

Logic - Notes

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1 Introduction

1.1 Alphabets & Strings

Definition 1.1 - Alphabet

An *Alphabet* is a set of symbols from which *Strings* can be created.

Definition 1.2 - String

A *String* over a set \mathcal{A} is any sequence $\alpha := \langle a_1, \dots, a_n \rangle$ where $a_1, \dots, a_n \in \mathcal{A}$.
N.B. Here we say α has *length* n and $\alpha \in \mathcal{A}^n$.

Definition 1.3 - Power Set

Let \mathcal{A} be an alphabet. We define

$$\mathcal{A}^* := \bigcup_{n \in \mathbb{N}} \mathcal{A}^n = \{ \langle a_1, \dots, a_n \rangle : n \in \mathbb{N}; a_1, \dots, a_n \in \mathcal{A} \}$$

This means \mathcal{A}^* is the set of all possible strings over alphabet \mathcal{A} .

Remark 1.1 - Concatenating Strings

Define *Strings* $\alpha := \langle a_1, \dots, a_n \rangle \in \mathcal{A}^n$ and $\beta := \langle b_1, \dots, b_m \rangle \in \mathcal{A}^m$.

We define *Concatenation* of α & β as $\alpha\beta := \langle a_1, \dots, a_n, b_1, \dots, b_m \rangle$ Note that

$$\alpha\beta \neq \langle \alpha, \beta \rangle = \langle \langle a_1, \dots, a_n \rangle, \langle b_1, \dots, b_m \rangle \rangle$$

N.B. Sometimes the following notation is used $\alpha * \beta$.

Example 1.1 - English Alphabet

If we define an alphabet $\mathcal{A} := \{ 'a', \dots, 'z' \}$ then $\langle 't', 'h', 'i', 's' \rangle$ is a *String* of \mathcal{A} .

Remark 1.2 - Ambiguity when using multiple Alphabets

Consider the *Alphabets* $\mathcal{A}_1 := \{0, 1, \dots, 9\}$ & $\mathcal{A}_2 := \mathbb{N}$.

Then we are unsure which of the following definitions of 123 is valid

$$\langle 123 \rangle, \langle 12, 3 \rangle, \langle 1, 23 \rangle, \langle 1, 2, 3 \rangle$$

Remark 1.3 - $\mathcal{A} := \{0, 1\}$ is sufficient to describe any language - binary

Remark 1.4 - Describing Formal Languages

When describing a *Formal Language* we need to provide two things

- i) An *Alphabet* which defines what symbols are allowed.
- ii) A *Grammar* which defines what combinations of symbols are allowed.

1.2 Countable Sets

Definition 1.1 - Countable Set

A set X is said to be *Countable* if

$$\begin{aligned} &\exists \text{ a surjection } f : \mathbb{N} \rightarrow X \\ &\exists \text{ an injection } f : X \rightarrow \mathbb{N} \end{aligned}$$

Definition 1.2 - Countably Infinite Set

A set X is said to be *Countably Infinite* if \exists a bijection $f : X \rightarrow \mathbb{N}$.

Theorem 1.1 - Power set is Countable

If set \mathcal{A} is *countable* then \mathcal{A}^* is *countable*.

Proof 1.1 - Theorem 1.1

Let $f : \mathcal{A} \rightarrow \mathbb{N}$ (This function exists trivially since we define \mathcal{A} to be countable).

Define the following function $g(\cdot) : \mathcal{A}^* \rightarrow \mathbb{N}$

$$g(\langle a_1, \dots, a_n \rangle) := p_1^{f(a_1)+1} \dots p_n^{f(a_n)+1}$$

where p_i is the i^{th} prime.

Since each natural number can be described by a unique composition of primes and since $f(\cdot)$ is injective, then $g(\cdot)$ is injective.

Thus there exists an injection from \mathcal{A}^* to \mathbb{N} , making \mathcal{A}^* countable.

Theorem 1.2 - If \mathcal{A} is countable, then so are $\mathcal{A}^*, (\mathcal{A}^*)^*, \dots$

2 First-Order Languages

Definition 2.1 - First-Order Language, \mathcal{L}

The *Alphabet* of a *First-Order Language*, comprises of the following, pairwise disjoint, categories (and nothing else)

- i) Negation, \neg , and implication, \rightarrow .
- ii) For all, \forall .
- iii) Infinitely many variables, $\{v_0, v_1, \dots\}$.
- iv) Parentheses, (\cdot) , and comma $,$.
- v) Equality, \equiv , which is the only logical predicate symbol with 2-arity.
- vi) A set of constant symbols, $\{c_1, c_2, \dots\}$. (Possibly empty)
- vii) For each $n \geq 1$, a set of n -arity function symbols $\{f_1^n, f_2^n, \dots\}$. (Possibly empty)
- viii) For each $n \geq 1$, a set of n -arity non-logical predicate symbols $\{P_1^n, P_2^n, \dots\}$. (Possibly empty)

N.B. We denote the set of variables by $Var := \{v_0, v_1, \dots\}$; denote a language as \mathcal{L} and the alphabet of \mathcal{L} as $\mathcal{A}_{\mathcal{L}}$.

N.B. In this course *Alphabets* are restricted to being *Countable*.

Definition 2.2 - Negation, \neg

Negation returns in the inverse of a predicate (DO I MEAN PREDICATE)

P	$\neg P$
T	F
F	T

Definition 2.3 - Implication, \rightarrow

Implication returns whether one predicate being true necessarily implies a second predicate being true

P	Q	$P \rightarrow Q$
T	T	T
T	F	F
F	T	T
F	F	T

Remark 2.1 - *First-Order Languages don't have \wedge , \vee , \exists*

Alphabets for *First-Order Languages* do not contain propositional connectives for AND, \wedge , OR, \vee , or EXISTS, \exists since they can be expressed as a combination of negation & implication.

$$\begin{aligned}
 P \vee Q &\iff \neg P \rightarrow Q \\
 P \wedge Q &\iff \neg(P \rightarrow \neg Q) \\
 \exists x \text{ st } P(x) \text{ is true} &\iff \neg(\forall x, \neg P(x))
 \end{aligned}$$

P	Q	$\neg P$	$\neg P \rightarrow Q$	P	Q	$\neg Q$	$P \rightarrow \neg Q$	$\neg(P \rightarrow \neg Q)$
T	T	F	T	T	T	F	F	T
T	F	F	T	T	F	T	T	F
F	T	F	T	F	T	F	T	F
F	F	F	F	F	F	T	T	F

Example 2.1 - *Recursive Definition*

Consider the following, normal, definition

$$x \text{ is a multiple of } 5 \iff \exists y \in \mathbb{Z} \text{ st } y \cdot 5 = x$$

We can instead use the recursive definition

- i) 0 is a multiple of 5.
- ii) If n is a multiple of 5 then $n + 5$ is a multiple of 5.

Definition 2.4 - *\mathcal{L} -Term & Complexity*

Let \mathcal{L} be a *First-Order Language*.

We define *\mathcal{L} -Terms & Complexity*, $cp(\cdot)$, together using the following *recursive definition*

- i) If s is a variable or a constant symbol, then s is an *\mathcal{L} -Term* with $cp(s) = 0$.
N.B. Terms with $cp(\cdot) = 0$ are called *Atomic Terms*.
- ii) If f is a function symbol with k -arity & if a_1, \dots, a_k are *\mathcal{L} -Terms* then $f(a_1, \dots, a_k)$ is an *\mathcal{L} -Term* with complexity

$$cp(f(a_1, \dots, a_k)) := \max\{cp(a_1), \dots, cp(a_k)\} + 1$$

N.B. Terms with $cp(\cdot) \geq 1$ are called *Compound Terms*.

- iii) Nothing else is an *\mathcal{L} -Term*

N.B. We denote the set of \mathcal{L} - Terms by $T_{\mathcal{M}_{\mathcal{L}}}$.

Example 2.2 - *\mathcal{L} -Term & Complexity*

Let $\{c, d, f, g, h, p\} \subseteq \mathbb{L}$ with c, d being constants, g, p being unary functions & f, h being binary functions.

Show that the following is an *\mathcal{L} -Term* & find its *Complexity*

$$h(g(f(x, c)), p(d))$$

- i) x is an *\mathcal{L} -Term* with $cp(x) = 0$ by (i).

- ii) c & d are \mathcal{L} -Terms with $cp(c) = 0 = cp(d)$ by (i).
- iii) $f(x, c)$ is an \mathcal{L} -Term with $cp(f) = \max 0, 0 + 1 = 1$ by (ii).
- iv) $p(d)$ is an \mathcal{L} -Term with $cp(f) = \max 0 + 1 = 1$ by (ii).
- v) $g(f(x, c), p(d))$ is an \mathcal{L} -Term with $cp(g) = \max 1, 1 + 1 = 2$ by (ii).
- vi) $h(g(f(x, c), p(d)))$ is an \mathcal{L} -Term with $cp(h) = \max 2 + 1 = 3$ by (ii).

Thus $h(g(f(x, c), p(d)))$ is an \mathcal{L} -Term with Complexity 3.

Notation 2.1 - More readable Functions

WE often write $x \circ y$ instead of $\circ(x, y)$ as it is more readable (even though the later is technically the only correct notation). Similarly, $x + y$ instead of $+(x, y)$.

Definition 2.5 - Atomic Formulae

Let \mathcal{L} be a First-Order Language.

The atomic \mathcal{L} -Formulae are those strings over $\mathcal{A}_{\mathcal{L}}$ of the form

$$R(t_1, \dots, t_n) \text{ for } n \in \mathbb{N}$$

where R is a predicate symbol of \mathcal{L} with n -arity and t_1, \dots, t_n are \mathcal{L} -terms.

N.B. $\equiv (t_1, t_2)$ is an Atomic \mathcal{L} -Formula for each \mathcal{L} terms t_1, t_2 .

Definition 2.6 - \mathcal{L} -Formulae & Complexity

We define \mathcal{L} -Formulae & Complexity, $cp(\cdot)$, together using the following recursive definition

- i) If $\phi \in \mathcal{A}_{\mathcal{L}}^*$ is an Atomic \mathcal{L} -Formula then ϕ is an \mathcal{L} -Formula with $cp(\phi) = 0$.
- ii) If ϕ is an \mathcal{L} -Formula with $cp(\phi) = n$ then $\neg\phi$ is an \mathcal{L} -Formula with $cp(\neg\phi) = n + 1$.
- iii) If ϕ & ψ are \mathcal{L} -Formulae then $\phi \rightarrow \psi$ is an \mathcal{L} -Formula with $cp(\phi \rightarrow \psi) = \max\{cp(\phi), cp(\psi)\} + 1$.
- iv) if ϕ is an \mathcal{L} -Formula then $\forall x\phi$ is an \mathcal{L} -Formula with $cp(\forall x\phi) = cp(\phi) + 1$, where x is a variable.

N.B. Complexity is just a measure of the syntactic complexity, not semantic. Notice how $cp(\neg\neg\phi) = cp(\phi) + 2$.

Remark 2.2 - Formulae are uniquely readable & parsable

Example 2.3 - \mathcal{L} -Formulae Complexity

Let $\{R, f\} \subset \mathcal{L}$ be binary operations.

Show that the following is an \mathcal{L} -Formula

$$\forall v_0 (\neg R(f(v_0, v_2), v_2) \longrightarrow \underbrace{\equiv (v_0, v_2)}_{v_0 \equiv v_2})$$

- i) v_0, v_2 are \mathcal{L} -Terms.
- ii) $f(v_0, v_2)$ is an \mathcal{L} -Term.
- iii) $R(f(v_0, v_2), v_2)$ is an \mathcal{L} -Formula with $cp(\cdot) = 0$.
- iv) $\neg R(f(v_0, v_2), v_2)$ is an \mathcal{L} -Formula with $cp(\cdot) = 0 + 1 = 1$.

v) $\equiv (v_0, v_2)$ is an \mathcal{L} -Formula with $cp(\cdot) = 0$.

vi) $\neq R(f(v_0, v_2), v_2) \longrightarrow \equiv (v_0, v_2)$ is an \mathcal{L} -Formula with $cp(\cdot) = \max\{0, 1\} + 1 = 2$.

vii) $\forall v_0 (\neg R(f(v_0, v_2), v_2) \longrightarrow \equiv (v_0, v_2))$ is an \mathcal{L} -Formula with $cp(\cdot) = 2 + 1 = 3$.

Notation 2.2 - *Convention for common operators*

To make formulae more readable we generally make the following allowances in notation

$$\begin{aligned} t_1 &\equiv t_2 & \text{for } &\equiv (t_1, t_2) \\ t_1 &\not\equiv t_2 & \text{for } &\neg \equiv (t_1, t_2) \\ t_1 &< t_2 & \text{for } &< (t_1, t_2) \\ t_1 &\not\equiv t_2 & \text{for } &\neg \equiv (t_1, t_2) \end{aligned}$$

Further, when a formula is encapsulated by parentheses then we will often suppress the outermost parentheses (only), as they do not affect anything.

$$\phi \longrightarrow (\psi \longrightarrow \theta) \text{ for } (\phi \longrightarrow (\psi \longrightarrow \theta))$$

Definition 2.7 - *More complex operators*

- AND, $(\phi \wedge \psi) := \neg(\phi \longrightarrow \neg\psi)$.
- OR, $(\phi \vee \psi) := (\neg\phi \longrightarrow \psi)$.
- IFF, $(\phi \longleftrightarrow \psi) := (\phi \longrightarrow \psi) \wedge (\psi \longrightarrow \phi)$.
- EXISTS, $(\exists x\phi) := \neg\forall x \neg\phi$.

Notation 2.3 - *Sets of \mathcal{L} Features*

- $T_{\mathcal{M}_{\mathcal{L}}} :=$ Set of \mathcal{L} -Terms.
- $F_{\mathcal{M}_{\mathcal{L}}} :=$ Set of \mathcal{L} -Formulae.
- $\text{Var} :=$ Set of Variables.

Proposition 2.1 - $T_{\mathcal{M}_{\mathcal{L}}}$ & $F_{\mathcal{M}_{\mathcal{L}}}$ are always countable in this course since we assume \mathcal{L} to be finite.

2.1 Induction of Terms & Formulae

Theorem 2.1 - *Inheritance of a Property - \mathcal{L} -Terms*

Let P be a property of \mathcal{L} -Terms.

Suppose the following to be true

- i) All Atomic \mathcal{L} -Terms have property P .
- ii) $\forall k \in \mathbb{N}, \forall$ function symbols f with k -arity: If \mathcal{L} -Terms t_1, \dots, t_k have property P then $f(t_1, \dots, t_k)$ has P .

Then every \mathcal{L} -Term has property P .

Proof 2.1 - *Theorem 2.1*

This is a proof by contradiction. Suppose that i) & ii) are true but there exists some \mathcal{L} -Term which does not have P .

Let t be an \mathcal{L} -Term with minimum complexity st t does not have P .

Then $cp(t) \neq 0$ otherwise i) would be untrue.

Thus $t \equiv f(t_1, \dots, t_k)$ by the minimality of $cp(t)$.

We know that t_1, \dots, t_k have P .

Thus $f(t_1, \dots, t_k)$ has P . This is a contradiction.

Theorem 2.2 - Inheritance of a Property - \mathcal{L} -Formulae

Let P be a property of \mathcal{L} -Formula.

Suppose the following to be true

- i) All Atomic \mathcal{L} -Formulae have property P .
- ii) If $\phi, \psi \in F_{\mathcal{M}_{\mathcal{L}}}$ have P then $\neg\phi, (\phi \rightarrow \psi)$ & $\forall x\phi$ have P to.

Then every \mathcal{L} -Formulae has property P .

Theorem 2.3 - Number of Parentheses

Every \mathcal{L} -Formula has as many left parentheses as right parentheses.

Every \mathcal{L} -Term has as many left parentheses as right parentheses.

Proof 2.2 - Theorem 2.3

This is a proof by induction.

Let P be the property “Has as many left parentheses as right”.

Base Case - When ϕ is an Atomic \mathcal{L} Formula it trivially has equal number of parentheses.

Inductive Case

Let ϕ & ψ be arbitrary \mathcal{L} -Formulae.

Assume that $P(\phi)$ & $P(\psi)$ hold.

We need to show that $P(\neg\phi)$, $P(\phi \rightarrow \psi)$ & $P(\forall x\phi)$ all hold.

We do not need to show $P(\neg\psi)$, $P(\psi \rightarrow \phi)$ & $P(\forall x\psi)$ hold as ϕ & ψ are arbitrary.

We have that $\neg\phi$ and $\forall x\phi$ don't add any brackets, so P holds.

We have that $(\phi \rightarrow \psi)$ add one left & one right parentheses (although they are often suppressed), thus P holds.

Thus by the process of mathematical induction P holds for all \mathcal{L} -Formulae.

N.B. The proof for \mathcal{L} -Terms is very similar.

2.2 Free Variables

Definition 2.1 - Variable Function, $var(\cdot)$

Define $var : \mathcal{A}_{\mathcal{L}}^* \rightarrow 2^{\text{Var}}$ st $var(s)$ is the set of all variables in string s .

Example 2.4 - $\text{Var}(\cdot)$

$$\begin{aligned} var(f(x, f(y, c))) &= \{x, y\} \\ var(f(c, f(c, c))) &= \emptyset \\ var(\equiv, \equiv, \equiv) &= \emptyset \text{ nonsense strings are acceptable} \end{aligned}$$

Definition 2.2 - Free Variables

Free Variables are variables whose value are ambiguous in an \mathcal{L} -Formula.

Definition 2.3 - Free Variable Function, $FV(\cdot)$

We recursively define $FV(\phi)$ for \mathcal{L} -Formulae as ϕ as follows

- i) $FV(\phi) = var(\phi)$ if ϕ is an Atomic \mathcal{L} -Formula.
- ii) $FV(\neg\phi) = FV(\phi)$.
- iii) $FV((\phi \rightarrow \psi)) = FV(\phi) \cup FV(\psi)$.

iv) $FV(\forall x\phi) = FV(\phi) \setminus \{x\}$.

Example 2.5 - Free Variable Function

$$\begin{aligned} FV(\forall x(P(y) \rightarrow Q(x))) &= FV(P(y) \rightarrow Q(x)) \setminus \{x\} \\ &= [FV(P(y)) \cup FV(Q(x))] \setminus \{x\} \\ &= [\{y\} \cup \{x\}] \setminus \{x\} \\ &= \{y\} \end{aligned}$$

Proposition 2.2 - Free Variable Function for more complex operators

$$\begin{aligned} FV(\phi \wedge \psi) &= FV(\neg(\phi \rightarrow \neg\psi)) \text{ by definition of } \wedge \\ &= FV(\phi) \cup FV(\psi) \\ FV(\phi \vee \psi) &= FV(\neg\phi \rightarrow \psi) \text{ by definition of } \vee \\ &= FV(\phi) \cup FV(\psi) \\ FV(\exists x\phi) &= FV(\neg\forall x\neg\phi) \text{ by definition of } \exists \\ &= FV(\phi) \setminus \{x\} \end{aligned}$$

Definition 2.4 - Closed \mathcal{L} -Term

Let t be an \mathcal{L} -Term.

If $var(t) = \emptyset$ then t is called a *Closed \mathcal{L} -Term*.

Definition 2.5 - \mathcal{L} -Sentence

Let ϕ be an \mathcal{L} -Formula.

If $FV(\phi) = \emptyset$ then ϕ is called an *\mathcal{L} -Sentence*.

Example 2.6 - \mathcal{L} -Sentence

$$\begin{aligned} FV(\forall x(P(x) \rightarrow \exists y R(y, x))) &= FV((P(x) \rightarrow \exists y R(y, x)) \setminus \{x\}) \\ &= FV(P(x)) \cup FV(\exists y R(y, x)) \setminus \{x\} \\ &= \{x\} \cup (FV(R(y, x) \setminus \{y\}) \setminus \{x\}) \\ &= \{x\} \cup (\{y, x\} \setminus \{y\}) \setminus \{x\} \\ &= \{x\} \cup \{x\} \setminus \{x\} \\ &= \emptyset \end{aligned}$$

Remark 2.3 - \mathcal{L} -Sentences have no Free Variables and thus no ambiguity in meaning.

3 Semantics of First-Order Languages

3.1 Structures, Variable Assignments & Satisfaction

Definition 3.1 - \mathcal{L} -Structure

Let \mathcal{L} be a first-order language.

An *\mathcal{L} -Structure* is an ordered pair $\mathfrak{M} = \langle D, \mathfrak{I} \rangle$

- i) D is a non-empty set.
- ii) \mathfrak{I} is a function on the non-logical symbols of \mathcal{L} st
 - For each predicate symbol $P \in \mathcal{L}$ with n -arity.

$$\mathfrak{I}(P) \subset D^n$$

- For each function symbol f of \mathcal{L} with n -arity

$$\mathfrak{I}(f) : D^n \rightarrow D$$

- For each constant symbol c of \mathcal{L}

$$\mathfrak{I}(c) \in D$$

N.B. D is the domain, \mathfrak{I} is the interpretation.

Notation 3.1 - \mathcal{L} -Structure

For ease we use the following notation wrt \mathcal{L} -Structure

$$|\mathfrak{M}| := D \quad f^{\mathfrak{M}} := \mathfrak{I}(f) \quad c^{\mathfrak{M}} := \mathfrak{I}(c) \quad p^{\mathfrak{M}} = \mathfrak{I}(p)$$

Example 3.1 - \mathcal{L} -Structure

Let $\mathcal{L}_{\text{Rng}} := \{\bar{0}, \bar{1}, \bar{+}, \bar{\cdot}\}$ where $\bar{+}$ & $\bar{\cdot}$ are binary functions and $\bar{0}$ & $\bar{1}$ are constants.

(This is the language for ring theory)

We use the overline to distinguish language symbols from standard symbols.

Define

$$\begin{aligned} D &:= \mathbb{R} \\ \mathfrak{I}(\bar{0}) &= 0 \in \mathbb{R} \\ \mathfrak{I}(\bar{1}) &= 1 \in \mathbb{R} \\ \mathfrak{I}(\bar{+}) &: \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R} \text{ with } (a, b) \mapsto (a + b) \\ \mathfrak{I}(\bar{\cdot}) &: \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R} \text{ with } (a, b) \mapsto (a \cdot b) \end{aligned}$$

We recall $\langle D, \mathfrak{I} \rangle$ is the standard model of the real field.

N.B. We can alternatively write $\langle D, \mathfrak{I} \rangle = \langle \mathbb{R}, 0, 1, +, \cdot \rangle$ for neatness.

Definition 3.2 - Variable Assignment

A *Variable Assignment* over an \mathcal{L} -Structure is a function which maps from the set of variables to the domain of the \mathcal{L} -Structure.

$$s : \text{Var} \rightarrow |\mathfrak{M}|$$

Definition 3.3 - Extension of Variable Assignment

Let \mathfrak{M} be an \mathcal{L} -Structure & s be a variable assignment over \mathfrak{M} .

The function $\bar{s} : T_{\mathfrak{M}_{\mathcal{L}}} \rightarrow |\mathfrak{M}|$ is defined using the following recursion

- i) $\bar{s}(t) = s(t)$ if $t \in \text{Var}$.
- ii) $\bar{s}(t) = t^{\mathfrak{M}}$ if t is a constant symbol.
- iii) $\bar{s}(f(t_1, \dots, t_k)) = f^{\mathfrak{M}}(\bar{s}(t_1), \dots, \bar{s}(t_k))$.

Example 3.2 - Variable Assignment

Let \mathfrak{M} be the standard model of the real field.

Let s be a variable assignment over \mathfrak{M} st $s(x) = s(y) = \pi$. Then

$$\begin{aligned} \bar{s}(x \bar{+} y) &= \bar{+}^{\mathfrak{M}}(\bar{s}(x), \bar{s}(y)) \\ &= \bar{+}^{\mathfrak{M}}(s(x), s(y)) \\ &= \bar{+}^{\mathfrak{M}}(\pi, \pi) \\ &= \pi + \pi \\ &= 2\pi \end{aligned}$$

Theorem 3.1 - Substitution

Let s be a variable assignment over \mathfrak{M} , $x \in \text{Var}$ & $d \in |\mathfrak{M}|$.

A new variable assignment $\frac{sd}{x}$ over \mathfrak{M} is defined as

$$\frac{sd}{x}(y) = \begin{cases} d & \text{if } y = x \\ s(y) & \text{otherwise} \end{cases}$$

Definition 3.4 - Satisfaction Relation

Let \mathfrak{M} be an \mathcal{L} -Structure & s be a variable assignment over \mathfrak{M} .

The *Satisfaction Relation*, $\mathfrak{M}, s \models \phi$ between \mathfrak{M}, s and \mathcal{L} -Formula ϕ is recursively defined as

- i) $\mathfrak{M}, s \models t_1 \equiv t_2$ iff $\bar{s}(t_1) = \bar{s}(t_2)$.
- ii) $\mathfrak{M}, s \models P(t_1, \dots, t_k)$ iff $\langle \bar{s}(t_1), \dots, \bar{s}(t_k) \rangle \in P^{\mathfrak{M}} \subset D^k$.
- iii) $\mathfrak{M}, s \models \neg \phi$ iff $\mathfrak{M}, s \not\models \phi$.
- iv) $\mathfrak{M}, s \models \phi \rightarrow \psi$ iff if $\mathfrak{M}, s \models \phi$ then $\mathfrak{M}, s \models \psi$.
- v) $\mathfrak{M}, s \models \forall x \phi$ iff for all $d \in |\mathfrak{M}|$, $\mathfrak{M}_{\frac{sd}{x}} \models \phi$.

Proposition 3.1 - Extension of Satisfaction Relation

$$\begin{array}{lll}
 \mathfrak{M}, s \models \phi \wedge \psi & \text{iff} & \mathfrak{M}, s \models \phi \text{ and } \mathfrak{M}, s \models \psi \\
 \mathfrak{M}, s \models \phi \vee \psi & \text{iff} & \mathfrak{M}, s \models \phi \text{ or } \mathfrak{M}, s \models \psi \\
 \mathfrak{M}, s \models \phi \leftrightarrow \psi & \text{iff} & \mathfrak{M}, s \models \phi \text{ iff } \mathfrak{M}, s \models \psi \\
 \mathfrak{M}, s \models \exists x \phi & \text{iff} & \mathfrak{M}, s_{\frac{d}{x}} \models \phi \text{ for some } d \in |\mathfrak{M}|
 \end{array}$$

Definition 3.5 - Model

Let $\Phi \subset \text{Fml}_{\mathcal{L}}$, a subset of formulae of a first order language \mathcal{L} .

\mathfrak{M}, s is a *model* of Φ iff $\mathfrak{M}, s \models \phi$ for all $\phi \in \Phi$.

N.B. This is denoted $\mathfrak{M}, s \models \Phi$.

Example 3.3 - Model

Let \mathfrak{M} be the standard model of ring theory & s be a variable assignment over \mathfrak{M} st $s(v_1) = 3$ & $s(v_2) = -\pi$.

$$\begin{aligned}
 & \mathfrak{M}, s \models \bar{0} < v_1 + v_2 \\
 \iff & <^{\mathfrak{M}}(\bar{s}(\bar{0}), \bar{s}(v_1 + v_2)) \\
 \iff & <^{\mathfrak{M}}(\bar{0}^{\mathfrak{M}}, \bar{+}^{\mathfrak{M}}(\bar{s}(v_1), \bar{s}(v_2))) \\
 \iff & <^{\mathfrak{M}}(0, \bar{+}^{\mathfrak{M}}(s(v_1), s(v_2))) \\
 \iff & 0 < 3 + (-\pi) \\
 \implies & \mathfrak{M}, s \not\models \bar{0} < v_1 + v_2
 \end{aligned}$$

$$\begin{aligned}
 & \mathfrak{M}, s \models \forall v_2 \exists v_0, v_2 < v_0 \\
 \iff & \text{for all } d \in \mathbb{R}, \mathfrak{M}_{s_{\frac{d}{v_2}}} \models \exists v_0, v_2 < v_0 \\
 \iff & \text{for all } e \in \mathbb{R}, \mathfrak{M}(s_{\frac{d}{v_2}})_{\frac{e}{v_0}} \models v_2 < v_0 \\
 \iff & \text{for all } d \in \mathbb{R} \text{ there is } e \in \mathbb{R}, s(\frac{d}{v_2})_{\frac{e}{v_0}} v_2 < (s_{\frac{d}{v_0}})_{\frac{e}{v_0}} v_0 \\
 \iff & \text{for all } d \in \mathbb{R} \text{ there is } e \in \mathbb{R} \text{ st } d < e
 \end{aligned}$$

Theorem 3.2 -

Let $\mathcal{L}_1, \mathcal{L}_2$ be first order languages.

Define models

$$\begin{aligned}
 \mathfrak{M}_1 &= \langle D, \mathcal{I}_1 \rangle : \mathcal{L}_1 \text{ structure} \\
 \mathfrak{M}_1 &= \langle D, \mathcal{I}_2 \rangle : \mathcal{L}_2 \text{ structure}
 \end{aligned}$$

Note that D is the same for both (*i.e.* Different languages, same world).

Let $\mathcal{L} := \mathcal{L}_1 \cup \mathcal{L}_2$

- i) For all \mathcal{L} -Terms, t , for all variable assignments s_1 over \mathfrak{M}_1 & s_2 over \mathfrak{M}_2 .
 If $\begin{cases} c^{\mathfrak{M}_1} = c^{\mathfrak{M}_2} & \text{for all constants symbols that occur in } t \\ f^{\mathfrak{M}_1} = f^{\mathfrak{M}_2} & \text{for all functions symbols that occur in } t \text{ then } \bar{s}_1(t) = \bar{s}_2(t). \\ s_1(x) = s_2(x) & \text{for all variable symbols that occur in } t \end{cases}$

- ii) For all $\phi \in \text{Fml}_{\mathcal{L}}$ & for all variable assignments s_1 over \mathfrak{M}_1 & s_2 over \mathfrak{M}_2 .
- If $\begin{cases} c^{\mathfrak{M}_1} = c^{\mathfrak{M}_2} & \text{for all constants symbols that occur in } \phi \\ f^{\mathfrak{M}_1} = f^{\mathfrak{M}_2} & \text{for all functions symbols that occur in } \phi \\ R^{\mathfrak{M}_1} = R^{\mathfrak{M}_2} & \text{for all predicate symbols that occur in } \phi \\ s_1(x) = s_2(x) & \text{for all variable symbols in } \text{FV}(\phi) \end{cases}$ then $\mathfrak{M}_1 s_1 \models \phi$ iff $\mathfrak{M}_2 \models \phi$.

N.B. If $\mathcal{L} = \emptyset$ it is not very interesting.

Proof 3.1 - Theorem 3.2 i)

This is a proof by induction on \mathcal{L} -Term.

Base Case

Let t be atomic then $\bar{s}_1(t) = \bar{s}_2(t)$ is trivial, $\bar{s}_1(x) = s_1(x) = s_2(x) = \bar{s}_2(x)$ and $\bar{s}_1(c) = c^{\mathfrak{M}_1} = c^{\mathfrak{M}_2} = \bar{s}_2(c)$.

Inductive Case

Let $t = f(t_1, \dots, t_k)$. Then

$$\begin{aligned} \bar{s}_1(t) &= f^{\mathfrak{M}_1}(\bar{s}_1(t_1), \dots, \bar{s}_1(t_k)) \\ &= f^{\mathfrak{M}_1}(\bar{s}_2(t_1), \dots, \bar{s}_2(t_k)) \text{ by inductive hypothesis} \\ &= f^{\mathfrak{M}_2}(\bar{s}_2(t_1), \dots, \bar{s}_2(t_k)) \\ &= \bar{s}_2(t) \end{aligned}$$

□

Proof 3.2 - Theorem 3.2 ii)

This is a proof by induction on \mathcal{L} -Formulae. **Base Case**

Let $\phi = R(t_1, \dots, t_k)$ be an atomic \mathcal{L} -formula (*i.e.* $\text{cp}(\phi) = 0$).

Note that $\text{FV}(\phi) = \text{var}(\phi)$, the conditions in **ii)** for ϕ imply the conditions of **i)** for t_1, \dots, t_k .

Therefore $\bar{s}_i(t_i) = \bar{s}_1(t_i) \forall i \in [1, k]$. Then

$$\begin{aligned} &\mathfrak{M}_1, s_1 \models \phi \\ \iff &\langle \bar{s}_1(t_1), \dots, \bar{s}_1(t_k) \rangle \in R^{\mathfrak{M}_1} \\ \iff &\langle \bar{s}_2(t_1), \dots, \bar{s}_2(t_k) \rangle \in R^{\mathfrak{M}_1} \\ \iff &\langle \bar{s}_2(t_1), \dots, \bar{s}_2(t_k) \rangle \in R^{\mathfrak{M}_2} \\ \iff &\mathfrak{M}_2, s_2 \models \phi \end{aligned}$$

Inductive Case

Let $\phi := \psi \rightarrow \theta$.

Since the conditions hold for ϕ they hold for ψ & θ . Then

$$\begin{aligned} &\mathfrak{M}_1, s_1 \models \phi \\ \iff &\text{if } \mathfrak{M}_1, s_1 \models \psi \text{ then } \mathfrak{M}_1, s_1 \models \theta \\ \stackrel{\text{by IH}}{\iff} &\text{if } \mathfrak{M}_2, s_2 \models \psi \text{ then } \mathfrak{M}_2, s_2 \models \theta \\ \iff &\mathfrak{M}_2, s_2 \models \phi \end{aligned}$$

Let $\phi := \neg\psi$.

By the inductive hypothesis the claim holds for ψ .

Since the conditions of **ii)** hold for ϕ they hold for ψ .

Note that $\text{FV}(\neg\psi) = \text{FV}(\psi)$. Hence

$$\begin{aligned} &\mathfrak{M}_1, s_1 \models \phi \\ \iff &\mathfrak{M}_1, s_1 \not\models \psi \\ \stackrel{\text{by IH}}{\iff} &\mathfrak{M}_2, s_2 \not\models \psi \\ \iff &\mathfrak{M}_2, s_2 \models \phi \end{aligned}$$

Let $\phi := \forall z\psi$.

By the last condition of **ii**) we have that $s_1(x) = s_2(x) \forall x \in FV(\phi)$.

Since $FV(\phi) \subset FV(\psi) \subset FV(\phi) \cup \{z\}$ it holds that $\forall d \in D$ that $s_1 \frac{d}{z}(x) = s_2 \frac{d}{z}(x) \forall x \in FV(\psi)$.
Meaning that $\forall d \in D$ the conditions of **ii**) hold for ψ wrt $s_1 \frac{d}{z}$ & $s_2 \frac{d}{z}$.

Hence

$$\begin{aligned} & \mathfrak{M}_1, s_1 \models \phi \\ \iff & \forall d \in D, \mathfrak{M}_1, s_1 \frac{d}{z} \models \psi \\ \stackrel{\text{by IH}}{\iff} & \forall d \in D, \mathfrak{M}_2, s_2 \frac{d}{z} \models \psi \\ \iff & \mathfrak{M}_2, s_2 \models \phi \end{aligned}$$

□

Theorem 3.3 -

Let \mathfrak{M} be an \mathcal{L} -Structure, t be a closed \mathcal{L} -Term, ϕ to be an \mathcal{L} -Sentence and s_1, s_2 to be variable assignments over \mathfrak{M} .

Then $\bar{s}_1(t) = \bar{s}_2(t)$ and $\mathfrak{M}, s_1 \models \phi$ iff $\mathfrak{M}, s_2 \models \phi$.

N.B. This is since t is closed and thus its semantic value is independent of variable assignment.

Notation 3.2 -

Let t be a closed \mathcal{L} -Term & ϕ an \mathcal{L} -Sentence.

We use the following notation

$$\begin{aligned} t^{\mathfrak{M}} &:= \text{the unique } d \in D \text{ st } \bar{s}(t) = d \text{ for **some** variable assignment } s \text{ over } \mathfrak{M} \\ &:= \text{the unique } d \in D \text{ st } \bar{s}(t) = d \text{ for **all** variable assignment } s \text{ over } \mathfrak{M} \\ \mathfrak{M} \models \phi &:= \mathfrak{M}, s \models \phi \text{ for **some** variable assignment } s \text{ over } \mathfrak{M} \\ &:= \mathfrak{M}, s \models \phi \text{ for **all** variable assignment } s \text{ over } \mathfrak{M} \end{aligned}$$

N.B. The s is dropped in $\mathfrak{M} \models \phi$.

3.2 Important Semantic Concepts

Remark 3.1 - Throughout this section \mathcal{L} will be a first-order language

Definition 3.1 - Logical Consequence

Let $\phi \in \text{Fml}_{\mathcal{L}}$ & $\Phi \subset \text{Fml}_{\mathcal{L}}$.

ϕ is said to be a *Logical Consequence* of Φ iff $\forall \mathcal{L}$ -Structures, \mathfrak{M} , and variable assignments, s , over \mathfrak{M} if $\mathfrak{M}, s \models \Phi$ then $\mathfrak{M}, s \models \phi$.

N.B. When this holds we say ϕ logically follows from Φ , denoted $\Phi \models \phi$.

Example 3.4 - Logical Consequence

TODO

Proposition 3.2 -

$\forall \phi, \psi \in \text{Fml}_{\mathcal{L}}$ and $\Phi \subset \text{Fml}_{\mathcal{L}}$

$$\underbrace{\Phi, \phi}_{\equiv \Phi \cup \{\phi\}} \models \psi \text{ iff } \Phi \models \phi \rightarrow \psi$$

Proof 3.3 - Proposition 3.2

$$\begin{aligned} & \Phi, \phi \models \psi \\ \iff & \forall \mathfrak{M}, s \text{ if } \mathfrak{M}, s \models \theta \forall \theta \in \Phi \cup \{\psi\} \text{ then } \mathfrak{M}, s \models \psi \\ \iff & \forall \mathfrak{M}, s \text{ if } \mathfrak{M}, s \models \theta \forall \theta \in \Phi \text{ and } \mathfrak{M}, s \models \phi \text{ then } \mathfrak{M}, s \models \psi \\ \iff & \text{if } \mathfrak{M}, s \models \theta \forall \theta \in \Phi \text{ then } \mathfrak{M}, s \models \phi \text{ implies } \mathfrak{M}, s \models \psi \\ \stackrel{\text{by def}}{\iff} & \Phi \models \phi \rightarrow \psi \end{aligned}$$

Definition 3.2 -

Let Λ be a set of \mathcal{L} -sentences

$$\mathfrak{M} \models \Lambda \text{ iff } \mathfrak{M} \models \sigma \ \forall \sigma \in \Lambda$$

Example 3.5 -

Let $\mathcal{L} = \mathcal{L}_{\text{GT}} = \{e, \cdot\}$ the language of group theory.

Let $\Phi = \{\forall x \forall y \forall z (x \cdot y) \cdot z \equiv x \cdot (y \cdot z), \forall x \cdot e \equiv x, \forall x \exists y x \cdot y \equiv e\}$.

Then $\Phi \models \forall x \exists y$ st $y \cdot x \equiv e$ but $\Phi \not\models \forall x \forall y (x \cdot y) \equiv (y \cdot x)$.

Remark 3.2 -

We always have either $\mathfrak{M}, s \models \phi$ or $\mathfrak{M}, s \not\models \phi$ since $\mathfrak{M}, s \models \phi \Leftrightarrow \mathfrak{M}, s \not\models \neg \phi$.

But it is not always the case that either $\Phi \models \phi$ or $\Phi \not\models \phi$. (There may be some elements in a group with fulfil a criteria by chance).

Definition 3.3 - Logically Valid

Let $\phi \in \text{Fml}_{\mathcal{L}}$.

ϕ is said to be *Logically Valid* iff $\mathfrak{M}, s \models \phi \ \forall \mathfrak{M}, s$.

N.B. This is also known as valid & logically true.

N.B. Denoted $\models \phi$ for short.

Example 3.6 - Logically Valid

- $\forall x \exists x \equiv y$ is *Logically Valid* since trivially true for $y = x$.

- $\exists x P(x)$ is not *Logically Valid*.

Consider the case where $|\mathfrak{M}| = \mathbb{N}$ & $P^{\mathfrak{M}} = \emptyset$ where $\mathfrak{M} \not\models \exists x P(x)$.

Definition 3.4 - Satisfiable

Let $\phi \in \text{Fml}_{\mathcal{L}}$ & $\Phi \subset \text{Fml}_{\mathcal{L}}$.

ϕ is *Satisfiable* iff $\mathfrak{M}, s \models \phi$ for some \mathfrak{M}, s .

Φ is *Satisfiable* iff $\mathfrak{M}, s \models \Phi$ for some \mathfrak{M}, s .

Example 3.7 - Satisfiable

- $\exists x P(x)$ is *Satisfiable*.

Since $|\mathfrak{M}| = \mathbb{N}$ & $P^{\mathfrak{M}} = |\mathfrak{M}| = \mathbb{N}$ satisfies $\exists x P(x)$.

- $x \neq x$ is not *Satisfiable* as $\bar{s}(x) = s(x) = s(x) = \bar{s}(x)$ always and so $\mathfrak{M}, s \models x \equiv x \ \forall \mathfrak{M}, s$.

Theorem 3.4 -

Let $\phi \in \text{Fml}_{\mathcal{L}}$ & $\Phi \subset \text{Fml}_{\mathcal{L}}$.

i) ϕ is *Logically Valid* iff $\emptyset \models \phi$.

ii) $\Phi \models \phi$ iff $\Phi \cup \{\neg \phi\}$ is *Unsatisfiable*.

iii) ϕ is logically valid iff $\neg \phi$ is *Unsatisfiable*.

Proof 3.4 - Theorem 3.4

i)

$$\begin{aligned} & \emptyset \models \phi \\ \iff & \text{for all } \mathfrak{M}, s \text{ if } \underbrace{\mathfrak{M}, s \models \theta \ \forall \theta \in \emptyset}_{\text{Vacuously true}} \text{ then } \mathfrak{M}, s \models \phi \\ \iff & \forall \mathfrak{M}, s \ \mathfrak{M}, s \models \phi \\ \iff & \phi \text{ is logically valid} \end{aligned}$$

ii)

$$\begin{aligned}
& \Phi \models \phi \\
\iff & \forall \mathfrak{M}, s \text{ if } \mathfrak{M}, s \models \phi \text{ then } \mathfrak{M}, s \models \phi \\
\iff & \text{there is no } \mathfrak{M}, s \text{ st } \mathfrak{M}, s \models \Phi \text{ and } \underbrace{\mathfrak{M}, s \not\models \phi}_{\mathfrak{M}, s \models \neg \phi}
\end{aligned}$$

iii) By i) & ii)

Definition 3.5 - Logically EquivalentLet $\phi, \psi \in \text{Fml}_{\mathcal{L}}$. ϕ is *Logically Equivalent* to ψ iff $\phi \models \psi$ & $\psi \models \phi$.N.B. Equivalently $\models \phi \leftrightarrow \psi$.**Proposition 3.3 - Logical Equivalence**Let $\phi, \psi \in \text{Fml}_{\mathcal{L}}$. ϕ & ψ are *Logically Equivalent* iff $\models \phi \rightarrow \psi$.i.e. $\phi \leftrightarrow \psi$ is *Logically True*.**Proof 3.5 - Logical Equivalence**Recall that $\Phi, \phi \models \psi$ iff $\Phi \models \phi \rightarrow \psi$. Thus

$$\begin{aligned}
& \phi \models \psi \\
\iff & \emptyset \cup \{\phi\} \models \\
\iff & \emptyset \models \phi \rightarrow \psi \\
\iff & \phi \rightarrow \psi
\end{aligned}$$

Similar for converse.

We have that $\phi \models \psi$ and $\psi \models \phi$ iff $\models \phi \rightarrow \psi$ and $\models \psi \rightarrow \phi$. $\iff \phi \models \psi$ and $\psi \models \phi$ iff $\models \phi \leftrightarrow \psi$.**Proposition 3.4 - Logical Equivalence**i) $((\phi \wedge \psi) \wedge \theta)$ is logically equivalent to $(\phi \wedge (\psi \wedge \theta))$.ii) $((\phi \vee \psi) \vee \theta)$ is logically equivalent to $(\phi \vee (\psi \vee \theta))$.iii) $\neg\neg\phi$ is logically equivalent to ϕ .iv) $\phi \wedge \psi$ is logically equivalent to $\neg(\neg\phi \vee \neg\psi)$.N.B. We write $\phi \wedge \psi \wedge \theta$ for $(\phi \wedge \psi) \wedge \theta$.**3.3 Substitution****Remark 3.3 -**If we have $P(x) \rightarrow Q(x)$ then $P(\bar{0}) \rightarrow Q(\bar{0})$ and $P(f(y)) \rightarrow Q(f(y))$.If we have $\forall x(P(\bar{0}) \rightarrow Q(\bar{0}))$ the $\forall x$ is redundant.**Definition 3.1 - Substitution in an \mathcal{L} -Term**Let $a, t \in T_{\mathcal{M}_{\mathcal{L}}}$. $[a]_{\frac{t}{x}}$ denotes the result of replacing all occurrences of x in a with t .

We define this substitution using the following recursive definition

i) When a is atomic:

$$[a]_{\frac{t}{x}} := \begin{cases} t & \text{if } a \equiv x \\ a & \text{otherwise} \end{cases}$$

ii) When $a = f(a_1, \dots, a_j)$ is a compound:

$$[a] \frac{t}{x} := f \left([a_1] \frac{t}{x}, \dots, [a_k] \frac{t}{x} \right)$$

Example 3.8 - *Substitution in an \mathcal{L} -Term*

$$\begin{aligned} & [(x + y) + z] \frac{\bar{0} \cdot \bar{0}}{y} \\ \iff & [x + y] \frac{\bar{0} \cdot \bar{0}}{y} \cdot [z] \frac{\bar{0} \cdot \bar{0}}{y} \\ \iff & \left([x] \frac{\bar{0} \cdot \bar{0}}{y} + [y] \frac{\bar{0} \cdot \bar{0}}{y} \right) \cdot z \\ \iff & (x + \bar{0} \cdot \bar{0}) \cdot z \end{aligned}$$

Definition 3.2 - *Substitution in an \mathcal{L} -Formula*

Let $t \in T_{\mathfrak{M}_{\mathcal{L}}}$, $x \in \text{Var}$ and $\phi \in \text{Fml}_{\mathcal{L}}$.

$[\phi] \frac{t}{x}$ denotes the result of replacing all occurrences of x in ϕ with t .

We define this substitution using the following recursive definition

i) When $\phi := (P(a_1, \dots, a_t))$ is atomic:

$$[\phi] \frac{t}{x} = P \left([a_1] \frac{t}{x}, \dots, [a_k] \frac{t}{x} \right)$$

ii) When $\phi := \neg\psi$:

$$[\phi] \frac{t}{x} := \neg[\psi] \frac{t}{x}$$

iii) When $\phi := \psi \rightarrow \theta$:

$$[\phi] \frac{t}{x} := [\psi] \frac{t}{x} \rightarrow [\theta] \frac{t}{x}$$

iv) When $\phi := \forall z\psi$:

$$[\phi] \frac{t}{x} = \begin{cases} \forall z[\psi] \frac{t}{x} & \text{if } x \neq z \\ \phi & \text{otherwise} \end{cases}$$

N.B. The **otherwise** case is due to all x variables being bounded.

Example 3.9 - *Substitution in an \mathcal{L} -Formula*

Let $x \neq y$

$$\begin{aligned} & [\forall x P(x) \rightarrow \forall y R(x, y)] \frac{c}{x} \\ = & [\forall x P(x)] \frac{c}{x} \rightarrow [\forall y R(x, y)] \frac{c}{x} \\ = & \forall x P(x) \rightarrow \forall y [R(x, y)] \frac{c}{x} \\ = & \forall x P(x) \rightarrow \forall y R([x] \frac{c}{x}, [y] \frac{c}{x}) \\ = & \forall x P(x) \rightarrow \forall y R(c, y) \end{aligned}$$

Proposition 3.5 -

For every $x \in \text{Var}$, $a, t \in T_{\mathfrak{M}_{\mathcal{L}}}$ and $\phi \in \text{Fml}_{\mathcal{L}}$ the following results hold

$$[a] \frac{x}{x} = a \quad \text{and} \quad [\phi] \frac{x}{x} = \phi$$

Proof 3.6 - *Proposition 3.5*

TODO

Induction on terms and then on formulae

Proposition 3.6 -

For every $x \in \text{Var}$, $a, t \in T_{\mathfrak{M}_{\mathcal{L}}}$ and $\phi \in \text{Fml}_{\mathcal{L}}$.

- i) If $x \notin \text{Var}(a)$ then $[a]_x^t = a$.
- ii) If $x \notin \text{FV}(\phi)$ then $[\phi]_x^t = \phi$.

Proof 3.7 - Proposition 3.6 ii)

Proof by induction on terms, and then on formulae.

Base Case

Let $\phi := P(a_1, \dots, a_k)$ be atomic.

Suppose $x \in \text{FV}(\phi)$

We have that $x \notin \text{FV}(\phi) = \underbrace{\text{Var}(\phi)}_{\phi \text{ is atomic}}$

Then $x \notin \text{Var}(a_i) \forall i \in [1, k]$.

$$\begin{aligned}
 & [\phi]_x^t \\
 &= P([a_1]_x^t, \dots, [a_k]_x^t) \\
 &= P(a_1, \dots, a_k) \\
 &= \phi
 \end{aligned}$$

Result holds for base case

Inductive Case Let $\phi = \forall z\psi$.

Suppose $x \notin \text{FV}(\phi)$.

Then $\text{FV}(\phi) = \text{FV}(\psi) \setminus \{z\}$.

$$\text{If } x \notin \text{FV}(\psi) \text{ then } [\phi]_x^t = \begin{cases} \phi & \text{if } x = z \\ \forall z [\underbrace{\phi]_x^t}_{\text{by IH}} & \text{otherwise.} \end{cases}$$

Otherwise $x = z$ and $[\phi]_x^t = \phi$.

Proposition 3.7 -

Let $a, t \in T_{\mathfrak{M}_{\mathcal{L}}}$, $\phi \in \text{Fml}_{\mathcal{L}}$, $x \in \text{Var}$ with $x \notin t$. Then

- i) $x \notin \text{Var}([a]_x^t)$.
- ii) $x \notin \text{FV}([\phi]_x^t) \subset ([\phi]_x^t)$.
- iii) $x \notin \text{Var}([\phi]_x^t)$.

N.B. Proof is done by induction on terms & formulae.

Remark 3.4 -

From $\forall x P(x)$ being true, we can infer that $P(y) \equiv [P(x)]_x^y$, $P(f(y)) = [P(x)]_x^{f(y)}$ & $P(\bar{0}) \equiv [P(\bar{0})]_x^{\bar{0}}$ are all true.

but $\forall x \phi$ being true does not mean that $[\phi]_x^y$ and $[\phi]_x^{\bar{0}}$ are true.

Example 3.10 - Remark 3.4

We have that $\forall x \exists y y \prec x$ is true, but $\exists y y \prec y$ is not true.

i.e. Everyone is liked by someone $\not\Rightarrow$ someone is liked by themselves.

Definition 3.3 - Substitutability

Refers to valid substitutions.

Let $x \in \text{Var}$, $t \in T_{\mathfrak{M}_{\mathcal{L}}}$ & $\phi \in \text{Fml}_{\mathcal{L}}$.

We use the notation $\text{SubSt}(t, x, \phi)$ to mean t is substitutable for x in ϕ .

We define *Substitutability* using the following recursive definition

- i) ϕ is atomic then

$$\text{SubSt}(t, x, \phi) \forall x \in \text{Var}, t \in T_{\mathfrak{M}_{\mathcal{L}}}$$

ii) $\phi = \not\psi$ then

$$\text{SubSt}(t, x, \phi) \text{ iff } \text{SubSt}(t, x, \psi)$$

iii) $\phi = \psi \rightarrow \theta$ then

$$\text{SubSt}(t, x, \phi) \text{ iff } \text{SubSt}(t, x, \psi) \wedge \text{SubSt}(t, x, \theta)$$

iv) $\phi = \forall z\psi$ then

$$\text{SubSt}(t, x, \phi) \text{ iff } \begin{cases} z \notin \text{Var}(t) \wedge \text{SubSt}(t, x, \psi) & \text{or} \\ x \notin FV(\phi) \end{cases}$$

N.B. the second case in *iv*) is a vacuous case since there are no variables to substitute.

Example 3.11 - Substitutability

Let $x \neq y \neq z$

- i) $\text{SubSt}(t, x, P(x)) \forall t \in T_{\mathfrak{M}_{\mathcal{L}}}$.
- ii) $\text{SubSt}(t, y, P(y)) \forall t \in T_{\mathfrak{M}_{\mathcal{L}}}$.
- iii) $\text{SubSt}(t, x, \neg P(z)) \forall t \in T_{\mathfrak{M}_{\mathcal{L}}}$.
- iv) $\text{SubSt}(t, x, P(x) \rightarrow \neg P(z)) \forall t \in T_{\mathfrak{M}_{\mathcal{L}}}$.
- v) $\text{SubSt}(t, x, \forall x (P(x) \rightarrow \neg P(z)) \forall t \in T_{\mathfrak{M}_{\mathcal{L}}}$ since $x, y \notin FV$.
- vi) $\text{SubSt}(t, y, \forall x (P(x) \rightarrow \neg P(z)) \forall t \in T_{\mathfrak{M}_{\mathcal{L}}}$ since $x, y \notin FV$.
- vii) $\neg \text{SubSt}(f(x), z, \forall x (P(x) \rightarrow \neg P(z))) \forall t \in T_{\mathfrak{M}_{\mathcal{L}}}$ since $z \notin FV$.

Theorem 3.5 -

- i) Every variable is substitutable for itself in any formula. $\text{SubSt}(x, x, \phi)$.
- ii) For $x \in \text{Var}$, $\phi \in F_{\mathfrak{M}_{\mathcal{L}}}$ with $x \notin FV(\phi)$, any $t \in T_{\mathfrak{M}_{\mathcal{L}}}$ is $\text{SubSt}(t, x, \phi)$.
- iii) For all $t \in T_{\mathfrak{M}_{\mathcal{L}}}$ and $\phi \in F_{\mathfrak{M}_{\mathcal{L}}}$ if $\text{Var}(t) \cup \text{Var}(\phi) = \emptyset$ then t is substitutable for every variable in ϕ .

3.4 Substitution Lemma

Proposition 3.8 -

Let $x_1, \dots, x_k \in \text{Var}$ be pairwise disjoint and π be a permutation of $(1, \dots, k)$.

Then, for all variable assignments s over \mathcal{L} -Structure \mathfrak{M} and $d_1, \dots, d_l \in |\mathfrak{M}|$

$$\left(\left(\dots \left(s \frac{d_1}{x_1} \right) \dots \right) \frac{d_{k-1}}{x_{k-1}} \right) \frac{d_k}{x_k} = \left(\left(\dots \left(s \frac{d_{\pi(1)}}{x_{\pi(1)}} \right) \dots \right) \frac{d_{\pi(k-1)}}{x_{\pi(k-1)}} \right) \frac{d_{\pi(k)}}{x_{\pi(k)}}$$

Example 3.12 - Proposition 3.8

$$\left(\left(\left(s \frac{d_1}{x_1} \right) \frac{d_2}{x_2} \right) \frac{d_3}{x_3} \right) y = \begin{cases} d_1 & \text{if } y = x_1 \\ d_2 & \text{if } y = x_2 \\ d_3 & \text{if } y = x_3 \\ s(y) & \text{otherwise} \end{cases} = \left(\left(\left(s \frac{d_2}{x_2} \right) \frac{d_3}{x_3} \right) \frac{d_1}{x_1} \right) y$$

Theorem 3.6 - Substitution Lemma

Let $\mathfrak{M} = \langle D, \mathcal{I} \rangle$ be an \mathcal{L} -Structure, and let x be a variable

i) Let a be an arbitrary \mathcal{L} -term.

For every variable assignment s over \mathfrak{M} , for every \mathcal{L} -term t

$$\bar{s}\left([a]\frac{t}{x}\right) = \overline{s\frac{\bar{s}(t)}{x}}(a)$$

ii) Let ϕ be an arbitrary \mathcal{L} -Formula.

For every variable assignment s over \mathfrak{M} , for every \mathcal{L} -term t
if t is substitutable for x in ϕ then we have

$$\mathfrak{M}, s \models \phi\frac{t}{x} \quad \text{iff} \quad \mathfrak{M}, s\frac{\bar{s}(t)}{x} \models \phi$$

Proof 3.8 - Substitution Lemma

This is a proof by induction on terms.

Let $\mathfrak{M} = \langle D, \mathfrak{I} \rangle$ and $x \in \text{Var}$. *Base*

Let a be a constant. Then

$$\begin{aligned} \bar{s}([a]\frac{t}{x}) &= \bar{s}(a) \\ &= a^{\mathfrak{M}} \\ &= \overline{s\frac{\bar{s}(t)}{x}}(a) \end{aligned}$$

Let a be a variable.

If $a = x$ then

$$\begin{aligned} \bar{s}([x]\frac{t}{x}) &= \bar{s}(t) \\ &= \overline{s\frac{\bar{s}(t)}{x}}(tx) \\ &= \overline{s\frac{\bar{s}(t)}{x}}(x) \end{aligned}$$

If $a \neq x$ then

$$\begin{aligned} \bar{s}([a]\frac{t}{x}) &= \bar{s}(a) \\ &= \overline{s\frac{\bar{s}(t)}{x}}(a) \end{aligned}$$

Induction

Let $a = f(a_1, \dots, a_k)$. Then

$$\begin{aligned} \bar{s}([a]\frac{t}{x}) &= \bar{s}(f([a_1]\frac{t}{x_1}, \dots, [a_k]\frac{t}{x_k})) \\ &= f^{\mathfrak{M}}(\bar{s}([a_1]\frac{t}{x_1}), \dots, \bar{s}([a_k]\frac{t}{x_k})) \\ &\stackrel{\text{by IH}}{=} f^{\mathfrak{M}}\left(\overline{s\frac{\bar{s}(t)}{x}}(a_1), \dots, \overline{s\frac{\bar{s}(t)}{x}}(a_k)\right) \\ &= \overline{s\frac{\bar{s}(t)}{x}}(f(a_1, \dots, a_k)) \end{aligned}$$

Now we prove ii) by induction on formulae.

If $\phi := a \equiv b$. Then

$$\begin{aligned} &\mathfrak{M}, s \models [a \equiv b]\frac{t}{x} \\ \iff &\mathfrak{M}, s \models [a]\frac{t}{x} \equiv [b]\frac{t}{x} \\ \iff &\bar{s}([a]\frac{t}{x}) = \bar{s}([b]\frac{t}{x}) \\ \iff &s\frac{\bar{s}(t)}{x}(a) = s\frac{\bar{s}(t)}{x}(b) \\ \iff &\mathfrak{M}, s\frac{\bar{s}(t)}{x} \models a \equiv b \end{aligned}$$

If $\phi := \neg\psi$. Then

$$\begin{aligned} &\mathfrak{M}, s \models [\phi]\frac{t}{x} \\ \iff &\mathfrak{M}, s \models \neg[\psi]\frac{t}{x} \\ \iff &\mathfrak{M}, s \not\models [\psi]\frac{t}{x} \\ \iff &\mathfrak{M}, s\frac{\bar{s}(t)}{x} \not\models \psi \\ \iff &\mathfrak{M}, s\frac{\bar{s}(t)}{x} \models \neg\psi \\ \iff &\mathfrak{M}, s\frac{\bar{s}(t)}{x} \models \phi \end{aligned}$$