Logic - Problem Sheet 3

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Question 1 b) - Show that the following is *not* the case for all languages $\mathcal{L}$ and $\mathcal{L}$-formulae ϕ, ψ.

$$\forall y \exists x \phi \vDash \exists x \forall y \psi$$

Answer 1 b)
Consider the scenario where $\phi := x \equiv y$ and $\psi := x \equiv y$ and are restricted to the natural numbers.
$\forall y \exists x \phi$ is satisfied for all $y \in \mathbb{N}$ since $x = y$ is a valid assignment.
However, there does not exist a natural number which is equivalent to all natural numbers.
Thus, $\exists x \forall y \psi$ does not hold for any x.
Thus $\forall y \exists x \phi \vDash \exists x \forall y \psi$ does not hold for all languages $\mathcal{L}$ and $\mathcal{L}$-formulae ϕ, ψ.

Question 2 b) - Show that the following is *not* the case for all languages $\mathcal{L}$ and $\mathcal{L}$-formulae ϕ, ψ.

$$\exists x(\phi \wedge \psi) \text{ is logically equivalent to } \exists x \phi \wedge \exists x \psi.$$

Answer 2 b)
Consider the case where $\phi := x \equiv 1$ and $\psi := x \equiv 2$.
There is no x which satisfies $\phi \wedge \psi$.
However $\exists x \phi \wedge \exists x \psi$ holds trivally.
Thus $\exists x(\phi \wedge \psi) \nvDash \exists x \phi \wedge \exists x \psi$. Further, these two are not logically equivalent for all languages $\mathcal{L}$ and $\mathcal{L}$-formulae, ϕ, ψ.

Question - 5.

Let $\{P, f, c\} \subset \mathcal{L}$ where P is a binary predicate symbol, f is a binary function symbol and c is a constant symbol. Perform the following substitutions.

NOTE

$$\begin{array}{rl}
& [\exists x P(x)]\frac{t}{x} \\
\Leftrightarrow & [\neg(\forall x \neg P(x)]\frac{t}{x} \\
\Leftrightarrow & \neg[\forall x \neg P(x)]\frac{t}{x} \\
\Leftrightarrow & \begin{cases} \neg\forall x[\neg P(x)]\frac{t}{x} & \text{if } x \not\equiv z \\ \neg(\forall x \neg P(x)) = P(x) & \text{otherwise} \end{cases} \\
\Leftrightarrow & \begin{cases} \exists x[P(x)]\frac{t}{x} & \text{if } x \not\equiv z \\ \exists x P(x) & \text{otherwise} \end{cases}
\end{array}$$

$$\begin{array}{rl}
& [P \vee Q]\frac{t}{x} \\
\Leftrightarrow & [\neg P \to Q]\frac{t}{x} \\
\Leftrightarrow & [\neg P]\frac{t}{x} \to [Q]\frac{t}{x} \\
\Leftrightarrow & \neg[P]\frac{t}{x} \to [Q]\frac{t}{x} \\
\Leftrightarrow & [P]\frac{t}{x} \vee [Q]\frac{t}{x}
\end{array}$$

$$\begin{array}{rl}
& [P \wedge Q]\frac{t}{x} \\
\Leftrightarrow & [\neg(P \to \neg Q)]\frac{t}{x} \\
\Leftrightarrow & \neg[P \to \neg Q]\frac{t}{x} \\
\Leftrightarrow & \neg\left([P]\frac{t}{x} \to [\neg Q]\frac{t}{x}\right) \\
\Leftrightarrow & \neg\left([P]\frac{t}{x} \to \neg[Q]\frac{t}{x}\right) \\
\Leftrightarrow & [P]\frac{t}{x} \wedge [Q]\frac{t}{x}
\end{array}$$

Question 5 a) - $\left[\exists z\big(P(x,z) \vee P(y,x)\big)\right]\frac{f(x,c)}{y}$

Answer 5 a)

$$\begin{aligned}
& \big[\exists z\big(P(x,z)\vee P(y,x)\big)\big]\,\frac{f(x,c)}{y} \\
\iff\; & \exists z\,[P(x,z)\vee P(y,x)]\,\frac{f(x,c)}{y} \\
\iff\; & \exists z[P(x,z)]\frac{f(x,c)}{y}\vee [P(y,x)]\frac{f(x,c)}{y} \\
\iff\; & \exists z\big(P(x,z)\vee P(f(x,c),x)\big)
\end{aligned}$$

Question 5 b) - $\big[\exists y\big(\neg P(x,y)\to \exists y\big(P(z,y)\wedge \exists z(P(x,z)\wedge P(y,x))\big)\big)\big]\,\frac{f(z,c)}{x}$

Answer 5 b)

$$\begin{aligned}
& \big[\exists y\big(\neg P(x,y)\to \exists y\big(P(z,y)\wedge \exists z(P(x,z)\wedge P(y,x))\big)\big)\big]\,\frac{f(z,c)}{x} \\
\iff\; & \exists y\,\big[(\neg P(x,y))\to \exists y\big(P(z,y)\wedge \exists z(P(x,z)\wedge P(y,x))\big)\big]\,\frac{f(z,c)}{x} \\
\iff\; & \exists y\,[(\neg P(x,y))]\,\frac{f(z,c)}{x}\to \big[\exists y\big(P(z,y)\wedge \exists z(P(x,z)\wedge P(y,x))\big)\big]\,\frac{f(z,c)}{x} \\
\iff\; & \exists y\left(\neg[P(x,y)]\frac{f(z,c)}{x}\right)\to \exists y\,[P(z,y)\wedge \exists z(P(x,z)\wedge P(y,x))]\,\frac{f(z,c)}{x} \\
\iff\; & \exists y\left(\neg[P(x,y)]\frac{f(z,c)}{x}\right)\to \exists y\left([P(z,y)]\,\frac{f(z,c)}{x}\wedge[\exists z(P(x,z)\wedge P(y,x))]\,\frac{f(z,c)}{x}\right) \\
\iff\; & \exists y\left(\neg[P(x,y)]\frac{f(z,c)}{x}\right)\to \exists y\left([P(z,y)]\,\frac{f(z,c)}{x}\wedge\exists z\,[P(x,z)\wedge P(y,x)]\,\frac{f(z,c)}{x}\right) \\
\iff\; & \exists y\left(\neg[P(x,y)]\frac{f(z,c)}{x}\right)\to \exists y\left([P(z,y)]\,\frac{f(z,c)}{x}\wedge\exists z[P(x,z)]\frac{f(z,c)}{x}\wedge[P(y,x)]\frac{f(z,c)}{x}\right) \\
\iff\; & \exists y\big(\neg P(f(z,c),y)\to \exists y\big(P(z,y)\wedge \exists z\big(P(f(z,c),z)\wedge P(y,f(z,c))\big)\big)\big)
\end{aligned}$$

Question 5 c) - $\big[\forall x\big(P(x,y)\to \exists y\big(P(z,y)\wedge \exists z(P(x,z)\vee P(y,x))\big)\big)\big]\,\frac{f(z,z)}{z}$

Answer 5 c)

$$\begin{aligned}
& \big[\forall x\big(P(x,y)\to \exists y\big(P(z,y)\wedge \exists z(P(x,z)\vee P(y,x))\big)\big)\big]\,\frac{f(z,z)}{z} \\
\iff\; & \forall x\,\big[P(x,y)\to \exists y\big(P(z,y)\wedge \exists z(P(x,z)\vee P(y,x))\big)\big]\,\frac{f(z,z)}{z} \\
\iff\; & \forall x\big([P(x,y)]\frac{f(z,z)}{z}\to \big[\exists y\big(P(z,y)\wedge \exists z(P(x,z)\vee P(y,x))\big)\big]\,\frac{f(z,z)}{z}\big) \\
\iff\; & \forall x\big(P(x,y)\to \exists y\,\big[\big(P(z,y)\wedge \exists z(P(x,z)\vee P(y,x))\big)\big]\,\frac{f(z,z)}{z}\big) \\
\iff\; & \forall x\big(P(x,y)\to \exists y\,\left([P(z,y)]\frac{f(z,z)}{z}\wedge\big[\exists z\big(P(x,z)\vee P(y,x)\big)\big]\,\frac{f(z,z)}{z}\right) \\
\iff\; & \forall x\,\big(P(x,y)\to \exists y\,\big(P(f(z,z),y)\wedge \exists z\big(P(x,z)\vee P(y,x)\big)\big)\big)
\end{aligned}$$

Question - 6.

In the substitutions performed in 5 *b*) & 5 *c*) state whether or not the terms you substituted are *substitutable* for the variables in the formulae; provide a proof.

Answer 6

5*b*) By definition of substitutability we have $\texttt{SubSt}(f(z,c),x,P(x,z))$ & $\texttt{SubSt}(f(z,c),x,P(y,z))$.
Thus $\texttt{SubSt}(f(z,c),x,P(x,z)\vee P(y,z))$.
However $z\in \text{Var}(f(z,c))=\{z\}$ and $x\in \text{FV}(\exists z(P(x,z)\vee P(y,x)))=\{x,y\}$.
Thus we do not have that $\texttt{SubSt}(f(z,c),x,\exists z(P(x,z)\vee P(y,z)))$.
Thus $f(z,c)$ is not substitutable for x in $\exists y\big(\neg P(x,y)\to\exists y\big(P(z,y)\wedge\exists z(P(x,z)\wedge P(y,x))\big)\big)$.

5*c*) We have $\texttt{SubSt}(f(z,z),z,P(x,z))$ & $\texttt{SubSt}(f(z,z),z,P(y,x))$.
Thus $\texttt{SubSt}(f(z,z),z,P(x,z)\wedge P(y,x))$.
Note that $z\notin FV(\exists(P(x,z)\vee P(y,x)))=\{x,y\}$.
Thus $\texttt{SubSt}(f(z,z),z,\exists z(P(x,z)\wedge P(y,x)))$.
We have $\texttt{SubSt}(f(z,z),z,P(z,y))$ so $\texttt{SubSt}(f(z,z),z,P(z,y)\wedge\exists z(P(x,z)\wedge P(y,x)))$.
Note that $y\notin \text{Var}(f(z,z))=\{z\}$.
Thus $\texttt{SubSt}\big(f(z,z),z,\exists y\big(P(z,y)\wedge\exists z(P(x,z)\wedge P(y,x))\big)\big)$.

Let $\phi := \exists y\big(P(z,y) \wedge \exists z(P(x,z) \wedge P(y,x))\big)$ noting $\texttt{SubSt}(f(z,z),z,\phi)$.
We have $\texttt{SubSt}(f(z,z),z,P(x,y))$.
Thus $\texttt{SubSt}(f(z,z),z,P(x,y) \to \phi)$.
Note that $x \not\in \text{Var}(f(z,z)) = \{z\}$.
Thus $\texttt{SubSt}(f(z,z),z,\forall x(P(x,y) \to \phi))$.
Thus $f(z,z)$ <u>is</u> substitutable for z in $\forall x\big(P(x,y) \to \exists y\big(P(z,y) \wedge \exists z(P(x,z) \vee P(y,x))\big)\big)$.