

Logic - Problem Sheet 1

Dom Hutchinson

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Question - 2.

Let $\mathcal{L} = \{c, f, g\}$ be a first-order language, where c is a constant symbol, f is a binary function symbol and g is a unary function symbol. Let x & y be variables.

Prove that the following strings over $\mathcal{A}_{\mathcal{L}}$ are \mathcal{L} -terms and calculate their complexity.

Question 2 a) - $g(f(x, c))$.

Answer 2 a)

x, c are atomic \mathcal{L} -terms with $cp(x) = cp(c) = 0$.

Thus $f(x, c)$ is an \mathcal{L} -term with $cp(f(x, c)) = \max\{cp(x), cp(c)\} + 1 = \max\{0, 0\} + 1 = 1$.

Thus $g(f(x, c))$ is an \mathcal{L} -term with $cp(g(f(x, c))) = \max\{cp(f(x, c))\} + 1 = 1 + 1 = 2$.

Question 2 b) - $f(f(x, c), f(x, f(x, y)))$.

Answer 2 b)

x, y, c are atomic \mathcal{L} -terms with $cp = 0$.

Thus $f(x, y)$ is an \mathcal{L} -term with $cp = \max\{cp(x), cp(y)\} + 1 = \max\{0, 0\} + 1 = 1$,

and $f(x, c)$ is an \mathcal{L} -term with $cp = \max\{cp(x), cp(c)\} + 1 = 0 + 1 = 1$.

Thus $f(x, f(x, y))$ is an \mathcal{L} -term with $cp = \max\{cp(x), cp(f(x, y))\} = 1 + 1 = 2$.

Thus $f(f(x, c), f(x, f(x, y)))$ is an \mathcal{L} -term with

$$cp(f(f(x, c), f(x, f(x, y)))) = \max\{cp(f(x, c)), cp(f(x, f(x, y)))\} = 2 + 1 = 3$$

Question - 3.

Let $\mathcal{L} = \{P, Q, c, f, g\}$ be a first-order language where c, f & g are as in Question 2, P is a unary predicate symbol and Q is a binary predicate symbol.

Prove that the following strings over $\mathcal{A}_{\mathcal{L}}$ are \mathcal{L} -formulae and calculate their complexity.

Question 3 a) - $\neg\forall x\neg\forall y(P(g(f(x, c))) \longrightarrow \equiv (y, y))$.

Answer 3 a)

$\equiv (y, y)$ and $P(g(f(x, c)))$ are atomic \mathcal{L} -Formulae with $cp = 0$.

Thus $\forall y(P(g(f(x, c))))$ is an \mathcal{L} -Formula with $cp = 0 + 1 = 1$.

Thus $\neg\forall y(P(g(f(x, c))))$ is an \mathcal{L} -Formula with $cp = 1 + 1 = 2$.

Thus $\forall x\neg\forall y(P(g(f(x, c))))$ is an \mathcal{L} -Formula with $cp = 2 + 1 = 3$.

Thus $\neg\forall x\neg\forall y(P(g(f(x, c))))$ is an \mathcal{L} -Formula with $cp = 3 + 1 = 4$.

Thus $\neg\forall x\neg\forall y(P(g(f(x, c))) \longrightarrow \equiv (y, y))$ is an \mathcal{L} -Formula with $cp = \max\{4, 0\} + 1 = 5$.

Question 3 b) - $(\forall x \neg P(f(x, c)) \rightarrow Q(f(x, c), f(f(x, c), f(x, f(x, y))))).$

Answer 3 b)

Thus $P(f(x, c))$ & $Q(f(x, c), f(f(x, c), f(x, f(x, y))))$ are atomic \mathcal{L} -Formulae with $cp = 0$.

Thus $\neg P(f(x, c))$ is an \mathcal{L} -Formula with $cp = 0 + 1 = 1$.

Thus $\forall x \neg P(f(x, c))$ is an \mathcal{L} -Formula with $cp = 1 + 1 = 2$.

Thus $(\forall x \neg P(f(x, c)) \rightarrow Q(f(x, c), f(f(x, c), f(x, f(x, y))))$ is an \mathcal{L} -Formula with $cp = \max\{2, 0\} + 1 = 3$.

Question - 5.

Let x, y, z be variables and $\mathcal{L} = \{f, P, Q, R\}$ where f is a unary function symbol, P is a binary predicate symbol, Q is a unary predicate symbol and R is a ternary predicate symbol.

For the following \mathcal{L} -formulae, ϕ , determine the corresponding set of variables that occur free in ϕ .

Question 5 a) - $\forall x \exists y (P(x, z) \rightarrow \neg Q(y)) \rightarrow \neg Q(y).$

Answer 5 a) -

$$\begin{aligned}
 & FV(\forall x \exists y (P(x, z) \rightarrow \neg Q(y)) \rightarrow \neg Q(y)) \\
 = & FV(\forall x \exists y (P(x, z) \rightarrow \neg Q(y))) \cup FV(\neg Q(y)) \\
 = & [FV((P(x, z) \rightarrow \neg Q(y)) \setminus \{x, y\}) \cup FV(Q(y))] \\
 = & [(FV(P(x, z)) \cup FV(\neg Q(y)) \setminus \{x, y\}) \cup \{y\}] \\
 = & [\{x, z\} \cup FV(Q(y)) \setminus \{x, y\}] \cup \{y\} \\
 = & [(\{x, z\} \cup \{y\}) \setminus \{x, y\}] \cup \{y\} \\
 = & [\{x, y, z\} \setminus \{x, y\}] \cup \{y\} \\
 = & \{z\} \cup \{y\} \\
 = & \{y, z\}
 \end{aligned}$$

Question 5 b) - $\forall x \forall y (Q(c) \wedge Q(f(x))) \rightarrow \forall y \forall x (Q(y) \wedge R(x, x, y)).$

Answer 5 b) -

$$\begin{aligned}
 & FV(\forall x \forall y (Q(c) \wedge Q(f(x))) \rightarrow \forall y \forall x (Q(y) \wedge R(c, x, y))) \\
 = & FV(\forall x \forall y (Q(c) \wedge Q(f(x)))) \cup FV(\forall y \forall x (Q(y) \wedge R(c, x, y))) \\
 = & [FV(Q(c) \wedge Q(f(x))) \setminus \{x, y\}] \cup [FV(Q(y) \wedge R(c, x, y)) \setminus \{x, y\}] \\
 = & [[FV(Q(c)) \cup FV(Q(f(x))) \setminus \{x, y\}] \cup [[FV(Q(y)) \cup FV(R(c, x, y))] \setminus \{x, y\}] \\
 = & [[\emptyset \cup \{x\}] \setminus \{x, y\}] \cup [\{y\} \cup \{x, y\}] \setminus \{x, y\} \\
 = & \emptyset \cup \emptyset \\
 = & \emptyset
 \end{aligned}$$

Question 5 c) - $Q(z) \longleftrightarrow \exists z (P(x, y) \wedge R(c, x, y)).$

Answer 5 c)

Consider

$$\begin{aligned}
 FV(\phi \longleftrightarrow \psi) &= FV[(\phi \rightarrow \psi) \wedge (\psi \rightarrow \phi)] \\
 &= FV(\phi \rightarrow \psi) \cup FV(\psi \rightarrow \phi) \\
 &= FV(\phi) \cup FV(\psi) \cup FV(\psi) \cup FV(\phi) \\
 &= FV(\phi) \cup FV(\psi)
 \end{aligned}$$

Thus

$$\begin{aligned}
 & FV(Q(z) \longleftrightarrow \exists z(P(x, y) \wedge R(c, x, y))) \\
 = & FV(Q(z)) \cup FV(\exists z(P(x, y) \wedge R(c, x, y))) \\
 = & \{z\} \cup [FV(P(x, y) \wedge R(c, x, y)) \setminus \{z\}] \\
 = & \{z\} \cup [FV(P(x, y)) \cup FV(R(c, x, y)) \setminus \{z\}] \\
 = & \{z\} \cup [\{x, y\} \cup \{x, y\} \setminus \{z\}] \\
 = & \{z\} \cup \{x, y\} \\
 = & \{x, y, z\}
 \end{aligned}$$

Question 5 d) - Which of these formulae are \mathcal{L} -sentences?

Answer 5 d) - Only the formula from (b) is an \mathcal{L} -sentence since it is the only one with no free variables.