Logic - Notes

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Contents

NOTES

- Not included any proofs.
- Not included any examples.

1 Syntax

1.1 General

Definition 1.1 - *Alphabet, $\mathcal{A}$*
An *Alphabet* is a set of characters, $\mathcal{A}$. These characters do not have any assigned values (yet).

Definition 1.2 - *String*
A *String*, $a := \langle a_1, \dots, a_n \rangle$, over an alphabet $\mathcal{A}$ is an element of $\mathcal{A}^n$ for $n \in \mathbb{N}$.
Here a is said to have *length* n.

Remark 1.1 - $\langle a, b \rangle = \langle \langle a_1, \dots, a_n \rangle, \langle b_1, \dots, b_m \rangle \neq \langle a_1, \dots, a_n, b_1, \dots, b_m \rangle$

Definition 1.3 - *Set of all Strings, $\mathcal{A}^*$*
Let $\mathcal{A}$ be an *Alphabet*.
We define the set of all strings, $\mathcal{A}^*$, over the alphabet as

$$\mathcal{A}^* := \{\langle a_1, \dots, a_n \rangle : n \in \mathbb{N} \text{ and } a_1, \dots, a_n \in \mathcal{A}\} \equiv \bigcup_{n \in \mathbb{N}} \mathcal{A}^n$$

Remark 1.2 - *If an alphabet, $\mathcal{A}$, is countable then $\mathcal{A}^*$ is countable*
Further $\mathcal{A}^*$ is countably infinite if $\mathcal{A} \neq \emptyset$.

Definition 1.4 - *Declarative Sentence*
A *Declarative Sentence* is a sentence which is either true or false.
N.B. These are the focus of mathematical logic.

1.2 First Order Languages

Definition 1.1 - *Common Components of Alphabets*
Below are some common classes of characters used in mathematical alphabets

i) Propositional Connectives (Describe locagical relations between predicates).
‘not’, ‘and’, ‘or’, ‘if...then...’.

ii) Quantifiers
‘for all’, ‘there is’.

iii) Variables
‘x’, ‘y’, ‘z’, ...

iv) Punctuation
‘(’,‘)’, ‘,’, ...

v) Equality
‘ $=$’.

vi) Constants
‘1’,‘2’,‘3’,‘e’,...

vii) Predicates
‘ $\prec$’.

viii) Functions
‘$\circ$’.

Definition 1.2 - *Alphabet of First-Order Language*
The *Alphabet* of a *First-Order Language* comprises the following elements

i) Propositional Connectives
$\neg, \rightarrow$

ii) Quantifiers
$\forall$

iii) Variables
$v_1, v_2, \dots$ (Infinetly many).

iv) Punctuation
( ) and ,

v) Equality
$\equiv$ (This is a 2-arity logical predicate)

vi) Constants
$c_1, c_2, \dots$ (Countable many since we use countable alphabets).

vii) Predicates
P_i^n is an n-arity predicate for $n \in \mathbb{N}$.

viii) Functions
f_i^n is an n-arity function for $n \in \mathbb{N}$.

i) - v) are *Logical Symbols* & vi) - viii) are *Non-Logical Symbols* of *First-Order Languages*. The *Non-Logical Symbols* will vary depending on the subject matter of the language.

Remark 1.3 - $\equiv$ *is the only logical predicate symbol in FOLs*

Definition 1.3 - *Negation, $\neg$, and Implication, $\rightarrow$*
Let P, Q be *Predicates*.

P	$\neg P$
T	**F**
T	**T**

P	Q	$P \rightarrow Q$
T	T	**T**
T	F	**F**
F	T	**T**
F	F	**T**

Proposition 1.1 - *Extension to Alphabet of First-Order Language*
For concisness of notation we usally allow the following extra propositional connectives & quantifiers to be used.

- Propositional Connectives
  $\wedge, \vee$

- Quantifiers
  $\exists$

Definition 1.4 - *And, $\wedge$, and Or, $\vee$*
Let P, Q be *Predicates*.

P	Q	$P \wedge Q$
T	T	**T**
T	F	**F**
F	T	**F**
F	F	**F**

P	Q	$P \vee Q$
T	T	**T**
T	F	**T**
F	T	**T**
F	F	**F**

Remark 1.4 - $P \wedge Q \Leftrightarrow \neg(P \to \neg Q)$ *and* $P \vee Q \Leftrightarrow (\neg P) \to Q$

Definition 1.5 - *$\mathcal{L}$-Term (and $\mathcal{L}$-Term Complexity)*
Let $\mathcal{L}$ be a FOL.
We define *$\mathcal{L}$-Terms* & *$\mathcal{L}$-Term Complexity* recursively.

T1 Let s be a variable or constant symbol.
s is an *$\mathcal{L}$-Term* with $cp(s) = 0$.

T2 Let f be a k-arity function symbol and $t_1, \dots, t_k$ be *$\mathcal{L}$-Terms*.
$f(t_1, \dots, t_k)$ is an *$\mathcal{L}$-Term* with $cp(f) = \max\{cp(t_1), \dots, cp(t_k)\} + 1 \geq 1$.

N.B. By this definition we cannot have infinitely long *$\mathcal{L}$-Terms*.

Definition 1.6 - *Atomic $\mathcal{L}$-Term*
Let $\mathcal{L}$ be a FOL and $t \in Tm_{\mathcal{L}}$.
t is an *Atomic $\mathcal{L}$-Term* iff $cp(t) = 0$.
i.e. An *Atomic $\mathcal{L}$-Term* is either a constant or variable symbol.

Definition 1.7 - *Compound $\mathcal{L}$-Term*
Let $t \in Tm_{\mathcal{L}}$.
t is a *Compound $\mathcal{L}$-Term* iff $cp(t) \geq 1$.
i.e. An *Atomic $\mathcal{L}$-Term* is function symbol.

Definition 1.8 - *Atomic Formulae*
Let $\mathcal{L}$ be a FOL, P be a k-arity predicate symbol of $\mathcal{L}$ and $t_1, \dots, t_k \in Tm_{\mathcal{L}}$.
An *Atomic Formulae* has the form

$$P(t_1, \dots, t_k)$$

i.e. Atomic Formulae are predicates on *$\mathcal{L}$-Terms*.

Definition 1.9 - *$\mathcal{L}$-Formula (and $\mathcal{L}$-Formulae Complexity)*
Let $\mathcal{L}$ be a FOL.
We define *$\mathcal{L}$-Formulae* & *$\mathcal{L}$-Formulae Complexity* recursively

F1 Let ϕ be an *Atomic $\mathcal{L}$-Formula*.
ϕ is an $\mathcal{L}$-Formula with $cp(\phi) = 0$.

F2 Let ϕ be an *$\mathcal{L}$-Formula*.
$\neg\phi$ is an $\mathcal{L}$-Formula with $cp(\neg\phi) = cp(\phi) + 1$.

F3 Let ϕ, ψ be a *$\mathcal{L}$-Formulae*.
$\phi \to \psi$ is an $\mathcal{L}$-Formula with $cp(\phi \to \psi) = \max\{cp(\phi), cp(\psi)\} + 1$.

F4 Let ϕ be an *$\mathcal{L}$-Formula* & x be any variable.
$\forall x\phi$ is an $\mathcal{L}$-Formula with $cp(\forall x\phi) = cp(\phi) + 1$.

N.B. By this definition we cannot have infinitely long *$\mathcal{L}$-Formulae*.

Remark 1.5 - *$\mathcal{L}$-Term & $\mathcal{L}$-Formulae complexity is a measure of syntactic complexity and is unrelated to any semantic meaning.*
$\mathcal{L}$-Formulae complexity is unrelated from the complexity of any terms in it.

Remark 1.6 - *F4 necessitates the use of parentheses*
Otherwise $\phi \to \psi \to \theta$ is ambiguous as it could be read as either $(\phi \to \psi) \to \theta$ or $\phi \to (\psi \to \theta)$

which don't necessarily have the same semantic meaning.

Definition 1.10 - *Compound $\mathcal{L}$-Formula*
Let $\phi \in Fml_{\mathcal{L}}$.
ϕ is a *Compound $\mathcal{L}$-Formula* iff $cp(\phi) \geq 1$.

1.3 Induction

Theorem 1.1 - *Induction on Terms*
Let $\mathcal{L}$ be a FOL and P be a property that $\mathcal{L}$-Terms may have.
If

i) All *Atomic $\mathcal{L}$-terms* have P; And,

ii) For all k-arity function symbols f of $\mathcal{L}$ and $t_1, \dots, t_k \in Tm_{\mathcal{L}}$ which have property P, $f(t_1, \dots, t_k)$ has P.

Then all $t \in Tm_{\mathcal{L}}$ have property P.

Theorem 1.2 - *Induction on Formulae*
Let $\mathcal{L}$ be a FOL and P be a property that $\mathcal{L}$-Formulae may have.
If

i) All *Atomic $\mathcal{L}$-Formulae* have P; And,

ii) ϕ, ψ have P then $\neg\phi$, $\phi \to \psi$ and $\forall x \phi$ (for all variables x) have property P.

Then all $\phi \in Fm_{\mathcal{L}}$ have property P.

1.4 Free Variables

Definition 1.1 - *Set of Variables,* $\text{Var}(\cdot)$
$\text{Var} : \mathcal{A}_{\mathcal{L}}^* \to 2^{\text{Var}}$ is a function which maps from a string to the set of variables in it.
Variables are defined by the *Alphabet* of the language being used.

Definition 1.2 - *Closed $\mathcal{L}$-Term*
Let $t \in Tm_{\mathcal{L}}$ for some FOL, $\mathcal{L}$.
If $\text{Var}(t) = \emptyset$ then t is said to be a *Closed $\mathcal{L}$-Term*.

Definition 1.3 - *Free Variables, $FV(\cdot)$*
Let $\mathcal{L}$ be a FOL.
Free Variables are unbounded variables in an *$\mathcal{L}$-Formula*.
We define the *Set of Free Variables* of an *$\mathcal{L}$-Formula* inductively

FV1 Let P be a k-arity *Predicate* & $t_1, \dots, t_k \in Tml_{\mathcal{L}}$.
$FV(P(t_1, \dots, t_k)) := \text{Var}(P(t_1, \dots, t_k))$.

FV2 Let $\phi \in Fml_{\mathcal{L}}$.
$FV(\neg\phi) := FV(\phi)$.

FV3 Let $\phi, \psi \in Fml_{\mathcal{L}}$.
$FV(\phi \to \psi) := FV(\phi) \cup FC(\psi)$.

FV4 Let $\phi \in Fml_{\mathcal{L}}$ and x be any variable.
$FV(\forall x \phi) := FV(\phi) \backslash \{x\}$.

FV-EXT1 Let $\phi, \psi \in Fml_{\mathcal{L}}$.
$FV(\phi \wedge \psi) := FV(\phi) \cup FV(\psi)$.

FV-EXT2 Let $\phi, \psi \in Fml_{\mathcal{L}}$.
$FV(\phi \vee \psi) := FV(\phi) \cup FV(\psi)$.

FV-EXT3 Let $\phi \in Fml_{\mathcal{L}}$ and x be any variable.
$FV(\exists x \phi) := FV(\phi) \backslash \{x\}$.

N.B. $FV(\cdot) : \mathcal{A}_{\mathcal{L}}^* \to 2^{\text{Var}}$.

Definition 1.4 - *$\mathcal{L}$-Sentence*
Let $\phi \in Fml_{\mathcal{L}}$ for some FOL, $\mathcal{L}$.
If $FV(\phi) = \emptyset$ then ϕ is said to be a *$\mathcal{L}$-Sentence*.

Remark 1.7 - *The meaning anthonof formulae depends on how we interpret their free variables*

2 Semantics

Definition 2.1 - *$\mathcal{L}$-Structure*
Let $\mathcal{L}$ be a FOL.
An *$\mathcal{L}$-Structure* assigns meaning to the *Non-Logical* symbols of $\mathcal{L}$.
An *$\mathcal{L}$-Structure* is an ordered pair $\mathfrak{M} := (D, \mathfrak{I})$ where

Domain D is a non-empty set.
Often $\mathbb{R}$ or similar.

Interpretation $\mathfrak{I}$ is a function over the non-logical symbols of $\mathcal{L}$.

$$\begin{array}{rcl} \mathfrak{I}(c) & \in & D \text{ where } c \text{ is a constant symbol of } \mathcal{L} \\ \mathfrak{I}(P) & \subset & D^n \text{ where } P \text{ is a } k\text{-arity predicate symbol of } \mathcal{L} \\ \mathfrak{I}(f) & : & D^n \to D \text{ where } f \text{ is a } k\text{-arity function symbol of } \mathcal{L} \end{array}$$

Remark 2.1 - *Interpretation, $\mathfrak{I}$*
The *Interpretation* is a function which assigns meaning to non-logical symbols.
$\mathfrak{I}(P)$ gives the property or relation on D by which P is interpreted.
$\mathfrak{I}(f)$ gives the function on D^n by which f is interpreted.
$\mathfrak{I}(P)$ gives the object in D which c denotes.

Definition 2.2 - *Variable Assignment, s*
Let $\mathcal{L}$ be a FOL and $\mathfrak{M} := (|\mathfrak{M}|, \mathfrak{I})$ be an $\mathcal{L}$-Structure.
A *Variable Assignment* maps variables to a value in the domain of $\mathfrak{M}$.

$$s : \text{Var} \to |\mathfrak{M}|$$

Definition 2.3 - *Variable Assignment for $\mathcal{L}$-Terms, $\bar{s}$*
Let $\mathcal{L}$ be a FOL and $\mathfrak{M} := (|\mathfrak{M}|, \mathfrak{I})$ be an $\mathcal{L}$-Structure.
We define *Variable Assignment* over *$\mathcal{L}$-Terms* recursively

V1 Let x be a variable symbol of $\mathcal{L}$.
$\bar{s}(x) := s(x)$

V2 Let c be a constant symbol of $\mathcal{L}$.
$\bar{s}(c) := c^{\mathfrak{M}}$

V3 Let f be a k-arity function symbol of $\mathcal{L}$ and $t_1,\dots,t_k$ be *$\mathcal{L}$-Terms*.
$\bar{s}(f(t_1,\dots,t_k)) := f^M(\bar{s}(t_1),\dots,\bar{s}(t_k))$

N.B. $\bar{s} : Tm_{\mathcal{L}} \to |\mathfrak{M}|$.

Remark 2.2 - *$\bar{s}(t)$ is the Semantic Value of term t in struture $\mathfrak{M}$ under assignement s.*
$\bar{s}(t)$ gives a description of what t designates in $\mathfrak{M}$ under the assignment s.

2.1 Satisfaction Relation

Definition 2.1 - *Satisfaction Relation,* $\vDash$
Let $\mathcal{L}$ be a FOL, $\mathfrak{M}$ be an *$\mathcal{L}$-Structure* and s a *Variable Assignment* over $\mathfrak{M}$.
The *Satisfaction Relation* (states whether a given formula is true under a given model)??
We define the *Satisfaction Relation*, $\vDash$, recursively

S1 Let $t_1, t_2 \in Tm_{\mathcal{L}}$.
$\mathfrak{M}, s, \vDash (t_1 \equiv t_2) :\Leftrightarrow \bar{s}(t_1) = \bar{s}(t_2)$.

S2 Let P be a k-arity predicate symbol of $\mathcal{L}$ and $t_1,\dots,t_k \in Tm_{\mathcal{L}}$.
$\mathfrak{M}, s, \vDash P(t_1,\dots,t_k) :\Leftrightarrow \langle \bar{s}(t_1),\dots,\bar{s}(t_k)\rangle \in P^{\mathfrak{M}}$.

S3 Let $\phi \in Fml_{\mathcal{L}}$.
$\mathfrak{M}, s, \vDash \neg\phi :\Leftrightarrow \mathfrak{M}, s \nvDash \phi$.

S4 Let $\phi, \psi \in Fml_{\mathcal{L}}$.
$\mathfrak{M}, s, \vDash (\phi \to \psi) :\Leftrightarrow$ if $\mathfrak{M}, s \vDash \phi$ then $\mathfrak{M}, s \vDash \psi$.

S5 Let $\phi \in Fml_{\mathcal{L}}$ and x be any variable.
$\mathfrak{M}, s, \vDash \forall x\phi :\Leftrightarrow \mathfrak{M}, s\frac{d}{x} \vDash \phi$ for all $d \in |\mathfrak{M}|$.

S-EXT1 Let $\phi, \psi \in Fml_{\mathcal{L}}$.
$\mathfrak{M}, s \vDash (\phi \land \psi) :\Leftrightarrow \mathfrak{M}, s \vDash \phi$ and $\mathfrak{M}, s \vDash \psi$.

S-EXT2 Let $\phi, \psi \in Fml_{\mathcal{L}}$.
$\mathfrak{M}, s \vDash (\phi \lor \psi) :\Leftrightarrow \mathfrak{M}, s \vDash \phi$ or $\mathfrak{M}, s \vDash \psi$.

S-EXT3 Let $\phi \in Fml_{\mathcal{L}}$ and x be any variable.
$\mathfrak{M}, s \vDash \exists x\phi :\Leftrightarrow \mathfrak{M}, s\frac{d}{x} \vDash \phi$ for at least one $d \in |\mathfrak{M}|$.

S-EXT2 Let $\phi, \psi \in Fml_{\mathcal{L}}$.
$\mathfrak{M}, s \vDash (\phi \leftrightarrow \psi) :\Leftrightarrow \mathfrak{M}, s \vDash \phi$ iff $\mathfrak{M}, s \vDash \psi$.

Remark 2.3 - *When $\mathfrak{M}, s \vDash \phi$ holds we say "ϕ is true in $\mathfrak{M}$ under s"*
Or, "ϕ is satisfied by $\mathfrak{M}$ under s".
Or, "$\mathfrak{M}, s$ models ϕ".

Definition 2.2 - *Model*
Let $\mathcal{L}$ be a FOL, $\Phi \subseteq Fml_{\mathcal{L}}$, $\mathfrak{M}$ be an *Ł-Structure* and s a *Variable Assignment*.
$\mathfrak{M}, s$ is a *Model* of Φ if $\mathfrak{M}, s \vDash \Phi$.

Remark 2.4 - *Semantic Value of a Term*
Let $t \in Tm_{\mathcal{L}}$ for some FOL, $\mathcal{L}$, and s be a *Variable Assignment*.
The semantic value of t, $\bar{s}(t)$, only depends on

i) The *Interpreation* of the constant & function symbols that occur in t. And,

ii) The *Assignment* of values to variables in t, given by s.

Remark 2.5 - *Truth of a Formula*
Let $\phi \in Fml_{\mathcal{L}}$ for some FOL, $\mathcal{L}$.
The truth of ϕ only depends on

i) The domain of discourse, $|\mathfrak{M}|$, over which the quantifiers range

ii) The *Interpretation* of the constants, functions & predicate symbols in ϕ.

iii) The *Assignment* of values to *Free Variables* in ϕ, given by s.

Theorem 2.1 - *Coincidence Lemma*
Let $\mathcal{L}_1, \mathcal{L}_2$ be unique FOLs, $\mathfrak{M}_1 := (D, \mathfrak{I}_1)$ be an $\mathcal{L}_1$-Structure and $\mathfrak{M}_2 := (D, \mathfrak{I}_2)$ be an $\mathcal{L}_2$-Structure.
Note that both structures have the same domain.
Let $\mathcal{L} := \mathcal{L}_1 \cap \mathcal{L}_2$. Then the following are true

i) $\forall\ t \in Tm_{\mathcal{L}}, \forall$ variable assignments s_1 over $\mathfrak{M}_2$ and s_2 over $\mathfrak{M}_2$

$$\text{If } \left\{\begin{matrix} c^{\mathfrak{M}_1} = c^{\mathfrak{M}_2}\ \forall\ c \text{ that occur in } t \\ f^{\mathfrak{M}_1} = f^{\mathfrak{M}_2}\ \forall\ f \text{ that occur in } t \\ s_1(x) = s_2(x)\ \forall\ x \text{ that occur in } t \end{matrix}\right\} \text{ then } \overline{s_1}(t) = \overline{s_2}(t).$$

i.e. If these conditions hold then t has the same semantic value under both variable assignments.

ii) $\forall\ \phi \in Fml_{\mathcal{L}}, \forall$ variable assignments s_1 over $\mathfrak{M}_2$ and s_2 over $\mathfrak{M}_2$

$$\text{If } \left\{\begin{matrix} c^{\mathfrak{M}_1} = c^{\mathfrak{M}_2}\ \forall\ c \text{ that occur in } \phi \\ f^{\mathfrak{M}_1} = f^{\mathfrak{M}_2}\ \forall\ f \text{ that occur in } \phi \\ P^{\mathfrak{M}_1} = P^{\mathfrak{M}_2}\ \forall\ P \text{ that occur in } \phi \\ s_1(x) = s_2(x)\ \forall\ x \text{ that occur in } \phi \end{matrix}\right\} \text{ then } \mathfrak{M}_1, s_2 \models \phi \text{ iff } \mathfrak{M}_2, s_2 \models \phi.$$

i.e. If these conditions hold ϕ is equivalent truth values under both $\mathcal{L}$-structures & variable assignemnts.

N.B. AKA *Reduct Property* of First-Order Logic.

Remark 2.6 - *Semantic Interpretations Closed $\mathcal{L}$-Terms & $\mathcal{L}$-Sentences*
Let $\mathcal{L}$ be a FOL, t be a *Closed $\mathcal{L}$-Term*, ϕ be an *$\mathcal{L}$-Sentence*, $\mathfrak{M}$ be an *$\mathcal{L}$-Structure.*
Let s_1, s_2 be arbitrary *Variable Assignments* over $\mathfrak{M}$. Then

$$\overline{s_1}(t) = \overline{s_2}(t) \text{ and } \mathfrak{M}, s_1 \models \phi \text{ iff } \mathfrak{M}, s_2 \models \phi$$

i.e. Choice of variable assignment does not affect semantic value of closed $\mathcal{L}$-Terms & $\mathcal{L}$-Sentences.

Definition 2.3 - *Logical Consequence,* $\Phi \models \phi$
Let $\mathcal{L}$ be a FOL, $\phi \in Fml_{\mathcal{L}}$ and $\Phi \subseteq Fml_{\mathcal{L}}$.
ϕ is a *Logical Consequence* of Φ iff

$\forall\ \mathcal{L}$-Structures $\mathfrak{M}$, $\forall$ variable assignments s over $\mathfrak{M}$ it holds that $(\mathfrak{M}, s \models \Phi) \to (\mathfrak{M}, s \models \phi)$.

N.B. When this is the case, it is denoted $\Phi \models \phi$.
N.B. AKA "ϕ logically follows from Φ" or "Φ logically implies ϕ".

Proposition 2.1 - *For unary predicates P, $P(x) \models P(x) \vee P(y)$*

Proposition 2.2 - $\forall\ \phi, \psi \in Fml_{\mathcal{L}}\ \&\ \Phi \subseteq Fml_{\mathcal{L}},\ \Phi, \phi \models \psi$ *iff* $\Phi \models \phi \to \psi$

Definition 2.4 - *Logically Valid,* $\models \phi$
Let $\mathcal{L}$ be a FOL, $\phi \in Fml_{\mathcal{L}}$.
ϕ is *Logically Valid* iff $\mathfrak{M}, s \models \phi$ for all $\mathcal{L}$-Structures $\mathfrak{M}$ and variable assignemnts s over $\mathfrak{M}$.
N.B. This is denoted $\models \phi$.

Definition 2.5 - *Satisfiable*
Let $\mathcal{L}$ be a FOL, $\phi \in Fml_{\mathcal{L}}$ and $\Phi \subseteq Fml_{\mathcal{L}}$.

ϕ is *Satisfiable* iff $\exists$ an $\mathcal{L}$-Structure $\mathfrak{M}$ and variable assignment s over $\mathfrak{M}$, st $\mathfrak{M}, s \models \phi$.
Φ is *Satisfiable* iff $\exists$ an $\mathcal{L}$-Structure $\mathfrak{M}$ and variable assignment s over $\mathfrak{M}$, st $\mathfrak{M}, s \models \Phi$.

Theorem 2.2 -
Let $\mathcal{L}$ be a FOL, $\phi \in Fml_{\mathcal{L}}$ and $\Phi \subseteq Fml_{\mathcal{L}}$.
Then

i) ϕ is *Logically Valid* iff $\emptyset \models \phi$.

ii) ϕ is *Logically Valid* iff $\neg\phi$ is not *Satisfiable.*

iii) $\Phi \models \phi$ iff $\Phi \cup \{\neg\phi\}$ is not *Satisfiable.*

Definition 2.6 - *Logical Equivalence*
Let $\mathcal{L}$ be a FOL and $\phi, \psi \in Fml_{\mathcal{L}}$.
ϕ is *Logically Equivalent* to ψ iff $\phi \models \psi$ and $\psi \models \phi$.
i.e. ϕ is *Logically Equivalent* to ψ iff $\models \phi \leftrightarrow \psi$.

Proposition 2.3 - *Logical Equivalences*
The following are *Logically Equivalent*

i) $((\phi \wedge \psi) \wedge \theta)$ is logically equivalent to $(\phi \wedge (\psi \wedge \theta))$.

ii) $((\phi \vee \psi) \vee \theta)$ is logically equivalent to $(\phi \vee (\psi \vee \theta))$.

iii) $\neg\neg\phi$ is logically equivalent to ϕ.

iv) $\phi \wedge \psi$ is logically equivalent to $\neg((\neg\phi) \vee (\neg\psi))$.

Definition 2.7 - *True of,* $\mathfrak{M} \models \phi[\![a_1, \dots, a_n]\!]$
Let $\mathcal{L}$ be a FOL, $\phi \in Fml_{\mathcal{L}}$ with $FV(\phi) \subset \{x_1, \dots, x_n\}$.
Let $\mathfrak{M}$ be an $\mathcal{L}$-Structure, s_1, s_2 be variable assignments over $\mathfrak{M}$ and $a_1, \dots, a_n \in |\mathfrak{M}|$.
By the *Conincidence Lemma*, **Theorem 2.1**

$$\text{if } s_1(x_i) = s_2(x_2)\ \forall\ i \in [1, n] \text{ then } \mathfrak{M}, s_1 \models \phi \Leftrightarrow \mathfrak{M}, s_2 \models \phi$$

Equivalently

$$\begin{array}{rl} & \mathfrak{M}, s \models \phi \text{ for } \textit{all} \text{ variable assignments } s \text{ over } \mathfrak{M} \text{ st } s(x_1) = a_1, \dots, s(x_n) = a_n \\ \Leftrightarrow & \mathfrak{M}, s \models \phi \text{ for } \textit{some} \text{ variable assignments } s \text{ over } \mathfrak{M} \text{ st } s(x_1) = a_1, \dots, s(x_n) = a_n \end{array}$$

We denote these holding by $\mathfrak{M} \models \phi[\![a_1, \dots, a_n]\!]$.
N.B. $\mathfrak{M} \models \phi[\![a_1, \dots, a_n]\!]$ means "ϕ is *true of* the objects $a_1, \dots, a_n \in \mathfrak{M}$".

2.2 Substitution

Definition 2.1 - *Substitution*
Substitution is the process of replacing one expression with another.
Substituting t for x in a is denoted by $[a]\frac{t}{x}$.
N.B. Usually t is an $\mathcal{L}$-term, x is a variable & a is an $\mathcal{L}$-term or $\mathcal{L}$-Formula.

Definition 2.2 - *Substitution of a Term for a Variable in a Term*
Let $\mathcal{L}$ be a FOL, $a, t \in Tm_\mathcal{L}$ and x be a variable.
We define the *Substitution* $[a]\frac{t}{x}$ recursively

Sub-T1 If a is an *Atomic $\mathcal{L}$-Term* then

$$[a]\frac{t}{x} := \begin{cases} t & \text{if } a = x \\ a & \text{if } a \neq x \end{cases}$$

Sub-T2 If a is a *Compound $\mathcal{L}$-Term* of the form $a := f(a_1, \dots, a_k)$ where $a_1, \dots, a_k \in Tm_\mathcal{L}$

$$[a]\frac{t}{x} := f\left([a_1]\frac{t}{x}, \dots, [a_k]\frac{t}{x}\right)$$

Remark 2.7 - *$[a]\frac{t}{x} = a$ for all constant symbols in a*

Definition 2.3 - *Substitution of a Term for a Variable in a Formula*
Let $\mathcal{L}$ be a FOL, $\phi, \psi \in Fml_\mathcal{L}$, $t \in Tm_\mathcal{L}$ and x, z be variables.
We define the *Substitution* $[\phi]\frac{t}{x}$ recursively

SUB1 If ϕ is an *Atomic $\mathcal{L}$-Formula* of the form $P(a_1, \dots, a_k)$ where $a_1, \dots, a_k \in Tm_\mathcal{L}$.
$[\phi]\frac{t}{x} := P\left([a_1]\frac{t}{x}, \dots, [a_k]\frac{t}{x}\right)$

SUB-F2 $[\neg\phi]\frac{t}{x} := \neg[\phi]\frac{t}{x}$.

SUB-F3 $[(\phi \to \psi)]\frac{t}{x} := [\phi]\frac{t}{x} \to [\psi]\frac{t}{x}$.

SUB-F4 $[\forall z\phi]\frac{t}{x} := \begin{cases} \forall z[\phi]\frac{t}{x} & \text{if } x \neq z \\ \forall z\phi & \text{if } x = z \end{cases}$.

SUB-F-EXT1 $[\phi \wedge \psi]\frac{t}{x} := [\phi]\frac{t}{x} \wedge [\psi]\frac{t}{x}$.

SUB-F-EXT2 $[\phi \vee \psi]\frac{t}{x} := [\phi]\frac{t}{x} \vee [\psi]\frac{t}{x}$.

SUB-F-EXT3 $[\exists x\phi]\frac{t}{x} := \begin{cases} \exists x[\phi]\frac{t}{x} & \text{if } x \not\equiv z \\ \exists x\phi & \text{otherwise} \end{cases}$.

N.B. We never substitute bound variables (only *Free Variables*).

Proposition 2.4 - $\forall\ t \in Tm_\mathcal{L},\ [t]\frac{x}{x} = t$

Proposition 2.5 - $\forall\ \phi \in Fml_\mathcal{L},\ [\phi]\frac{x}{x} = \phi$

Proposition 2.6 - *If $x \notin var(t)$ then $[a]\frac{a}{x} = t$*

Proposition 2.7 - *If $x \notin FV(\phi)$ then $[\phi]\frac{a}{x} = \phi$*

Proposition 2.8 - *Let $x \notin var(a)$ then $x \notin var([t]\frac{a}{x})$ and $x \notin FV([\phi]\frac{a}{x})$*

Definition 2.4 - *Substitutable*
Let $\mathcal{L}$ be a FOL, $t \in Tm_{\mathcal{L}}$ and x be a variable.
Let $\phi, \psi \in Fml_{\mathcal{L}}$.
We define whether t is *Substitutable* for a variable x in a formula ϕ recursively

SU1 If ϕ is an *Atomic $\mathcal{L}$-Formula*. Then $\texttt{SubSt}(t, x, \phi)$ <u>always</u>.

SU2 $\texttt{SubSt}(t, x, \neg\phi)$ <u>iff</u> $\texttt{SubSt}(t, x, \phi)$.

SU3 $\texttt{SubSt}(t, x, \phi \to \psi)$ <u>iff</u> $\texttt{SubSt}(t, x, \phi)$ <u>and</u> $\texttt{SubSt}(t, x, \psi)$.

SU4 $\texttt{SubSt}(t, x, \forall z\phi)$ if $\begin{cases} & z \notin var(t) \text{ \underline{and} } mathttSubSt(t, x, \phi) \\ \text{or} & x \notin FV(\phi) \end{cases}$

SU-EXT1 $\texttt{SubSt}(t, x, \phi \wedge \psi)$ <u>iff</u> $\texttt{SubSt}(t, x, \phi)$ <u>and</u> $\texttt{SubSt}(t, x, \psi)$.

SU-EXT2 $\texttt{SubSt}(t, x, \phi \vee \psi)$ <u>iff</u> $\texttt{SubSt}(t, x, \phi)$ <u>and</u> $\texttt{SubSt}(t, x, \psi)$.

SU-EXT3 $\texttt{SubSt}(t, x, \exists z\phi)$ <u>if</u> $\begin{cases} & z \in var(t) \text{ \underline{and} } \texttt{SubSt}(\phi) \\ \text{or} & x \notin FV(\exists z\phi) \end{cases}$

N.B. If $\texttt{SubSt}(t, x, \phi)$, t is said to be *Free* for x in ϕ.

Proposition 2.9 - *Every variable is Substitutable for itself, in all formulae*

Proposition 2.10 - *If $x \notin FV(\phi)$ all $t \in Tm_{\mathcal{L}}$ are Substitutable for x in Φ*

Proposition 2.11 - *If $var(t) \cap var(\phi) = \emptyset$ then t is substitutable for any variable in ϕ.*
Notably, every closed $\mathcal{L}$-term is substitutable for any variable in any formula.

Proposition 2.12 - *Substitution order doesn't matter*
Let $\mathcal{L}$ be a FOL, $\mathfrak{M}$ be an $\mathcal{L}$-structure, s be a variable assignment and $d_1, \dots, d_k \in |\mathfrak{M}|$.
Let $x_1, \dots, x_k$ be distinct variables and π be a permutation over k. Then

$$\left(\left(\dots\left(s\tfrac{d_1}{x_1}\right)\dots\right)\tfrac{d_{k-1}}{x_{k-1}}\right)\tfrac{d_k}{x_k} = \left(\left(\dots\left(s\tfrac{d_{\pi(1)}}{x_{\pi(1)}}\right)\dots\right)\tfrac{d_{\pi(k-1)}}{x_{\pi(k-1)}}\right)\tfrac{d_{\pi(k)}}{x_{\pi(k)}}$$

Theorem 2.3 - *Substitution Lemma*
Let $\mathcal{L}$ be a FOL, $\mathfrak{M}$ be an $\mathcal{L}$-Structure, $t \in Tm_{\mathcal{L}}$, $\phi \in Fml_{\mathcal{L}}$ and x be a variable.

i) For every variable assignment s over $\mathfrak{M}$, $\forall\ a \in Tm_{\mathcal{L}}$

$$\overline{s}\left([a]\tfrac{t}{x}\right) = \overline{s\tfrac{\overline{s}(t)}{x}}(a)$$

ii) For every variable assignment s over $\mathfrak{M}$, $\forall\ a \in Tm_{\mathcal{L}}$ where a is *Substitutable* for x in ϕ

$$\mathfrak{M}, s \models \phi\tfrac{t}{x} \quad \text{iff} \quad \mathfrak{M}, s\frac{\overline{s}(t)}{x} \models \phi$$

Proposition 2.13 - $\forall\ t \in Tm_{\mathcal{L}}$ *if t is substitutable for x then* $\models \left(\forall x\phi \to [\phi]\tfrac{t}{x}\right)$ *for all $t \in Tm_{\mathcal{L}}$*

Proposition 2.14 - $\forall\ \phi \in Fml_{\mathcal{L}}$ *if t is substitutable for x then* $\models \left([\phi]\tfrac{t}{x} \to \exists x\phi\right)$ *for all $t \in Tm_{\mathcal{L}}$*

2.3 Homomorphism

Definition 2.1 - *Homomorphism*
Let $\mathcal{L}$ be a FOL and $\mathfrak{M}_1, \mathfrak{M}_2$ be $\mathcal{L}$-Structures.
A function $H : \mathfrak{M}_1 \to \mathfrak{M}_2$ is a *Homomorphism* if it fulfils the following

- $H(c^{\mathfrak{M}_1}) = c^{\mathfrak{M}_2}$ for all constant symbols, c, of $\mathcal{L}$.
- $H(f^{\mathfrak{M}_1}(t_1, \dots, t_k) = f^{\mathfrak{M}_2}(H(t_1), \dots, H(t_k))$ for all k-arity function symbols f of $\mathcal{L}$ and $t_1, \dots, t_k \in |\mathfrak{M}_1|$.
- $\langle t_1, \dots, t_k \rangle \in P^{\mathfrak{M}_1} \Leftrightarrow \langle H(t_1), \dots, H(t_k) \rangle \in P^{\mathfrak{M}_2}$ for all k-arity predicates symbols P of $\mathcal{L}$ and $t_1, \dots, t_k \in |\mathfrak{M}_1|$.
  i.e. $\langle t_1, \dots, t_k \rangle$ has property $P^{\mathfrak{M}_1}$ <u>iff</u> $\langle H(t_1), \dots, H(t_k) \rangle$ has property $P^{\mathfrak{M}_2}$.

Theorem 2.4 - *Semantic Value of a Homomorphism*
Let $\mathcal{L}$ be a FOL, $\mathfrak{M}_1, \mathfrak{M}_2$ be $\mathcal{L}$-Structures and s be a variable assignment over $\mathfrak{M}_1$.
Let H be a *Homomorphism* from $\mathfrak{M}_1$ to $\mathfrak{M}_2$.
Then, $\forall\ t \in Tm_{\mathcal{L}}$
$$H \circ \overline{s}(t) = \overline{H \circ s}(t)$$

Definition 2.2 - *Isomorphism*
Let H be a *Homomorphism.*
H is an *Isomorhpism* if it is *Bijective.*
N.B. If there exists an *Isomorphism* between $\mathfrak{M}_1$ and $\mathfrak{M}_2$ they are said to be *Isomorphic.*

Definition 2.3 - *Substructure*
Let $\mathcal{L}$ be a FOL and $\mathfrak{M}_1, \mathfrak{M}_2$ be $\mathcal{L}$-Structures.
$\mathfrak{M}_1$ is a *Substructure* of $\mathfrak{M}_2$ if

- $|\mathfrak{M}_1| \subset |\mathfrak{M}_2|$. And,
- The function $i(d) = d\ \forall\ d \in |\mathfrak{M}_1|$ is a *Homomorphism*

N.B. $\mathfrak{M}_2$ is called an *Extension* of $\mathfrak{M}_1$.

Definition 2.4 - *Elementary Equivalence*
Let $\mathcal{L}$ be a FOL and $\mathfrak{M}_1, \mathfrak{M}_2$ be $\mathcal{L}$-Structures.
$\mathfrak{M}_1$ and $\mathfrak{M}_2$ are *Elementary Equivalent* if
$$\mathfrak{M}_1 \vDash \sigma \Leftrightarrow \mathfrak{M}_2 \vDash \sigma \quad \forall\ \sigma \in Sent_{\mathcal{L}}$$

Proposition 2.15 - *Isomorphic $\mathcal{L}$-Structures are Elementary Equivalence*

Definition 2.5 - *Elementary Embedding*
Let $\mathcal{L}$ be a FOL and $\mathfrak{M}_1, \mathfrak{M}_2$ be $\mathcal{L}$-Structures.
An *Elementary Embedding* of $\mathfrak{M}_1$ in $\mathfrak{M}_2$ is a function $H : |\mathfrak{M}_1| \to |\mathfrak{M}_2|$ st
$$\forall\ \phi \in Fml_{\mathcal{L}}, \forall \text{ variable assignments } s \text{ over } \mathfrak{M}_1 \quad \mathfrak{M}_1, s \vDash \phi \Leftrightarrow \mathfrak{M}_2, H \circ s \vDash \phi$$
N.B. $H \circ s : \text{Var} \to |\mathfrak{M}_2|$.
N.B. If there exists an *Elementary Embedding* of $\mathfrak{M}_1$ in $\mathfrak{M}_2$, then $\mathfrak{M}_1$ and $\mathfrak{M}_2$ are *Elementary Equivalent.*

Proposition 2.16 - *An Isomorphism is an Elementary Embedding*

Proposition 2.17 - *An Elemenetary Embedding of $\mathfrak{M}_1$ in $\mathfrak{M}_2$ is an Injective Homomorphism from $\mathfrak{M}_1$ to $\mathfrak{M}_2$*
N.B. The converse may not be true.

2.4 Definable

Definition 2.1 - *Definable*
Let $\mathcal{L}$ be a FOL, $\mathfrak{M}$ be an $\mathcal{L}$-Structure and $\mathcal{R}$ be a k-arity relation on $|\mathfrak{M}|$.
$\mathcal{R}$ is *Definable* in $\mathfrak{M}$ if $\exists\ \phi \in Fml_{\mathcal{L}}$ where $FV(\phi) \subset \{x_1, \dots, x_k\}$ and $\forall\ a_1, \dots, a_k \in |\mathfrak{M}|$ it holds that

$$\langle a_1, \dots, a_k \rangle \in \mathcal{R} \quad \text{iff} \quad \mathfrak{M} \vDash \phi[\![a_1, \dots, a_k]\!]$$

N.B. $\mathcal{R}$ is **not** a predicate and is **not** related to any symbols in $\mathfrak{M}$.
N.B. We say $\mathcal{R}$ is *defined by* ϕ in $\mathfrak{M}$.

Proposition 2.18 - *$\mathcal{R}$ is defined by ϕ in $\mathfrak{M}$ iff $\mathfrak{M}, s \vDash \phi \Leftrightarrow \langle s(x_1), \dots, s(x_k) \rangle \in \mathcal{R}$ for all variable assignments s over $\mathfrak{M}$.*

0 Appendix

0.1 Standard Models

Definition 0.1 - *Standard Model of Arithmetic*
Let language of arithmetic is $\mathcal{L}_\mathbb{N} := \{\bar{<}, S, \bar{+}, \bar{\cdot}, E, \bar{0}\}$ where

- $\bar{<}$ is a binary relation symbol.
- S is a unary function symbol.
- $\bar{+}, \bar{\cdot}, E$ are binary function symbols.
- $\bar{0}$ is the constant symbol for $0 \in \mathbb{N}$.

Let $\mathfrak{M}$ be a $\mathcal{L}_\mathbb{N}$-Structure with the domain $|\mathfrak{M}| = \mathbb{N}$ defined as

- $\bar{<}$ is interpreted as the usual '*less-than*' relation on $\mathbb{N}$.

$$i.e.\ \langle x, y\rangle \in \bar{<}^{\mathfrak{M}} \Leftrightarrow x < y$$

- S is interpreted as the *successor function* '+1' on $\mathbb{N}$.

$$i.e. S^{\mathfrak{M}}(n) = n + 1$$

- $\bar{+}, \bar{\cdot}, E$ are interpreted as the usual '*addition*', '*multiplication*' and '*exponentitation*' on $\mathbb{N}$ respectively.

$$i.e. E^{\mathfrak{M}}(n, m) = n^m$$

- $\bar{0}$ is interpreted as the natural numbers 0.

0.2 Notation

Proposition 0.1 - *Formal Notation*

Notation	Use
$\langle a_1, \dots, a_n \rangle$	A string of length n
$\langle a, b \rangle$	Two consecutive strings
$a * b$	Concatenation of two strings
$\mathcal{A}^*$	Set of all strings over alphabet $\mathcal{A}$
$\mathcal{A}_\mathcal{L}$	Alphabet of language $\mathcal{L}$
$Tm_\mathcal{L}$	Set of $\mathcal{L}$-Terms of language $\mathcal{L}$
$Fml_\mathcal{L}$	Set of $\mathcal{L}$-Formulae of language $\mathcal{L}$
Var	Set of variables in the alphabet??
$Sent_\mathcal{L}$	Set of $\mathcal{L}$-Sentences of language $\mathcal{L}$
$\to$	Implication
$\leftrightarrow$	Equivalence
$\vee$	Or
$\wedge$	And
$\forall$	For all
$\exists$	There exists
$\exists!$	There exists a unique
$\bar{+}$	Syntactic $+$, has no semantic value.
	$^{-}$signals this for all symbols
$:\Leftrightarrow$	Defined to have same logical value (true or false)
$\Phi \vDash \phi$	$\phi \in Fml_\mathcal{L}$ is a logical consequence of $\Phi \subseteq Fml_\mathcal{L}$.
$\vDash \phi$	$\phi \in Fml_\mathcal{L}$ is logically valid.
$\mathfrak{M}_1 \cong \mathfrak{M}_2$	Structures $\mathfrak{M}_1$ and $\mathfrak{M}_2$ are isomorphic.
$\mathfrak{M} \vDash \phi[\![a_1, \dots, a_n]\!]$	ϕ is true for the objects $a_1, \dots, a_n \in \lvert\mathfrak{M}\rvert$.

Proposition 0.2 - *Convential Notation*

Notation	Use
$\mathcal{A}$	Alphabet
$\mathcal{L}$	Language (First-Order)
t	Term
ϕ, ψ, θ, χ	Formulae
$x \circ y$	$\circ(x, y)$ where $\circ$ is a function or predicate
$c \not\prec d$	$\neg \prec (c, d)$
$\mathfrak{M}$	$\mathcal{L}$-Structure
$\mathfrak{I}$	Interpreation from an $\mathcal{L}$-Structure
D or $\lvert\mathfrak{M}\rvert$	Domain of an $\mathcal{L}$-Structure
$P^\mathfrak{M}$	$\mathfrak{I}(P)$
$f^\mathfrak{M}$	$\mathfrak{I}(f)$
$c^\mathfrak{M}$	$\mathfrak{I}(c)$
$\mathfrak{M} \vDash \phi$	$\mathfrak{M}, s \vDash \phi\ \forall\ s$ over $\mathfrak{M}$ since ϕ <u>is an $\mathcal{L}$-sentence</u>.
$t^\mathfrak{M}$	$d \in \lvert\mathfrak{M}\rvert$ st $\bar{s}(t) = d\ \forall\ s$ over $\mathfrak{M}$ since t <u>is a Closed $\mathcal{L}$-Term</u>.
$\texttt{SubSt}(t, x, \phi)$	$t \in Tm_\mathcal{L}$ is substitutable for $x \in$ Var in $\phi \in Fml_\mathcal{L}$.

0.3 Definitions

Definition 0.1 - *Arity*
The *Arity* of a function is the number of arguments it takes.
N.B. Unary, Binary, Ternary, Quaternary, ...

Definition 0.2 - *Countable Set*
Let X be a set.

X is *Countable* if

$$\begin{aligned} &\exists\ f : \mathbb{N} \to X \text{ st } f \text{ is surjective.} \\ \text{Or} \quad &\exists\ f : X \to \mathbb{N} \text{ st } f \text{ is injective.} \end{aligned}$$

Definition 0.3 - *Predicate*
A *Predicate* is an expression over a set of variables and returns a logical conclustion (*i.e.* True or False).
N.B. Practically a function from set of variables to true or false.

0.4 Indentities

Theorem 0.1 - *Complex Connectives & Quantifiers in terms of FOL*

Term	**In FOL**
$\exists x,\ P(x)$	$\neg(\forall\ x,\ \neg P(x))$
$P \vee Q$	$(\neg P) \to Q$
$P \wedge Q$	$\neg(P \to \neg Q)$
$P \leftrightarrow Q$	$(P \to Q) \wedge (Q \to P)$ $\Leftrightarrow\ \neg((P \to Q) \to \neg(Q \to P))$

0.5 Techniques

Proposition 0.3 - *Induction on Terms*
Proposition 0.4 - *Induction on Formulae*