# Logic - Problem Sheet 2

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## Question - 1.

Let $\mathcal{L} = \{P, R, f, g, c_0, c_1\}$ where $P$ is a unary predicate, $R$ is a binary prediate and $f$ & $g$ are binary function symbols. Let $\mathfrak{M}$ be an $\mathcal{L}$-Structure such that $|\mathfrak{M}| = \mathbb{R}$, $P^{\mathfrak{M}} = \mathbb{N}$, $R^{\mathfrak{M}}$ is the usual greater-than ($>$) relation on $R$, $f^{\mathfrak{M}}$ is the usual addition on $\mathbb{R}$, $g^{\mathfrak{M}}$ is the usual multiplicaiton on $\mathbb{R}$, $c_0^{\mathfrak{M}} = 0$ and $c_1^{\mathfrak{M}} = 1$. Finally, let $s$ be a variable assignment over $\mathfrak{M}$ with $s(x) = 5$ and $s(y) = 3$ where $x$ and $y$ are distinct variables.

**Question 1 b)** - Determine the semantic value of the following $\mathcal{L}$-term in $\mathfrak{M}$ under $s$

$$f(g(x,y), g(x,c_1))$$

**Answer 1 b)**

$$\begin{array}{rl} & \bar{s}(f(g(x,y), g(x,c_1))) \\ \iff & f^{\mathfrak{M}}(\bar{s}(g(x,y)), \bar{s}(g(x,c_1))) \\ \iff & f^{\mathfrak{M}}(g^{\mathfrak{M}}(\bar{s}(x), \bar{s}(y)), g^{\mathfrak{M}}(\bar{s}(x), \bar{s}(c_1))) \\ \iff & f^{\mathfrak{M}}(\cdot^{\mathfrak{M}}(s(x), s(y)), \cdot^{\mathfrak{M}}(s(x), c_1^{\mathfrak{M}})) \\ \iff & +(\cdot(5,3), \cdot(5,1)) \\ \iff & +(15,5) \\ \iff & 20 \end{array}$$

**Question 1 c)** - Answer whether or not the $\mathcal{L}$-formula below is satisfied in $\mathfrak{M}$ under $s$

$$\forall x \forall y (R(x,c_0) \to \exists z (P(z) \land R(g(z,x),y)))$$

**Answer 1 c)**

$$\begin{array}{rl} & \mathfrak{M}, s \vDash \forall x \forall y (R(x,c_0) \to \exists z (P(z) \land R(g(z,x),y))) \\ \iff & \text{for all } d_0, d_1 \in |\mathfrak{M}|,\ \mathfrak{M}, s\frac{d_0}{x}\frac{d_1}{y} \vDash (R(x,c_0) \to \exists z (P(z) \land R(g(z,x),y))) \\ \iff & \text{for all } d_0, d_1 \in \mathbb{R} \text{ if } \mathfrak{M}, s\frac{d_0}{x}\frac{d_1}{y} \vDash R(x,c_0) \text{ then } \mathfrak{M}, s\frac{d_0}{x}\frac{d_1}{y} \vDash \exists z (P(z) \land R(g(z,x),y)) \\ \iff & \text{for all } d_0, d_1 \in \mathbb{R} \text{ if } d_0 > 0 \text{ then exists } d_2 \in |\mathfrak{M}| \text{ st } \mathfrak{M}, s\frac{d_0}{x}\frac{d_1}{y}\frac{d_2}{z} \vDash (P(z) \land R(g(z,x),y)) \\ \iff & \text{for all } d_0, d_1 \in \mathbb{R} \text{ if } d_0 > 0 \text{ then exists } d_2 \in \mathbb{R} \text{ st } \mathfrak{M}, s\frac{d_0}{x}\frac{d_1}{y}\frac{d_2}{z} \vDash P(z) \text{ and } \mathfrak{M}, s\frac{d_0}{x}\frac{d_1}{y}\frac{d_2}{z} \vDash R(g(z,x),y) \\ \iff & \text{for all } d_0, d_1 \in \mathbb{R} \text{ if } d_0 > 0 \text{ then exists } d_2 \in \mathbb{R} \text{ st } d_2 \in \mathbb{N} \text{ and } \mathfrak{M}, s\frac{d_0}{x}\frac{d_1}{y}\frac{d_2}{z} \vDash g(z,x) > y \\ \iff & \text{for all } d_0, d_1 \in \mathbb{R} \text{ if } d_0 > 0 \text{ then exists } d_2 \in \mathbb{N} \text{ st } d_2 \cdot d_0 > d_1 \\ \iff & \texttt{true} \end{array}$$

## Question - 2.

Let $\mathcal{L} = \{P, f\}$ where $P$ is a unary predicate and $f$ is a binary function symbol.
For each of the following $\mathcal{L}$-formulae find an $\mathcal{L}$-structure $\mathfrak{M}_0$ and a variable assignemnt $s_0$ over $\mathfrak{M}_0$ in which the formula is true, and find an $\mathcal{L}$-structure $\mathfrak{M}_1$ and a variable assignment $s_1$ over

$\mathfrak{M}_1$ in which the formula is false:

**Question 2 b)** - $\exists v_2 \forall v_1 f(v_2, v_1) \equiv v_2$.

**Answer 2 b)**
$\mathfrak{M}_0 = (\mathbb{R}, \cdot),\ \bar{s}_0(v_2) = 0$.
$\mathfrak{M}_1 = (\mathbb{R}, \cdot),\ \bar{s}_1(v_2) = 1$.

**Question 2 c)** - $\exists v_2 (P(v_2) \wedge \forall v_1 P(f(v_2, v_1)))$.

**Answer 2 c)**
$\mathfrak{M}_0 = (\mathbb{R}, \equiv 0, \cdot),\ \bar{s}_0(v_2) = 0$.
$\mathfrak{M}_1 = (\mathbb{R}, \equiv 0, \cdot),\ \bar{s}_1(v_2) = 1$.

## Question - 5.

Show the following

**Question 5 a)** - $\models (\phi \to \forall x \phi)$ if $x \notin FV(\phi)$.

**Answer 5 a)**
We have that

$$\models (\phi \to \forall x \phi) \text{ if } x \notin FV(\phi) \iff \text{if } \models \phi \text{ then } \models \forall x \phi \text{ if } x \notin FV(\phi)$$

Whether $\phi$ is satisfied depends on its free variables.
Since $x \notin FV(\phi)$, whether $\phi$ is satisified is independent of the value of $x$.
Under the condition that $\phi$ is logically valid, it holds that $\forall x \phi$ is logically valid for $x \notin FV(\phi)$.
Thus the statement holds.

**Question 5 b)** - $\models \forall x (\phi \to \psi) \to (\forall x \phi \to \forall x \psi)$.

**Answer 5 b)**

$$\begin{aligned}
& \models \forall x (\phi \to \psi) \\
\iff & \text{ for all } x \in |\mathfrak{M}| \models (\phi \to \psi) \\
\iff & \text{ for all } x \in |\mathfrak{M}| \text{ if } \models \phi \text{ then} \models \psi \\
\iff & \text{ if for all } x \in |\mathfrak{M}| \models \phi \text{ then for all } x \in |\mathfrak{M}| \models \psi \\
\iff & \models (\forall x \phi \to \forall x \psi) \\
\implies & \models \forall x (\phi \to \psi) \to \models (\forall x \phi \to \forall x \psi) \\
\iff & \models \forall x (\phi \to \psi) \to (\forall x \phi \to \forall x \psi)
\end{aligned}$$

**Question 5 c)** - if $\Gamma \models \phi$ and $x \notin FV(\Gamma) := \bigcup\limits_{\chi \in \Gamma} FV(\chi)$ then $\Gamma \models \forall x \phi$ where $\Gamma \subset \text{Fml}_{\mathcal{L}}$.

**Answer 5 c)**
Assume that $\Gamma \models \phi$.
Then, for all models $\mathfrak{M}, s$ of $\Gamma$, $\mathfrak{M}, s \models \phi$.
Thus, for all models $\mathfrak{M}, s$ where $\mathfrak{M}, s \models \gamma$ for all $\gamma \in \Gamma$, $\mathfrak{M}, s \models \phi$.
Thus, if $\mathfrak{M}, s \models \gamma$ for all $\gamma \in \Gamma$ then $\mathfrak{M}, s \models \phi$.
Let $x \notin FV(\Gamma)$.
Then, if $\mathfrak{M}, s \models \gamma$ then $\mathfrak{M}, s \models \forall x \gamma$ by **Question 5 a)**.
Thus, if $\mathfrak{M}, s \models \gamma$ for all $\gamma \in \Gamma$ then $\mathfrak{M}, s \models (\phi \wedge \forall x \gamma)$ for all $\gamma \in \Gamma$.

Thus, if $\mathfrak{M}, s \models \gamma$ for all $\gamma \in \Gamma$ then for all $x \notin FV(\Gamma)$, $\mathfrak{M}, s \models (\phi \wedge \gamma)$ for all $\gamma \in \Gamma$.
Thus, if $\mathfrak{M}, s \models \gamma$ for all $\gamma \in \Gamma$ then for all $x \notin FV(\Gamma)$, $\mathfrak{M}, s \models (\phi)$.
Thus, if $\mathfrak{M}, s \models \gamma$ for all $\gamma \in \Gamma$ then $\mathfrak{M}, s \models (\forall x \phi)$ with $x \notin FV(\Gamma)$.