Financial Mathematics - Assessed Problem Sheet 3

Dom Hutchinson

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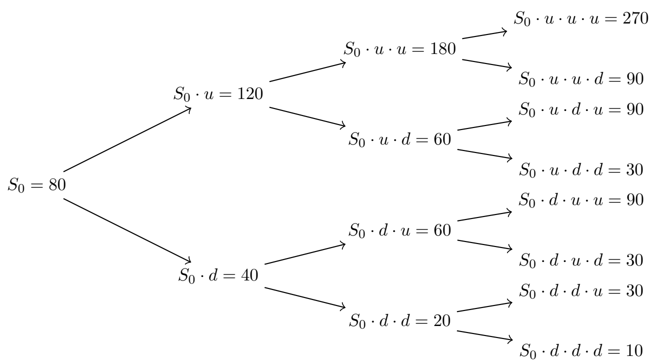
Answer 1. a) i.
Define stopping time $\tau = \inf\{t : S_t \leq b\}$, this represents the first time the stock price falls below the knockout price b.

The payoff process $\{Y_t\}$ for the Down-and-Out call can be expressed as

$$Y_t = \begin{cases} \{S_t - K\}_+ & \text{if } t < \tau \\ R & \text{if } t \geq \tau \end{cases}$$

Consider the European Claim $X = Y_\tau \dfrac{B_T}{B_\tau}$ which corresponds to exercising the Down-and-Out call at time τ and then accumulating interest from the bank account until the expiry date of the claim at time T.

Answer 1. a) ii.
Consider the tree below which shows the possible evolutions of the price process S_t for each time-point and event.



The risk-neutral probability measure for a Cox-Ross-Rubinstein model satisfies

$$\mathbb{Q}(S_t = S_0 u^n d^{t-n}) = \binom{t}{n} q^n (1-q)^{t-n} \text{ where } q = \frac{1+r-d}{u-d} \text{ for } n = 0, \dots, t$$

where n is the number of up steps taken in the first t time-periods.

Under this specification of the Cox-Ross-Rubinstein model

$$q = \frac{1 + 0.1 - 0.5}{1.5 - 0.5} = \frac{3}{5}$$

Thus

$$\mathbb{Q}(S_t = S_0 u^n d^{t-n}) = \binom{t}{n} \frac{3^n 2^{t-n}}{5^t} \text{ for } n = 0, \dots, t$$

By inspecting the tree of stock prices above we can determine the possible prices at each time-point, and thus the risk-neutral probability of each node.

At time $t = 0$

$$\mathbb{Q}(S_0 = 80) = \mathbb{Q}(S_0 = S_0) = 1$$

At time $t = 1$

$$\begin{aligned}
\mathbb{Q}(S_1 = 120) &= \mathbb{Q}(S_1 = S_0u) = \binom{1}{1} \cdot \frac{3}{5} = \frac{3}{5} \\
\mathbb{Q}(S_1 = 40) &= \mathbb{Q}(S_1 = S_0d) = \binom{1}{0} \cdot \frac{2}{5} = \frac{2}{5}
\end{aligned}$$

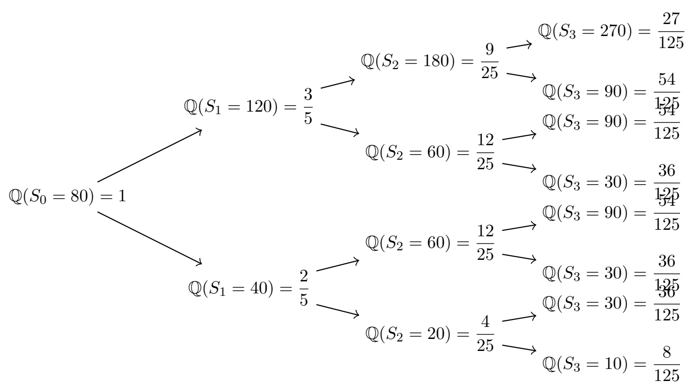
At time $t = 2$

$$\begin{aligned}
\mathbb{Q}(S_2 = 180) &= \mathbb{Q}(S_2 = S_0u^2) = \binom{2}{2} \cdot \frac{3^2}{5^2} = \frac{9}{25} \\
\mathbb{Q}(S_2 = 60) &= \mathbb{Q}(S_2 = S_0ud) = \binom{2}{1} \cdot \frac{3 \cdot 2}{5^2} = 2 \cdot \frac{6}{25} = \frac{12}{25} \\
\mathbb{Q}(S_2 = 40) &= \mathbb{Q}(S_2 = S_0d^2) = \binom{2}{0} \cdot \frac{2^2}{5^2} = \frac{4}{25}
\end{aligned}$$

At time $t = 3$

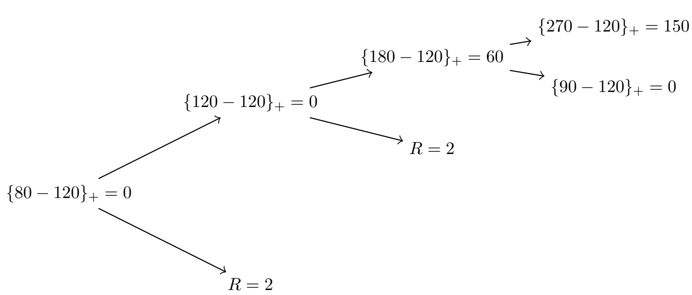
$$\begin{aligned}
\mathbb{Q}(S_3 = 270) &= \mathbb{Q}(S_3 = S_0u^3) &= \binom{3}{3} \cdot \frac{3^3}{5^3} = \frac{27}{125} \\
\mathbb{Q}(S_3 = 90) &= \mathbb{Q}(S_3 = S_0u^2d) &= \binom{3}{2} \cdot \frac{3^2 \cdot 2}{5^3} = 3 \cdot \frac{18}{125} = \frac{54}{125} \\
\mathbb{Q}(S_3 = 30) &= \mathbb{Q}(S_3 = S_0ud^2) &= \binom{3}{1} \cdot \frac{3 \cdot 2^2}{5^3} = 3 \cdot \frac{12}{125} = \frac{36}{125} \\
\mathbb{Q}(S_3 = 10) &= \mathbb{Q}(S_3 = S_03^3) &= \binom{3}{0} \cdot \frac{2^3}{5^3} = \frac{8}{125}
\end{aligned}$$

I summarise these values in the tree below



Answer 1. a) iii.

The tree below specifies the pay-out process $\{Y_t\}$ of the down-and-out call option is exercised at each possible time-point and sequence of events



I construct a Snell Envelope $\{Z_t\}$ to determine the the value of the down-and-out option at each time-point t and state ω. At time-point $t=3$

$$Z_3(\omega)=Y_3(\omega)\ \forall\ \omega$$

At time-point $t=2$ and state ω_{uu} (ie Two up steps have occurred)

$$\mathbb{E}[Z_3|\mathcal{F}_2]=\frac{1}{1.1}\left(150q+0\cdot(1-q)\right)=81.8182$$

At time-point $t=2$ and state ω_{ud} (ie One up step and one down step have occurred)

$$\mathbb{E}[Z_3|\mathcal{F}_2]=\frac{1}{1.1}\left(2q+2\cdot(1-q)\right)=1.8182$$

At time-point $t=2$ and state ω_{dd} (ie Two down steps have occurred)

$$\mathbb{E}[Z_3|\mathcal{F}_2]=\frac{1}{1.1}\left(2q+2\cdot(1-q)\right)=1.8182$$

Thus

$$\begin{array}{rcl}
Z_2(\omega) & = & \max(\mathbb{E}[Z_3|\mathcal{F}_2],Y_2(\omega)) \\
 & = & \begin{cases}\max(60,81.8182) & \text{if } \omega=\omega_{uu}\\ \max(2,1.8182) & \text{otherwise}\end{cases} \\
 & = & \begin{cases}81.8182 & \text{if } \omega=\omega_{uu}\\ 2 & \text{otherwise}\end{cases}
\end{array}$$

At time-point $t=1$ and state ω_u (ie One up steps has occurred)

$$\mathbb{E}[Z_2|\mathcal{F}_1]=\frac{1}{1.1}\left(81.8182\cdot q+2\cdot(1-q)\right)=45.3554$$

At time-point $t=1$ and state ω_d (ie One down step has occurred)

$$\mathbb{E}[Z_2|\mathcal{F}_1]=\frac{1}{1.1}\left(2q+2\cdot(1-q)\right)=1.8182$$

Thus

$$\begin{array}{rcl}
Z_1(\omega) & = & \max(\mathbb{E}[Z_2|\mathcal{F}_1],Y_1(\omega)) \\
 & = & \begin{cases}\max(45.3554,0) & \text{if } \omega=\omega_u\\ \max(2,1.8182) & \text{if } \omega=\omega_d\end{cases} \\
 & = & \begin{cases}45.3554 & \text{if } \omega=\omega_u\\ 2 & \text{if } \omega=\omega_d\end{cases}
\end{array}$$

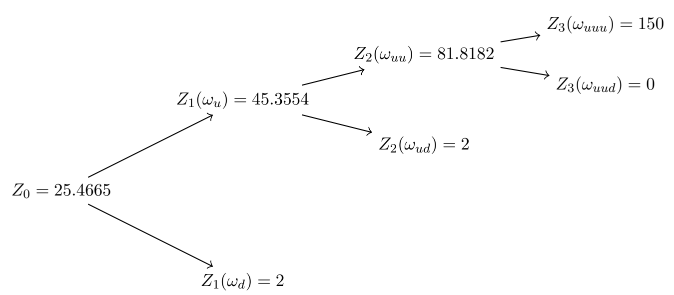
At time-point $t=0$

$$\begin{array}{rcl}
\mathbb{E}[Z_1|\mathcal{F}_0] & = & \mathbb{E}[Z_1] \\
 & = & \frac{1}{1.1}\left(45.3554\cdot q+2(1-q)\right) \\
 & = & 25.4666
\end{array}$$

Thus

$$Z_0=\max(\mathbb{E}[Z_1|\mathcal{F}_0],Y_0(\omega))=\max(25.4666,0)=25.4666$$

I summarise the value of this call option at each time-point and event in the tree below



Using the optimal stopping theorem, the optimal stopping strategy $\{\tau(t)\}_t$ is

$$\begin{aligned}
\tau(3)(\omega) &= 3\ \forall\ \omega \\
\tau(2)(\omega) &= \begin{cases} 3 & \text{if } \omega \in \{\omega_{uuu}, \omega_{uud}\} \\ 2 & \text{otherwise} \end{cases} \\
\tau(1)(\omega) &= \begin{cases} 3 & \text{if } \omega \in \{\omega_{uuu}, \omega_{uud}\} \\ 2 & \text{if } \omega \in \{\omega_{udu}, \omega_{udd}\} \\ 1 & \text{otherwise} \end{cases} \\
\tau(0)(\omega) &= \begin{cases} 3 & \text{if } \omega \in \{\omega_{uuu}, \omega_{uud}\} \\ 2 & \text{if } \omega \in \{\omega_{udu}, \omega_{udd}\} \\ 1 & \text{otherwise} \end{cases}
\end{aligned}$$

I now find a replicating strategy $\{H(t)\}$ for $\tau(0)$. In the third-time period the following equations must be satisfied

$$\begin{aligned}
H_0(3) + 270H_1(3) &= Z_3(\omega_{uuu}) = 150 \\
H_0(3) + 90H_1(3) &= Z_3(\omega_{uud}) = 0 \\
\implies \quad H_1(3) &= \frac{150}{180} = \frac{5}{6} \\
\implies \quad H_0(3) &= 0 - 90 \cdot \frac{5}{6} = -75
\end{aligned}$$

Thus $H(3)(\omega) = (-75, 5/6)$ if $\omega \in \{\omega_{uuu}, \omega_u ud\}$. We do not consider any other states as the option would have already been exercised by time-point $t = 3$.

In the second-time period the following equations must be satisfied

$$\begin{aligned}
H_0(2) + 180H_1(2) &= Z_2(\omega_{uu}) = 81.8182 \\
H_0(2) + 60H_1(2) &= Z_2(\omega_{ud}) = 2 \\
\implies \quad H_1(2) &= \frac{79.8182}{120} = 0.6652 \\
\implies \quad H_1(2) &= 2 - 60 \cdot 0.6651 = -37.906
\end{aligned}$$

Thus $H(2)(\omega) = (-37.906, 11/15)$ if $\omega \in \{\omega_{uu}, \omega_u d\}$. We do not consider any other states as the option would have already been exercised by time-point $t = 2$.

In the first-time period the following equations must be satisfied

$$\begin{aligned}
H_0(1) + 120H_1(1) &= Z_1(\omega_u) = 45.3554 \\
H_0(1) + 40H_1(1) &= Z_1(\omega_d) = 2 \\
\implies \quad H_1(1) &= \frac{45.3554}{80} = 0.5669 \\
\implies \quad H_1(1) &= 2 - 40 \cdot 0.5669 = -20.676
\end{aligned}$$

Thus $H(1)(\omega) = (-20.676, 33/50)$ for all ω.

Answer 1. a) iv.

The time $t = 0$ of a European call option in a Cox-Ross-Rubinstein model is

$$\Pi(0) = \frac{1}{(1+r)^T} \sum_{n=0}^{T} \binom{T}{n} q^n (1-q)^{T-n} \{S_0 u^n d^{T-n} - K\}_+$$

More specifically, for the model in this question, a European call option with exercise price

$K = 120$ and exercise date $T = 3$

$$\begin{array}{rcl}\Pi(0) & = & \dfrac{1}{1.1^3}\displaystyle\sum_{n=0}^{3}\binom{3}{n}\frac{3^n2^{3-n}}{5^3}\left\{80\cdot\frac{3^n\cdot 1^{3-n}}{2^3}-120\right\}_+\\ & = & \dfrac{1}{1.1^3}\left\{\binom{3}{0}\cdot\dfrac{2^3}{5^3}\{10-120\}_+ + \binom{3}{1}\cdot\dfrac{3\cdot 2^2}{5^3}\{30-120\}_+\right.\\ & & \left.+\binom{3}{2}\cdot\dfrac{3^2\cdot 2}{5^3}\{90-120\}_+ + \binom{3}{3}\cdot\dfrac{3^3}{5^3}\{270-120\}_+\right\}\\ & = & \dfrac{1}{1.1^3}\cdot\dfrac{3^3}{5^3}\cdot 150\\ & = & 24.34259\dots\end{array}$$

Answer 1. b) i.
Let $a \leq b$ and $\{W_t\}_t$ be standard Brownian motion.
Note that $W_T \sim \text{Normal}(0,T)$, thus $\dfrac{1}{\sqrt{T}}W_T \sim \Phi$. Thus

$$\begin{array}{rcl}\mathbb{P}(W_T \in [a,b]) & = & \mathbb{P}(W_T \leq b) - \mathbb{P}(W_T \leq a)\\ & = & \mathbb{P}\left(\dfrac{1}{\sqrt{T}}W_T \leq \dfrac{b}{\sqrt{T}}\right) - \mathbb{P}\left(\dfrac{1}{\sqrt{T}}W_T \leq \dfrac{a}{\sqrt{T}}\right)\\ & = & \Phi(b/\sqrt{T}) - \Phi(a/\sqrt{T})\end{array}$$

Consider the expected value of e^{cW_T} for $c \in \mathbb{R}$

$$\begin{array}{rcl}\mathbb{E}\left[e^{cW_T}\right] & = & \displaystyle\int e^{cx}f_{W_T}(x)dx\\ & = & \displaystyle\int e^{cx}\cdot\frac{1}{\sqrt{2\pi T}}e^{-\frac{1}{2}\cdot\frac{x^2}{T}}dx\\ & = & \displaystyle\int \frac{1}{\sqrt{2\pi T}}\exp\left\{-\frac{1}{2}\cdot\frac{(x-c)^2+c^2}{T}\right\}dx\\ & = & e^{c^2/T}\displaystyle\int \frac{1}{\sqrt{2\pi T}}\exp\left\{-\frac{1}{2}\cdot\frac{(x-c)^2}{T}\right\}dx\\ & = & e^{c^2/T}\cdot\mathbb{E}\left[\text{Normal}(c,T)\right]\\ & = & ce^{c^2/T}\end{array}$$

Answer 1. b) ii.
Let $\{W_t^{(1)}\}_t\{W_t^{(2)}\}$ be independent standard Brownian motions and define stochastic process $\{X_t\}_t$ as

$$X_t = \gamma W_t^{(1)} + W_t^{(2)}\sqrt{1-\gamma^2} \text{ with } \gamma \in [-1,1]$$

There are four properties I need to show for X_t to be a standard Brownian motion

i). *That* $X_0 = 0$.

$$\begin{array}{rcl}X_0 & = & \gamma W_0^{(1)} + W_t^{(2)}\sqrt{1-\gamma^2}\\ & = & \gamma\cdot 0 + 0\cdot\sqrt{1-\gamma^2}\\ & = & 0\end{array}$$

X_t has this property.

ii). *Increments of* X_t *are independent.*

Consider the increment $(X_{t+u} - X_t)$ for $t, u \geq 0$

$$\begin{aligned}(X_{t+u} - X_t) &= \left(\gamma W^{(1)}_{t+u} + W^{(2)}_{t+u}\sqrt{1-\gamma^2}\right) - \left(\gamma W^{(1)}_t + W^{(2)}_t\sqrt{1-\gamma^2}\right) \\ &= \gamma(W^{(1)}_{t+u} - W^{(1)}_t) + \sqrt{1-\gamma^2}(W^{(1)}_{t+u} - W^{(1)}_t)\end{aligned}$$

Since $W^{(1)}_t, W^{(2)}_t$ are standard Brownian motions then their increments $(W^{(1)}_{t+u} - W^{(1)}_t), (W^{(2)}_{t+u} - W^{(2)}_t)$ are both independent of $\mathcal{F}_t$. Thus, by linearity the increment of X_t is independent of $\mathcal{F}_t$.

iii). *X_t has stationary Gaussian increments.*

Consider the increment $(X_{t+u} - X_t)$ for $t, u \geq 0$

$$(X_{t+u} - X_t) = \gamma(W^{(1)}_{t+u} - W^{(1)}_t) + \sqrt{1-\gamma^2}(W^{(1)}_{t+u} - W^{(1)}_t)$$

As the increments of $W^{(1)}_t, W^{(2)}_t$ have gaussian distributions, the increments of X_t have gaussian distributions to. Now consider the mean and variance of these increments

$$\begin{aligned}\mathbb{E}[X_{t+u} - X_t] &= \gamma\mathbb{E}[W^{(1)}_{t+u} - W^{(1)}_t] + \sqrt{1-\gamma^2}\mathbb{E}[W^{(2)}_{t+u} - W^{(2)}_t] \\ &= \gamma \cdot 0 + \sqrt{1-\gamma^2} \cdot 0 \\ &= 0\end{aligned}$$

$$\begin{aligned}\text{Var}[X_{t+u} - X_t] &= \gamma^2\text{Var}[W^{(1)}_{t+u} - W^{(1)}_t] + (\sqrt{1-\gamma^2})^2\text{Var}[W^{(2)}_{t+u} - W^{(2)}_t] \\ &= \gamma^2 \cdot u + (1-\gamma^2)u \\ &= u\end{aligned}$$

Thus

$$(X_{t+u} - X_t) \sim \text{Normal}(0, u) \text{ for all } t, u \geq 0$$

iv). *X_t has continuous paths.*

We know that $W^{(1)}_t(\omega), W^{(2)}_t(\omega)$ are continuous functions of t for all ω.

Thus, by linearity, $X_t(\omega)$ is a continuous function of t for all ω.

Since X_t has all four properties, it is a standard Brownian motion.