuation of Contingent Claims - I*
Consider a stock which is trading for £10 at time $t = 0$, and we know that at time $t = 0$ it will be worth either £25 or £5.

In order to find a *Risk-Neutral Probability Measure* $\mathbb{Q}$ we need to find strictly positive numbers $\mathbb{Q}(\{\omega_1\})$ and $\mathbb{Q}(\{\omega_2\})$ st $\mathbb{E}_\mathbb{Q}[S_n^*(1)] = S_n^*(0)$ is satisfied. That is

$$\begin{array}{rrcl} & \mathbb{E}_\mathbb{Q}[S_0^*(1)] & = & S_0^*(0) \\ \implies & S_1^*(\omega_1)\mathbb{Q}(\{\omega_1\}) + S_1^*(\omega_2)\mathbb{Q}(\{\omega_2\}) & = & S_0 \\ \implies & \frac{25}{1+r}\mathbb{Q}(\{\omega_1\}) + \frac{5}{1+r}\mathbb{Q}(\{\omega_2\}) & = & 10 \end{array}$$

As $\mathbb{Q}$ is a *Probability Measure*, we must have that

$$1 = \mathbb{Q}(\{\omega_1\}) + \mathbb{Q}(\{\omega_2\})$$

We deduce the values $q_1 = \mathbb{Q}(\{\omega_1\}), q_2 = \mathbb{Q}(\{\omega_2\})$ as follows

$$\begin{array}{rrcl} & \frac{25}{1+r}q_1 + \frac{5}{1+r}q_2 & = & 10 \\ \& & q_1 + q_2 & = & 1 \\ \implies & q_1 & = & 1 - q_2 \\ \implies & \frac{5}{1+r}\left(5(1-q_2) + q_2\right) & = & 1- \\ \implies & 5 - 4q & = & 2(1+r) \\ \implies & q_2 & = & \frac{3-2r}{4} \\ \implies & q_1 & = & 1 - \frac{3-2r}{4} = \frac{1+2r}{4} \end{array}$$

The values $\mathbb{Q}(\{\omega_1\}) = \frac{1+2r}{4}, \mathbb{Q}(\{\omega_2\}) = \frac{3-2r}{4}$ satisfy both these equations, so this is a *Risk-Neutral Probability Measure* and by `Theorem 1.5` (The "No-Arbitrage Principle") there cannot be any *Arbitrage Opportunities*.

[11]So value is independent of the choice of $\mathbb{Q}$

In particular for $r = 0$ (ie No interest), we recover $\mathbb{Q}(\{\omega_1\}) = \frac{1}{4}$ which is the same as a previous example.

Example 1.7 - *Valuation of Contingent Claims - II*
Continuing from `Example 1.6`.

Suppose X is a *Contingent Claim* with $X(\omega_1) = 4$ and $X(\omega_2) = 0$. If X is *Attainable*, then the value of X at time $t = 0$ is

$$\mathbb{E}_{\mathbb{Q}}[X/B_1] = \frac{1+2r}{4} \cdot \frac{4}{1+r} + 0 = \frac{1+2r}{1+r}$$

Check whether X can be generated by solving for $H = (H_0, H_1)$[12][13]

$$\begin{array}{rcl} 4 &=& H_0(1+r) + H_1 \cdot 25 \\ 0 &=& H_0(1+r) + H_1 \cdot 5 \end{array}$$

Solving gives

$$\begin{array}{rrcl} & (4-0) &=& \big((1+r)-(1+r)\big)H_0 + (25-5)H_1 \\ \implies & 4 &=& 20H_1 \\ \implies & H_1 &=& 1/5 \\ \implies & 0 &=& H_0(1+r)+1 \\ \implies & H_0 &=& -1/(1+r) \end{array}$$

This means that X is indeed *Attainable*.

We check the value of the *Replicating Portfolio* H at time $t = 0$

$$V_0 = H_0 + S_1 \cdot H_1 = -\frac{1}{1+r} + 10(1/5) = \frac{1+2r}{1+r}$$

1.4 Complete & Incomplete Markets

Definition 1.25 - *Complete & Incomplete Models*
A model is said to be *Complete* if every *Contingent Claim* X can be generated by some trading strategy. Otherwise, the model is said to be *In-Complete*

Remark 1.6 - *Checking if a Model is Complete*
There are simple ways to check whether a model is *Complete*. One way is to define $K \times (N+1)$-matrix A as

$$A = \begin{pmatrix} B_1(\omega_1) & S_1(1)(\omega_1) & \dots & S_N(1)(\omega_1) \\ B_1(\omega_2) & S_1(1)(\omega_2) & \dots & S_N(1)(\omega_2) \\ \vdots & \vdots & \ddots & \vdots \\ B_1(\omega_K) & S_1(1)(\omega_K) & \dots & S_N(1)(\omega_K) \end{pmatrix}$$

If a particular *Contingent Claim* X is attainable, then $AH = X$ will have a solution for H.

A model is *Complete* $\underline{\text{iff}}$ $\forall\ X \in \mathbb{R}^K\ \exists H$ st $AH = X$.

NB - If the bank process B is deterministic then the first column will be identical for all rows.

Theorem 1.8 - *Uniqueness of Risk-Neutral Probability Measure*

[12] H_0 is amount invested in our bank account and H_1 is amount invested in the stock.
[13] We fine (H_0, H_1) by solving $AH = X$ for defined A, X.

Suppose there are no *Arbitrage Opportunities*. Then a model is *Complete* iff there exists a unique *Risk-Neutral Probability Measure*.

Proof 1.8 - *Theorem 1.8*
We denote by $\mathbb{M}$ the set of all *Risk-Neutral Probability Measures*. Since there are no *Arbitrage Opportunities* we know that $\mathbb{M} \neq \emptyset$. We consider the statement in both directions

$\implies$ For the sake of contradiction, assume that the model is *Complete*, but $\mathbb{M}$ contains two distinct *Risk-Neutral Probability Measures* $\mathbb{Q}, \hat{\mathbb{Q}}$. In this case there must exists some state ω_k with $\mathbb{Q}(\omega_k) \neq \hat{\mathbb{Q}}(\omega_k)$ so take the contingent claim X defined by

$$X(\omega) = \begin{cases} B_1(\omega_k) & \text{if } \omega = \omega_k \\ 0 & \text{otherwise} \end{cases} = B_1 \mathbb{1}\{\omega = \omega_k\}$$

Then

$$\mathbb{E}_{\mathbb{Q}}(X/B_1) = \mathbb{E}_{\mathbb{Q}}(\mathbb{1}\{\omega = \omega_k\}) = \mathbb{Q}(\{\omega_k\}) \neq \hat{\mathbb{Q}}(\omega_k) = \mathbb{E}_{\hat{\mathbb{Q}}}(X/B_1)$$

But, this contradicts the calculation in `Proof 1.7` where we saw that if *Contingent Claim* X is *Attainable* then $\mathbb{E}_{\mathbb{Q}}[X/B_1]$ has the same value $\forall\ \mathbb{Q} \in \mathbb{M}$.

$\impliedby$ For the sake of contradiction, assume that there is only one *Risk-Neutral Probabiltiy Measure* $\hat{\mathbb{Q}}$, but there exists a *Contingent Claim* X which is not *Attainable*.

Then there is no solution to the system $AH = X$. By *Separating Hyperplane Theorem* it follows that there must exists a row vector $\pi \in \mathbb{R}^K$ st

$$\pi^T A = 0^{[14]}, \pi^T X > 0$$

Let $\lambda > 0$ be small enough st

$$\mathbb{Q}(\{\omega_j\}) = \hat{\mathbb{Q}}(\{\omega_j\}) + \lambda \pi_j \cdot B_1(\omega_j) > 0\ \forall\ j \in \{1, \dots, K\}$$

Since B_1 is the first column of A and $\pi^T A = 0$, the quantity $\mathbb{Q}$ is defined above is actually a *Probabiltiy Measure.*[15] Moreover, for any *Discounted Price Process* $S^* = (S_1^*, \dots, S_N^*)$ and for every $n \in \{1, \dots, N\}$ we have

$$\begin{aligned} \mathbb{E}_{\mathbb{Q}}[S_n^*(1)] &= \sum_{j=1}^K \mathbb{Q}(\{\omega_j\}) S_n(1, \omega_j)/B_1(\omega_j) \\ &= \sum_{j=1}^K \hat{\mathbb{Q}}(\{\omega_j\}) S_n(1, \omega_j)/B_1(\omega_j) + \lambda \underbrace{\sum_{j=1}^K \pi_j S_n(1, \omega_j)}_{=0} \\ &= \sum_{j=1}^K \hat{\mathbb{Q}}(\{\omega_j\}) S_n^*(1, \omega_j) \\ &= \mathbb{E}_{\hat{\mathbb{Q}}}(S_n^*(1)) \\ &= S_n^*(0)^{[16]} \end{aligned}$$

Thus $\mathbb{Q} \in \mathbb{M}$ which is a contradiction to the uniqueness of $\hat{\mathbb{Q}}$.

[14] π is *Orthogonal* to A.
[15] i.e. $\sum_j \mathbb{Q}(\{\omega_j\}) = \underbrace{\sum_j \hat{\mathbb{Q}}(\{\omega_j\})}_{=1} + \underbrace{\sum_j \lambda \pi B_1}_{=0} = 1$
[16] As $\hat{\mathbb{Q}}$ is a *Risk-Neutral Probability Measure.*

□

Example 1.8 - *Complete Markets*
Consider the situation in `Example 1.6`.

Here is another way to check that X is *Attainable* is to check if the model is *Complete*.

Consider the matrix A

$$A = \begin{pmatrix} 1+r & 25 \\ 1+r & 5 \end{pmatrix}$$

Since A has two linearly-independent columns, AH can take all values in $\mathbb{R}^2$.

Hence the model is *Complete*.

2 Stochastic Processes in Discrete Time

2.1 Multi-Period Models

Remark 2.1 - *Multi-Period vs Single-Period Model*
The *Multi-Period Model* for securities markets are much more realistic than *Single-Period Models*.

Definition 2.1 - *Multi-Period Model*
A *Multi-Period Model* is a model for a financial market with the following elements

- $T+1$ trading dates $\{0, 1, \dots, T\}$.
- A finite sample space Ω where each element corresponds to a certain state of the world.

$$\Omega = \{\omega_1, \dots, \omega_K\} \text{ with } K < \infty$$

- A probability measure $\mathbb{P}$ on Ω with $\mathbb{P}(\omega) > 0 \ \forall \ \omega \in \Omega$.

Definition 2.2 - *Filtration $\mathcal{F}$*
A *Filtration* $\mathcal{F} := \{\mathcal{F}_t : t = 0, 1, \dots, T\}$[17] is a sub-model describing how the formation about the security prices is revealed to investors.

Definition 2.3 - *Bank Process B*
A *Bank Process* $B := \{B_t : t = 0, 1, \dots, T\}$ where B is a stochastic process with $B_0 = 1$ and $B_t(\omega) > 0$.

The *Bank Process* at time t B_t should be thought of as the time t value of a savings account when 1 unit of currency was deposited at time $t = 0$.

B is usually a non-decreasing process and the (possibly random) quantity $r_t := \frac{B_t - B_{t+1}}{B_{t+1}} \geq 0, \ t = 1, \dots, T$ should be though of as the interest rate for time-interval $(t-1, t)$.

Definition 2.4 - *Price Process S*
A *Price Process* $S = \{S(t) : t = 0, 1, \dots, T\}$ where each element is multi-dimensional[18] $S(t) = (S_1(t), \dots, S_N(t))$ and $S_i(t)$ is a non-negative stochastic process for each $i \in [1, N]$.

[17] Each $\mathcal{F}_t$ is a σ-*Algebra*

[18] One dimension for each security in the market.

The $S_i(t)$ should be though of as the price of the i^{th} security in time-period t.

2.2 Information Structures

Definition 2.5 - *Partition $\mathcal{P}$*
A set $A = \{a_1, \dots, a_n\}$ forms a partition $\mathcal{P}$ if each element of A is disjoint and the union of all the elements of A is the whole sample space Ω.

$$a_i \cap a_j = \emptyset \ \forall \ a_i, a_j \in A \quad \text{and} \quad \bigcup_{i=1}^{n} a_i = \Omega$$

Example 2.1 - *Information Structures*
Consider a multi-period model with $T = 3$ time-periods and whether the stock either increases or decreases in value is the only event recorded in each time-period. The sample space has $K = 2^3 = 8$ different outcomes

ω_1	ω_2	ω_3	ω_4	ω_5	ω_6	ω_7	ω_8
UUU	UUD	UDU	UDD	DUU	DUD	DDU	DDD

Initially, all of these outcomes are possible (ie $A_0 = \Omega$). After the first move occurs there are only $2^2 = 4$ possible outcomes, after the second there are only $2^1 = 2$ possible outcomes and after the third there is only 1 possible outcome.

The set of possible subsets A_t at time t form a *Partition* $\mathcal{P}_t$.

Remark 2.2 - *Flow of Information*
At time $t = 0$ every state $\omega \in \Omega$ is a possible outcome (although usually not with the same probability). And, at time $t = T$, the true state of the world is known. Each new piece of information removes certain states.

One can view the evolution of information as a random sequence $\{A_t\}$ of subsets of Ω where

$$A_0 = \Omega, \ A_T = \{\omega\} \text{ and } A_0 \supseteq A_1 \supseteq \dots \supseteq A_T$$

There exists K possible *Information Sequences* $\{A_t\}_{t\in[1,T]}$ of subsets, corresponding to the K elements of Ω. At time $t = 0$ the investors are aware of all these sequences, but they do not know which one is the true sequence.

Definition 2.6 - *Information Sequence*
The *Information Sequence* is fully describe by a sequence $\mathcal{P}_0, \dots, \mathcal{P}_T$ of partitions of Ω with

- $\mathcal{P}_0 = \{\Omega\}$.
- $\mathcal{P}_T = \big\{\{\omega_1\}, \{\omega_2\}, \{\omega_3\}, \{\omega_4\}, \{\omega_5\}, \{\omega_6\}, \{\omega_7\}, \{\omega_8\}\big\}$
- And, $\forall \ t < T$ each $A \in \mathcal{P}_t$ is equal to the union of some elements of $\mathcal{P}_{t+1}$.

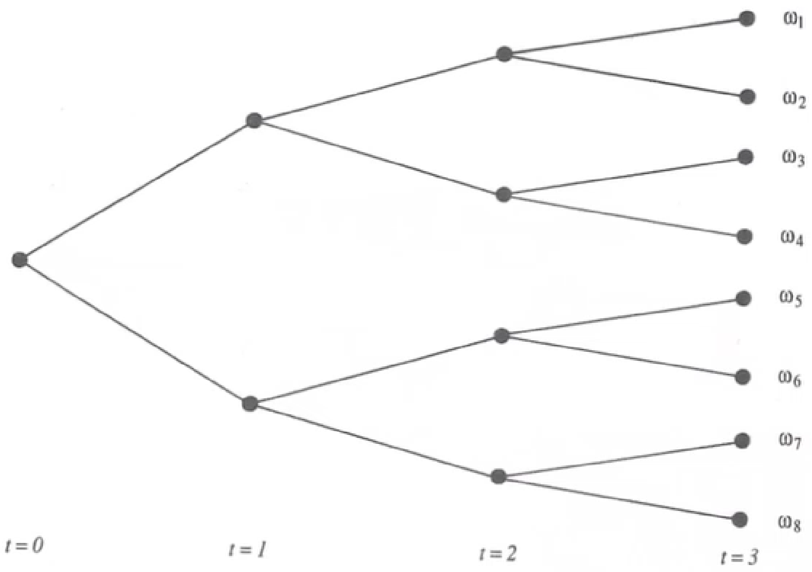

Figure 1: Tree diagram for the structure of information from `Example 2.1.`

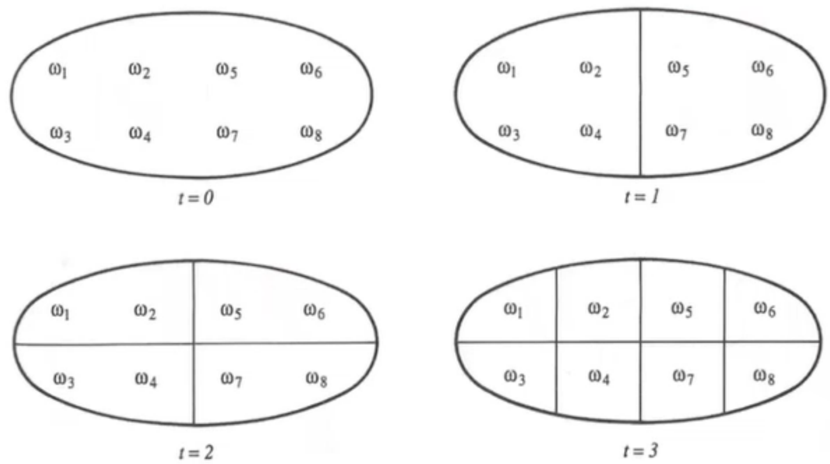

Figure 2: A sequence of pictures demonstrating the structure of information `Example 2.1.`

Example 2.2 -
Consider an arbitrary sequence $\{\hat{A}_t\}_{t\in[1,T]}$ and some time-point $s<T$. For each $\omega\in\hat{A}_s$ the sequence $\{A_t\}_{t\in[1,T]}$ with $A_T=\{\omega\}$ will coincide with $\{\hat{A}_t\}$ at least until time $t=s$.

The collection of subsets $\{A_{s+1}\}$ that can follow $\hat{A}_s$ forms a partition of $\hat{A}_s$, that is, a collection of disjoint subsets whose union equals $\hat{A}_s$.

In particular, taking $s=0$, we see that the collection $\{A_1\}$ of all possible time-period $t=1$ subsets forms a partition of Ω. This partition is denoted $\mathcal{P}_1$.

Moreover, the collection $\{A_2\}$ of all possible time-period $t=2$ subsets also forms a partition of Ω, denoted $\mathcal{P}_2$. Consider `Example 2.1` where $K=8$ and $T=3$, the partitions are

$$\begin{array}{rcl}
\mathcal{P}_0 &=& \big\{\{\omega_1,\omega_2,\omega_3,\omega_4,\omega_5,\omega_6,\omega_7,\omega_8\}\big\}\\
\mathcal{P}_1 &=& \big\{\{\omega_1,\omega_2,\omega_3,\omega_4\},\{\omega_5,\omega_6,\omega_7,\omega_8\}\big\}\\
\mathcal{P}_2 &=& \big\{\{\omega_1,\omega_2\},\{\omega_3,\omega_4\},\{\omega_5,\omega_6\},\{\omega_7,\omega_8\}\big\}\\
\mathcal{P}_3 &=& \big\{\{\omega_1\},\{\omega_2\},\{\omega_3\},\{\omega_4\},\{\omega_5\},\{\omega_6\},\{\omega_7\},\{\omega_8\}\big\}
\end{array}$$

Remark 2.3 - *Visualising Information Structure*
There are two popular ways to visualise information structure

i). A *Tree Diagram* where each node corresponds to an element A_t of the time t partition and where there is edge arc going from this node to each node corresponding to some $A_{t+1} \subseteq A_t$. (See `Figure 1`).

ii). A *Sequence of Pictures* of the same space. (See `Figure 2`).

Definition 2.7 - *σ-Algebra $\mathcal{F}_t$*
A collection $\mathcal{F}$ of subsets of Ω is called a *σ-Algebra* on sample space Ω if

i). $\Omega \in \mathcal{F}$.

ii). $\forall\ F \in \mathcal{F},\ F^c \in \mathcal{F}$.

iii). $\forall\ F, G \in \mathcal{F},\ (F \cup G) \in \mathcal{F}$.

Proposition 2.1 - *Generated σ-Algebra*
For any partition $\mathcal{P}$ of Ω we can generate *σ-Algebra* $\mathcal{F}$ by letting $\mathcal{F}$ be the collection of all unions of elements of $\mathcal{P}$ together with the complements of all such unions.

Hence, the sub-models of the information structure can be organised as a sequence $\{\mathcal{F}_t\}_{t\in[1,T]}$ of *σ-Algebras*.

Definition 2.8 - *Filtration $\mathcal{F}$*
A *Filtration* is a family of *σ-Algebras* $\mathcal{F} := \{\mathcal{F}_t : t = 0, 1, \dots, T\}$ where

i). $\mathcal{F}_0 = \{\emptyset, \Omega\}$.

ii). $\mathcal{F}_T$ consists of all subsets of Ω.

iii). $\mathcal{F}_n \subset \mathcal{F}_{n+1}$ for all $n < T$, by which we mean that each subset of $\mathcal{F}_n$ must be an element of $\mathcal{F}_{n+1}$.

Example 2.3 - *σ-Algebra*
Consider the context of `Example 2.1`. The corresponding filtration is given by

$$\begin{array}{rcl}
\mathcal{F}_0 & = & \big\{\emptyset, \Omega\big\} \\
\mathcal{F}_1 & = & \big\{\emptyset, \Omega, \{\omega_1, \omega_2, \omega_3, \omega_4\}, \{\omega_5, \omega_6, \omega_7, \omega_8\}\big\} \\
\mathcal{F}_2 & = & \big\{\emptyset, \Omega, \{\omega_1, \omega_2\}, \{\omega_3, \omega_4\}, \{\omega_5, \omega_6\}, \{\omega_7, \omega_8\}, \{\omega_1, \omega_2, \omega_3, \omega_4\}, \\
& & \{\omega_5, \omega_6, \omega_7, \omega_8\}, \{\omega_1, \omega_2, \omega_5, \omega_6\}, \{\omega_1, \omega_2, \omega_7, \omega_8\}, \{\omega_3, \omega_4, \omega_5, \omega_6\}, \\
& & \{\omega_3, \omega_4, \omega_7, \omega_8\}, \{\omega_1, \omega_2, \omega_3, \omega_4, \omega_5, \omega_6\}, \{\omega_1, \omega_2, \omega_3, \omega_4, \omega_7, \omega_8\}, \\
& & \{\omega_1, \omega_2, \omega_5, \omega_6, \omega_7, \omega_8\}, \{\omega_3, \omega_4, \omega_5, \omega_6, \omega_7, \omega_8\}\big\}
\end{array}$$

and $\mathcal{F}_3$ contains all the subsets of Ω (ie $\mathcal{F}_3 = 2^\Omega$).

2.3 Stochastic Processes in Discrete Time

Definition 2.9 - *Stochastic Process S*
A *Stochastic Process* S is a real-valued function $S(t)(\omega)$ of two variables, t and ω.

For each fixed $\omega \in \Omega$ the function mapping $t \to S(t)(\omega)$ is called the *Sample Path*.

For each fixed $t \in [0, T]$ the function mapping $\omega \to S(t)(\omega)$ is a *Random Variable*.

NB - For simplicity, we assume $S(0)$ is constant.

Definition 2.10 - *Measurable Function*
A function $\omega \to W(\omega)$ is said to be *Measurable* wrt the *σ-Algebra* $\mathcal{F}$ if

$$\forall\ x \in \mathbb{R} \text{ where } W^{-1}(x) \subset \mathcal{F} \text{ it is true that } W^{-1}(x) := \big\{\omega \in \Omega : W(\omega) = x\big\}.$$ [19]

Example 2.4 - *Measurable Function*
Consider random variables X, Y defined as

$$\begin{aligned} X(\omega) &= \begin{cases} 5 & ,\omega \in \{\omega_1, \omega_2, \omega_3, \omega_4\} \\ 7 & ,\omega \in \{\omega_5, \omega_6, \omega_7, \omega_8\} \end{cases} \\ Y(\omega) &= \begin{cases} 8 & ,\omega \in \{\omega_1, \omega_3, \omega_5, \omega_7\} \\ 6 & ,\omega \in \{\omega_2, \omega_4, \omega_6, \omega_8\} \end{cases} \end{aligned}$$

Then, by the notation of `Example 2.1`, we have that

- X is *Measurable* wrt $\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3$ as $\big\{\{\omega_1, \omega_2, \omega_3, \omega_4\}, \{\omega_5, \omega_6, \omega_7, \omega_8\}, \emptyset\big\} \subset \mathcal{F}_1$.
- Y is <u>not</u> *Measurable* wrt $\mathcal{F}_1, \mathcal{F}_2$, but is measurable wrt $\mathcal{F}_3$, as $\{\omega_1, \omega_3, \omega_5, \omega_7\} \notin \mathcal{F}_1$.

Remark 2.4 - *Measurable*
If a function X is *Measurable* wrt $\mathcal{F}_t$ then it will be *Measurable* wrt $\mathcal{F}_{t+1}$, as $\mathcal{F}_t \subseteq \mathcal{F}_{t+1}$.

Definition 2.11 - *Adapted*
A *Stochastic Process* $S = \{S(t) : t = 0, 1, \dots, T\}$ is said to be *Adapted* to the *Filtration* $\mathcal{F} = \{\mathcal{F}_t : t = 0, 1, \dots, T\}$ if for every $t = 0, \dots, T$ the random variable $S(t)$ is *Measurable* wrt $\mathcal{F}_t$.

Remark 2.5 - *Adapted Filtrations in Practice*
In practice we often define the *Stochastic Process* S first and use the so-called "*Natural Filtration*", defined as

i). For each $t = 0, 1, \dots, T$ let $\mathcal{P}_t$ be the partition of Ω st the *Stochastic Process* $\{S(0), \dots, S(t)\}$ takes the same value for each $\omega \in A$, for each subset $A \in \mathcal{P}_t$.

ii). Let $\mathcal{F}_t$ be the *σ-Algebra* generated by $\mathcal{P}_t$.

iii). Then $\mathcal{F} := \{\mathcal{F}_t : t = 0, 1, \dots, T\}$ is called the filtration *Generated* by the *Stochastic Process* S.

Example 2.5 -
Consider the setting it `Example 2.1` and let $S(t)$ be the upwards movements in the value of the asset by time t. Then $S(t)$ generates the *Filtration* above.

This can be summarised in the table

[19] Equivalently, if we know which set of the *σ-algebra* ω is in, then we know the value of $W(\omega)$.

ω_k	$t=0$	$t=1$	$t=2$	$t=3$
ω_1	$S(t)=0$	$S(t)=1$	$S(t)=2$	$S(t)=3$
ω_2	$S(t)=0$	$S(t)=1$	$S(t)=2$	$S(t)=2$
ω_3	$S(t)=0$	$S(t)=1$	$S(t)=1$	$S(t)=2$
ω_4	$S(t)=0$	$S(t)=1$	$S(t)=1$	$S(t)=1$
ω_5	$S(t)=0$	$S(t)=0$	$S(t)=1$	$S(t)=2$
ω_6	$S(t)=0$	$S(t)=0$	$S(t)=1$	$S(t)=1$
ω_7	$S(t)=0$	$S(t)=0$	$S(t)=0$	$S(t)=1$
ω_8	$S(t)=0$	$S(t)=0$	$S(t)=0$	$S(t)=0$

Definition 2.12 - *Random Walk*
Let $X_t := X_0 + Y_1 + \dots + Y_t$ where $Y_1, \dots, Y_t$ are iid random variables with finite variance σ^2 and mean *mean*. Then $\{X_t : t \geq 0\}$ is a *Random Walk*.

We say that $\{X_t : t \geq 0\}$ is a *Simple Random Walk* if Y_i takes only the values 1 with probability p and -1 with probability $1-p$.

Proposition 2.2 - *Distribution of Simple Random Walk Values*
A *Simple Random Walk* takes values y at time-point t iff exactly $\frac{t+y}{2}$ of $Y_1, \dots, Y_t$ are equal to 1, and the remaining $\frac{t-y}{2}$ equal -1.

$$\forall\ t \geq 0, y \in \{-t, -t+2, \dots, t-2, t\}, \quad \mathbb{P}(X_t = y) = \binom{t}{\frac{t+y}{2}} p^{(t+y)/2}(1-p)^{(t-y)/2}$$

2.4 Conditional Expectations

Definition 2.13 - *Conditional Expectation* $\mathbb{E}[\cdot|\cdot]$
For a finite *Sample Space* Ω the *Conditional Expectation* of a discrete RV Y given the event A $\mathbb{E}[Y|A]$ is defined as

$$\mathbb{E}[Y|A] = \sum_y y\mathbb{P}(Y = y|A)$$

This *Conditional Expectation* maps from the events A to the real numbers.

Remark 2.6 - *Rewriting Conditional Expectation*
We can rewrite the *Conditional Expectation* as

$$\begin{array}{rcl} \mathbb{E}[Y|A] & = & \sum_y y \frac{\mathbb{P}(Y(\omega)=y, A)}{\mathbb{P}(A)} \\ & = & \sum_{\omega \in A} Y(\omega) \frac{\mathbb{P}(\omega)}{\mathbb{P}(A)} \end{array}$$

Example 2.6 - *Conditional Expectation*
Consider `Example 2.5` with ω_i having probability 1/8.

- If $A := \{\omega_1, \omega_2, \omega_3, \omega_4\}$ then
$$\mathbb{E}[S_3|A] = \frac{(3+2+2+1)(1/8)}{1/2} = 2$$
- If $A := \{\omega_5, \omega_6, \omega_7, \omega_8\}$ then
$$\mathbb{E}[S_3|A] = \frac{(2+1+1+0)(1/8)}{1/2} = 1$$

Definition 2.14 - *Conditional Expectation /w σ-Algebra*
Let $\mathcal{F}$ be a σ-*algebra* and $\mathcal{P}$ be the corresponding *Partition* of Ω.

We define the *Conditional Expectation* of RV Y give σ-*algebra* $\mathcal{F}$ as

$$\mathbb{E}[Y|\mathcal{F}] = \sum_{A\in\mathcal{P}} \mathbb{E}[Y|A]\mathbb{1}_A$$

This is a random-variable[20] which is *Measurable* wrt $\mathcal{F}$. And,

$$\forall\ \omega \in A,\ \mathbb{E}[Y|\mathcal{F}](\omega) = \mathbb{E}[Y|A]$$

Example 2.7 - *Conditional Expectation /w σ-Algebras*
Consider `Example 2.5` with ω_i having probability $1/8$.

Recall that $\mathcal{F}_1 = \big\{\emptyset, \Omega, \{\omega_1,\omega_2,\omega_3,\omega_4\}, \{\omega_5,\omega_6,\omega_7,\omega_8\}\big\}$. Then

$$\mathbb{E}[S_3|A] = \begin{cases} 2 & \text{if } A = \{\omega_1,\omega_2,\omega_3,\omega_4\} \\ 1 & \text{if } A = \{\omega_5,\omega_6,\omega_7,\omega_8\} \end{cases}$$

Hence $\mathbb{E}[S_3|\mathcal{F}_1]$ is a random variable with

$$\mathbb{E}[S_3|\mathcal{F}](\omega_i) = \begin{cases} 2 & \text{if } i = 1,2,3,4 \\ 1 & \text{if } i = 5,6,7,8 \end{cases}$$

This random variable is $\mathcal{F}_1$-*Measurable.*

Observer that

$$\begin{array}{rcl} \mathbb{E}[\mathbb{E}[S_3|\mathcal{F}_1]] & = & \sum_i \mathbb{P}(\omega_i)\mathbb{E}[S_3|\mathcal{F}_1](\omega_i) \\ & = & 2\cdot\frac{4}{8} + 1\cdot\frac{4}{8} = \frac{3}{2} = \mathbb{E}[S_3]^{[21]} \end{array}$$

Note that if

$$Y(\omega_i) = \begin{cases} 7 & \text{if } i \leq 4 \\ 1 & \text{if } i \geq 5 \end{cases} \implies \mathbb{E}[Y|\mathcal{F}_1] = Y$$

Moreover, If $Z = \mathbb{1}\{\text{3rd move is up}\}$ then | is independent of $\mathcal{F}_1$ and

$$\mathbb{E}[Z|\mathcal{F}_1] = \frac{1}{2} = \mathbb{E}[Z]$$

Theorem 2.1 - *Properties of Conditional Expectation*
Let Y be a random variable and $\mathcal{F}, \mathcal{F}_1, \mathcal{F}_2$ be σ-*algebras* with $\mathcal{F}_1 \subset \mathcal{F}_2$.

Conditional Expectations satisfy the following properties:

i). $\mathbb{E}[\mathbb{E}[Y|\mathcal{F}]] = \mathbb{E}[Y]$. (The "Tower Law")

ii). $\mathbb{E}[\mathbb{E}[Y|\mathcal{F}_2]|\mathcal{F}_1] = \mathbb{E}[Y|\mathcal{F}_1] = \mathbb{E}[\mathbb{E}[Y|\mathcal{F}_1]\mathcal{F}_2]$. (The "Generalised Tower Law")

iii). If X is a random variable which is *Measurable* wrt $\mathcal{F}$ then $\mathbb{E}[XY|\mathcal{F}] = X\mathbb{E}[Y|\mathcal{F}]$ and $\mathbb{E}[X|\mathcal{F}] = X$.[22]

[20]Not a sum, as there is one A st $\mathbb{1}_A = 1$ (the rest equal zero)
[21]This is the "Tower Law"
[22]As X is measurable wrt $\mathcal{F}$ all information is known about it, and thus can treat it as a scalar.

iv). If Y is independent of $\mathcal{F}$[23] then $\mathbb{E}[Y|\mathcal{F}]=\mathbb{E}[Y]$.

Proof 2.1 - *Theorem 2.1 i)*

$$\begin{aligned}
\mathbb{E}[\mathbb{E}[Y|\mathcal{F}]] &= \mathbb{E}\left[\sum_{A\in\mathcal{P}}\mathbb{E}[Y|A]\mathbb{1}_A\right] \text{ by def. cond. expectation}\\
&= \sum_{A\in\mathcal{P}}\mathbb{E}[\mathbb{1}_A]\mathbb{E}[Y|A]\\
&= \sum_{A\in\mathcal{P}}\left(\sum_{\omega\in\Omega}\mathbb{1}_A(\omega)\mathbb{P}(\omega)\right)\cdot\left(\sum_{\omega\in A}\frac{Y(\omega)\mathbb{P}(\omega)}{\mathbb{P}(A)}\right)\\
&= \sum_{A\in\mathcal{P}}\mathbb{P}(A)\left(\sum_{\omega\in A}\frac{Y(\omega)\mathbb{P}(\omega)}{\mathbb{P}(A)}\right)\\
&= \sum_{A\in\mathcal{P}}\sum_{\omega\in A}Y(\omega)\mathbb{P}(\omega)\\
&= \mathbb{E}[Y]
\end{aligned}$$

□

Proof 2.2 - *Theorem 2.1 ii)*

$$\begin{aligned}
\mathbb{E}[\mathbb{E}[Y|\mathcal{F}_2]\mathcal{F}_1] &= \mathbb{E}\left(\sum_{B\in\mathcal{P}_2}\mathbb{E}[Y|B]\mathbb{1}_B\middle|\mathcal{F}_1\right)\\
&= \sum_{B\in\mathcal{P}_2}\mathbb{E}[Y|B]\mathbb{E}[\mathbb{1}_B|\mathcal{F}_1]\\
&= \sum_{B\in\mathcal{P}_2}\mathbb{E}[Y|B]\sum_{A\in\mathcal{P}_1}\mathbb{E}[1_B|A]\mathbb{1}_A\\
&= \sum_{A\in\mathcal{P}_1}\sum_{B\in\mathcal{P}_2}\mathbb{E}[Y|B]\mathbb{E}[1_B|A]\mathbb{1}_A\\
&= \sum_{A\in\mathcal{P}_1}\sum_{B\in\mathcal{P}_2}\mathbb{E}[Y|B]\frac{\mathbb{P}(A\cap B)}{\mathbb{P}(A)}\mathbb{1}_A
\end{aligned}$$

Since the partition $\mathcal{P}_2$ is finer than $\mathcal{P}_1$ for all $B\in\mathcal{P}_2$ and $A\in\mathcal{P}_1$, either $B\subset A$ or $B\cap A=\emptyset$. Thus

$$\begin{aligned}
\mathbb{E}[\mathbb{E}[Y|\mathcal{F}_2]|\mathcal{F}_1] &= \sum_{A\in\mathcal{P}_1}\sum_{B\in\mathcal{P}_2,B\subset A}\mathbb{E}[Y|B]\frac{\mathbb{P}(B)}{\mathbb{P}(A)}\mathbb{1}_A\\
&= \sum_{A\in\mathcal{P}_1}\sum_{B\in\mathcal{P}_2,B\subset A}\left(\sum_{\omega\in B}Y(\omega)\frac{\mathbb{P}(\omega)}{\mathbb{P}(B)}\right)\frac{\mathbb{P}(B)}{\mathbb{P}(A)}\mathbb{1}_A\\
&= \sum_{A\in\mathcal{P}_1}\sum_{\omega\in A}Y(\omega)\frac{\mathbb{P}(\omega)}{\mathbb{P}(A)}\mathbb{1}_A\\
&= \sum_{A\in\mathcal{P}_1}\mathbb{E}[Y|A]\mathbb{1}_A\\
&= \mathbb{E}[Y|\mathcal{F}_1]
\end{aligned}$$

□

Proof 2.3 - *Theorem 2.1 iii)*
Since S is *Measurable* it is constant on sets of $\mathcal{P}$ so we can write

$$X=\sum_{A\in\mathcal{P}}x_A\mathbb{1}_A$$

[23] $\forall\ A\in\mathcal{P}$ the distribution of Y is independent of A

with suitable scalars $x_A \in \mathbb{R}$. Then

$$\begin{aligned}
\mathbb{E}[XY|\mathcal{F}] &= \sum_{A\in\mathcal{P}} \mathbb{E}[XY|A]\mathbb{1}_A \\
&= \sum_{A\in\mathcal{P}} \mathbb{E}[x_A Y|A]\mathbb{1}_A \\
&= \sum_{A\in\mathcal{P}} x_A \mathbb{E}[Y|A]\mathbb{1}_A \\
&= \sum_{A\in\mathcal{P}} X\mathbb{E}[Y|A]\mathbb{1}_A{}^{[24]} \\
&= X\sum_{A\in\mathcal{P}} \mathbb{E}[Y|A]\mathbb{1}_A \\
&= X\mathbb{E}[Y|\mathcal{F}]
\end{aligned}$$

Consider the special case where $Y = 1$, then it follows that $\mathbb{E}[X|\mathcal{F}] = X$. $\square$

Proof 2.4 - `Theorem 2.1 iv)`
If Y is independent of $\mathcal{F}$ then for all $A \in \mathcal{F}$

$$\begin{aligned}
\mathbb{E}[Y|A] &= \sum_y y\mathbb{P}(Y = y|A) \\
&= \sum_y y\mathbb{P}(Y = y) \\
&= \mathbb{E}[Y]
\end{aligned}$$

$\square$

Proposition 2.3 - *Conditional Expectation for Non-Finite Sample Spaces*
Let Y be a RV, then the *Conditional Expectation* $\mathbb{E}[Y|\mathcal{F}]$ is the unique random variable st

i). $\mathbb{E}[Y|\mathcal{F}]$ is $\mathcal{F}$-Measurable.

ii). $\forall\ A \in \mathcal{F},\ \mathbb{E}[\mathbb{E}[Y|\mathcal{F}] \cdot \mathbb{1}_A] = \mathbb{E}[Y\mathbb{1}_A]$.

Proof 2.5 - `Proposition 2.3`
For an arbitrary event $A \in \mathcal{F}$ the indicator function $\mathbb{1}_A$ is *$\mathcal{F}$-Measurable* and thus $\mathbb{E}[\mathbb{1}_A Y|\mathcal{F}] = \mathbb{1}_A\mathbb{E}[Y|\mathcal{F}]$ by `Theorem 2.1`.
Hence, for all $A \in \mathcal{F}$

$$\mathbb{E}[\mathbb{E}[Y|\mathcal{F}] \cdot \mathbb{1}_A] = \mathbb{E}[\mathbb{E}[\mathbb{1}_A Y|\mathcal{F}]] = \mathbb{E}[\mathbb{1}_A Y]$$

On the other hand, suppose X is *$\mathcal{F}$-Measurable* and satisfies

$$\mathbb{E}[X\mathbb{1}_A] = \mathbb{E}[Y1_A]\ \forall\ A \in \mathcal{F}$$

Then write X as in `Proof 2.3` $X = \sum_{A\in\mathcal{P}} x_A\mathbb{1}_A$.
It follows that $\forall\ A \in \mathcal{P},\ \mathbb{E}[X\mathbb{1}_A] = x_A\mathbb{P}(A)$ and thus

$$\begin{aligned}
\mathbb{E}[Y\mathbb{1}_A] &= \textstyle\sum_{\omega\in A} Y(\omega)\mathbb{P}(\omega) \\
&= \tfrac{\mathbb{P}(A)}{\mathbb{P}(A)}\textstyle\sum_{\omega\in A} Y(\omega)\mathbb{P}(\omega) \\
&= \mathbb{P}(A)\textstyle\sum_{\omega\in A} \tfrac{Y(\omega)\mathbb{P}(\omega)}{\mathbb{P}(A)} \\
&= \mathbb{P}(A)\mathbb{E}[Y|A]
\end{aligned}$$

[24] This holds as $\mathbb{1}_A = 1$ for only one specific A.

Thus

$$x\mathbb{P}(A) = \mathbb{E}[X\mathbb{1}_A] = \mathbb{E}[Y\mathbb{1}_A] = \mathbb{P}(A)\mathbb{E}[Y|A]$$

Therefore, we have found that $x_A = \mathbb{E}[Y|A] \ \forall \ A \in \mathcal{P}$ where $\mathcal{P}$ is the partition which corresponds to *σ-algebra* $\mathcal{F}$.

Moreover, $X = \mathbb{E}[Y|\mathcal{F}]$.

2.5 Martingales

Definition 2.15 - *Martingale Z*
Let $Z := \{Z_t : t = 0, 1, \dots, T\}$ be an *Adapted Stochastic Process* defined on a *Sample Space* Ω with a filtration $\{\mathcal{F}_t\}$.

The process Z is said to be a *Martingale* if

$$\mathbb{E}[Z_t|\mathcal{F}_{t-1}] = Z_{t-1} \ \forall \ t \geq 1$$

Definition 2.16 - *Super-Martingale Z*
Let $Z := \{Z_t : t = 0, 1, \dots, T\}$ be an *Adapted Stochastic Process* defined on a *Sample Space* Ω with a filtration $\{\mathcal{F}_t\}$.

The process Z is said to be a *Super-Martingale* if

$$\mathbb{E}[Z_t|\mathcal{F}_{t-1}] \leq Z_{t-1} \ \forall \ t \geq 1$$

Definition 2.17 - *Sub-Martingale Z*
Let $Z := \{Z_t : t = 0, 1, \dots, T\}$ be an *Adapted Stochastic Process* defined on a *Sample Space* Ω with a filtration $\{\mathcal{F}_t\}$.

The process Z is said to be a *Sub-Martingale* if

$$\mathbb{E}[Z_t|\mathcal{F}_{t-1}] \geq Z_{t-1} \ \forall \ t \geq 1$$

Remark 2.7 - *Super- vs Sub-Martingale*
You should think of a *Super-Martingale* as a process where the current value provides an upper-bound on the next value, and a *Sub-Martingale* as a process where the current value provides a lower-bound on the next value.

Theorem 2.2 - *When is an Adapted Stochastic Process a Martingale?*
An *Adapted Stochastic Process Z* is a *Martingale* iff

$$\mathbb{E}[Z_t|\mathcal{F}_s] = Z_s \ \forall \ t \geq s$$

Corresponding results (with equality swapped out) hold for super- and sub-martingales.

Proof 2.6 - `Theorem 2.2`

$\Longleftarrow$ Clearly, if the equation in `Theorem 2.2` holds then `Definition 2.15` holds.

$\implies$ Assume Z is a *Martingale*. Then `Theorem 2.1` implies that

$$\begin{aligned}\mathbb{E}[Z_t|\mathcal{F}_s] &= \mathbb{E}\big[\mathbb{E}[Z_t|\mathcal{F}_{t-1}]|\mathcal{F}_s\big]\\ &= \mathbb{E}[Z_{t-1}|\mathcal{F}_s] \text{ as } Z \text{ is a martingale}\\ &= \mathbb{E}[Z_s|\mathcal{F}_s] \text{ by repetition}\\ &= \mathbb{E}[Z_s] = Z_s\end{aligned}$$

A similar proof is done for super- and sub-martingales.

Example 2.8 - *Martingales*
Let $\{X_t\}_{t\geq 0}$ be a simple random walk with parameter p and $\mathcal{F}_t$ be the σ-*algebra* generated by (X_t). Then

i). $\{X_t\}_{t\geq 0}$ is a *Martingale* if $p = 1/2$ as there is an equal probability of stepping up and stepping down.

ii). $\{X_t\}_{t\geq 0}$ is a *Super-Martingale* if $p \leq 1/2$ as there is a greater probability of stepping down than stepping up.

iii). $\{X_t\}_{t\geq 0}$ is a *Sub-Martingale* if $p \geq 1/2$ as there is a greater probability of stepping up than stepping down.

iv). If $p = 1/2$ then the process $\{Z_t\}_{t\geq 0}$ defined st $Z_t := X_t^2 - t$ for $t = 0, 1, \dots$ is a *Martingale*

v). If $p \neq 1/2$ then the processes $\{L_t\}_{t\geq 0}$ & $\{M_t\}_{t\geq 0}$ defined by $L_0 = 1,\ L_t = \left(\frac{1-p}{p}\right)^{X_t}$ and $M_t = X_t - t(2p-1)$ are both *Martingales*[25].

Proof 2.7 - `Example 2.8` *i)-iii)*
Since $\mathcal{F}_t$ is the natural filtration, then $\{X_t\}$ is $\mathcal{F}_t$-measurable and Y_t is independent of $\mathcal{F}_{t-1}$. Therefore by `Theorem 2.1` we have that

$$\begin{aligned}\mathbb{E}[X_t|\mathcal{F}_{t-1}] &= \mathbb{E}[X_{t-1} + Y_t|\mathcal{F}_{t-1}]\\ &= \mathbb{E}[X_{t-1}|\mathcal{F}_{t-1}] + \mathbb{E}[Y_t|\mathcal{F}_{t-1}]\\ &= X_{t-1} + \mathbb{E}[Y_t]\end{aligned}$$

- If $p = 1/2$ then $\mathbb{E}[Y_t] = 0 \implies \mathbb{E}[X_t|\mathcal{F}_{t-1}] = X_{t-1}$, the definition of a *Martingale.*
- If $p \leq 1/2$ then $\mathbb{E}[Y_t] \leq 0 \implies \mathbb{E}[X_t|\mathcal{F}_{t-1}] \leq X_{t-1}$, the definition of a *Super-Martingale.*
- If $p \geq 1/2$ then $\mathbb{E}[Y_t] \geq 0 \implies \mathbb{E}[X_t|\mathcal{F}_{t-1}] \geq X_{t-1}$, the definition of a *Sub-Martingale.*

Proof 2.8 - `Example 2.8` *iv)*
Note that

$$\begin{aligned}\mathbb{E}[Z_t|\mathcal{F}_{t-1}] &= \mathbb{E}[X_t^2 - t|\mathcal{F}_{t-1}]\\ &= \mathbb{E}[(X_{t-1} + Y_t)^2|\mathcal{F}_{t-1}] - t\\ &= \mathbb{E}[X_{t-1}^2|\mathcal{F}_{t-1}] + 2\mathbb{E}[X_tY_t|\mathcal{F}_{t-1}] + \mathbb{E}[Y_t^2|\mathcal{F}_{t-1}]t\\ &= X_{t-1}^2 + 2X_{t-1}\underbrace{\mathbb{E}[Y_t|\mathcal{F}_{t-1}]}_{=\mathbb{E}[Y_t]=0} + \underbrace{\mathbb{E}[Y_t^2]}_{=1^{[26]}} - t\\ &= X_{t-1}^2 + 0 + 1 - t\\ &= X_{t-1}^2 - (t-1) = Z_{t-1}\end{aligned}$$

This shows that the Z_t we defined fulfils the definition of a *Martingale.*

[25]Proved in a homework
[26]Y_t only takes values $\{-1, 1\}$ so $Y_t^2 = 1$ always.

2.6 Stopping Times τ

Remark 2.8 - *Stopping Times & Finance*
Stopping Times are useful for analysing American options.

Definition 2.18 - *Stopping Time τ*
Let Ω be a *Sample Space* with a filtration $\{\mathcal{F}_t\}_{t\in\mathbb{N}_0}$.

A *Stopping Time* is a random variable τ which takes values in the set $\{0,1,\dots,\infty\}$[27] st each event of the form $\{\tau \leq t\}$ from some t is an element of the *σ-algebra* $\mathcal{F}_t$.[28]

A *Stopping Time* is said to be *Bounded* if $\exists\ k$ st $\mathbb{P}(\tau < t) = 1$.

Example 2.9 - *Stopping Time*
Consider the following events

- "RBS shares hit 100p." - This is a *Stopping Time* event.
- "RBS shares hit their maximum." - This is not a *Stopping Time* event.

Theorem 2.3 - *Stopping Times & σ-Algebras*
A random variable τ is a *Stopping Time* iff each event of the form $\{\tau = t\}$ for some t is an element of the *σ-Algebra* $\mathcal{F}_t$.

Proof 2.9 - `Theorem 2.3`

$$\Longleftarrow\ \{\tau = t\} = \Big(\underbrace{\{\tau \leq t\}}_{\in\mathcal{F}_t}\setminus\underbrace{\{\tau \leq t-1\}}_{\in\mathcal{F}_t}\Big) \in \mathcal{F}_t$$

$$\Longrightarrow\ \{\tau \leq t\} = \bigcup_{k\leq t}\underbrace{\{\tau = k\}}_{\in\mathcal{F}_t} \in \mathcal{F}_t$$

Thus the result holds in both directions.

Theorem 2.4 - *Stopping Time for an Adapted Stochastic Process*
Let $\{X_t\}_{t\in\mathbb{N}_0}$ be an *Adapted Stochastic Process* and $c \in \mathbb{R}$.

A *Stopping Time* τ_c can be defined as $\tau_c = \inf\{t \geq 0 : X_t \geq\}$[29].

Proposition 2.4 - `Theorem 2.4`
We note that $\tau_c \leq t$ iff $\exists\ k \leq t$ st $X_k \geq c$.

Therefore

$$\{\tau_c \leq t\} = \bigcup_{k\leq t}\underbrace{\{X_k \geq c\}}_{\in\mathcal{F}_t} \in \mathcal{F}_t$$

Thus τ_c is a *Stopping Time*.

Theorem 2.5 - *Optional Sampling Theorem*[30] *- Martingales*

[27] ∞ is used for events which never occur.
[28] Thus, we can determine whether $\{\tau \leq t\}$ has occurred just by observing $\mathcal{F}_t$ (ie all the information available at time-point t).
[29] The event which stops the moment X_t reaches c
[30] AKA *Optional Stopping Theorem*

Let τ be a *Bounded Stopping Time* and $X=\{X_t\}_{t\in\mathbb{N}_0}$ be a *Martingale*. Then

$$\mathbb{E}[X_\tau]=\mathbb{E}[X_0]=X_0$$

Theorem 2.6 - *Optional Sampling Theorem - Super-Martingales*
Let τ be a *Bounded Stopping Time* and $X=\{X_t\}_{t\in\mathbb{N}_0}$ be a *Super-Martingale*. Then

$$\mathbb{E}[X_\tau]=\leq\mathbb{E}[X_0]$$

Proof 2.10 - `Theorem 2.5`
Assume that $\tau\leq K$ and write

$$X_{\tau(\omega)}(\omega)=\sum_{t=0}^K X_t(\omega)\mathbb{1}\{\tau(\omega)=t\}^{[31]}$$

Then

$$\begin{array}{rcl}\mathbb{E}[X_\tau]&=&\mathbb{E}\left[\sum_{t=0}^K X_t\mathbb{1}\{\tau=t\}\right]\\&=&\sum_{t=0}^K\mathbb{E}[X_t\mathbb{1}\{\tau=t\}]\\&=&\sum_{t=0}^K\mathbb{E}[\mathbb{E}[X_K|\mathcal{F}_t]\mathbb{1}\{\tau=t\}]\end{array}$$

Using that since τ is a *Stopping Time*, then the event $\{\tau=t\}$ is measurable wrt $\mathcal{F}_t$. Thus $\mathbb{E}[X_K|\mathcal{F}_t]\mathbb{1}\{\tau=t\}=\mathbb{E}[X_K\mathbb{1}\{\tau=t\}|\mathcal{F}_t]$.

$$\begin{array}{rl}=&\sum_{t=0}^K\mathbb{E}[\mathbb{E}[X_K\mathbb{1}\{\tau=t\}|\mathcal{F}_t]]\\=&\sum_{t=0}^K\mathbb{E}[X_K\mathbb{1}\{\tau=t\}]\text{ by Tower Law}\\=&\mathbb{E}\left[X_K\sum_{t=0}^K\mathbb{1}\{\tau=t\}\right]\\=&\mathbb{E}[X_K\cdot 1]=\mathbb{E}[X_K]\\=&\mathbb{E}[X_0]=X_0\end{array}$$

Example 2.10 - *Gambler's Ruin*
Consider a gambler with an initial wealth of £C. He gambles by guessing whether a coin flip results in heads or tails. If he guess correctly then he receives £1 and he is wrong he loses £1. The game ends when he either becomes bankrupt or he reaches a wealth of £$C+G$ where $G>0$.

Proposition 2.5 - *Gambler's Ruin*

[31] This is not really a sum, due to the indicator function $\mathbb{1}$ meaning only one value is non-zero.

Let $\{X_t\}_{t\geq 0}$ be a *Simple Random Walk* with parameter p and $X_0 = 0$, and $C, G > 0$. Define the *Stopping-Time* event $\tau = \inf\{t : X_t = G \text{ or } X_t = -C\}$[32].

If $p = 1/2$ then

$$\begin{array}{rcl}\mathbb{P}(X_\tau = G) & = & \dfrac{C}{C+G}\\ \mathbb{P}(X_\tau = -C) & = & \dfrac{G}{C+G}\\ \mathbb{E}[\tau] & = & CG\end{array}$$

Else, if $p \neq 1/2$ then

$$\begin{array}{rcl}\mathbb{P}(X_\tau = G) & = & \dfrac{1-\left(\frac{p}{1-p}\right)^C}{\left(\frac{p}{1-p}\right)^G - \left(\frac{p}{1-p}\right)^C}\\ \mathbb{P}(X_\tau = -C) & = & 1 - \mathbb{P}(X_\tau = G)\\ \mathbb{E}[\tau] & = & \dfrac{G\mathbb{P}(X_\tau = G) + (-C)\mathbb{P}(X_\tau) = -C}{2p-1}\end{array}$$

Proof 2.11 - ***Proposition 2.5***

We want to apply the *Optitional Stopping Theorem* (`Theorem 2.5`).

The *Stopping Time* τ is not bounded, but X_τ is bounded. Therefore we can use one of the alternative versions of the *Optional Stopping Theorem* provided that $\mathbb{P}(\tau < \infty = 1)$.

Note that whenever there is a run of at least $k = C + G$ successive 1's in the process Y which defines X, the process will stop and $\tau < \infty$. Thus for all m

$$\begin{array}{rrcl} & \mathbb{P}(\tau > km) & = & \mathbb{P}(\text{No run of k 1's in } Y_1 \text{ to } Y_{mk})\\ & & = & \displaystyle\prod_{j=0}^{n-1}\mathbb{P}(\text{No run of k 1's in } Y_{jk+1} \text{ to } Y_{(j+1)k})\\ & & = & (1-p^k)^m\\ \implies & \mathbb{P}(\tau < \infty) & = & 1\end{array}$$

i). Consider the case $p = 1/2$. Using the *Optional Stopping Theorem* we can deduce that

$$\begin{array}{rrcl} & 0 & = & \mathbb{E}[X_\tau]\\ & & = & G\mathbb{P}(X_\tau = G) + (-C)\mathbb{P}(X_\tau = -C)\\ & & = & G\mathbb{P}(X_\tau = G) + (-C)(1 - \mathbb{P}(X_\tau = G))\\ \implies & C & = & (G+C)\mathbb{P}(X_\tau = G)\\ \implies & \mathbb{P}(X_\tau = G) & = & \dfrac{C}{(G+C)}\\ \text{and} & \mathbb{P}(X_\tau = -C) & = & 1 - \dfrac{C}{(G+C)} = \dfrac{G}{G+C}\end{array}$$

We determine $\mathbb{E}[X_\tau]$ by applying the *Optional Stopping Theorem* to the process $\{Z_t\}_{t\geq 0}$ where $Z_t := X_t^2 - t$. We know $\{Z_t\}_{t\geq 0}$ is a *Martingale* from `Example 2.8 iv)`.

As $\{Z_t\}_{t\geq 0}$ is a *Martingale* it holds that

$$0 = \mathbb{E}[Z_0] = \mathbb{E}[Z_\tau] = \mathbb{E}[X_\tau^2 - \tau]$$

[32]The event the game in `Example 2.10`

Thus, we obtain that

$$\begin{aligned}\mathbb{E}[\tau] &= \mathbb{E}[X_\tau^2] \\ &= G^2\mathbb{P}(X_\tau = G) + C^2\mathbb{P}(X_\tau = -C) \\ &= G^2\frac{C}{C+G} + C^2\frac{G}{C+G} \\ &= CG\end{aligned}$$

ii). Consider the case $p \neq 1/2$. Using the *Optional Stopping Theorem* on the process $\{L_t\}_{t\geq 0}$ where $L_t := \left(\frac{1-p}{p}\right)^{X_t}$. We know $\{L_t\}_{t\geq 0}$ is a *Martingale* from `Example 2.8 v)`.

$$1 = \mathbb{E}[L_0] = \left(\frac{1-p}{p}\right)^G \mathbb{P}(X_\tau = G) + \left(\frac{1-p}{p}\right)^C \mathbb{P}(X_\tau = -C)$$

Note that $\mathbb{P}(X_\tau = G) + \mathbb{P}(X_\tau = -C) = 1$, so we can derive each probability as

$$\begin{aligned}1 &= \left(\frac{1-p}{p}\right)^G \mathbb{P}(X_\tau = G) + \left(\frac{1-p}{p}\right)^C (1 - \mathbb{P}(X_\tau = G)) \\ &= \left(\frac{1-p}{p}\right)^C + \left[\left(\frac{1-p}{p}\right)^G - \left(\frac{1-p}{p}\right)^C\right]\mathbb{P}(X_\tau = G) \\ \implies \quad \mathbb{P}(X_\tau = G) &= \frac{1 - \left(\frac{1-p}{p}\right)^C}{\left(\frac{1-p}{p}\right)^G - \left(\frac{1-p}{p}\right)^C}\end{aligned}$$

We determine $\mathbb{E}[X_\tau]$ by apply the *Optional Stopping Theorem* to the process $\{M_t\}_{t\geq 0}$ where $M_\tau := X_t - t(2p-1)$. We know $\{M_t\}_{t\geq 0}$ is a *Martingale* from `Example 2.8 v)`. Thus

$$0 = \mathbb{E}[M_\tau] = G\mathbb{P}(X_\tau = G) + (-C)\mathbb{P}(X_\tau = -C) - \mathbb{E}[\tau](2p-1)$$

Re-arranging this we can find an expression for $\mathbb{E}[\tau]$

$$\begin{aligned}0 &= G\mathbb{P}(X_\tau = G) + (-C)\mathbb{P}(X_\tau = -C) - \mathbb{E}[\tau](2p-1) \\ \implies \quad \mathbb{E}[\tau](2p-1) &= G\mathbb{P}(X_\tau = G) + (-C)\mathbb{P}(X_\tau = -C) \\ \implies \quad \mathbb{E}[\tau] &= \frac{G\mathbb{P}(X_\tau = G) + (-C)\mathbb{P}(X_\tau = -C)}{(2p-1)}\end{aligned}$$

$\square$

3 Multi-Period Models

3.1 Trading Strategies

3.2 Arbitrage in Multi-Period Model and Martingale Measures

3.3 Valuation of Contingent Claims

3.4 American Claims

3.5 The Cox-Ross Rubinstein Model

3.6 The Cox-Ross Rubinstein Model and the Black-Scholes Formula

4 Stochastic Processes in Continuous Time

4.1 The Brownian Motion

4.2 Stochastic Integration

4.3 Itô's Lemma

5 Financial Market Models in Continuous Time

5.1 The Financial Market Model in Continuous Time

5.2 Trading Strategies

5.3 Arbitrage in the Continuous Time Model

5.4 The Black-Scholes Model

5.5 Equivalent Martingale Measures in the Black-Scholes Model

5.6 Pricing in the Black-Scholes Model

5.7 Replicating Strategies and the Black-Scholes-Merton Equation

0 Reference

1 Notation

Notation 1.1 - *General Mathematical Notation*

Notation	Description
$\{x\}_+$	Only the positive part of x (i.e. $\max\{0, x\}$).