Logic - Reviewed Notes

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NOTES

- Not included any proofs.
- Not included any examples.

1 Syntax

1.1 General

Definition 1.1 - *Alphabet, $\mathcal{A}$*
An *Alphabet* is a set of characters, $\mathcal{A}$. These characters do not have any assigned values (yet).

Definition 1.2 - *String*
A *String*, $a := \langle a_1, \dots, a_n \rangle$, over an alphabet $\mathcal{A}$ is an element of $\mathcal{A}^n$ for $n \in \mathbb{N}$.
Here a is said to have *length* n.

Remark 1.1 - $\langle a, b \rangle = \langle \langle a_1, \dots, a_n \rangle, \langle b_1, \dots, b_m \rangle \neq \langle a_1, \dots, a_n, b_1, \dots, b_m \rangle$

Definition 1.3 - *Set of all Strings, $\mathcal{A}^*$*
Let $\mathcal{A}$ be an *Alphabet*.
We define the set of all strings, $\mathcal{A}^*$, over the alphabet as

$$\mathcal{A}^* := \{\langle a_1, \dots, a_n \rangle : n \in \mathbb{N} \text{ and } a_1, \dots, a_n \in \mathcal{A}\} \equiv \bigcup_{n \in \mathbb{N}} \mathcal{A}^n$$

Remark 1.2 - *If an alphabet, $\mathcal{A}$, is countable then $\mathcal{A}^*$ is countable*
Further $\mathcal{A}^*$ is countably infinite if $\mathcal{A} \neq \emptyset$.

Definition 1.4 - *Declarative Sentence*
A *Declarative Sentence* is a sentence which is either true or false.
N.B. These are the focus of mathematical logic.

Definition 1.5 - *Characteristic Function, χ_R*
Let $R \subset \mathbb{N}^m$ be an m-arity *Relation*.
The *Characteristic Function* $\chi_R : \mathbb{N}^m \to \mathbb{N}$ is defined as

$$\chi_R(n_1, \dots, n_m) := \begin{cases} 0 & \text{if } \langle n_1, \dots, n_m \rangle \in R \\ 1 & \text{otherwise} \end{cases}$$

N.B. R is *Recursive* if χ_R is recursive by **Definition 4.5**.

1.2 First Order Languages

Definition 1.6 - *Common Components of Alphabets*
Below are some common classes of characters used in mathematical alphabets

i) Propositional Connectives (Describe locagical relations between predicates).
   'not', 'and', 'or', 'if...then...'.

ii) Quantifiers
   'for all', 'there is'.

iii) Variables
   'x', 'y', 'z', ...

iv) Punctuation
   '(','‚)', ',', ...

v) Equality
   ' $=$'.

vi) Constants
‘1’,‘2’,‘3’,‘e’,...

vii) Predicates
‘ $\prec$’.

viii) Functions
‘$\circ$’.

Definition 1.7 - *Alphabet of First-Order Language*
The *Alphabet* of a *First-Order Language* comprises the following elements

i) Propositional Connectives
$\neg$, $\to$

ii) Quantifiers
$\forall$

iii) Variables
$v_1, v_2, \dots$ (Infinetly many).

iv) Punctuation
( ) and ,

v) Equality
$\equiv$ (This is a 2-arity logical predicate)

vi) Constants
$c_1, c_2, \dots$ (Countable many since we use countable alphabets).

vii) Predicates
P_i^n is an n-arity predicate for $n \in \mathbb{N}$.

viii) Functions
f_i^n is an n-arity function for $n \in \mathbb{N}$.

i) - v) are *Logical Symbols* & vi) - viii) are *Non-Logical Symbols* of *First-Order Languages.*
The *Non-Logical Symbols* will vary depending on the subject matter of the language.

Remark 1.3 - $\equiv$ *is the only* <u>*logical*</u> *predicate symbol in FOLs*

Definition 1.8 - *Negation,* $\neg$*, and Implication,* $\to$
Let P, Q be *Predicates.*

P	$\neg P$
T	**F**
T	**T**

P	Q	$P \to Q$
T	T	**T**
T	F	**F**
F	T	**T**
F	F	**T**

Proposition 1.1 - *Extension to Alphabet of First-Order Language*
For concisness of notation we usally allow the following extra propositional connectives & quantifiers to be used.

- Propositional Connectives
$\wedge$, $\vee$

- Quantifiers
  $\exists$

Definition 1.9 - *And,* $\wedge$*, and Or,* $\vee$
Let P, Q be *Predicates.*

P	Q	$P \wedge Q$
T	T	**T**
T	F	**F**
F	T	**F**
F	F	**F**

P	Q	$P \vee Q$
T	T	**T**
T	F	**T**
F	T	**T**
F	F	**F**

Remark 1.4 - $P \wedge Q \Leftrightarrow \neg(P \to \neg Q)$ *and* $P \vee Q \Leftrightarrow (\neg P) \to Q$

Definition 1.10 - *$\mathcal{L}$-Term (and $\mathcal{L}$-Term Complexity)*
Let $\mathcal{L}$ be a FOL.
We define *$\mathcal{L}$-Terms* & *$\mathcal{L}$-Term Complexity* recursively.

T1 Let s be a variable or constant symbol.
s is an *$\mathcal{L}$-Term* with $cp(s) = 0$.

T2 Let f be a k-arity function symbol and $t_1, \dots, t_k$ be *$\mathcal{L}$-Terms.*
$f(t_1, \dots, t_k)$ is an *$\mathcal{L}$-Term* with $cp(f) = \max\{cp(t_1), \dots, cp(t_k)\} + 1 \geq 1$.

N.B. By this definition we cannot have infinitely long *$\mathcal{L}$-Terms.*

Definition 1.11 - *Atomic $\mathcal{L}$-Term*
Let $\mathcal{L}$ be a FOL and $t \in Tm_{\mathcal{L}}$.
t is an *Atomic $\mathcal{L}$-Term* iff $cp(t) = 0$.
i.e. An *Atomic $\mathcal{L}$-Term* is either a constant or variable symbol.

Definition 1.12 - *Compound $\mathcal{L}$-Term*
Let $t \in Tm_{\mathcal{L}}$.
t is a *Compound $\mathcal{L}$-Term* iff $cp(t) \geq 1$.
i.e. An *Atomic $\mathcal{L}$-Term* is function symbol.

Definition 1.13 - *Atomic Formulae*
Let $\mathcal{L}$ be a FOL, P be a k-arity predicate symbol of $\mathcal{L}$ and $t_1, \dots, t_k \in Tm_{\mathcal{L}}$.
An *Atomic Formulae* has the form

$$P(t_1, \dots, t_k)$$

i.e. Atomic Formulae are predicates on *$\mathcal{L}$-Terms.*

Definition 1.14 - *$\mathcal{L}$-Formula (and $\mathcal{L}$-Formulae Complexity)*
Let $\mathcal{L}$ be a FOL.
We define *$\mathcal{L}$-Formulae* & *$\mathcal{L}$-Formulae Complexity* recursively

F1 Let ϕ be an *Atomic $\mathcal{L}$-Formula.*
ϕ is an $\mathcal{L}$-Formula with $cp(\phi) = 0$.

F2 Let ϕ be an *$\mathcal{L}$-Formula.*
$\neg\phi$ is an $\mathcal{L}$-Formula with $cp(\neg\phi) = cp(\phi) + 1$.

F3 Let ϕ, ψ be a *$\mathcal{L}$-Formulae.*
$\phi \to \psi$ is an $\mathcal{L}$-Formula with $cp(\phi \to \psi) = \max\{cp(\phi), cp(\psi)\} + 1$.

F4 Let ϕ be an *$\mathcal{L}$-Formula* & x be any variable.
$\forall x\phi$ is an $\mathcal{L}$-Formula with $cp(\forall x\phi) = cp(\phi) + 1$.

N.B. By this definition we cannot have infinitely long *$\mathcal{L}$-Formulae.*

Remark 1.5 - *$\mathcal{L}$-Term & $\mathcal{L}$-Formulae complexity is a measure of syntactic complexity and is unrelated to any semantic meaning.*
$\mathcal{L}$-Formulae complexity is unrelated from the complexity of any terms in it.

Remark 1.6 - *F4 necessitates the use of parentheses*
Otherwise $\phi \to \psi \to \theta$ is ambiguous as it could be read as either $(\phi \to \psi) \to \theta$ or $\phi \to (\psi \to \theta)$ which don't necessarily have the same semantic meaning.

Definition 1.15 - *Compound $\mathcal{L}$-Formula*
Let $\phi \in Fml_{\mathcal{L}}$.
ϕ is a *Compound $\mathcal{L}$-Formula* iff $cp(\phi) \geq 1$.

1.3 Induction

Theorem 1.1 - *Induction on Terms*
Let $\mathcal{L}$ be a FOL and P be a property that $\mathcal{L}$-Terms may have.
If

i) All *Atomic $\mathcal{L}$-terms* have P; And,

ii) For all k-arity function symbols f of $\mathcal{L}$ and $t_1, \dots, t_k \in Tm_{\mathcal{L}}$ which have property P, $f(t_1, \dots, t_k)$ has P.

Then all $t \in Tm_{\mathcal{L}}$ have property P.

Theorem 1.2 - *Induction on Formulae*
Let $\mathcal{L}$ be a FOL and P be a property that $\mathcal{L}$-Formulae may have.
If

i) All *Atomic $\mathcal{L}$-Formulae* have P; And,

ii) ϕ, ψ have P then $\neg\phi$, $\phi \to \psi$ and $\forall x\phi$ (for all variables x) have property P.

Then all $\phi \in Fm_{\mathcal{L}}$ have property P.

1.4 Free Variables

Definition 1.16 - *Set of Variables,* $\text{Var}(\cdot)$
$\text{Var} : \mathcal{A}_{\mathcal{L}}^* \to 2^{\text{Var}}$ is a function which maps from a string to the set of variables in it.
Variables are defined by the *Alphabet* of the language being used.

Definition 1.17 - *Closed $\mathcal{L}$-Term*
Let $t \in Tm_{\mathcal{L}}$ for some FOL, $\mathcal{L}$.
If $\text{Var}(t) = \emptyset$ then t is said to be a *Closed $\mathcal{L}$-Term.*

Definition 1.18 - *Free Variables, $FV(\cdot)$*
Let $\mathcal{L}$ be a FOL.
Free Variables are unbounded variables in an *$\mathcal{L}$-Formula.*
We define the *Set of Free Variables* of an *$\mathcal{L}$-Formula* inductively

FV1 Let P be a k-arity *Predicate* & $t_1, \dots, t_k \in Tml_{\mathcal{L}}$.
$FV(P(t_1, \dots, t_k)) := \text{Var}(P(t_1, \dots, t_k))$.

FV2 Let $\phi \in Fml_{\mathcal{L}}$.
$FV(\neg\phi) := FV(\phi)$.

FV3 Let $\phi, \psi \in Fml_{\mathcal{L}}$.
$FV(\phi \to \psi) := FV(\phi) \cup FC(\psi)$.

FV4 Let $\phi \in Fml_{\mathcal{L}}$ and x be any variable.
$FV(\forall x\phi) := FV(\phi) \backslash \{x\}$.

FV-EXT1 Let $\phi, \psi \in Fml_{\mathcal{L}}$.
$FV(\phi \land \psi) := FV(\phi) \cup FV(\psi)$.

FV-EXT2 Let $\phi, \psi \in Fml_{\mathcal{L}}$.
$FV(\phi \lor \psi) := FV(\phi) \cup FV(\psi)$.

FV-EXT3 Let $\phi \in Fml_{\mathcal{L}}$ and x be any variable.
$FV(\exists x\phi) := FV(\phi) \backslash \{x\}$.

N.B. $FV(\cdot) : \mathcal{A}^*_{\mathcal{L}} \to 2^{\text{Var}}$.

Definition 1.19 - *$\mathcal{L}$-Sentence*
Let $\phi \in Fml_{\mathcal{L}}$ for some FOL, $\mathcal{L}$.
If $FV(\phi) = \emptyset$ then ϕ is said to be a *$\mathcal{L}$-Sentence.*

Remark 1.7 - *The meaning anthonof formulae depends on how we interpret their free variables*

1.5 Consistency

Definition 1.20 - *Consistent*
Let $\mathcal{L}$ be a FOL, $\phi \in Fml_{\mathcal{L}}$ and $\Phi \subset Fml_{\mathcal{L}}$.

- ϕ is *Consistent* iff $\nexists\ \phi \in Fml_{\mathcal{L}}$ st $\phi \vdash \psi$ and $\phi \vdash \neg\psi$.
- Φ is *Consistent* iff $\nexists\ \phi \in Fml_{\mathcal{L}}$ st $\Phi \vdash \psi$ and $\Phi \vdash \neg\psi$.

Proposition 1.2 - Φ *is Consistent iff* $\exists\ \phi \in Fml_{\mathcal{L}}$ *st* $\Phi \nvdash \psi$

Proposition 1.3 - Φ *is Consistent iff* $\forall\ \Sigma \subset \Phi$, Σ *is Consistent*

Proposition 1.4 - *If* $\Gamma \vdash \phi$ *then* $\exists\ \Sigma \subset \Gamma$ *st* $\Sigma \vdash \phi$.

Theorem 1.3 - *Inconsistency*
Let $\mathcal{L}$ be a FOL, $\phi \in Fml_{\mathcal{L}}$ and $\Phi \subset Fml_{\mathcal{L}}$.

i) $\Phi \vdash \phi$ iff $\Phi \cup \{\neg\phi\}$ is *Inconsistent.*

ii) $\Phi \cup \{\phi\}$ is *Inconsistent* iff $\Phi \cup \{\neg\neg\phi\}$ is *Inconsistent.*

iii) $\Phi \vdash \neg\phi$ iff $\Phi \cup \{\phi\}$ is *Inconsistent.*

Theorem 1.4 - *If* $\Gamma \cup \{\neg\phi\}$ *is Satisfiable then* $\Gamma \nvdash \phi$.

Theorem 1.5 - *Chain of Consistency*
Let $\mathcal{L}$ be a FOL and $\Gamma_0 \subset \cdots \subset \Gamma_n \subset \cdots \subset Fml_{\mathcal{L}}$.

$$\text{If } \forall i,\ \Gamma_i \text{ is } \textit{Consistent}, \text{ then } \Gamma := \cup_i \Gamma_i \text{ is } \textit{Consistent}.$$

Definition 1.21 - *Maximally Consistent*
Let $\mathcal{L}$ be a FOL and $\Gamma \subset Fml_{\mathcal{L}}$.
Γ is *Maximally Consistent* if Γ is *Consistent* <u>and</u> $\Gamma \not\subset \Delta$ where Δ is any consistent set of $Fml_{\mathcal{L}}$.

Proposition 1.5 -
Let $\mathcal{L}$ be a FOL and $\Sigma \subset Fml_L L$.
If Σ is *Consistent* then $\exists\ \Delta \subset Fml_{\mathcal{L}}$ st $\Delta \supset \Sigma$ and Δ is *Maximally Consistent.*

Definition 1.22 - *Henkin*
Let $\mathcal{L}$ be a FOL and $\Gamma \subset Fml_{\mathcal{L}}$.
Γ is *Henkin* if $\forall$ formula of the form $\exists x\phi \in Fml_{\mathcal{L}}, \exists\, t \in Tm_{\mathcal{L}}$ st $\texttt{SubSt}(t, x, \phi)$ <u>and</u> $\Gamma \vdash (\exists x\phi \to [\phi]\frac{t}{x})$.

Proposition 1.6 - *Let* $\Gamma \subset \Sigma \subset Fml_{\mathcal{L}}$ *if* Γ *is Henkin then* Σ *is Henkin.*

Theorem 1.6 -
Let $\mathcal{L}$ be a FOL and $\Gamma \subseteq Fml_{\mathcal{L}}$ be *Consistent* st $FV(\Gamma)$ is finite. Then

$$\exists\ \Sigma \subseteq Fml_{\mathcal{L}} \text{ st } \Gamma \subset \Sigma \text{ and } \Sigma \text{is a consistent Henkin.}$$

Proposition 1.7 - *If* $\Gamma \subset Fml_{\mathcal{L}}$ *is Consistent and* $\Gamma \vdash \gamma \implies \Gamma \cup \{\phi\}$ *is Consistent.*

Proposition 1.8 - *If* $\Gamma \subset Fml_{\mathcal{L}}$ *is Consistent then* Γ *is Maximally Consistent* <u>*iff*</u> $\psi \in \Gamma$ <u>*or*</u> $\neg\psi \in \Gamma$ *holds* $\forall\ \psi \in Fml_{\mathcal{L}}$.

Proposition 1.9 - *If* Γ *is Maximally Consistent then* $\Gamma \vdash \phi$ <u>*iff*</u> $\phi \in \Gamma$.
Hence, by **Proposition 1.6**, if Γ is *Maximally Consistent* then $\forall\ \psi \in Fml_{\mathcal{L}}$ either $\Gamma \vdash \psi$ <u>xor</u> $\Gamma \vdash \neg\psi$ holds.

Proposition 1.10 - *Properties Maximally Consistent Henkin Sets*
Let $\mathcal{L}$ be a FOL, $\Gamma \subseteq Fml_{\mathcal{L}}$ <u>be</u> *Maximally Consistent Henkin* and $\phi, \psi \in Fml_{\mathcal{L}}$.

$$\begin{aligned}
\Gamma \vdash \neg\phi &\iff \Gamma \nvdash \phi \\
\Gamma \vdash (\phi \to \psi) &\iff \text{if } \Gamma \vdash \phi \text{ then } \Gamma \vdash \psi \\
\Gamma \vdash \forall x\phi &\iff \forall\ t \in Tm_{\mathcal{L}} \text{ if } \texttt{SubSt}(t, x, \phi) \text{ then } \Gamma \vdash [\phi]\tfrac{t}{x}
\end{aligned}$$

Proposition 1.11 -
Let $\mathcal{L}$ be a FOL and $\Gamma \subset Fml_{\mathcal{L}}$ be *Consistent* st $FV(\Gamma)$ is finite. Then

$$\exists\ \Sigma \subset \Delta \subset Fml_{\mathcal{L}} \text{ st } \Delta \text{ is Maximially Consistent Henkin set}$$

Theorem 1.7 - *Henkin's Theorem*
Let $\mathcal{L}$ be a FOL and $\Gamma \subset Fml_{\mathcal{L}}$ be a *Maximally Consistent Henkin* set. Then

$$\forall \phi \in Fml_{\mathcal{L}},\ \mathfrak{M}^{\Gamma}, s^{\Gamma} \vDash \phi \ \underline{\text{iff}}\ \Gamma \vdash \phi$$

N.B. $\Gamma \vdash \phi$ iff $\phi \in \Gamma$ since Γ is *Maximally Consistent.*

1.6 Alphabetic Variants

Theorem 1.8 - *Alphabetic Variants*
$\forall\ \phi \in Fml_{\mathcal{L}},\ t \in Tm_{\mathcal{L}},\ z \in \text{Var},\ \exists\psi \in Fml_{\mathcal{L}}$ with $cp(\psi) = cp(\phi)$ st

$$\vdash (\phi \leftrightarrow \psi) \quad \text{and} \quad \texttt{SubSt}(t, z, \psi)$$

N.B. This ψ is called an *Alphabetic Variant* of ϕ.

2 Semantics

Definition 2.1 - *$\mathcal{L}$-Structure*
Let $\mathcal{L}$ be a FOL.
An *$\mathcal{L}$-Structure* assigns meaning to the *Non-Logical* symbols of $\mathcal{L}$.
An *$\mathcal{L}$-Structure* is an ordered pair $\mathfrak{M} := (D, \mathfrak{I})$ where

Domain D is a non-empty set.
Often $\mathbb{R}$ or similar.

Interpretation $\mathfrak{I}$ is a function over the non-logical symbols of $\mathcal{L}$.

$$\begin{array}{rcl} \mathfrak{I}(c) & \in & D \text{ where } c \text{ is a constant symbol of } \mathcal{L} \\ \mathfrak{I}(P) & \subset & D^n \text{ where } P \text{ is a } k\text{-arity predicate symbol of } \mathcal{L} \\ \mathfrak{I}(f) & : & D^n \to D \text{ where } f \text{ is a } k\text{-arity function symbol of } \mathcal{L} \end{array}$$

Remark 2.1 - *Interpretation, $\mathfrak{I}$*
The *Interpretation* is a function which assigns meaning to non-logical symbols.
$\mathfrak{I}(P)$ gives the property or relation on D by which P is interpreted.
$\mathfrak{I}(f)$ gives the function on D^n by which f is interpreted.
$\mathfrak{I}(P)$ gives the object in D which c denotes.

Definition 2.2 - *Variable Assignment, s*
Let $\mathcal{L}$ be a FOL and $\mathfrak{M} := (|\mathfrak{M}|, \mathfrak{I})$ be an $\mathcal{L}$-Structure.
A *Variable Assignment* maps variables to a value in the domain of $\mathfrak{M}$.

$$s : \text{Var} \to |\mathfrak{M}|$$

Definition 2.3 - *Variable Assignment for $\mathcal{L}$-Terms, $\bar{s}$*
Let $\mathcal{L}$ be a FOL and $\mathfrak{M} := (|\mathfrak{M}|, \mathfrak{I})$ be an $\mathcal{L}$-Structure.
We define *Variable Assignment* over *$\mathcal{L}$-Terms* recursively

V1 Let x be a variable symbol of $\mathcal{L}$.
$\bar{s}(x) := s(x)$

V2 Let c be a constant symbol of $\mathcal{L}$.
$\bar{s}(c) := c^{\mathfrak{M}}$

V3 Let f be a k-arity function symbol of $\mathcal{L}$ and $t_1, \dots, t_k$ be *$\mathcal{L}$-Terms*.
$\bar{s}(f(t_1, \dots, t_k)) := f^M(\bar{s}(t_1), \dots, \bar{s}(t_k))$

N.B. $\bar{s} : Tm_{\mathcal{L}} \to |\mathfrak{M}|$.

Remark 2.2 - *$\bar{s}(t)$ is the <u>Semantic Value</u> of term t in struture $\mathfrak{M}$ under assignement s.*
$\bar{s}(t)$ gives a description of what t designates in $\mathfrak{M}$ under the assignment s.

2.1 Satisfaction Relation

Definition 2.4 - *Satisfaction Relation, $\models$*
Let $\mathcal{L}$ be a FOL, $\mathfrak{M}$ be an *$\mathcal{L}$-Structure* and s a *Variable Assignment* over $\mathfrak{M}$.
The *Satisfaction Relation* (states whether a given formula is true under a given model)??
We define the *Satisfaction Relation*, $\models$, recursively

S1 Let $t_1, t_2 \in Tm_{\mathcal{L}}$.
$\mathfrak{M}, s, \models (t_1 \equiv t_2) :\Leftrightarrow \bar{s}(t_1) = \bar{s}(t_2)$.

S2 Let P be a k-arity predicate symbol of $\mathcal{L}$ and $t_1, \dots, t_k \in Tm_{\mathcal{L}}$.
$\mathfrak{M}, s, \models P(t_1, \dots, t_k) :\Leftrightarrow \langle \bar{s}(t_1), \dots, \bar{s}(t_k) \rangle \in P^{\mathfrak{M}}$.

S3 Let $\phi \in Fml_{\mathcal{L}}$.
$\mathfrak{M}, s, \models \neg\phi :\Leftrightarrow \mathfrak{M}, s \not\models \phi$.

S4 Let $\phi, \psi \in Fml_{\mathcal{L}}$.
$\mathfrak{M}, s, \models (\phi \to \psi) :\Leftrightarrow$ if $\mathfrak{M}, s \models \phi$ then $\mathfrak{M}, s \models \psi$.

S5 Let $\phi \in Fml_{\mathcal{L}}$ and x be any variable.
$\mathfrak{M}, s, \models \forall x\phi :\Leftrightarrow \mathfrak{M}, s\frac{d}{x} \models \phi$ for all $d \in |\mathfrak{M}|$.

S-EXT1 Let $\phi, \psi \in Fml_{\mathcal{L}}$.
$\mathfrak{M}, s \models (\phi \wedge \psi) :\Leftrightarrow \mathfrak{M}, s \models \phi$ and $\mathfrak{M}, s \models \psi$.

S-EXT2 Let $\phi, \psi \in Fml_{\mathcal{L}}$.
$\mathfrak{M}, s \models (\phi \vee \psi) :\Leftrightarrow \mathfrak{M}, s \models \phi$ or $\mathfrak{M}, s \models \psi$.

S-EXT3 Let $\phi \in Fml_{\mathcal{L}}$ and x be any variable.
$\mathfrak{M}, s \models \exists x\phi :\Leftrightarrow \mathfrak{M}, s\frac{d}{x} \models \phi$ for at least one $d \in |\mathfrak{M}|$.

S-EXT2 Let $\phi, \psi \in Fml_{\mathcal{L}}$.
$\mathfrak{M}, s \models (\phi \leftrightarrow \psi) :\Leftrightarrow \mathfrak{M}, s \models \phi$ iff $\mathfrak{M}, s \models \psi$.

Remark 2.3 - *When $\mathfrak{M}, s \models \phi$ holds we say "ϕ is true in $\mathfrak{M}$ under s"*
Or, "ϕ is satisfied by $\mathfrak{M}$ under s".
Or, "$\mathfrak{M}, s$ models ϕ".

Definition 2.5 - *Model*
Let $\mathcal{L}$ be a FOL, $\Phi \subseteq Fml_{\mathcal{L}}$, $\mathfrak{M}$ be an *L-Structure* and s a *Variable Assignment*.
$\mathfrak{M}, s$ is a *Model* of Φ if $\mathfrak{M}, s \models \Phi$.

Remark 2.4 - *Semantic Value of a Term*
Let $t \in Tm_{\mathcal{L}}$ for some FOL, $\mathcal{L}$, and s be a *Variable Assignment*.
The semantic value of t, $\bar{s}(t)$, only depends on

i) The *Interpreation* of the constant & function symbols that occur in t. And,

ii) The *Assignment* of values to variables in t, given by s.

Remark 2.5 - *Truth of a Formula*
Let $\phi \in Fml_{\mathcal{L}}$ for some FOL, $\mathcal{L}$.
The truth of ϕ only depends on

i) The domain of discourse, $|\mathfrak{M}|$, over which the quantifiers range

ii) The *Interpretation* of the constants, functions & predicate symbols in ϕ.

iii) The *Assignment* of values to *Free Variables* in ϕ, given by s.

Theorem 2.1 - *Coincidence Lemma*
Let $\mathcal{L}_1, \mathcal{L}_2$ be unique FOLs, $\mathfrak{M}_1 := (D, \mathfrak{I}_1)$ be an $\mathcal{L}_1$-Structure and $\mathfrak{M}_2 := (D, \mathfrak{I}_2)$ be an $\mathcal{L}_2$-Structure.
Note that both structures have the same domain.
Let $\mathcal{L} := \mathcal{L}_1 \cap \mathcal{L}_2$. Then the following are true

i) $\forall\ t \in Tm_{\mathcal{L}}, \forall$ variable assignments s_1 over $\mathfrak{M}_2$ and s_2 over $\mathfrak{M}_2$

$$\text{If} \left\{ \begin{array}{c} c^{\mathfrak{M}_1} = c^{\mathfrak{M}_2} \ \forall \ c \text{ that occur in } t \\ f^{\mathfrak{M}_1} = f^{\mathfrak{M}_2} \ \forall \ f \text{ that occur in } t \\ s_1(x) = s_2(x) \ \forall \ x \text{ that occur in } t \end{array} \right\} \text{ then } \overline{s_1}(t) = \overline{s_2}(t).$$

i.e. If these conditions hold then t has the same semantic value under both variable assignments.

ii) $\forall \ \phi \in Fml_{\mathcal{L}}, \forall$ variable assignments s_1 over $\mathfrak{M}_2$ and s_2 over $\mathfrak{M}_2$

$$\text{If} \left\{ \begin{array}{c} c^{\mathfrak{M}_1} = c^{\mathfrak{M}_2} \ \forall \ c \text{ that occur in } \phi \\ f^{\mathfrak{M}_1} = f^{\mathfrak{M}_2} \ \forall \ f \text{ that occur in } \phi \\ P^{\mathfrak{M}_1} = P^{\mathfrak{M}_2} \ \forall \ P \text{ that occur in } \phi \\ s_1(x) = s_2(x) \ \forall \ x \text{ that occur in } \phi \end{array} \right\} \text{ then } \mathfrak{M}_1, s_2 \models \phi \text{ iff } \mathfrak{M}_2, s_2 \models \phi.$$

i.e. If these conditions hold ϕ is equivalent truth values under both $\mathcal{L}$-structures & variable assignemnts.

N.B. AKA *Reduct Property* of First-Order Logic.

Remark 2.6 - *Semantic Interpretations Closed $\mathcal{L}$-Terms & $\mathcal{L}$-Sentences*
Let $\mathcal{L}$ be a FOL, t be a *Closed $\mathcal{L}$-Term*, ϕ be an *$\mathcal{L}$-Sentence*, $\mathfrak{M}$ be an *$\mathcal{L}$-Structure.*
Let s_1, s_2 be arbitrary *Variable Assignments* over $\mathfrak{M}$. Then

$$\overline{s_1}(t) = \overline{s_2}(t) \text{ and } \mathfrak{M}, s_1 \models \phi \text{ iff } \mathfrak{M}, s_2 \models \phi$$

i.e. Choice of variable assignment does not affect semantic value of closed $\mathcal{L}$-Terms & $\mathcal{L}$-Sentences.

Definition 2.6 - *Logical Consequence, $\Phi \models \phi$*
Let $\mathcal{L}$ be a FOL, $\phi \in Fml_{\mathcal{L}}$ and $\Phi \subseteq Fml_{\mathcal{L}}$.
ϕ is a *Logical Consequence* of Φ iff

$\forall$ $\mathcal{L}$-Structures $\mathfrak{M}$, $\forall$ variable assignments s over $\mathfrak{M}$ it holds that $(\mathfrak{M}, s \models \Phi) \to (\mathfrak{M}, s \models \phi)$.

N.B. When this is the case, it is denoted $\Phi \models \phi$.
N.B. AKA "ϕ logically follows from Φ" or "Φ logically implies ϕ".

Proposition 2.1 - *For unary predicates P, $P(x) \models P(x) \vee P(y)$*

Proposition 2.2 - $\forall \ \phi, \psi \in Fml_{\mathcal{L}}$ & $\Phi \subseteq Fml_{\mathcal{L}}$, $\Phi, \phi \models \psi$ *iff* $\Phi \models \phi \to \psi$

Definition 2.7 - *Logically Valid, $\models \phi$*
Let $\mathcal{L}$ be a FOL, $\phi \in Fml_{\mathcal{L}}$.
ϕ is *Logically Valid* iff $\mathfrak{M}, s \models \phi$ for all $\mathcal{L}$-Structures $\mathfrak{M}$ and variable assignemnts s over $\mathfrak{M}$.
N.B. This is denoted $\models \phi$.

Definition 2.8 - *Satisfiable*
Let $\mathcal{L}$ be a FOL, $\phi \in Fml_{\mathcal{L}}$ and $\Phi \subseteq Fml_{\mathcal{L}}$.

ϕ is *Satisfiable* iff $\exists$ an $\mathcal{L}$-Structure $\mathfrak{M}$ and variable assignment s over $\mathfrak{M}$, st $\mathfrak{M}, s \models \phi$.
Φ is *Satisfiable* iff $\exists$ an $\mathcal{L}$-Structure $\mathfrak{M}$ and variable assignment s over $\mathfrak{M}$, st $\mathfrak{M}, s \models \Phi$.

Theorem 2.2 -
Let $\mathcal{L}$ be a FOL, $\phi \in Fml_{\mathcal{L}}$ and $\Phi \subseteq Fml_{\mathcal{L}}$.
Then

i) ϕ is *Logically Valid* iff $\emptyset \models \phi$.

ii) ϕ is *Logically Valid* iff $\neg\phi$ is not *Satisfiable*.

iii) $\Phi \models \phi$ iff $\Phi \cup \{\neg\phi\}$ is not *Satisfiable*.

Definition 2.9 - *Logical Equivalence*
Let $\mathcal{L}$ be a FOL and $\phi, \psi \in Fml_{\mathcal{L}}$.
ϕ is *Logically Equivalent* to ψ iff $\phi \models \psi$ and $\psi \models \phi$.
i.e. ϕ is *Logically Equivalent* to ψ iff $\models \phi \leftrightarrow \psi$.
N.B. For formulae this is the *Equivalence Relation.*

Proposition 2.3 - *Logical Equivalences*
The following are *Logically Equivalent*

i) $((\phi \wedge \psi) \wedge \theta)$ is logically equivalent to $(\phi \wedge (\psi \wedge \theta))$.

ii) $((\phi \vee \psi) \vee \theta)$ is logically equivalent to $(\phi \vee (\psi \vee \theta))$.

iii) $\neg\neg\phi$ is logically equivalent to ϕ.

iv) $\phi \wedge \psi$ is logically equivalent to $\neg((\neg\phi) \vee (\neg\psi))$.

Definition 2.10 - *True of,* $\mathfrak{M} \models \phi[\![a_1, \dots, a_n]\!]$
Let $\mathcal{L}$ be a FOL, $\phi \in Fml_{\mathcal{L}}$ with $FV(\phi) \subset \{x_1, \dots, x_n\}$.
Let $\mathfrak{M}$ be an $\mathcal{L}$-Structure, s_1, s_2 be variable assignments over $\mathfrak{M}$ and $a_1, \dots, a_n \in |\mathfrak{M}|$.
By the *Conincidence Lemma*, **Theorem 2.1**

$$\text{if } s_1(x_i) = s_2(x_2)\ \forall\ i \in [1, n] \text{ then } \mathfrak{M}, s_1 \models \phi \Leftrightarrow \mathfrak{M}, s_2 \models \phi$$

Equivalently

$$\begin{aligned} & \mathfrak{M}, s \models \phi \text{ for } \textit{all} \text{ variable assignments } s \text{ over } \mathfrak{M} \text{ st } s(x_1) = a_1, \dots, s(x_n) = a_n \\ \Leftrightarrow\ & \mathfrak{M}, s \models \phi \text{ for } \textit{some} \text{ variable assignments } s \text{ over } \mathfrak{M} \text{ st } s(x_1) = a_1, \dots, s(x_n) = a_n \end{aligned}$$

We denote these holding by $\mathfrak{M} \models \phi[\![a_1, \dots, a_n]\!]$.
N.B. $\mathfrak{M} \models \phi[\![a_1, \dots, a_n]\!]$ means "ϕ is *true of* the objects $a_1, \dots, a_n \in \mathfrak{M}$".

2.2 Substitution

Definition 2.11 - *Substitution*
Substitution is the process of replacing one expression with another.
Substituting t for x in a is denoted by $[a]\frac{t}{x}$.
N.B. Usually t is an $\mathcal{L}$-term, x is a variable & a is an $\mathcal{L}$-term or $\mathcal{L}$-Formula.

Definition 2.12 - *Substitution of a Term for a Variable in a Term*
Let $\mathcal{L}$ be a FOL, $a, t \in Tm_{\mathcal{L}}$ and x be a variable.
We define the *Substitution* $[a]\frac{t}{x}$ recursively

Sub-T1 If a is an *Atomic $\mathcal{L}$-Term* then

$$[a]\frac{t}{x} := \begin{cases} t & \text{if } a = x \\ a & \text{if } a \neq x \end{cases}$$

Sub-T2 If a is a *Compound $\mathcal{L}$-Term* of the form $a := f(a_1, \dots, a_k)$ where $a_1, \dots, a_k \in Tm_{\mathcal{L}}$

$$[a]\frac{t}{x} := f\left([a_1]\frac{t}{x}, \dots, [a_k]\frac{t}{x}\right)$$

Remark 2.7 - $[a]\frac{t}{x} = a$ *for all constant symbols in a*

Definition 2.13 - *Substitution of a Term for a Variable in a Formula*
Let $\mathcal{L}$ be a FOL, $\phi, \psi \in Fml_{\mathcal{L}}$, $t \in Tm_{\mathcal{L}}$ and x, z be variables.
We define the *Substitution* $[\phi]\frac{t}{x}$ recursively

SUB1 If ϕ is an *Atomic $\mathcal{L}$-Formula* of the form $P(a_1, \dots, a_k)$ where $a_1, \dots, a_k \in Tm_{\mathcal{L}}$.
$[\phi]\frac{t}{x} := P\left([a_1]\frac{t}{x}, \dots, [a_k]\frac{t}{x}\right)$

SUB-F2 $[\neg\phi]\frac{t}{x} := \neg[\phi]\frac{t}{x}$.

SUB-F3 $[(\phi \to \psi)]\frac{t}{x} := [\phi]\frac{t}{x} \to [\psi]\frac{t}{x}$.

SUB-F4 $[\forall z\phi]\frac{t}{x} := \begin{cases} \forall z[\phi]\frac{t}{x} & \text{if } x \neq z \\ \forall z\phi & \text{if } x = z \end{cases}$.

SUB-F-EXT1 $[\phi \wedge \psi]\frac{t}{x} := [\phi]\frac{t}{x} \wedge [\psi]\frac{t}{x}$.

SUB-F-EXT2 $[\phi \vee \psi]\frac{t}{x} := [\phi]\frac{t}{x} \vee [\psi]\frac{t}{x}$.

SUB-F-EXT3 $[\exists x\phi]\frac{t}{x} := \begin{cases} \exists x[\phi]\frac{t}{x} & \text{if } x \not\equiv z \\ \exists x\phi & \text{otherwise} \end{cases}$.

N.B. We never substitute bound variables (only *Free Variables*).

Proposition 2.4 - $\forall\ t \in Tm_{\mathcal{L}},\ [t]\frac{x}{x} = t$

Proposition 2.5 - $\forall\ \phi \in Fml_{\mathcal{L}},\ [\phi]\frac{x}{x} = \phi$

Proposition 2.6 - *If* $x \not\in var(t)$ *then* $[a]\frac{a}{x} = t$

Proposition 2.7 - *If* $x \not\in FV(\phi)$ *then* $[\phi]\frac{a}{x} = \phi$

Proposition 2.8 - *Let* $x \not\in var(a)$ *then* $x \not\in var([t]\frac{a}{x})$ *and* $x \not\in FV([\phi]\frac{a}{x})$

Proposition 2.9 - *Substitution of Free Variables*
Let $x, y \in Var$ with $x \neq y,\ t \in Tm_{\mathcal{L}}$ and $\phi \in Fml_{\mathcal{L}}$.

- If $y \not\in \text{Var}(t)$ then $\left[[t]\frac{y}{x}\right]\frac{y}{x} = t$.
- If $y \not\in \text{Var}(\phi)$ then $\texttt{SubSt}\left(x, y, [\phi]\frac{x}{y}\right)$ <u>and</u> $\left[[\phi]\frac{y}{x}\right]\frac{x}{y} = \phi$.

Definition 2.14 - *Substitutable*
Let $\mathcal{L}$ be a FOL, $t \in Tm_{\mathcal{L}}$ and x be a variable.
Let $\phi, \psi \in Fml_{\mathcal{L}}$.
We define whether t is *Substitutable* for a variable x in a formula ϕ recursively

SU1 If ϕ is an *Atomic $\mathcal{L}$-Formula*. Then $\texttt{SubSt}(t, x, \phi)$ <u>always</u>.

SU2 $\texttt{SubSt}(t, x, \neg\phi)$ <u>iff</u> $\texttt{SubSt}(t, x, \phi)$.

SU3 $\texttt{SubSt}(t, x, \phi \to \psi)$ <u>iff</u> $\texttt{SubSt}(t, x, \phi)$ <u>and</u> $\texttt{SubSt}(t, x, \psi)$.

SU4 $\texttt{SubSt}(t, x, \forall z\phi)$ if $\begin{cases} & z \not\in var(t) \ \underline{\text{and}}\ mathttSubSt(t, x, \phi) \\ \text{or} & x \not\in FV(\phi) \end{cases}$

SU-EXT1 $\texttt{SubSt}(t, x, \phi \wedge \psi)$ <u>iff</u> $\texttt{SubSt}(t, x, \phi)$ <u>and</u> $\texttt{SubSt}(t, x, \psi)$.

SU-EXT2 $\texttt{SubSt}(t, x, \phi \vee \psi)$ <u>iff</u> $\texttt{SubSt}(t, x, \phi)$ <u>and</u> $\texttt{SubSt}(t, x, \psi)$.

SU-EXT3 $\texttt{SubSt}(t, x, \exists z \phi)$ <u>if</u> $\begin{cases} & z \in var(t) \ \underline{\text{and}}\ \texttt{SubSt}(\phi) \\ \text{or} & x \notin FV(\exists z \phi) \end{cases}$

N.B. If $\texttt{SubSt}(t, x, \phi)$, t is said to be *Free* for x in ϕ.

Proposition 2.10 - *Every variable is Substitutable for itself, in all formulae*

Proposition 2.11 - *If $x \notin FV(\phi)$ all $t \in Tm_{\mathcal{L}}$ are Substitutable for x in Φ*

Proposition 2.12 - *If $var(t) \cap var(\phi) = \emptyset$ then t is substitutable for any variable in ϕ.*
Notably, every closed $\mathcal{L}$-term is substitutable for any variable in any formula.

Proposition 2.13 - *Substitution order doesn't matter*
Let $\mathcal{L}$ be a FOL, $\mathfrak{M}$ be an $\mathcal{L}$-structure, s be a variable assignment and $d_1, \dots, d_k \in |\mathfrak{M}|$.
Let $x_1, \dots, x_k$ be distinct variables and π be a permutation over k. Then

$$\left(\left(\dots\left(s\frac{d_1}{x_1}\right)\dots\right)\frac{d_{k-1}}{x_{k-1}}\right)\frac{d_k}{x_k} = \left(\left(\dots\left(s\frac{d_{\pi(1)}}{x_{\pi(1)}}\right)\dots\right)\frac{d_{\pi(k-1)}}{x_{\pi(k-1)}}\right)\frac{d_{\pi(k)}}{x_{\pi(k)}}$$

Theorem 2.3 - *Substitution Lemma*
Let $\mathcal{L}$ be a FOL, $\mathfrak{M}$ be an $\mathcal{L}$-Structure, $t \in Tm_{\mathcal{L}}$, $\phi \in Fml_{\mathcal{L}}$ and x be a variable.

i) For every variable assignment s over $\mathfrak{M}$, $\forall\ a \in Tm_{\mathcal{L}}$

$$\overline{s}\left([a]\frac{t}{x}\right) = \overline{s\frac{\overline{s}(t)}{x}}(a)$$

ii) For every variable assignment s over $\mathfrak{M}$, $\forall\ a \in Tm_{\mathcal{L}}$ where a is *Substitutable* for x in ϕ

$$\mathfrak{M}, s \models \phi\frac{t}{x} \quad \text{iff} \quad \mathfrak{M}, s\frac{\overline{s}(t)}{x} \models \phi$$

Proposition 2.14 - $\forall\ t \in Tm_{\mathcal{L}}$ *if t is substitutable for x then* $\models \left(\forall x \phi \to [\phi]\frac{t}{x}\right)$ *for all $t \in Tm_{\mathcal{L}}$*

Proposition 2.15 - $\forall\ \phi \in Fml_{\mathcal{L}}$ *if t is substitutable for x then* $\models \left([\phi]\frac{t}{x} \to \exists x \phi\right)$ *for all $t \in Tm_{\mathcal{L}}$*

2.3 Homomorphism

Definition 2.15 - *Homomorphism*
Let $\mathcal{L}$ be a FOL and $\mathfrak{M}_1, \mathfrak{M}_2$ be $\mathcal{L}$-Structures.
A function $H : \mathfrak{M}_1 \to \mathfrak{M}_2$ is a *Homomorphism* if it fulfils the following

- $H(c^{\mathfrak{M}_1}) = c^{\mathfrak{M}_2}$ for all constant symbols, c, of $\mathcal{L}$.
- $H(f^{\mathfrak{M}_1}(t_1, \dots, t_k) = f^{\mathfrak{M}_2}(H(t_1), \dots, H(t_k))$ for all k-arity function symbols f of $\mathcal{L}$ and $t_1, \dots, t_k \in |\mathfrak{M}_1|$.
- $\langle t_1, \dots, t_k \rangle \in P^{\mathfrak{M}_1} \Leftrightarrow \langle H(t_1), \dots, H(t_k) \rangle \in P^{\mathfrak{M}_2}$ for all k-arity predicates symbols P of $\mathcal{L}$ and $t_1, \dots, t_k \in |\mathfrak{M}_1|$.
  i.e. $\langle t_1, \dots, t_k \rangle$ has property $P^{\mathfrak{M}_1}$ <u>iff</u> $\langle H(t_1), \dots, H(t_k) \rangle$ has property $P^{\mathfrak{M}_2}$.

Theorem 2.4 - *Semantic Value of a Homomorphism*
Let $\mathcal{L}$ be a FOL, $\mathfrak{M}_1, \mathfrak{M}_2$ be $\mathcal{L}$-Structures and s be a variable assignment over $\mathfrak{M}_1$.
Let H be a *Homomorphism* from $\mathfrak{M}_1$ to $\mathfrak{M}_2$.
Then, $\forall\ t \in Tm_{\mathcal{L}}$

$$H \circ \overline{s}(t) = \overline{H \circ s}(t)$$

Definition 2.16 - *Isomorphism*
Let H be a *Homomorphism.*
H is an *Isomorhpism* if it is *Bijective.*
N.B. If there exists an *Isomorphism* between $\mathfrak{M}_1$ and $\mathfrak{M}_2$ they are said to be *Isomorphic.*

Definition 2.17 - *Substructure*
Let $\mathcal{L}$ be a FOL and $\mathfrak{M}_1, \mathfrak{M}_2$ be $\mathcal{L}$-Structures.
$\mathfrak{M}_1$ is a *Substructure* of $\mathfrak{M}_2$ if

- $|\mathfrak{M}_1| \subset |\mathfrak{M}_2|$. And,
- The function $i(d) = d\ \forall\ d \in |\mathfrak{M}_1|$ is a *Homomorphism*

N.B. $\mathfrak{M}_2$ is called an *Extension* of $\mathfrak{M}_1$.

Definition 2.18 - *Elementary Equivalence*
Let $\mathcal{L}$ be a FOL and $\mathfrak{M}_1, \mathfrak{M}_2$ be $\mathcal{L}$-Structures.
$\mathfrak{M}_1$ and $\mathfrak{M}_2$ are *Elementary Equivalent* if

$$\mathfrak{M}_1 \vDash \sigma \Leftrightarrow \mathfrak{M}_2 \vDash \sigma \quad \forall\ \sigma \in Sent_{\mathcal{L}}$$

Proposition 2.16 - *Isomorphic $\mathcal{L}$-Structures are Elementary Equivalence*

Definition 2.19 - *Elementary Embedding*
Let $\mathcal{L}$ be a FOL and $\mathfrak{M}_1, \mathfrak{M}_2$ be $\mathcal{L}$-Structures.
An *Elementary Embedding* of $\mathfrak{M}_1$ in $\mathfrak{M}_2$ is a function $H : |\mathfrak{M}_1| \to |\mathfrak{M}_2|$ st

$$\forall\ \phi \in Fml_{\mathcal{L}}, \forall \text{ variable assignments } s \text{ over } \mathfrak{M}_1 \quad \mathfrak{M}_1, s \vDash \phi \Leftrightarrow \mathfrak{M}_2, H \circ s \vDash \phi$$

N.B. $H \circ s : \text{Var} \to |\mathfrak{M}_2|$.
N.B. If there exists an *Elementary Embedding* of $\mathfrak{M}_1$ in $\mathfrak{M}_2$, then $\mathfrak{M}_1$ and $\mathfrak{M}_2$ are *Elementary Equivalent.*

Proposition 2.17 - *An Isomorphism is an Elementary Embedding*

Proposition 2.18 - *An Elemenetary Embedding of $\mathfrak{M}_1$ in $\mathfrak{M}_2$ is an Injective Homomorphism from $\mathfrak{M}_1$ to $\mathfrak{M}_2$*
N.B. The converse may not be true.

2.4 Definable

Definition 2.20 - *Definable*
Let $\mathcal{L}$ be a FOL, $\mathfrak{M}$ be an $\mathcal{L}$-Structure and $\mathcal{R}$ be a k-arity relation on $|\mathfrak{M}|$.
$\mathcal{R}$ is *Definable* in $\mathfrak{M}$ if $\exists\ \phi \in Fml_{\mathcal{L}}$ where $FV(\phi) \subset \{x_1, \dots, x_k\}$ and $\forall\ a_1, \dots, a_k \in |\mathfrak{M}|$ it holds that

$$\langle a_1, \dots, a_k \rangle \in \mathcal{R} \quad \text{iff} \quad \mathfrak{M} \vDash \phi[\![a_1, \dots, a_k]\!]$$

N.B. $\mathcal{R}$ is **not** a predicate and is **not** related to any symbols in $\mathfrak{M}$.
N.B. We say $\mathcal{R}$ is *defined by* ϕ in $\mathfrak{M}$.

Proposition 2.19 - *$\mathcal{R}$ is defined by ϕ in $\mathfrak{M}$ iff $\mathfrak{M}, s \vDash \phi \Leftrightarrow \langle s(x_1), \dots, s(x_k) \rangle \in \mathcal{R}$ for all variable assignments s over $\mathfrak{M}$.*

3 Deductive Reasoning

Remark 3.1 - *Structure of Deductive Mathematical Proofs*
Deductive mathematical proofs take (roughly) the following structure

i) *Assumptions* - Axioms, definitions & proved theorems.
N.B. These depend on the subject matter.

ii) *Deduction Steps.*

iii) *Theorem* - The consequent of the deductions.

Remark 3.2 - *Logical Axioms are assumptions in almost all mathemtical proofs*

Definition 3.1 - *Generalisation*
Let L be a FOL & $\phi, \psi \in Fml_{\mathcal{L}}$.
ϕ is a *Generalisation* of ψ if

$$\phi = \psi;$$

$$\text{Or, } \phi = \forall x_1, \dots, \forall x_n \psi \text{ for some } x_1, \dots, x_n \in \text{Var}.$$

N.B. Every *$\mathcal{L}$-Formula* is a *Generalisaiton* of itself.

3.1 Hilbert Calculus

Definition 3.2 - *Hilbert Calculus*
Hilbert Calculus is a formal system of deductive logic, used in mathematical proofs.

Definition 3.3 - *Logical Axioms of Hilbert Calculus, $\Lambda_{\mathcal{L}}$*
Let $\mathcal{L}$ be a FOL, $\phi, \psi, \theta \in Fml_{\mathcal{L}}$, $t, t_0, t_1 \in Tm_{\mathcal{L}}$ and x is an arbitrary variable.
The *Logical Axioms of Hilbert Calculus* over $\mathcal{L}$ comprises all *Generalisations* of the following forms of an *$\mathcal{L}$-Formula*:

H1 $\phi \to (\psi \to \theta)$.

H2 $(\phi \to (\psi \to \theta)) \to ((\phi \to \psi) \to (\phi \to \theta))$.

H3 $(\neg\phi \to \neg\psi) \to (\psi \to \phi)$.

H4 $\forall x \phi \to [\phi]\frac{t}{x}$ where $\texttt{SubSt}(t, x, \phi)$.

H5 $\phi \to \forall x \psi$ for $x \notin FV(\phi)$.

H6 $[\forall x(\phi \to \psi)] \to [\forall x \phi \to \forall x \psi]$.

H7 $t \equiv t$.

H8 $t_0 \equiv t_1 \to \left([\phi]\frac{t_0}{x} \to [\phi]\frac{t_1}{x}\right)$ where $\texttt{SubSt}(t_0, x, \phi)$ <u>and</u> $\texttt{SubSt}(t_1, x, \phi)$.

N.B. This set is denoted as $\Lambda_{\mathcal{L}}$.

Remark 3.3 - *Logical Axioms are formulae*

Definition 3.4 - *Deduction in Hilbert Calculus*
Let $\mathcal{L}$ be a FOL & $\Gamma \subset Fml_{\mathcal{L}}$ in *Hilbert Calculus.*
A *Deduction*, $\mathcal{D}$ from Γ is a finite sequence, $\langle \phi_1, \dots, \phi_n \rangle$, of *$\mathcal{L}$-Formulae* where $\forall\ k \in [1, n]$:

$\phi_k \in \Lambda_{\mathcal{L}} \cup \Gamma$.
i.e. ϕ_k is assumed to be true.

or, $\exists i, j, k$ with $i, j < k$ st $\phi_j = \phi_i \to \phi_k$.
i.e. ϕ_k is true by implication.

N.B. We say $\mathcal{D}$ is a *Deduction* of ϕ_n since ϕ_n is the last formula.

Proposition 3.1 - *Deductions from Deductions*
Let $\mathcal{L}$ be a FOL, $\Gamma \subset Fml_{\mathcal{L}}$ & $\mathcal{D} := \langle \phi_1, \dots, \phi_n \rangle$ be a *Deduction* from Γ.

- $\forall\ m \leq n,\ \mathcal{D} \restriction_m := \langle \phi_1, \dots, \phi_m \rangle$ is a *Deduction* of θ_m from Γ.
  i.e. All subsequences of a *Deduction* are *Deductions.*

- $\forall\ \Sigma \supset \Gamma,\ \mathcal{D}$ is a *Deduction* of ϕ_n from Σ.

- For all deductions $\mathcal{D}' := \langle \psi_1, \dots, \phi_m \rangle$ from Γ

$$\mathcal{D} * \mathcal{D} = \langle \phi_1, \dots, \phi_n, \psi_1, \dots, \psi_m \rangle \text{ is a deduction of } \theta_m \text{ from } \Gamma.$$

Definition 3.5 - *Deducibility in Hilbert Style*
Let $\mathcal{L}$ be a FOL, $\phi \in Fml_{\mathcal{L}}$ & $\Gamma \subset Fml_{\mathcal{L}}$.
ϕ is *Deducible* from Γ if $\exists$ a deduction of ϕ from Γ.
N.B. This is denoted $\Gamma \vdash \phi$ (If $\Gamma \equiv \emptyset$ then we write $\vdash \phi$).

Remark 3.4 - *If* $\Gamma \vdash \phi$ *we say* ϕ *is a* <u>*Theorem*</u> *of* Γ

Definition 3.6 - *Modus Ponens*
Let $\mathcal{L}$ be a FOL, $\phi, \psi \in Fml_{\mathcal{L}}$ & $\Gamma, \Sigma \subset Fml_{\mathcal{L}}$.
If $\Gamma \vdash \psi$ is obtained from $\Gamma \vdash (\phi \to \psi)$ <u>and</u> $\Gamma \vdash \phi$ we say

$$\Gamma \vdash \phi \text{ is obtained by appyling } \textit{Modus Ponens} \text{ to } \Gamma \vdash (\phi \to \psi) \text{ and } \Gamma \vdash \phi.$$

N.B. "Modus Ponens" translates to *"Putting the limit"*.

Theorem 3.1 - *Monotonicity of Deducibility*
Let $\mathcal{L}$ be a FOL, $\phi \in Fml_{\mathcal{L}}$ & $\Gamma, \Sigma \subset Fml_{\mathcal{L}}$.

$$\text{If } \Gamma \vdash \phi \text{ \underline{and} } \Sigma \supset \Lambda \text{ then } \Sigma \vdash \phi.$$

Theorem 3.2 - *Generalisation Theorem*
Let $\mathcal{L}$ be a FOL, $\phi \in Fml_{\mathcal{L}}$ & $\Gamma, \Sigma \subset Fml_{\mathcal{L}}$.

$$\text{If } \Gamma \vdash \phi \text{ \underline{and} } x \notin FV(\Gamma) \text{ then } \Gamma \vdash \forall x \phi \text{ where } FV(\Gamma) := \bigcup_{\psi \in \Gamma} FV(\psi)$$

Proposition 3.2 - *Alternative Expressions of prev theorems*
Let $\mathcal{L}$ be a FOL, $\phi, \psi \in Fml_{\mathcal{L}}$ & $\Gamma, \Sigma \subset Fml_{\mathcal{L}}$.

AX $\Gamma \vdash \phi\ \forall\ \phi \in \Gamma \cup \Lambda_{\mathcal{L}}$.

MP If $\Gamma \vdash (\phi \to \psi)$ <u>and</u> $\Gamma \vdash \phi$ then $\Gamma \vdash \psi$.

GEN If $\Gamma \vdash \phi$ <u>and</u> $x \notin FV(\Gamma)$ then $\Gamma \vdash \forall x \phi$.

3.2 Deduction Theorem

Theorem 3.3 - *Law of Excluded Middle* - $\forall\ \phi \in Fml_{\mathcal{L}},\ \vdash \phi \vee \neg\phi$
Since $(\phi \vee \neg\phi) \Leftrightarrow (\neg\phi \to \neg\phi)$.

Theorem 3.4 - *Deduction Theorem*
Let $\mathcal{L}$ be a FOL, $\phi, \psi \in Fml_{\mathcal{L}}$ & $\Gamma, \Sigma \subset Fml_{\mathcal{L}}$.

$$\Gamma, \phi \vdash \psi \text{ iff } \Gamma \vdash (\phi \to \psi).$$

Theorem 3.5 - *Transitivity of Conditional*
Let $\mathcal{L}$ be a FOL, $\phi, \psi, \theta \in Fml_{\mathcal{L}}$ & $\Gamma, \Sigma \subset Fml_{\mathcal{L}}$.

$$\text{If } \Gamma \vdash (\phi \to \psi) \ \underline{\text{and}}\ \Gamma \vdash (\psi \to \theta) \text{ then } \Gamma \vdash (\phi \to \theta)$$

Theorem 3.6 - $\to$ *Exchange*
Let $\mathcal{L}$ be a FOL, $\phi, \psi, \theta \in Fml_{\mathcal{L}}$ & $\Gamma, \Sigma \subset Fml_{\mathcal{L}}$.

$$\text{If } \Gamma \vdash (\phi \to (\psi \to \theta)) \text{ then } \Gamma \vdash (\psi \to (\phi \to \theta)).$$

Theorem 3.7 - *Ex Falso Quodlibet*
Let $\mathcal{L}$ be a FOL, $\phi \in Fml_{\mathcal{L}}$ & $\Gamma, \Sigma \subset Fml_{\mathcal{L}}$.

$$\text{If } \Gamma \vdash \phi \ \underline{\text{and}}\ \Gamma \vdash (\neg\phi) \text{ then } \Gamma \vdash \psi\ \forall\ \psi \in Fml_{\mathcal{L}}.$$

N.B. "Ex Falso Quodlibet" translates to *"From a false proposition"*.

Theorem 3.8 - *Double Negation Elimination*
Let $\mathcal{L}$ be a FOL.

$$\forall\ \phi \in Fml_{\mathcal{L}},\ \vdash ((\neg\neg\phi) \to \phi)$$

Theorem 3.9 - *Double Negation Introduction*
Let $\mathcal{L}$ be a FOL.

$$\forall\ \phi \in Fml_{\mathcal{L}},\ \vdash (\phi \to (\neg\neg\phi))$$

Theorem 3.10 - *Contraposition*
Let $\mathcal{L}$ be a FOL and $\Gamma \subset Fml_{\mathcal{L}}$.

$$\forall\ \phi, \psi \in Fml_{\mathcal{L}},\ \Gamma \vdash (\phi \to \psi) \text{ iff } \Gamma \vdash (\neg\psi \to \neg\phi).$$

Theorem 3.11 - *Reductio ad Absurdum - 1*
Let $\mathcal{L}$ be a FOL, $\Gamma \subset Fml_{\mathcal{L}}$ and $\phi \in Fml_{\mathcal{L}}$.

$$\text{If } \exists\ \psi \in Fml_{\mathcal{L}} \text{ st } \Gamma, \neg\phi \vdash \psi \ \underline{\text{and}}\ \Gamma, \neg\phi \vdash \neg\psi \text{ then } \Gamma \vdash \phi.$$

N.B. "Reductio ad Absurdum" translates to *"Reduction to absurdity"*.

Theorem 3.12 - *Reductio ad Absurdum - 2*
Let $\mathcal{L}$ be a FOL, $\Gamma \subset Fml_{\mathcal{L}}$ and $\phi \in Fml_{\mathcal{L}}$.

$$\text{If } \exists\ \psi \in Fml_{\mathcal{L}} \text{ st } \Gamma, \phi \vdash \psi \ \underline{\text{and}}\ \Gamma, \phi \vdash \neg\psi \text{ then } \Gamma \vdash \neg\phi.$$

Theorem 3.13 - *Left* $\forall$ *Introduction*
Let $\mathcal{L}$ be a FOL, $\Gamma \subset Fml_{\mathcal{L}}$, $t \in Tm_{\mathcal{L}}$ and $\phi, \psi \in Fml_{\mathcal{L}}$.

$$\text{If } \Gamma, [\phi]\tfrac{t}{x} \vdash \psi \text{ and } \texttt{SubSt}(t, x, \phi) \text{ then } \Gamma, \forall x\phi \vdash \psi.$$

Theorem 3.14 - *Right* $\forall$ *Elimination*
Let $\mathcal{L}$ be a FOL, $\Gamma \subset Fml_{\mathcal{L}}$, $t \in Tm_{\mathcal{L}}$ and $\phi, \psi \in Fml_{\mathcal{L}}$.

$$\text{If } \Gamma \vdash \forall x\phi \text{ and } \texttt{SubSt}(t, x, \phi) \text{ then } \Gamma \vdash [\phi]\tfrac{t}{x}.$$

3.2.1 Facts

Proposition 3.3 - $\forall\ \phi, \psi \in Fml_{\mathcal{L}},\ \vdash ((\phi \to \psi) \to (\neg\psi \to \neg\phi))$

Proposition 3.4 - $\vdash (\phi \to \theta) \to ((\psi \to \theta) \to ((\phi \vee \psi) \to \theta))$

Proposition 3.5 - $\vdash ((\phi \wedge \psi) \to \phi)$

Proposition 3.6 - $\vdash ((\phi \wedge \psi) \to \psi)$

Proposition 3.7 - $\vdash (\phi \to (\psi \to (\phi \wedge \psi)))$

Theorem 3.15 - *Left $\wedge$ Introduction*
Let $\mathcal{L}$ be a FOL, $\Gamma \subset Fml_{\mathcal{L}}$, $t \in Tm_{\mathcal{L}}$ and $\phi, \psi, \theta \in Fml_{\mathcal{L}}$.

- If $\Gamma, \phi \vdash \theta$ then $\Lambda, (\phi \wedge \psi) \vdash \theta$.
- If $\Gamma, \psi \vdash \theta$ then $\Lambda, (\phi \wedge \psi) \vdash \theta$.

Theorem 3.16 - *Right $\wedge$*
Let $\mathcal{L}$ be a FOL, $\Gamma \subset Fml_{\mathcal{L}}$, $t \in Tm_{\mathcal{L}}$ and $\phi, \psi \in Fml_{\mathcal{L}}$.

Introduction If $\Gamma \vdash \phi$ and $\Gamma \vdash \psi$ then $\Gamma \vdash (\phi \wedge \psi)$.

Elmination If $\Gamma \vdash (\phi \wedge \psi)$ then $\Gamma \vdash \phi$ and $\Gamma \vdash \psi$.

Proposition 3.8 -
Let $\mathcal{L}$ be a FOL, $\Gamma \subset Fml_{\mathcal{L}}$, $t \in Tm_{\mathcal{L}}$ and $\phi \in Fml_{\mathcal{L}}$.

- If $\exists\ \psi \in Fml_{\mathcal{L}}$ st $\Gamma, \neg\phi \vdash (\psi \wedge \neg\psi)$ then $\Gamma \vdash \phi$.
- If $\exists\ \psi \in Fml_{\mathcal{L}}$ st $\Gamma, \phi \vdash (\psi \wedge \neg\psi)$ then $\Gamma \vdash \neg\phi$.

Proposition 3.9 - *If* $\mathtt{t}, \mathtt{x}, \phi$ *then* $\vdash ([\psi]\frac{t}{x} \to \exists x\phi)$

Proposition 3.10 - *If* $x \notin FV(\phi)$ *then* $\vdash (\exists x\phi \to \phi)$

Proposition 3.11 - $\vdash (\forall x(\phi \to \psi) \to (\exists x\phi \to \exists x\psi))$

Proposition 3.12 - $\vdash (t_0 \equiv t_1) \to (t_1 \equiv t_0)$

Proposition 3.13 - $\vdash (t_0 \equiv t_1) \leftrightarrow (t_1 \equiv t_0)$

Proposition 3.14 - $\vdash (t_0 \equiv t_1) \to ((t_1 \equiv t_2) \to (t_0 \equiv t_2))$

Proposition 3.15 - $\{\langle t_0, t_1\rangle \in Tm_{\mathcal{L}} \times Tm_{\mathcal{L}} : \Gamma \vdash (t_0 \equiv t_1)\}$ *is an Equivalence Relation.*

3.3 Completeness & Soundness Theorems

Theorem 3.17 - *Every Logical Axiom is logically true* Let $\mathcal{L}$ be a FOL.

$$\forall\ \lambda \in \Lambda_{\mathcal{L}},\ \vDash \lambda\ (\textit{i.e. } \emptyset \vDash \lambda)$$

Theorem 3.18 - *Soundess Theorem* Let $\mathcal{L}$ be a FOL.

$$\forall\ \Gamma \subset Fml_{\mathcal{L}},\ \forall\ \phi \in Fml_{\mathcal{L}} \text{ if } \Gamma \vdash \phi \text{ then } \Gamma \vDash \phi.$$

Equivalently

$$\forall\, \Gamma \subset Fml_{\mathcal{L}} \text{ if } \Gamma \text{ is } \textit{Satisfiable}\text{, then } \Gamma \text{ is } \textit{Consistent.}$$

Theorem 3.19 - *Completeness Theorem*
Let $\mathcal{L}$ be a FOL.

$$\begin{aligned} & \forall\, \Gamma \subset Fml_{\mathcal{L}},\, \phi \in Fml_{\mathcal{L}} \text{ if } \Gamma \vDash \phi \text{ then } \Gamma \vdash \phi \\ \iff & \forall\, \Gamma \subset Fml_{\mathcal{L}},\text{ if } \Gamma \text{ is consistent, then } \Gamma \text{ is satisfiable.} \end{aligned}$$

This is the coverse of the *Soundess Theorem* (**Theorem 3.17**).

Remark 3.5 - *The Completeness Theorem makes Hilbert Calculus unnecessary.*
Proving $(\vdash \phi) \leftrightarrow (\neg\neg\phi)$ is not obvious but $(\vDash \phi) \leftrightarrow (\neg\neg\phi)$ is obvious.

Theorem 3.20 - *Restricted Completeness Theorem*
Let $\mathcal{L}$ be a FOL and $\Gamma \subset Fml_{\mathcal{L}}$.
If Γ is *Consistent* and $FV(\Gamma)$ is finite then

$$\exists \text{ a model } \mathfrak{M}, s \text{ of } \Gamma \text{ st } |\mathfrak{M}| \text{ is } \textit{Countable}$$

Remark 3.6 - *Combining Soundness & Completeness Theorem*
Let $\mathcal{L}$ be a FOL.
As a result of the *Soundness Theorem* and *Completeness Theorem* we have that

$$\forall\, \Gamma \subset Fml_{\mathcal{L}} \text{ and } \phi \in Fml_{\mathcal{L}},\, \Gamma \vDash \phi \ \underline{\text{iff}}\ \Gamma \vdash \phi$$

This means that logical implication and deducibility are equivalent.

Definition 3.7 - *Eqivalence Relation* $\sim_\Gamma$
Let $\mathcal{L}$ be a FOL, $\Gamma \subset Fml_{\mathcal{L}}$ and $t, t' \in Tm_{\mathcal{L}}$.
We define the relation $\sim_\Gamma$ as

$$t \sim_\Gamma t' :\Leftrightarrow \Gamma \vdash (t \equiv t')$$

N.B. This is an equivalence relation by **Proposition 3.15**.

Proposition 3.16 - *Equivalence Relation for Functions and Predicates*
Let $\mathcal{L}$ be a FOL, f be a function symbol of arity n, P be a predicate symbol of arity n and $t_1, \dots, t_n, t'_1, \dots, t'_n \in Tm_{\mathcal{L}}$.
Then

$$\begin{aligned} t_1 \sim_\Gamma t'_1, \dots, t_n \sim_\Gamma t'_n &\Rightarrow f(t_1, \dots, t_n) \sim_\Gamma f(t'_1, \dots, t'_n) \\ &\Rightarrow (\Gamma \vdash P(t_1, \dots, t_n) \Leftrightarrow \Gamma \vdash P(t'_1, \dots, t'_n)) \end{aligned}$$

Definition 3.8 - *Canonical Model* $\mathfrak{M}^\Gamma$ *and Canonical Variable Assignment* s^Γ
Let $\mathcal{L}$ be a FOL, $\Gamma \subset Fml_{\mathcal{L}}$ and $t_1, \dots, t_n \in Tm_{\mathcal{L}}$.
The *Canonical Model* $\mathfrak{M}^\Gamma$ and *Canonical Variable Assignment* s^Γ over $\mathfrak{M}^\Gamma$ are defined as

$$\begin{aligned} |\mathfrak{M}^\Gamma| &:= \{[t]_{\sim_\Gamma} : t \in Tm_{\mathcal{L}}\} \\ c^{\mathfrak{M}^\Gamma} &:= [c]_{\sim_\Gamma} \\ f^{\mathfrak{M}^\Gamma}([t_1]_{\sim_\Gamma}, \dots, [t_n]_{\sim_\Gamma}) &:= [f(t_1, \dots, t_n)]_{\sim_\Gamma} \\ P^{\mathfrak{M}^\Gamma} &:= \{\langle [t_1]_{\sim_\Gamma}, \dots, [t_n]_{\sim_\Gamma} \rangle : \Gamma \vdash (t_1, \dots, t_n)\} \\ s^\Gamma(x) &:= [x]_{\sim_\Gamma} \end{aligned}$$

Remark 3.7 - $|\mathfrak{M}^\Gamma|$ *is Countable*

Proposition 3.17 - *Properties of Canonical Model*
Let $\mathcal{L}$ be a FOL and $\Gamma \subseteq Fml_{\mathcal{L}}$ (<u>not</u> necessarily *Maximally Consistent* or a *Henkin*).

- $\forall\ t \in Tm_{\mathcal{L}},\ \overline{s^{\Gamma}}(t) = [t]_{\sim_\Gamma}$
- $\forall$ atomic formulae $\phi \in Fml_{\mathcal{L}},\ \mathfrak{M}^{\Gamma}, s^{\Gamma} \models \phi$ iff $\Gamma \vdash \phi$

3.4 Consequences & Applications of Completeness Theorem

Theorem 3.21 - *Downward Löwenheim-Skolem Theorem*
Let $\mathcal{L}$ be a FOL and $\Gamma \subset Fml_{\mathcal{L}}$.
If Γ is *Satisfiable* then Γ has a *Countable* model $\mathfrak{M}$.

Theorem 3.22 - *Compactness Theorem*
Let $\mathcal{L}$ be a FOL and $\Gamma \subset Fml_{\mathcal{L}}$.
Then Γ is *Satisfiable* <u>iff</u> every finite subset $\Delta \subseteq \Gamma$ is *Satisfiable.*

Definition 3.9 - *Theory of* $\mathfrak{M}$
Let $\mathcal{L}$ be a FOL and $\mathfrak{M}$ be an $\mathcal{L}$-Strucutre.
The *Theory of* $\mathfrak{M}$ is defined as

$$Th(\mathfrak{M}) := \{\phi \in Sent_{\mathcal{L}} : \mathfrak{M} \models \phi\}$$

Definition 3.10 - *Consequences of* Γ
Let $\mathcal{L}$ be a FOL, $\mathfrak{M}$ be an $\mathcal{L}$-Strucutre and $\Gamma \subset Fml_{\mathcal{L}}$.
The *Consequences of* Γ is defined as

$$Cn(\Gamma) := \{\phi \in Sent_{\mathcal{L}} : \Gamma \vdash \phi\}$$

4 Gödel's Incompleteness Theorem

Remark 4.1 - *In this chapter we focus on a specific language* $\mathcal{L}_{\mathbb{N}} = \{0, S, +, \cdot, E, <\}$
This is just for simplicity's sake, everything applies to many other langauges.

Definition 4.1 - *Gödel Numbering*
Let $\mathcal{L}$ be a FOL.
A *Gödel Numbering* of $\mathcal{L}$ is a *'Computable Encoding'* of the symbols $\mathcal{A}_{\mathcal{L}}$, expressions $\mathcal{A}^*_{\mathcal{L}}$ and finite sequences of expressions $(\mathcal{A}^*_{\mathcal{L}})^*$ of $\mathcal{L}$ to natural numbers $\mathbb{N}$.

Definition 4.2 - *Gödel Numbers for Symbols,* $\#(\cdot) : A_{\mathcal{L}_{\mathbb{N}}} \to \mathbb{N}$
This is an injection $\# : \mathcal{A}_{\mathcal{L}_{\mathbb{N}}} \to \mathbb{N}$

Logical Symbols	Non-Logical Symbols
$(\mapsto 2$	$0 \mapsto 1$
$) \mapsto 4$	$S \mapsto 3$
$, \mapsto 6$	$+ \mapsto 5$
$\equiv \mapsto 8$	$\cdot \mapsto 7$
$\neg \mapsto 10$	$E \mapsto 9$
$\to \mapsto 12$	$< \mapsto 11$
$\forall \mapsto 14$	
$v_i \mapsto 16 + 2i$	

Definition 4.3 - *Gödel Numbers for Expressions,* $\#^*(\cdot) : A^*_{\mathcal{L}_{\mathbb{N}}} \to \mathbb{N}$
Let $\mathcal{L}$ be a FOL and $e_1, \dots, e_n \in \mathcal{A}_{\mathcal{L}}$. Define

$$\#^*(\langle e_1, \dots, e_n \rangle) = p_1^{\#(e_1)+2} \cdot \dots \cdot p_n^{\#(e_n)+2}$$

where p_i is the i^{th} prime number.

Definition 4.4 - *Gödel Numbers for Sequences of Expressions, $\#^{**}(\cdot) : A^{**}_{\mathcal{L}_\mathbb{N}} \to \mathbb{N}$*
Let $\mathcal{L}$ be a FOL and $s_1, \dots, s_n \in \mathcal{A}^*_\mathcal{L}$. Define

$$\#^{**}(\langle s_1, \dots, s_n \rangle) = p_2^{\#^*(s_1)} \cdot \dots \cdot p_{n+1}^{\#^* s_n}$$

Remark 4.2 - *The ranges of $\#$, $\#^*$ & $\#^{**}$ are disjoint*
Further, $\# \cup \#^* \cup \#^{**}$ is still injective.

Definition 4.5 - $\#R : \mathbb{N}^m$
Let $R \subset \mathbb{N}^m$ be a relation on $\mathcal{A}_L, \mathcal{A}^*_L$ or $(\mathcal{A}^*_L)^*$. Then

$$\#R := \{\langle \#(x_1), \dots, \#(x_m) : \langle x_1, \dots, x_m \rangle \in R\}$$

Definition 4.6 - $\#F(\cdot) : \mathbb{N}^m \to \mathbb{N}$
Let $F : \mathbb{N}^m \to \mathbb{N}$ be a suntactic operation on $\mathcal{A}_L, \mathcal{A}^*_L$ or $(\mathcal{A}^*_L)^*$. Then

$$\#F(\#(x_1), \dots, \#(x_m)) := \#(F(x_1, \dots, x_m))$$

Definition 4.7 - *Total Recursive Functions*
The class of *Total Recursive Functions* on $\mathbb{N}$ is defined by induction.

Base Functions The following are *Recursive Function*

- Successor function $S(n) := n + 1$
- Addition $+$
- Multiplication $\cdot$
- Exponential function $E(n, m) := n^m$
- Characteristic function of equality relation $\chi_=(n_1, n_2) = \begin{cases} 0 & \text{if } n_1 = n_2 \\ 0 & \text{otherwise} \end{cases}$
- Characteristic function of less-than relation $\chi_<(n_1, n_2) = \begin{cases} 0 & \text{if } n_1 < n_2 \\ 0 & \text{otherwise} \end{cases}$
- Constant function $c_m(n) =: m$
- Projection functions $I^m_i(n_1, \dots, n_m) := n_i$

Substitution Let $G : \mathbb{N}^k \to \mathbb{N}$ and $H_1 L \mathbb{N}^m \to \mathbb{N}, \dots, H_k : \mathbb{N}^m \to \mathbb{N}$ be *Recursive Function.* Then

$$F(n_1, \dots, n_m) := G(H_1(n_1, \dots, n_m), \dots, H_k(n_1, \dots, n_m)) \text{ is recursive.}$$

Minimalisation Let $G : \mathbb{N}^{m+1} \to \mathbb{N}$ be a *Recursive Function* st $\forall\ n_1, \dots, n_m \in \mathbb{N}\ \exists n \in \mathbb{N}$ st $G(n_1, \dots, n_m, n) = 0$.
Define $\mu[G] : \mathbb{N}^m \to \mathbb{N}$ as

$$\mu[G](n_1, \dots, n_m) = \min\{k : G(n_1, \dots, n_m, k) = 0\}$$

$\mu[G]$ is *Recursive.*

Remark 4.3 - *A relation R is Recursive if χ_R is Recursive*

Notation 4.1 -
Let $R \subset \mathbb{N}^{m+1}$ be a relation.
If $\forall\ n_1, \dots, n_m \in \mathbb{N}\ \exists\ n$ st $\langle n_1, \dots, n_m, n \rangle \in R$ we denote $\mu[\chi_R]$ by $\mu[R]$.

N.B. $\mu[R]:\mathbb{N}^m\to\mathbb{N}$ and $\mu[R](n_1,\dots,n_{m+1})=\min\{\langle n_1,\dots,n_m,l\rangle\in R\}$.

Remark 4.4 - *If R is Recursive then $\mu[R]$ is Recursive*

Theorem 4.1 - *Church-Turing Thesis*

- A function $F:\mathbb{N}^m\to\mathbb{N}$ has an effective computation procedure (which effectively gives the value of F on each given arguments $n_1,\dots,n_m\in\mathbb{N}$) iff F is *Recursive.*

- A relation $R\subset\mathbb{N}^m$ is effectively deciable (*i.e.* has a decision procedure which determines whether or not $n_1,\dots,n_m$ satisfy $R(n_1,\dots,n_m)$) iff R is *Recursive.*

Remark 4.5 -
A syntactic relation R or operation F is effective deciable iff $\#R$ or $\#F$ is *Recursive*, respectively.
Definition 4.8 - *Representation*
Let $\mathcal{L}$ be a FOL and $R\subset\mathbb{N}^m$ be an m-arity relation.
We say $\phi\in Fml_{\mathcal{L}}$ *Represents* R in $\Gamma\subset Fml_{\mathcal{L}_{\mathbb{N}}}$ when $\exists$ distinct $x_1,\dots,x_m$ st

- $FV(\phi)\subset\{x_1,\dots,x_m\}$.
- $\forall\ n_1,\dots,n_m\in\mathbb{N}$, if $R(n_1,\dots,n_m)$ holds then $\Gamma\vdash[\phi]\frac{\overline{n_1},\dots,\overline{n_m}}{x_1,\dots,x_m}$
- $\forall\ n_1,\dots,n_m\in\mathbb{N}$, if $R(n_1,\dots,n_m)$ does not hold then $\Gamma\vdash[\neg\phi]\frac{\overline{n_1},\dots,\overline{n_m}}{x_1,\dots,x_m}$

R is *Representable* in $\Gamma\subset Fml_{\mathcal{L}}$ when R is *Represented* by some $\phi\in\Gamma$.

Definition 4.9 - *Functional Representation*
Let $F:\mathbb{N}^m\to\mathbb{N}$ and $\phi\in Fml_{\mathcal{L}}$.
F is *Functionally Represented* by ϕ where $\exists$ distinct $x_1,\dots,x_m,z$ st

- $FV(\phi)\subset\{x_1,\dots,x_m,z\}$.
- $\forall\ n_1,\dots,n_m\in\mathbb{N}\ A_E\vdash\forall z(\phi(\overline{n_1},\dots,\overline{n_m},z))\leftrightarrow(z\equiv\overline{F(n_1,n_m)})$.
  N.B. $\overline{F(n_1,\dots,n_m)}$ denotes $\underbrace{S\dots S}_{F(\vec{n})-\text{times}}\ \overline{0}$.

4.1 Theory A_E of Arithmetic with Exponentiation

Definition 4.10 - *Theory A_E of Arithmetic with Exponentiation*
The axiom set A_E comprises the following axioms

(S1) $\forall x\ S(x)\not\equiv 0$

(S2) $\forall x\forall y\ (S(x)\equiv S(y)\to(x\equiv y))$

(L1) $\forall x\forall y\ (x<S(y)\leftrightarrow(x<y\wedge x\equiv y))$

(L2) $\forall x\ \neg(x<0)$

(L3) $\forall x\forall y\ (x<y\wedge x\equiv y\wedge y<x)$

(A1) $\forall x\ x+0\equiv x$

(A2) $\forall x\forall y\ x+S(y)\equiv S(x+y)$

(M1) $\forall x\ x\cdot 0\equiv 0$

(M2) $\forall x \forall y\ x \cdot S(y) \equiv (x \cdot y) + x$

(E1) $\forall x\ E(x, 0) \equiv S(0)$

(E2) $\forall x \forall y\ E(x, S(y)) \equiv E(x, y) \cdot x$

Theorem 4.2 - $\mathfrak{N} \models A_E$. *Hence, if* $A_E \vdash \phi$ *then* $\mathfrak{N} \models \phi$

Proposition 4.1 -
Let $R \subset \mathbb{N}^m$.
If $\phi \in Fml_{\mathcal{L}_\mathbb{N}}$ *Represents* R in A_E then ϕ defines R in $\mathfrak{N}$.
Hence, if a relation over $\mathbb{N}$ is *Representable* in A_E, then it is definable in $\mathfrak{N}$.

4.2 Fixed-Point Lemma

TODO from here if happy with rest

0 Appendix

0.1 Standard Models

Definition 0.1 - *Standard Model of Arithmetic*
Let language of arithmetic is $\mathcal{L}_\mathbb{N} := \{\overline{<}, S, \overline{+}, \bar{\cdot}, E, \overline{0}\}$ where

- $\overline{<}$ is a binary relation symbol.
- S is a unary function symbol.
- $\overline{+}, \bar{\cdot}, E$ are binary function symbols.
- $\overline{0}$ is the constant symbol for $0 \in \mathbb{N}$.

Let $\mathfrak{M}$ be a $\mathcal{L}_\mathbb{N}$-Structure with the domain $|\mathfrak{M}| = \mathbb{N}$ defined as

- $\overline{<}$ is interpreted as the usual '*less-than*' relation on $\mathbb{N}$.
$$i.e.\ \langle x, y\rangle \in \overline{<}^\mathfrak{M} \Leftrightarrow x < y$$
- S is interpreted as the *successor function* '+1' on $\mathbb{N}$.
$$i.e. S^\mathfrak{M}(n) = n + 1$$
- $\overline{+}, \bar{\cdot}, E$ are interpreted as the usual '*addition*', '*multiplication*' and '*exponentitation*' on $\mathbb{N}$ respectively.
$$i.e. E^\mathfrak{M}(n, m) = n^m$$
- $\overline{0}$ is interpreted as the natural numbers 0.

0.2 Notation

Proposition 0.1 - *Formal Notation*

Notation	**Use**
$\langle a_1, \dots, a_n \rangle$	A string of length n
$\langle a, b \rangle$	Two consecutive strings
$a * b$	Concatenation of two strings
$\mathcal{A}^*$	Set of all strings over alphabet $\mathcal{A}$
$\mathcal{A}_{\mathcal{L}}$	Alphabet of language $\mathcal{L}$
$Tm_{\mathcal{L}}$	Set of $\mathcal{L}$-Terms of language $\mathcal{L}$
$Fml_{\mathcal{L}}$	Set of $\mathcal{L}$-Formulae of language $\mathcal{L}$
Var	Set of variables in the alphabet??
$Sent_{\mathcal{L}}$	Set of $\mathcal{L}$-Sentences of language $\mathcal{L}$
$\to$	Implication
$\leftrightarrow$	Equivalence
$\vee$	Or
$\wedge$	And
$\forall$	For all
$\exists$	There exists
$\exists!$	There exists a unique
$\bar{+}$	Syntactic $+$, has no semantic value. $\bar{\ }$signals this for all symbols
$:\Leftrightarrow$	Defined to have same logical value (true or false)
$\Phi \models \phi$	$\phi \in Fml_{\mathcal{L}}$ is a logical consequence of $\Phi \subseteq Fml_{\mathcal{L}}$.
$\models \phi$	$\phi \in Fml_{\mathcal{L}}$ is logically valid.
$\mathfrak{M}_1 \cong \mathfrak{M}_2$	Structures $\mathfrak{M}_1$ and $\mathfrak{M}_2$ are isomorphic.
$\mathfrak{M} \models \phi[\![a_1, \dots, a_n]\!]$	ϕ is true for the objects $a_1, \dots, a_n \in \lvert\mathfrak{M}\rvert$.
$\Lambda_{\mathcal{L}}$	Logical axioms for Hilbert Calculus.
$\Gamma \vdash \phi$	ϕ is deducible from Γ.
$t \sim_\Gamma t'$	$t \equiv t'$ is deducible from Γ.
$[t]_{\sim_\Gamma}$	Equivalence class of t modulo$\sim_\Gamma$.
χ_R	Characteristic Equation of Relation R.
$\mu[G]$	Minimalisation of recursive function G.
$\#(\cdot), \#^*(\cdot), \#^* * (\cdot)$	Gödel Number injections.

Proposition 0.2 - *Convential Notation*

Notation	Use
$\mathcal{A}$	Alphabet
$\mathcal{L}$	Language (First-Order)
t	Term
$\phi, \psi, \dots$	Formulae (Lower case greek).
$\Lambda, \Gamma, \dots$	Sets of Formulae (Upper case greek).
$x \circ y$	$\circ(x, y)$ where $\circ$ is a function or predicate
$c \not\prec d$	$\neg \prec (c, d)$
$\mathfrak{M}$	$\mathcal{L}$-Structure
$Th(\mathfrak{M})$	Theory of $\mathfrak{M}$
$Cn(\Gamma)$	Consequences of Γ
$\mathfrak{I}$	Interpreation from an $\mathcal{L}$-Structure
D or $\lvert\mathfrak{M}\rvert$	Domain of an $\mathcal{L}$-Structure
$P^{\mathfrak{M}}$	$\mathfrak{I}(P)$
$f^{\mathfrak{M}}$	$\mathfrak{I}(f)$
$c^{\mathfrak{M}}$	$\mathfrak{I}(c)$
$\mathfrak{M} \models \phi$	$\mathfrak{M}, s \models \phi\ \forall\ s$ over $\mathfrak{M}$ since ϕ <u>is an $\mathcal{L}$-sentence</u>.
$t^{\mathfrak{M}}$	$d \in \lvert\mathfrak{M}\rvert$ st $\bar{s}(t) = d\ \forall\ s$ over $\mathfrak{M}$ since t <u>is a Closed $\mathcal{L}$-Term</u>.
$\texttt{SubSt}(t, x, \phi)$	$t \in Tm_{\mathcal{L}}$ is substitutable for $x \in$ Var in $\phi \in Fml_{\mathcal{L}}$.
$\Gamma, \phi \vdash \psi$	$\Gamma \cup \{\phi\} \vdash \psi$.
$\Gamma, \phi \models \psi$	$\Gamma \cup \{\phi\} \models \psi$.

0.3 Definitions

Definition 0.2 - *Arity*
The *Arity* of a function is the number of arguments it takes.
N.B. Unary, Binary, Ternary, Quaternary, ...

Definition 0.3 - *Countable Set*
Let X be a set.
X is *Countable* if

$$\begin{array}{rl} & \exists\ f : \mathbb{N} \to X \text{ st } f \text{ is surjective.} \\ \text{Or} & \exists\ f : X \to \mathbb{N} \text{ st } f \text{ is injective.} \end{array}$$

Definition 0.4 - *Predicate*
A *Predicate* is an expression over a set of variables and returns a logical conclustion (*i.e.* True or False).
N.B. Practically a function from set of variables to true or false.

0.4 Indentities

Theorem 0.1 - *Complex Connectives & Quantifiers in terms of FOL*

Term	In FOL
$\exists x,\ P(x)$	$\neg(\forall\ x,\ \neg P(x))$
$P \vee Q$	$(\neg P) \to Q$
$P \wedge Q$	$\neg(P \to \neg Q)$
$P \leftrightarrow Q$	$(P \to Q) \wedge (Q \to P)$ $\Leftrightarrow \neg((P \to Q) \to \neg(Q \to P))$

0.5 Techniques

Proposition 0.3 - *Induction on Terms*
Proposition 0.4 - *Induction on Formulae*