

Linear Algebra & Geometry - Notes

Dom Hutchinson

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1 Euclidean Plane, Vectors, Cartesian Co-Ordinates & Complex Numbers

1.1 Vectors

Definition 1.01 - Vectors

Vectors are ordered sets of real numbers.

Denoted by $\mathbf{v} = (v_1, v_2, v_3, \dots)$.

Definition 1.02 - Euclidean Plane

The set of two dimensional vectors, with real componenets, is called the Euclidean Plane.

Denoted by \mathbb{R}^2 .

Definition 1.03 - Vector Addition

Let $\mathbf{v}, \mathbf{w} \in \mathbb{R}^2$ such that $\mathbf{v} = (v_1, v_2)$ and $\mathbf{w} = (w_1, w_2)$. Then

$$\mathbf{v} + \mathbf{w} = (v_1 + w_1, v_2 + w_2)$$

Definition 1.03 - Scalar Multiplication of Vectors

Let $\mathbf{v} \in \mathbb{R}^2$ and $\lambda \in \mathbb{R}$ such that $\mathbf{v} = (v_1, v_2)$. Then

$$\lambda \mathbf{v} = (\lambda v_1, \lambda v_2)$$

Definition 1.04 - Norm of vectors

The norm of a vector is its length from the origin.

$$\|\mathbf{v}\| := \sqrt{v_1^2 + v_2^2}, \quad \mathbf{v} \in \mathbb{R}^2$$

Theorem 1.05 - Properties of the Norm

Let $\mathbf{v}, \mathbf{w} \in \mathbb{R}^2$ and $\lambda \in \mathbb{R}$ such that $\mathbf{v} = (v_1, v_2)$ and $\mathbf{w} = (w_1, w_2)$. Then

$$\|\mathbf{v}\| = 0 \text{ iff } \mathbf{v} = \mathbf{0}$$

$$\|\lambda \mathbf{v}\| = \sqrt{\lambda^2 v_1^2 + \lambda^2 v_2^2}$$

$$= |\lambda| \cdot \|\mathbf{v}\|$$

$$\|\mathbf{v} + \mathbf{w}\| \leq \|\mathbf{v}\| + \|\mathbf{w}\|$$

Definition 1.06 - Unit Vector

A vector can be described by its length & direction.

Let $\mathbf{v} \in \mathbb{R}^2 \setminus \{\mathbf{0}\}$.

Then $\mathbf{v} = \|\mathbf{v}\| \cdot \mathbf{u}$ where \mathbf{u} is the unit vector, $\mathbf{u} = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$

Thus $\forall \mathbf{v} \in \mathbb{R}^2, \mathbf{v} = \begin{pmatrix} \lambda \cos\theta \\ \lambda \sin\theta \end{pmatrix}$ for some $\lambda \in \mathbb{R}$ & $\mathbf{w} = (w_1, w_2)$.

Definition 1.07 - Dot Product

Let $\mathbf{v}, \mathbf{w} \in \mathbb{R}^2$ and $\lambda \in \mathbb{R}$ where $\mathbf{v} = (v_1, v_2)$ & $\mathbf{w} = (w_1, w_2)$.

Then $\mathbf{v} \cdot \mathbf{w} = v_1 \cdot w_1 + v_2 \cdot w_2$.

Remark 1.08 - Positivity of Dot Product

Let $\mathbf{v} \in \mathbb{R}^2$.

Then $\mathbf{v} \cdot \mathbf{v} = v_1^2 + v_2^2 = \|\mathbf{v}\|^2 \geq 0$.

Remark 1.09 - *Angle between vectors in Euclidean Plane*

Let $\mathbf{v}, \mathbf{w} \in \mathbb{R}^2$.

Set θ to be the angle between \mathbf{v} & \mathbf{w} .

Then

$$\cos(\theta) = \frac{\mathbf{v} \cdot \mathbf{w}}{\|\mathbf{v}\| \|\mathbf{w}\|}$$

.

Theorem 1.10 - *Cauchy-Schwarz Inequality*

Let $\mathbf{v}, \mathbf{w} \in \mathbb{R}^2$.

Then

$$|\mathbf{v} \cdot \mathbf{w}| \leq \|\mathbf{v}\| \|\mathbf{w}\|$$

Proof

$$\begin{aligned} \frac{v_1 w_1}{\|\mathbf{v}\| \|\mathbf{w}\|} + \frac{v_2 w_2}{\|\mathbf{v}\| \|\mathbf{w}\|} &\leq \frac{1}{2} \left(\frac{v_1^2}{\|\mathbf{v}\|^2} + \frac{w_1^2}{\|\mathbf{w}\|^2} \right) + \frac{1}{2} \left(\frac{v_2^2}{\|\mathbf{v}\|^2} + \frac{w_2^2}{\|\mathbf{w}\|^2} \right) \\ &\leq \frac{1}{2} \left(\frac{v_1^2 + v_2^2}{\|\mathbf{v}\|^2} + \frac{w_1^2 + w_2^2}{\|\mathbf{w}\|^2} \right) \\ &\leq \frac{1}{2} (1 + 1) \\ &\leq 1 \\ \Rightarrow |v_1 w_1 + v_2 w_2| &\leq \|\mathbf{v}\| \|\mathbf{w}\| \\ |\mathbf{v} \cdot \mathbf{w}| &\leq \|\mathbf{v}\| \|\mathbf{w}\| \end{aligned}$$

1.2 Complex Numbers**Definition 1.11** - i

$$\begin{aligned} i^2 &= -1 \\ i &= \sqrt{-1} \end{aligned}$$

Definition 1.12 - *Complex Number Set*

The set of *complex numbers* contains all numbers with an imaginary part.

$$\mathbb{C} := \{x + iy; x, y \in \mathbb{R}\}$$

Complex numbers are often denoted by

$$z = x + iy$$

and we say x is the real part of z and y the imaginary part.

Definition 1.13 - *Complex Conjugate*

Let $z \in \mathbb{C}$ st $z = x + iy$. Then

$$\bar{z} := x - iy$$

Theorem 1.14 - *Operations on Complex Numbers*

Let $z_1, z_2 \in \mathbb{C}$ st $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$. Then

$$\begin{aligned} z_1 + z_2 &:= (x_1 + x_2) + i(y_1 + y_2) \\ z_1 \cdot z_2 &:= (x_1 + iy_1)(x_2 + iy_2) \\ &:= x_1 \cdot x_2 - y_1 \cdot y_2 + i(x_1 \cdot y_2 + x_2 \cdot y_1) \end{aligned}$$

N.B. When dividing by a complex number, multiply top and bottom by the complex conjugate.

Definition 1.15 - *Modulus of Complex Numbers*

The *modulus* of a complex number is the distance of the number, from the origin, on an Argand diagram. Let $z \in \mathbb{C}$ st $z = x + iy$. Then

$$|z| := \sqrt{x^2 + y^2} = \sqrt{\bar{z}z}$$

N.B. *Amplitude* is an alternative name for the modulus

Definition 1.16 - *Phase of Complex Numbers*

The phase of a complex number is the angle between the positive real axis and the line subtended from the origin and the number, on an Argand diagram.

$$z = |z| \cdot (\cos\theta + i \cdot \sin\theta), \quad \theta = \text{Phase}$$

N.B. (Phase of \bar{z}) = - (Phase of z)

Theorem 1.17 - *de Moivre's Formula*

$$z^n = (\cos(\theta) + i \cdot \sin(\theta))^n = \cos(n\theta) + i \cdot \sin(n\theta)$$

Theorem 1.18 - *Euler's Formula*

$$e^{i\theta} = \cos(\theta) + i \cdot \sin(\theta)$$

Remark 1.19

Using Euler's formula we can express all complex numbers in terms of e . Thus many properties of the exponential remain true:

$$\begin{aligned} z &= \lambda e^{i\theta}, & \lambda \in \mathbb{R}, \theta \in [0, 2\pi) \\ \Rightarrow z_1 + z_2 &= \lambda_1 \cdot \lambda_2 \cdot e^{i(\theta_1 + \theta_2)} \\ \&, \frac{z_1}{z_2} &= \frac{\lambda_1}{\lambda_2} \cdot e^{i(\theta_1 - \theta_2)} \end{aligned}$$

2 Euclidean Space, \mathbb{R}^n

Definition 2.01 - *Euclidean Space*

Let $n \in \mathbb{N}$ then $\forall \mathbf{x} = (x_1, x_2, \dots, x_n)$ with $x_1, x_2, \dots, x_n \in \mathbb{R}$ then $\mathbf{x} \in \mathbb{R}^n$.

Theorem 2.02 - *Operations in Euclidean Space*

Let $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ and $\lambda \in \mathbb{R}$. Then

$$\mathbf{x} + \mathbf{y} = (x_1 + y_1, \dots, x_n + y_n)$$

And

$$\mathbf{x} + \lambda \cdot \mathbf{y} = (x_1 + \lambda \cdot y_1, \dots, x_n + \lambda \cdot y_n)$$

Definition 2.03 - *Cartesian Product*

Let $A, B \in \mathbb{R}^n$ be non-empty sets.

Then

$$A \times B := \{(a, b); a \in A, b \in B\}$$

2.1 Dot Product

Definition 2.04 - Dot Product

Let $\mathbf{v}, \mathbf{w} \in \mathbb{R}^n$. Then

$$\begin{aligned}\mathbf{v} \cdot \mathbf{w} &:= v_1.w_1 + \dots + v_n.w_n \\ &:= \sum_{j=1}^n v_j.w_j\end{aligned}$$

Theorem 2.05 - Properties of the Dot Product

Let $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^n$.

Linearity:

$$(\mathbf{u} + \lambda \mathbf{v}) \cdot \mathbf{w} = \mathbf{u} \cdot \mathbf{w} + \lambda(\mathbf{v} \cdot \mathbf{w})$$

Symmetry:

$$\mathbf{v} \cdot \mathbf{w} = \mathbf{w} \cdot \mathbf{v}$$

Positivity:

$$\mathbf{v} \cdot \mathbf{v} = v_1^2 + v_2^2 + \dots + v_n^2 \geq 0$$

Definition 2.06 - Orthogonality

Let $\mathbf{v}, \mathbf{w} \in \mathbb{R}^n$.

It is said that \mathbf{v}, \mathbf{w} are *orthogonal* to each other if $\mathbf{v} \cdot \mathbf{w} = 0$

N.B. Orthogonal vectors are perpendicular to each other.

Definition 2.07 - The Norm

Let $\mathbf{x} \in \mathbb{R}^n$. Then

$$\|\mathbf{x}\| = \sqrt{\mathbf{x} \cdot \mathbf{x}} = \sqrt{\sum_{i=1}^n x_i^2}$$

Theorem 2.08 - Properties of the Norm

Let $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ and $\lambda \in \mathbb{R}$. Then

$$\begin{aligned}\|\mathbf{x}\| &\geq 0 \\ \|\mathbf{x}\| &= 0 \text{ iff } \mathbf{x} = \mathbf{0} \\ \|\lambda \mathbf{x}\| &= |\lambda| \|\mathbf{x}\| \\ \|\mathbf{x} + \mathbf{y}\| &\leq \|\mathbf{x}\| + \|\mathbf{y}\|\end{aligned}$$

Theorem 2.09 - Dot Product and Norm

Let $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$.

$$|\mathbf{x} \cdot \mathbf{y}| \leq \|\mathbf{x}\| \|\mathbf{y}\|$$

N.B. $|\mathbf{x} \cdot \mathbf{y}| = \|\mathbf{x}\| \|\mathbf{y}\|$ iff \mathbf{x} & \mathbf{y} are orthogonal.

Theorem 2.10 - Angle between Vectors

Let $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$. Then

$$\cos(\theta) = \frac{\mathbf{x} \cdot \mathbf{y}}{\|\mathbf{x}\| \|\mathbf{y}\|}$$

2.2 Linear Subspaces

Definition 2.11 - Linear Subspace

Let $V \subset \mathbb{R}^n$.

V is a *Linear Subspace* if

- i) $V \neq \emptyset$;
- ii) $\forall \mathbf{v}, \mathbf{w} \in V$ then $\mathbf{v} + \mathbf{w} \in V$; and
- iii) $\forall \lambda \in \mathbb{R}, \mathbf{v} \in V$ then $\lambda \mathbf{v} \in V$.

Definition 2.12 - Span

Let $\mathbf{x}_1, \dots, \mathbf{x}_k \in \mathbb{R}^n, k \in \mathbb{N}$. Then

$$\text{span}\{\mathbf{x}_1, \dots, \mathbf{x}_k\} := \{\lambda_1 \mathbf{x}_1 + \dots + \lambda_k \mathbf{x}_k; \lambda_i \in \mathbb{R}, 0 \leq i \leq k\}$$

Theorem 2.13 - Spans are Subspaces

Let $\mathbf{x}_1, \dots, \mathbf{x}_k \in \mathbb{R}^n; k \in \mathbb{N}$. Then $\text{span}\{\mathbf{x}_1, \dots, \mathbf{x}_k\}$ is a linear subspace of \mathbb{R}^n .

Theorem 2.14

$$W_{\mathbf{a}} := \{\mathbf{x} \in \mathbb{R}^n; \mathbf{x} \cdot \mathbf{a} = 0\} \text{ is a subspace.}$$

Definition 2.15 - Orthogonal Complement

Let $V \subset \mathbb{R}^n$. Then

$$V^\perp := \{\mathbf{x} \in \mathbb{R}^n; \mathbf{x} \cdot \mathbf{y} = 0 \forall \mathbf{y} \in V\}$$

N.B. $V^\perp \subset \mathbb{R}^n$.

Theorem 2.16 - Relationship of Subspaces

Let $V, W \subset \mathbb{R}^n$. Then

$$V \cap W \text{ is a subspace and}$$

$$V + W := \{\mathbf{v} + \mathbf{w}; \mathbf{v} \in V, \mathbf{w} \in W\} \text{ is a subspace.}$$

Definition 2.17 - Direct Sum

Let V_1, V_2, W be subspaces of \mathbb{R}^n . Then W is said to be a *direct sum* if

- i) $W = V_1 + V_2$; and,
- ii) $V_1 \cap V_2 = \emptyset$.

3 Linear Equations & Matrices

3.1 Linear Equations

Definition 3.01 - Multi-Variable Linear Equations

Linear equations produce a straight line and can have multiple variables.

Examples $x = 3, y = x + 3, z = 5x - 2y$

Definition 3.02 - Systems of Linear Equations

Let $\mathbf{a}, \mathbf{x} \in \mathbb{R}^n$ & $b \in \mathbb{R}$ such that $\mathbf{a} \cdot \mathbf{x} = b$.

$\mathbf{a} \cdot \mathbf{x} = b$ is a linear equation in S with $S(\mathbf{a}, b) = \{\mathbf{x}; \mathbf{a} \cdot \mathbf{x} = b\}$ as the set of solutions.

N.B. If $b = 0$ then $S(\mathbf{a}, 0)$ is a subspace.

3.2 Matrices

Definition 3.03 - Matrix

Let $m, n \in \mathbb{N}$, then a $m \times n$ grid of numbers form an ' m by n ' matrix. Each element of the matrix can be reference by a_{ij} with $i = 1, \dots, m$ and $j = 1, \dots, n$.

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}$$

N.B. m & i = rows; n & j = columns

Definition 3.04 - Sets of Matrices

$M_{m,n}(\mathbb{R})$ is the set of $m \times n$ matrices containing only real elements.

$M_{m,n}(\mathbb{Z})$ is the set of $m \times n$ matrices containing only integer elements.

$M_n(\mathbb{R})$ is the set square matrices, size n , containing only real elements.

Definition 3.05 - Transpose Vectors

Let $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$ then $\mathbf{x}^t = (x_1 \ x_2 \ \dots \ x_n)$

Definition 3.06 - Vector-Matrix Multiplication

Let $A \in \mathbb{R}_{m,n}$ and $\mathbf{x} \in \mathbb{R}^n$ then

$$A\mathbf{x} := \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n \end{pmatrix} = \begin{pmatrix} \mathbf{a}_1^t \cdot \mathbf{x} \\ \mathbf{a}_2^t \cdot \mathbf{x} \\ \vdots \\ \mathbf{a}_m^t \cdot \mathbf{x} \end{pmatrix} \in \mathbb{R}^m$$

This can be simplified to

$$\mathbf{y} = A\mathbf{x} \text{ with } y_i = \sum_{j=1}^n a_{ij}x_j$$

Theorem 3.07 - Operations on Matrices with Vectors

$$\text{i) } A(\mathbf{x} + \mathbf{y}) = A\mathbf{x} + A\mathbf{y}, \quad \forall \mathbf{x}, \mathbf{y} \in \mathbb{R}^n.$$

$$\text{ii) } A(\lambda\mathbf{x}) = \lambda(A\mathbf{x}), \quad \forall \mathbf{x} \in \mathbb{R}^n, \lambda \in \mathbb{R}.$$

Theorem 3.08 - Composition of Matrices

Let $A = (a_{ij}) \in M_{m,n}(\mathbb{R})$ and $B = (b_{ij}) \in M_{l,m}(\mathbb{R})$.

Then there exists a $C = (c_{ij}) \in M_{l,n}(\mathbb{R})$ such that

$$C\mathbf{x} = B(A\mathbf{x}), \quad \forall \mathbf{x} \in \mathbb{R}^n$$

$$\text{N.B. } c_{ij} = \sum_{k=1}^m b_{ik}a_{kj}$$

Theorem 3.09 - Operations with Matrices

Let $A, B \in M_{m,n}$ and $C \in M_{l,m}$

- i) $C(A + B) = CA + CB$;
- ii) $(A + B)C = AC + BC$; and,
- iii) Let $D \in M_{m,n}$, $E \in M_{n,l}$ & $F \in M_{l,k}$ then

$$E(FG) = (EF)G$$

N.B. $AB \neq BA$

Definition 3.10 - *Types of Matrix*

Upper Triangle - $\begin{pmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{pmatrix}$, $a_{ij} = 0$ if $i > j$.

Lower Triangle - $\begin{pmatrix} 1 & 0 & 0 \\ 2 & 3 & 0 \\ 4 & 5 & 6 \end{pmatrix}$, $a_{ij} = 0$ if $i < j$.

Symmetric Matrix - $\begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 0 \\ 3 & 0 & 1 \end{pmatrix}$, $a_{ij} = a_{ji}$.

Anti-Symmetric - $\begin{pmatrix} 1 & -2 & -3 \\ 2 & 0 & -4 \\ 3 & 4 & -1 \end{pmatrix}$, $a_{ij} = -a_{ji}$.

Definition 3.11 - *Transposed Matrices*

Let $A = (a_{ij}) \in M_{m,n}(\mathbb{R})$ then the transpose of A , A^t , is an element of $M_{n,m}(\mathbb{R})$.

$$A^t := (a_{ji})$$

Theorem 3.12 - *Transpose Matrix Multiplication*

Let $A \in M_{m,n}(\mathbb{R})$, $\mathbf{x} \in \mathbb{R}^n$ & $\mathbf{y} \in \mathbb{R}^m$. Then

$$\mathbf{y} \cdot A\mathbf{x} = (A^t\mathbf{y}) \cdot \mathbf{x}$$

Theorem 3.10 - *Transposing Multiplied Matrices*

$$(AB)^t = B^t A^t$$

3.3 Structure of Set of Solutions

Definition 3.13 - *Set of Solutions*

Let $A \in M_{m,n}(\mathbb{R})$ and $\mathbf{b} \in \mathbb{R}^m$. Then

$$S(A, \mathbf{b}) := \{\mathbf{x} \in \mathbb{R}^n; A\mathbf{x} = \mathbf{b}\}$$

Definition 3.14 - *Homogenous Solutions*

The system of $S(A, \mathbf{0})$ is said to be *homogenous*.

All other systems are *inhomogenous*. N.B. - $S(A, \mathbf{0})$ is a linear subspace.

Theorem 3.15 - *Using Homogenous Solutions*

Let $A \in M_{m,n}(\mathbb{R})$ and $\mathbf{b} \in \mathbb{R}^m$. Let $\mathbf{x}_0 \in \mathbb{R}^n$ such that $A\mathbf{x}_0 = \mathbf{b}$, then

$$S(A, \mathbf{b}) = \mathbf{x}_0 + S(A, \mathbf{0})$$

Remark 3.16 - *Systems of Linear Equations as Matrices*

The system of linear equations $3x + z = 0, y - z = 1$ & $3x + y = 1$ can be represented by a matrix and a vector.

$$A = \begin{pmatrix} 3 & 0 & 1 \\ 0 & 1 & -1 \\ 3 & 1 & 0 \end{pmatrix}, \mathbf{b} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

3.4 Solving Systems of Linear Equations

Systems of linear equations can be displayed as matrices which can be reduced and solved by a technique called *Gaussian Elimination*.

Theorem 3.17 - *Operations on Linear Equations*

There are certain operations that can be performed on a system of linear equations without changing the result:

- i) Multiply an equation by a non-zero constant;
- ii) Add a multiple of any equation to another equation; and,
- iii) Swap any two equations.

Definition 3.18 - *Augmented Matrices*

Let $A\mathbf{x} = \mathbf{b}$ be a system of linear equations.

The associated *Augmented Matrix* is

$$(A \ \mathbf{b}) \in M_{m,n+1}(\mathbb{R})$$

Theorem 3.19 - *Elementary Row Operations*

From *Theorem 3.17* we can deduce certain operations that can be performed on an *Augmented Matrix* which do not alter the solutions:

- i) Multiply a row by a non-zero constant, $\text{row } i \rightarrow \lambda(\text{row } i)$;
- ii) Add a multiple of any row to another row, $\text{row } i \rightarrow \text{row } i + \lambda(\text{row } j)$; and,
- iii) Swap two rows, $\text{row } i \leftrightarrow \text{row } j$.

Definition 3.20 - *Row Echelon Form*

A matrix is in *Row Echelon Form* if:

- i) The left-most non-zero value in each row is 1; And,
- ii) The leading 1 in each row is one place to the right of the leading 1 in the row below.

Example

$$\begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix}$$

Definition 3.20 - *Reduced Row Echelon Form*

A matrix is in *Reduced Row Echelon Form* if:

- i) The matrix is in *row echelon form*; and,

- ii) All values in a row, except the leading 1, are 0.

Example

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Theorem 3.21 - Gaussian Elimination

Gaussian Elimination is a technique used to solve systems of linear equations. *Example*
Solve $x + y + 2z = 9$, $2x + 4y - 3z = 1$, $3x + 6y - 5z = 0$.

$$\begin{aligned} \text{Augmented Matrix} &= \begin{pmatrix} 1 & 1 & 2 & 9 \\ 2 & 4 & -3 & 1 \\ 3 & 6 & -5 & 0 \end{pmatrix} \\ \text{By EROS} &= \begin{pmatrix} 1 & 1 & 2 & 9 \\ 2 & 4 & -3 & 1 \\ 3 & 6 & -5 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 2 & 9 \\ 0 & 2 & -7 & -17 \\ 0 & 3 & -11 & 27 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 1 & 2 & 9 \\ 0 & 2 & -7 & -17 \\ 0 & 1 & -4 & -10 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 1 & 2 & 9 \\ 0 & 1 & -4 & -10 \\ 0 & 2 & -7 & -17 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 1 & 2 & 9 \\ 0 & 1 & -4 & -10 \\ 0 & 0 & 1 & 3 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 & 6 & 19 \\ 0 & 1 & -4 & -10 \\ 0 & 0 & 1 & 3 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{pmatrix} \\ &=> \underline{x = 1, y = 2, z = 3} \end{aligned}$$

3.5 Elementary Matrices & Inverting Matrices

Definition 3.22 - Invertible Matrices

A matrix, $A \in M_{m,n}(\mathbb{R})$, is said to be *invertible* if there exists $A^{-1} \in M_{n,m}(\mathbb{R})$ such that

$$AA^{-1} = I$$

N.B. - If a matrix is not invertible then it is *singular*.

Definition 3.23 - Elementary Matrices

A matrix, $E \in M_{m,n}(\mathbb{R})$, is said to be an *Elementary Matrix* if it can be obtained by performing Elementary Row Operations on a square identity matrix.

Example $\begin{pmatrix} 0 & \lambda \\ \mu & 0 \end{pmatrix}$

Remark 3.24

All elementary matrices are invertible.

Remark 3.25

Let A be a matrix, and B be a matrix which can be obtained from A by elementary row operations. Then there exists an elementary matrix, E , such that

$$B = EA$$

Theorem 3.26 - Finding A^{-1}

Let $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$, $B = \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix}$, $I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

Then by using EROS to change $(A \ I) \rightarrow (I \ B)$, B is the inverse of A .

Theorem 3.27 - Inverse of a 2×2 Matrix

Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ then

$$A^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \frac{1}{\det(A)} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

4 Linear Independence, Bases & Dimensions

4.1 Linear Independence & Dependence

Definition 4.01 - Linear Independence & Dependence

Vectors, $\mathbf{x}_1, \dots, \mathbf{x}_n \in \mathbb{R}^k$, are said to be *linearly dependent* if there exists non-zero real numbers, $\lambda_1, \dots, \lambda_n$, such that

$$\lambda_1 \cdot \mathbf{x}_1 + \dots + \lambda_n \cdot \mathbf{x}_n = \mathbf{0}$$

N.B. - If this is only true if $\lambda_1 = \dots = \lambda_n = 0$ then the vectors are said to be *linearly independent*.

Remark 4.02

Vectors are *linearly dependent* if at least one of them lies in the span of the rest.

4.2 Bases & Dimensions

Definition 4.03 - Basis

A *basis* is a set of vectors, $\mathbf{v}_1, \dots, \mathbf{v}_n \in V$ such that

- i) $V = \text{span}\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$; and,
- ii) $\mathbf{v}_1, \dots, \mathbf{v}_n$ are linearly independent.

Definition 4.04 - Standard Basis

The *standard basis* for a vector space is the set fewest unit vectors which span it.

Example - $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ are the standard basis for \mathbb{R}^3 .

Theorem 4.05 - Basis of a Linear Subspace

For all elements, \mathbf{v} , of a linear subspace, $V \subset \mathbb{R}^n$, there exists a unique set of numbers, $\lambda_1, \dots, \lambda_n$, such that

$$\mathbf{v} = \lambda_1 \cdot \mathbf{v}_1 + \dots + \lambda_n \cdot \mathbf{v}_n$$

Theorem 4.06 - Linear Independence and Bases

Let $V \subset \mathbb{R}^n$ be a linear subspace with basis $\mathbf{v}_1, \dots, \mathbf{v}_n$.

Suppose $\mathbf{w}_1, \dots, \mathbf{w}_k \in V$ are linearly independent, then $k \leq n$.

Definition 4.07 - Dimension

Let $V \subset \mathbb{R}^n$ be a linear subspace then the *dimension* of V , $\dim(V)$, is the fewest number vectors required to form a basis for V .

4.3 Orthogonal Bases

Definition 4.08 - Orthogonal

Let $V \subset \mathbb{R}^n$ be a linear subspace with $\{\mathbf{v}_1, \dots, \mathbf{v}_k\}$ as its basis.

This basis is an *orthogonal basis* if it satisfies

i) $\mathbf{v}_i \cdot \mathbf{v}_j = 0$ if $i \neq j$; and,

ii) $\mathbf{v}_i \cdot \mathbf{v}_i = 1$, $i = 1, \dots, k$.

N.B. - This can be generalised to $\mathbf{v}_i \cdot \mathbf{v}_k = \delta_{ij}$ with $\delta_{ij} := \begin{cases} 1, & i = j \\ 0, & \text{otherwise} \end{cases}$

Theorem 4.09

Let $V \subset \mathbb{R}^n$ be a linear subspace with an orthogonal basis $\{\mathbf{v}_1, \dots, \mathbf{v}_k\}$.

Then for all $\mathbf{u} \in V$

$$\mathbf{u} = (\mathbf{v}_1 \cdot \mathbf{u})\mathbf{v}_1, \dots, (\mathbf{v}_k \cdot \mathbf{u})\mathbf{v}_k$$

5 Linear Maps

Definition 5.01 - Linear Map

A map, $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is a *linear map* if

i) $T(\mathbf{x} + \mathbf{y}) = T(\mathbf{x}) + T(\mathbf{y})$, $\forall \mathbf{x}, \mathbf{y} \in \mathbb{R}^n$; and,

ii) $T(\lambda \mathbf{x}) = \lambda T(\mathbf{x})$, $\forall \mathbf{x} \in \mathbb{R}^n, \lambda \in \mathbb{R}$.

N.B. - If $m = n$ then T is referred to as a *linear operator*.

Theorem 5.02 - Properties of Linear Maps

Let $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ be a linear map. Then $T(\mathbf{0}) = \mathbf{0}$.

Definition 5.03 - Linear Maps as Matrices

Let $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ be a linear map. Then the associated Matrix is defined as

$$M_T := (t_{ij}) \in M_{m,n}(\mathbb{R})$$

with the elements of M_T defined by

$$t_{ij} := \mathbf{e}_i \cdot T(\mathbf{e}_j)$$

Theorem 5.04 - Solutions to Linear Maps from Matrices

Let $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ be a linear map and M_T be the associated matrix. Then

$$T(\mathbf{x}) = M_T \mathbf{x}, \quad \forall \mathbf{x} \in \mathbb{R}^n$$

5.1 Abstract Properties of Linear Maps

Theorem 5.05 - Relationship between Linear Maps

Let $S : \mathbb{R}^n \rightarrow \mathbb{R}^m$, $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ & $U : \mathbb{R}^m \rightarrow \mathbb{R}^k$ be linear maps, $\mathbf{x} \in \mathbb{R}^n$ and $\lambda \in \mathbb{R}$. Then

- i) $(\lambda T)(\mathbf{x}) = \lambda T(\mathbf{x})$;
- ii) $(S + T)(\mathbf{x}) = S(\mathbf{x}) + T(\mathbf{x})$; and,
- iii) $(U \circ S)(\mathbf{x}) = U(S(\mathbf{x}))$.

Definition 5.06 - Image & Kernel

Let $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ be a linear map. Then

The *image* of T is defined to be

$$Im(T) := \{\mathbf{y} \in \mathbb{R}^m : \exists \mathbf{x} \in \mathbb{R}^n \text{ st } T(\mathbf{x}) = \mathbf{y}\}$$

The *kernel* of T is defined to be

$$Ker(T) := \{\mathbf{x} \in \mathbb{R}^n : T(\mathbf{x}) = \mathbf{0}\}$$

Theorem 5.07 - Image & Kernel are Linear Subspaces

Let $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ be a linear map then $Im(T)$ is a linear subspace of \mathbb{R}^m and $Ker(T)$ is a linear subspace of \mathbb{R}^n

Remark 5.08

Let $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ be a linear map. Then

- i) T is surjective if $Im(T) = \mathbb{R}^m$; and,
- ii) T is injective if $Ker(T) = \{0\}$.

5.2 Matrices

Definition 5.09 - Linear Maps as Matrices

Let $S : \mathbb{R}^n \rightarrow \mathbb{R}^m$, $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ & $U : \mathbb{R}^m \rightarrow \mathbb{R}^k$ be linear maps and $\lambda \in \mathbb{R}$ with M_S, M_T & M_U as the corresponding matrices. Then

- i) $M_{\lambda T} = \lambda M_T = (\lambda t_{ij})$;
- ii) $M_{S+T} = (s_{ij} + t_{ij}) = M_S + M_T$; and,
- iii) $M_{U \circ S} = (r_{ij})$ where $r_{ik} = \sum_{j=1}^m s_{ij} t_{jk}$.

5.3 Rank & Nullity

Definition 5.10 - Rank & Nullity

Let $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ be a linear map. Then we define *Rank* of T by

$$rank(T) := dim(Im(T))$$

and we define *Nullity* of T by

$$nullity(T) := dim(Ker(T))$$

N.B. - For all linear maps, $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$,

$$nullity(T) + rank(T) = n$$

Remark 5.11

Let $T : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a linear map. Then T is invertible if

- i) $\text{rank}(T) = n$, or
- ii) $\text{nullity}(T) = 0$.

Theorem 5.12 - Relationship of Rank & Nullity between Linear Maps

Let $S : \mathbb{R}^n \rightarrow \mathbb{R}^m$ & $T : \mathbb{R}^k \rightarrow \mathbb{R}^n$ be linear maps. Then

- i) $S \circ T = 0$ iff $\text{Im}(T) \subset \text{Ker}(S)$;
- ii) $\text{rank}(S \circ T) \leq \text{rank}(T)$ and $\text{rank}(S \circ T) \leq \text{rank}(S)$;
- iii) $\text{nullity}(S \circ T) \geq \text{nullity}(T)$ and $\text{nullity}(S \circ T) \geq \text{nullity}(S) + k - n$; and,
- iv) S is invertible then $\text{rank}(S \circ T) = \text{rank}(T)$ and $\text{nullity}(S \circ T) = \text{nullity}(T)$.

6 Determinants

6.1 Definition & Basic Properties

Definition 6.01 - Determinant Function

A determinant function $d_n : \mathbb{R}^n \times \dots \times \mathbb{R}^n \rightarrow \mathbb{R}$ is a function which satisfies three conditions:

- i) *Multilinearity* - $d_2(\lambda \mathbf{a}_1 + \mu \mathbf{b}, \mathbf{a}_2) = \lambda d_2(\mathbf{a}_1, \mathbf{a}_2) + \mu d_2(\mathbf{b}, \mathbf{a}_2)$;
- ii) *Antisymmetry* - $d_2(\mathbf{a}_1, \mathbf{a}_2) = -d_2(\mathbf{a}_2, \mathbf{a}_1)$; and,
- iii) *Normalisation* - $d_2(\mathbf{e}_1, \mathbf{e}_2) = 1$.

N.B. - Determinant functions only exist for *square matrices*.

Theorem 6.02 - Properties of Determinant

- i) $\det[\dots, \mathbf{a}_i + \lambda \mathbf{a}_i, \dots] = \det[\dots, \mathbf{a}_i, \dots] + \lambda \det[\dots, \mathbf{a}_i, \dots]$;
- ii) If A has two identical columns then $\det(A) = 0$;
- iii) If A has an all zero column then $\det(A) = 0$; and,
- iv) $\det[\dots \mathbf{a}_i \dots \mathbf{a}_j \dots] = \det[\dots (\mathbf{a}_i + \lambda \mathbf{a}_j) \dots \mathbf{a}_j \dots]$

Theorem 6.03

Let $f_n : \mathbb{R}^n \times \dots \times \mathbb{R}^n \rightarrow \mathbb{R}$ be a function which is *multilinear* & *Antisymmetric* then

$$f_n(A) = C \cdot \det(A)$$

where C is a constant such that $C = f_n(\mathbf{e}_1, \dots, \mathbf{e}_n)$.

Theorem 6.04 - Determinant of a Triangle Matrix

Let $A = (a_{ij}) \in M_n(\mathbb{R})$ be an upper triangle matrix, so $a_{ij} = 0$ if $i > j$. Then

$$\det(A) = a_{11} \cdot a_{22} \cdot \dots \cdot a_{nn}$$

N.B. - The same is true for lower triangle matrices.

Theorem 6.05 - Relationship between Determinants

Let $A, B \in M_n(\mathbb{R})$ then

$$\det(AB) = \det(A) \cdot \det(B)$$

but usually

$$\det(A + B) \neq \det(A) + \det(B)$$

Theorem 6.06 - Determinant & the Inverse Matrix

If $\det(A) = 0$ then A^{-1} does not exist.

Theorem 6.07 - Leibniz Formula

Let $A = (a_{ij}) \in M_n(\mathbb{R})$ then the *Leibniz Formula* states that

$$\det(A) := \sum_{\sigma \in S_n} \text{sign}(\sigma) \prod_{j=1}^n a_{\sigma(j),j}$$

Where

- S_n is the group of symmetries for a regular n-sided polygons;
- $\text{sign}(\sigma)$ is the sign function which returns +1 for even permutations and -1 for odd permutations.
A permutation is even if a even number of permutations (swaps) are required to change the identity permutation to the given permutation, σ .

Remark 6.08 - Determinant of Transpose

Let A be a square matrix, then

$$\det(A) = \det(A^t)$$

6.2 Computing Determinant

Theorem 6.09 - Laplace's Rule

Let $A \in M_n$ then

$$\det(A) = \sum_{j=1}^n a_{ij} \cdot (-1)^{i+j} \cdot \det(A_{ij})$$

where A_{ij} is the $(n-1) \times (n-1)$ matrix formed when row i and column j are removed from A .

Example Let $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ then $A_{11} = (4)$ and $A_{12} = (2)$

Definition 6.10 - Adjunct Matrices

Let $A, B \in M_n$ be defined such that $b_{ij} = (-1)^{i+j} \cdot \det(A_{ij})$ then B is said to be *adjunct* to A .

This means

$$AB = \begin{pmatrix} \det(A) & 0 & \dots & 0 \\ 0 & \det(A) & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \det(A) \end{pmatrix} = \det(A)I$$

Remark 6.11 - Determinant of Triangle Matrices

If A is an upper triangle matrix ($a_{ij} = 0$ if $i > j$) then

$$\det(A) = a_{11} \times \dots \times a_{nn}$$

If A is a lower triangle matrix ($a_{ij} = 0$ if $i < j$) then

$$\det(A) = a_{11} \times \dots \times a_{nn}$$

6.3 Applications of Determinant

Theorem 6.12 - Linear Equations as Matrices

A system of m linear equations, each with n variables, can be written as

$$A\mathbf{x} = \mathbf{b}, \quad A \in M_{mn}(\mathbb{R}), \mathbf{x} \in \mathbb{R}^n, \mathbf{b} \in \mathbb{R}^m$$

If $\det(A) \neq 0$ then we can find an $A^{-1} \in M_{n,m}$ such that

$$\mathbf{x} = A^{-1}\mathbf{b}$$

Theorem 6.13

Let $A \in M_n(\mathbb{R})$ where $\det(A) \neq 0$ then

$$A^{-1} = \frac{1}{\det(A)} \text{adj } A$$

Theorem 6.14 - Cramer's Rule

Consider $A\mathbf{x} = \mathbf{b}$ then

$$x_j = \frac{\det(A_j)}{\det(A)}$$

where A_j is the matrix A , but the j^{th} column has been replaced by \mathbf{b} .

Definition 6.15 - Cross Product

Let $\mathbf{x}, \mathbf{y} \in \mathbb{R}^3$ be in the same plane then we define the *cross product* by

$$\mathbf{x} \times \mathbf{y} := \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix} = \begin{pmatrix} x_2 y_3 - x_3 y_2 \\ x_3 y_1 - x_1 y_3 \\ x_1 y_2 - x_2 y_1 \end{pmatrix}$$

Theorem 6.16 - Properties of Cross Product

- i) $\mathbf{x} \cdot (\mathbf{y} \times \mathbf{z}) = \mathbf{z} \cdot (\mathbf{x} \times \mathbf{y}) = \mathbf{y} \cdot (\mathbf{z} \times \mathbf{x})$;
- ii) $\mathbf{x} \times \mathbf{y} = -\mathbf{y} \times \mathbf{x}$;
- iii) $\mathbf{x} \times \mathbf{x} = \mathbf{0}$;
- iv) $(\mathbf{x} + \lambda \mathbf{y}) \times \mathbf{z} = (\mathbf{x} \times \mathbf{z}) + (\lambda \mathbf{y} \times \mathbf{z})$; and,
- v) $\|\mathbf{x} \times \mathbf{y}\|^2 = \|\mathbf{x}\|^2 \|\mathbf{y}\|^2 - (\mathbf{x} \cdot \mathbf{y})^2$.

Theorem 6.17 - Cross Product and Angle between vectors

Let θ be the angle between two vectors then

$$\|\mathbf{x} \times \mathbf{y}\|^2 = \|\mathbf{x}\|^2 \|\mathbf{y}\|^2 \sin^2(\theta)$$

Theorem 6.18 - Cross Product with Matrices

Let $A \in M_n(\mathbb{R})$ where $\det(A) \neq 0$ then

$$(A\mathbf{x}) \times (A\mathbf{y}) = [\det(A)](A^t)^{-1}(\mathbf{x} \cdot \mathbf{y})$$

7 Vector Spaces

7.1 Groups & Fields

Definition 7.01 - Group

A group, G , is a combination of a set and a map from $G \times G \rightarrow G$. The map must obey the following rules:

- i) *Associativity* - $f * (g * h) = (f * g) * h$;
- ii) *Identity Element* - $\exists e \in G$ st $\forall g \in G, eg = ge = g$; and,
- iii) *Inverse* - $\forall g \in G \exists g^{-1} \in G$ st $gg^{-1} = e = g^{-1}g$.

Definition 7.02 - Matrix Groups

The *General Linear Group*, $GL(n, \mathbb{R})$, is a group defined by

$$GL(n, \mathbb{R}) = \{A \in M_n(\mathbb{R}) : \det(A) \neq 0\}$$

The *identity element* is $I \in M_n$ and inverse is A^{-1} .

The *Special Linear Group*, $SL(n, \mathbb{R})$, is a group defined by

$$SL(n, \mathbb{R}) = \{A \in M_n(\mathbb{R}) : \det(A) = 1\}$$

The *Orthogonal Group*, $O(n, \mathbb{R})$, is a group defined by

$$O(n, \mathbb{R}) = \{A \in M_n(\mathbb{R}) : A^t = A^{-1}\}$$

The *Special Orthogonal Group*, $SO(n, \mathbb{R})$, is a group defined by

$$SO(n, \mathbb{R}) = \{A \in O(n, \mathbb{R}) : \det(A) = \pm 1\}$$

The *Borel Matrix*, $B(n, \mathbb{R})$, is the group of upper triangle matrices with non-zero values on the main diagonal.

The *Permutations Group*, $S(n, \mathbb{R})$, is a group of permutations of $\{1, 2, \dots, n\}$ defined by $n \times n$ matrix

$$\text{Example } \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}$$

Theorem 7.03 - Abelian Groups

Let G be a group. If $\forall g, h \in G, gh = hg$ then G is commutative and is called an *Abelian Group*. N.B. $e = 0$ is the identity element of all Abelian groups.

Definition 7.04 - Direct/Cartesian Product of a Group

Let G, H be groups with the same map. Then

$$G \times H := \{(g, h) : g \in G, h \in H\}$$

Definition 7.05 - Fields, \mathbb{F}

A field, \mathbb{F} , is a set with two binary operations: addition & multiplication.

Theorem 7.06 - Properties of Fields

- i) \mathbb{F} is an abelian group w.r.t addition;
- ii) $\mathbb{F} \setminus \{0\}$ is an abelian group w.r.t multiplication;
- iii) $(x + y).z = x.z + y.z$; and,
- iv) A field always contains 0 & 1.

7.2 Vector Spaces

Definition 7.07 - Vector Space, \mathbb{V}

\mathbb{V} is a (linear) vector space over a field, \mathbb{F} if:

- i) \mathbb{V} is an abelian group w.r.t addition;
- ii) $\forall \mathbf{v} \in \mathbb{V} \ \& \ \lambda \in \mathbb{F}, \lambda \mathbf{v} \in \mathbb{V}$;
- iii) $\lambda(\mathbf{u} + \mathbf{v}) = \lambda \mathbf{u} + \lambda \mathbf{v}$;
- iv) $\lambda(\mu \mathbf{v}) = (\lambda \mu) \mathbf{v}$; and,
- v) $1 \cdot \mathbf{v} = \mathbf{v}$.

Theorem 7.08 - Vector Spaces over Fields

Let W be a vector space over a field, \mathbb{F} , and U be a set. Then define

$$F(U, W) := f : U \rightarrow W$$

$F(U, W)$ is a vector space over \mathbb{F} .

This means $F(U, W)$ is linear so $\forall \lambda \in \mathbb{F} \ \& \ f, g \in F(U, W)$ then

$$(f + g)(u) = f(u) + g(u), \quad (\lambda f)(u) = \lambda f(u)$$

7.3 Subspace, Linear Combinations & Span

Definition 7.09 - Subspace

Let \mathbb{V} be a vector space over a field \mathbb{F} and $W \subset \mathbb{V}$, W is a subspace if it is a vector space for the operations inherited from \mathbb{V} .

Theorem 7.10 - Properties of Subspaces

Let \mathbb{V} be a vector space and $U \subset \mathbb{V}$ be a subspace, then U has the following properties:

- i) Not empty - $U \neq \emptyset$;
- ii) Closed under addition - $\forall u, v \in U; (u + v) \in U$; and,
- iii) Closed under multiplication - $\forall \lambda \in \mathbb{F}, u \in U; \lambda u \in U$.

Theorem 7.11 - Subsets of Subspaces

Let \mathbb{V} be a vector space over \mathbb{F} and $U, W \subset \mathbb{V}$ be subspaces.

Then $U \cap W$ is a subspace of \mathbb{V} .

Remark 7.12 - Linear Independence and Span

Let \mathbb{V} be a vector space over field, \mathbb{F} , and $S \subset \mathbb{V}$.

S is linearly dependent if there exists $v \in \mathbb{V}$ such that $\text{span}(S) = \text{span}(S \setminus \{v\})$.

Definition 7.13 - Finite Dimensional

Let \mathbb{V} be a vector space over \mathbb{F} .

\mathbb{V} is finitely dimensional if it is a span of a finite set, $S \subset \mathbb{V}$, of vectors.

N.B. - If a vector space is not finite dimensional, then it is *infinitely dimensional*.

Theorem 7.14

Let \mathbb{V} be a vector space over \mathbb{F} with $\mathbb{B}, U \subset \mathbb{V}$.

If \mathbb{B} is a basis for \mathbb{V} , with $|\mathbb{B}| < \infty$, and U is linearly independent then

$$|U| \leq |\mathbb{B}|$$

Theorem 7.15 - Linearly Independent Sets as Bases

Let \mathbb{V} be a vector space over \mathbb{F} with $U \subset \mathbb{V}$ as a linearly independent set. Then U can be extended to form a basis of \mathbb{V} .

7.4 Direct Sums**Definition 7.16 - Direct Sum**

Let \mathbb{V} be a vector space over \mathbb{F} and $U, W \subset V$ be subspaces with $U \cap W = \emptyset$ then

$$U \oplus W := U + W$$

This is the *direct sum* of U and W .

Theorem 7.17 - Dimension of Direct Sum

Let \mathbb{V} be a vector space over \mathbb{F} and $U, W \subset V$ be subspaces with $U \cap W = \emptyset$ then

$$\dim(U \oplus W) = \dim(U) + \dim(W)$$

Theorem 7.18 - Complement

Let \mathbb{V} be a vector space over \mathbb{F} and $U, W \subset V$ be subspaces with $U \cap W = \emptyset$ if

$$U \oplus W = V$$

then W is said to be the complement of U in V .

7.5 Rank-Nullity Theorem**Definition 7.19 - Rank & Nullity**

Let \mathbb{V} & \mathbb{W} be vector spaces over \mathbb{F} and $T : \mathbb{V} \rightarrow \mathbb{W}$ be a linear map. Then

$$\text{rank}(T) := \dim(\text{Im}(T)), \quad \text{nullity}(T) := \dim(\text{Ker}(T))$$

Theorem 7.20 - Rank-Nullity Theorem

Let \mathbb{V} & \mathbb{W} be vector spaces over \mathbb{F} and $T : \mathbb{V} \rightarrow \mathbb{W}$ be a linear map, with $\dim(\mathbb{V}) < \infty$ then

$$\text{Rank}(T) + \dim(\text{Ker}(T)) = \dim(\mathbb{V})$$

7.6 Projection**Definition 7.21 - Projection**

A linear map $P : \mathbb{V} \rightarrow \mathbb{V}$ is called a projection if $P^2 = P$.

Theorem 7.22 - Image of Projection

Let $P : V \rightarrow V$ be a projection then $v \in \text{Im}(P)$ iff $P(v) = v$.

Theorem 7.23 - Direct Sum of Projection

Let $P : V \rightarrow V$ be a projection then

$$V = \text{Ker}(P) \oplus \text{Im}(P)$$

7.7 Isomorphisms

Definition 7.24 - Isomorphisms

Let \mathbb{V} & \mathbb{W} be vector spaces over \mathbb{F} .

We say that the map $T : \mathbb{V} \rightarrow \mathbb{W}$ is an *isomorphism* between \mathbb{V} & \mathbb{W} if

- i) T is linear; and,
- ii) T is bijective.

N.B. - If an *isomorphism* exists between V & W , then they are said to be isomorphic.

Theorem 7.25 - Dimension of Isomorphic Spaces

Let V be a finitely dimensional vector space over \mathbb{F} .

If W is isomorphic to V then

$$\dim(V) = \dim(W)$$

This definition can be extended to say

If two vector spaces have the same dimension, then they are isomorphic.

Proposition 7.26 - Multiple Bases

Let $A = \{\mathbf{a}_1, \dots, \mathbf{a}_n\}$ and $B = \{\mathbf{b}_1, \dots, \mathbf{b}_n\}$ be different bases for V .

Define $T_A : \mathbb{F}^n \rightarrow V$ and $T_B : \mathbb{F}^n \rightarrow V$ such that

$$T_A(x_1, \dots, x_n) = x_1 \cdot \mathbf{a}_1 + \dots + x_n \cdot \mathbf{a}_n; \quad T_B(x_1, \dots, x_n) = x_1 \cdot \mathbf{b}_1 + \dots + x_n \cdot \mathbf{b}_n$$

Then for all $\mathbf{v} \in V$ there are two ways of expressing \mathbf{v} .

$$x_1 \cdot \mathbf{a}_1 + \dots + x_n \cdot \mathbf{a}_n = \mathbf{v} = x_1 \cdot \mathbf{b}_1 + \dots + x_n \cdot \mathbf{b}_n$$

Unless $A = B$ then $x_i \neq y_i$ for at least one $i \in \mathbb{N}, i \leq n$.

Theorem 7.27 - Conversion Matrices

Let A & B be different bases for vector space V , with $\dim(V) = n$.

Then an $n \times n$ matrix, C_{AB} , can be used to convert elements given in basis A to now be given in basis B .

Let $\mathbf{v} \in V$ and $\mathbf{x} = T_A(\mathbf{x})$ & $\mathbf{y} = T_B(\mathbf{x})$ then

$$\mathbf{y} = C_{AB}\mathbf{x}$$

Theorem 7.28 - General Relationship between Bases

Let V be a vector space over \mathbb{F} , with $\dim(V) = n$.

Let A & B be different bases for V with $A = \{\mathbf{a}_1, \dots, \mathbf{a}_n\}$ & $B = \{\mathbf{b}_1, \dots, \mathbf{b}_n\}$.

Then $\forall \mathbf{v} \in V$ we have that

$$\mathbf{v} = \sum_{i=1}^n v_i \cdot \mathbf{a}_i = \sum_{i=1}^n v_i \cdot \mathbf{b}_i$$

Let $C_{AB} = (c_{ij})$ be the conversion matrix from A to B then

$$v_j = \sum_{i=1}^n c_{ij} \mathbf{b}_i$$

Theorem 7.29 - Properties of Transition Matrices

Let $A, B, C \subset V$ all be different bases for V . Then

- i) $C_{AA} = I$;

ii) $C_{AB}C_{BA} = I$; and,

iii) $C_{CA}C_{AB} = C_{CB}$.

Theorem 7.30 - *Linear Maps between Vector Spaces as Matrices*

Let \mathbb{V} & \mathbb{W} both be vector spaces over \mathbb{F} , with $\dim(\mathbb{V}) = n$ and $\dim(\mathbb{W}) = m$, and $T : \mathbb{V} \rightarrow \mathbb{W}$ be a linear map.

Let $A = \{\mathbf{a}_1, \dots, \mathbf{a}_n\} \subset \mathbb{V}$ and $B = \{\mathbf{b}_1, \dots, \mathbf{b}_m\} \subset \mathbb{W}$ be bases for \mathbb{V} & \mathbb{W} respectively. Then we can define an $n \times m$ matrix

$$M_{AB}(T) = (m_{ij}) \in M_{n,m}(\mathbb{F})$$

Where m_{ij} are defined to satisfy

$$T(\mathbf{a}_j) = \sum_{i=1}^m m_{ij} \mathbf{b}_i$$

Then

$$\mathbf{w} = M_{AB}(T)\mathbf{v}$$

For $\mathbf{v} \in \mathbb{V}$ & $\mathbf{w} \in \mathbb{W}$.

Theorem 7.31 - *Change Basis of Linear Map*

Let V be a vector space over F and $U, W \subset V$ be different bases for V .

Define $T : V \rightarrow V$ be a linear map and C to be the transition matrix from basis $U \rightarrow W$.

Then C^{-1} is the transition matrix from $W \rightarrow U$.

Set A to be the matrix representation of T in basis U . Then

$$A' = C^{-1}AC$$

Where A' is the matrix representation of T in basis W .

8 Eigenvalues & Eigenvectors

8.1 Characteristic Polynomial

Definition 8.01 - *Eigenvectors & Eigenvalues*

Let $\mathbf{v} \in V \setminus \{\mathbf{0}\}$ and $T : V \rightarrow V$ be a linear operator.

\mathbf{v} is called an *eigenvector* of T if

$$T(\mathbf{v}) = \lambda \mathbf{v}, \quad \lambda \in \mathbb{F}$$

This λ is the associated *eigenvalue* for \mathbf{v} .

Definition 8.02 - *Spectrum*

The set of eigenvectors of a linear operator $T : V \rightarrow V$ is called the *spectrum* of T , generally denoted as

$$\text{Spec}(T) := \{\mathbf{v} \in V : T(\mathbf{v}) = \lambda \mathbf{v}, \lambda \in \mathbb{F}\}$$

Definition 8.03 - *Diagonalisable*

A linear operator is *diagonalisable* if there exists a basis of eigenvectors for it.

Remark 8.04 - *Finding Eigenvalues*

Let A be the matrix which represents a linear operator T , and \mathbf{x} be a general eigenvector for T

$$T(\mathbf{x}) = A\mathbf{x} = \lambda \mathbf{x} \Rightarrow (A - \lambda I)\mathbf{x} = \mathbf{0}$$

Then λ is an eigenvalue if it satisfies

$$\det(A - \lambda.I) = 0$$

Definition 8.05 - Characteristic Polynomial

The polynomial which is equivalent to $\det(A - \lambda.I)$ is called the *characteristic polynomial* of A .

$$p_A(\lambda) := \det(A - \lambda.I)$$

N.B. - λ is an eigenvalue for A if $p_A(\lambda) = 0$

Definition 8.06 - Eigenspace

Let $\lambda \in \mathbb{F}$ be an eigenvalue of T , then the corresponding *eigenspace* is defined as

$$V_\lambda := \ker(T - \lambda.I)$$

Remark 8.07 - Finding Eigenvectors

Once we have found all $\lambda_1, \dots, \lambda_k$ that satisfy $p_A(\lambda_i) = 0$ then we can find the eigenvectors, \mathbf{x}_i , of A

$$(A - \lambda.I)\mathbf{x}_i = \mathbf{0}$$

A good way to start is to produce the linear equations

$$\sum_{j=1}^n (A - \lambda.I)_{ij} \cdot x_j = 0$$

For all $i \leq n$. Then solve these, as a series of simultaneous equations, to find the values x_j which produce the eigenvector \mathbf{x} .

Repeat this process for all $\lambda_1, \dots, \lambda_k$ to find all eigenvectors for A .

Theorem 8.08 - Similar Characteristic Polynomial

Let C be an invertible matrix.

Define $A' = C^{-1}AC$ where A & A' are conjugate or similar.

Then $p_A(\lambda) = p_{A'}(\lambda)$.

Theorem 8.09 - Characteristic Polynomial & Basis

The characteristic polynomial for T is the same, regardless of the basis of T .

Definition 8.10 - Trace

Let $A \in M_n(\mathbb{F})$.

Then the *trace* of A is defined as

$$\text{Tr}(A) := \sum_{i=1}^n a_{ii}$$

N.B. - *Trace* is sometimes called *Spur*.

Remark 8.11

As the terms after the first term of the determinant of a matrix do not contribute to the powers of λ in the characteristic equation then we can ignore them. This means we can deduce that

$$p_T(\lambda) = (-1)^n \lambda^n + (-1)^{n-1}(\text{Tr}(A)) + \dots + \det(A)$$

Theorem 8.12 - Diagonalised Matrix

Let T be a *diagonalisable* matrix with eigenvalues $\{\lambda_1, \lambda_2, \dots, \lambda_n\}$.

Then T can be represented by

$$\Delta = \begin{pmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & & \vdots \\ \vdots & & \ddots & \vdots \\ 0 & \dots & \dots & \lambda_n \end{pmatrix}$$

N.B. - T can also be represented in any basis with, C as the transition matrix, by $C^{-1}\Delta C$.

Theorem 8.13 - *Relationship between Matrix and its Diagonalised Form*

Let T be a matrix and Δ be its diagonalised form, then

$$\text{Det}(T) = \text{Det}(\Delta) = \prod_{j=1}^n \lambda_j$$

And

$$\text{Tr}(T) = \text{Tr}(\Delta) = \sum_{j=1}^n \lambda_j$$

Theorem 8.14 - *Distinct Eigenvectors and Diagonalisability*

Eigenvectors, which correspond to distinct eigen values, are linearly independent.

Thus if a matrix, A , has only distinct eigenvalues then it is diagonalisable.

8.2 Roots of Characteristic Polynomial

Remark 8.15 - *Degree of Characteristic Equation*

Eigenvalues are roots of $p_A(\lambda) = 0$ where p_A is an equation of degree $\dim(A)$.

Remark 8.16 - *Non-Distinct Roots of Characteristic Equation*

If the roots of $P_A(\lambda)$ are not distinct then A may be diagonalisable depending on how many eigenvectors are found.

Theorem 8.17 - *Vieta's Theorem*

If $\lambda_1, \dots, \lambda_n$ are roots of the Polynomial

$$\lambda^n + a_1\lambda^{n-1} + \dots + a_n = 0 \equiv p(\lambda) = 0$$

Then $p(\lambda) = (\lambda - \lambda_1)(\lambda - \lambda_2) \dots (\lambda - \lambda_n)$.

So $p(\lambda)$ factorises in the product $\prod_{i=1}^n (\lambda - \lambda_i)$ but the λ_i s are not necessarily distinct.

Definition 8.18 - *Multiplicity of Roots*

Let $\lambda_1 \in \mathbb{C}$ of characteristic polynomial, $p(\lambda)$.

λ_1 has multiplicity $m_1 \in \mathbb{N}$ if

$$p(\lambda_1) = \frac{dp}{d\lambda}(\lambda_1) = \dots = \frac{d^{m_1-1}p}{d\lambda^{m_1-1}}(\lambda_1) = 0$$

This means that $(\lambda - \lambda_1)^{m_1}$ is a factor of $p(\lambda)$.

Definition 8.19 - *Geometric & Algebraic Multiplicity*

Let $\lambda \in \text{spec}(T)$ and V_λ be the corresponding eigenspace.

i) λ has *geometric multiplicity*, $m_g(\lambda) \in \mathbb{N}$, if $\dim(V_\lambda) = m_g(\lambda)$; and,

ii) λ has *algebraic multiplicity*, $m_a(\lambda) \in \mathbb{N}$, if λ has multiplicity m_a of $p_T(\lambda)$.

Theorem 8.20 - Relationship between Geometric & Algebraic Multiplicity

Let $\lambda \in \text{spec}(T)$ then

$$m_g(\lambda) \leq m_a(\lambda)$$

Theorem 8.21 -

Let T be a linear operator on an n dimensional space over \mathbb{C} or \mathbb{R} , with eigenvalues $\lambda_1, \dots, \lambda_n$, which are not necessarily distinct. Then

$$\det(T) = \prod_{i=1}^n \lambda_i \quad \& \quad \text{tr}(T) = \sum_{i=1}^n \lambda_i$$

9 Inner Product Spaces

9.1 Inner Product, Norm & Orthogonality

Definition 9.01 - Inner Product (Complex)

Let V be a vector space over \mathbb{C} .

An *inner product* on V is a map, $\langle V, V \rangle : V \times V \rightarrow \mathbb{C}$, with the following properties:

- i) $\langle v, v \rangle \geq 0$;
- ii) $\langle v, w \rangle = \overline{\langle v, w \rangle}$;
- iii) $\langle u, v + w \rangle = \langle u, v \rangle + \langle u, w \rangle$; and,
- iv) $\langle \lambda.u, v \rangle = \lambda \langle u, v \rangle$.

Where $u, v, w \in V$ and $\lambda \in \mathbb{C}$.

Definition 9.02 - Inner Product (Real)

Let V be a vector space over \mathbb{R} .

An *inner product* on V is a map, $\langle, \rangle : V \times V \rightarrow \mathbb{C}$, with the following properties:

- i) $\langle v, v \rangle \geq 0$;
- ii) $\langle v, w \rangle = \langle w, v \rangle$;
- iii) $\langle u, v + w \rangle = \langle u, v \rangle + \langle u, w \rangle$; and,
- iv) $\langle \lambda.u, v \rangle = \lambda \langle u, v \rangle$.

Where $u, v, w \in V$ and $\lambda \in \mathbb{C}$.

Definition 9.03 - Inner Product Space

Let V be a vector space with \langle, \rangle as a defined inner product are called an *inner product space*, denoted by

$$(V, \langle, \rangle)$$

N.B. - If V is over \mathbb{C} then this is called a *complex inner product space*. If V is over \mathbb{R} then this is called a *real inner product space*.

Definition 9.04 - Norm

Let (V, \langle, \rangle) be an inner product space, then we define the associated norm as

$$\|v\| := \sqrt{\langle v, v \rangle}, \quad v \in V$$

Definition 9.05 - Orthogonal

Let (V, \langle, \rangle) be an inner product space, then

- i) $v, w \in V$ are *orthogonal*, $v \perp w$, if $\langle v, w \rangle = 0$; and,
- ii) $U, W \subset V$ are *orthogonal*, $U \perp W$, if $u \perp w \forall u \in U \ \& \ v \in V$.

Definition 9.06 - Orthogonal Complement

Let (V, \langle, \rangle) be an inner product space and $W \subset V$.

The *orthogonal complement* is defined as

$$W^\perp := \{v \in V : v \perp w \forall w \in W\}$$

Theorem 9.07 - Norm of Orthogonal Elements

Let (V, \langle, \rangle) be an inner product space and $v, w \in V$ with $v \perp w$, then

$$\|v + w\|^2 = \|v\|^2 + \|w\|^2$$

Definition 9.08 - Orthonormal Basis

Let (V, \langle, \rangle) be an inner product space.

A basis, $\mathbb{B} = \{v_1, \dots, v_n\}$, is called an *orthonormal basis* if

$$\langle v_i, v_j \rangle = \delta_{ij} := \begin{cases} 1 & i = j; \\ 0 & i \neq j. \end{cases}$$

Theorem 9.09 - Properties of Orthogonal Basis

Let (V, \langle, \rangle) be an inner product space and $\mathbb{B} = \{v_1, \dots, v_n\}$ an orthonormal basis.

Then $\forall v, w \in V$,

- i) $v = \sum_{i=1}^n \langle v_i, v \rangle v_i$;
- ii) $\langle v, w \rangle = \sum_{i=1}^n \overline{\langle v_i, v \rangle} \langle v_i, w \rangle$; and,
- iii) $\|v\| = [\sum_{i=1}^n |\langle v_i, v \rangle|^2]^{1/2}$.

9.2 Construction of Orthonormal Basis**Theorem 9.10 - Inner Product of Vectors and Orthonormal Basis Elements**

Let $\{v_1, \dots, v_n\}$ be an orthonormal basis.

Define v such that $v = \sum_{j=1}^n x_j v_j$. Then

$$\langle v_i, v \rangle = \sum_{j=1}^n x_j \langle v_i, v_j \rangle = x_i$$

Theorem 9.11 - Inner Product of Two Vectors Over the Same Orthonormal Basis

Let $\{v_1, \dots, v_n\}$ be an orthonormal basis.

Define v & w such that $v = \sum_{j=1}^n x_j v_j$ & $w = \sum_{j=1}^n y_j v_j$. Then

$$\langle v, w \rangle = \sum_{i,j=1}^n \bar{x}_j \cdot y_i \cdot \langle v_j, v_i \rangle = \sum_{i=1}^n \bar{x}_i \cdot y_i$$

N.B. - This is the same formula as the dot product of x & y .

Definition 9.12 - Orthogonal Projection

Let (V, \langle, \rangle) be an inner product space & $P : V \rightarrow V$ be a linear operation.

P is called an *orthogonal projection* if

i) $P^2 = P$; and,

ii) $\langle Pv, w \rangle = \langle v, PW \rangle \forall v, w \in V$.

Proposition 9.13 - Common Orthogonal Projection

Let (V, \langle, \rangle) be an inner product space, $W \subset V$ be a subspace and $w_1, \dots, w_k \in W$ form an orthogonal basis.

Then a common orthogonal projection, $P_W : V \rightarrow V$, is defined by

$$P_W(v) := \sum_{i=1}^k \langle w_i, v \rangle w_i$$

Theorem 9.14 - Value of Inner Product

Let (V, \langle, \rangle) be an inner product space. Then

$$|\langle v, w \rangle| \leq \|v\| \cdot \|w\|$$

Theorem 9.15 - Forming an Orthogonal Basis

let (V, \langle, \rangle) be an inner product space, $\dim(V) = n$ and $u_1, \dots, u_n \in V$.

Then an orthogonal basis, $\{v_1, \dots, v_n\}$, can be formed following

$$v_1 = \frac{1}{\|u_1\|} \cdot u_1, \quad v_2 = \frac{1}{\|u_2 - \langle v_1, u_2 \rangle v_1\|} \cdot (u_2 - \langle v_1, u_2 \rangle v_1)$$

$$v_n = \frac{1}{\|u_n - (\sum_{i=1}^{n-1} \langle v_i, u_n \rangle v_i)\|} \cdot \left(u_n - \left(\sum_{i=1}^{n-1} \langle v_i, u_n \rangle v_i \right) \right)$$