Logic - Notes

Dom Hutchinson

March 9, 2020

Contents

1 Introduction

1.1 Alphabets & Strings

Definition 1.1 - *Alphabet*
An *Alphabet* is a set of symbols from which *Strings* can be created.

Definition 1.2 - *String*
A *String* over a set $\mathcal{A}$ is any sequence $\alpha := \langle a_1, \dots, a_n\rangle$ where $a_1, \dots, a_n \in \mathcal{A}$.
N.B. Here we say α has *length* n and $\alpha \in \mathcal{A}^n$.

Definition 1.3 - *Power Set*
Let $\mathcal{A}$ be an alphabet. We define

$$\mathcal{A}^* := \bigcup_{n\in\mathbb{N}} \mathcal{A}^n = \{\langle a_1, \dots, a_n\rangle : n \in \mathbb{N}; a_1, \dots, a_n \in \mathcal{A}\}$$

This means $\mathcal{A}^*$ is the set of all possible strings over alphabet $\mathcal{A}$.

Remark 1.1 - *Concatenating Strings*
Define *Strings* $\alpha := \langle a_1, \dots, a_n\rangle \in \mathcal{A}^n$ and $\beta := \langle b_1, \dots, b_m\rangle \in \mathcal{A}^m$.
We define *Concatenation* of α & β as $\alpha\beta := \langle a_1, \dots, a_n, b_1, \dots, b_m\rangle$ Note that

$$\alpha\beta \neq \langle \alpha, \beta\rangle = \langle\langle a_1, \dots, a_n\rangle, \langle b_1, \dots, b_m\rangle\rangle$$

N.B. Sometimes the following notation is used $\alpha * \beta$.
Example 1.1 - *English Alphabet*
If we define an alphabet $\mathcal{A} := \{`a`, \dots, `z`\}$ then $\langle `t`, `h`, `i`, `s`\rangle$ is a *String* of $\mathcal{A}$.

Remark 1.2 - *Ambiguity when using multiple Alphabets*
Consider the *Alphabets* $\mathcal{A}_1 := \{0, 1, \dots, 9\}$ & $\mathcal{A}_2 := \mathbb{N}$.
Then we are unsure which of the following definitions of 123 is valid

$$\langle 123\rangle,\ \langle 12, 3\rangle,\ \langle 1, 23\rangle, \langle 1, 2, 3\rangle$$

Remark 1.3 - $\mathcal{A} := \{0, 1\}$ *is sufficient to describe any language - binary*

Remark 1.4 - *Describing Formal Languages*
When describing a *Formal Language* we need to provide two things

i) An *Alphabet* which defines what symbols are allowed.

ii) A *Grammar* which defines what combinations of symbols are allowed.

1.2 Countable Sets

Definition 1.1 - *Countable Set*
A set X is said to be *Countable* if

$$\begin{array}{l}\exists \text{ a surjection } f : \mathbb{N} \to X \\ \exists \text{ an injection } f : X \to \mathbb{N}\end{array}$$

Definition 1.2 - *Countably Infinite Set*
A set X is said to be *Countably Infinite* if $\exists$ a bijection $f : X \to \mathbb{N}$.

Theorem 1.1 - *Power set is Countable*
If set $\mathcal{A}$ is *countable* then $\mathcal{A}^*$ is *countable.*

Proof 1.1 - *Theorem 1.1*
Let $f : \mathcal{A} \longrightarrow \mathbb{N}$ (This function exists trivally since we define $\mathcal{A}$ to be countable).
Define the following function $g(\cdot) : \mathcal{A}^* \longrightarrow \mathbb{N}$

$$g(\langle a_1, \dots, a_n\rangle) := p_1^{f(a_1)+1} \cdot \dots \cdot p_n^{f(a_n)+1}$$

where p_i is the i^{th} prime.
Since each natural number can be described by a unique composition of primes and since $f(\dot{)}$ is injective, then $g(\cdot)$ is injective.
Thus there exists an injection from $\mathcal{A}^*$ to $\mathbb{N}$, making $\mathcal{A}^*$ countable.

Theorem 1.2 - *If $\mathcal{A}$ is countable, then so are $\mathcal{A}^*, (\mathcal{A}^*)^*, \dots$*

2 First-Order Languages

Definition 2.1 - *First-Order Language, $\mathcal{L}$*
The *Alphabet* of a *First-Order Language*, comprises of the following, pairwise disjoint, categories (and nothing else)

i) Negation, $\neg$, and implication, $\longrightarrow$.
ii) For all, $\forall$.
iii) Infinitely many variables, $\{v_0, v_1, \dots\}$.
iv) Parentheses, '(' ')', and comman ','.
v) Equality, $\equiv$, which is the only logical predicate symbol with 2-arity.
vi) A set of constant symbols, $\{c_1, c_2, \dots\}$. (Possibly empty)
vii) For each $n \geq 1$, a set of n-arity function symbols $\{f_1^n, f_2^n, \dots\}$. (Possibly empty)
viii) For each $n \geq 1$, a set of n-arity non-logical predicate symbols $\{P_1^n, P_2^n, \dots\}$. (Possibly empty)

N.B. We denote the set of variables by $Var := \{v_0, v_1, \dots\}$; denote a language as $\mathcal{L}$ and the alphabet of $\mathcal{L}$ as $\mathcal{A}_{\mathcal{L}}$.
N.B. In this course *Alphabets* are restricted to being *Countable.*

Definition 2.2 - *Negation, $\neg$*
Negation returns in the inverse of a predicate (DO I MEAN PREDICATE)

P	$\neg P$
T	F
F	T

Definition 2.3 - *Implication, $\longrightarrow$*
Implication returns whether one predicate being true necessarily implies a second predicate being true

P	Q	$P \to Q$
T	T	T
T	F	F
F	T	T
F	F	T

Remark 2.1 - *First-Order Languages don't have* $\wedge,\ \vee,\ \exists$
Alphabets for *First-Order Languages* do not contain propositional connectives for AND, $\wedge$, OR, $\vee$, or EXISTS,$\exists$ since they can be expressed as a combination of negation & implication.

$$\begin{array}{rcl} P \vee Q & \iff & \neg P \to Q \\ P \wedge Q & \iff & \neg(P \to \neg Q) \\ \exists\ x \text{ st } P(x) \text{ is true} & \iff & \neg(\forall\ x,\ \neg P(x)) \end{array}$$

P	Q	$\neg P$	$\neg P \to Q$
T	T	F	T
T	F	F	T
F	T	F	T
F	F	F	F

P	Q	$\neg Q$	$P \to \neg Q$	$\neg(P \to \neg Q)$
T	T	F	F	T
T	F	T	T	F
F	T	F	T	F
F	F	T	T	F

Example 2.1 - *Recursive Defintion*
Consider the following, normal, deifition

$$x \text{ is a multiple of } 5 \iff \exists\ y \in \mathbb{Z} \text{ st } y.5 = x$$

We can instead use the recursive definition

i) 0 is a multiple of 5.

ii) If n is a multiple of 5 then $n+5$ is a multiple of 5.

Definition 2.4 - *$\mathcal{L}$-Term & Complexity*
Let $\mathcal{L}$ be a *First-Order Language.*
We define *$\mathcal{L}$-Terms* & *Complexity*, $cp(\cdot)$, together using the following *recursive definition*

i) If s is a variable or a constant symbol, then s is an *$\mathcal{L}$-Term* with $cp(s) = 0$.
N.B. Terms with $cp(\cdot) = 0$ are called *Atomic Terms.*

ii) If f is a fucntion symbol with k-arity & if $a_1, \dots, a_k$ are *$\mathcal{L}$-Terms* then $f(a_1, \dots, f_k)$ is an *$\mathcal{L}$-Term* with complexity

$$cp(f(a_1, \dots, a_k)) := \max\{cp(a_1), \dots, cp(a_k)\} + 1$$

N.B. Terms with $cp(\cdot) \geq 1$ are called *Compound Terms.*

iii) Nothing else is an *$\mathcal{L}$-Term*

N.B. We denote the set of $\mathcal{L}$ − Terms by $T_{\mathcal{M}_\mathcal{L}}$.

Example 2.2 - *$\mathcal{L}$-Term & Complexity*
Let $\{c, d, f, g, h, p\} \subseteq$ Ł with c, d being constants, g, p being uniary functions & f, h being binary functions.
Show that the following is an *$\mathcal{L}$-Term* & find its *Complexity*

$$h(g(f(x, c)), p(d))$$

i) x is an *$\mathcal{L}$-Term* with $cp(x) = 0$ by (i).

ii) c & d are *$\mathcal{L}$-Terms* with *cp*$(c) = 0 =$ *cp*(d) by (i).

iii) $f(x, c)$ is an *$\mathcal{L}$-Term* with *cp*$(f) = \max 0, 0 + 1 = 1$ by (ii).

iv) $p(d)$ is an *$\mathcal{L}$-Term* with *cp*$(f) = \max 0 + 1 = 1$ by (ii).

v) $g(f(x, c), p(d))$ is an *$\mathcal{L}$-Term* with *cp*$(g) = \max 1, 1 + 1 = 2$ by (ii).

vi) $h(g(f(x, c), p(d)))$ is an *$\mathcal{L}$-Term* with *cp*$(h) = \max 2 + 1 = 3$ by (ii).

Thus $h(g(f(x, c), p(d)))$ is an *$\mathcal{L}$-Term* with *Complexity* 3.

Notation 2.1 - *More readble Functions*
WE often write $x \circ y$ instead of $\circ(x, y)$ as it is more readable (even though the later is technically the only correct notation). Similarly, $x + y$ instead of $+(x, y)$.

Definition 2.5 - *Atomic Formulae*
Let $\mathcal{L}$ be a *First-Order Language.*
The atomic *$\mathcal{L}$-Formulae* are those strings over $\mathcal{A}_{\mathcal{L}}$ of the form

$$R(t_1, \dots, t_n) \text{ for } n \in \mathbb{N}$$

where R is a predicate symbol of $\mathcal{L}$ with n-arity and $t_1, \dots, t_n$ are *$\mathcal{L}$-terms.*
N.B. $\equiv (t_1, t_2)$ is an *Atomic $\mathcal{L}$-Formula* for each $\mathcal{L}$ terms t_1, t_2.

Definition 2.6 - *$\mathcal{L}$-Formulae & Complexity*
We define *$\mathcal{L}$-Formulae* & *Complexity*, *cp*$(\cdot)$, together using the following recursive definition

i) If $\phi \in \mathcal{A}^*_{\mathcal{L}}$ is an *Atomic $\mathcal{L}$-Formula* then ϕ is an *$\mathcal{L}$-Formula* with *cp*$(\phi) = 0$.

ii) If ϕ is an *$\mathcal{L}$-Formula* with *cp*$(\phi) = n$ then $\neg\phi$ is an *$\mathcal{L}$-Formula* with *cp*$(\neg\phi) = n + 1$.

iii) If ϕ & ψ are *$\mathcal{L}$-Formulae* then $\phi \to \psi$ is an *$\mathcal{L}$-Formula* with *cp*$(\phi \to \psi) = \max\{$*cp*$(\phi),$ *cp*$(\psi)\} + 1$.

iv) if ϕ is an *$\mathcal{L}$-Formula* then $\forall x\phi$ is an *$\mathcal{L}$-Formula* with *cp*$(\forall x\phi) =$ *cp*$(\phi) + 1$, where x is a variable.

N.B. Complexity is just a measure of the syntactic complexity, not semantic. Notice how *cp*$(\neg\neg\phi) =$ *cp*$(\phi) + 2$.

Remark 2.2 - *Formulae are uniquely readable & parsable*

Example 2.3 - *$\mathcal{L}$-Formulae Complexity*
Let $\{R, f\} \subset \mathcal{L}$ be binary operations.
Show that the following is an *$\mathcal{L}$-Formula*

$$\forall\ v_0\ (\neg R(f(v_0, v_2), v_2) \longrightarrow \underbrace{\equiv (v_0, v_2)}_{v_0 \equiv v_2}$$

i) v_0, v_2 are $\mathcal{L}$-Terms.

ii) $f(v_0, v_2)$ is an $\mathcal{L}$-Term.

iii) $R(f(v_0, v_2), v_2)$ is an $\mathcal{L}$-Formula with *cp*$(\cdot) = 0$.

iv) $\neg R(f(v_0, v_2), v_2)$ is an $\mathcal{L}$-Formula with *cp*$(\cdot) = 0 + 1 = 1$.

v) $\equiv (v_0, v_2)$ is an $\mathcal{L}$-Formula with $cp(\cdot) = 0$.

vi) $\neq R(f(v_0, v_2), v_2) \longrightarrow \equiv (v_0, v_2)$ is an $\mathcal{L}$-Formula with $cp(\cdot) = \max\{0, 1\} + 1 = 2$.

vii) $\forall\ v_0\ (\neg R(f(v_0, v_2), v_2) \longrightarrow \equiv (v_0, v_2))$ is an $\mathcal{L}$-Formula with $cp(\dot{)} = 2 + 1 = 3$.

Notation 2.2 - *Convention for common operators*
To make formulae more readble we general make the following allowances in notation

$$\begin{array}{lll} t_1 \equiv t_2 & \text{for} & \equiv (t_1, t_2) \\ t_1 \not\equiv t_2 & \text{for} & \neg \equiv (t_1, t_2) \\ t_1 < t_2 & \text{for} & < (t_1, t_2) \\ t_1 \not\equiv t_2 & \text{for} & \neg \equiv (t_1, t_2) \end{array}$$

Further, when a formula is encapsulated by parantheses then we will often surpress the outermost parentheses (only), as they do not affect anything.

$$\phi \longrightarrow (\psi \longrightarrow \theta) \text{ for } (\phi \longrightarrow (\psi \longrightarrow \theta))$$

Definition 2.7 - *More complex operators*

- AND, $(\phi \wedge \psi) := \neg(\phi \longrightarrow \neg\psi)$.
- OR, $(\phi \vee \psi) := (\neg\phi \longrightarrow \psi)$.
- IFF, $(\phi \longleftrightarrow \psi) := (\phi \longrightarrow \psi) \wedge (\psi \longrightarrow \phi)$.
- EXISTS, $(\exists\ x\phi) := \neg\forall\ x\ \neg\phi$.

Notation 2.3 - *Sets of $\mathcal{L}$ Features*

- $T_{\mathcal{M}_\mathcal{L}}$:= Set of $\mathcal{L}$-Terms.
- $F_{\mathcal{M}_\mathcal{L}}$:= Set of $\mathcal{L}$-Formulae.
- Var := Set of Variables.

Proposition 2.1 - *$T_{\mathcal{M}_\mathcal{L}}$ & $F_{\mathcal{M}_\mathcal{L}}$ are always countable in this course since we assume $\mathcal{L}$ to be finite.*

2.1 Induction of Terms & Formulae

Theorem 2.1 - *Inheritance of a Proeprty - $\mathcal{L}$-Terms*
Let P be a property of $\mathcal{L}$-Terms.
Suppose the following to be true

i) All *Atomic $\mathcal{L}$-Terms* have property P.

ii) $\forall\ k \in \mathbb{N}$, $\forall$ function symbols f with k-arity: If $\mathcal{L}$-Terms $t_1, \dots, t_k$ have property P then $f(t_1, \dots, t_k)$ has P.

Then every $\mathcal{L}$-Term has property P.

Proof 2.1 - *Theorem 2.1*
This is a proof by contradiction. Suppose that *i)* & *ii)* are true but there exists some $\mathcal{L}$-Term which does not have P.
Let t be an $\mathcal{L}$-Term with minimum complexity st t does not have P.
Then $cp(t) \neq 0$ otherwise *i)* would be untrue.

Thus $t \equiv f(t_1, \dots, t_k)$ by the minimlaity of $cp(t)$.
We know that $t_1, \dots, t_k$ have P.
Thus $f(t_1, \dots, t_k)$ has P. This is a contradiction.

Theorem 2.2 - *Inheritance of a Proeprty - $\mathcal{L}$-Formulae*
Let P be a property of $\mathcal{L}$-Formula.
Suppose the following to be true

i) All *Atomic $\mathcal{L}$-Formulae* have property P.

ii) If $\phi, \psi \in F_{\mathcal{M}_\mathcal{L}}$ have P then $\neg\phi, (\phi \to \psi)$ & $\forall\ x\phi$ have P to.

Then every $\mathcal{L}$-Formulae has property P.

Theorem 2.3 - *Number of Parenthese*
Ever *$\mathcal{L}$-Formula* has as many left parentheses as right parentheses.
Ever *$\mathcal{L}$-Term* has as many left parentheses as right parentheses.

Proof 2.2 - *Theorem 2.3*
This is a proof by induction.
Let P be the property *"Has as many left parenthese as right"*.
Base Case - When ϕ is an *Atomic $\mathcal{L}$ Formula* it trivially has equal number of parenthese.
Inductive Case
Let ϕ & ψ be arbitrary $\mathcal{L}$-Formulae.
Assume that $P(\phi)$ & $P(\psi)$ hold.
We need to show that $P(\neg\phi),\ P(\phi \to \psi)$ & $P(\forall\ x\phi)$ all hold.
We do not need to show $P(\neg\psi),\ P(\psi \to \phi)$ & $P(\forall\ x\psi)$ hold as ϕ & ψ are arbitrary.
We have that $\neg\phi$ and $\forall\ x\phi$ don't add any brackets, so P holds.
We have that $(\phi \to \psi)$ add one left & one right parentheses (although they are often surpressed), thus P holds.
Thus by the process of mathematical induction P holds forall $\mathcal{L}$-Formulae.
N.B. The proof for $\mathcal{L}$-Terms is very similar.

2.2 Free Variables

Definition 2.1 - *Variable Function, $var(\cdot)$*
Define $var : \mathcal{A_L}^* \to 2^{\text{Var}}$ st $var(s)$ is the set of all variables in string s.

Example 2.4 - Var$(\cdot)$

$$\begin{array}{rcl} var(f(x, f(y, c)) & = & \{x, y\} \\ var(f(c, f(c, c)) & = & \emptyset \\ var(\equiv, \equiv, \equiv) & = & \emptyset \text{ nonsense strins are acceptable} \end{array}$$

Definition 2.2 - *Free Variables*
Free Variables are variables whose value are ambiguous in an *$\mathcal{L}$-Formula.*

Definition 2.3 - *Free Variable Function, $FV(\cdot)$*
We recursively define $FV(\phi)$ for *$\mathcal{L}$-Formulae* as ϕ as follows

i) $FV(\phi) = var(\phi)$ if ϕ is an *Atomic $\mathcal{L}$-Formula.*

ii) $FV(\neg\phi) = FV(\phi)$.

iii) $FV((\phi \to \psi)) = FV(\phi) \cup FV(\psi)$.

iv) $FV(\forall\ x\phi) = FV(\phi)\backslash\{x\}$.

Example 2.5 - *Free Variable Function*

$$\begin{array}{rcl} FV(\forall x(P(y) \to Q(x))) & = & FV(P(y) \to Q(x))\backslash\{x\} \\ & = & [FV(P(y)) \cup FV(Q(x))]\backslash\{x\} \\ & = & [\{y\} \cup \{x\}]\backslash\{x\} \\ & = & \{y\} \end{array}$$

Proposition 2.2 - *Free Variable Function for more complex operators*

$$\begin{array}{rcl} FV(\phi \wedge \psi) & = & FV(\neg(\phi \to \neg\psi)) \text{ by definition of } \wedge \\ & = & FV(\phi) \cup FV(\psi) \\ FV(\phi \vee \psi) & = & FV(\neg\phi \to \psi) \text{ by definition of } \vee \\ & = & FV(\phi) \cup FV(\psi) \\ FV(\exists x\phi) & = & FV(\neg\forall x\neg\phi) \text{ by definition of } \exists \\ & = & FV(\phi)\backslash\{x\} \end{array}$$

Definition 2.4 - *Closed $\mathcal{L}$-Term*
Let t be an $\mathcal{L}$-Term.
If $var(t) = \emptyset$ then t is called a *Closed $\mathcal{L}$-Term.*

Definition 2.5 - *$\mathcal{L}$-Sentence*
Let ϕ be an $\mathcal{L}$-Formula.
If $FV(\phi) = \emptyset$ then ϕ is called an *$\mathcal{L}$-Sentence.*

Example 2.6 - *$\mathcal{L}$-Sentence*

$$\begin{array}{rcl} FV(\forall x(P(x) \to \exists y\ R(y,x)) & = & FV((P(x) \to \exists y\ R(y,x))\backslash\{x\} \\ & = & FV(P(x)) \cup FC(\exists y\ R(y,x))\backslash\{x\} \\ & = & \{x\} \cup (FV(R(y,x)\backslash\{y\})\backslash\{x\} \\ & = & \{x\} \cup (\{y,x\}\backslash\{y\})\backslash\{x\} \\ & = & \{x\} \cup \{x\}\backslash\{x\} \\ & = & \emptyset \end{array}$$

Remark 2.3 - *$\mathcal{L}$-Sentences have no Free Variables and thus no ambiguity in meaning.*

3 Semantics of First-Order Languages

3.1 Structures, Variable Assignments & Satisfaction

Definition 3.1 - *$\mathcal{L}$-Structure*
Let $\mathcal{L}$ be a first-order language.
An *$\mathcal{L}$-Structure* is an ordered pair $\mathfrak{M} = \langle D, \mathfrak{I}\rangle$

i) D is a non-empty set.

ii) $\mathfrak{I}$ is a function on the non-logical symbols of $\mathcal{L}$ st

- For each predicate symbol $P \in \mathcal{L}$ with n-arity.

$$\mathfrak{I}(P) \subset D^n$$

- For each function symbol f of $\mathcal{L}$ with n-arity

$$\mathfrak{I}(f) : D^n \to D$$

- For each constant symbol c of $\mathcal{L}$
$$\mathfrak{I}(c) \in D$$

N.B. D is the domain, $\mathfrak{I}$ is the interpretation.

Notation 3.1 - *$\mathcal{L}$-Structure*
For ease we use the following notation wrt *$\mathcal{L}$-Structure*

$$|\mathfrak{M}| := D \quad f^{\mathfrak{M}} := \mathfrak{I}(f) \quad c^{\mathfrak{M}} := \mathfrak{I}(c) \quad p^{\mathfrak{M}} = \mathfrak{I}(p)$$

Example 3.1 - *$\mathcal{L}$-Structure*
Let $\mathcal{L}_{\text{Rng}} := \{\bar{0}, \bar{1}\bar{+}, \bar{\cdot}\}$ where $\bar{+}$ & $\bar{\cdot}$ are binary functions and $\bar{0}$ & $\bar{1}$ are constants.
(This is the language for ring theory)
We use the overline to distringuish language symbols from standard symbols.
Define

$$\begin{array}{rcl}
D & := & \mathbb{R} \\
\mathfrak{I}(\bar{0}) & = & 0 \in \mathbb{R} \\
\mathfrak{I}(\bar{1}) & = & 1 \in \mathbb{R} \\
\mathfrak{I}(\bar{+}) & : & \mathbb{R} \times \mathbb{R} \to \mathbb{R} \text{ with}(a, b) \mapsto (a + b) \\
\mathfrak{I}(\bar{\cdot}) & : & \mathbb{R} \times \mathbb{R} \to \mathbb{R} \text{ with}(a, b) \mapsto (a \cdot b)
\end{array}$$

We recall $\langle D, \mathfrak{I} \rangle$ is the standard model of the real field.
N.B. We can alternatively write $\langle D, \mathfrak{I} \rangle = \langle \mathbb{R}, 0, 1, +, \cdot \rangle$ for neatness.

Definition 3.2 - *Variable Assignment*
A *Variable Assignment* over an $\mathcal{L}$-Structure is a function which maps from the set of variables to the domain of the $\mathcal{L}$-Structure.

$$s : \text{Var} \to |\mathfrak{M}|$$

Definition 3.3 - *Extension of Variable Assignment*
Let $\mathfrak{M}$ be an $\mathcal{L}$-Structure & s be a variable assignment over $\mathfrak{M}$.
The function $\bar{s} : T_{\mathfrak{M}_{\mathcal{L}}} \to |\mathfrak{M}|$ is defined using the following recursion

i) $\bar{s}(t) = s(t)$ if $t \in \text{Var}$.

ii) $\bar{s}(t) = t^{\mathfrak{M}}$ if t is a constant symbol.

iii) $\bar{s}(f(t_1, \dots, t_k)) = f^{\mathfrak{M}}(\bar{s}(t_1), \dots, \bar{s}(t_k))$.

Example 3.2 - *Variable Assignment*
Let $\mathfrak{M}$ be the standard model of the real field.
Let s bea variable assignment over $\mathfrak{M}$ st $s(x) = s(y) = \pi$. Then

$$\begin{array}{rcl}
\bar{s}(x\bar{+}y) & = & \bar{+}^{\mathfrak{M}}(\bar{s}(x), \bar{s}(y)) \\
& = & \bar{+}^{\mathfrak{M}}(s(x), s(y)) \\
& = & \bar{+}^{\mathfrak{M}}(\pi, \pi) \\
& = & \pi + \pi \\
& = & 2\pi
\end{array}$$

Theorem 3.1 - *Substitution*
Let s be a variable assignment over $\mathfrak{M}$, $x \in \text{Var}$ & $d \in |\mathfrak{M}|$.
A new variable assignment $\frac{sd}{x}$ over $\mathfrak{M}$ is defined as

$$\frac{sd}{x}(y) = \begin{cases} d & \text{if } y = x \\ s(y) & \text{otherwise} \end{cases}$$

Definition 3.4 - *Satisfaction Relation*
Let $\mathfrak{M}$ be an $\mathcal{L}$-Structure & s be a variable assignment over $\mathfrak{M}$.
The *Satiscation Relation*, $\mathfrak{M}, s \vDash \phi$ between $\mathfrak{M}, s$ and $\mathcal{L}$-Formula ϕ is recursively defined as

i) $\mathfrak{M}, s \vDash t_1 \equiv t_2$ iff $\bar{s}(t_1) = \bar{s}(t_2)$.

ii) $\mathfrak{M}, s \vDash P(t_1, \dots, t_k)$ iff $\langle \bar{s}(t_1), \dots, \bar{s}(t_k) \rangle \in P^{\mathfrak{M}} \subset D^k$.

iii) $\mathfrak{M}, s \vDash \neg\phi$ iff $\mathfrak{M}, s \nvDash \phi$.

iv) $\mathfrak{M}, s \vDash \phi \to \psi$ iff if $\mathfrak{M}, s \vDash \phi$ then $\mathfrak{M}, s \vDash \psi$.

v) $\mathfrak{M}, s \vDash \forall x\phi$ iff for all $d \in |\mathfrak{M}|$, $\mathfrak{M}\frac{sd}{x} \vDash \phi$.

Proposition 3.1 - *Extension of Satisfaction Relation*

$$\begin{array}{rcl} \mathfrak{M}, s \vDash \phi \wedge \psi & \text{iff} & \mathfrak{M}, s \vDash \phi \text{ and } \mathfrak{M}, s \vDash \psi \\ \mathfrak{M}, s \vDash \phi \vee \psi & \text{iff} & \mathfrak{M}, s \vDash \phi \text{ or } \mathfrak{M}, s \vDash \psi \\ \mathfrak{M}, s \vDash \phi \leftrightarrow \psi & \text{iff} & \mathfrak{M}, s \vDash \phi \text{ iff } \mathfrak{M}, s \vDash \psi \\ \mathfrak{M}, s \vDash \exists x\phi & \text{iff} & \mathfrak{M}, s\frac{d}{x} \vDash \phi \text{ for some } d \in |\mathfrak{M}| \end{array}$$

Definition 3.5 - *Model*
Let $\Phi \subset \text{Fml}_{\mathcal{L}}$, a subset of formulae of a first order language $\mathcal{L}$.
$\mathfrak{M}, s$ is a *model* of Φ iff $\mathfrak{M}, s \propto \phi$ for all$\phi \in \Phi$.
N.B. This is denoted $\mathfrak{M}, s \vDash \Phi$.

Example 3.3 - *Model*
Let $\mathfrak{M}$ be the standard model of ring theory & s be a variable assignment over $\mathfrak{M}$ st $s(v_1) = 3$ & $s(v_2) = -\pi$.

$$\begin{array}{rl} & \mathfrak{M}, s \vDash \bar{0} \bar{<} v_1 \bar{+} v_2 \\ \iff & \bar{<}^{\mathfrak{M}}(\bar{s}(\bar{0}), \bar{s}(v_1 \bar{+} v_2)) \\ \iff & \bar{<}^{\mathfrak{M}}(\bar{0}^{\mathfrak{M}}, \bar{+}^{\mathfrak{M}}(\bar{s}(v_1), \bar{s}(v_2))) \\ \iff & \bar{<}^{\mathfrak{M}}(0, \bar{+}^{\mathfrak{M}}(s(v_1), s(v_2))) \\ \iff & 0 < 3 + (-\pi) \\ \implies & \mathfrak{M}, s \nvDash \bar{0} \bar{<} v_1 \bar{+} v_2 \end{array}$$

$$\begin{array}{rl} & \mathfrak{M}, s \vDash \forall v_2 \exists v_0,\ v_2 \bar{<} v_0 \\ \iff & \text{for all } d \in \mathbb{R},\ \mathfrak{M}s\frac{d}{v_2} \vDash \exists v_0,\ v_2 \bar{<} v_0 \\ \iff & \text{for all } e \in \mathbb{R},\ \mathfrak{M}(s\frac{d}{v_2})\frac{e}{v_0} \vDash v_2 \bar{<} v_0 \\ \iff & \text{for all } d \in \mathbb{R} \text{ there is } e \in \mathbb{R},\ s(\frac{d}{v_2})\frac{e}{v_)} v_2 < (s\frac{d}{v_0})\frac{e}{v_0} v_0 \\ \iff & \text{for all } d \in \mathbb{R} \text{ there is } e \in \mathbb{R} \text{ st } d < e \end{array}$$

Theorem 3.2 -
Let $\mathcal{L}_1, \mathcal{L}_2$ be first order langauges.
Define models

$$\begin{array}{rcl} \mathfrak{M}_1 & = & \langle D, \mathcal{I}_1 \rangle : \mathcal{L}_1 \text{ structure} \\ \mathfrak{M}_1 & = & \langle D, \mathcal{I}_2 \rangle : \mathcal{L}_2 \text{ structure} \end{array}$$

Note that D is the same for both (*i.e.* Different languages, same world).
Let $\mathcal{L} := \mathcal{L}_1 \cup \mathcal{L}_2$

i) For all $\mathcal{L}$-Terms, t, for all variable assignements s_1 over $\mathfrak{M}_1$ & s_2 over $\mathfrak{M}_2$.
$$\text{If } \begin{cases} c^{\mathfrak{M}_1} = c^{\mathfrak{M}_2} & \text{for all constants symbols that occur in } t \\ f^{\mathfrak{M}_1} = f^{\mathfrak{M}_2} & \text{for all functions symbols that occur in } t \\ s_1(x) = s_2(x) & \text{for all variable symbols that occur in } t \end{cases} \text{ then } \bar{s}_1(t) = \bar{s}_2(t).$$

ii) For all $\phi \in \text{Fml}_{\mathcal{L}}$ & for all variable assignemnts s_1 over $\mathfrak{M}_1$ & s_2 over $\mathfrak{M}_2$.

$$\text{If } \begin{cases} c^{\mathfrak{M}_1} = c^{\mathfrak{M}_2} & \text{for all constants symbols that occur in } \phi \\ f^{\mathfrak{M}_1} = f^{\mathfrak{M}_2} & \text{for all functions symbols that occur in } \phi \\ R^{\mathfrak{M}_1} = R^{\mathfrak{M}_2} & \text{for all predicate symbols that occur in } \phi \\ s_1(x) = s_2(x) & \text{for all variable symbols in FV}(\phi) \end{cases} \text{ then } \mathfrak{M}_1 s_1 \models \phi \text{ iff } \mathfrak{M}_2 \models \phi.$$

N.B. If $\mathcal{L} = \emptyset$ it is not very interesting.

Proof 3.1 - *Theorem 3.2 i)*
This is a proof by induction on $\mathcal{L}$-Term.
Base Case
Let t be atomic then $\bar{s}_1(t) = \bar{s}_2(t)$ is trivial, $\bar{s}_1(x) = s_1(x) = s_2(x) = \bar{s}_2(x)$ and $\bar{s}_1(c) = c^{\mathfrak{M}_1} = c^{\mathfrak{M}_2} = \bar{s}_2(c)$.
Inductive Case
Let $t = f(t_1, \dots, t_k)$. Then

$$\begin{array}{rcl} \bar{s}_1(t) & = & f^{\mathfrak{M}_1}(\bar{s}_1(t_1), \dots, \bar{s}_1(t_k)) \\ & = & f^{\mathfrak{M}_1}(\bar{s}_2(t_1), \dots, \bar{s}_2(t_k)) \text{ by inductive hypothesis} \\ & = & f^{\mathfrak{M}_2}(\bar{s}_2(t_1), \dots, \bar{s}_2(t_k)) \\ & = & \bar{s}_2(t) \end{array}$$

□

Proof 3.2 - *Theorem 3.2 ii)*
This is a proof by induction on $\mathcal{L}$-Formulae. **Base Case**
Let $\phi = R(t_1, \dots, t_k)$ be an atomic $\mathcal{L}$-formula (*i.e.* $cp(\phi) = 0$).
Note that $FV(\phi) = \text{var}(\phi)$, thu conditions in **ii)** for ϕ imply the conditions of **i)** for $t_1, \dots, t_k$.
Therefore $\bar{s}_i(t_i) = \bar{s}_1(t_i)\ \forall\ i \in [1, k]$. Then

$$\begin{array}{rl} & \mathfrak{M}_1, s_1 \models \phi \\ \iff & \langle \bar{s}_1(t_1), \dots, \bar{s}_1(t_k) \rangle \in R^{\mathfrak{M}_1} \\ \iff & \langle \bar{s}_2(t_1), \dots, \bar{s}_2(t_k) \rangle \in R^{\mathfrak{M}_1} \\ \iff & \langle \bar{s}_2(t_1), \dots, \bar{s}_2(t_k) \rangle \in R^{\mathfrak{M}_2} \\ \iff & \mathfrak{M}_2, s_2 \models \phi \end{array}$$

Inductive Case
Let $\phi := \psi \to \theta$.
Since the conditions hold for ϕ they hold for ψ & θ. Then

$$\begin{array}{rl} & \mathfrak{M}_1, s_1 \models \phi \\ \iff & \text{if } \mathfrak{M}_1, s_1 \models \phi \text{ then } \mathfrak{M}_1, s_1 \models \theta \\ \overset{\text{by IH}}{\iff} & \text{if } \mathfrak{M}_2, s_2 \models \phi \text{ then } \mathfrak{M}_2, s_2 \models \theta \\ \iff & \mathfrak{M}_2, s_2 \models \phi \end{array}$$

Let $\phi := \neg\psi$¿
By the inductive hypothesis the claim holds for ψ.
Since the conditions of **ii)** hold for ϕ they hold for ψ.
Note that $FV(\neg\psi) = FV(\psi)$. Hence

$$\begin{array}{rl} & \mathfrak{M}_1, s_1 \models \phi \\ \iff & \mathfrak{M}_1, s_1 \not\models \psi \\ \overset{\text{by IH}}{\iff} & \mathfrak{M}_2, s_2 \not\models \psi \\ \iff & \mathfrak{M}_2, s_2 \models \phi \end{array}$$

Let $\phi := \forall z\psi$.
By the last condition of **ii)** we have that $s_1(x) = s_2(x) \; \forall \; x \in FV(\phi)$.
Since $FV(\phi) \subset FV(\psi) \subset FV(\phi) \cup \{z\}$ it holds that $\forall \; d \in D$ that $s_1\frac{d}{z}(x) = s_2\frac{d}{z}(x) \; \forall \; x \in FV(\psi)$.
Meaning that $\forall \; d \in D$ the conditions of **ii)** hold for ψ wrt $s_1\frac{d}{z}$ & $s_2\frac{d}{z}$.
Hence

$$\begin{array}{rl} & \mathfrak{M}_1, s_1 \models \phi \\ \iff & \forall \; d \in D, \; \mathfrak{M}_1, s_1\frac{d}{z} \models \psi \\ \overset{\text{by IH}}{\iff} & \forall \; d \in D, \; \mathfrak{M}_2, s_2\frac{d}{z} \models \psi \\ \iff & \mathfrak{M}_2, s_2 \models \phi \end{array}$$

□

Theorem 3.3 -
Let $\mathfrak{M}$ be an $\mathcal{L}$-Strucutre, t be a closed $\mathcal{L}$-Term, ϕ to be an $\mathcal{L}$-Sentence and $s_1, s)2$ to be variable assignments over $\mathfrak{M}$.
Then $\bar{s}_1(t) = \bar{s}_2(t)$ and $\mathfrak{M}, s_1 \models \phi$ iff $\mathfrak{M}, s_2 \models \phi$.
N.B. This is since t is closed and thus its sematnic value is indepedent of variable assignment.

Notation 3.2 -
Let t be a closed $\mathcal{L}$-Term & ϕ an $\mathcal{L}$-Sentence.
We use the following notation

$$\begin{array}{rcl} t^{\mathfrak{M}} & := & \text{the unique } d \in D \text{ st } \; \bar{s}(t) = d \text{ for } \textbf{some} \text{ variable assignment } s \text{ over } \mathfrak{M} \\ & := & \text{the unique } d \in D \text{ st } \; \bar{s}(t) = d \text{ for } \textbf{all} \text{ variable assignment } s \text{ over } \mathfrak{M} \\ \mathfrak{M} \models \phi & := & \mathfrak{M}, s \models \phi \text{ for } \textbf{some} \text{ variable assignment } s \text{ over } \mathfrak{M} \\ & := & \mathfrak{M}, s \models \phi \text{ for } \textbf{all} \text{ variable assignment } s \text{ over } \mathfrak{M} \end{array}$$

N.B. The s is dropped in $\mathfrak{M} \models \phi$.

3.2 Important Semantic Concepts

Remark 3.1 - *Throughout this section $\mathcal{L}$ will be a first-order language*

Definition 3.1 - *Logical Consequence*
Let $\phi \in \text{Fml}_{\mathcal{L}}$ & $\Phi \subset \text{Fml}_{\mathcal{L}}$.
ϕ is said to be a *Logical Consequence* of Φ iff $\forall$ $\mathcal{L}$-Structures, $\mathfrak{M}$, and variable assignments, s, over $\mathfrak{M}$ if $\mathfrak{M}, s \models \Phi$ then $\mathfrak{M}, s \models \phi$.
N.B. When this holds we say ϕ logically follows from Φ, denoted $\Phi \models \phi$.

Example 3.4 - *Logical Consequence*
TODO

Proposition 3.2 -
$\forall \; \phi, \psi \in \text{Fml}_{\mathcal{L}}$ and $\Phi \subset \text{Fml}_{\mathcal{L}}$

$$\underbrace{\Phi, \phi}_{\equiv \Phi \cup \{\phi\}} \models \psi \text{ iff } \Phi \models \phi \to \psi$$

Proof 3.3 - *Proposition 3.2*

$$\begin{array}{rl} & \Phi, \phi \models \psi \\ \iff & \forall \mathfrak{M}, s \text{ if } \mathfrak{M}s \models \theta \; \forall \; \theta \in \Phi \cup \{\psi\} \text{ then } \mathfrak{M}, s \models \psi \\ \iff & \forall \mathfrak{M}, s \text{ if } \mathfrak{M}s \models \theta \; \forall \; \theta \in \Phi \text{ and } \mathfrak{M}, s \models \phi \text{ then } \mathfrak{M}, s \models \psi \\ \iff & \text{if } \mathfrak{M}, s \models \theta \; \forall \; \theta \in \Phi \text{ then } \mathfrak{M}, s \models \phi \text{ implies } \mathfrak{M}, s \models \psi \\ \overset{\text{by def}}{\iff} & \Phi \models \phi \to \psi \end{array}$$

Definition 3.2 -
Let Λ be a set of $\mathcal{L}$-sentences

$$\mathfrak{M} \models \Lambda \text{ iff } \mathfrak{M} \models \sigma \ \forall\ \sigma \in \Lambda$$

Example 3.5 -
Let $\mathcal{L} = \mathcal{L}_{\text{GT}} = \{e, \cdot\}$ the language of group theory.
Let $\Phi = \{\forall x \forall y \forall z\ (x \cdot y) \cdot z \equiv x \cdot (y \cdot z),\ \forall\ x \cdot e \equiv x,\ \forall x \exists y\ x \cdot y \equiv e\}$.
Then $\Phi \models \forall\ x \exists y$ st $y \cdot x \equiv e$ but $\Phi \not\models \forall x \forall y\ (x \cdot y) \equiv (y \cdot x)$.

Remark 3.2 -
We always have either $\mathfrak{M}, s \models \phi$ or $\mathfrak{M}s \not\models \phi$ since $\mathfrak{M}s \models \phi \Leftrightarrow \mathfrak{M}s \not\models \phi$.
But it is not always the case that either $\Phi \models \phi$ or $\Phi \not\models \phi$. (There may be some elements in a group with fulfil a criteria by chance).

Definition 3.3 - *Logically Valid*
Let $\phi \in \text{Fml}_{\mathcal{L}}$.
ϕ is said to be *Logically Valid* iff $\mathfrak{M}, s \models \phi\ \forall\ \mathfrak{M}, s$.
N.B. This is also known as valid & logically true.
N.B. Denoted $\models \phi$ for short.

Example 3.6 - *Logically Valid*

- $\forall x \exists\ x \equiv y$ is *Logically Valid* since trivially true for $y = x$.
- $\exists x P(x)$ is <u>not</u> *Logically Valid.*
  Consider the case where $|\mathfrak{M}| = \mathbb{N}$ & $P^{\mathfrak{M}} = \emptyset$ where $\mathfrak{M} \not\models \exists x P(x)$.

Definition 3.4 - *Satisfiable*
Let $\phi \in \text{Fml}_{\mathcal{L}}$ & $\Phi \subset \text{Fml}_{\mathcal{L}}$.
ϕ is *Satisfiable* iff $\mathfrak{M}, s \models \phi$ for some $\mathfrak{M}, s$.
Φ is *Satisfiable* iff $\mathfrak{M}, s \models \Phi$ for some $\mathfrak{M}, s$.

Example 3.7 - *Satisfiable*

- $\exists x P(x)$ is *Satisfiable.*
  Since $|\mathfrak{M}| = \mathbb{N}$ & $P^{\mathfrak{M}} = |\mathfrak{M}| = \mathbb{N}$ satisfies $\exists x P(x)$.
- $x \not\equiv x$ is <u>not</u> *Satisfiable* as $\bar{s}(x) = s(x) = s(x) = \bar{s}(x)$ always and so $\mathfrak{M}, s \models x \equiv x\ \forall\ \mathfrak{M}, s$.

Theorem 3.4 -
Let $\phi \in \text{Fml}_{\mathcal{L}}$ & $\Phi \subset \text{Fml}_{\mathcal{L}}$.

i) ϕ is *Logically Valid* iff $\emptyset \models \phi$.

ii) $\Phi \models \phi$ iff $\Phi \cup \{\neg\phi\}$ is *Unsatisfiable.*

iii) ϕ is logically valid iff $\neg\phi$ is *Unsatisfiable.*

Proof 3.4 - *Theorem 3.4*

i)

$$\begin{aligned}
& \emptyset \models \phi \\
\iff & \text{for all } \mathfrak{M}, s \underbrace{\text{ if } \mathfrak{M}, s \models \theta\ \forall\ \theta \in \emptyset}_{\text{Vacuously true}} \text{ then } \mathfrak{M}, s \models \phi \\
\iff & \forall\ M, s\ \mathfrak{M}, s \models \phi \\
\iff & \phi \text{ is logically valid}
\end{aligned}$$

ii)

$$\begin{aligned} & \Phi \models \phi \\ \iff & \forall\, \mathfrak{M}, s \text{ if } \mathfrak{M}, s \models \phi \text{ then } \mathfrak{M}, s \models \phi \\ \iff & \text{ there is no } \mathfrak{M}, s \text{ st } \mathfrak{M}, s \models \Phi \text{ and } \underbrace{\mathfrak{M}, s \not\models \phi}_{\mathfrak{M}, s \models \neg\phi} \end{aligned}$$

iii) By $i)$ & $ii)$

Definition 3.5 - *Logically Equivalent*
Let $\phi, \psi \in \text{Fml}_{\mathcal{L}}$.
ϕ is *Logically Equivalent* to ψ iff $\phi \models \psi$ & $\psi \models \phi$.
N.B. Equivalently $\models \phi \leftrightarrow \psi$.
Proposition 3.3 - *Logical Equivalence*
Let $\phi, \psi \in \text{Fml}_{\mathcal{L}}$.
ϕ & ψ are *Logically Equivalent* iff $\models \phi \to \psi$.
i.e. $\phi \leftrightarrow \psi$ is *Logically True*.

Proof 3.5 - *Logical Equivalence*
Recall that $\Phi, \phi \models \psi$ iff $\Phi \models \phi \to \psi$. Thus

$$\begin{aligned} & \phi \models \psi \\ \iff & \emptyset \cup \{\phi\} \models \\ \iff & \emptyset \models \phi \to \psi \\ \iff & \phi \to \psi \end{aligned}$$

Similar for converse.
We have that $\phi \models \psi$ and $\psi \models \phi$ iff $\models \phi \to \psi$ and $\models \psi \to \phi$.
$\iff \phi \models \psi$ and $\psi \models \phi$ iff $\models \phi \leftrightarrow \psi$.

Proposition 3.4 - *Logical Equivalence*

i) $((\phi \wedge \psi) \wedge \theta)$ is logically equivalent to $(\phi \wedge (\psi \wedge \theta))$.

ii) $((\phi \vee \psi) \vee \theta)$ is logically equivalent to $(\phi \vee (\psi \vee \theta))$.

iii) $\neg\neg\phi$ is logically equivalent to ϕ.

iv) $\phi \wedge \psi$ is logically equivalent to $\neg(\neg\phi \vee \neg\psi$.

N.B. We write $\phi \wedge \psi \wedge \theta$ for $(\phi \wedge \psi) \wedge \theta$.

3.3 Substitution

Remark 3.3 -
If we have $P(x) \to Q(x)$ then $P(\bar{0}) \to Q(\bar{0})$ and $P(f(y)) \to Q(f(y))$.
If we have $\forall x(P(\bar{0}) \to Q(\bar{0}))$ the $\forall x$ is redundent.

Definition 3.1 - *Substitution in an $\mathcal{L}$-Term*
Let $a, t \in T_{\mathfrak{M}_{\mathcal{L}}}$.
$[a]\frac{t}{x}$ denotes the result of replacing all occurences of x in a with t.
We define this substitution using the following recursive definition

i) When a is atomic:

$$[a]\frac{t}{x} := \begin{cases} t & \text{if } a \equiv x \\ a & \text{otherwise} \end{cases}$$

ii) When $a = f(a_1, \dots, a_j)$ is a compound:

$$[a]\frac{t}{x} := f\left([a_1]\frac{t}{x}, \dots, [a_k]\frac{t}{x}\right)$$

Example 3.8 - *Substitution in an $\mathcal{L}$-Term*

$$\begin{array}{rl} & [(x+y)+z]\frac{\bar{0}\cdot\bar{0}}{y} \\ \iff & [x+y]\frac{\bar{0}\cdot\bar{0}}{y}\cdot[z]\frac{\bar{0}\cdot\bar{0}}{y} \\ \iff & \left([x]\frac{\bar{0}\cdot\bar{0}}{y}+[y]\frac{\bar{0}\cdot\bar{0}}{y}\right)\cdot z \\ \iff & (x+\bar{0}\cdot\bar{0})\cdot z \end{array}$$

Definition 3.2 - *Substitution in an $\mathcal{L}$-Formula*
Let $t \in T_{\mathfrak{M}_\mathcal{L}},\ x \in \text{Var}$ and $\phi \in \text{Fml}_\mathcal{L}$.
$[\phi]\frac{t}{x}$ denotes the result of replacing all occurences of x in ϕ with t.
We define this substitution using the following recursive definition

i) When $\phi := (P(a_1, \dots, a_t)$ is atomic:

$$[\phi]\frac{t}{x} = P\left([a_1]\frac{t}{x}, \dots, [a_k]\frac{t}{x}\right)$$

ii) When $\phi := \neg\psi$:

$$[\phi]\frac{t}{x} := \neg[\psi]\frac{t}{x}$$

iii) When $\phi := \psi \to \theta$:

$$[\phi]\frac{t}{x} := [\psi]\frac{t}{x} \to [\theta]\frac{t}{x}$$

iv) When $\phi := \forall z\psi$:

$$[\phi]\frac{t}{x} = \begin{cases} \forall z[\psi]\frac{t}{x} & \text{if } x \not\equiv z \\ \phi & \text{otherwise} \end{cases}$$

N.B. The `otherwise` case is due to all x variables being bounded.

Example 3.9 - *Substitution in an $\mathcal{L}$-Formula*
Let $x \not\equiv y$

$$\begin{array}{rl} & [\forall xP(x) \to \forall yR(x,y)]\frac{c}{x} \\ = & [\forall xP(x)]\frac{c}{x} \to [\forall yR(x,y)]\frac{c}{x} \\ = & \forall xP(x) \to \forall y[R(x,y)]\frac{c}{x} \\ = & \forall xP(x) \to \forall yR([x]\frac{c}{x},[y]\frac{c}{x}) \\ = & \forall xP(x) \to \forall yR(c,y) \end{array}$$

Definition 3.3 - *Extension of Substitution*

$$\begin{array}{rl} & [\exists xP(x)]\frac{t}{x} \\ \Leftrightarrow & [\neg(\forall x\neg P(x)]\frac{t}{x} \\ \Leftrightarrow & \neg[\forall x\neg P(x)]\frac{t}{x} \\ \Leftrightarrow & \begin{cases} \neg\forall x[\neg P(x)]\frac{t}{x} & \text{if } x \not\equiv z \\ \neg(\forall x\neg P(x)) = P(x) & \text{otherwise} \end{cases} \\ \Leftrightarrow & \begin{cases} \exists x[P(x)]\frac{t}{x} & \text{if } x \not\equiv z \\ \exists xP(x) & \text{otherwise} \end{cases} \end{array}$$

$$\begin{array}{rl} & [P \vee Q]\frac{t}{x} \\ \Leftrightarrow & [\neg P \to Q]\frac{t}{x} \\ \Leftrightarrow & [\neg P]\frac{t}{x} \to [Q]\frac{t}{x} \\ \Leftrightarrow & \neg[P]\frac{t}{x} \to [Q]\frac{t}{x} \\ \Leftrightarrow & [P]\frac{t}{x} \vee [Q]\frac{t}{x} \end{array}$$

$$\begin{array}{rl} & [P \wedge Q]\frac{t}{x} \\ \Leftrightarrow & [\neg(P \to \neg Q)]\frac{t}{x} \\ \Leftrightarrow & \neg[P \to \neg Q]\frac{t}{x} \\ \Leftrightarrow & \neg\left([P]\frac{t}{x} \to [\neg Q]\frac{t}{x}\right) \\ \Leftrightarrow & \neg\left([P]\frac{t}{x} \to \neg[Q]\frac{t}{x}\right) \\ \Leftrightarrow & [P]\frac{t}{x} \wedge [Q]\frac{t}{x} \end{array}$$

Proposition 3.5 -
For every $x \in \text{Var},\ a, t \in T_{\mathfrak{M}_\mathcal{L}}$ and $\phi \in \text{Fml}_\mathcal{L}$ the following results hold

$$[a]\frac{x}{x} = a \quad \text{and} \quad [\phi]\frac{x}{x} = \phi$$

Proof 3.6 - *Proposition 3.5*
TODO
Induction on terms and then on formulae

Proposition 3.6 -
For every $x \in \text{Var},\ a, t \in T_{\mathfrak{M}_\mathcal{L}}$ and $\phi \in \text{Fml}_\mathcal{L}$.

i) If $x \notin \text{Var}(a)$ then $[a]\frac{t}{x} = a$.

ii) If $x \notin \text{FV}(\phi)$ then $[\phi]\frac{t}{x} = \phi$.

Proof 3.7 - *Proposition 3.6 ii)*
Proof by induction on terms, and then on formulae.
Base Case
Let $\phi := P(a_1, \dots, a_k)$ be atomic.
Suppose $x \in \text{FV}(\phi)$
We have that $x \notin FV(\phi) \underbrace{= \text{Var}(\phi)}_{\phi \text{ is atomic}}$
Then $x \notin \text{Var}(a_i)\ \forall\ i \in [1, k]$.

$$\begin{array}{rl} & [\phi]\frac{t}{x} \\ = & P([a_1]\frac{t}{x}, \dots, [a_k]\frac{t}{x}) \\ = & P(a_1, \dots, a_k) \\ = & \phi \end{array}$$

Result holds for base case

Inductive Case Let $\phi = \forall z \psi$.
Suppose $x \notin FV(\phi)$.
Then $FV(\phi) = FV(\psi) \backslash \{z\}$.
If $x \notin FV(\psi)$ then $[\phi]\frac{t}{x} = \begin{cases} \phi & \text{if } x = z \\ \forall z [\phi]\frac{t}{x} \underbrace{= \forall z \phi}_{\text{by IH}} & \text{otherwise} \end{cases}.$
Otherwise $x = z$ and $[\phi]\frac{t}{x} = \phi$.

Proposition 3.7 -
Let $a, t \in T_{\mathfrak{M}_\mathcal{L}},\ \phi \in \text{Fml}_\mathcal{L},\ x \in \text{Var}$ with $x \notin t$. Then

i) $x \notin \text{Var}\left([a]\frac{t}{x}\right)$.

ii) $x \notin FV\left([\phi]\frac{t}{x}\right) \subset \left([\phi]\frac{t}{x}\right)$.

iii) $x \notin \text{Var}\left([\phi]\frac{t}{x}\right)$.

N.B. Proof is done by induction on terms & formulae.

Remark 3.4 -
From $\forall x P(x)$ being true, we can infer that $P(y) \equiv [P(x)]\frac{y}{x},\ P(f(y)) = [P(x)]\frac{f(y)}{x}$ & $P(\bar{0}) \equiv [P(\bar{0})]\frac{\bar{0}}{x}$ are all true.
<u>but</u> $\forall x \phi$ being true does not mean that $[\phi]\frac{y}{x}$ and $[\phi]\frac{\bar{0}}{x}$ are true.

Example 3.10 - *Remark 3.4*
We have that $\forall x \exists y\ y \bar{<} x$ is true, but $\exists y\ y \bar{<} y$ is not true.
i.e. Everyone is liked by someone $\not\Longrightarrow$ someone is liked by themselves.

Definition 3.4 - *Substitutablity*
Refers to valid substitutions.
Let $x \in \text{Var},\ t \in T_{\mathfrak{M}_\mathcal{L}}$ & $\phi \in \text{Fml}_\mathcal{L}$.
We use the notation $\texttt{SubSt}(t, x, \phi)$ to mean t is substitutable for x in ϕ.
We define *Substitutability* using the following recursive definition

i) ϕ is atomic then
$$\texttt{SubSt}(t, x, \phi)\ \forall\ x \in \text{Var}, t \in T_{\mathfrak{M}_\mathcal{L}}$$

ii) $\phi = \not\psi$ then
$$\texttt{SubSt}(t, x, \phi) \text{ iff } \texttt{SubSt}(t, x, \psi)$$

iii) $\phi = \psi \to \theta$ then
$$\texttt{SubSt}(t, x, \phi) \text{ iff } \texttt{SubSt}(t, x, \psi) \wedge \texttt{SubSt}(t, x, \theta)$$

iv) $\phi = \forall z \psi$ then
$$\texttt{SubSt}(t, x, \phi) \text{ iff } \begin{cases} z \notin \text{Var}(t) \wedge \texttt{SubSt}(t, x, \psi) & \text{or} \\ x \notin FV(\phi) \end{cases}$$

N.B. the second case in *iv*) is a vacuous case since there are no variables to substitute.

Proposition 3.8 - *Extension to Substitutability*

$$\begin{aligned} & \texttt{SubSt}(t, x, P \vee Q) \\ \Leftrightarrow\ & \texttt{SubSt}(t, x, \neg P \to Q) \\ \Leftrightarrow\ & \texttt{SubSt}(t, x, \neg P) \wedge \texttt{SubSt}(t, x, Q) \\ \Leftrightarrow\ & \texttt{SubSt}(t, x, P) \wedge \texttt{SubSt}(t, x, Q) \end{aligned}$$

$$\begin{aligned} & \texttt{SubSt}(t, x, P \wedge Q) \\ \Leftrightarrow\ & \texttt{SubSt}(t, x, \neg(P \to \neg Q)) \\ \Leftrightarrow\ & \texttt{SubSt}(t, x, P \to \neg Q) \\ \Leftrightarrow\ & \texttt{SubSt}(t, x, P) \wedge \texttt{SubSt}(t, x, \neg Q) \\ \Leftrightarrow\ & \texttt{SubSt}(t, x, P) \wedge \texttt{SubSt}(t, x, Q) \end{aligned}$$

$$\begin{aligned} & \texttt{SubSt}(t, x, \exists P(z)) \\ \Leftrightarrow\ & \texttt{SubSt}\big(t, x, \neg(\forall z \neg P(z))\big) \\ \Leftrightarrow\ & \texttt{SubSt}\big(t, x, \forall z \neg P(z)\big) \\ \Leftrightarrow\ & \begin{cases} & z \in \text{Var}(t) \wedge \texttt{SubSt}(\neg P(z)) \\ \text{or} & x \notin FV(\forall z P(z)) \end{cases} \\ \Leftrightarrow\ & \begin{cases} & z \in \text{Var}(t) \wedge \texttt{SubSt}(P(z)) \\ \text{or} & x \notin FV(\forall z P(z)) \end{cases} \end{aligned}$$

Example 3.11 - *Substitutability*
Let $x \neq y \neq z$

i) $\texttt{SubSt}(t, x, P(x))\ \forall\ t \in T_{\mathfrak{M}_\mathcal{L}}$.

ii) $\texttt{SubSt}(t, y, P(y))\ \forall\ t \in T_{\mathfrak{M}_\mathcal{L}}$.

iii) $\texttt{SubSt}(t, x, \neg P(z))\ \forall\ t \in T_{\mathfrak{M}_\mathcal{L}}$.

iv) $\texttt{SubSt}(t, x, P(x) \to \neg P(z))\ \forall\ t \in T_{\mathfrak{M}_\mathcal{L}}$.

v) $\texttt{SubSt}(t, x, \forall x\ (P(x) \to \neg P(z))\ \forall\ t \in T_{\mathfrak{M}_\mathcal{L}}$ since $x, y \notin FV$.

vi) $\texttt{SubSt}(t, y, \forall x\ (P(x) \to \neg P(z))\ \forall\ t \in T_{\mathfrak{M}_\mathcal{L}}$ since $x, y \notin FV$.

vii) $\neg\texttt{SubSt}(f(x), z, \forall x(P(x) \to \neg P(z)))\ \forall\ t \in T_{\mathfrak{M}_\mathcal{L}}$ since $z \notin FV$.

Theorem 3.5 -

i) Every variable is substitutable for itself in any formula. $\texttt{SubSt}(x, x, \phi)$.

ii) For $x \in \text{Var}$, $\phi \in F_{\mathfrak{M}_\mathcal{L}}$ with $x \notin FV(\phi)$, any $t \in T_{\mathfrak{M}_\mathcal{L}}$ is $\texttt{SubSt}(t, x, \phi)$.

iii) For all $t \in T_{\mathfrak{M}_\mathcal{L}}$ and $\phi \in F_{\mathfrak{M}_\mathcal{L}}$ if $\text{Var}(t) \cup \text{Var}(\phi) = \emptyset$ then
t is substitutable for every variable in ϕ.

3.4 Substitution Lemma

Proposition 3.9 -
Let $x_1, \dots, x_k \in \text{Var}$ be pairwise disjoint and π be a permutation of $(1, \dots, k)$.
Then, for all variable assignments s over $\mathcal{L}$-Structure $\mathfrak{M}$ and $d_1, \dots, d_l \in |\mathfrak{M}|$

$$\left(\left(\dots\left(s\frac{d_1}{x_1}\right)\dots\right)\frac{d_{k-1}}{x_{k-1}}\right)\frac{d_k}{x_k} = \left(\left(\dots\left(s\frac{d_{\pi(1)}}{x_{\pi(1)}}\right)\dots\right)\frac{d_{\pi(k-1)}}{x_{\pi(k-1)}}\right)\frac{d_{\pi(k)}}{x_{\pi(k)}}$$

Example 3.12 - *Proposition 3.8*

$$\left(\left(\left(s\frac{d_1}{x_1}\right)\frac{d_2}{x_2}\right)\frac{d_3}{x_3}\right)y = \begin{cases} d_1 & \text{if} y = x_1 \\ d_2 & \text{if} y = x_2 \\ d_3 & \text{if} y = x_3 \\ s(y) & \text{otherwise} \end{cases} = \left(\left(\left(s\frac{d_2}{x_2}\right)\frac{d_3}{x_3}\right)\frac{d_1}{x_1}\right)y$$

Theorem 3.6 - *Substitution Lemma*
Let $\mathfrak{M} = \langle D, \mathfrak{I}\rangle$ be an $\mathcal{L}$-Structure, and let x be a variable

i) Let a be an arbitrary $\mathcal{L}$-term.
For every variable assignment s over $\mathfrak{M}$, for every $\mathcal{L}$-term t

$$\bar{s}\left([a]\frac{t}{x}\right) = \overline{s\frac{\bar{s}(t)}{x}(a)}$$

ii) Let ϕ be an arbitrary $\mathcal{L}$-Formula.
For every variable assignment s over $\mathfrak{M}$, for every $\mathcal{L}$-term t
if t is substitutable for x in ϕ then we have

$$\mathfrak{M}, s \models \phi\frac{t}{x} \quad \text{iff} \quad \mathfrak{M}, s\frac{\bar{s}(t)}{x} \models \phi$$

Proof 3.8 - *Substitution Lemma*
This is a proof by induction on terms.
Let $\mathfrak{M} = \langle D, \mathfrak{I}\rangle$ and $x \in \text{Var}$. *Base*
Let a be a constant. Then

$$\begin{array}{rcl} \bar{s}([a]\frac{t}{x}) & = & \bar{s}(a) \\ & = & a^{\mathfrak{M}} \\ & = & \overline{s\frac{\bar{s}(t)}{x}(a)} \end{array}$$

Let a be a variable.
If $a = x$ then

$$\begin{aligned}\bar{s}([x]\tfrac{t}{x}) &= \bar{s}(t)\\ &= s\tfrac{\bar{s}(t)}{x}(tx\\ &= \overline{s\tfrac{\bar{s}(t)}{x}(x)}\end{aligned}$$

If $a \neq x$ then

$$\begin{aligned}\bar{s}([a]\tfrac{t}{x}) &= \bar{s}(a)\\ &= \overline{s\tfrac{\bar{s}(t)}{x}(a)}\end{aligned}$$

Induction
Let $a = f(a_1, \dots, a_k)$. Then

$$\begin{aligned}\bar{s}([a]\tfrac{t}{x}) &= \bar{s}(f([a_1]\tfrac{t}{x_1}, \dots, [a_k]\tfrac{t}{x_k})\\ &= f^{\mathfrak{M}}(\bar{s}([a_1]\tfrac{t}{x_1}), \dots, \bar{s}([a_k]\tfrac{t}{x_k}))\\ &\overset{\text{by IH}}{=} f^M\left(\overline{s\tfrac{\bar{s}(t)}{x}}(a_1), \dots, \overline{s\tfrac{\bar{s}(t)}{x}}(a_k)\right)\\ &= \overline{s\tfrac{\bar{s}(t)}{x}})(f(a_1, \dots, a_k))\end{aligned}$$

Now we prove ii) by induction on formulae.
If $\phi := a \equiv b$.Then

$$\begin{aligned}&\mathfrak{M}, s \models [a \equiv b]\tfrac{t}{x}\\ \iff &\mathfrak{M}, s \models [a]\tfrac{t}{x} \equiv [b]\tfrac{t}{x}\\ \iff &\bar{s}([a]\tfrac{t}{x}) = \bar{s}([b]\tfrac{t}{x})\\ \iff &s\tfrac{\bar{s}(t)}{x}(a) = s\tfrac{\bar{s}(t)}{x}(b)\\ \iff &\mathfrak{M}, s\tfrac{\bar{s}(t)}{x} \models a \equiv b\end{aligned}$$

If $\phi := \neg\psi$. Then

$$\begin{aligned}&\mathfrak{M}, s \models [\phi]\tfrac{t}{x}\\ \iff &\mathfrak{M}, s \models \neg[\psi]\tfrac{t}{x}\\ \iff &\mathfrak{M}, s \not\models [\psi]\tfrac{t}{x}\\ \overset{\text{by IH}}{\iff} &\mathfrak{M}, s\tfrac{\bar{s}(t)}{x} \not\models \psi\\ \iff &\mathfrak{M}, s\tfrac{\bar{s}(t)}{x} \models \neg\psi\\ \iff &\mathfrak{M}, s\tfrac{\bar{s}(t)}{x} \models \phi\end{aligned}$$

Similar derivation for $phi = \psi \to \theta$.

Remark 3.5 - *I have missed some from 3.3.14 but have covered in ReviewedNotesLOG*

4 Deductive Systems for First-Order Predicate Logic

Remark 4.1 - *Structure of Mathetmical Proofs*
Mathetmatical Proofs take the following, rough, structure

- Assumptions: Axioms, definitions, proved theorem,...
  N.B. These depend on the subject matter. The *Logical Axioms* appear in every proof.
- Deduction Steps
- Theorem - Consequent of a deduction from the assumptions & *Logical Axioms.*

Definition 4.1 - *Generalisation*
Let $\mathcal{L}$ be a FOL and $\phi, psi \in Fml_{\mathcal{L}}$.
ϕ is a *Generalisation* of ψ $\begin{cases} & \phi = \psi \\ \text{or} & \phi = \forall x_1 \dots \forall x_n \psi \end{cases}$ for some variables $x_1, \dots, x_n \in var$.
N.B. Every $\mathcal{L}$-Formula is a generalisation of itself.

4.1 Hilbert Calculus

Definition 4.1 - *Logical Axioms of Hilbert Calculus*
Let $\mathcal{L}$ be a FOL.
The set of *Logical Axioms* over $\mathcal{L}$ comprises all generalisation of the following forms of $\mathcal{L}$-Formulae (and nothing else)

i) $\phi(\to \psi \to \theta)$.

ii) $(\phi(\to \psi \to \theta)) \to ((\phi \to \psi) \to (\phi \to \theta))$.

iii) $(\neg\phi \to \neg\psi) \to (\psi \to \phi)$.

iv) $\forall x\phi \to [\phi]\frac{t}{x}$ where $\texttt{SubSt}(t, x, \phi)$.

v) $\phi \to \forall x\phi$ where $x \notin FV(\phi)$.

vi) $\forall x(\phi \to \psi) \to (\forall x\phi \to \forall x\psi)$.

vii) $t \equiv t$.

viii) $t_0 \equiv t_1 \to \left([\phi]\frac{t_0}{x} \to [\phi]\frac{t_1}{x}\right)$ where $\texttt{SubSt}(t_0, x, \phi)$ <u>and</u> $\texttt{SubSt}(t_1, x, \phi)$.

N.B. $\phi, \psi, \theta \in Fml_{\mathcal{L}}$, x is any variable and $t, t_0, t_1 \in Tm_{\mathcal{L}}$.
N.B. This set of axioms is denoted $\Lambda_{\mathcal{L}}$ or Λ.

Definition 4.2 - *Deduction in Hilbert Calculus*
A *Deduction*, D, from a set of $\mathcal{L}$-Formulae, Γ, in *Hilbert Calculus* is a finite sequence $\langle \phi_1, \dots, \phi_n \rangle$ of $\mathcal{L}$-Formulae st either of the following holds $\forall i \in [1, n]$

i) $\phi_i \in \Gamma \cup \Lambda$; or,

ii) $\exists j, k \in [1, i]$ st $\phi_j = \phi_k \to \phi_i$.

ϕ_n is the final output of $\mathcal{D}$.
N.B. We say that $\mathcal{D}$ is a deduction of ϕ_n from Γ.

Proposition 4.1 -
Let $\mathcal{D} := \langle \phi_1, \dots, \phi_n \rangle$ be a deduction of ϕ_n from Γ.

i) $\mathcal{D}_m := \langle \phi_1, \phi_m \rangle$ is a deduction of ϕ_m from Γ for $m \in [1, n]$

ii) $\mathcal{D}$ is a deduction of ϕ_n from Σ for all $\Sigma \supset \Gamma$.

iii) For any deduction $\mathcal{D} := \langle \phi'_1, \dots, \phi'_m \rangle$ from Γ, the concatenation $\mathcal{D} * \mathcal{D}' = \langle \phi_1, \dots, \phi_n, \phi'_1, \dots, \phi'_m \rangle$ is a deduction of θ'_m from Γ.

Definition 4.3 - *Deducibility in Hilbert Style*
Let $\mathcal{L}$ be a FOL, $\phi \in Fml_{\mathcal{L}}$ and Γ be a set of axioms.
ϕ is *Deducible* from Γ, if there exists a deduction of ϕ from Γ.
N.B. This is denoted $\Gamma \vdash \phi$ and we say ϕ is a *Theorem* of Γ.
N.B. If $\Gamma := \emptyset$ then we denote it as $\vdash \phi$.

Theorem 4.1 - *Monotonicity of Deducibility*
If $\Gamma \vdash \phi$ and $\Gamma \subset \Sigma$ then $\Sigma \vdash \phi$.

Theorem 4.2 - *Generalisation Theorem*
If $\Gamma \vdash \phi$ and $x \notin FV(\Gamma)$ then $\Gamma \vdash \forall x\phi$.
Here $FV(\Gamma) := \bigcup_{\phi\in\Gamma} FV(\phi)$.

Theorem 4.3 -

- Ax $\Gamma \vdash \phi \forall\ \phi \in \Gamma \cup \Lambda$.
- MP If $\Gamma \vdash \phi \to \psi$ and $\Gamma \vdash \phi$ then $\Gamma \vdash \psi$.
- Gen If $\Gamma \vdash \phi$ and $x \notin FV(\Gamma)$, then $\Gamma \vdash \forall x\phi$