Financial Mathematics - Notes

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1　Financial Terminology & Single-Period Models

Proposition 1.1 - *Types of Financial Objects*
In this course we consider two types of financial objects

   i). *Underlying Traded Assets* (e.g. Oil, interest rates & exchange rates).

  ii). *Derivative Securities* (contracts based on *Underlying Traded Assets*)

Proposition 1.2 - *Derivative Securities*
Derivative Securities can act as insurance, by paying the holder when the value of the *Underlying Traded Asset* goes in the opposite direction.

e.g. "I will pay you £1mn for every dollar the price of oil is over £80 on 1st December 2025."

Our question is what such a contract is worth. To answer this we consider both *Arbitrage* and *Modelling*.

Definition 1.1 - *Modelling*
Modelling is the practice of forecasting the future value of an *Underlying Traded Asset*. Classical models for price processes involve *Brownian Motion*.

Definition 1.2 - *Arbitrage*
Arbitrage is the possibility of being able to make a profit on a transaction without being exposed to the risk of incurring a loss. Traders who try to lock in riskless profit are called *Arbitrageurs*.

Generally realised by buying and selling the same asset in different markets, with the asset having a different value in each market.

(2)

Example 1.1 - *Arbitrage*
Consider a stock that is traded on both the NYSE and the LSE. Suppose the sotck price is $189 on the NYSE and £100 on the LSE, and the exchange rate is $1.87=£1. The following *Arbitrage Opportunity* exists:

   i).　Buy 100 shares in London.

  ii).　Sell all these shares in NY.

 iii).　Exchange dollars to sterling.

This has a riskless profit of $200=£106.95 (assuming negligible transaction costs).

Proposition 1.3 - *"No Arbitrage Principle"*
The *"No Arbitrage Principle"* is the principle that *Arbitrage* opportunity do not exist (for long) in real life markets. If they did then the market activity caused by agents exploiting the *Arbitrage* opportunity[1] would raise the cost of buying and thus remove the *Arbitrage* opportunity.

Remark 1.1 - *Arbitrage & Valuing Contracts*
When valuing contracts we assume there is no arbitrage (The *"No Arbitrage Principle"*). This means we can decide a single price for a contract, as at any other price there would be

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[1]The forces of *Supply-and-Demand.*

arbitrage.

Definition 1.3 - *Short Selling*
Short Selling is the practice of borrowing an asset, selling it and then at some point in the future buying an equivalent asset to reimburse whoever lent you the original asset. If the value of the asset has fallen in this time then you make a profit.

Example 1.2 -
Consider a share which trades at £10 on 1st Jan. Suppose we know that on 1st July there is a 2/3 chance it will be worth £25 and 1/3 chance it will be worth £5. Suppose the following contract is offer on the 1st Jan:

  i). "If the share goes up, I will pay you £4. If the share price goes down, I pay you nothing."

What is this contract worth? (Assuming your bank account pays no interest).

You could argue that the contract should be worth the expected payout of the contract , $\frac{2}{3} \times$ £4 $+ \frac{1}{3} \times$ £0 $=$ £2.66.

However, consider creating a replicating portfolio which buys $\frac{1}{5}$ of a unit of the stock and borrows £1 from the bank. On 1st Jan this portfolio is worth $(1/5) \times$ £10 $-$ £1 $=$ £1.

  - If the stock goes up, the portfolio is worth $\frac{1}{5} \times$ £25 $-$ £1 $=$ £4.

  - If the stock goes down, the portfolio is worth $\frac{1}{5} \times$ £5 $-$ £1 $=$ £0.

These two outcomes show that this replicating portfolio has exactly the same payouts as the contract being offer. This means that whatever the portfolio costs, the contract must cost the same.

Suppose the portfolio costs £1 and the contract costs £1.50. Then you would sell the contract, buy the portfolio and make £0.50 profit independent of the price of the share on 1st July. (and visa-versa).

This valuation is independent of the probability of the asset's value increasing, but there is an interesting "phantom probability" $q = \frac{1}{4}$. The price of the contract is the expected payout of the contract, if the probability of the share price increasing was q, $\mathbb{E}[\text{payout}] = \frac{1}{4} \times$ £4 $+ \frac{3}{4} \times 0 =$ £1. Also, using this probability the expected value of the stock on July 1st is $\frac{1}{4} \times$ £25 $+ \frac{3}{4} \times$ £5 $=$ £10, the same as it was on Jan 1st. (This is how we find q).

This q is known as the *Equivalent Martingale Measure*

Definition 1.4 - *Equivalent Martingale Measure*
TODO

Remark 1.2 - *Value of Money*
These problems are more interesting when we consider that the value of money is not constant due to inflation & interest rates.

Remark 1.3 - *Risk-Free*
When referring to an activity being "Risk-Free" we mean that the loses & profits from it are known, not that there are no loses.

Definition 1.5 - *Bank Process, B_t*

Bank Process, B_t, is a measure of how much the value of money has changed over t time-periods.

Assume the existence of a risk-free bank account with known interest rate r (assumed to be constant over interval $[0, T]$). This means the initial deposit of one unit becomes the following after t time-steps

inuous Time Model $B_t = e^{rt}$

Multi-Period Model $B_t = (1 + r)^t$

We must consider the performance of our portfolio relative to the *Bank Process*, by discounting our profits by factor B_t.

N.B. - The *Bank Process* is also known as a *Bond* or a *Numeraire*.

Definition 1.6 - *Derivative Security*
A *Derivative Security* is a contract whose value at expiration date T is a function of the values of the assets within the time interval $[0, T]$. Often, the value is just a function of the value at time T.

Definition 1.7 - *Forward Contract*
A *Forward Contract* is an agreement to buy or sell an asset S at future *Delivery Date*[2] T for *Delivery Price* K. Both parties are obliged to fulfil the contract.

The agent who agrees to buy the underlying asset is said to have a *Long Position*, the other agent has a *Short Position*.

The *Forward Price* $f(t; T)$ is the delivery price which would make the contract have zero value at time t.

Remark 1.4 - *Usefulness of Forward Contracts*
Forward Contracts allow you to agree terms of a future purchase/sale in advance of actually transacting. This means you know the price you will receive/pay and thus can plan accordingly. (e.g. Farmers may agree to price per tonne of corn well before the harvest).

Definition 1.8 - *Option Contract*
An *Option* is a financial instrument giving one the right, but <u>not</u> the obligation, to buy or sell an asset S at (or before) a specified date T for an agreed *Strike Price*[3] K. There are two classes of *Option Contract*

- *Call Option* - The holder has the right to buy.

- *Put Option* - The holder has the right to sell

N.B. Only one party (the holder) decides whether to exercise the contract or not, the other (the writer) has to do what the former wishes.

Definition 1.9 - *European & American Options*
European vs *American Options* determine when the *Option Contract* can be exercised.

- *European Option* can only be executed on the expiry date.

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[2]AKA *Settlement Date*
[3]AKA *Exercise Price*

- *American Option* can be executed on any date up to and inc. the expiry date.

Example 1.3 - *Derivative*
Consider the following call option

"The holder has the right to buy 1,000 litres of petrol for £1,000 next Jan 1st"

If the market price S_T on 1st Jan is greater than £1,000 it is profitable to exercise the option. You would make profit $S_T - 1000$. However, if S_T was less than £1,000 then it is better to buy petrol at the market price.

The value on Jan 1st of the option will be $\{S_T - 1000\}_+$, a function of the random price S_T.

N.B. - a put option would the opposite analysis with profit $1000 - S_T$.

Definition 1.10 - *Dividend*
A *Dividend* is a one-off payment provided made to the holder of an underlying asset at a certain time. Not all underlying assets provide a *Dividend*.

Proposition 1.4 - *Fair Price of a Forward Contract with a Dividend*
Consider a forward contract on an asset with current price S_0 which provides a known dividend D at time point $t_0 \in (0, T)$ and has delivery date T.

Assume the existence of a risk-free bank account with constant interest rate r during the interval $[0, T]$. This means an initial deposit of one unit grows to $B_t = e^{rt}$ up to time point t.

If $D = 0$ (ie no dividend is paid) then the fair delivery price for the forward contract is $K = S_0 e^{rT}$. Else, if $D > 0$ (ie a dividend is paid) then the fair delivery price for the forward contract is $K = (S_0 - I)e^{rT}$ where $I = De^{-rt_0}$.

Proof 1.1 - *Proposition 1.4*
We use the "no-arbitrage principle" to prove that this is the fair price.

First assume that the price of the contract is $K > (S_0 - I)e^{rT}$ where $I = De^{-rt_0}$. Then an arbitrageur will adopt a short position on this contract by doing the following

i). Borrow £S_0 at an interest rate of r.

ii). Buy the underlying asset.

iii). Take a short position in the forward contract (i.e. agree to sell the asset for K at time T).

At time point T_0 we use the dividend to partially repay the loan. Finally, at time point T we sell the asset for K and repay the outstanding balance of the loan. The riskless profit is $K - (S_0 e^{rt_0} D)e^{(T-t_0)} = K - (S_0 - I)e^{rT}$, regardless of the price of the underlying asset at time point T.

Now assume the converse, the price of the contract is $K - (S_0 - I)e^{rT} > 0$ where $I = De^{-rt_0}$. Then an arbitrageur will adopt a long position on this contract by doing the following

i). Short sell the underlying asset (N.B. you are obliged to pay dividends to the lender).

ii). Invest the proceeds of S_0 at the risk-free interest rate of r.

iii). Take a long position in the forward contract (i.e. agree to buy the asset for K at time T).

At time t_0 we have to pay the dividend from our bank account. On the delivery date the arbitrageur buys the asset for K and makes a riskless profit of $(S_0 e^{rt_0} - D)e^{r(T-t_0)} - K = (S_0 - I)e^{rT} - K > 0$ where $I = De^{-rt_0}$.[4]

Theorem 1.1 - *Equivalent Contract Valuations over Time*
Consider two combinations of financial derivatives that both have the same value $V = W$ at time point T. Then their prices V_t and W_t at time point $t < T$ must also coincide.

Proof 1.2 - *Theorem 1.1*
We use the "no-arbitrage principle" and assume WLOG that $V_t > W_t$. Then, at time t, we would do the following

i). Sell or short the first combination.

ii). Buy the second combination.

iii). Invest the difference $V_t - W_t > 0$.

At time point T we would do the following

i). Buy the second combination for W.

ii). Sell the first combination for $V = W$.

The risk-free profit, assuming a risk-free interest rate of r, is $(V_t - W_t)e^{r(T-t)} > 0$

Proposition 1.5 - *Put-Call Parity*
We apply `Theorem 1.1` to *European Put & Call Options*.

Consider a *European Put Option* and a *European Call Option* for the same underlying asset, both with strike price K and expiry date T. Assume that S_T is the price of the underlying asset at time point T then the payoff of the *Call Option* is C and the *Put Option* is P at time T, where
$$\begin{aligned} C &= \{S_T - K\}_+ \\ P &= \{K - S_T\}_+ \end{aligned}$$

For the first combination choose the *Underlying Asset* and *European Put Option*. The value of this combination at time point T is $V = S_T + P = \max\{S_T, K\}$.

For the second combination choose the *European Call Option* and a bond which matures at time point T with a price of K. The value of the second combination is $W = K + C = \max\{S_T, K\}$

Theorem 1.2 - *Put-Call Parity - Formal*
Let S_t be the price of the asset at time point t, $Ke^{-r(T-t)}$ the discounted value of the bond and C_t, P_t be the prices of the *European Call* and *Put Options* at time t, respectively. Then

$$S_t + P_t - C_t = Ke^{-r(T-t)} \quad \forall t \in [0, T]$$

If, at some time t, this relationship does not hold then an *Arbitrage Opportunity* exists at this time.

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[4]D is subtracted as you have to pay the lender the dividend, but as you have already sold the asset you do not receive the dividend from the asset.

Theorem 1.3 - *Lower Bound for a European Call Option*
We can use *Put-Call Parity* to determine a lower bound for a *European Call Option*

$$
\begin{aligned}
S_t + P_t - C_t &= K e^{-r(T-t)} \\
\implies C_t &= S_t + P_t - K e^{-r(T-t)} \\
\implies C_t \geq \{S_t - K e^{-r(T-t)}\}_+ \text{ as } P_t \geq 0
\end{aligned}
$$

Theorem 1.4 - *American Call Options*
Let C_A be the price of an *American Call Option* and C_E be the price of a *European Call Option* for the same underlying asset, with the same strike price and expiry date.

Then, for a <u>non-dividend paying</u> stock we have that

$$C_A = C_E$$

This means that, for non-dividend paying stock, it is suboptimal to exercise an American call optional early.

Proof 1.3 - *Theorem 1.4*
First, note that exercising the American call early at time $t < T$ generates an income of $S_t - K$. However, from the inequality above, we know that selling the call options yields a cash-flow of $\{S_t - K e^{-r(T-t)}\}_+ \geq S_t - K e^{-r(T-t)}$. Since $e^{-r(T-t)} < 1$, exercising the call at any $t < T$ (i.e. early) is suboptimal.

1.1 Single-Period Model

Definition 1.11 - *Sample Space, Ω*
The *Sample Space* Ω is the set consisting of all elementary outcomes.

Definition 1.12 - *Random Variable*
A *Random Variable X* is a function from the *Sample Space* to real numbers

$$X : \Omega \to \mathbb{R}$$

Definition 1.13 - *Single-Period Model*
The *Single-Period Model* is a model for a financial market with the following components

- Initial date $t = 0$ and terminal date $t = 1$, with trading and consumption only allowed on these two dates.

- A finite *Sample Space* Ω with $|\Omega| = K < \infty$.

$$\Omega = \{\omega_1, \ldots, \omega_k\}$$

with each element corresponding to some state of the world.

- A *Probability Measure* $\mathbb{P}$ on Ω with $\mathbb{P}(\omega_i) > 0 \; \forall \; \omega_i \in \Omega$.

Definition 1.14 - *Bank Account Process B*
A *Bank Account Process* $B = \{B_t : t = 0, 1\}$ where $B_0 = 1$ and B_1 is a *Random Variable*.

The *Bank Account Process* is distinguished from other securities because its price $B_1(\omega)$ at time $t = 1$ is assumed to be strictly positive for all $\omega \in \Omega$. Usually, in fact, $B_1 \geq 1$ in which case B_1 should be though of as the value of the bank account at time $t = 1$, if 1 unit of currency was deposited at time $t = 0$. And, $r = B_t - 1 \geq 0$ should be thought of as the *Interest Rate*. In many applications r and B_1 are deterministic scalars.

Definition 1.15 - *Price Process*[5]
A *Price Process* $S = \{S(t) : t = 0, 1\}$ where $S(t) = (S_1(t), S_2(t), \ldots, S_N(t))$, $N < \infty$ and $S_i(t)$ is the price of the ith security at time t. In many applications these N risky securities are stocks.

The prices at time $t = 0$ are positive scalars that are known to the investors, whereas the prices at time $t = 1$ are non-negative random variables whose value only become known to investors at time $t = 1$.

Definition 1.16 - *Trading Strategy H*
A *Trading Strategy* $H = (H_0, \ldots, H_N)$ describes an investor's portfolio as carried from time $t = 0$ to time $t = 1$. Specifically, H_0 is the number of units of the currency invested in the *Savings Account* and H_i with $i \geq 1$ is a scalar of the number of units invested in the ith security.

Note that H_i can be positive or negative. Positive means you have bought/invested and negative means you are borrowing or short selling.

Example 1.4 - *Single-Period Model (Zero Interest)*
Consider a share which trades on 1st Jan for £10. Suppose we know with probability $p_1 = \frac{2}{3}$ it will be worth £25 on 1st July, and with probability $p_2 = \frac{1}{3}$ it will be worth £5 on 1st July. You also have access to a bank account which pays no interest ($r = 0$), which you can pay into or borrow from. Create a *Replicating Portfolio* as follows: buy $\frac{1}{5}$ unit of the stock, and borrow £1 from the bank.

We can rephrase this as a *Single-Period Model* for a single period ($T = 1$).

- *Bank Account Process* - $B_0 = B_1 = 1$. As no interest rate.

- *Sample Space* - $\Omega = \{\omega_1, \omega_2\}$ with ω_1 being the event the stock rises to £25 and ω_2 being the event the stock falls to £5.

- There are $N = 1$ stocks. So the *Price Process* is

$$
\begin{aligned}
S_1(0)(\omega_i) &= 10 \text{ for } i = 1, 2 \\
S_1(1)(\omega_i) &= \begin{cases} 25 & \text{if } i = 1 \\ 5 & \text{if } i = 2 \end{cases}
\end{aligned}
$$

- We can write the *Replicating Portfolio* as a vector (H_0, H_1).

Definition 1.17 - *Value Process, V*
A *Value Process* $V = \{V_t : t = 0, 1\}$ describes the total value of the portfolio at each point in time

$$V_t = H_0 B_t + \sum_{n=1}^{N} H_n S_n(t)$$

---

[5]AKA *Stock Process*

Definition 1.18 - *Gains Process, G*
The *Gains Process* G is a random variable that describes the total profit or loss generated by the portfolio between times 0 and 1.

$$G = H_0 r + \sum_{n=1}^{N} H_n \Delta S_n$$

where $\Delta S_n = S_n(1) - S_n(0)$ is the change in price of the nth asset.

Remark 1.5 - *Normalising Prices*
It is convenient to normalize the prices, so that the bank account becomes constant by defining discounted versions of the processes defined earlier. See `Definitions 1.19, 1.20, 1.21`.

Definition 1.19 - *Discounted Price Process, S^**
A *Discounted Price Process* S^* is the *Price Process* S normalised by the *Bank Process* B_t

$$S^* := \{S_t^* : t = 0, 1\} \text{ with } S_n^*(t) := \frac{S_n(t)}{B_t} \text{ for } n \in \{1, \ldots, N\}, \ t = \{0, 1\}$$

Definition 1.20 - *Discounted Value Process, V^**
A *Discounted Value Process* V_t^* is the *Value Process* V_t normalised by the *Bank Process* B_t

$$V_t^* := \frac{V_t}{B_t} = H_0 + \sum_{n=1}^{N} H_n S_n^*(t) \text{ for } t = 0, 1$$

Definition 1.21 - *Discounted Gains Process, G^**
A *Discounted Gains Process* G^* is the *Gains Process* G normalised by the *Bank Process* B_t

$$G^* := \frac{G}{B_t} = \sum_{n=1}^{N} H_n \Delta S_n^* \text{ with } \Delta S_n^* := S_n^*(1) - S_n^*(0)$$

Example 1.5 - *Single-Period Model (Non-Zero Interest)*
Consider a share which trades on 1st Jan for £10. Suppose we know with probability $p_1 = \frac{2}{3}$ it will be worth £25 on 1st July, and with probability $p_2 = \frac{1}{3}$ it will be worth £5 on 1st July. You also have access to a bank account which pays interest $r \in \mathbb{R}$, which you can pay into or borrow from.

We can rephrase this as a *Single-Period Model* for a single period $(T = 1)$.

- Bank Account Process - $B_0 = 1, \ B_1 = 1 + r$.

- Probability Space - $\Omega = \{\omega_1, \omega_2\}$ where ω_1 indicates the stock going up and ω_2 indicates the stock going down.

- There is $N = 1$ units of the stock, so we have the following price process

$$
\begin{aligned}
S_1(0)(\omega_i) &= 10 \text{ for } i \in \{1, 2\} \\
S_1(1)(\omega_1) &= 25 \\
S_1(1)(\omega_2) &= 5
\end{aligned}
$$

And discounted price process

$$
\begin{aligned}
S_1^*(0)(\omega_i) &= 10 \text{ for } i \in \{1, 2\} \\
S_1^*(1)(\omega_1) &= \tfrac{25}{1+r} \\
S_1^*(1)(\omega_2) &= \tfrac{5}{1+r}
\end{aligned}
$$

- For an arbitrary strategy H, our portfolio has initial value $V_0 = V_0^* = H_0 + 10H_1$. Thus, we can determine the gains and future values of this portfolio

$$
\begin{aligned}
V_1(\omega_1) &= (1+r)H_0 + 25H_1 \\
V_1^*(\omega_1) &= H_0 + \tfrac{25}{1+r}H_1 \\
G(\omega_1) &= rH_0 + 15H_1 \\
G^*(\omega_1) &= \left(\tfrac{25}{1+r} - 10\right)H_1 \\
V_1(\omega_2) &= (1+r)H_0 + 5H_1 \\
V_1^*(\omega_2) &= H_0 + \tfrac{5}{1+r}H_1 \\
G(\omega_2) &= rH_0 - 5H_1 \\
G^*(\omega_2) &= \left(\tfrac{5}{1+r} - 10\right)H_1
\end{aligned}
$$

Definition 1.22 - *Arbitrage Opportunity*
An *Arbitrage Opportunity* is a *Trading Strategy* H with the following three properties

  i). $V_0 = 0$.

  ii). $V_1(\omega) \geq 0 \ \forall \ \omega \in \Omega$.

  iii). $\mathbb{P}(V1(\omega) \geq 0) > 0 \ \forall \ \omega \in \Omega$.[6]

Proposition 1.6 - *Gains Process & Arbitrage Opportunity*
There exists an *Arbitrage Opportunity* iff there is some trading strategy H st $G^* \geq 0$ and $\mathbb{E}[G^*] > 0$.

This means that *Gains* is always non-negative (so not losses), and as its expected value is strictly positive there must be at least one positive value of G^* which has a non-zero probability.

Proof 1.4 - *Proposition 1.6*
As this is an <u>iff</u> statement, I prove the statement in both directions. First consider the forwards direction. Let H be an *Arbitrage Opportunity*. Since $G^* = V_1^* - V_0^*$ and $B_t > 0 \ \forall \ t, \omega$, by the definition of an *Arbitrage Opportunity*, we find that $G^* \geq 0$ and thus $\mathbb{E}[G^*] = \mathbb{E}[V_1^*] > 0$.

Now consider the other direction, suppose H satisfies $G^* \geq 0$ and $\mathbb{E}[G^*] > 0$. Define $\hat{H} := (\hat{H}_0, H_1 \dots, H_N)$ where $\hat{H}_0 := -\sum_{n=1}^{N} H_n S_n^*(0)$[7]. Under $\hat{H}$ one has $V_0^* = 0$ and $V_1^* = V_0^* + G^* = G^*$.

Hence $V_1^* \geq 0$ and $\mathbb{E}[V_1^*] = \mathbb{E}[G^*] > 0$, meaning $\hat{H}$ is an arbitrage opportunity.

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[6]Equivalently, $\mathbb{E}[V_1] > 0$.

[7]Any money used to buy a stock is borrowed from the bank, and any money made from short selling is deposited into the bank. This means the total value of the portfolio at $t = 0$ is 0 (requirement of an *Arbitrage Opportunity*).

1.2   Risk-Neutral Probability Measures

Definition 1.23 - *Risk-Neutral Probability Measure*
A probability measure $\mathbb{Q}$ on Ω is said to be a *Risk-Neutral Probability Measure* if both the following hold

   i). $\mathbb{Q}(\{\omega\}) > 0 \ \forall \ \omega \in \Omega$.

   ii). $\mathbb{E}_{\mathbb{Q}}(S_n^*(1)) = S_n^*(0)$ for $n \in \{1, \ldots, N\}$.

Theorem 1.5 - *No-Arbitrage Theorem*
There are no *Arbitrage Opportunities* <u>iff</u> there exists a *Risk-Neutral Probability Measure* $\mathbb{Q}$.

Theorem 1.6 - *Separating Hyperplane Theorem*[8]
Let $\mathbb{W}$ be a linear subspace of $\mathbb{R}^K$ and $\mathbb{K}$ be a compact convex subset in $\mathbb{R}^K$ which is disjoint from $\mathbb{W}$. Then we can separate $\mathbb{W}$ and $\mathbb{K}$ strictly by a hyperplane containing $\mathbb{W}$ (ie $\exists \ v \in \mathbb{R}^K$ which is *Orthogonal* to $\mathbb{W}$[9]) such that

$$u^T v > 0 \ \forall \ u \in \mathbb{K}$$

N.B. - Proof of this is beyond the scope of this course.

Proof 1.5 - *No-Arbitrage Theorem*
Consider the three following sets

   i). $\mathbb{W} = \left\{ X \in \mathbb{R}^K : X = G^* \text{ for some } Trading\ Strategy\,H \right\}$.

   This can be considered the set of random variables which each represent a possible $t = 1$ discounted wealth when the initial value of the investment is zero. Here, $\mathbb{W}$ is a linear subspace of $\mathbb{R}^{K}$[10].

   ii). $\mathbb{A} = \left\{ X \in \mathbb{R}^K : X \geq 0, X \neq 0 \right\}$. ($\mathbb{A}$ is not compact, so can not be used for $\mathbb{K}$ in *Separating Hyperplane Theorem*).

   Note there exists an arbitrage opportunity <u>iff</u> $\mathbb{W} \cap \mathbb{A} \neq \emptyset$.

   iii). $\mathbb{A}^+ = \left\{ X \in \mathbb{R}^N : X \geq 0, X \neq 0, \sum_{i=1}^K X_i = 1 \right\}$.

   $\mathbb{A}^+$ is a convex and compact subset of $\mathbb{R}^K$.

Assume that there is no *Arbitrage Opportunity*, then $\mathbb{W} \cap \mathbb{A} \neq \emptyset$ (They are disjoint). By the *Separating Hyperplane Theorem* $\exists \ Y \in \mathbb{R}^K$ which is *Orthogonal* to $\mathbb{W}$ ($X^T Y = 0 \ \forall \ X \in \mathbb{W}$) st

$$X^T Y > 0 \ \forall \ X \in \mathbb{A}^+$$

For each $k \in \{1, \ldots, K\}$ the kth unit vector e_k is an element of $\mathbb{A}^+$. Therefore, $\forall \ k \in \{1, \ldots, K\}$ we have that

$$Y_K := e_K^T Y > 0$$

which means all entries of Y are strictly positive.

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[8]This is a consequence of the *Hahn-Banach Theorem* from functional analysis.
[9]$u^T v = 0 \ \forall \ u \in \mathbb{W}$
[10]Proved by showing it is complete under: addition, and scalar multiplication.

Define a probability measure $\mathbb{Q}$ by setting

$$\mathbb{Q}(\{\omega_k\}) = \frac{Y(\omega_k)}{Y(\omega_1) + \cdots + Y(\omega_k)}$$

Furthermore, $\Delta S_n^* \in \mathbb{W}$ for all n because $\Delta S_n^* := S_n^*(1) - S_n^*(0)$ is the discounted wealth for the portfolio $H = e_n$ which consists of one unit of the nth asset only. Since Y is orthogonal to $\mathbb{W}$ we can conclude that

$$\mathbb{E}_{\mathbb{Q}}[\Delta S_n^*] = \sum_{k=1}^{K} \Delta S_n^*(\omega_k) \mathbb{Q}(\{\omega_k\}) = 0 \ \forall \ n$$

In other words

$$\mathbb{E}_{\mathbb{Q}}[S_n^*(1)] = S_n^*(0) \ \forall n$$

Thus $\mathbb{Q}$ is a *Risk-Neutral Probability Measure.*

For the converse, assume that $\mathbb{Q}$ is a *Risk-Neutral Probability Measure.* Then for an arbitrary *Trading Strategy H* we have

$$\mathbb{E}_{\mathbb{Q}}[G^*] = \mathbb{E}_{\mathbb{Q}}\left[\sum_{n=1}^{N} H_n \Delta S_n^*\right] = \sum_{n=1}^{N} H_n \mathbb{E}_{\mathbb{Q}}[\Delta S_n^*] = 0$$

and, in particular

$$\sum_{k=1}^{K} G^*(\omega_k) \mathbb{Q}(\{\omega_k\}) = 0$$

which shows that either $G^*(\omega_k < 0)$ for some k or $G^* = 0$, but then $\mathbb{E}_{\mathbb{P}}[G^*] = 0$. Hence, by `Proposition 1.6` , there cannot be any arbitrage opportunities. $\qquad \square$

1.3   Valuation of Contingent Claims

Definition 1.24 - *Contingent Claim*
A *Contingent Claim* in the *Single-Period Model* is a random variable X which represents the payoff at time $t = 1$. A contingent claim X is said to be "attainable" (or "marketable") if there exists a trading strategy H st $V_1 = X$. In this case one says that H generates X and H is called the *Replicating Portfolio.*

Proposition 1.7 - *Fair Price of a Contingent Claim?*
Consider a *Contingent Claim* which is "attainable", and suppose that H is its *Replicating Portfolio.* The value V_0 of H at time $t = 0$ is the fair price of the *Contingent Claim.*

Proof 1.6 - *Proposition 1.7*
Suppose the fair price of the *Contingent Claim* p is not equal to the value V_0 of H at time $t = 0$ (ie $p \neq V_0$). Then we have two cases

$p > V_0$ - An *Arbitrage Opportunity* arises because an arbitrageur would sell the *Contingent Claim* for p at time 0, follow the trading strategy H at a time $t = 0$ cost of V_0 and pocket the difference $p - V_0$.

At time $t = 1$ the value V_1 of the portfolio matches the obligation X of the *Contingent Claim*. We make profit $(p - V_0)e^{rT}$.

$p < V_0$ - An arbitrageur would follow the trading strategy H and purchase the *Contingent Claim*, again making a riskless profit.

Theorem 1.7 - *Risk-Neutral Valuation Principle*
If the *Single Period Model* is free of *Arbitrage Opportunities*, then the time $t = 0$ value of an *Attainable Contingent Claim* X is $\mathbb{E}_{\mathbb{Q}}[X/B_1]$ where $\mathbb{Q}$ is any *Risk-Neutral Probability Measure*[11].

Proof 1.7 - *Risk-Neutral Valuation Principle*
Suppose there exists a second trading strategy $\hat{H}$ st $\hat{V}_1 = X$, but $\hat{V}_0 \neq V_0$.

Let $\mathbb{Q}$ be a *Risk-Neutral Probability Measure*, then for any trading strategy H we have seen that $\mathbb{E}_{\mathbb{Q}}[G^*] = 0$. Consequently

$$V_0 = V_0^* = \mathbb{E}_{\mathbb{Q}}[V_0^*] = \mathbb{E}_{\mathbb{Q}}[V_1^* - G^*] = \mathbb{E}_{\mathbb{Q}}[V_1^*] - \mathbb{E}_{\mathbb{Q}}[G^*] = \mathbb{E}_{\mathbb{Q}}[V_1^*] = \mathbb{E}_{\mathbb{Q}}\left[\frac{V_1}{B_1}\right]$$

In particular, any replicating strategy with $V_1 = X$ has time $t = 0$ value

$$V_0 = \mathbb{E}_{\mathbb{Q}}\left[\frac{X}{B_1}\right]$$

As this holds for all $\mathbb{Q}$, this calculation does not depend on the choice of $\mathbb{Q}$ and therefore $\mathbb{E}_{\mathbb{Q}}[V_1^*]$ is constant even if there are two or more *Risk-Neutral Probability Measures*.

This means all *Replicating Porfolios* for the same *Attainable Contingent Claim* have the same values in time $t = 0$ and $t = 1$.

Example 1.6 - *Valuation of Contingent Claims - I*
Consider a stock which is trading for £10 at time $t = 0$, and we know that at time $t = 0$ it will be worth either £25 or £5.

In order to find a *Risk-Neutral Probability Measure* $\mathbb{Q}$ we need to find strictly positive numbers $\mathbb{Q}(\{\omega_1\})$ and $\mathbb{Q}(\{\omega_2\})$ st $\mathbb{E}_{\mathbb{Q}}[S_n^*(1)] = S_n^*(0)$ is satisfied. That is

$$
\begin{aligned}
\mathbb{E}_{\mathbb{Q}}[S_0^*(1)] &= S_0^*(0) \\
\implies S_1^*(\omega_1)\mathbb{Q}(\{\omega_1\}) + S_1^*(\omega_2)\mathbb{Q}(\{\omega_2\}) &= S_0 \\
\implies \tfrac{25}{1+r}\mathbb{Q}(\{\omega_1\}) + \tfrac{5}{1+r}\mathbb{Q}(\{\omega_2\}) &= 10
\end{aligned}
$$

As $\mathbb{Q}$ is a *Probability Measure*, we must have that

$$1 = \mathbb{Q}(\{\omega_1\}) + \mathbb{Q}(\{\omega_2\})$$

We deduce the values $q_1 = \mathbb{Q}(\{\omega_1\}), q_2 = \mathbb{Q}(\{\omega_2\})$ as follows

$$
\begin{aligned}
\tfrac{25}{1+r}q_1 + \tfrac{5}{1+r}q_2 &= 10 \\
\& \qquad q_1 + q_2 &= 1 \\
\implies q_1 &= 1 - q_2 \\
\implies \tfrac{5}{1+r}\left(5(1 - q_2) + q_2\right) &= 1- \\
\implies 5 - 4q &= 2(1 + r) \\
\implies q_2 &= \tfrac{3-2r}{4} \\
\implies q_1 &= 1 - \tfrac{3-2r}{4} = \tfrac{1+2r}{4}
\end{aligned}
$$

The values $\mathbb{Q}(\{\omega_1\}) = \tfrac{1+2r}{4}, \mathbb{Q}(\{\omega_2\}) = \tfrac{3-2r}{4}$ satisfy both these equations, so this is a *Risk-Neutral Probability Measure* and by `Theorem 1.5` (The "No-Arbitrage Principle") there cannot be any *Arbitrage Opportunities*.

---

[11]So value is independent of the choice of $\mathbb{Q}$

In particular for $r = 0$ (ie No interest), we recover $\mathbb{Q}(\{\omega_1\}) = \frac{1}{4}$ which is the same as a previous example.

Example 1.7 - *Valuation of Contingent Claims - II*
Continuing from `Example 1.6`.

Suppose X is a *Contingent Claim* with $X(\omega_1) = 4$ and $X(\omega_2) = 0$. If X is *Attainable*, then the value of X at time $t = 0$ is

$$\mathbb{E}_{\mathbb{Q}}[X/B_1] = \frac{1 + 2r}{4} \cdot \frac{4}{1 + r} + 0 = \frac{1 + 2r}{1 + r}$$

Check whether X can be generated by solving for $H = (H_0, H_1)^{[12][13]}$

$$\begin{aligned} 4 &= H_0(1 + r) + H_1 \cdot 25 \\ 0 &= H_0(1 + r) + H_1 \cdot 5 \end{aligned}$$

Solving gives
$$\begin{aligned} (4 - 0) &= \big((1 + r) - (1 + r)\big)H_0 + (25 - 5)H_1 \\ \implies \quad 4 &= 20H_1 \\ \implies \quad H_1 &= 1/5 \\ \implies \quad 0 &= H_0(1 + r) + 1 \\ \implies \quad H_0 &= -1/(1 + r) \end{aligned}$$

This means that X is indeed *Attainable*.

We check the value of the *Replicating Portfolio H* at time $t = 0$

$$V_0 = H_0 + S_1 \cdot H_1 = -\frac{1}{1 + r} + 10(1/5) = \frac{1 + 2r}{1 + r}$$

1.4   Complete & Incomplete Markets

Definition 1.25 - *Complete & Incomplete Models*
A model is said to be *Complete* if every *Contingent Claim X* can be generated by some trading strategy. Otherwise, the model is said to be *In-Complete*

Remark 1.6 - *Checking if a Model is Complete*
There are simple ways to check whether a model is *Complete*. One way is to define $K \times (N + 1)$-matrix A as
$$A = \begin{pmatrix} B_1(\omega_1) & S_1(1)(\omega_1) & \dots & S_N(1)(\omega_1) \\ B_1(\omega_2) & S_1(1)(\omega_2) & \dots & S_N(1)(\omega_2) \\ \vdots & \vdots & \ddots & \vdots \\ B_1(\omega_K) & S_1(1)(\omega_K) & \dots & S_N(1)(\omega_K) \end{pmatrix}$$

If a particular *Contingent Claim X* is attainable, then $AH = X$ will have a solution for H.

A model is *Complete* <u>iff</u> $\forall\, X \in \mathbb{R}^K\ \exists H$ st $AH = X$.

NB - If the bank process B is deterministic then the first column will be identical for all rows.

Theorem 1.8 - *Uniqueness of Risk-Neutral Probability Measure*

---

[12] H_0 is amount invested in our bank account and H_1 is amount invested in the stock.
[13] We fine (H_0, H_1) by solving $AH = X$ for defined A, X.

Suppose there are no *Arbitrage Opportunities*. Then a model is *Complete* iff there exists a unique *Risk-Neutral Probability Measure*.

Proof 1.8 - *Theorem 1.8*
We denote by $\mathbb{M}$ the set of all *Risk-Neutral Probability Measures*. Since there are no *Arbitrage Opportunities* we know that $\mathbb{M} \neq \emptyset$. We consider the statement in both directions

$\implies$ For the sake of contradiction, assume that the model is *Complete*, but $\mathbb{M}$ contains two distinct *Risk-Neutral Probability Measures* $\mathbb{Q}, \hat{\mathbb{Q}}$. In this case there must exists some state ω_k with $\mathbb{Q}(\omega_k) \neq \hat{\mathbb{Q}}(\omega_k)$ so take the contingent claim X defined by

$$X(\omega) = \begin{cases} B_1(\omega_k) & \text{if } \omega = \omega_k \\ 0 & \text{otherwise} \end{cases} = B_1 \mathbb{1}\{\omega = \omega_k\}$$

Then

$$\mathbb{E}_{\mathbb{Q}}(X/B_1) = \mathbb{E}_{\mathbb{Q}}(\mathbb{1}\{\omega = \omega_k\}) = \mathbb{Q}(\{\omega_k\}) \neq \hat{\mathbb{Q}}(\omega_k) = \mathbb{E}_{\hat{\mathbb{Q}}}(X/B_1)$$

But, this contradicts the calculation in `Proof 1.7` where we saw that if *Contingent Claim* X is *Attainable* then $\mathbb{E}_{\mathbb{Q}}[X/B_1]$ has the same value $\forall \mathbb{Q} \in \mathbb{M}$.

$\impliedby$ For the sake of contradiction, assume that there is only one *Risk-Neutral Probabiltiy Measure* $\hat{\mathbb{Q}}$, but there exists a *Contingent Claim* X which is not *Attainable*.

Then there is no solution to the system $AH = X$. By *Separating Hyperplane Theorem* it follows that there must exists a row vector $\pi \in \mathbb{R}^K$ st

$$\pi^T A = 0^{[14]}, \pi^T X > 0$$

Let $\lambda > 0$ be small enough st

$$\mathbb{Q}(\{\omega_j\}) = \hat{\mathbb{Q}}(\{\omega_j\}) + \lambda \pi_j \cdot B_1(\omega_j) > 0 \ \forall \ j \in \{1, \dots, K\}$$

Since B_1 is the first column of A and $\pi^T A = 0$, the quantity $\mathbb{Q}$ is defined above is actually a *Probabiltiy Measure*.[15] Moreover, for any *Discounted Price Process* $S^* = (S_1^*, \dots, S_N^*)$ and for every $n \in \{1, \dots, N\}$ we have

$$
\begin{aligned}
\mathbb{E}_{\mathbb{Q}}[S_n^*(1)] &= \sum_{j=1}^K \mathbb{Q}(\{\omega_j\}) S_n(1, \omega_j)/B_1(\omega_j) \\
&= \sum_{j=1}^K \hat{\mathbb{Q}}(\{\omega_j\}) S_n(1, \omega_j)/B_1(\omega_j) + \lambda \underbrace{\sum_{j=1}^K \pi_j S_n(1, \omega_j)}_{=0} \\
&= \sum_{j=1}^K \hat{\mathbb{Q}}(\{\omega_j\}) S_n^*(1, \omega_j) \\
&= \mathbb{E}_{\hat{\mathbb{Q}}}(S_n^*(1)) \\
&= S_n^*(0)^{[16]}
\end{aligned}
$$

Thus $\mathbb{Q} \in \mathbb{M}$ which is a contradiction to the uniqueness of $\hat{\mathbb{Q}}$.

---

[14] π is *Orthogonal* to A.
[15] i.e. $\displaystyle\sum_j \mathbb{Q}(\{\omega_j\}) = \underbrace{\sum_j \hat{\mathbb{Q}}(\{\omega_j\})}_{=1} + \underbrace{\sum_j \lambda \pi B_1}_{=0} = 1$
[16] As $\hat{\mathbb{Q}}$ is a *Risk-Neutral Probability Measure*.

$\square$

Example 1.8 - *Complete Markets*
Consider the situation in `Example 1.6`.

Here is another way to check that X is *Attainable* is to check if the model is *Complete*.

Consider the matrix A

$$A = \begin{pmatrix} 1+r & 25 \\ 1+r & 5 \end{pmatrix}$$

Since A has two linearly-independent columns, AH can take all values in $\mathbb{R}^2$.

Hence the model is *Complete*.

2 Stochastic Processes in Discrete Time

2.1 Multi-Period Models

Remark 2.1 - *Multi-Period vs Single-Period Model*
The *Multi-Period Model* for securities markets are much more realistic than *Single-Period Models*.

Definition 2.1 - *Multi-Period Model*
A *Multi-Period Model* is a model for a financial market with the following elements

- $T + 1$ trading dates $\{0, 1, \ldots, T\}$.

- A finite sample space Ω where each element corresponds to a certain state of the world.

$$\Omega = \{\omega_1, \ldots, \omega_K\} \text{ with } K < \infty$$

- A probability measure $\mathbb{P}$ on Ω with $\mathbb{P}(\omega) > 0 \ \forall \ \omega \in \Omega$.

Definition 2.2 - *Filtration $\mathcal{F}$*
A *Filtration* $\mathcal{F} := \{\mathcal{F}_t : t = 0, 1, \ldots, T\}^{[17]}$ is a sub-model describing how the formation about the security prices is revealed to investors.

Definition 2.3 - *Bank Process B*
A *Bank Process* $B := \{B_t : t = 0, 1, \ldots, T\}$ where B is a stochastic process with $B_0 = 1$ and $B_t(\omega) > 0$.

The *Bank Process* at time t B_t should be thought of as the time t value of a savings account when 1 unit of currency was deposited at time $t = 0$.

B is usually a non-decreasing process and the (possibly random) quantity $r_t := \frac{B_t - B_{t+1}}{B_{t+1}} \geq 0$, $t = 1, \ldots, T$ should be though of as the interest rate for time-interval $(t-1, t)$.

Definition 2.4 - *Price Process S*
A *Price Process* $S = \{S(t) : t = 0, 1, \ldots, T\}$ where each element is multi-dimensional[18] $S(t) = (S_1(t), \ldots, S_N(t))$ and $S_i(t)$ is a non-negative stochastic process for each $i \in [1, N]$.

---

[17]Each $\mathcal{F}_t$ is a σ-*Algebra*

[18]One dimension for each security in the market.

The $S_i(t)$ should be though of as the price of the ith security in time-period t.

2.2　Information Structures

Definition 2.5 - *Partition $\mathcal{P}$*
A set $A = \{a_1, \ldots, a_n\}$ forms a partition $\mathcal{P}$ if each element of A is disjoint and the union of all the elements of A is the whole sample space Ω.

$$a_i \cap a_j = \emptyset \ \forall \ a_i, a_j \in A \quad \text{and} \quad \bigcup_{i=1}^n a_i = \Omega$$

Example 2.1 - *Information Structures*
Consider a multi-period model with $T = 3$ time-periods and whether the stock either increases or decreases in value is the only event recorded in each time-period. The sample space has $K = 2^3 = 8$ different outcomes

ω_1	ω_2	ω_3	ω_4	ω_5	ω_6	ω_7	ω_8
UUU	UUD	UDU	UDD	DUU	DUD	DDU	DDD

Initially, all of these outcomes are possible (ie $A_0 = \Omega$). After the first move occurs there are only $2^2 = 4$ possible outcomes, after the second there are only $2^1 = 2$ possible outcomes and after the third there is only 1 possible outcome.

The set of possible subsets A_t at time t form a *Partition $\mathcal{P}_t$*.

Remark 2.2 - *Flow of Information*
At time $t = 0$ every state $\omega \in \Omega$ is a possible outcome (although usually not with the same probability). And, at time $t = T$, the true state of the world is known. Each new piece of information removes certain states.

One can view the evolution of information as a random sequence $\{A_t\}$ of subsets of Ω where

$$A_0 = \Omega, \ A_T = \{\omega\} \text{ and } A_0 \supseteq A_1 \supseteq \cdots \supseteq A_T$$

There exists K possible *Information Sequences* $\{A_t\}_{t \in [1,T]}$ of subsets, corresponding to the K elements of Ω. At time $t = 0$ the investors are aware of all these sequences, but they do not know which one is the true sequence.

Definition 2.6 - *Information Sequence*
The *Information Sequence* is fully describe by a sequence $\mathcal{P}_0, \ldots, \mathcal{P}_T$ of partitions of Ω with

- $\mathcal{P}_0 = \{\Omega\}$.

- $\mathcal{P}_T = \big\{ \{\omega_1\}, \{\omega_2\}, \{\omega_3\}, \{\omega_4\}, \{\omega_5\}, \{\omega_6\}, \{\omega_7\}, \{\omega_8\} \big\}$

- And, $\forall \ t < T$ each $A \in \mathcal{P}_t$ is equal to the union of some elements of $\mathcal{P}_{t+1}$.
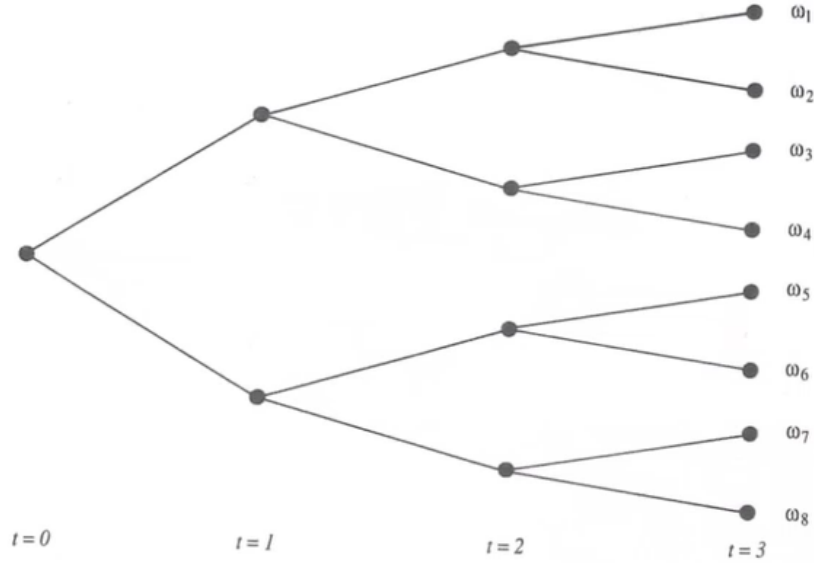
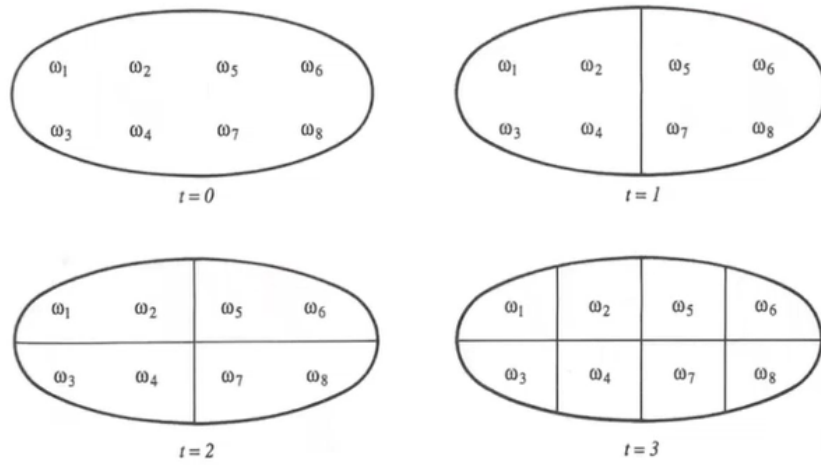Figure 1: Tree diagram for the structure of information from `Example 2.1`.

Figure 2: A sequence of pictures demonstrating the structure of information `Example 2.1`.

Example 2.2 -
Consider an arbitrary sequence $\{\hat{A}_t\}_{t \in [1,T]}$ and some time-point $s < T$. For each $\omega \in \hat{A}_s$ the sequence $\{A_t\}_{t \in [1,T]}$ with $A_T = \{\omega\}$ will coincide with $\{\hat{A}_t\}$ at least until time $t = s$.

The collection of subsets $\{A_{s+1}\}$ that can follow $\hat{A}_s$ forms a partition of $\hat{A}_s$, that is, a collection of disjoint subsets whose union equals $\hat{A}_s$.

In particular, taking $s = 0$, we see that the collection $\{A_1\}$ of all possible time-period $t = 1$ subsets forms a partition of Ω. This partition is denoted $\mathcal{P}_1$.

Moreover, the collection $\{A_2\}$ of all possible time-period $t = 2$ subsets also forms a partition of Ω, denoted $\mathcal{P}_2$. Consider `Example 2.1` where $K = 8$ and $T = 3$, the partitions are

$$
\begin{aligned}
\mathcal{P}_0 &= \{\{\omega_1, \omega_2, \omega_3, \omega_4, \omega_5, \omega_6, \omega_7, \omega_8\}\} \\
\mathcal{P}_1 &= \{\{\omega_1, \omega_2, \omega_3, \omega_4\}, \{\omega_5, \omega_6, \omega_7, \omega_8\}\} \\
\mathcal{P}_2 &= \{\{\omega_1, \omega_2\}, \{\omega_3, \omega_4\}, \{\omega_5, \omega_6\}, \{\omega_7, \omega_8\}\} \\
\mathcal{P}_3 &= \{\{\omega_1\}, \{\omega_2\}, \{\omega_3\}, \{\omega_4\}, \{\omega_5\}, \{\omega_6\}, \{\omega_7\}, \{\omega_8\}\}
\end{aligned}
$$

Remark 2.3 - *Visualising Information Structure*
There are two popular ways to visualise information structure

   i). A *Tree Diagram* where each node corresponds to an element A_t of the time t partition and where there is edge arc going from this node to each node corresponding to some $A_{t+1} \subseteq A_t$. (See `Figure 1`).

   ii). A *Sequence of Pictures* of the same space. (See `Figure 2`).

Definition 2.7 - σ*-Algebra* $\mathcal{F}_t$
A collection $\mathcal{F}$ of subsets of Ω is called a σ*-Algebra* on sample space Ω if

   i). $\Omega \in \mathcal{F}$.

   ii). $\forall\ F \in \mathcal{F},\ F^c \in \mathcal{F}$.

   iii). $\forall\ F, G \in \mathcal{F},\ (F \cup G) \in \mathcal{F}$.

Proposition 2.1 - *Generated* σ*-Algebra*
For any partition $\mathcal{P}$ of Ω we can generate σ*-Algebra* $\mathcal{F}$ by letting $\mathcal{F}$ be the collection of all unions of elements of $\mathcal{P}$ together with the complements of all such unions.

Hence, the sub-models of the information structure can be organised as a sequence $\{\mathcal{F}_t\}_{t\in[1,T]}$ of σ*-Algebras.*

Definition 2.8 - *Filtration* $\mathcal{F}$
A *Filtration* is a family of σ*-Algebras* $\mathcal{F} := \{\mathcal{F}_t : t = 0, 1, \ldots, T\}$ where

   i). $\mathcal{F}_0 = \{\emptyset, \Omega\}$.

   ii). $\mathcal{F}_T$ consists of all subsets of Ω.

   iii). $\mathcal{F}_n \subset \mathcal{F}_{n+1}$ for all $n < T$, by which we mean that each subset of $\mathcal{F}_n$ must be an element of $\mathcal{F}_{n+1}$.

Example 2.3 - σ*-Algebra*
Consider the context of `Example 2.1`. The corresponding filtration is given by

$$
\begin{aligned}
\mathcal{F}_0 &= \big\{\emptyset, \Omega\big\} \\
\mathcal{F}_1 &= \big\{\emptyset, \Omega, \{\omega_1, \omega_2, \omega_3, \omega_4\}, \{\omega_5, \omega_6, \omega_7, \omega_8\}\big\} \\
\mathcal{F}_2 &= \big\{\emptyset, \Omega, \{\omega_1, \omega_2\}, \{\omega_3, \omega_4\}, \{\omega_5, \omega_6\}, \{\omega_7, \omega_8\}, \{\omega_1, \omega_2, \omega_3, \omega_4\}, \\
&\quad \{\omega_5, \omega_6, \omega_7, \omega_8\}, \{\omega_1, \omega_2, \omega_5, \omega_6\}, \{\omega_1, \omega_2, \omega_7, \omega_8\}, \{\omega_3, \omega_4, \omega_5, \omega_6\}, \\
&\quad \{\omega_3, \omega_4, \omega_7, \omega_8\}, \{\omega_1, \omega_2, \omega_3, \omega_4, \omega_5, \omega_6\}, \{\omega_1, \omega_2, \omega_3, \omega_4, \omega_7, \omega_8\}, \\
&\quad \{\omega_1, \omega_2, \omega_5, \omega_6, \omega_7, \omega_8\}, \{\omega_3, \omega_4, \omega_5, \omega_6, \omega_7, \omega_8\}\big\}
\end{aligned}
$$

and $\mathcal{F}_3$ contains all the subsets of Ω (ie $\mathcal{F}_3 = 2^\Omega$).

2.3   Stochastic Processes in Discrete Time

Definition 2.9 - *Stochastic Process* S
A *Stochastic Process* S is a real-valued function $S(t)(\omega)$ of two variables, t and ω.

For each fixed $\omega \in \Omega$ the function mapping $t \to S(t)(\omega)$ is called the *Sample Path.*

For each fixed $t \in [0, T]$ the function mapping $\omega \to S(t)(\omega)$ is a *Random Variable.*

NB - For simplicity, we assume $S(0)$ is constant.

Definition 2.10 - *Measurable Function*
A function $\omega \to W(\omega)$ is said to be *Measurable* wrt the σ-*Algebra* $\mathcal{F}$ if

$$\forall\, x \in \mathbb{R} \text{ where } W^{-1}(x) \subset \mathcal{F} \text{ it is true that } W^{-1}(x) := \{\omega \in \Omega : W(\omega) = x\}.^{[19]}$$

Example 2.4 - *Measurable Function*
Consider random variables X, Y defined as

$$
\begin{aligned}
X(\omega) &= \begin{cases} 5 & , \omega \in \{\omega_1, \omega_2, \omega_3, \omega_4\} \\ 7 & , \omega \in \{\omega_5, \omega_6, \omega_7, \omega_8\} \end{cases} \\
Y(\omega) &= \begin{cases} 8 & , \omega \in \{\omega_1, \omega_3, \omega_5, \omega_7\} \\ 6 & , \omega \in \{\omega_2, \omega_4, \omega_6, \omega_8\} \end{cases}
\end{aligned}
$$

Then, by the notation of `Example 2.1`, we have that

- X is *Measurable* wrt $\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3$ as $\big\{\{\omega_1, \omega_2, \omega_3, \omega_4\}, \{\omega_5, \omega_6, \omega_7, \omega_8\}, \emptyset\big\} \subset \mathcal{F}_1$.

- Y is <u>not</u> *Measurable* wrt $\mathcal{F}_1, \mathcal{F}_2$, but is measurable wrt $\mathcal{F}_3$, as $\{\omega_1, \omega_3, \omega_5, \omega_7\} \notin \mathcal{F}_1$.

Remark 2.4 - *Measurable*
If a function X is *Measurable* wrt $\mathcal{F}_t$ then it will be *Measurable* wrt $\mathcal{F}_{t+1}$, as $\mathcal{F}_t \subseteq \mathcal{F}_{t+1}$.

Definition 2.11 - *Adapted*
A *Stochastic Process* $S = \{S(t) : t = 0, 1, \ldots, T\}$ is said to be *Adapted* to the *Filtration* $\mathcal{F} = \{\mathcal{F}_t : t = 0, 1, \ldots, T\}$ if for every $t = 0, \ldots, T$ the random variable $S(t)$ is *Measurable* wrt $\mathcal{F}_t$.

Remark 2.5 - *Adapted Filtrations in Practice*
In practice we often define the *Stochastic Process* S first and use the so-called "*Natural Filtration*", defined as

  i). For each $t = 0, 1, \ldots, T$ let $\mathcal{P}_t$ be the partition of Ω st the *Stochastic Process* $\{S(0), \ldots, S(t)\}$ takes the same value for each $\omega \in A$, for each subset $A \in \mathcal{P}_t$.

  ii). Let $\mathcal{F}_t$ be the σ-*Algebra* generated by $\mathcal{P}_t$.

  iii). Then $\mathcal{F} := \{\mathcal{F}_t : t = 0, 1, \ldots, T\}$ is called the filtration *Generated* by the *Stochastic Process* S.

Example 2.5 -
Consider the setting it `Example 2.1` and let $S(t)$ be the upwards movements in the value of the asset by time t. Then $S(t)$ generates the *Filtration* above.

This can be summarised in the table

---

[19] Equivalently, if we know which set of the σ-*algebra* ω is in, then we know the value of $W(\omega)$.

ω_k	$t = 0$	$t = 1$	$t = 2$	$t = 3$
ω_1	$S(t) = 0$	$S(t) = 1$	$S(t) = 2$	$S(t) = 3$
ω_2	$S(t) = 0$	$S(t) = 1$	$S(t) = 2$	$S(t) = 2$
ω_3	$S(t) = 0$	$S(t) = 1$	$S(t) = 1$	$S(t) = 2$
ω_4	$S(t) = 0$	$S(t) = 1$	$S(t) = 1$	$S(t) = 1$
ω_5	$S(t) = 0$	$S(t) = 0$	$S(t) = 1$	$S(t) = 2$
ω_6	$S(t) = 0$	$S(t) = 0$	$S(t) = 1$	$S(t) = 1$
ω_7	$S(t) = 0$	$S(t) = 0$	$S(t) = 0$	$S(t) = 1$
ω_8	$S(t) = 0$	$S(t) = 0$	$S(t) = 0$	$S(t) = 0$

Definition 2.12 - *Random Walk*
Let $X_t := X_0 + Y_1 + \cdots + Y_t$ where $Y_1, \ldots, Y_t$ are iid random variables with finite variance σ^2 and mean *mean*. Then $\{X_t : t \geq 0\}$ is a *Random Walk*.

We say that $\{X_t : t \geq 0\}$ is a *Simple Random Walk* if Y_i takes only the values 1 with probability p and -1 with probability $1 - p$.

Proposition 2.2 - *Distribution of Simple Random Walk Values*
A *Simple Random Walk* takes values y at time-point t iff exactly $\frac{t+y}{2}$ of $Y_1, \ldots, Y_t$ are equal to 1, and the remaining $\frac{t-y}{2}$ equal -1.

$$\forall \, t \geq 0, y \in \{-t, -t+2, \ldots, t-2, t\}, \quad \mathbb{P}(X_t = y) = \binom{t}{\frac{t+y}{2}} p^{(t+y)/2}(1-p)^{(t-y)/2}$$

2.4   Conditional Expectations

Definition 2.13 - *Conditional Expectation* $\mathbb{E}[\cdot|\cdot]$
For a finite *Sample Space* Ω the *Conditional Expectation* of a discrete RV Y given the event A $\mathbb{E}[Y|A]$ is defined as
$$\mathbb{E}[Y|A] = \sum_y y\mathbb{P}(Y = y|A)$$

This *Conditional Expectation* maps from the events A to the real numbers.

Remark 2.6 - *Rewriting Conditional Expectation*
We can rewrite the *Conditional Expectation* as
$$\begin{aligned} \mathbb{E}[Y|A] &= \sum_y y\frac{\mathbb{P}(Y(\omega)=y,A)}{\mathbb{P}(A)} \\ &= \sum_{\omega \in A} Y(\omega)\frac{\mathbb{P}(\omega)}{\mathbb{P}(A)} \end{aligned}$$

Example 2.6 - *Conditional Expectation*
Consider `Example 2.5` with ω_i having probability $1/8$.

- If $A := \{\omega_1, \omega_2, \omega_3, \omega_4\}$ then
$$\mathbb{E}[S_3|A] = \frac{(3 + 2 + 2 + 1)(1/8)}{1/2} = 2$$

- If $A := \{\omega_5, \omega_6, \omega_7, \omega_8\}$ then
$$\mathbb{E}[S_3|A] = \frac{(2 + 1 + 1 + 0)(1/8)}{1/2} = 1$$

Definition 2.14 - *Conditional Expectation /w σ-Algebra*
Let $\mathcal{F}$ be a σ-*algebra* and $\mathcal{P}$ be the corresponding *Partition* of Ω.

We define the *Conditional Expectation* of RV Y give σ-*algebra* $\mathcal{F}$ as

$$\mathbb{E}[Y|\mathcal{F}] = \sum_{A \in \mathcal{P}} \mathbb{E}[Y|A]\mathbb{1}_A$$

This is a random-variable[20] which is *Measurable* wrt $\mathcal{F}$. And,

$$\forall \; \omega \in A, \; \mathbb{E}[Y|\mathcal{F}](\omega) = \mathbb{E}[Y|A]$$

Example 2.7 - *Conditional Expectation /w σ-Algebras*
Consider `Example 2.5` with ω_i having probability $1/8$.

Recall that $\mathcal{F}_1 = \big\{\emptyset, \Omega, \{\omega_1, \omega_2, \omega_3, \omega_4\}, \{\omega_5, \omega_6, \omega_7, \omega_8\}\big\}$. Then

$$\mathbb{E}[S_3|A] = \begin{cases} 2 & \text{if } A = \{\omega_1, \omega_2, \omega_3, \omega_4\} \\ 1 & \text{if } A = \{\omega_5, \omega_6, \omega_7, \omega_8\} \end{cases}$$

Hence $\mathbb{E}[S_3|\mathcal{F}_1]$ is a random variable with

$$\mathbb{E}[S_3|\mathcal{F}](\omega_i) = \begin{cases} 2 & \text{if } i = 1, 2, 3, 4 \\ 1 & \text{if } i = 5, 6, 7, 8 \end{cases}$$

This random variable is $\mathcal{F}_1$-*Measurable.*

Observer that
$$\begin{aligned} \mathbb{E}[\mathbb{E}[S_3|\mathcal{F}_1]] &= \textstyle\sum_i \mathbb{P}(\omega_i)\mathbb{E}[S_3|\mathcal{F}_1](\omega_i) \\ &= 2 \cdot \tfrac{4}{8} + 1 \cdot \tfrac{4}{8} = \tfrac{3}{2} = \mathbb{E}[S_3]^{[21]} \end{aligned}$$

Note that if

$$Y(\omega_i) = \begin{cases} 7 & \text{if } i \leq 4 \\ 1 & \text{if } i \geq 5 \end{cases} \implies \mathbb{E}[Y|\mathcal{F}_1] = Y$$

Moreover, If $Z = \mathbb{1}\{\text{3rd move is up}\}$ then $|$ is independent of $\mathcal{F}_1$ and

$$\mathbb{E}[Z|\mathcal{F}_1] = \frac{1}{2} = \mathbb{E}[Z]$$

Theorem 2.1 - *Properties of Conditional Expectation*
Let Y be a random variable and $\mathcal{F}, \mathcal{F}_1, \mathcal{F}_2$ be σ-*algebras* with $\mathcal{F}_1 \subset \mathcal{F}_2$.

Conditional Expectations satisfy the following properties:

   i). $\mathbb{E}[\mathbb{E}[Y|\mathcal{F}]] = \mathbb{E}[Y]$. (The "Tower Law")

   ii). $\mathbb{E}[\mathbb{E}[Y|\mathcal{F}_2]|\mathcal{F}_1] = \mathbb{E}[Y|\mathcal{F}_1] = \mathbb{E}[\mathbb{E}[Y|\mathcal{F}_1]\mathcal{F}_2]$. (The "Generalised Tower Law")

   iii). If X is a random variable which is *Measurable* wrt $\mathcal{F}$ then $\mathbb{E}[XY|\mathcal{F}] = X\mathbb{E}[Y|\mathcal{F}]$ and $\mathbb{E}[X|\mathcal{F}] = X$.[22]

---

[20]Not a sum, as there is one A st $\mathbb{1}_A = 1$ (the rest equal zero)
[21]This is the "Tower Law"
[22]As X is measurable wrt $\mathcal{F}$ all information is known about it, and thus can treat it as a scalar.

iv). If Y is independent of $\mathcal{F}$[23] then $\mathbb{E}[Y|\mathcal{F}] = \mathbb{E}[Y]$.

Proof 2.1 - *Theorem 2.1 i)*

$$
\begin{aligned}
\mathbb{E}[\mathbb{E}[Y|\mathcal{F}]] &= \mathbb{E}\left[\sum_{A\in\mathcal{P}}\mathbb{E}[Y|A]\mathbb{1}_A\right] \text{ by def. cond. expectation}\\
&= \sum_{A\in\mathcal{P}}\mathbb{E}[\mathbb{1}_A]\mathbb{E}[Y|A]\\
&= \sum_{A\in\mathcal{P}}\left(\sum_{\omega\in\Omega}\mathbb{1}_A(\omega)\mathbb{P}(\omega)\right)\cdot\left(\sum_{\omega\in A}\frac{Y(\omega)\mathbb{P}(\omega)}{\mathbb{P}(A)}\right)\\
&= \sum_{A\in\mathcal{P}}\mathbb{P}(A)\left(\sum_{\omega\in A}\frac{Y(\omega)\mathbb{P}(\omega)}{\mathbb{P}(A)}\right)\\
&= \sum_{A\in\mathcal{P}}\sum_{\omega\in A}Y(\omega)\mathbb{P}(\omega)\\
&= \mathbb{E}[Y]
\end{aligned}
$$

$\square$

Proof 2.2 - *Theorem 2.1 ii)*

$$
\begin{aligned}
\mathbb{E}[\mathbb{E}[Y|\mathcal{F}_2]\mathcal{F}_1] &= \mathbb{E}\left(\sum_{B\in\mathcal{P}_2}\mathbb{E}[Y|B]\mathbb{1}_B\Big|\mathcal{F}_1\right)\\
&= \sum_{B\in\mathcal{P}_2}\mathbb{E}[Y|B]\mathbb{E}[\mathbb{1}_B|\mathcal{F}_1]\\
&= \sum_{B\in\mathcal{P}_2}\mathbb{E}[Y|B]\sum_{A\in\mathcal{P}_1}\mathbb{E}[1_B|A]\mathbb{1}_A\\
&= \sum_{A\in\mathcal{P}_1}\sum_{B\in\mathcal{P}_2}\mathbb{E}[Y|B]\mathbb{E}[1_B|A]\mathbb{1}_A\\
&= \sum_{A\in\mathcal{P}_1}\sum_{B\in\mathcal{P}_2}\mathbb{E}[Y|B]\frac{\mathbb{P}(A\cap B)}{\mathbb{P}(A)}\mathbb{1}_A
\end{aligned}
$$

Since the partition $\mathcal{P}_2$ is finer than $\mathcal{P}_1$ for all $B\in\mathcal{P}_2$ and $A\in\mathcal{P}_1$, either $B\subset A$ or $B\cap A=\emptyset$. Thus

$$
\begin{aligned}
\mathbb{E}[\mathbb{E}[Y|\mathcal{F}_2]|\mathcal{F}_1] &= \sum_{A\in\mathcal{P}_1}\sum_{B\in\mathcal{P}_2,B\subset A}\mathbb{E}[Y|B]\frac{\mathbb{P}(B)}{\mathbb{P}(A)}\mathbb{1}_A\\
&= \sum_{A\in\mathcal{P}_1}\sum_{B\in\mathcal{P}_2,B\subset A}\left(\sum_{\omega\in B}Y(\omega)\frac{\mathbb{P}(\omega)}{\mathbb{P}(B)}\right)\frac{\mathbb{P}(B)}{\mathbb{P}(A)}\mathbb{1}_A\\
&= \sum_{A\in\mathcal{P}_1}\sum_{\omega\in A}Y(\omega)\frac{\mathbb{P}(\omega)}{\mathbb{P}(A)}\mathbb{1}_A\\
&= \sum_{A\in\mathcal{P}_1}\mathbb{E}[Y|A]\mathbb{1}_A\\
&= \mathbb{E}[Y|\mathcal{F}_1]
\end{aligned}
$$

$\square$

Proof 2.3 - *Theorem 2.1 iii)*
Since S is *Measurable* it is constant on sets of $\mathcal{P}$ so we can write

$$X = \sum_{A\in\mathcal{P}}x_A\mathbb{1}_A$$

---

[23]$\forall\ A\in\mathcal{P}$ the distribution of Y is independent of A

with suitable scalars $x_A \in \mathbb{R}$. Then

$$
\begin{aligned}
\mathbb{E}[XY|\mathcal{F}] &= \sum_{A\in\mathcal{P}} \mathbb{E}[XY|A]\mathbb{1}_A \\
&= \sum_{A\in\mathcal{P}} \mathbb{E}[x_A Y|A]\mathbb{1}_A \\
&= \sum_{A\in\mathcal{P}} x_A\mathbb{E}[Y|A]\mathbb{1}_A \\
&= \sum_{A\in\mathcal{P}} X\mathbb{E}[Y|A]\mathbb{1}_A{}^{[24]} \\
&= X\sum_{A\in\mathcal{P}} \mathbb{E}[Y|A]\mathbb{1}_A \\
&= X\mathbb{E}[Y|\mathcal{F}]
\end{aligned}
$$

Consider the special case where $Y = 1$, then it follows that $\mathbb{E}[X|\mathcal{F}] = X$. □

Proof 2.4 - *Theorem 2.1 iv)*
If Y is independent of $\mathcal{F}$ then for all $A \in \mathcal{F}$

$$
\begin{aligned}
\mathbb{E}[Y|A] &= \sum_y y\mathbb{P}(Y = y|A) \\
&= \sum_y y\mathbb{P}(Y = y) \\
&= \mathbb{E}[Y]
\end{aligned}
$$

□

Proposition 2.3 - *Conditional Expectation for Non-Finite Sample Spaces*
Let Y be a RV, then the *Conditional Expectation* $\mathbb{E}[Y|\mathcal{F}]$ is the unique random variable st

i). $\mathbb{E}[Y|\mathcal{F}]$ is $\mathcal{F}$-Measurable.

ii). $\forall\, A \in \mathcal{F},\ \mathbb{E}[\mathbb{E}[Y|\mathcal{F}] \cdot \mathbb{1}_A] = \mathbb{E}[Y\mathbb{1}_A]$.

Proof 2.5 - *Proposition 2.3*
For an arbitrary event $A \in \mathcal{F}$ the indicator function $\mathbb{1}_A$ is $\mathcal{F}$-*Measurable* and thus $\mathbb{E}[\mathbb{1}_A Y|\mathcal{F}] = \mathbb{1}_A\mathbb{E}[Y|\mathcal{F}]$ by `Theorem 2.1`.

Hence, for all $A \in \mathcal{F}$

$$\mathbb{E}[\mathbb{E}[Y|\mathcal{F}] \cdot \mathbb{1}_A] = \mathbb{E}[\mathbb{E}[\mathbb{1}_A Y|\mathcal{F}]] = \mathbb{E}[\mathbb{1}_A Y]$$

On the other hand, suppose X is $\mathcal{F}$-*Measurable* and satisfies

$$\mathbb{E}[X\mathbb{1}_A] = \mathbb{E}[Y\mathbb{1}_A] \ \forall\, A \in \mathcal{F}$$

Then write X as in `Proof 2.3` $X = \sum_{A\in\mathcal{P}} x_A\mathbb{1}_A$.
It follows that $\forall\, A \in \mathcal{P},\ \mathbb{E}[X\mathbb{1}_A] = x_A\mathbb{P}(A)$ and thus

$$
\begin{aligned}
\mathbb{E}[Y\mathbb{1}_A] &= \sum_{\omega\in A} Y(\omega)\mathbb{P}(\omega) \\
&= \frac{\mathbb{P}(A)}{\mathbb{P}(A)}\sum_{\omega\in A} Y(\omega)\mathbb{P}(\omega) \\
&= \mathbb{P}(A)\sum_{\omega\in A} \frac{Y(\omega)\mathbb{P}(\omega)}{\mathbb{P}(A)} \\
&= \mathbb{P}(A)\mathbb{E}[Y|A]
\end{aligned}
$$

---

[24]This holds as $\mathbb{1}_A = 1$ for only one specific A.

Thus
$$x\mathbb{P}(A) = \mathbb{E}[X \mathbb{1}_A] = \mathbb{E}[Y \mathbb{1}_A] = \mathbb{P}(A)\mathbb{E}[Y|A]$$

Therefore, we have found that $x_A = \mathbb{E}[Y|A] \; \forall \; A \in \mathcal{P}$ where $\mathcal{P}$ is the partition which corresponds to σ-algebra $\mathcal{F}$.

Moreover, $X = \mathbb{E}[Y|\mathcal{F}]$.

2.5   Martingales

Definition 2.15 - *Martingale Z*
Let $Z := \{Z_t : t = 0, 1, \ldots, T\}$ be an *Adapted Stochastic Process* defined on a *Sample Space* Ω with a filtration $\{\mathcal{F}_t\}$.

The process Z is said to be a *Martingale* if

$$\mathbb{E}[Z_t|\mathcal{F}_{t-1}] = Z_{t-1} \; \forall \; t \geq 1$$

Definition 2.16 - *Super-Martingale Z*
Let $Z := \{Z_t : t = 0, 1, \ldots, T\}$ be an *Adapted Stochastic Process* defined on a *Sample Space* Ω with a filtration $\{\mathcal{F}_t\}$.

The process Z is said to be a *Super-Martingale* if

$$\mathbb{E}[Z_t|\mathcal{F}_{t-1}] \leq Z_{t-1} \; \forall \; t \geq 1$$

Definition 2.17 - *Sub-Martingale Z*
Let $Z := \{Z_t : t = 0, 1, \ldots, T\}$ be an *Adapted Stochastic Process* defined on a *Sample Space* Ω with a filtration $\{\mathcal{F}_t\}$.

The process Z is said to be a *Sub-Martingale* if

$$\mathbb{E}[Z_t|\mathcal{F}_{t-1}] \geq Z_{t-1} \; \forall \; t \geq 1$$

Remark 2.7 - *Super- vs Sub-Martingale*
You should think of a *Super-Martingale* as a process where the current value provides an upper-bound on the next value, and a *Sub-Martingale* as a process where the current value provides a lower-bound on the next value.

Theorem 2.2 - *When is an Adapted Stochastic Process a Martingale?*
An *Adapted Stochastic Process* Z is a *Martingale* iff

$$\mathbb{E}[Z_t|\mathcal{F}_s] = Z_s \; \forall \; t \geq s$$

Corresponding results (with equality swapped out) hold for super- and sub-martingales.

Proof 2.6 - *Theorem 2.2*

$\Longleftarrow$ Clearly, if the equation in `Theorem 2.2` holds then `Definition 2.15` holds.

$\implies$ Assume Z is a *Martingale*. Then `Theorem 2.1` implies that

$$
\begin{aligned}
\mathbb{E}[Z_t|\mathcal{F}_s] &= \mathbb{E}\big[\mathbb{E}[Z_t|\mathcal{F}_{t-1}]|\mathcal{F}_s\big] \\
&= \mathbb{E}[Z_{t-1}|\mathcal{F}_s] \text{ as } Z \text{ is a martingale} \\
&= \mathbb{E}[Z_s|\mathcal{F}_s] \text{ by repetition} \\
&= \mathbb{E}[Z_s] = Z_s
\end{aligned}
$$

A similar proof is done for super- and sub-martingales.

Example 2.8 - *Martingales*
Let $\{X_t\}_{t\geq 0}$ be a simple random walk with parameter p and $\mathcal{F}_t$ be the σ-*algebra* generated by (X_t). Then

i). $\{X_t\}_{t\geq 0}$ is a *Martingale* if $p = 1/2$ as there is an equal probability of stepping up and stepping down.

ii). $\{X_t\}_{t\geq 0}$ is a *Super-Martingale* if $p \leq 1/2$ as there is a greater probability of stepping down than stepping up.

iii). $\{X_t\}_{t\geq 0}$ is a *Sub-Martingale* if $p \geq 1/2$ as there is a greater probability of stepping up than stepping down.

iv). If $p = 1/2$ then the process $\{Z_t\}_{t\geq 0}$ defined st $Z_t := X_t^2 - t$ for $t = 0, 1, \dots$ is a *Martingale*

v). If $p \neq 1/2$ then the processes $\{L_t\}_{t\geq 0}$ & $\{M_t\}_{t\geq 0}$ defined by $L_0 = 1$, $L_t = \left(\frac{1-p}{p}\right)^{X_t}$ and $M_t = X_t - t(2p - 1)$ are both *Martingales*[25].

Proof 2.7 - *Example 2.8 i)-iii)*
Since $\mathcal{F}_t$ is the natural filtration, then $\{X_t\}$ is $\mathcal{F}_t$-measurable and Y_t is independent of $\mathcal{F}_{t-1}$. Therefore by `Theorem 2.1` we have that

$$
\begin{aligned}
\mathbb{E}[X_t|\mathcal{F}_{t-1}] &= \mathbb{E}[X_{t-1} + Y_t|\mathcal{F}_{t-1}] \\
&= \mathbb{E}[X_{t-1}|\mathcal{F}_{t-1}] + \mathbb{E}[Y_t|\mathcal{F}_{t-1}] \\
&= X_{t-1} + \mathbb{E}[Y_t]
\end{aligned}
$$

- If $p = 1/2$ then $\mathbb{E}[Y_t] = 0 \implies \mathbb{E}[X_t|\mathcal{F}_{t-1}] = X_{t-1}$, the definition of a *Martingale*.

- If $p \leq 1/2$ then $\mathbb{E}[Y_t] \leq 0 \implies \mathbb{E}[X_t|\mathcal{F}_{t-1}] \leq X_{t-1}$, the definition of a *Super-Martingale*.

- If $p \geq 1/2$ then $\mathbb{E}[Y_t] \geq 0 \implies \mathbb{E}[X_t|\mathcal{F}_{t-1}] \geq X_{t-1}$, the definition of a *Sub-Martingale*.

Proof 2.8 - *Example 2.8 iv)*
Note that
$$
\begin{aligned}
\mathbb{E}[Z_t|\mathcal{F}_{t-1}] &= \mathbb{E}[X_t^2 - t|\mathcal{F}_{t-1}] \\
&= \mathbb{E}[(X_{t-1} + Y_t)^2|\mathcal{F}_{t-1}] - t \\
&= \mathbb{E}[X_{t-1}^2|\mathcal{F}_{t-1}] + 2\mathbb{E}[X_t Y_t|\mathcal{F}_{t-1}] + \mathbb{E}[Y_t^2|\mathcal{F}_{t-1}]t \\
&= X_{t-1}^2 + 2X_{t-1} \underbrace{\mathbb{E}[Y_t|\mathcal{F}_{t-1}]}_{=\mathbb{E}[Y_t]=0} + \underbrace{\mathbb{E}[Y_t^2]}_{=1\,[26]} - t \\
&= X_{t-1}^2 + 0 + 1 - t \\
&= X_{t-1}^2 - (t-1) = Z_{t-1}
\end{aligned}
$$

This shows that the Z_t we defined fulfils the definition of a *Martingale*.

---

[25]Proved in a homework
[26]Y_t only takes values $\{-1, 1\}$ so $Y_t^2 = 1$ always.

2.6   Stopping Times τ

Remark 2.8 - *Stopping Times & Finance*
Stopping Times are useful for analysing American options.

Definition 2.18 - *Stopping Time τ*
Let Ω be a *Sample Space* with a filtration $\{\mathcal{F}_t\}_{t \in \mathbb{N}_0}$.

A *Stopping Time* is a random variable τ which takes values in the set $\{0, 1, \ldots, \infty\}$[27] st each event of the form $\{\tau \leq t\}$ from some t is an element of the *σ-algebra* $\mathcal{F}_t$.[28]

A *Stopping Time* is said to be *Bounded* if $\exists \ k$ st $\mathbb{P}(\tau < t) = 1$.

Example 2.9 - *Stopping Time*
Consider the following events

- "RBS shares hit 100p." - This is a *Stopping Time* event.

- "RBS shares hit their maximum." - This is <u>not</u> a *Stopping Time* event.

Theorem 2.3 - *Stopping Times & σ-Algebras*
A random variable τ is a *Stopping Time* <u>iff</u> each event of the form $\{\tau = t\}$ for some t is an element of the *σ-Algebra* $\mathcal{F}_t$.

Proof 2.9 - *Theorem 2.3*

$$\Longleftarrow \ \{\tau = t\} = \Big(\underbrace{\{\tau \leq t\}}_{\in \mathcal{F}_t} \setminus \underbrace{\{\tau \leq t-1\}}_{\in \mathcal{F}_t} \Big) \in \mathcal{F}_t$$

$$\Longrightarrow \ \{\tau \leq t\} = \bigcup_{k \leq t} \underbrace{\{\tau = k\}}_{\in \mathcal{F}_t} \in \mathcal{F}_t$$

Thus the result holds in both directions.

Theorem 2.4 - *Stopping Time for an Adapted Stochastic Process*
Let $\{X_t\}_{t \in \mathbb{N}_0}$ be an *Adapted Stochastic Process* and $c \in \mathbb{R}$.

A *Stopping Time* τ_c can be defined as $\tau_c = \inf\{t \geq 0 : X_t \geq \}$[29].

Proof 2.10 - *Theorem 2.4*
We note that $\tau_c \leq t$ <u>iff</u> $\exists \ k \leq t$ st $X_k \geq c$.

Therefore

$$\{\tau_c \leq t\} = \bigcup_{k \leq t} \underbrace{\{X_k \geq c\}}_{\in \mathcal{F}_t} \in \mathcal{F}_t$$

Thus τ_c is a *Stopping Time*.

Theorem 2.5 - *Optional Sampling Theorem*[30] - *Martingales*

---

[27]∞ is used for events which never occur.
[28]Thus, we can determine whether $\{\tau \leq t\}$ has occurred just by observing $\mathcal{F}_t$ (ie all the information available at time-point t).
[29]The event which stops the moment X_t reaches c
[30]AKA *Optional Stopping Theorem*

Let τ be a *Bounded Stopping Time* and $X = \{X_t\}_{t \in \mathbb{N}_0}$ be a *Martingale*. Then

$$\mathbb{E}[X_\tau] = \mathbb{E}[X_0] = X_0$$

Theorem 2.6 - *Optional Sampling Theorem - Super-Martingales*
Let τ be a *Bounded Stopping Time* and $X = \{X_t\}_{t \in \mathbb{N}_0}$ be a *Super-Martingale*. Then

$$\mathbb{E}[X_\tau] = \leq \mathbb{E}[X_0]$$

Proof 2.11 - `Theorem 2.5`
Assume that $\tau \leq K$ and write

$$X_{\tau(\omega)}(\omega) = \sum_{t=0}^{K} X_t(\omega)\mathbb{1}\{\tau(\omega) = t\}^{[31]}$$

Then

$$
\begin{aligned}
\mathbb{E}[X_\tau] &= \mathbb{E}\left[\sum_{t=0}^{K} X_t \mathbb{1}\{\tau = t\}\right] \\
&= \sum_{t=0}^{K} \mathbb{E}[X_t \mathbb{1}\{\tau = t\}] \\
&= \sum_{t=0}^{K} \mathbb{E}[\mathbb{E}[X_K|\mathcal{F}_t]\mathbb{1}\{\tau = t\}]
\end{aligned}
$$

Using that since τ is a *Stopping Time*, then the event $\{\tau = t\}$ is measurable wrt $\mathcal{F}_t$. Thus $\mathbb{E}[X_K|\mathcal{F}_t]\mathbb{1}\{\tau = t\} = \mathbb{E}[X_K\mathbb{1}\{\tau = t\}|\mathcal{F}_t]$.

$$
\begin{aligned}
&= \sum_{t=0}^{K} \mathbb{E}[\mathbb{E}[X_K\mathbb{1}\{\tau = t\}|\mathcal{F}_t]] \\
&= \sum_{t=0}^{K} \mathbb{E}[X_K\mathbb{1}\{\tau = t\}] \text{ by Tower Law} \\
&= \mathbb{E}\left[X_K \sum_{t=0}^{K} \mathbb{1}\{\tau = t\}\right] \\
&= \mathbb{E}[X_K \cdot 1] = \mathbb{E}[X_K] \\
&= \mathbb{E}[X_0] = X_0
\end{aligned}
$$

Example 2.10 - *Gambler's Ruin*
Consider a gambler with an initial wealth of £C. He gambles by guessing whether a coin flip results in heads or tails. If he guess correctly then he receives £1 and he is wrong he loses £1. The game ends when he either becomes bankrupt or he reaches a wealth of £$C + G$ where $G > 0$.

Proposition 2.4 - *Gambler's Ruin*

---

[31]This is not really a sum, due to the indicator function $\mathbb{1}$ meaning only one value is non-zero.

Let $\{X_t\}_{t\geq 0}$ be a *Simple Random Walk* with parameter p and $X_0 = 0$, and $C, G > 0$. Define the *Stopping-Time* event $\tau = \inf\{t : X_t = G \text{ or } X_t = -C\}$[32].

If $p = 1/2$ then

$$
\begin{aligned}
\mathbb{P}(X_\tau = G) &= \frac{C}{C + G} \\
\mathbb{P}(X_\tau = -C) &= \frac{G}{C + G} \\
\mathbb{E}[\tau] &= CG
\end{aligned}
$$

Else, if $p \neq 1/2$ then

$$
\begin{aligned}
\mathbb{P}(X_\tau = G) &= \frac{1 - \left(\frac{p}{1-p}\right)^C}{\left(\frac{p}{1-p}\right)^G - \left(\frac{p}{1-p}\right)^C} \\
\mathbb{P}(X_\tau = -C) &= 1 - \mathbb{P}(X_\tau = G) \\
\mathbb{E}[\tau] &= \frac{G\mathbb{P}(X_\tau = G) + (-C)\mathbb{P}(X_\tau) = -C}{2p - 1}
\end{aligned}
$$

Proof 2.12 - *Proposition 2.4*
We want to apply the *Optitional Stopping Theorem* (`Theorem 2.5`).

The *Stopping Time* τ is not bounded, but X_τ is bounded. Therefore we can use one of the alternative versions of the *Optional Stopping Theorem* provided that $\mathbb{P}(\tau < \infty = 1)$.

Note that whenever there is a run of at least $k = C + G$ successive 1's in the process Y which defines X, the process will stop and $\tau < \infty$. Thus for all m

$$
\begin{aligned}
\mathbb{P}(\tau > km) &= \mathbb{P}(\text{No run of k 1's in } Y_1 \text{ to } Y_{mk}) \\
&= \prod_{j=0}^{n-1} \mathbb{P}(\text{No run of k 1's in } Y_{jk+1} \text{ to } Y_{(j+1)k}) \\
&= (1 - p^k)^m \\
\implies \mathbb{P}(\tau < \infty) &= 1
\end{aligned}
$$

i). Consider the case $p = 1/2$. Using the *Optional Stopping Theorem* we can deduce that

$$
\begin{aligned}
0 &= \mathbb{E}[X_\tau] \\
&= G\mathbb{P}(X_\tau = G) + (-C)\mathbb{P}(X_\tau = -C) \\
&= G\mathbb{P}(X_\tau = G) + (-C)(1 - \mathbb{P}(X_\tau = G)) \\
\implies C &= (G + C)\mathbb{P}(X_\tau = G) \\
\implies \mathbb{P}(X_\tau = G) &= \frac{C}{(G + C)} \\
\text{and} \quad \mathbb{P}(X_\tau = -C) &= 1 - \frac{C}{(G + C)} = \frac{G}{G + C}
\end{aligned}
$$

We determine $\mathbb{E}[X_\tau]$ by applying the *Optional Stopping Theorem* to the process $\{Z_t\}_{t\geq 0}$ where $Z_t := X_t^2 - t$. We know $\{Z_t\}_{t\geq 0}$ is a *Martingale* from `Example 2.8 iv)`.

As $\{Z_t\}_{t\geq 0}$ is a *Martingale* it holds that

$$
0 = \mathbb{E}[Z_0] = \mathbb{E}[Z_\tau] = \mathbb{E}[X_\tau^2 - \tau]
$$

---

[32]The event the game in `Example 2.10`

Thus, we obtain that

$$
\begin{aligned}
\mathbb{E}[\tau] &= \mathbb{E}[X_\tau^2] \\
&= G^2 \mathbb{P}(X_\tau = G) + C^2 \mathbb{P}(X_\tau = -C) \\
&= G^2 \frac{C}{C+G} + C^2 \frac{G}{C+G} \\
&= CG
\end{aligned}
$$

ii). Consider the case $p \neq 1/2$. Using the *Optional Stopping Theorem* on the process $\{L_t\}_{t \geq 0}$ where $L_t := \left(\frac{1-p}{p}\right)^{X_t}$. We know $\{L_t\}_{t \geq 0}$ is a *Martingale* from `Example 2.8 v)`.

$$
1 = \mathbb{E}[L_0] = \left(\frac{1-p}{p}\right)^G \mathbb{P}(X_\tau = G) + \left(\frac{1-p}{p}\right)^C \mathbb{P}(X_\tau = -C)
$$

Note that $\mathbb{P}(X_\tau = G) + \mathbb{P}(X_\tau = -C) = 1$, so we can derive each probability as

$$
\begin{aligned}
1 &= \left(\frac{1-p}{p}\right)^G \mathbb{P}(X_\tau = G) + \left(\frac{1-p}{p}\right)^C (1 - \mathbb{P}(X_\tau = G)) \\
&= \left(\frac{1-p}{p}\right)^C + \left[\left(\frac{1-p}{p}\right)^G - \left(\frac{1-p}{p}\right)^C\right] \mathbb{P}(X_\tau = G) \\
\implies \mathbb{P}(X_\tau = G) &= \frac{1 - \left(\frac{1-p}{p}\right)^C}{\left(\frac{1-p}{p}\right)^G - \left(\frac{1-p}{p}\right)^C}
\end{aligned}
$$

We determine $\mathbb{E}[X_\tau]$ by apply the *Optional Stopping Theorem* to the process $\{M_t\}_{t \geq 0}$ where $M_\tau := X_t - t(2p-1)$. We know $\{M_t\}_{t \geq 0}$ is a *Martingale* from `Example 2.8 v)`. Thus

$$
0 = \mathbb{E}[M_\tau] = G\mathbb{P}(X_\tau = G) + (-C)\mathbb{P}(X_\tau = -C) - \mathbb{E}[\tau](2p-1)
$$

Re-arranging this we can find an expression for $\mathbb{E}[\tau]$

$$
\begin{aligned}
0 &= G\mathbb{P}(X_\tau = G) + (-C)\mathbb{P}(X_\tau = -C) - \mathbb{E}[\tau](2p-1) \\
\implies \mathbb{E}[\tau](2p-1) &= G\mathbb{P}(X_\tau = G) + (-C)\mathbb{P}(X_\tau = -C) \\
\implies \mathbb{E}[\tau] &= \frac{G\mathbb{P}(X_\tau = G) + (-C)\mathbb{P}(X_\tau = -C)}{(2p-1)}
\end{aligned}
$$

$\square$

3   Multi-Period Models

3.1   Trading Strategies

Definition 3.1 - *Predictable Stochastic Process*
A *Stochastic Process* $\{X_t\}_{t \in \mathbb{N}_0}$ is called *Predictable* if it is *Measurable* wrt σ-*Algebra* $\mathcal{F}_{t-1}$.

Definition 3.2 - *Trading Strategy H*
A vector of *Stochastic Processes* $H := (H_0, H_1, \ldots, H_N)$ is called a *Trading Strategy* if, for all n, the process H_n is *Predictable*.

Note that $H_n(t) \in \mathbb{Z}$ denotes the number of units[33] the investor carries forward of the nth security from time $t-1$ to t. And, $H_0(t)B_{t-1}$ is the amount of money in the *Bank Account* at time $t-1$.
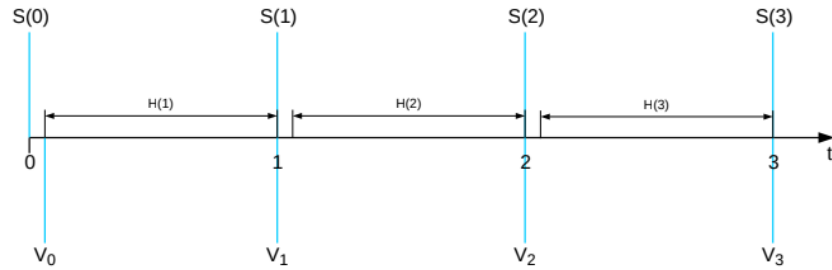

Figure 3: Diagram to show relationship between trading strategies H, values processes V and stock processes S.

Definition 3.3 - *Value Process V*
The *Value Process* $\{V_t\}_{t\in[0,T]}$ is a stochastic process defined by

$$V_t = \begin{cases} H_0(1)B_0 + \sum_{n=1}^{N} H_n(1)S_n(0) & \text{if } t = 0 \\ H_0(t)B_t + \sum_{n=1}^{N} H_n(t)S_n(t) & \text{if } t \geq 1 \end{cases}$$

See `Figure 3`.

Definition 3.4 - *Gains Process G*
The *Gains Process* $\{G_t\}_{t\in[0,T]}$ of a *Trading Strategy H* is given by

$$G_t = \left(\sum_{u=1}^{t} H_0(u)\Delta B_u \right) + \sum_{n=1}^{N}\sum_{u=1}^{t} H_n(u)\Delta S_n(u) \text{ where } \Delta S_n(t) := S_n(t) - S_n(t-1) \text{ for } t \geq 1$$

This is an example of a stochastic integral of the trading strategy H wrt to the price process S.

Definition 3.5 - *Self-Financing Trading Strategies*
A *Trading Strategy H* is *Self-Financing* if

$$V_t = H_0(t+1)B_t + \sum_{n=1}^{N} H_n(t+1)S_n(t)$$

You cannot introduce money into or take money out of a *Self-Financing Trading Strategy* at any time-points t, except the first $t = 0$ and last $t = T$.

Definition 3.6 - *Discounted Processes* S_n^*, V^*
The *Discounte3d Price Process* $S_n^* = \{S_n^*(t)\}_{t\in[0,T]}$ for $n \in [1, N]$ is defined as

$$S_n^*(t) = \frac{S_n(t)}{B_t}$$

The *Discounted Value Process* $V^* = \{V_t^*\}_{t\in[0,T]}$ is defined as

$$V_t^* = \begin{cases} H_0(1) + \sum_{n=1}^{n} H_n(1)S_n^*(0) & \text{if } t = 0 \\ H_0(t) + \sum_{n=1}^{n} H_n(t)S_n^*(t) & \text{if } t \geq 1 \end{cases}$$

---

[33]Negative values indicate a short position.

The *Discounted Gains Process* $\{G_t^*\}_{t \in [1,T]}$ is defined as

$$G_t^* = \sum_{n=1}^{N} \sum_{u=1}^{t} H_n(u) \Delta S_n^*(u) \text{ for } t \geq 1$$

Proposition 3.1 - *Self-Financing Trading Strategy and Value Process*
A *Trading Strategy* H is *Self-Financing* <u>iff</u> $V_t^* = V_0^* + G_t^*$ for $t = 1, \ldots, T$.

Proof 3.1 - `Proposition 3.1`
For all $t = 1, \ldots, T$ it holds that

$$G_t^* = G_{t-1}^* + \sum_{n=1}^{N} H_n(t) \Delta S_n^*(t)$$

For convenience we define $G_0^* = 0$.

I prove the statement in both directions

$\implies$ Assume that H is *Self-Financing.*

By the definitions of *Self-Financing, Discounted Processes* and the above result, we can show the following for all $t = 1, \ldots, T$

$$
\begin{aligned}
V_t^* - G_t^* &= H_0(t) + \left(\sum_{n=1}^{N} H_n(t) S_n^*(t) \right) - \left(\sum_{n=1}^{N} H_n(t) \Delta S_n^*(t) \right) - G_{t-1}^* \\
&= H_0(t) + \left(\sum_{n=1}^{N} H_n(t)(S_n^*(t) - \Delta S_n^*(t)) \right) - G_{t-1}^* \\
&= H_0(t) + \left(\sum_{n=1}^{N} H_n(t) S_n^*(t-1) \right) - G_{t-1}^* \\
&= V_{t-1}^* - G_{t-1}^*
\end{aligned}
$$

By recursion we find that $V_t^* - G_t^* = V_0^*$.

$\impliedby$ Assume that $V_t^* = V_0^* + G_t^*$ for all $t = 1, \ldots, T$.

Then, for all $t = 1, \ldots, T_1$ we have the following

$$
\begin{aligned}
V_t^* - V_{t+1}^* &= V_0^* + G_t^* - (V_0^* + G_{t+1}^*) \\
&= G_t^* - G_{t-1}^*
\end{aligned}
$$

Therefore, by the definitions of discounted process and the result at the start of this proof

$$
\begin{aligned}
V_t^* &= V_{t+1}^* - (G_{t+1}^* - G_t^*) \\
&= H_0(t+1) + \sum_{n=1}^{N} H_n(t+1) S_n^*(t+1) - \sum_{n=1}^{N} H_n(t+1) \Delta S_n^*(t+1) \\
&= H_0(t+1) + \sum_{n=1}^{N} H_n(t+1)
\end{aligned}
$$

Thus H is *Self-Financing.*

3.2   Arbitrage in Multi-Period Model and Martingale Measures

Example 3.1 -
Consider a model with $N = 1$ stocks over $T = 2$ time-periods and $K = 4$ possible states.

For simplicity we change the notation of the price process to S_t instead of $S_1(t)$. The process is given by the following table

$$
S_t(\omega) = \quad
\begin{array}{c|ccc}
\omega \setminus t & 0 & 1 & 2 \\
\hline
\omega_1 & 6 & 9 & 10 \\
\omega_2 & 6 & 9 & 7 \\
\omega_3 & 6 & 3 & 7 \\
\omega_4 & 6 & 3 & 1
\end{array}
$$

We further assume that the interest-rate r is constant in both periods. Therefore the bank account process is $B_t = (1+r)^t$.

The filtration generated by this price process is

$$
\begin{aligned}
\mathcal{F}_0 &= \{\emptyset, \Omega\} \\
\mathcal{F}_1 &= \{\emptyset, \{\omega_1, \omega_2\}, \{\omega_3, \omega_4\}, \Omega\} \\
\mathcal{F}_2 &= \{\emptyset, \{\omega_1\}, \{\omega_2\}, \{\omega_3\}, \{\omega_4\}, \{\omega_1, \omega_2\}, \{\omega_1, \omega_3\}, \{\omega_1, \omega_4\}, \{\omega_2, \omega_3\}, \{\omega_3, \omega_4\}, \{\omega_3, \omega_4\}, \\
&\quad \{\omega_1, \omega_2, \omega_3\}, \{\omega_1, \omega_2, \omega_4\}, \{\omega_1, \omega_3, \omega_4\}, \{\omega_2, \omega_3, \omega_4\}, \Omega\}
\end{aligned}
$$

Recall that the trading strategy has components $H_n(t)(\omega)$ giving the number of units of a stock held between time periods $t-1$ and t.

Let H be an arbitrary trading strategy, then for the value process for this problem we have

$$
V_0(\omega) = H_0(1)(\omega) + 6H_1(1)(\omega)
$$

Further

$$
V_1(\omega) = \begin{cases}
(1+r)H_0(1)(\omega) + 9H_1(1)(\omega) & \text{if } \omega \in \{\omega_1, \omega_2\} \\
(1+r)H_0(1)(\omega) + 3H_1(1)(\omega) & \text{if } \omega \in \{\omega_3, \omega_4\}
\end{cases}
$$

$$
V_2(\omega) = \begin{cases}
(1+r)^2 H_0(2)(\omega) + 10H_1(2)(\omega) & \text{if } \omega = \omega_1 \\
(1+r)^2 H_0(2)(\omega) + 7H_1(2)(\omega) & \text{if } \omega \in \{\omega_2, \omega_3\} \\
(1+r)^2 H_0(2)(\omega) + 1H_1(2)(\omega) & \text{if } \omega = \omega_4
\end{cases}
$$

The *Trading Strategy* H is *Predictable* so $H_i(1)(\omega)$ is constant for all ω, $H_i(2)(\omega_1) = H_i(2)(\omega_2)$ and $H_i(2)(\omega_3) = H_i(2)(\omega_4)$.

Now consider the gains process in this model

$$
G_1(\omega) = \begin{cases}
rH_0(1) + 3H_1(1) & \text{if } \omega \in \{\omega_1, \omega_2\} \\
rH_0(1) - 3H_1(1) & \text{if } \omega \in \{\omega_3, \omega_4\}
\end{cases}
$$

For brevity, we write $H_n(t)$ instead of $H_n(t)(\omega)$

$$
G_2(\omega) = \begin{cases}
rH_0(1) + 3H_1(1) + r(1+r)H_0(2) + H_1(2) & \text{if } \omega = \omega_1 \\
rH_0(1) + 3H_1(1) + r(1+r)H_0(2) - 2H_1(2) & \text{if } \omega = \omega_2 \\
rH_0(1) - 3H_1(1) + r(1+r)H_0(2) + 4H_1(2) & \text{if } \omega = \omega_3 \\
rH_0(1) - 3H_1(1) + r(1+r)H_0(2) - 2H_1(2) & \text{if } \omega = \omega_4
\end{cases}
$$

For the *Trading Strategy* H to be *Self-Financing*, we must have that at time-point $t = 1$ in states ω_1, ω_2

$$
V_1 = (1+r)H_0(1) + 9H_1(1) = {}^{[34]} (1+r)H_0(2) + 9H_1(2)
$$

In states ω_3, ω_4 we require

$$
V_1 = (1+r)H_0(1) + 3H_1(1) = {}^{[35]} (1+r)H_0(2) + 3H_1(2)
$$

We consider possible arbitrage opportunities without considering the existence of martingale measures for the moment. If the interest rate is $r = 1/9$, then a possible trading strategy is as follows

---

[34] Condition for H to be *Self-Financing*.
[35] Condition for H to be *Self-Financing*.

- Start with an amount of £0.

- If $S_1 = 3$ do nothing at time points $t = 0, 1$.

- If $S_1 = 9$ at time point $t = 1$ short sell one share of the risky asset ($H_1(2) = -1$) and put the received proceeds into the bank account ($H_0(2) = 9/(1+r)$).

- Then, at time-point $t = 2$ the value of the portfolio is

$$V_2 = \begin{cases} (1+r)^2 H_0(2) + 10 H_1(2) = 10 - 10 = 0 & \text{if } \omega = \omega_1 \\ (1+r)^2 H_0(2) + 7 H_1(2) = 10 - 7 > 0 & \text{if } \omega = \omega_2 \\ 0 & \text{if } \omega \in \{\omega_3, \omega_4\} \end{cases}$$

If, on the other-hand, $r = 0$ then there are no arbitrage opportunities.

We now consider two ways to compute an equivalent martingale measure.

i). Recall that `Definition 3.8` states that

$$\mathbb{E}_{\mathbb{Q}}\left[\frac{S_n(t+s)}{B_{t+s}} \middle| \mathcal{F}_t \right] = \frac{S_n(t)}{B_t} \ \forall \ t, s \geq 0$$

Let $q_i := \mathbb{Q}(\{\omega_i\})$ for $i = 1, 2, 3, 4$. Further, we know that $q_1 + q_2 + q_3 + q_4 = 1$ must hold and from the above result we have that

$$\begin{array}{lll} (t = 0, s = 1) \implies & 6(1+r) & = \ 9(q_1 + 1_2) + 3(q_3 + q_4) \\ (t = 0, s = 2) \implies & 6(1+r)^2 & = \ 10q_1 + 7q_2 + 7q_3 + q_4 \\ (t = 1, s = 1) \implies & 9(1+r) & = \ (10q_1 + 7q_2)/(q_1 + q_2) \\ (t = 1, s = 1) \implies & 3(1+r) & = \ (7q_3 + q_4)/(q_3 + q_4) \end{array}$$

This gives five equations with four unknowns which could be solved to find $\mathbb{Q}$.

ii). We can make calculations quicker by considering conditional probabilities.

At time $t = 0$ we can obtain $p_1 := \mathbb{Q}(S_1 = 9)$ by solving

$$6(1+r) = 9p_1 + 3(1 - p_1)$$

Which produces that $\mathbb{Q}(S_1 = 9) = \dfrac{1 + 2r}{2}$ and $\mathbb{Q}(S_1 = 3) = \dfrac{1 - 2r}{2}$.

At time $t = 1$ is $\omega \in \{\omega_1, \omega_2\}$ we can write

$$p_2 := \mathbb{Q}(S_2 = 10 | S_1 = 9) \text{ and } 9(1+r) = 10p_2 + 7(1 - p_2)$$

This produces that $\mathbb{Q}(S_2 = 10 | S_1 = 9) = (2 + 9r)/3$ and $Q(S_2 = 7 | S_1 = 9) = (1 - 9r)/3$.

At time $t = 1$ if $\omega \in \{\omega_3, \omega_4\}$ we can solve

$$p_3 := \mathbb{Q}(S_2 = 7 | S_1 = 3) \text{ and } 3(1+r) = 7p_3 + (1 - p_3)$$

to find that $\mathbb{Q}(S_2 = 7 | S_1 = 3) = (2 + 3r)/6$ and $\mathbb{Q}(S_2 = 1 | S_1 = 3) = (4 - 3r)/6$.

By multiplying and conditional probabilities along the path leading to all four states in Ω, we obtain

$$\begin{array}{lll} \mathbb{Q}(\{\omega_1\}) & = & \dfrac{1 + 2r}{2} \cdot \dfrac{2 - 9r}{3} = \dfrac{1}{6}(1 + 2r)(2 + 9r) \\ \mathbb{Q}(\{\omega_2\}) & = & \dfrac{1 + 2r}{2} \cdot \dfrac{1 - 9r}{3} = \dfrac{1}{6}(1 + 2r)(1 - 9r) \\ \mathbb{Q}(\{\omega_3\}) & = & \dfrac{1}{12}(1 - 2r)(2 + 3r) \\ \mathbb{Q}(\{\omega_4\}) & = & \dfrac{1}{12}(1 - 2r)(4 - 3r) \end{array}$$

All these probabilities are greater than 0 if $r < 1/9$, thus we can conclude that a *Martingale Measure* exists iff $r \in [0, 1/9)$.

Definition 3.7 - *Arbitrage Opportunity - Muliti-Period Model*
An *Arbitrage Opportunity* in the case of a multi-period model is a *Trading Strategy H* st

i). $V_0 = 0$ - It starts with zero money.

ii). $V_T \geq 0$ - It does note loose money at the end.[36]

iii). $\mathbb{E}[V_T] > 0$ - It is expect that you always make money; And,

iv). H is *Self-Financing.*

Proposition 3.2 - *Self-Financing Strategies and Arbitrage Opportunity*
A *Self-Financing Strategy H* is an *Arbitrage Opportunity* <u>iff</u> all the following hold

$$G_T^* \geq 0 \quad \mathbb{E}[G_T^*] > 0 \quad V_0^* = 0$$

Definition 3.8 - *Martingale Measure* $\mathbb{Q}$
A *Martingale Measure* is a probability measure $\mathbb{Q}$ st[37]

i). $\mathbb{Q}(\{\omega\}) > 0 \ \forall \ \omega \in \Omega$; and,

ii). The *Discounted Price Process* S_n^* is a *Martingale* under $\mathbb{Q}$ for every $n = 1, \ldots, N$

$$\mathbb{E}_{\mathbb{Q}}\left[S_n^*(t+s)|\mathcal{F}_t\right] = S_n^*(t) \ \forall \ t, s \geq 0$$

More precisely, a *Martingale Measure* $\mathbb{Q}$ must satisfy

$$\mathbb{E}_{\mathbb{Q}}[S_n(t+s)/B_{t+s}|\mathcal{F}_t] = S_n(t) \ \forall \ t, s \geq 0$$

Theorem 3.1 - *Extension of No-Arbitrage Principle*
There are no *Arbitrage Opportunities* <u>iff</u> there exists a *Martingale Measure* $\mathbb{Q}$.[38]

Proposition 3.3 - *Finance Processes which are Martingales*[39]
If $\mathbb{Q}$ is a *Martingale Measure* and H is a *Self-Financing Strategy* then

i). The *Discounted Value Process* V^*, and

ii). The *Discounted Gains Process* G^*

are *Martingales* under $\mathbb{Q}$.

Proof 3.2 - `Theorem 3.3`
We have to show that $\mathbb{E}_{\mathbb{Q}}\left[V_{t+1}^*|\mathcal{F}_t\right] = V_t^* \ \forall \ t > 0$.
This is equivalent to showing that $\mathbb{E}_{\mathbb{Q}}[G_{t+1}^*|\mathcal{F}_t] = G_t^*$ using `Proposition 3.1`.

---

[36]It is possible for $V_t < 0$ for $t \in (0, T)$.

[37]This is similar to a *Risk-Neutral Probability Measure.*

[38]This is an extension of the "No-Arbitrage Principle" from the single-period model to the multi-period model.

[39]This proposition is useful for proving the Extension of No-Arbitrage Principle

Using the expression for $G_{t+1}^* - G_t^*$ derived in `Proof 3.1` we can conclude that

$$
\begin{aligned}
\mathbb{E}_\mathbb{Q}\left[G_{t+1}^* - G_t^*|\mathcal{F}_t\right] &= \mathbb{E}_\mathbb{Q}\left[\sum_{n=1}^N H_n(t+1)\Delta S_n^*(t+1)\Big|\mathcal{F}_t\right] \\
&= \sum_{n=1}^N \mathbb{E}_\mathbb{Q}\big[\underbrace{H_n(t+1)}_{\in \mathcal{F}_t}{}^{[40]}\Delta S_n^*(t+1)|\mathcal{F}_t\big] \\
&= \sum_{n=1}^N H_n(t+1)\mathbb{E}_\mathbb{Q}\left[\Delta S_n^*(t+1)|\mathcal{F}_t\right]
\end{aligned}
$$

The last result is due to H_n being a *Trading Strategy* and thus *Predictable*.

Furthermore, $\mathbb{E}_\mathbb{Q}\left[\Delta S_n^*(t+1)|\mathcal{F}_t\right] =$ for all $t > 0$ since S_n^* is a under $\mathbb{Q}$.

Hence G_t^*, V_t^* are *Martingales*. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ □

Proposition 3.4 - *Arbitrage Opportunities between Single-Period and Multi-Period Models*
If the multi-period model does <u>not</u> have any arbitrage opportunities, then all of the underlying single period model <u>have</u> arbitrage opportunities.

Proof 3.3 - *Proposition 3.4*
For each $t < T$ and for each $A \in \mathcal{P}_t$ there is one underlying single-period model where

- *Initial Time Discounted Price* is $S_n^*(t, \omega)$ for an arbitrary $\omega \in A$ since $S_n^*(t, \omega)$ are constant on A.

- *Sample Space* contains one state for each cell $A' \in \mathcal{P}_{t+1}$ st $A' \subset A$.

- *Terminal Time Discounted Price* is $S_n^*(t+1, \omega)$ for each $n = 1, \ldots, N$ for some $\omega \in A$.

If any underlying single-period model has an arbitrage opportunity in the single-period sense, then the multi-period model must have an arbitrage opportunity in the multi-period sense.

To see this, suppose there exists an *Arbitrage Opportunity* $\hat{H}$ for the single period model corresponding to some $A \in \mathcal{P}_t$ for $t < T$. This means that the discounted gain, given as

$$\hat{H}_1\Delta S_n^*(t+1) + \cdots + \hat{H}_N\Delta S_N^*(t+1)$$

, is *non-negative* and <u>not</u> identical to zero on the event A.

We now construct a multi-period *Trading Strategy* H which is an *Arbitrage Opportunity*.

$$
H_n(s, \omega) = \begin{cases}
0 & \text{if } s \leq t \text{ or } \omega \notin A \\
\hat{H}_n & \text{if } s = t+1, \ \omega \in A, \text{ and } n = 1, \ldots, N \\
-\sum_{i=1}^N \hat{H}_i S_i^*(t) & \text{if } s = t+1, \ \omega \in A \text{ and } n = 0 \\
\sum_{i=1}^N \hat{H}_1\Delta S_1^*(t+1) & \text{if } s > t+1, \ \omega \in A \text{ and } n = 0 \\
0 & \text{if } s > t+1, \ \text{ and } n = 0
\end{cases}
$$

This strategy starts with zero money and does nothing unless the event A occurs at time t, in which case at time t the position $\hat{H}_n$ is taken in the nth risky security while the position in the bank account is used to self-finance.

---

[40]This is due to $H_n(t+1)$ being the strategy we are building for time-step $t+1$ and thus we use all the information available in time-step t.

Subsequently, no position is taken in any of the risky securities, and non-zero value of the portfolio is reflected by a position in the bank account.

THis *Trading Strategy H* is an *Arbitrage Opportunity* in the multi-period model. □

Proof 3.4 - *Theorem 3.1*

⟵ We first show that there can be <u>no</u> *Arbitrage Opportunities* provided the existence of a *Martingale Measure* $\mathbb{Q}$.

Suppose H is any *Self-Finacing Trading Strategy* with

$$V_T^* \geq 0 \text{ and } \mathbb{E}[V_T^*] > 0$$

This implies
$$\mathbb{E}_{\mathbb{Q}}[V_T^*] > 0$$

Since V^* is a *Martingale* under $\mathbb{Q}$ by **Theorem 3.3**, it follows that

$$V_0^* = \mathbb{E}_{\mathbb{Q}}[V_T^*] > 0$$

Hence H cannot be an *Arbitrage Opportunity*, nor can any other trading strategies be *Arbitrage Opportunities* due to H being chosen arbitrarily.

⟶[41] Using **Proposition 3.4**, we have that for each $t < T$ and each $A \in \mathcal{P}_t$ there is a *risk-neutral probability measure* $\mathbb{Q}(t, A)$ for the underlying single-period model.

This probability measure gives positive mass to each cell $A' \in \mathcal{P}_{t+1}$ st it sums to 1 over all such cells and it satisfies

$$\mathbb{E}_{\mathbb{Q}(t,A)}[\Delta S_n^*(t+1)] = 0 \text{ for } n = 1, \dots, N$$

Notice that $\mathbb{Q}(t, A)$ puts probability on each branch in the information tree which emerges from the node corresponding to (t, A).

We can calculate a *Martingale Measure* $\mathbb{Q}$ for the multi-period model from these probabilities by setting $\mathbb{Q}(\{\omega\})$ equal to the product of the conditional probabilities along the path from the node at $t = 0$ to the node at (T, ω).

Then

- $\sum_{\omega \in \Omega} \mathbb{Q}(\{\omega\}) = 1$.
- $\mathbb{Q}(\{\omega\}) > 0$ for every $\omega \in \Omega$ because all the conditional risk neutral probabilities are strictly positive.
- And, $\mathbb{E}_{\mathbb{Q}}[S_n^*(t+1)|\mathcal{F}_t] = S_n^*(t)$ for all t and n.

Thus $\mathbb{Q}$ is indeed a *Martingale Measure*. □

3.3   Valuation of Contingent Claims

Definition 3.9 - *Contingent Claim*[42]
A *Contingent Claim* in the multi-period model is a random variable X which represents a payoff at the terminal time $t = T$.

A *Contingent Claim X* is said to be *Attainable*[43] if there exists a *Self-Financing Trading Strategy H* st $V_T(\omega) = X(\omega) \ \forall \ \omega \in \Omega$. In this case one says that H "generates" X and H is

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[42]This is an extension of **Definition 1.24** to the multi-period model.
[43]AKA *Marketable.*

called the *Replicating Portfolio*.

Theorem 3.2 - *Risk-Neutral Valuation Principle*[44]
The time t value of an *Attainable Contingent Claim* X is equal to V_t where V_t is the time t value of the portfolio which replicates X.

Moreover,

$$V_t^* = \frac{V_t}{B_t} = \mathbb{E}_{\mathbb{Q}}[X/B_t|\mathcal{F}_t] \text{ for } t = 0, \dots, T \text{ and all } \mathbb{Q}$$

Proof 3.5 - `Theorem 3.2`
Let P_t denote the actual price of the *Contingent Claim* at time point t.

By `Theorem 1.1`, $P_t = V_t$ is the only possibility to avoid arbitrage opportunities.

Now, let $\mathbb{Q}$ be an arbitrary *Martingale Measure* then for every $t < T$ we have that

$$V_t^* := \mathbb{E}_{\mathbb{Q}}[V_t^*|\mathcal{F}_t]$$

as V_t^* is a *Martingale* by `Proposition 3.3`.

Moreover, since V_T^* is the discounted value of the portfolio which replicated X we have that

$$\mathbb{E}_{\mathbb{Q}}[V_T^*|\mathcal{F}_t] = \mathbb{E}_{\mathbb{Q}}[X/B_T|\mathcal{F}_t]$$

Thus $V_t^* = \mathbb{E}_{\mathbb{Q}}[X/B_T|\mathcal{F}_t]$ independent of the choice of *Martingale Measure* $\mathbb{Q}$.

Remark 3.1 - *Computing a Generating a Trading Strategy H*
If we already know the *Value Process* $\{V_t\}_{t \in \mathbb{N}_0}$ for the replicating portfolio, we need to solve for the trading strategy H using the linear equations in the definition of the value process (keeping in mind that H is *Predictable*).

$$V_t(\omega_i) = H_0(t)B_t + \sum_{n=1}^{N} H_n(t)S_n(t)(\omega_i) \ \forall \ \omega_i \in \Omega$$

However, if we only know the *Contingent Claim* X we need to work backwards in time, deriving V and H simultaneously.

Since $V_T = X$ we must first solve the following for $H(T)$.

$$X(\omega_i) = H_0(T)B)T + \sum_{n=1}^{N} H_n(T)S_n(T)(\omega_i)$$

Since H is self-financing, we can calculate V_{T-1}.

$$V_{T-1} = H_0(T)B_{T-1} + \sum_{n=1}^{N} H_n(T)S_n(T-1)(\omega_i)$$

Therefore, our next step is to solve the following for $H(T_1)$.

$$V_{T-1} = H_0(T_1)B_{T-1} + \sum_{n=1}^{N} H_n(T-1)S_n(T-1)(\omega_i)$$

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[44]This is an extension of `Theorem 1.7` to the multi-period model.

We can now continue by calculating V_{T-2} etc. until we end up with V_0.

Example 3.2 - *Finding a Replicating Portfolio*

Consider a European call option with exercise price $e = 5$.

The corresponding contingent claim is $X = \{S_2 - 5\}_+$ and takes the values $\{5, 2, 2, 0\}$ in states $\{\omega_1, \omega_2, \omega_3, \omega_4\}$ respectively.

Suppose $r = 0$ then $\mathbb{Q} = (1/3, 1/6, 1/6, 1/3)$.

Since the martingale measure is unique, `Proposition 3.5` will show below that the model is complete and therefore every *Contingent Claim* is *Marketbale* (inc. the European option).

At time $t = 0$ the contingent claim's value must be

$$V_0 = \mathbb{E}_\mathbb{Q}[X] = (1/3) \cdot 5 + (1/6) \cdot 2 + (1/6) \cdot 2 + (1/3) \cdot 0 = 7/3$$

At time $t = 1$ the contingent claim's value in states ω_1, ω_2 is

$$V_1(\omega_1) = V_1(\omega_2) = \mathbb{E}_\mathbb{Q}[X|S_1 = 9] = (2/3) \cdot 5 + (1/3) \cdot 2 = 4$$

At time $t = 1$ the contingent claim's value in states ω_3, ω_4 is

$$V_1(\omega_3) = V_1(\omega_4) = \mathbb{E}_\mathbb{Q}[X|S_1 = 3] = (1/3) \cdot 2 + (2/3) \cdot 0 = 2/3$$

To find a replicating portfolio we start at time $t = 2$ and find that

$$\begin{aligned} V_2(\omega_1) &= 5 = H_0(2)(\omega_1, \omega_2) \cdot B_2 + H_1(2)(\omega_1, \omega_2) \cdot 10 \\ V_2(\omega_2) &= 2 = H_0(2)(\omega_1, \omega_2) \cdot B_2 + H_1(2)(\omega_1, \omega_2) \cdot 7 \end{aligned}$$

Noting that $B_2 = 1$, we obtain that $H_0(2)(\omega_1, \omega_2) = -5$ and $H_1(2)(\omega_1, \omega_2) = 1$.

Similarly

$$\begin{aligned} V_2(\omega_3) &= 2 = H_0(2)(\omega_3, \omega_4) \cdot 1 + H_1(2)(\omega_3, \omega_4) \cdot 7 \\ V_2(\omega_4) &= 0 = H_0(2)(\omega_3, \omega_4) \cdot 1 + H_1(2)(\omega_3, \omega_4) \cdot 1 \end{aligned}$$

and obtain that $H_0(2)(\omega_3, \omega_4) = -1/3$ and $H_1(2)(\omega_3, \omega_4) = 1/3$.

Now we have two ways to proceed

i). Since we have already calculated the value of the option at time-point $t = 1$ we can proceed as in the second period.

Writing $H_0(1)$ for $H_0(1)(\omega_1, \ldots, \omega_4)$ and $H_1(1)$ for $H_1(1)(\omega_1, \ldots, \omega_4)$, we have

$$\begin{aligned} V_1(\omega) &= 4 = H_0(1) \cdot B_1 + H_1(1) \cdot 9 \text{ for } \omega \in \{\omega_1, \omega_2\} \\ V_1(\omega) &= 2/3 = H_0(1) \cdot B_1 + H_1(1) \cdot 3 \text{ for } \omega \in \{\omega_3, \omega_4\} \end{aligned}$$

Solving these equations, and noting $B_1 = 1$, gives $H_0(1) = -1$ and $H_1(1) = 5/9$.

Note that this confirms our calculation for V_0, since this portfolio costs $-1 + 6(5/9) = 7/3$ at time $t = 0$.

ii). Suppose we had not already calculated the value of the option at time $t = 1$.

Using that H is *Self-Financing* we calculate the value of the portfolio at time-point $t = 1$.

So we assume that we have already calculated $H_0(2)$ and $H_1(2)$. Then

$$V_1 = H_0(2) \cdot 1 + H_1(2) \cdot 9 = -5 \cdot 1 + 1 \cdot 9 = 4 \text{ in } \omega_1, \omega_2$$

Similarly

$$V_1 = (-1/3) \cdot 1 + (1/3) \cdot 3 = 2/3 \text{ in } \omega_3, \omega_4$$

We proceed as above, obtaining

$$H_0(1) = -1 \quad H_1(1) = 5/9$$

Proposition 3.5 - *Complete Markets and Martingale Measures*
An arbitrage-free multi-period model is complete <u>iff</u> there exists a unique martingale measure
$\mathbb{Q}$.

Proof 3.6 - *Proposition 3.5*

- If the multi-period model is complete, for any claim X we can work backwards in time to
  compute the trading strategy that generates X. Hence, for each underlying single-period
  model the matrix A must have independent columns and the model is complete.

- Conversely, if every underlying single-period model is complete then the computational
  procedure for the multi-period model succeeds.

Therefore, completeness of the multi-period model is equivalent to completeness of all underlying
single-period models.

In particular the multi-period model is complete <u>iff</u> each underlying model has a unique risk-
neutral probability measure.

On the other hand, uniqueness of the martingale measure $\mathbb{Q}$ is equivalent to uniqueness of the
risk-neutral probability measures of the underlying single-period models.

Obviously, the existence of several risk-neutral probability measures for a single-period model
leads to several multi-period martingale measures.

However, assume there are two multi-period martingale measures. Then the conditional
probability must be different for at least one specific single-period model.   □

3.4   American Claims

Definition 3.10 - *American Claim Y_τ*
Let $\{Y_t\}_{t \in T}$ be a stochastic process describing the payoff of a financial derivative if exercised at
time t and let τ be the stopping time representing the exercise date in an *Exercise Strategy*.

Then Y_τ is called the *American Claim* wrt to Y_t and τ.[45]

Definition 3.11 - *Attainable American Claim*
The *American Claim Y_τ* is said to be *Attainable*[46] if there exists a *Self-Financing Trading
Strategy* H st the corresponding portfolio value V satisfies $V_\tau = Y_\tau$.

An *American Pay-off Process* $\{Y_t\}$ is said to be attainable if for every *Stopping Time τ* the
American Claim Y_τ is *Attainable*.

Proposition 3.6 - *Complete Markets & Attainable American Claims*
If a financial market is *Complete* then <u>every</u> *American Claim* is *Attainable*.

Proof 3.7 - *Proposition 3.6*
Let $\{Y_t\}_{t \in T}$ be a payoff-process and τ be an exercise strategy.

We have to find a self-financing trading strategy H st $V_\tau = Y_\tau$.

---

[45]Stopping times needs to be $\mathcal{F}_t$ measurable so we will know at time t whether we will exercise the *American
Claim*.
[46]AKA "Marketable"

Consider the *European Claim* $X = Y_\tau(B_T/B_\tau)$ which corresponds to someone exercising the *American Claim* Y at time-point $t = \tau$ and then earning interest from a bank-account until time-point T. Since the model is complete, there must be a replicating trading strategy H st $V_T = X = Y_\tau(B_T/B_\tau)$.

This portfolio which starts at time τ with the amount of Y_τ all of which is put into and kept in the bank account until time T, has the same value at time T as H.

We conclude that $V_\tau = Y_\tau$. $\hfill \square$

Definition 3.12 - *Snell Envelope*
Let $\{X_t\}_{t\in T}$ be a stochastic process adapted to some filtration $\mathcal{F}_t$.

Then the process $\{Z_t\}_{t\in T}$, defined below, is called the *Snell Envelope* of X.

$$Z_t = \begin{cases} X_T & \text{if } t = T \\ \max\{X_t, \mathbb{E}[Z_{t+1}|\mathcal{F}_t]\} & \text{if } t < T \end{cases}$$

Example 3.3 - *Snell Envelope*
Consider rolling a six-sided die three times. After the ith roll, you can either keep the score X_i on the die, or roll again. Whoever gets the highest score wins. What should your strategy be?

First, note that we have to keep the third score if we roll the die three times. Moreover, after the second roll we take

$$Z_2 = \max\{X_2, \mathbb{E}[Z_3|\mathcal{F}_2]\}^{[47]} = \max\{X_2, \mathbb{E}[X_3]\} = \max\{X_2, 7/2\}$$

This means we should stick with the second roll if it is greater than or equal to 4, otherwise re-roll. Thus

$$\mathbb{E}[Z_2] = \mathbb{E}[\max\{X_2, 7/2\}] = \frac{1}{6} \cdot \left(\frac{7}{2} + \frac{7}{2} + \frac{7}{2} + 4 + 5 + 6\right) = \frac{17}{4}$$

After the first roll, we take

$$Z_1 = \max\{X_1, \mathbb{E}[Z_2|\mathcal{F}_1] = \max\{X_1, \mathbb{E}[Z_2]\} = \max\left\{X_1, \frac{17}{4}\right\}$$

This means we should stick with the first roll if it is greater than or equal to 5, otherwise re-roll.

Theorem 3.3 - *Snell Envelope is the Smallest Super-Martingale*
The *Snell Envelope* $\{Z_t\}$ of X is the smallest *Super-Martingale* which dominates X.

(i.e. $Z_t \geq X_t \; \forall \; t$).

Proof 3.8 - *Theorem 3.3*
First, $Z_t \geq \mathbb{E}[Z_{t+1}|\mathcal{F}_t]$ and $Z_t \geq X_t$, so $\{Z_t\}$ is a super-martingale and dominates X.

Next, let $\{U_t\}_{t\in T}$ be any other supermartinale which dominates X. Since, by definition, $Z_T = X_T$ and U dominates X we must have $U_T \geq Z_T$. Assume inductively that $U_t \geq Z_t$. Then

$$\begin{aligned} U_{t-1} &\geq& \mathbb{E}[U_t|\mathcal{F}_{t-1}] \text{ since } U_t \text{ is a supermartingale} \\ &\geq& \mathbb{E}[Z_t|\mathcal{F}_{t-1}] \end{aligned}$$

---

[47]In this game $\mathcal{F}_t$ is not helpful as each roll is independent.

and U dominates X

$$U_{t-1} \geq X_{t-1}$$

Combining

$$U_{t-1} \geq \max\{X_{t-1}\mathbb{E}[Z_t|\mathcal{F}_{t-1}]$$

By repeating this argument we get $U_t \geq Z_t \ \forall \ t$. □

Proposition 3.7 - *Snell Envelope as Discounted Value Process*
Consider a financial market with a *Martingale Measure* $\mathbb{Q}$ and an attainable *American Payoff Process* $\{Y_t\}$.

Then the *Snell Envelope* $\{Z_t\}$ of the discounted payoff process $\{Y_t/B_t\}$ is the discounted value process for Y.

Proof 3.9 - `Proposition 3.7`
This proof uses the Optimal Stopping Theorem (`Theorem 3.4`).

Let p denote the time t price of the American claim wrt the payoff process (Y_t/B_t).

Suppose, first, that $p < Z_t$ then:

- We buy the option for p.

- If $\tau(t) = t$[48] then $Z_t = Y_t/B_t$ and so we can exercise the option immediately for $(Y_t/B_t) > p$ to make a riskless profit.

- If $\tau(t) > t$ we undertake the negative of the trading strategy that replicates $Y_{\tau(t)}/B_{\tau(t)}$, as we want to short sell. The price of the replicating strategy is $\mathbb{E}_Q[Y_{\tau(t)}/B_{\tau(t)}|\mathcal{F}_t]$ by the *Risk-Netural Valuation Principle* (`Theorem 3.2`), but by the *Optimal Stopping Theorem* (`Theorem 3.4`) this is equal to Z_t and so we can invest the different $Z_t - p$ in the bank account.

- Later at time $\tau(t)$ we exercise the option and liquidate the replicating portfolio at the same time. The amount we collect from the option seller is equal to our liability on the portfolio[49]. Meanwhile, we have $(Z_t - p) \cdot (B_{\tau(t)}/B_t) > 0$ in the bank account.

This shows that we make a riskless profit in any case.

Now consider the case where $p > Z_t$ then

- We sell the option for p.

Consider in detail the case where $\tau(t) > t$. Then

- We undertake the trading strategy that replicates $Y_{\tau(t)}/B_{\tau(t)}$. Again we find by using *Risk-Netural Valuation Principle* (`Theorem 3.2`) and the *Optimal Stopping Theorem* (`Theorem 3.4`) that the price of building up the portfolio is

$$Z_t = \mathbb{E}_{\mathbb{Q}}[Y_{\tau(t)/B_{\tau(t)}}|\mathcal{F}_t]$$

- Therefore there is a profit of $p - Z_t$ which we put in the bank account.

How we proceed will depend on when the buyer exercises the option.

---

[48]This is unlikely as the arbitrage opportunity is obvious.
[49]Meaning the cash flow at this time-period is net 0

i). If the buyer exercises the option at time-point $s \leq \tau(t)$ then

- We pay the buyer the payoff Y_s/B_s.
- We liquidate our portfolio and using risk-neutral valuation the value of our portfolio at time-point s is

$$\mathbb{E}_{\mathbb{Q}}[Y_{\tau(t)}/B_{\tau(t)}|\mathcal{F}_s]$$

Since $s \in [t, \tau(t)]$ we see from the definition of $\tau(t)$ that $\tau(t) = \tau(s)$.
Using this and the *Optimal Stopping Theorem* (`Theorem 3.4`) that the value of our portfolio at time s is

$$\mathbb{E}_{\mathbb{Q}}[Y_{\tau(t)}/B_{\tau(t)}|\mathcal{F}_s] = \mathbb{E}_{\mathbb{Q}}[Y_{\tau(s)}/B_{\tau(s)}|\mathcal{F}_s] = Z_s \geq (Y_s/B_s)$$

- Therefore all transaction at time-point s will only add to our portfolio and our total profit is strictly positive.

ii). If the buyer does not exercise by time $s = \tau(t)$ where $\tau(t) < T$ then:

- We repeat the process, undertaking the trading strategy that replicates $Y_{\tau(s+1)}/B_{\tau(s+1)}$.
- The value of the portfolio to be built up is equal to

$$\mathbb{E}_{\mathbb{Q}}[Y_{\tau(s+1)}/B_{\tau(s+1)}|\mathcal{F}_s] \leq \mathbb{E}_{\mathbb{Q}}[Y_{\tau(s)}/B_{\tau(s)}|\mathcal{F}_s] = Z_s$$

Therefore the change of the portfolio will only pay us some money which we put in the bank account.

- As before, if the option buyer exercises at some time $u \leq \tau(s+1)$ then the value of the portfolio will be enough to cover the payoff Y_u.

iii). If the buyer has not exercised by time $\tau(s+1)$ then we repeat this process again, and so forth. There will always be enough money in the portfolio to cover the payoff. Our overall profit will be at least $p - Z_t > 0/$

Finally, we consider the case where $\tau(t)$. Then the optimal strategy would be to exercise the option immediately.

- If the buyer indeed exercises at time t, then we pay them $(Y_t/B_t) = Z_t < p$ and make a riskless-profit.

- If note, then we proceed as in the previous case where the buyer does not exercise by the optimal stopping time and undertake the trading strategy which replicates $(Y_{\tau(t+1)}/B_{\tau(t+1)})$ and so forth making again profit of at least $p - Z_t$.

$\square$

Theorem 3.4 - *Optimal Stopping Theorem*
This is <u>*NOT*</u> *the "Optional Stopping Theorem"* (`Theorem 2.5`).

Let $\{X_t\}_{t\in T}$ be a stochastic process adapted to some *Filtration* $\mathcal{F}_t$ and Z_t be the *Snell Envelope* of $\{X_t\}_{t\in T}$.

For any $t = 0, \ldots, T$ we define a stopping time by $\tau(t) = \min_{s \geq t}\{Z_s = X_s\}$, then the optimal stopping rule is

$$Z_t = \mathbb{E}[X_{\tau(t)}|\mathcal{F}_t] = \max_t \{\mathbb{E}[X_\tau|\mathcal{F}_t] : \text{ all stopping times } t \leq \tau \leq T\} \text{ for all } t = 0, \ldots, T \quad (1)$$

In particular

$$Z_0 = \mathbb{E}[X_{\tau(0)}] = \max_t \left\{ \mathbb{E}[X_\tau)|\mathcal{F}_t] : \text{ all stopping times } \tau \leq T \right\}$$

Proof 3.10 - *Theorem 3.4*
This proof is a backwards induction through time.

Base Case

Note first that Eq. 1 is clearly true for $t = T$ because, by definition of the *Snell Envelope*, $Z_T = X_T$ and therefore the stopping time $\tau(T)$ stops at T.

Inducitve Step

Now assume that Eq. 1 is satisfied for some t, we now need to show it holds for $t - 1$. Let τ be an arbitrary stopping time between $t - 1$ and T.

Define another stopping time $\tau' = \max\{\tau, t\}$[50]. Then, since $\tau \geq t \implies \tau' = \tau$

$$
\begin{aligned}
\mathbb{E}[X_\tau|\mathcal{F}_{t-1}] &= \mathbb{E}[\mathbb{1}\{\tau = t - 1\}X_{t-1} + \mathbb{1}\{\tau > t - 1\}X_\tau|\mathcal{F}_{t-1}] \\
&= \mathbb{E}[\mathbb{1}\{\tau = t - 1\}X_{t-1} + \mathbb{1}\{\tau > t - 1\}X_{\tau'}|\mathcal{F}_{t-1}] \\
&= \mathbb{1}\{\tau = t - 1\}X_{t-1} + \mathbb{1}\{\tau > t - 1\}\mathbb{E}[X_{\tau'}|\mathcal{F}_{t-1}] \\
&= \mathbb{1}\{\tau = t - 1\}X_{t-1} + \mathbb{1}\{\tau > t - 1\}\mathbb{E}[\mathbb{E}[X_{\tau'}|\mathcal{F}_t]|\mathcal{F}_{t-1}] \text{ by Tower Law}
\end{aligned}
$$

Since τ' is a stopping time st $\tau' \in [t, T]$, we find that $\mathbb{E}[X_{\tau'}|\mathcal{F}_t] \leq Z_t$ because we have assumed Eq. 1 holds for t.

Using the definition of Z_{t-1} we see that

$$
\begin{aligned}
\mathbb{E}[X_\tau|\mathcal{F}_{t-1}] &\leq \mathbb{1}\{\tau = t - 1\}X_{t-1} + \mathbb{1}\{\tau > t - 1\}\mathbb{E}[Z_t|\mathcal{F}_{t-1}] \\
&\leq Z_{t-1}\text{by def. Snell Envelope}
\end{aligned}
$$

In the special case where $\tau = \tau(t - 1)$ we find that $\tau = (t - 1)$ stops in $t - 1$ <u>iff</u> $Z_{t-1} = X_{t-1}$, otherwise $Z_{t-1} > X_{t-1}$[51] and $\tau(t - 1) = \tau(t)$. Hence

$$
\begin{aligned}
\mathbb{E}[X_{\tau(t-1)}|\mathcal{F}_{t-1}] &= \mathbb{1}\{Z_{t-1} = X_{t-1}\}X_{t-1} + \mathbb{1}\{Z_{t-1} > X_{t-1}\}\mathbb{E}[\mathbb{E}[X_{\tau(t)}|\mathcal{F}_t]|\mathcal{F}_{t-1}] \\
&= \mathbb{1}\{Z_{t-1} = X_{t-1}\}X_{t-1} + \mathbb{1}\{Z_{t-1} > X_{t-1}\}\mathbb{E}[Z_t|\mathcal{F}_{t-1}] \\
&= Z_{t-1} \text{ by def. Snell Envelope}
\end{aligned}
$$

Example 3.4 - *Snell Envelope - #1*
Consider a market where there is a bank account with interest-rate $r = 0$ and a single security with the following price process

$$
S_t(\omega) = \quad
\begin{array}{c|ccc}
\omega \setminus t & 0 & 1 & 2 \\
\hline
\omega_1 & 6 & 9 & 10 \\
\omega_2 & 6 & 9 & 7 \\
\omega_3 & 6 & 3 & 7 \\
\omega_4 & 6 & 3 & 1 \\
\end{array}
$$

For this model, $\mathbb{Q}(\omega) := (1/3, 1/6, 1/6, 1/3)$ is a risk-neutral probability measure.

Consider an American call option with the exercise price of £5. Then its payoff process is

---

[50]This is similar to τ but can never take the value $t - 1$.
[51]As $\{Z_t\}$ dominates $\{X_t\}$.

$$Y_t(\omega) = \begin{array}{c|ccc} \omega \setminus t & 0 & 1 & 2 \\ \hline \omega_1 & 1 & 4 & 5 \\ \omega_2 & 1 & 4 & 2 \\ \omega_3 & 1 & 0 & 2 \\ \omega_4 & 1 & 0 & 0 \end{array}$$

In states ω_3, ω_4 the price

$$\begin{aligned} Z_1 &= \max\{Y_1, \mathbb{E}_\mathbb{Q}[Z_2|\mathcal{F}_1]\} \\ &= \max\{0, 2 \cdot (1/3) + 0 \cdot (2/3)\} \\ &= 2/3 \end{aligned}$$

This is equal to the European Option above $V_1(\omega_3, \omega_4)$.

In states ω_1, ω_2 the price

$$\begin{aligned} Z_1 &= \max\{Y_1, \mathbb{E}_\mathbb{Q}[Z_2|\mathcal{F}_1]\} \\ &= \max\{4, 5 \cdot (2/3) + 2 \cdot (1/3)\} \\ &= 4 \end{aligned}$$

This is equal to the European Option above $V_1(\omega_1, \omega_2)$.

so $Z_0 = \max\{1, (2/3) \cdot (1/2) + 4 \cdot (1/2)\} = 7/3$. The time $t = 0$ price of the European call option of

Example 3.5 - *Snell Envelope - #2*
Consider the same scenario as `Example 3.4` except that the payoff process of the American call option is

$$Y_t(\omega) = \begin{array}{c|ccc} \omega \setminus t & 0 & 1 & 2 \\ \hline \omega_1 & 1 & 5 & 5 \\ \omega_2 & 1 & 5 & 2 \\ \omega_3 & 1 & 0 & 2 \\ \omega_4 & 1 & 0 & 0 \end{array}$$

At time-point $t = 2$ it still holds that $Z_2 = Y_2$. Moreover, $\mathbb{E}_\mathbb{Q}[Z_2|\mathcal{F}_1] = \mathbb{E}_\mathbb{Q}[Y_2|\mathcal{F}_1]$ which is the same as `Example 3.4`.

So

$$\begin{aligned} \mathbb{E}_Q[Z_2|\mathcal{F}_1] &= \begin{cases} (2/3) \cdot 5 + (1/3) \cdot 2 & \text{if in } \omega_1, \omega_2 \\ (1/3) \cdot 2 + (2/3) \cdot 0 & \text{if in } \omega_3, \omega_4 \end{cases} \\ &= \begin{cases} 4 & \text{if in } \omega_1, \omega_2 \\ 2/3 & \text{if in } \omega_3, \omega_4 \end{cases} \end{aligned}$$

Hence

$$\begin{aligned} Z_1 &= \max\{Y_1 \mathbb{E}_\mathbb{Q}[Z_2|\mathcal{F}_1] \\ &= \begin{cases} \max\{4, 5\} = 5 & \text{if in } \omega_1, \omega_2 \\ \max\{0, 2/3\} = 5 & \text{if in } \omega_3, \omega_4 \end{cases} \end{aligned}$$

For time $t = 0$ we have

$$\begin{aligned} \mathbb{E}_\mathbb{Q}[Z_1|\mathcal{F}_0] &= \mathbb{E}_\mathbb{Q}[Z_1] \\ &= (1/2) \cdot 5 + (1/2) \cdot (2/3) \\ &= 17/6 \end{aligned}$$

in which case $Z_0 = \max\{Y_0, \mathbb{E}_\mathbb{Q}[Z_1]\} = 17/6$.

The optimal exercise strategies are

$$\tau(0)(\omega) = \tau(1)(\omega) = \begin{cases} 1 & \text{if in } \omega_1, \omega_2 \\ 2 & \text{if in } \omega_3, \omega_4 \end{cases}$$

and, of course, $\tau(2)(\omega) = 2 \ \forall \ \omega \in \Omega$.

Finally, we compute a replicating trading strategy for $\tau(0)$.

In states ω_3, ω_4 the stopping time $\tau(0)$ stops in time 2 so the portfolio in the second period must satisfy

$$\begin{aligned} H_0(2) + 7 \cdot H_1(2) &= Y_2(\omega_3) = 2 \\ H_0(2) + 1 \cdot H_1(2) &= Y_2(\omega_4) = 0 \end{aligned}$$

Let $H_i(2) := H_i(2)(\omega_3, \omega_4)$.

Then the solution is that $H_0(2) = -1/3$ and $H_1(2) = 1/3$ in states ω_3, ω_4.

The value of this portfolio at time $t = 1$ is $-(1/3) + 3 \cdot (1/3) = 2/3$.

The values of $H_0(2)$ and $H_1(2)$ in states ω_1, ω_2 do not matter.

In the first period, the portfolio must satisfy

$$\begin{aligned} H_0(1) + 9H_1(1) &= Z_1(\omega_1) = Z_1(\omega_2) = 5 \\ H_0(1) + 3H_1(1) &= Z_1(\omega_3) = Z_1(\omega_4) = 2/3 \end{aligned}$$

which has solutions $H_0(1) = -3/2$ and $H_1(1) = 13/18$.

The American option is attainable with time $t = 0$ value of $1 \cdot (-3/2) + 6 \cdot (13/18) = 17/6$ which confirms previous calculations[52].

3.5   The Cox-Ross-Rubinstein Model

Definition 3.13 - *Bernoulli Process*
Let $\{X_t : t = 1, 2, \dots\}$, $\{N_t : t = 1, 2, \dots\}$ be a *Stochastic Process*.

$\{X_t\}_t$ is a *Bernoulli Process* with parameter p if the random variable $X_1, X_2, \dots$ are independent and

$$\mathbb{P}(X_t = 1) = 1 - \mathbb{P}(X_t = 0) = p \ \forall \ t$$

$\{N_t\}_t$ is a *Random Walk Process* when $N_t := X_1 + \dots + X_t$ and $\{X_t\}_t$ is a *Bernoulli Process*.

Definition 3.14 - *Cox-Ross-Rubinstein Model*[53]
The *Cox-Ross Rubinstein Model*[54] consists of a bank account with constant interest rate $r > 0$ and <u>one</u> risky asset with price process S_t defined by

$$S_t = S_0 u^{N_t} d^{t - N_t} \text{ for } t = 1, \dots, T$$

Here $N_t = X_1 + \dots + X_t$ is the *Random Walk Process* associated with a *Bernoulli Process* X_t associated with a *Bernoulli Process* X_t with parameter $p \in (0, 1)$; u is the ratio of the price after it has increases[55]; d is the ratio of the price after it has decreased; and S_0 is the start price.

---

[52]Showing this is a replicating portfolio.

[53]AKA "Binomial Model"

[54]This simplifies a special case of the general multi-period model where prices move up and down by a fixed ratio and moves occur with fixed probability, wrt t.

[55]$u = S_{t+1}/S_t$ when $S_{t+1} > S_t$. u is constant for all t.

Note u, d, S_0 are all positive real values and in well-defined models $0 < d < 1 < u$.

Remark 3.2 - *Events in Cox-Ross-Rubinstein Model*
In each time-period, two events are possible under the *Cox-Ross-Rubinstein Model*

   i). Price increases by a factor of u. This occurs with probabilty p.

   ii). Price decreases by a factor of d. This occurs with probabilty $1 - p$.

Remark 3.3 - *Logarithms and Cox-Ross-Rubinstein Model*
Due to the multiplicative factors in the price process of a *Cox-Ross-Rubinstein Model* we often think in terms of the logarithm of the price.

Moreover, the logarithm of the price process S_t behaves like an asymmetric random walk with drift.

Theorem 3.5 - *Arbitrage in Cox-Ross-Rubinstein Model*
The *Cox-Ross-Rubinstein Model* has no arbitrage opportunities <u>iff</u> $d < 1 + r < u$.

In this case the model is complete and the unique *Martingale Measure* $\mathbb{Q}$ is defined by

$$\mathbb{Q}(\{\omega\}) = q^n (1 - q)^{T-n}$$

where $\omega \in \Omega$ is a state which corresponds to n upwards steps and $(T - n)$ downwards steps; and

$$q = \frac{1 + r - d}{u - d}$$

In particular, adding all contributions from states with exactly n upwards steps and $(T - n)$ downwards steps we get

$$\mathbb{Q}\left(S_t = S_0 u^n d^{T-n}\right) = \binom{t}{n} q^n (1 - q)^{t-n} \text{ for } n = 0, \ldots, t$$

Moreover, if X is a *Contingent Claim* of the form $X = g(S_T)$ for some real-valued function g, then the time $t = 0$ value of X is given by

$$V_0 = (1 + r)^{-T} \sum_{n=0}^{T} \binom{T}{n} q^n (1 - q)^{T-n} g(S_0 u^n d^{T-n})$$

More generally, the value of contingent claim X at time-point t is

$$\Pi(t) = \frac{1}{(1 + r)^{T-t}} \sum_{n=0}^{T-t} \binom{T-t}{n} q^n (1 - q)^{T-t-n} g(S_t U^n d^{T-t-n})$$

Proof 3.11 - `Theorem 3.5`
The *No-Arbitrage Theorem* (`Theorem 3.1`) shows the *Cox-Ross-Rubinstein Model* is free of arbitrage <u>iff</u> there is a *Martingale Measure* $\mathbb{Q}$ with

$$S_t = \frac{1}{1 + r} \mathbb{E}_{\mathbb{Q}}[S_{t+1}|\mathcal{F}_t] \ \forall \ t < T$$

Let $q := \mathbb{Q}(X_{t+1} = 1) = \mathbb{Q}(S_{t+1}/S_t = u)$ which means that $\mathbb{Q}(S_{t+1}/S_t = d) = 1 - q$.

S_t is known at time-point t, so we can put it inside the conditional expectation wrt $\mathcal{F}_t$ in the expression for S_t above. This means we can divide by S_t and get

$$1 = \frac{1}{1+r}\mathbb{E}_{\mathbb{Q}}\left[S_{t+1}/S_t | \mathcal{F}_t\right] = \frac{1}{1+r}\left(uq + d(1-q)\right)$$

By rearranging, we see the *No-Arbitrage Principle* is satisfied <u>iff</u> $qu + (1-q)d = 1+r$. Thus

$$q = \frac{1+r-d}{u-d}$$

By its definition, $\mathbb{Q}$ must be positive everywhere meaning $q \in (0,1)$. Equivalently $d < 1+r < u$ must be satisfied.

We can now state the form of the *Martingale Measure* $\mathbb{Q}$ by multiplying the conditional probabilities along the paths that lead to each state ω_i. This is made easier since the *Cox-Ross-Rubinstein Model* assumes that the probability of each movement is constant at all time-period.

By the uniqueness of q we can determine that the *Cox-Ross-Rubinstein Model* is complete.

The expression for $\mathbb{Q}\left(S_t = S_0 u^n d^{T-n}\right)$ is straightforward.

The time $t = 0$ value of the contingent claim X is calculated as the expectation wrt $\mathbb{Q}$.

We think of $\Pi(t)$ as a time-shifted version of the formula for the time $t = 0$ price[56], replacing T by $(T - t)$ and S_0 by S_t. $\square$

Example 3.6 - *Cox-Ross-Rubinstein Model*
Consider a *Cox-Ross-Rubinstein Model* with $T = 3$ periods, initial price $S_0 = 8$, interest rate $r = 0.1$ and parameters $u - 2, d = 1/2$[57]

No arbitrage opportunities exist since $d = 0.5 < 1.1 = 1 + r < 2 = u$.

We can calculate $q = (1 + r - d)/(u - d) = (1.1 - 0.5)/(1.5) = 0.4$.

After 3 periods, the possible values of S_T are $1, 4, 16, 64$.

In each case, the value of a *European Call* with exercise price 4 is $60, 12, 0, 0$.

The time $t = 0$ price of this European call option is therefore

$$
\begin{aligned}
\Pi_K(0) &= \frac{1}{(1+r)^T}\sum_{n=0}^{T}\binom{T}{n}q^n(1-q)^{T-n}(S_0 u^n d^{T-n} - k) \\
&= \frac{1}{1.1^3}\sum_{n=2}^{3}\binom{3}{n}q^n(1-q)^{3-n}(S_0 u^n d^{3-n} - 4) \quad [58] \\
&= \frac{1}{1.1^3}\left(60 \cdot 1 \cdot q^3(1-q)^0 + 12 \cdot 3 \cdot q^2(1-q)\right) \\
&= 5.4816
\end{aligned}
$$

Proposition 3.8 - *Value of European Call Option in Cox-Ross-Rubinstein Model*
Consider a *Cox-Ross-Rubinstein Model* with T periods, starting price S_0, interest rate r and parameters d, u.

Then the time t price of a *European Call Option* with exercise price K is

$$\Pi_K(t) = \frac{1}{(1+r)^{T-t}}\sum_{n=0}^{T-t}\binom{(T-t)}{n}q^n(1-q)^{T-t-n}\left\{S_0 u^n d^{T-t-n} - K\right\}_+$$

where $q = \frac{1+r-d}{u-d}$ as usual.

---

[56]Effectively assume that the model starts at time t and runs for $T - t$ steps.
[57]Intuitively this means the price doubles or halves each time-period.
[58]The sum starts at $n = 2$ as the payoff is zero if $n < 2$.

3.6   The Cox-Ross-Rubinstein Model and the Black-Scholes Formula

Remark 3.4 - *Motivation*
Consider a market where we observe two assets in a continuous time-interval $[0, U]$:

- A riskless bond;

- And, a stock.

We can try to approximate them using the *Cox-Ross-Rubinstein Model* with T periods, where trading is only possible at the following $T + 1$ time-points

$$0, \frac{1 \cdot U}{T}, \frac{2 \cdot U}{T}, \ldots, \frac{(T-1) \cdot U}{T}, U$$

Since we assume continuous compounding componding we define the interest rate of the *Cox-Ross-Rubinstein Model* as $r_T = e^{\frac{rU}{T}} - 1$, at time point $\frac{jU}{T}$, in the discrete model the bank account process has value

$$(1 + r_T)^j = e^{jrU}T$$

In the special case where $u_T = e^{\sigma\sqrt{U/T}} = 1/d_T$ we obtain the *Black-Scholes Formula* as the limit of the prices of a *European Call Option* in our *Cox-Ross-Rubinstein Models* if $T \to \infty$.

Theorem 3.6 - *Binomial Distribution and Standard Normal*
Let $Y_n \sim \text{Binomial}(n, \pi_n)$. Then

$$\tilde{Y}_n := \frac{Y_n - n\pi_n}{\sqrt{n\pi_n(1 - \pi_n)}}$$

converges in distribution to the standard normal distribution as $n \to \infty$.

Theorem 3.7 - *Black-Scholes Formula*
Consider a *European Call Option* with exercise price K and matures at time U.

Let $\Pi_K^{(t)}(0)$ denote its fair-price in a *Cox-Ross-Rubinstein Model* with $T + 1$ time-points $\{0, \frac{U}{T}, \ldots, U\}$, constant interest rate $r_T = e^{-\frac{rU}{T}} - 1$ and $u_T = e^{\sigma\sqrt{U/T}} = \frac{1}{d_T}$. Then

$$\lim_{T \to \infty} \Pi_K^{(T)}(0) = \Pi_K^{BS}(0)$$

where

$$\Pi_K^{BS}(0) = S_0\Phi\big(d_1(S_0, U)\big) - Ke^{-rU}\big(d_2(S_0, U)\big)$$
$$\text{with}$$
$$d_1(s, u) = \frac{\ln(s/K) + \big(r + (\sigma^2/2)\big)U}{\sigma\sqrt{U}}$$
$$d_2(s, u) = \frac{\ln(s/K) + \big(r - (\sigma^2/2)\big)U}{\sigma\sqrt{U}}$$

Φ is the CDF of the standard Normal distribution.

Proof 3.12 - *Theorem 3.7*
Let $\alpha_T := \min\left\{n : S_0u_T^n d_T^{T-n} > l\right\}$. This allows us to consider only terms in `Proposition 3.8` which are positive.

We rewrite the time $t = 0$ price $\Pi_K^{(T)}(0)$ in the Tth *Cox–Ross–Rubinstein Model* as

$$
\begin{aligned}
\Pi_K^{(T)}(0) &= (1 + r_T)^{-T} \sum_{n=\alpha_T}^{T} \binom{T}{n} q_T^n (1 - q_T)^{T-n} (S_0 u_T^n d_T^{T-n} - K) \\
&= S_0 \left(\sum_{n=\alpha_T}^{T} \binom{T}{n} \left(\frac{q_T u_T}{1 + r_T} \right)^n \left(\frac{(1 - q_T) d_T}{1 + r_T} \right)^{T-n} \right) \\
&\quad - (1 + r_T)^{-T} K \left(\sum_{n=\alpha_T}^{T} \binom{T}{n} q_T^n (1 - q_T)^{T-n} \right)
\end{aligned}
\qquad [1]
$$

We can identify terms involved in the second sum as the density of a $\text{Bin}(T, q_T)$ distribution. For notational ease we define

$$
\hat{q}_T = \frac{q_T u_T}{1 + r_T}
$$

This $\hat{q}_T$ is a probability[59] because

$$
\begin{aligned}
0 < \hat{q}_T \\
&= \frac{q_T u_T}{1 + r_T} \\
&= \frac{\left(\frac{1 + r_T - d_T}{u_T - d_T} \right) \cdot u_T}{q + r_T} \\
&= \frac{u_T - \frac{d_T u_T}{1 + r_T}}{u_T - d_T} \\
&< \frac{u_T - \frac{d_T u_T}{u_T}}{u_T - d_T} \quad \text{since } u_T > 1 + r_T \\
&= 1
\end{aligned}
$$

Moreover, we see from the definition of q_T that

$$
\begin{aligned}
1 - \hat{q}_T &= \frac{1 + r_T - q_T u_T}{1 + r_T} \\
&= \frac{1 + r_T - (1 + r_T - d_T) - q_T d_T}{1 + r_T} \\
&= \frac{(1 - q_T) d_T}{1 + r_T}
\end{aligned}
$$

Thus we identify the first sum in [1] as the density of $Bin(T, \hat{q}_T)$ distribution.

Now, let $Y_T \sim \text{Bin}(T, q_T)$ and $\hat{Y}_T \sim \text{Bin}(T, \hat{q}_T)$. We can rewrite [1] as

$$
\Pi_K^{(T)}(0) = S_0 \mathbb{P} \left(\hat{Y} > d_T - 1 \right) - K(1 + r_T)^{-T} \mathbb{P}(Y_T > d_T - 1)
$$

Thus we need to show that

i). $\displaystyle\lim_{T \to \infty} \mathbb{P} \left(\hat{Y}_T > \alpha_T - 1 \right) = \Phi \left(d_1(S_0, U) \right).$ **Proof 3.13 -** *This result is not proved in detail*

ii). $\displaystyle\lim_{T \to \infty} \mathbb{P} \left(Y_T > \alpha_T - 1 \right) = \Phi \left(d_2(S_0, U) \right).$

Consider ii) first

$$
\mathbb{P}(Y_T > \alpha_T - 1) = \mathbb{P} \left(\frac{Y_T - T q_t}{\sqrt{T q_T (1 - q_T)}} > \frac{d_T - 1 - T q_T}{\sqrt{T q_T (1 - q_T)}} \right)
$$

---

[59]ie $\hat{q}_T \in (0, 1)$

We want to use `Theorem 3.6` and therefore determine the convergence of the term $\frac{\alpha_T - 1 - Tq_T}{\sqrt{Tq_T(1 - q_T)}}$.

Note that we defined

$$u_T := \exp\{\sigma\sqrt{U/T}\} = 1/d_T \quad \text{and} \quad r_T = \exp\{rU/T\} - 1$$

Using a *Taylor Decomposition* we see that

$$
\begin{aligned}
q_T &= \frac{1 + r_T - d_T}{u_T - d_T} \\
&= \frac{\exp\{rU/T\} - e^{-\sqrt{U/T}}}{e^{\sigma\sqrt{U/T}} - e^{-\sigma\sqrt{U/T}}} \\
&= \frac{\{1 + (rU/T) + o(1/T)\} + \{-1\sigma\sqrt{U/T} - \frac{1}{2}\sigma^2(U/T) - o(1/T)\}}{\{1 + (\sigma\sqrt{U/T}) + \frac{1}{2}\sigma^2(U/T) + o(1/T)\} + \{-1 + \sigma\sqrt{U/T} - \frac{1}{2}\sigma^2(U/T) - o(1/T)\}} \\
&= \frac{(rU/T) + \sigma\sqrt{U/T} - \frac{1}{2}\sigma^2(U/T) + o(1/T)}{2\sigma\sqrt{U/T} + o(1/T)} \\
&= \frac{1}{2}\left\{\frac{r(U/T)}{\sigma\sqrt{U/T}} + \frac{\sigma\sqrt{U/T}}{\sigma\sqrt{U/T}} - \frac{(1/2)\sigma^2(U/T)}{\sigma\sqrt{U/T}}\right\} + o(1/\sqrt{T}) \\
&= \frac{1}{2}\left(\left(\frac{r}{\sigma} - \frac{\sigma}{2}\right)\sqrt{U/T} + 1\right) + o(1/\sqrt{T})
\end{aligned}
$$

Thus we obtain the limiting relations

$$\lim_{T\to\infty} q_T = \frac{1}{2} \quad \text{and} \quad \lim_{T\to\infty}(1 - 2q_T)\sqrt{UT} = -U\left(\frac{r}{\sigma} - \frac{\sigma}{2}\right)$$

Finally, we see from the definition of α_T that for some $|\gamma_T| < 1$

$$
\begin{aligned}
\alpha_T &= \frac{\ln(K/S_0 d_T^T)}{\ln(u_T/d_T)} + \gamma_T \\
&= \frac{\ln(k/S_0) - T\ln(d_T)}{\ln(u_T^2)} + \gamma_T \\
&= \frac{\ln(K/S_0) + T\sigma\sqrt{U/T}}{2\sigma\sqrt{U/T}} + \gamma_T
\end{aligned}
$$

Hence

$$
\begin{aligned}
&\lim_{T\to\infty} \frac{\alpha_T - 1 - Tq_T}{\sqrt{Tq_T(1 - q_T)}} \\
&= \lim_{T\to\infty} \frac{\left(\frac{\ln(K/S_0) + T\sigma\sqrt{U/T}}{2\sigma\sqrt{U/T}} + \gamma_T\right) - 1 - Tq_T}{\sqrt{Tq_T(1 - q_T)}} \\
&= \lim_{T\to\infty} \frac{\ln(K/S_0) + \sigma\sqrt{U_T}(1 - 2q_T)}{2\sigma\sqrt{Uq_T(1 - q_T)}} {}_{[60]} \\
&= \frac{\ln(K/S_0) - (r - \sigma^2/2)U}{\sigma\sqrt{U}} \\
&= -d_2(S_0, U)
\end{aligned}
$$

By `Theorem 3.6` we conclude that

$$\lim_{T\to\infty} \mathbb{P}(Y_T > \alpha_T - 1) = 1 - \Phi(-d_2(S_0, U)) = \Phi(d_2(S_0, u))$$

---

[60]γ_T disappears due to taking the limit.

This proves result ii).

To prove result i) a very similar argument is used.

Using the limiting relations

$$\lim_{T \to \infty} \hat{q}_T = \frac{1}{2} \text{ and } \lim_{T \to \infty} (1 - 2\hat{q}_T)\sqrt{UT} = -U\left(\frac{r}{\sigma} + \frac{\sigma}{2}\right)$$

we can conlude that

$$\lim_{T \to \infty} \frac{\alpha_T - 1 - T\hat{q}_T}{\sqrt{T\hat{q}_T(1 - \hat{q}_T)}} = \frac{\ln(k/S_0)0(r + \sigma^2/2)U}{\sigma\sqrt{U}} = -d_1(S_0, U)$$

Hence

$$\lim_{T \to \infty} \mathbb{P}(\hat{Y}_T > \alpha_T - 1) = 1 - \Phi(-d_1(S_0, U)) = \Phi(d_1(S_0, U))$$

$\square$

4 Stochastic Processes in Continuous Time

4.1 The Brownian Motion

Definition 4.1 - *Stochastic Process*
A *Stochastic Process* $X := \{X_t\}_{t \geq 0}$ is a family of random variables $X_t : \Omega \to \mathbb{R}$.

For a given $\omega \in \Omega$ and time-point t, the value $X(t, \omega)$ is called the *Trajectory* of the process X.
We say X is *Adapted* to a *Filtration* $\{\mathcal{F}_t\}_{t \geq 0}$, if X_t is $\mathcal{F}_t$-measurable for all t.

Definition 4.2 - *Martingale*
A *Stochastic Process* $X := \{X_t\}_{t \geq 0}$ is a *Martingale* wrt *Filtration* $\mathcal{F} := \{\mathcal{F}_t\}_{t \geq 0}$ if

i). X is adapted to the filtration $\mathcal{F}$.

ii). $\mathbb{E}[|X_t|] < \infty \ \forall \ t \geq 0$.

iii). $\mathbb{E}[X_t | \mathcal{F}_s] = X_s \ \forall \ s \in [0, t]$.

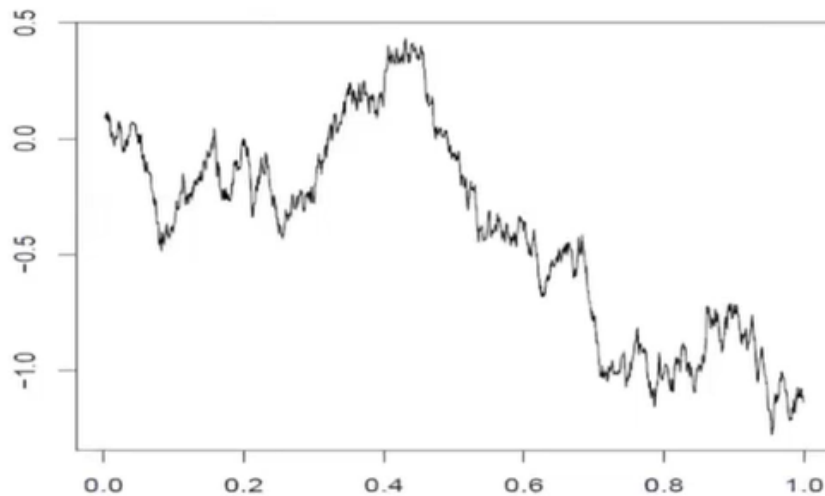The definitions for *Super-* and *Sub-Martingales* are similar.

Figure 4: An example of Brownian Motion.

Remark 4.1 - *Relevance Brownian Motion*

Brownian Motion can be used to construct a good model for prices (See `Figure 4`). Brownian motion is not smooth but generally shows a particular trend/drift.

Definition 4.3 - $S_t^{(n)}$

Let $\{S_j\}_{j \in \mathbb{N}}$ be a *Random Walk* defined by $S_0 = 0$ and $S_n = S_{n-1} + X_n$ where $X_1, X_2, \ldots$ are IID random variables with mean 0 and variance 1.

Define a process $\{S_t^{(n)}\}_{t \in [0,1]}$ on $[0,1]$ by setting $S_t^{(n)} = S_j/\sqrt{n}$ for every $t = j/n$ and linearly interpolate in between.

Proposition 4.1 - *Properties of $S_t^{(n)}$*

Here are some properties of $\{S_t^{(n)}\}_{t \in [0,1]}$

i). $S_0^{(n)} = S_0/\sqrt{n} = 0$.

ii). Taking $t = j/n$ and $u = k/n$ gives

$$S_{t+u}^{(n)} - S_t^{(n)} = (S_{j+k} - S_j)/\sqrt{n} = (X_{j+1} + \cdots + X_{j+k})/\sqrt{n}$$

As the Xs are independent, this shows that the change in value of a given period is independent of both the start and end points.

iii). Taking $t = j/n$ and $u = k/n$, for large n. Consider this expression for the change in value over a time period

$$S_{t+u}^{(n)} - S_t^{(n)} = \frac{X_{j+1} + \cdots + X_{j+k}}{\sqrt{n}} = \frac{\sqrt{k}}{\sqrt{n}} \left(\sum_{i=j+1}^{j+k} \frac{X_i}{\sqrt{k}} \right)$$

By the *Central Limit Theorem* this tends to $\text{Normal}(0, k/n) = \text{Normal}(0, u)$.

iv). $S_t^{(n)}(\omega)$ is continuous as a function of t, for all n and all $\omega \in \Omega$.

v). As $n \to \infty$, $S_t^{(n)}$ tends to a process W_t known as *Brownian Motion*.

Theorem 4.1 - *Theorem of Donsher*[61]

NON-EXAMINABLE

For every bounded and continuous function $h : C(0,1) \to \mathbb{R}$

$$\mathbb{E}\left[h(\{S_t^{(n)}\}_{t \in [0,1]}) \right] \longrightarrow \mathbb{E}\left[h(\{W_t\}_{t \in [0,1]}) \right]$$

Definition 4.4 - *Brownian Motion*[62]

Let $\mathcal{F}_t$ be a *Filtration*.

An *Adapted Stochastic Process* $W := \{W_t\}_{t \geq 0}$ is a standard one-dimensional *Brownian Motion* if

i). $W_0 = 0$ (Almost surely)

---

[61]AKA *Functional Central Limit Theorem*
[62]AKA *Wiener Process*

ii). W has independent increments. (i.e. $W_{t+u} - W_t$ is independent of $\mathcal{F}_t$ for all $u, t \geq 0$.)

iii). W has stationary Gaussian increments. (i.e. $W_{t+u} - W_t$ is normally distributed with mean 0 and variance u.)

iv). W has continuous paths. ($W_t(\omega)$ is a continuous function of t for all $\omega \in \Omega$.)

Theorem 4.2 - *Wiener*
The existence of *Brownian Motion* was proved by *Wiener* in 1923. That's the theorem.

Proposition 4.2 - *Properties of Standard Brownian Motion*
Let W_t be *Standard Brownian Motion*. Then

i). $\mathbb{E}[X_t] = 0$, $\operatorname{Var}(W_t) = t \ \forall \ t$.

ii). $\forall \ s, t \ \operatorname{Cov}(W_s, W_t) = \min\{s, t\}$.

iii). $-W_t$ is *Standard Brownian Motion*.

iv). For fixed t, $X_s := W_{s+t} - W_t$ is *Standard Brownian Motion*. (i.e. You can move the start point).

v). For any α, $y_s := W_{\alpha s}/\sqrt{\alpha}$ is *Standard Brownian Motion*. (i.e. If you zoom in to see a day, rather than a week, the same properties hold).

Proof 4.1 - *Theorem 4.2 i)-v)*

i). Follows from properties i) & iii) of `Definition 4.4`.

ii). By the previous result we can conclude that

$$
\begin{aligned}
\operatorname{Cov}(W_s, W_t) &= \mathbb{E}[W_s \cdot W_t] \\
&= \mathbb{E}[W_s \cdot (W_t - W_s)] + \mathbb{E}[W_s^2] \\
&= \mathbb{E}[W_s \cdot (W_t - W_s)] + \underbrace{\operatorname{Var}(W_s)}_{=s} \\
&= \underbrace{\mathbb{E}[W_s]}_{=0} \mathbb{E}[W_t - W_s] + s \\
&= s
\end{aligned}
$$

iii). $\mathbb{E}[-W_t] = -\mathbb{E}[W_t] = 0$ and increments occur in the t-direction, not the X-direction, so the distribution of $W_{t+u} - W_t$ is unaffected.

iv). In the case $t = 0$ we find that $X_0 = W_t - W_t = 0$. The other properties in `Defintion 4.4` are shift-invariant and therefore follow as well.

v). The main part to check here is propert iii) of `Defintion 4.4`. Indeed

$$
\begin{aligned}
Y_{t+s} - Y_t &= \underbrace{(W_{\alpha(t+s)} - W_{\alpha t})}_{\sim \operatorname{Nor}(0, \alpha s)}/\sqrt{\alpha} \\
&\sim \ \operatorname{Normal}(0, s)
\end{aligned}
$$

$\square$

Proposition 4.3 - *More properties of Brownian Motion*
Let $\{W_t\}_{t \geq 0}$ be *Standard Brownian Motion* wrt *Filtration* $\{\mathcal{F}_t\}$. Then

i). $\{W_t\}_{t\geq 0}$ is a *Martingale*.

ii). $\{W_t^2 - t\}_{t\geq 0}$ is a *Martingale*.

iii). $\forall\, \lambda \in \mathbb{R}$, the process $\{Z_t\}_{t\geq 0}$ defined as

$$Z_t = \exp\left\{\lambda W_t - \frac{1}{2}\lambda^2 t\right\}$$

is a *Martingale*.

More generally, the *Geometric Brownian Motion* with volatility $\sigma > 0$ and drift $a \in \mathbb{R}$ can be defined as

$$\tilde{Z}_t = \exp\{\sigma W_t + at\}$$

and is a *Martingale* <u>iff</u> $a = -\sigma^2/2$.

Proof 4.2 - *Proposition 4.3*

i). We can write $W_t = (W_t - W_s) + W_s$. As $(W_t - W_s)$ is independent of filtration $\mathcal{F}_s$ and has zero mean, we can conclude that

$$\mathbb{E}[W_t|\mathcal{F}_s] = \mathbb{E}[W_t - W_s|\mathcal{F}_s] + \mathbb{E}[W_s|\mathcal{F}_s] = \underbrace{\mathbb{E}[W_t - W_s]}_{=0} + W_s = W_s$$

ii). Similarly, we can write

$$W_t^2 = (W_t - W_s)^2 + 2W_s(W_t - W_s) + W_s^2$$

Then

$$
\begin{aligned}
\mathbb{E}[W_t^2 - t|\mathcal{F}_s] &= \left\{\mathbb{E}[(W_t - W_s)^2|\mathcal{F}_s] + 2W_s\mathbb{E}[W_t - W_s|\mathcal{F}_s] + W_s^2\right\} - t \\
&= \operatorname{Var}(W_t - W_s) + 0 + W_s^2 - t \\
&= t - s + W_s^2 - t \\
&= W_s^2 - s
\end{aligned}
$$

iii). We find that

$$
\begin{aligned}
\mathbb{E}[\tilde{Z}_t|\mathcal{F}_s] &= \mathbb{E}[\exp\{\sigma W_t + at\}|\mathcal{F}_s] \\
&= \exp\{at\}\mathbb{E}[\exp\{\sigma(W_t - W_s + W_s)\}|\mathcal{F}_s] \\
&= \exp\{\sigma W_s + at\} \cdot \mathbb{E}[\exp\{\sigma \overbrace{(W_t - W_s)}^{:=N}\}|\mathcal{F}_s] \\
&= \exp\{\sigma W_s + at\}\mathbb{E}[\exp\{\sigma N\}|\mathcal{F}_s] \\
&= \exp\{\sigma W_s + at\}\mathbb{E}[\exp\{\sigma N\}]
\end{aligned}
$$

where $N \sim \operatorname{Normal}(0, t - s)$. The MGF of N is $\mathbb{E}[e^{\sigma N}] = e^{(t-s)\sigma^2/2}$, therefore

$$
\begin{aligned}
\mathbb{E}[\tilde{Z}_t|\mathcal{F}_s] &= \exp\{\sigma W_s + at\}\mathbb{E}[\exp\{\sigma N\}] \\
&= \tilde{Z}_s e^{a(t-s)}e^{(t-s)\sigma^2/2} \\
&= \tilde{Z}_s \exp\left\{(t-s)\left(a + \frac{\sigma^2}{2}\right)\right\}
\end{aligned}
$$

We conclude that $\tilde{Z}_t$ is a *Martingale* <u>iff</u> $a = -\sigma^{2[63]}$.

$\square$

---

[63] As we require $\mathbb{E}[\tilde{Z}_t|\mathcal{F}_s] = \tilde{Z}_s$ which only occurs <u>iff</u> $a + \frac{\sigma^2}{2} = 0$.

4.2   Stochastic Integration

Remark 4.2 - *Motivation*
Ideally we could evaluate our trading strategy $H(t)$ by integrating it over all possible paths of
the stock price (e.g. Brownian Motion $W(t)$)

$$\int_0^T H(t)dW(t)$$

For a differentiable function $g(\cdot)$ it holds that

$$\int_0^T H(t)dg(t) = \int_0^T H(t)g'(t)dt$$

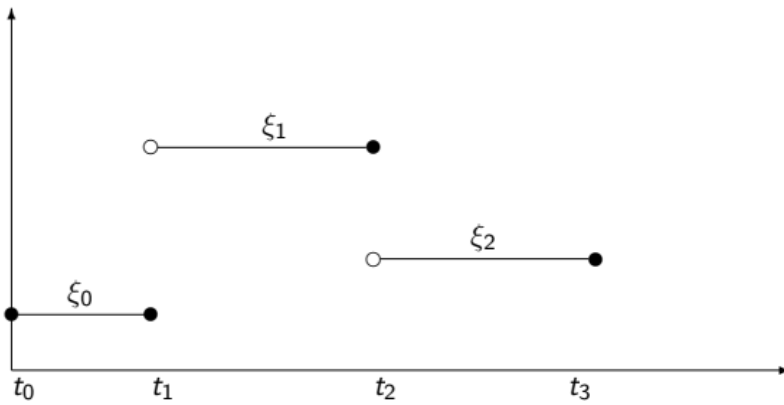However, the price process $W(t)$ is <u>not</u> differentiable and thus we have to take another
approach.

Figure 5: Example of Simple Stochastic Process

Definition 4.5 - *Simple Stochastic Process*
A *Stochastic Process* $X := \{X_t\}_{t \in [0,T]}$ is said to be *Simple* if there exists a partition

$$0 = t_0 < t_1 < \cdots < t_n = T$$

and a set of $\mathcal{F}_{t_k}$-measurable random variables $\{\xi_k\}_{k \in [0,n]}$, which all satisfy $\mathbb{E}[|\xi_k|] < \infty$, st $X_t(\omega)$
can be written in the form

$$X_t(\omega) = \xi_0(\omega)\mathbb{1}\{t = 0\} + \sum_{i=0}^{n-1} \xi_i(\omega)\mathbb{1}\{t \in [t_i, t_{i+1}]\}$$

This produces a stepped function (See `Figure 5`).

Definition 4.6 - *Stochastic Integral*
Let $X := \{X_t\}_{0 \in [0,T]}$ be a *Simple Stochastic Process.*
We define the *Stochastic Integral* $I_t(\cdot)$ of X as

$$I_t(X) := \int_0^t XdW = \sum_{i=0}^{n-1} \xi_i \cdot \left(W_{\min(t,t_{i+1})} - W_{\min(t,t_i)}\right)$$

where W_t is *Brownian Motion.*

Remark 4.3 - *Stochastic Integral is a Stochastic Process*
For each t, $I_t(X)$ is a random variable.
Thus, $\{I_t(X)\}_t$ is a *Stochastic Process.*

Remark 4.4 - *Stochastic Integral Expanding*
If $t \in [t_k, t_{k+1}]$ for some $k \in [0, n-1]$ then

$$I_t(X) = \left\{ \sum_{i=0}^{k-1} \xi_i \cdot (W_{t_{i+1}} - W_{t_i}) \right\} + \xi_k \cdot (W_t - W_{t_k})$$

Example 4.1 - *Stochastic Integrals*

i). Consider the simple stochastic process $\{X_t\}_{t \in [0,T]}$ where $X_t = c \, \forall \, t$. This is arguably the simplest random process possible and means $n = 1$ in the partition. Then

$$I_t(X) = cW_t$$

ii). Suppose each $X_t = X$ which is random, but fixed for all t. This means $n = 1$ in the partition and
$$I_t(X) = XW_t$$

iii). Suppose $X_t = c$ for $t \leq 1/2$ and $X_t = d$ for $t > 1/2$. Then

$$I_t(X) = \begin{cases} cW_t & \text{if } t < 1/2 \\ cW_{1/2} + d(W_t - W_{1/2}) & \text{if } t \geq 1/2 \end{cases}$$

Theorem 4.3 - *Properties of Simple Stochastic Processes*
Let $X := \{X_t\}_{t \in [0,T]}$ be a *Simple Stochastic Process*. Then

i). The expectation of its *Stochastic Process* is zero

$$\mathbb{E}[I_t(X)] = 0$$

ii). The *Stochastic Process* satisfies Itô's isometry

$$\mathbb{E}\left[(I_t(X))^2 \right] = \int_0^t \mathbb{E}[X_s^2 ds]$$

iii). $I_t(X)$ is a continuous *Martingale* wrt the *Natural Brownian Motion Filtration* $\mathcal{F}_t$ for all $0 \leq s \leq t \leq T$.
$$\mathbb{E}\left[I_t(X) | \mathcal{F}_s \right] = I_s(X)$$

iv). *Linearity* - Let Y be another *Simple Stochastic Process* and a, b be arbitrary real numbers. Then
$$I_t(aX + bY) = aI_t(X) + bI_t(Y)$$

v). The stochastic process $I_t(X)$ has continuous sample paths.

Proof 4.3 - *Theorem 4.3*

i). Note that since ξ_i is $\mathcal{F}_{t_i}$ measurable, then ξ_i & $(W_{t_{i+1}} - W_{t_i})$ are independent for all i. Also $\mathbb{E}[W_{t_{i+1}} - W_{t_i}] = 0$, thus

$$\mathbb{E}[\xi_i(W_{t_{i+1}} - W_{t_i})] = \mathbb{E}[\xi_i]\mathbb{E}[W_{t_{i+1}} - W_{t_i}] = 0$$

Meaning
$$\mathbb{E}[I_t(X)] = 0$$

ii). Consider a partition of $[0, t]$ st $0 = t_0 < \cdots < t_k = t$. Then

$$\mathbb{E}[(I_t(X))^2] = \sum_{i=0}^{k-1}\sum_{j=0}^{k-1} \mathbb{E}\left[\xi_i(W_{t_{i+1}} - W_{t_i})\xi_j(W_{t_{j+1}} - W_{t_j})\right]$$

If $i > j$ then $(W_{t_{i+1}} - W_{t_i})$ is independent of all the other factors and has expectation 0

$$\mathbb{E}\left[\xi_i(W_{t_{i+1}} - W_{t_i})\xi_j(W_{t_{j+1}} - W_{t_j})\right]$$
$$= \underbrace{\mathbb{E}[W_{t_{i+1}} - W_{t_i}]}_{\substack{\sim N(0, t_{i+1}-t_i) \\ =0}} \mathbb{E}[\xi_i\xi_j(W_{t_{j+1}} - W_{t_j})]$$

So all terms with $i > j$ and, similarly $i < j$, disappear and we conclude that

$$\begin{aligned}
\mathbb{E}[I_t(X)^2] &= \sum_{i=0}^{k-1}\mathbb{E}[\xi_i(W_{t_{i+1}} - W_{t_i})\xi_i(W_{t_{i+1}} - W_{t_i})] \\
&= \sum_{i=0}^{k-1}\mathbb{E}[\xi_i^2]\mathbb{E}[(W_{t_{i+1}} - W_{t_i})^2] \\
&= \sum_{i=0}^{k-1}\mathbb{E}[\xi_i^2](t_{i+1} - t_i)
\end{aligned}$$

The last step comes from considering the variance of $(W_{t_{i+1}} - W_{t_i})$.

The RHS is the usual Riemann Integral $\int_0^T f(s)ds$ of the step function $f(x) = \mathbb{E}[X_s^2]$ which coincides with $\mathbb{E}[\xi^2]$ for $s \in [t_i, t_{i+1}]$.

iii). Adaptedness of $I(X)$ follows since at time t all ξ_i and $(W_{t_{i+1}} - W_{t_i})$ contributing to $I_t(X)$ are functions of *Brownian Motion* up to time t.

The condition $\mathbb{E}[|I_t(X)|]$ follows from the isometry property ii).

It remains to show that $\mathbb{E}[I_t(X)|\mathcal{F}_s]$ for all $s < t$.

First, assume that $s < t$ and $s, t \in [t_k, t_{k+1}]$. Notice that

$$\begin{aligned}
I_t(X) &= I_{t_k}(X) + \xi_k(W_t - W_{t_k}) \\
I_s(X) &= I_{t_k}(X) + \xi_k(W_s - W_{t_k})
\end{aligned}$$

Hence $I_t(X) = I_s(X) + \xi_k(W_t - W_s)$ where $I_s(X)$ and ξ_k are known at time s, and $(W_t - W_s)$ is independent of $\mathcal{F}_s$ and has mean zero. Hence

$$\mathbb{E}[I_t(X)|\mathcal{F}_s] = I_s(X) + \xi_k\underbrace{\mathbb{E}[W_t - W_s]}_{=0} = I_s(X)$$

The case where $s < t_k < t$ can be handled analogously.

iv). Assume that the simple process X is defined using the partition

$$0 = t_0 < t_1 < \cdots < t_n = T$$

and Y uses the partition

$$0 = s_0 < s_1 < \cdots < s_m = T$$

Note that these partitions can be different lengths and take different values.

Consider the joint-partition

$$0 = u_0 < u_1 < \cdots < u_l = T$$

which combines all the points from the other partitions, ensuring the correct ordering of values. The values of $I_t(X)$ and $I_t(Y)$ wrt the finer partition $\{u_0, \ldots, u_l\}$ remain the same.

Linearity follows from the linearity of the underlying sums.

v). This follows from the definition of $I_t(X)$ since

$$I_t(X) = I_{t_{k-1}}(X) + \xi_k(W_t - W_{t_{k-1}}) \ \forall \ t \in [t_{k-1}, t_k]$$

The only unfixed term in this expression is the *Brownian Motion* W_t. We know that *Brownian Motion* has continuous sample paths, thus $I_t(X)$ has continuous sample paths.

Theorem 4.4 -
Let $X := \{X_t\}_{t \in [0,T]}$ be a *Stochastic Process* which is adapted to *Brownian Motion*[64].

Then one can find a sequence $X^{(n)}$ of simple processes st

$$\int_0^T \mathbb{E}\left[(X_s - X_s^{(n)})^2\right] ds \overset{n \to \infty}{\longrightarrow} 0$$

Moreover, there is some process $I(X)$ on $[0, T]$ st

$$\mathbb{E}\left[\sup_{t \in [0,T]} \left(I_t(X) - I_t(X^{(n)})\right)^2\right] \overset{n \to \infty}{\longrightarrow} 0$$

Definition 4.7 - *Itô Stochastic Integral*
The mean square limit $I(X)$ in `Theorem 4.4` is called the *Itô Stochastic Integral* of X.

It is denoted by

$$I_t(X) := \int_0^t X_s dW_s \quad t \in [0, T]$$

Remark 4.5 - *Rule of Thumb for Itô Stochastic Integral*
The *Itô Stochastic Integral* $\{I_t(X)\}_{t \in [0,T]}$ constitute a *Stochastic Process*. For a given partition $0 = t_0 < t_1 < \cdots < t_n = T$ and $t \in [t_k, t_{k+1}]$ the random variable $I_t(X)$ is approximately

$$I_t(X) \simeq \sum_{i=0}^{k-1} \left\{X_{t_i}(W_{t_{i-1}} - W_{t_i})\right\} + X_{t_k}(W_t - W_{t_k})$$

---

[64]i.e. X_t is a function of W_s for $s \leq$ which satisfies that $\int_0^T \mathbb{E}[X_t^2] < \infty$

This approximation is closer to the value of $I_t(X)$ the denser the partition is in $[0, T]$.

Theorem 4.5 - *Properties of Itô Stochastic Integral*[65]
Let $X := \{X_t\}_{t \in [0,T]}$ be a stochsatic process which satisfies the requirements of `Theorem 4.4`.
Then

   i). The stochastic integral $I_t(X)$ has expectation zero.

  ii). The stochastic integral $I_t(X)$ satisfies the Itô Isometry Property (`Theorem 4.3`).

 iii). The stochastic integral $I_t(X)$ is a *Martingale* wrt the *Natural Brownian Filtration*.

 iv). The stochastic integral $I_t(X)$ is *Linear*.

  v). The stochastic integral $I_t(X)$ has continuous sample paths.

Example 4.2 - *Stochastic Integral of Brownian Motion*
We calculate the stochastic integral of Brownian motion

$$I_t(W) = \int_0^t W_s dW_s$$

We start by approximating the integrand W_s by a sequence of simple functions

$$X_u^{(n)} = \begin{cases} W_0 = 0 & \text{if } u \in [0, t_1] \\ W_{t_1} & \text{if } u \in [t_1, t_2] \\ \vdots \\ W_{t_{n-1}} & \text{if } u \in [t_{n-1}, t] \end{cases} \quad \text{where } t_k = \frac{tk}{n} \text{ for } k = 0, 1, \dots, n^{[66]}$$

Then

$$\begin{aligned} I_t(X^{(n)}) &= \sum_{k=0}^{n-1} X_{t_k}^{(n)}(W_{t_{k+1}} - W_{t_k}) \\ &= \sum_{k=0}^{n-1} W_{t_k}(W_{t_{k+1}} - W_{t_k}) \\ &= \sum_{k=0}^{n-1} \left\{ W_{t_{k+1}} W_{t_k} - W_{t_k}^2 \right\} \\ &= \left\{ \sum_{k=0}^{n-1} W_{t_{k+1}} W_{t_k} \right\} - \frac{1}{2} \left\{ \sum_{k=0}^{n-1} (W_{t_{k+1}}^2 + W_{t_k}^2) \right\} + \frac{1}{2} W_{t_n}^2 \\ &= \frac{1}{2} W_t^2 - \frac{1}{2} \sum_{k=0}^{n-1} (W_{t_{k+1}} - W_{t_k})^2 \end{aligned}$$

Since the increments $(W_{t_{k+1}} - W_{t_k})$ are independent and normally distributed with mean 0 and variance $t_{k+1} - t_k = t/n$, the sum of the RHS converges to t as $n \to \infty$. Thus we obtain

$$I_t(W) = \int_0^t W_s dW_s = \frac{1}{2} W_t^2 - \frac{1}{2}$$

---

[65]The proof for these properties is not covered in this course.
[66]The partition is equally spaced.

Remark 4.6 - *Example 4.2*
The argument used in `Example 4.2` leads to a general principle.

If we write $\Delta W_{t_k} := W_{t_{k+1}} - W_{t_k}$ and $\Delta t_k := t_{k+1} - t_k$ then we can derive the following mean square limit properties

$$\sum_{k=0}^{n-1}(\Delta W_{t_k})^2 \stackrel{n\to\infty}{\Longrightarrow} t \quad \sum_{k=0}^{n-1} \Delta W_t \Delta t_k \stackrel{n\to\infty}{\Longrightarrow} 0$$
$$\sum_{k=0}^{n-1}(\Delta t_k)^2 \stackrel{n\to\infty}{\Longrightarrow} 0$$

These are often written as the formal multiplication rules

$$
\begin{aligned}
dW \cdot dW &= dt \\
dt \cdot dW &= 0 \\
dt \cdot dt &= 0
\end{aligned}
$$

4.3   Itô's Lemma

Theorem 4.6 - *Itô's Lemma - Special Case*
This is a special case of Itô's Lemma.

Let $f(x)$ be a twice continuously differentiable function. Then for any $t > 0$

$$f(W_t) - f(W_0) = \underbrace{\int_0^t f'(W_u)dW_u}_{\text{Standard Integral}} + \underbrace{\frac{1}{2}\int_0^t f''(W_u)du}_{\text{Stochastic Integral}}$$

or in differential form

$$df(W_t) = f'(W_t)dW_t + \underbrace{\frac{1}{2}f''(W_t)dt}_{\text{Itô's Correction Term}}$$

Proof 4.4 - *Theorem 4.6*
Consider a partition $\{t_0, \ldots, t_n\}$ of $[0,t]$ with $0 = t_0 < \cdots < t_n = t$.

By Taylor's formula we obtain

$$
\begin{aligned}
f(W_t) - f(W_0) &= \sum_{i=0}^{n-1} f(W_{t_{i+1}}) - f(W_{t_i}) \\
&= \sum_{i=0}^{n-1} f'(W_{t_i})(W_{t_{i+1}} - w_{t_i}) \\
&+ \frac{1}{2}\sum_{i=0}^{n-1} f''(W_{t_i} + \theta_i(W_{t_{i+1}} - W_{t_i}))(W_{t_{i+1}} - w_{t_i})^2 \text{ with } \theta_i \in (0,1)
\end{aligned}
$$

The first sum is an approximating sequence of a stochastic integral. Indeed, we find

$$\sum_{i=0}^{n-1} f'(W_{t_i})(W_{t_{i+1}} - W_{t_i}) \stackrel{n\to\infty}{\Longrightarrow} \int_0^t f'(W_u)dW_u$$

We also know that

$$\lim_{n\to\infty} \sum_{i=0}^{n-1}(W_{t_{i+1}} - W_{t_i})^2 = t$$

and with a little more effort we can prove that[67]

$$\lim_{n\to\infty} \sum_{i=0}^{n-1} f''(W_{t_i} + \theta_i(W_{t_{i+1}} - W_{t_i}))(W_{t_{i+1}} - W_{t_i})^2 = \int_0^t f''(W_u)du$$

---

[67]Proof is beyond scope of course. Won't be asked to reproduce this step.

Example 4.3 - *Theorem 4.6*
Consider the function $f(x) = x^2$. Then

$$f'(x) = 2x \quad f''(x) = 2$$

By `Theorem 4.6` we have

$$W_t^2 = f(W_t) - f(W_0) = \int_0^t 2WudW_u + \frac{1}{2}\int_0^t 2du = \int_0^t 2W_u dW_u + t$$

Hence

$$\frac{1}{2}(W_t^2 - t) = \int_0^t W_u dW_u$$

as in `Example 4.2`

Theorem 4.7 - *Itô's Lemma - More General Case*
This is a more general case of Itô's Lemma than `Theorem 4.6`*, but not as general as* `Theorem 4.8`*.*
Let $f(t, x)$ be a function which is continuously differentiable once in its first argument (the time parameter t) and twice in its second argument x. Then

$$f(t, W_t) - f(0, W_0) = \int_0^t f_t(u, W_u) + \frac{1}{2}f_{xx}(u, W_u)du + \int_0^t f_x(u, W_i)dW_u$$

where $f_t := \frac{\partial f}{\partial t}$, $f_{xx} := \frac{\partial^2 f}{\partial x^2}$ and $f_x := \frac{\partial f}{\partial x}$.
Or, in differentiable form

$$df(t, W_t) = (f_t(t, W_t) + \frac{1}{2}f_{xx}(t, W_t))dt + f_x(t, W_t)dW_t$$

Proof 4.5 - *Theorem 4.7*
By the Taylor expansion of a smooth function of several variables we get for t close to t_0 that

$$
\begin{aligned}
f(t, W_t) \quad = \quad & f(t_0, W_{t_0}) + (t - t_0)f_t(t_0, W_{t_0}) + (W_t - W_{t_0})f_x(t_0, W_{t_0}) \\
+ \quad & \frac{1}{2}(t - t_0)^2 f_{tt}(t_0, W_{t_0}) + \frac{1}{2}(W_t - W_{t_0})^2 f_{xx}(t_0, W_{t_0}) \\
+ \quad & (t - T_0)(W_t - W_{t_0})f_{tx}(t_0, W_{t_0}) + \text{Higher order terms}
\end{aligned}
$$

This can be written symbolically as

$$df = f_t dt + fx dW + \frac{1}{2}f_{tt}(dt)^2 + f_{tx}dtdW + \frac{1}{2}f_{xx}(dW)^2 + \dots$$

Note that $(dt)^2 = 0$. Now using the formal multiplication rules

$$dt \cdot dt = 0 \quad dt \cdot dW = 0 \quad dW \cdot dW = dt$$

We get

$$df = f_t d_t + f_x dW + \frac{1}{2}f_{xx}dt = (f_t + \frac{1}{2}f_{xx})df + f_x dW$$

Example 4.4 - *Itô's Exponential*
Consider the function

$$f(t, x) = \exp\left\{x - \frac{t}{2}\right\}$$

Then

$$f_t = -\frac{f(t,x)}{2} \quad f_x(t,x) = f(t,x) \quad f_{xx}(t,x) = f(t,x)$$

Thus, by `Theorem 4.7` , since $f(0, W_0) = 1$.

$$\begin{aligned}
f(t, W_t) - 1 &= \int_0^t -\frac{1}{2}f(u, W_u) + \frac{1}{2}f(u, W_u)du + \int_0^t f(u, W_u)dW_u \\
&= \int_0^t f(u, W_u)dW_u
\end{aligned}$$

Note that this is a stochastic integral.

Hence f satisfies the differential equation $df = fdW$ in the Itô sense and is therefore called the Itô exponential.

Example 4.5 - *Geometric Brownian Motion*
Consider the function

$$f(t, x) = \exp\left\{\sigma x + (\mu - \frac{\sigma^2}{2})t\right\}$$

Then

$$f_t(t, x) = \left(\mu - \frac{\sigma^2}{2}\right)f(t, x) \quad f_x(t, x) = \sigma f(t, x) \quad f_{xx}(t, x) = \sigma^2 f(t, x)$$

Thus $f_t + \frac{f_{xx}}{2} = \mu f(t, x)$. So by `Theorem 4.20`

$$f(t, W_t) - f(0, W_0) = \int_0^t \mu f(u, W_u)du + \int_0^t \sigma f(u, W_u)dW_u$$

The process $X_t = f(t, W_t)$ satisfies the linear stochastic differential equation

$$X_t - X_0 = \mu \int_0^t X_s ds + \sigma \int_0^t X_s dW_s$$

In differential notation this is

$$dX = \mu X ds + \sigma X dW = X(\mu ds + \sigma dW)$$

This is useful for geometric Brownian motion when μ is the drift term and σ is the volatility.

Definition 4.8 - *Itô Process*
Let $X = \{X_t\}_{t \in [0,T]}$ be a stochastic process such that

$$X_t = X_0 + \int_0^t b_u du + \int_0^t \sigma_u dW_u$$

where both b, σ are functions which are adapted to Brownian motion. This X is called an *Itô Process*.

We also say that the process X has a stochastic differential

$$dX_t = b_t d_t + \sigma_t dW_t$$

Theorem 4.8 - *Itô's Lemma - Most General*
Let X be an *Itô Process* and $f(t,x)$ be a function whose second order partial derivatives are continuous. Then for any $t > 0$

$$f(t, X_t) - f(0, X_0) = \int_0^t f_t(u, X_u) + b_u f_x(u, X_u) + \frac{1}{2}\sigma_u^2 f_{xx}(u, X_u)du + \int_0^t \sigma_u f_x(u, X_u)dW_u$$

Or in differential form

$$df = \left(f_t + b_t f_X + \frac{1}{2}\sigma_t^2 f_{xx} \right)dt + \sigma_t f_x dW_t$$

Proof 4.6 - *Theorem 4.8*
We proceed as in the preceding version of Itô's formula and consider a Taylor expansion of $f(t, X_t)$ which is in differential notation

$$df = f_t dt + f_x dX + \frac{1}{2}f_{tt}(dt)^2 + f_{tx}dtdx + \frac{1}{2}f_{xx}(dX)^2 + \text{High order terms}$$

Now we substitute $dX = bdt + \sigma dW$ and obtain

$$df = f_t dt + f_x(bd_t + \sigma dW) + \tfrac{1}{2}f_{tt}(dt)^2 + f_{tx}dt(bd_t + \sigma dW) + \tfrac{1}{2}f_{xx}(bd_t + \sigma dW) + \text{High order terms}$$

Again, neglecting all $(ft)^2)$ and $dtdW$ terms as well as the high order terms we obtain

$$
\begin{aligned}
df &= f_t d_t + f_x(bd_t + \sigma dW) + \tfrac{1}{2}f_{xx}(\sigma dW)^2 \\
&= (f_t + f_x b + \tfrac{1}{2}\sigma^2 f_{xx})dt + f_x \sigma dW
\end{aligned}
$$

Theorem 4.9 - *Product Rule for Stochastic Calculus*
We can derive the produce rule for stochastic calculus from `Theorem 4.8.`

Suppose two process X_t, Y_t are adapated to the same Brownian motion

$$
\begin{aligned}
dX_t &= \sigma_t dW_t + \mu_t dt \\
dY_t &= \rho_t dW_t + \nu_t dt
\end{aligned}
$$

Then

$$d(X_t Y_t) = X_t dY_t + Y_t dX_t + \sigma_t \rho_t dt$$

Proof 4.7 - *Theorem 4.9*
Since $d(X_t + Y_t) = (\sigma_t + \rho_t)dW_t + (\mu_t + \sigma_t)dt$, using `Theorem 4.8` applied to $f(t,x) = x^2$ we obtain that

$$
\begin{aligned}
d(X_t^2) &= (2\mu_t X_t + \sigma_t^2)dt + 2\sigma_t X_t dW_t \\
d(Y_t^2) &= (2u\nu_t Y_t + \rho_t^2)dt + 2\rho_t Y_t dW_t \\
d((X_t + Y_t)^2) &= (2(\mu_t + \nu_t)(X_t + Y_t) + (\sigma_t + \rho_t)^2)dt \\
&\quad + 2(\sigma_t + \rho_t)(X_t + Y_t)dW_t
\end{aligned}
$$

Subtracting, the result follows since

$$
\begin{aligned}
2d(X_t Y_t) = d((X_t + Y_t)^2 - X_t^2 - Y_t^2) \\
&= 2(\mu_t Y_t + \nu_t X_t)dt + 2\sigma_t \rho_t dt + 2(\sigma_t Y_t + \rho_t X_t)dW_t \\
&= 2Y_t dX_t + 2X_t dY_t + 2\sigma_t \rho_t dt
\end{aligned}
$$

5   Financial Market Models in Continuous Time

5.1   The Financial Market Model in Continuous Time

5.2   Trading Strategies

5.3   Arbitrage in the Continuous Time Model

5.4   The Black-Scholes Model

5.5   Equivalent Martingale Measures in the Black-Scholes Model

5.6   Pricing in the Black-Scholes Model

5.7   Replicating Strategies and the Black-Scholes-Merton Equation

0   Reference

1   Notation

Notation 1.1 - *General Mathematical Notation*

Notation	Description
$\{x\}_+$	Only the positive part of x (i.e. $\max\{0, x\}$).