Logic - Problem Sheet 1

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Question - 1.

Prove

$$\text{If the sets } M_0, M_1, M_2, \dots \text{ are countable, then } \bigcup_{n\in\mathbb{N}} M_n \text{ is countable as well.}$$

Answer 1

Question - 2.

Let $\mathcal{L} = \{c, f, g\}$ be a first-order language, where c is a constant symbol, f is a binary function symbol and g is a unary function symbol. Let x & y be variables.

Prove that the following strings over $\mathcal{A}_\mathcal{L}$ are $\mathcal{L}$-terms and calculate their complexity.

Question 2 a) - $g(f(x, c))$.

Answer 2 a)
x, c are atomic $\mathcal{L}$-terms with $cp(x) = cp(c) = 0$.
Thus $f(x, c)$ is an $\mathcal{L}$-term with $cp(f(x, c)) = \max\{cp(x), cp(c)\} + 1 = \max\{0, 0\} + 1 = 1$.
Thus $g(f(x, c))$ is an $\mathcal{L}$-term with $cp(g(f(x, c))) = \max\{cp(f(x, c)\} + 1 = 1 + 1 = 2$.

Question 2 b) - $f(f(x, c), f(x, f(x, y)))$.

Answer 2 b)
x, y, c are atomic $\mathcal{L}$-terms with $cp = 0$.
Thus $f(x, y)$ is an $\mathcal{L}$-term with $cp = \max\{cp(x), cp(y)\} + 1 = \max\{0, 0\} + 1 = 1$,
and $f(x, c)$ is an $\mathcal{L}$-term with $cp = \max\{cp(x), cp(c)\} + 1 = 0 + 1 = 1$.
Thus $f(x, f(x, y))$ is an $\mathcal{L}$-term with $cp = \max\{cp(x), cp(f(x, y))\} = 1 + 1 = 2$.
Thus $f(f(x, c), f(x, f(x, y)))$ is an $\mathcal{L}$-term with

$$cp(f(f(x, c), f(x, f(x, y)))) = \max\{cp(f(x, c)), cp(f(x, f(x, y)))\} = 2 + 1 = 3$$

Question - 3.

Let $\mathcal{L} = \{P, Q, c, f, g\}$ be a first-order language where c, f & g are as in `Question2`, P is a unary predicate symbol and Q is a binary predicate symbol.

Prove that the following strings over $\mathcal{A}_\mathcal{L}$ are $\mathcal{L}$-formulae and calculate their complexity.

Question 3 a) - $\neg\forall x\neg\forall y(P(g(f(x, c))) \longrightarrow \equiv (y, y)$.

Answer 3 a)
x, y, c are $\mathcal{L}$-Terms with $cp = 0$.
Thus $f(x,c)$ & $\equiv (y,y)$ are $\mathcal{L}$-Terms with $cp = 0$.
Thus $g(f(x,c))$ is an $\mathcal{L}$-Term with $cp = 0$.
Thus $P(g(f(x,c))$ is an $\mathcal{L}$-Formula with $cp = 0$.
Thus $\forall y(P(g(f(x,c)))$ is an $\mathcal{L}$-Formula with $cp = 0 + 1 = 1$.
Thus $\neg\forall y(P(g(f(x,c)))$ is an $\mathcal{L}$-Formula with $cp = 1 + 1 = 2$.
Thus $\forall x\neg\forall y(P(g(f(x,c)))$ is an $\mathcal{L}$-Formula with $cp = 2 + 1 = 3$.
Thus $\neg\forall x\neg\forall y(P(g(f(x,c)))$ is an $\mathcal{L}$-Formula with $cp = 3 + 1 = 4$.
Thus $\neg\forall x\neg\forall y(P(g(f(x,c))) \longrightarrow\equiv (y,y)$ is an $\mathcal{L}$-Formula with $cp = \max\{4,0\} + 1 = 5$.

Question 3 b) - $\big(\forall x\neg P\big(f(x,c)\big) \to Q\big(f(x,c), f(f(x,c), f(x, f(x,y))))\big)$.

Answer 3 b)
x, y, c are $\mathcal{L}$-Terms with $cp = 0$.
Thus $f(x,c)$ & $f(x,y)$ are $\mathcal{L}$-Terms with $cp = 0$.
Thus $f(x, f(x,y))$ is an $\mathcal{L}$-Term with $cp = 0$.
Thus $f(f(x,c), f(x, f(x,y)))$ is an $\mathcal{L}$-Term with $cp = 0$.
Thus $P(f(x,c))$ & $Q\big(f(x,c), f(f(x,c), f(x, f(x,y)))$ are $\mathcal{L}$-Terms with $cp = 0$.
Thus $\neg P(f(x,c))$ is an $\mathcal{L}$-Formula with $cp = 0 + 1 = 1$.
Thus $\forall x\neg P(f(x,c))$ is an $\mathcal{L}$-Formula with $cp = 1 + 1 = 2$.
Thus $\big(\forall x\neg P\big(f(x,c)\big) \to Q\big(f(x,c), f(f(x,c), f(x, f(x,y))))\big)$ is an $\mathcal{L}$-Formula with $cp = \max\{2,0\} + 1 = 3$.

Question - 4.

Prove

Every $\mathcal{L}$-formula contains as many left parenthese as right parentheses.

Answer 4

Question - 5.

Let x, y, z be variables and $\mathcal{L} = \{f, P, Q, R\}$ where f is a unary function symbol, P is a binary predicate symbol, Q is a unary predicate symbol and R is a ternary predicate symbol.
For the following $\mathcal{L}$-formulae, ϕ, determine the corresponding set of variables that occur free in ϕ.

Question 5 a) - $\forall x\exists y(P(x,z) \to \neg Q(y)) \to \neg Q(y)$.

Answer 5 a) -

$$\begin{aligned}
& FV\big(\forall x\exists y(P(x,z) \to \neg Q(y)) \to \neg Q(y)\big) \\
= \ & FV((P(x,z) \to \neg Q(y)) \to \neg Q(y))\backslash\{x,y\} \\
= \ & [FV(P(x,z) \to \neg Q(y)) \cup FV(\neg Q(y))]\backslash\{x,y\} \\
= \ & [FV(P(x,z)) \cup FV(\neg Q(y)) \cup \{y\}]\backslash\{x,y\} \\
= \ & [\{x,z\} \cup FV(Q(y)) \cup \{y\}]\backslash\{x,y\} \\
= \ & [\{x,z\} \cup \{y\} \cup \{y\}]\backslash\{x,y\} \\
= \ & \{x,y,z\}\backslash\{x,y\} \\
= \ & \{z\}
\end{aligned}$$

Question 5 b) - $\forall x\forall y(Q(c) \wedge Q(f(x))) \to \forall y\forall x(Q(y) \wedge R(x,x,y))$.

Answer 5 b) -

$$
\begin{aligned}
& FV\big(\forall x\forall y(Q(c)\wedge Q(f(x)))\to\forall y\forall x(Q(y)\wedge R(x,x,y))\big)\\
= & FV(\forall x\forall y(Q(c)\wedge Q(f(x))))\cup FV(\forall y\forall x(Q(y)\wedge R(x,x,y)))\\
= & [FV(Q(c)\wedge Q(f(x)))\backslash\{x,y\}]\cup[FV(Q(y)\wedge R(x,x,y))\backslash\{x,y\}]\\
= & [[FV(Q(c))\cup FV(Q(f(x)))]\backslash\{x,y\}]\cup[[FV(Q(y))\cup FV(R(x,x,y))]\backslash\{x,y\}]\\
= & [[\emptyset\cup\{x\}]\backslash\{x,y\}]\cup[\{y\}\cup\{x,y\}]\backslash\{x,y\}]\\
= & \emptyset\cup\emptyset\\
= & \emptyset
\end{aligned}
$$

Question 5 c) - $Q(z)\longleftrightarrow\exists z(P(x,y)\wedge R(c,x,y))$.

Answer 5 c)
Consider

$$
\begin{aligned}
FV(\phi\longleftrightarrow\psi) &= FV[(\phi\to\psi)\wedge(\psi\to\phi)]\\
&= FV(\phi\to\psi)\cup FV(\psi\to\phi)\\
&= FV(\phi)\cup FV(\psi)\cup FV(\psi)\cup FV(\phi)\\
&= FV(\phi)\cup FV(\psi)
\end{aligned}
$$

Thus

$$
\begin{aligned}
& FV\big(Q(z)\longleftrightarrow\exists z(P(x,y)\wedge R(c,x,y))\big)\\
= & FV(Q(z))\cup FV(\exists z(P(x,y)\wedge R(c,x,y)))\\
= & \{z\}\cup[FV(P(x,y)\wedge R(c,x,y))\backslash\{z\}]\\
= & \{z\}\cup[FV(P(x,y))\cup FV(R(c,x,y))\backslash\{z\}]\\
= & \{z\}\cup\big[[\{x,y\}\cup\{x,y\}]\backslash\{z\}\big]\\
= & \{z\}\cup\{x,y\}\\
= & \{x,y,z\}
\end{aligned}
$$

Question 5 d) - Which of these formulae are $\mathcal{L}$-sentences?

Answer 5 d) - Only the formula from (b) is an $\mathcal{L}$-sentence since it is the only one with no free variables.