Calculus 1 - Notes

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Contents

1 Before Calculus

1.1 Fundamental Theorem of Calculus

Definition 1.01 - *Fundamental Theorem of Calculus*
The Fundamental Theorem of Calculus states

$$\frac{d}{dx}\int_a^x f(t)dt = f(x)$$

Definition 1.02 - *Common Sets of Numbers*
Natural Numbers, set of positive integers - $\mathbb{N} := \{1, 2, 3, ...\}$.
Whole Numbers, set of all integers - $\mathbb{Z} := \{..., -2, -1, 0, 1, 2, ...\}$.
Rational Numbers, set of fractions - $\mathbb{Q} := \left\{\frac{p}{q} : p \in \mathbb{Z}, q \in \mathbb{N}\right\}$.
Real Numbers, set of all rational & irrational numbers - $\mathbb{R}$.

1.2 Intervals

Definition 1.03 - *Intervals*
Sets of real numbers that fulfil in given ranges.
Notation

$$\begin{aligned}
[a, b] &:= \{x \in \mathbb{R} : a \leq x \leq b\} \\
(a, b] &:= \{x \in \mathbb{R} : a < x \leq b\} \\
[a, b) &:= \{x \in \mathbb{R} : a \leq x < b\} \\
(a, b) &:= \{x \in \mathbb{R} : a < x < b\}
\end{aligned}$$

Example
In what interval does x lie such that:

$$|3x + 4| < |2x - 1|$$

Solution

$$\begin{aligned}
\text{Case 1} : x \geq \frac{1}{2}& \\
&=> 1 - 2x < 3x + 4 < 2x - 1 \\
&=> 1 - 2x < 3x + 4 \\
&=> x > \frac{-3}{5} \\
\text{And,} &=> 3x + 4 < 2x - 1 \\
&=> x < -5 \\
\text{There are no real solutions in this range.}& \\
\text{Case 2} : x < \frac{1}{2}& \\
&=> 2x - 1 < 3x + 4 < 1 - 2x \\
&=> 2x - 1 < 3x + 4 \\
&=> -5 < x
\end{aligned}$$

$$\begin{aligned} \text{And,} &=> 3x + 4 < 1 - 2x \\ &=> 5x < -3 \\ &=> x < \frac{-3}{5} \\ \\ &=> -5 < x < \frac{-3}{5},\ \underline{x \in \left(-5, \frac{-3}{5}\right)} \end{aligned}$$

Definition 1.04 - *Functions*
Functions map values between fields of numbers. The signature of a function is defined by

$$f : A \to B$$

Where f is the name of the function, A is the domain and B is the co-domain.
The *Domain* of a function is the set of numbers it can take as an input.
The *Co-Domain* is the set of numbers that the domain is mapped to.

N.B. - A function is valid iff it maps each value in the domain to a single value in the co-domain.

Definition 1.05 - *Maximal Domain*
The *Maximal Domain* of a function is the largest set of values which can serve as the domain of a function.

Remark 1.06 - *Types of Function*
Let $f : A \to B$
Polynomials

$$f(x) = a_0 + a_1 x + ... + a_n x^n$$

Rational

$$f(x) = \frac{p(x)}{q(x)}, \quad q(x) \neq 0 \ \forall \ x \in A$$

Trigonometric

$$sin(x),\ cos(x),\ tan(x) \text{ etc.}$$

2 Limits

2.1 Limits

Definition 2.01 - *Limits*
A limit is the value a function tends to, for a given x.
i.e. The value f(x) has at it gets very close to x.

Formally We say L is the limit of $f(x)$ as x tends to x_0 if

$$\forall \ \varepsilon > 0,\ \exists \ \delta > 0 \text{ st if } x \in A \text{ and } |x - x_0| < \delta => |f(x) - L| < \varepsilon$$

Notation

$$\lim_{x \to x_0} f(x) = L$$

Definition 2.02 - *Directional Limits*
Sometimes the value of a limit depends on which direction you approach it from.
$\lim_{x \to x_0+}$ is used when approaching from values greater than x_0.
$\lim_{x \to x_0-}$ is used when approaching from values less than x_0.

Theorem 2.03 - *Operations with limits*
Let $\lim_{x \to x_0} f(x) = L_f$ and $\lim_{x \to x_0} g(x) = L_g$ Then

$$\begin{aligned}
\lim_{x \to x_0} [f(x) + g(x)] &= L_f + L_g \\
\lim_{x \to x_0} f(x).g(x) &= L_f.L_g \\
\lim_{x \to x_0} \frac{f(x)}{g(x)} &= \frac{L_f}{L_g} \quad L_g \neq 0
\end{aligned}$$

2.2 Exponential Function

Definition 2.04 - *Exponential Function*

$$e := \lim_{x \to \infty} \left(1 + \frac{1}{n}\right)^n \simeq 2.7182818...$$

Theorem 2.05 - *Binomial Expansion*
A techique for expanding binomial expressions

$$\begin{aligned}
\left(1 + \frac{x}{n}\right)^n &= \sum_{i=0}^{n} \binom{i}{n} .1^{(n-i)}. \left(\frac{x}{n}\right)^i \\
&= 1 + x + \frac{n-1}{2n}.x^2 + ... + \frac{x^n}{n^n}
\end{aligned}$$

3 The Derivative

Definition 3.01 - *Differentiable Equations*
Let $f : A \to B$ and $x_0 \in A$.
f is differentiable at x_0 if $\exists\ L \in B$ such that

$$L = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

If this limit exists $\forall\ x \in A$ then we can define the derivative of $f(x)$

$$f'(x) := \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}$$

Definition 3.02 - *Notation for Differentiation*
There are two ways to denote the derivative of an equation

$$f'(x) \iff \frac{df}{dx}, f''(x) \iff \frac{d^2 f}{dx^2}, ..., f^{(n)}(x) \iff \frac{d^n f}{dx^n}$$

N.B. - Using $\frac{df}{dx}$ is more informative, especially for equations with multiple variables.

3.1 Techniques for finding derivative

Theorem 3.03 - *Sum Rule*
Let f, g be differentiable with respect to x.

$$(f+g)' = f' + g'$$

Theorem 3.04 - *Product Rule*
Let f, g be differentiable with respect to x.

$$(fg)' = f'g + fg'$$

Theorem 3.05 - *Quotient Rule*
Let f, g be differentiable with respect to x.

$$\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$$

Definition 3.06 - *Composite Functions*
Let $f : B \to C$ and $g : A \to B$ Then

$$(f \circ g)(x) = f(g(x))$$

Theorem 3.07 - *Chain Rule*
Let f, g be differentiable with respect to x.

$$\frac{d}{dx} f(g(x)) = f'(g(x)).g'(x)$$

3.2 Implicit Differentiation

Definition 3.08 - *Implicit Differentiation*
Sometimes it is hard to isolate variables in multi-variable equations, in these cases differentiate both sides with respect to the same variable.
Remembering

$$\frac{d}{dx}(x) = 1 \text{ and } \frac{d}{dx}(y) = \frac{dy}{dx} = y'$$

Example
Find y if $x^3 + y^3 = 6xy$

$$\begin{aligned} \frac{d}{dx}\left(x^3 + y^3\right) &= \frac{d}{dx}(6xy) \\ =>3x^2 + 3y^2.y' &= 6y + 6x.y' \\ =>y'(3y^2 - 6x) &= 6y - 3x^2 \\ =>y' &= \frac{2y - x^2}{y^2 - 2x} \end{aligned}$$

3.3 Applications of The Derivative

Thoerem 3.09 - *Netwon's Method*
Let f be differentiable. Using *Newton's Method* we can approximate a solution to $f(x) = 0$.

i) Take an inital guess, x_0;

ii) Find the value of x where the tangent to x_0 on $f(x)$ intercepts the x-axis;

iii) Use this value as the next guess;

iv) Repeat until the value of x reduces little.

The equation for the tangent is

$$y = f(x_0) + (x - x_0)f'(x_0)$$

so a simplified equation for the process can be deduced

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

Theorem 3.10 - *Angle between Intersecting Curves*
Let $y = f_1(x)$ and $y = f_2(x)$ be two curves which intersect at (x_0, y_0).
Then $y_0 = f_1(x_0) = f_2(x_0)$
Let m_1, m_2 be the gradient of the tangents to f_1 & f_2 at x_0.
Then $\theta_i := tan^{-1}(m_i)$ for $i = 1, 2$.
Let $\phi = |\theta_1 - \theta_2|$, then

$$\phi = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$$

Theorem 3.11 - *L'Hospital's Rule*
For two equations equations, f, g with limit of $-\infty, 0$ or ∞ as x tends to a, it is hard to solve

$$\lim_{x\to a} \frac{f(x)}{g(x)}$$

Provided the limit exists, L'Hospital's Rule states that

$$\lim_{x\to a} \frac{f(x)}{g(x)} \iff \lim_{x\to a} \frac{f'(x)}{g'(x)}$$

3.4 Sketching Curves

Remark 3.11 - *Sketching Curves*
Evaluating the derivative of a curve can make it easier to sketch:

i) When $f'(x) > 0$ the curve is heading upwards;

ii) When $f'(x) < 0$ the curve is heading downwards;

iii) When $f'(x) = 0$ the curve is flat;

iv) When $f'(x) = \infty, -\infty$ there are assymptotes.

Definition 3.12 - *Even Functions*
If $f(x) = f(-x)$ then the function is symmetrical and said to be *even.*
Examples - $x^2, cos(x), |x|$

Definition 3.13 - *Odd Functions*
If $f(x) = -f(-x)$ then the function is said to be *odd.*
Examples - $x, sin(x), x.cos(x)$

Remark 3.14
Some functions are neither *odd* nor *even.*
Example - $x + x^2$

4 Inegration

4.1 The Primitive

Definition 4.01 - *The Primitive*
A function, $F : A \to \mathbb{R}$, is a primative for the function $f : A \to \mathbb{R}$ if F is differentiable and

$$\frac{d}{dx}F = f$$

N.B. - Primitives are also called *Indefinite Integral* or *Anti-Derivative.*

Remark 4.02 - *Area Under a Curve*
Let $f : [a, b] \to \mathbb{R}$ be a continuous function. Then the area between the curve and the x-axis is found by integration.

$$A := \int_a^b f(x)dx$$

Definition 4.03 - *Convergent Improper Integrals*
Let $b > a$ and define a function, $f : [a, \infty) \to \mathbb{R}$, which is continuous in $[a, b]$ Then

$$\int_0^\infty f(x)dx := \lim_{b\to\infty} \int_a^b f(x)dx$$

If this limit exists then the improper integral is *convergent*, otherwise it is *divergent.*

Definition 4.04 - *Definite Integral*
Let F be the primative for the function f. Then

$$\int_b^a f(x)dx = F(a) - F(b)$$

Notation - $F(x)\Big|_a^b = F(b) - F(a)$

Remark 4.05 - *Summing Definite Inegrals*
For all $a < c < b$

$$\int_a^b f(x)dx = \int_c^a f(x)dx + \int_b^c f(x)dx$$

$$\int_b^a f(x)d := -\int_a^b f(x)dx$$

Theorem 4.06 - *Taylor Series*
Functions can be expanded into polynomial form with degree n, T_n, and remainder R_n such that $f(x) = T_n(x) + R_n(x)$.

$$T_n(x) = f(a) + (x-a)f'(a) + ... + \frac{1}{n}.(x-a)^n.f^n(a)$$

$$R_n(x) = \frac{1}{n}\int_a^x (x-t)^n.f^{(n+1)}(t)dt$$

5 Parametric Curves & Arc-Length

5.1 Parametric Curves

Definition 5.01 - *Parametric Curves*
Parametric equations are an alternative to Cartesian equations, for representing curves. They can also represent a point in 3D space.

$$\boldsymbol{p} = \begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix}$$

Theorem 5.02 - *Parametric to Cartesian Equations*
As all equations in a Parametric series have a common variable, substition can be used to form a single equation.
Example Let $\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} t-2 \\ \frac{t}{t-2} \end{pmatrix}$.

$$\begin{aligned} x &= t-2 \\ =>t &= x+2 \\ =>y &= \frac{x+2}{(x+2)-2} \\ &= \frac{x+2}{x} \\ y &= \underline{1+\frac{2}{x}} \end{aligned}$$

5.2 Tangent of a Curve

Theorem 5.02 - *Tangent to a Parametric Curve*
Let $(x(t), y(t))$ be a parametric equation. If we want to find the tangent at a point on the line, (a, b), we need to find the value t_0 such that $x(t_0) = a$ & $y(t_0) = b$.
Then by using the chain rule we can deduce the following equation for the tangent when $t = t_0$:

$$\frac{dy(t_0)}{dx(t_0)} = \frac{y - y(t_0)}{x - x(t_0)}$$

. Similarly we can deduce the equation for the normal when $t = t_0$:

$$-\frac{dx(t_0)}{dy(t_0)} = \frac{y - y(t_0)}{x - x(t_0)}$$

5.3 Arc-Length

Theorem 5.03 - *Arc-Length*
Arc-Length is the length of a curve, following a function, between two points. For a cartesian equation, $y = f(x)$, between the points x and $x + dx$ is

$$ds = \sqrt{dx^2 + dy^2}$$

So for a set of parametric equations, $(x(t), y(t)),\ a \leq t \leq b$,

$$ds = \sqrt{\frac{dx^2}{dt} + \frac{dy^2}{dt}}$$

To find the length of a curve between points a and b

$$s = \int_a^b \sqrt{\frac{dx^2}{dt} + \frac{dy^2}{dt}} dt$$

Definition 5.04 - *Curvature*
Curvature measures how fast the unit tangent vector to a curve rotates. Curvature of a curve, $y = f(x)$, can be found using the equation:

$$K(x) = \frac{|y''(x)|}{[1 + (y'(x))^2]^{\frac{3}{2}}}$$

For a set of parametric equations, $(x(t), y(t))$, it can be found using:

$$K(t_0) = \frac{y''(t_0).x'(t_0) - y'(t_0).x''(t_0)}{[(x'(t_0))^2 + (y'(t_0))^2]^{\frac{3}{2}}}$$

5.4 Level Curves

Definition 5.05 - *Level Curves*
Let $f : \mathbb{R}^d \to \mathbb{R}$ be a function with $d \geq 2,\ d \in \mathbb{N}$. A level curve for f is the set of real solutions for $f(\boldsymbol{x}) = c, c \in \mathbb{R}$.
N.B - $f(\boldsymbol{x}) = c$ is often written as $f = c$.

6 Differential Equations

Definition 6.01 - *Differential Equations*
Differential equations take the form

$$f(x, y, \frac{dx}{dy}, ..., \frac{d^{(n)}y}{dx^{(n)}}) = 0,\ x \in I$$

6.1 First Order Differential Equations

Definition 6.02 - *First Order*
First order differential equations are equations of form $f(x, y, \frac{dx}{dy}) = 0$.

Definition 6.03 - *Seperable Equations*
An equation, f, is said to be seperable if there exists two equations, $M(x),\ N(y)$, such that

$$f(x, y, y') = y' - M(x).N(y)$$

Thus

$$\begin{aligned}
y' &= M(x).N(y) \\
=> \frac{y'}{N(y)} &= M(x) \\
=> \int \frac{1}{N(y)} dy &= \int M(x) dx
\end{aligned}$$

After integration, the equation can be rearranged to be in terms of y.

6.2 Integrating Factor

Theorem 6.04 - *Integrating Factor*
Consider the equation $y' + f(x)y + g(x)$. Let $F(x) = \int f(x)dx$. Thus

$$\begin{aligned}
e^{F(x)}.y' + e^{F(x)}.y &= e^{F(x)}.g(x) \\
=> \frac{d}{dx}\left(e^{F(x)}.y\right) &= e^{F(x)}.g(x) \\
=> e^{F(x)}.y &= \int e^{F(x)}.g(x)\ dx \\
=> y &= e^{-F(x)} \int e^{F(x)}.g(x)\ dx
\end{aligned}$$

6.3 Second Order Differential Equations

Definition 6.05 - *Linear Differential Equations*
A differential equation is said to be *linear* if it can be written in the form

$$Ay(x) := a_n(x).y^{(n)}(x) + ... + a_1(x).y'(x) + a_0(x).y(x) = b(x)$$

We define the set of solutions as

$$S(A, b) := y : I \to \mathbb{R}; Ay = b$$

If the only solution is $b = 0$ then the system is homogenous, otherwise it is inhomogenous.

Definition 6.06 - *Particular & Complimentary Solutions*
When solving a differential equation, $Ay(x) = b(x)$, we need to find two functions in order to find the final solution.

i) Complementary Function, y_c - The homogenous case of the equation, $Ay(x) = 0$;

ii) Particular Function, y_p - The inhomogenous case of the equation, $Ay(x) = b(x)$ for a given $b(x)$.

Then $y = y_c + y_p$ is the final solution for $Ay(x) = b(x)$.

Theorem 6.07 - *Complementary Functon of LDEs with Constant Coefficients*
Take a linear differential equation

$$a_n.y^{(n)}(x) + ... + a_1.y'(x) + a_0.y(x) = b(x)$$

where $a_n, ..., a_1, a_0 \in \mathbb{R}$ & $b(x) : \mathbb{R} \to \mathbb{R}$ are all constant.
To find the *Complementary Function* we solve the equation

$$a_n.\lambda^n + ... + a_1.\lambda^n + a_0 = 0$$

. to get solutions $\lambda_1, ..., \lambda_k$ and then produce the complimentary function

$$y_c(x) = \mu_1 e^{\lambda_1 x} + ... + \mu_k e^{\lambda_k x}$$

Where $\mu_1, ..., \mu_k$ are constants to be found later, by comparing with $b(x)$.
Remark 6.08 - *Complementary Function*
The complementary function, y_c, for differential equations with constant coefficients depends upon the $\lambda_1, ..., \lambda_k$ we find, due to Euler's Formula.

i) $\lambda_i = c, \quad y_{c_i} = \mu_i e^{\lambda_i x}$;

ii) $\lambda_i = \pm ib, \quad y_{c_i} = \mu_{i_1} cos(bx) + \mu_{i_2} sin(bx)$;

iii) $\lambda_i = a \pm ib, \quad y_{c_i} = e^{ax}[\mu_{i_1} cos(bx) + \mu_{i_2} sin(bx)]$.

Then $y_c = \sum_{j=1}^{k} y_{c_j}$.

Remark 6.09 - *Particular Function*
The particular function, y_p, for a differential equation with constand coefficients, $Ay(x) = b(x)$, depends on the form of $b(x)$.

i) $b(x) = a_n x^n + ... + a_1 x + a_0, \quad y_p = b_n x^n + ... + b_1 x + b_0$;

ii) $b(x) = ae^{bx}, \quad y_p = \alpha e^{\beta x}$;

iii) $b(x) = a.sin(bx) + c.cos(dx), \quad y_p = \alpha sin(\beta x) + \gamma cos(\delta x)$.

Where the constants of y_p are values to be found, when given certain conditions.
Theorem 6.10 - *Particular Functon of LDEs with Constant Coefficients*
Take a linear differential equation

$$a_n.y^{(n)}(x) + ... + a_1.y'(x) + a_0.y(x) = b(x)$$

where $a_n, ..., a_1, a_0 \in \mathbb{R}$ & $b(x) : \mathbb{R} \to \mathbb{R}$ are all constant.
Deduce the particular function for the differential equation, given $b(x)$, and then differentiate y_p n times.
Substitute in these values, in place of the ys, into the original equation and solve to find values for the constants in y_p.

Example
Solve $y'' - y' + y = x^2$.

2**Complementary Function**

$$\text{Let } \lambda^2 - \lambda + 1 = 0$$

$$=> \lambda = \frac{1 \pm \sqrt{1-4}}{2}$$

$$
\begin{aligned}
&= \frac{1}{2} \pm i\frac{\sqrt{3}}{2} \\
=> y_c &= e^{\frac{x}{2}}[Acos(x\frac{\sqrt{3}}{2}) + Bsin(\frac{\sqrt{3}}{2})]
\end{aligned}
$$

Particular Function

$$
\begin{aligned}
\text{Let } y_p &= \alpha x^2 + \beta x + \gamma \\
=> y_p'(x) &= 2\alpha x + \beta, \& \\
=> y_p''(x) &= 2\alpha \\
=> (2\alpha) - (2\alpha x + \beta) + (\alpha x^2 + \beta x + \gamma) &= x^2 \\
=> x^2[\alpha] + x[\beta - 2\alpha] + [2\alpha - \beta + \gamma] &= x^2
\end{aligned}
$$

$$
\begin{aligned}
[x^2] : \alpha &= 1 \\
[x] : \beta - 2\alpha &= 0 \\
=> \beta &= 2 \\
[x^0] : 2\alpha + \gamma - \beta &= 0 \\
=> \gamma &= 0 \\
=> y_p &= x^2 + 2x
\end{aligned}
$$

$$
=> \underline{y = x^2 + 2x + e^{\frac{x}{2}}[Acos(x\frac{\sqrt{3}}{2}) + Bsin(\frac{\sqrt{3}}{2})]}
$$

6.4 Wronskian

Definition 6.11 - *The Wronskian*
The *Wronskian*, $W[y_1, y_2]$, of two differentiable functions is defined by

$$
W[y_1, y_2](x) = y_1(x).y_2'(x) - y_1'(x).y_2(x)
$$

The notation for this is

$$
\begin{vmatrix} a & b \\ c & d \end{vmatrix} := ad - bc
$$

So

$$
W[y_1, y_2] = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}
$$

Remark 6.12
If $W[y_1, y_2] \neq 0$ then y_1, y_2 are linearly independent.

6.5 Variation of Constants

Theorem 6.12
This is a technique for solving all differential equations, not just ones with constant coefficients. Consider the equation

$$
Ay(x) := y''(x) + \beta(x)y'(x) + \gamma(x)y(x) = b(x), \quad \text{for a known } b(x)
$$

We can deduce a complementary function in the form

$$
y_c = \lambda_1(x)y_1(x) + \lambda_2(x)y_2(x)
$$

where y_1 & y_2 are linearly independent, thus $W[y_1, y_2] \neq 0$.

WIP