

Calculus 1 - Notes

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1 Before Calculus

1.1 Fundamental Theorem of Calculus

Definition 1.01 - *Fundamental Theorem of Calculus*

The Fundamental Theorem of Calculus states

$$\frac{d}{dx} \int_a^x f(t)dt = f(x)$$

Definition 1.02 - *Common Sets of Numbers*

Natural Numbers, set of positive integers - $\mathbb{N} := \{1, 2, 3, \dots\}$.

Whole Numbers, set of all integers - $\mathbb{Z} := \{\dots, -2, -1, 0, 1, 2, \dots\}$.

Rational Numbers, set of fractions - $\mathbb{Q} := \left\{\frac{p}{q} : p \in \mathbb{Z}, q \in \mathbb{N}\right\}$.

Real Numbers, set of all rational & irrational numbers - \mathbb{R} .

1.2 Intervals

Definition 1.03 - *Intervals*

Sets of real numbers that fulfil in given ranges.

Notation

$$[a, b] := \{x \in \mathbb{R} : a \leq x \leq b\}$$

$$(a, b] := \{x \in \mathbb{R} : a < x \leq b\}$$

$$[a, b) := \{x \in \mathbb{R} : a \leq x < b\}$$

$$(a, b) := \{x \in \mathbb{R} : a < x < b\}$$

Example

In what interval does x lie such that:

$$|3x + 4| < |2x - 1|$$

Solution

$$\text{Case 1 : } x \geq \frac{1}{2}$$

$$\Rightarrow 1 - 2x < 3x + 4 < 2x - 1$$

$$\Rightarrow 1 - 2x < 3x + 4$$

$$\Rightarrow x > \frac{-3}{5}$$

$$\text{And, } \Rightarrow 3x + 4 < 2x - 1$$

$$\Rightarrow x < -5$$

There are no real solutions in this range.

$$\text{Case 2 : } x < \frac{1}{2}$$

$$\Rightarrow 2x - 1 < 3x + 4 < 1 - 2x$$

$$\Rightarrow 2x - 1 < 3x + 4$$

$$\Rightarrow -5 < x$$

$$\text{And, } \Rightarrow 3x + 4 < 1 - 2x$$

$$\Rightarrow 5x < -3$$

$$\Rightarrow x < \frac{-3}{5}$$

$$\Rightarrow -5 < x < \frac{-3}{5}, \quad x \in \left(-5, \frac{-3}{5} \right)$$

Definition 1.04 - Functions

Functions map values between fields of numbers. The signature of a function is defined by

$$f : A \rightarrow B$$

Where f is the name of the function, A is the domain and B is the co-domain.

The *Domain* of a function is the set of numbers it can take as an input.

The *Co-Domain* is the set of numbers that the domain is mapped to.

N.B. - A function is valid iff it maps each value in the domain to a single value in the co-domain.

Definition 1.05 - Maximal Domain

The *Maximal Domain* of a function is the largest set of values which can serve as the domain of a function.

Remark 1.06 - Types of Function

Let $f : A \rightarrow B$

Polynomials

$$f(x) = a_0 + a_1x + \dots + a_nx^n$$

Rational

$$f(x) = \frac{p(x)}{q(x)}, \quad q(x) \neq 0 \quad \forall x \in A$$

Trigonometric

$$\sin(x), \cos(x), \tan(x) \text{ etc.}$$

2 Limits

2.1 Limits

Definition 2.01 - Limits

A limit is the value a function tends to, for a given x .

i.e. The value $f(x)$ has at it gets very close to x .

Formally We say L is the limit of $f(x)$ as x tends to x_0 if

$$\forall \varepsilon > 0, \exists \delta > 0 \text{ st if } x \in A \text{ and } |x - x_0| < \delta \Rightarrow |f(x) - L| < \varepsilon$$

Notation

$$\lim_{x \rightarrow x_0} f(x) = L$$

Definition 2.02 - Directional Limits

Sometimes the value of a limit depends on which direction you approach it from.

$\lim_{x \rightarrow x_0+}$ is used when approaching from values greater than x_0 .

$\lim_{x \rightarrow x_0-}$ is used when approaching from values less than x_0 .

Theorem 2.03 - Operations with limits

Let $\lim_{x \rightarrow x_0} f(x) = L_f$ and $\lim_{x \rightarrow x_0} g(x) = L_g$ Then

$$\begin{aligned}\lim_{x \rightarrow x_0} [f(x) + g(x)] &= L_f + L_g \\ \lim_{x \rightarrow x_0} f(x) \cdot g(x) &= L_f \cdot L_g \\ \lim_{x \rightarrow x_0} \frac{f(x)}{g(x)} &= \frac{L_f}{L_g} \quad L_g \neq 0\end{aligned}$$

2.2 Exponential Function**Definition 2.04 - Exponential Function**

$$e := \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^n \simeq 2.7182818...$$

Theorem 2.05 - Binomial Expansion

A technique for expanding binomial expressions

$$\begin{aligned}\left(1 + \frac{x}{n}\right)^n &= \sum_{i=0}^n \binom{n}{i} \cdot 1^{(n-i)} \cdot \left(\frac{x}{n}\right)^i \\ &= 1 + x + \frac{n-1}{2n} \cdot x^2 + \dots + \frac{x^n}{n^n}\end{aligned}$$

3 The Derivative**Definition 3.01 - Differentiable Equations**

Let $f : A \rightarrow B$ and $x_0 \in A$.

f is differentiable at x_0 if $\exists L \in B$ such that

$$L = \lim_{h \rightarrow 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

If this limit exists $\forall x \in A$ then we can define the derivative of $f(x)$

$$f'(x) := \lim_{h \rightarrow 0} \frac{f(x + h) - f(x)}{h}$$

Definition 3.02 - Notation for Differentiation

There are two ways to denote the derivative of an equation

$$f'(x) \iff \frac{df}{dx}, f''(x) \iff \frac{d^2f}{dx^2}, \dots, f^{(n)}(x) \iff \frac{d^n f}{dx^n}$$

N.B. - Using $\frac{df}{dx}$ is more informative, especially for equations with multiple variables.

3.1 Techniques for finding derivative

Theorem 3.03 - Sum Rule

Let f, g be differentiable with respect to x .

$$(f + g)' = f' + g'$$

Theorem 3.04 - Product Rule

Let f, g be differentiable with respect to x .

$$(fg)' = f'g + fg'$$

Theorem 3.05 - Quotient Rule

Let f, g be differentiable with respect to x .

$$\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$$

Definition 3.06 - Composite Functions

Let $f : B \rightarrow C$ and $g : A \rightarrow B$ Then

$$(f \circ g)(x) = f(g(x))$$

Theorem 3.07 - Chain Rule

Let f, g be differentiable with respect to x .

$$\frac{d}{dx}f(g(x)) = f'(g(x)) \cdot g'(x)$$

3.2 Implicit Differentiation

Definition 3.08 - Implicit Differentiation

Sometimes it is hard to isolate variables in multi-variable equations, in these cases differentiate both sides with respect to the same variable.

Remembering

$$\frac{d}{dx}(x) = 1 \text{ and } \frac{d}{dx}(y) = \frac{dy}{dx} = y'$$

Example

Find y if $x^3 + y^3 = 6xy$

$$\begin{aligned} \frac{d}{dx}(x^3 + y^3) &= \frac{d}{dx}(6xy) \\ \Rightarrow 3x^2 + 3y^2 \cdot y' &= 6y + 6x \cdot y' \\ \Rightarrow y'(3y^2 - 6x) &= 6y - 3x^2 \\ \Rightarrow y' &= \frac{2y - x^2}{y^2 - 2x} \end{aligned}$$

3.3 Applications of The Derivative

Theorem 3.09 - Newton's Method

Let f be differentiable. Using *Newton's Method* we can approximate a solution to $f(x) = 0$.

- i) Take an initial guess, x_0 ;
- ii) Find the value of x where the tangent to x_0 on $f(x)$ intercepts the x-axis;
- iii) Use this value as the next guess;
- iv) Repeat until the value of x reduces little.

The equation for the tangent is

$$y = f(x_0) + (x - x_0)f'(x_0)$$

so a simplified equation for the process can be deduced

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

Theorem 3.10 - Angle between Intersecting Curves

Let $y = f_1(x)$ and $y = f_2(x)$ be two curves which intersect at (x_0, y_0) .

Then $y_0 = f_1(x_0) = f_2(x_0)$

Let m_1, m_2 be the gradient of the tangents to f_1 & f_2 at x_0 .

Then $\theta_i := \tan^{-1}(m_i)$ for $i = 1, 2$.

Let $\phi = |\theta_1 - \theta_2|$, then

$$\phi = \left| \frac{m_1 - m_2}{1 + m_1 m_2} \right|$$

Theorem 3.11 - L'Hospital's Rule

For two equations f, g with limit of $-\infty, 0$ or ∞ as x tends to a , it is hard to solve

$$\lim_{x \rightarrow a} \frac{f(x)}{g(x)}$$

Provided the limit exists, L'Hospital's Rule states that

$$\lim_{x \rightarrow a} \frac{f(x)}{g(x)} \iff \lim_{x \rightarrow a} \frac{f'(x)}{g'(x)}$$

3.4 Sketching Curves

Remark 3.11 - Sketching Curves

Evaluating the derivative of a curve can make it easier to sketch:

- i) When $f'(x) > 0$ the curve is heading upwards;
- ii) When $f'(x) < 0$ the curve is heading downwards;
- iii) When $f'(x) = 0$ the curve is flat;
- iv) When $f'(x) = \infty, -\infty$ there are asymptotes.

Definition 3.12 - Even Functions

If $f(x) = f(-x)$ then the function is symmetrical and said to be *even*.

Examples - $x^2, \cos(x), |x|$

Definition 3.13 - Odd Functions

If $f(x) = -f(-x)$ then the function is said to be *odd*.

Examples - $x, \sin(x), x.\cos(x)$

Remark 3.14

Some functions are neither *odd* nor *even*.

Example - $x + x^2$

4 Inegration

4.1 The Primitive

Definition 4.01 - The Primitive

A function, $F : A \rightarrow \mathbb{R}$, is a primitive for the function $f : A \rightarrow \mathbb{R}$ if F is differentiable and

$$\frac{d}{dx}F = f$$

N.B. - Primitives are also called *Indefinite Integral* or *Anti-Derivative*.

Remark 4.02 - Area Under a Curve

Let $f : [a, b] \rightarrow \mathbb{R}$ be a continuous function. Then the area between the curve and the x-axis is found by integration.

$$A := \int_a^b f(x)dx$$

Definition 4.03 - Convergent Improper Integrals

Let $b > a$ and define a function, $f : [a, \infty) \rightarrow \mathbb{R}$, which is continuous in $[a, b]$ Then

$$\int_a^\infty f(x)dx := \lim_{b \rightarrow \infty} \int_a^b f(x)dx$$

If this limit exists then the improper integral is *convergent*, otherwise it is *divergent*.

Definition 4.04 - Definite Integral

Let F be the primitive for the function f . Then

$$\int_b^a f(x)dx = F(a) - F(b)$$

Notation - $F(x)|_a^b = F(b) - F(a)$

Remark 4.05 - Summing Definite Integrals

For all $a < c < b$

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$$

$$\int_b^a f(x)dx := - \int_a^b f(x)dx$$

Theorem 4.06 - Taylor Series

Functions can be expanded into polynomial form with degree n , T_n , and remainder R_n such that $f(x) = T_n(x) + R_n(x)$.

$$T_n(x) = f(a) + (x-a)f'(a) + \dots + \frac{1}{n!}(x-a)^n \cdot f^n(a)$$

$$R_n(x) = \frac{1}{n!} \int_a^x (x-t)^n \cdot f^{(n+1)}(t) dt$$

5 Parametric Curves & Arc-Length

5.1 Parametric Curves

Definition 5.01 - Parametric Curves

Parametric equations are an alternative to Cartesian equations, for representing curves. They can also represent a point in 3D space.

$$\mathbf{p} = \begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix}$$

Theorem 5.02 - Parametric to Cartesian Equations

As all equations in a Parametric series have a common variable, substitution can be used to form a single equation.

Example Let $\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} t-2 \\ \frac{t}{t-2} \end{pmatrix}$.

$$\begin{aligned} x &= t - 2 \\ \Rightarrow t &= x + 2 \\ \Rightarrow y &= \frac{x+2}{(x+2)-2} \\ &= \frac{x+2}{x} \\ y &= 1 + \frac{2}{x} \end{aligned}$$

5.2 Tangent of a Curve

Theorem 5.02 - Tangent to a Parametric Curve

Let $(x(t), y(t))$ be a parametric equation. If we want to find the tangent at a point on the line, (a, b) , we need to find the value t_0 such that $x(t_0) = a$ & $y(t_0) = b$.

Then by using the chain rule we can deduce the following equation for the tangent when $t = t_0$:

$$\frac{dy(t_0)}{dx(t_0)} = \frac{y - y(t_0)}{x - x(t_0)}$$

. Similarly we can deduce the equation for the normal when $t = t_0$:

$$-\frac{dx(t_0)}{dy(t_0)} = \frac{y - y(t_0)}{x - x(t_0)}$$

5.3 Arc-Length

Theorem 5.03 - Arc-Length

Arc-Length is the length of a curve, following a function, between two points. For a cartesian equation, $y = f(x)$, between the points x and $x + dx$ is

$$ds = \sqrt{dx^2 + dy^2}$$

So for a set of parametric equations, $(x(t), y(t))$, $a \leq t \leq b$,

$$ds = \sqrt{\frac{dx^2}{dt} + \frac{dy^2}{dt}}$$

To find the length of a curve between points a and b

$$s = \int_a^b \sqrt{\frac{dx^2}{dt} + \frac{dy^2}{dt}} dt$$

Definition 5.04 - Curvature

Curvature measures how fast the unit tangent vector to a curve rotates. Curvature of a curve, $y = f(x)$, can be found using the equation:

$$K(x) = \frac{|y''(x)|}{[1 + (y'(x))^2]^{\frac{3}{2}}}$$

For a set of parametric equations, $(x(t), y(t))$, it can be found using:

$$K(t_0) = \frac{y''(t_0).x'(t_0) - y'(t_0).x''(t_0)}{[(x'(t_0))^2 + (y'(t_0))^2]^{\frac{3}{2}}}$$

5.4 Level Curves

Definition 5.05 - Level Curves

Let $f : \mathbb{R}^d \rightarrow \mathbb{R}$ be a function with $d \geq 2$, $d \in \mathbb{N}$. A level curve for f is the set of real solutions for $f(\mathbf{x}) = c$, $c \in \mathbb{R}$.

N.B - $f(\mathbf{x}) = c$ is often written as $f = c$.

6 Differential Equations

Definition 6.01 - Differential Equations

Differential equations take the form

$$f(x, y, \frac{dx}{dy}, \dots, \frac{d^{(n)}y}{dx^{(n)}}) = 0, \quad x \in I$$

6.1 First Order Differential Equations

Definition 6.02 - First Order

First order differential equations are equations of form $f(x, y, \frac{dx}{dy}) = 0$.

Definition 6.03 - Seperable Equations

An equation, f , is said to be seperable if there exists two equations, $M(x)$, $N(y)$, such that

$$f(x, y, y') = y' - M(x).N(y)$$

Thus

$$\begin{aligned} y' &= M(x).N(y) \\ \Rightarrow \frac{y'}{N(y)} &= M(x) \\ \Rightarrow \int \frac{1}{N(y)} dy &= \int M(x) dx \end{aligned}$$

After integration, the equation can be rearranged to be in terms of y .

6.2 Integrating Factor

Theorem 6.04 - Integrating Factor

Consider the equation $y' + f(x)y + g(x)$. Let $F(x) = \int f(x)dx$. Thus

$$\begin{aligned} e^{F(x)}.y' + e^{F(x)}.y &= e^{F(x)}.g(x) \\ \Rightarrow \frac{d}{dx} (e^{F(x)}.y) &= e^{F(x)}.g(x) \\ \Rightarrow e^{F(x)}.y &= \int e^{F(x)}.g(x) dx \\ \Rightarrow y &= e^{-F(x)} \int e^{F(x)}.g(x) dx \end{aligned}$$

6.3 Second Order Differential Equations

Definition 6.05 - Linear Differential Equations

A differential equation is said to be *linear* if it can be written in the form

$$Ay(x) := a_n(x).y^{(n)}(x) + \dots + a_1(x).y'(x) + a_0(x).y(x) = b(x)$$

We define the set of solutions as

$$S(A, b) := y : I \rightarrow \mathbb{R}; Ay = b$$

If the only solution is $b = 0$ then the system is homogenous, otherwise it is inhomogenous.

Definition 6.06 - Particular & Complimentary Solutions

When solving a differential equation, $Ay(x) = b(x)$, we need to find two functions in order to find the final solution.

- i) Complementary Function, y_c - The homogenous case of the equation, $Ay(x) = 0$;
- ii) Particular Function, y_p - The inhomogenous case of the equation, $Ay(x) = b(x)$ for a given $b(x)$.

Then $y = y_c + y_p$ is the final solution for $Ay(x) = b(x)$.

Theorem 6.07 - Complementary Function of LDEs with Constant Coefficients

Take a linear differential equation

$$a_n \cdot y^{(n)}(x) + \dots + a_1 \cdot y'(x) + a_0 \cdot y(x) = b(x)$$

where $a_n, \dots, a_1, a_0 \in \mathbb{R}$ & $b(x) : \mathbb{R} \rightarrow \mathbb{R}$ are all constant.

To find the *Complementary Function* we solve the equation

$$a_n \cdot \lambda^n + \dots + a_1 \cdot \lambda + a_0 = 0$$

. to get solutions $\lambda_1, \dots, \lambda_k$ and then produce the complimentary function

$$y_c(x) = \mu_1 e^{\lambda_1 x} + \dots + \mu_k e^{\lambda_k x}$$

Where μ_1, \dots, μ_k are constants to be found later, by comparing with $b(x)$.

Remark 6.08 - Complementary Function

The complementary function, y_c , for differential equations with constant coefficients depends upon the $\lambda_1, \dots, \lambda_k$ we find, due to Euler's Formula.

- i) $\lambda_i = c, \quad y_{c_i} = \mu_i e^{\lambda_i x};$
- ii) $\lambda_i = \pm ib, \quad y_{c_i} = \mu_{i_1} \cos(bx) + \mu_{i_2} \sin(bx);$
- iii) $\lambda_i = a \pm ib, \quad y_{c_i} = e^{ax} [\mu_{i_1} \cos(bx) + \mu_{i_2} \sin(bx)].$

Then $y_c = \sum_{j=1}^k y_{c_j}$.

Remark 6.09 - Particular Function

The particular function, y_p , for a differential equation with constant coefficients, $Ay(x) = b(x)$, depends on the form of $b(x)$.

- i) $b(x) = a_n x^n + \dots + a_1 x + a_0, \quad y_p = b_n x^n + \dots + b_1 x + b_0;$
- ii) $b(x) = a e^{bx}, \quad y_p = \alpha e^{\beta x};$
- iii) $b(x) = a \sin(bx) + c \cos(dx), \quad y_p = \alpha \sin(\beta x) + \gamma \cos(\delta x).$

Where the constants of y_p are values to be found, when given certain conditions.

Theorem 6.10 - Particular Function of LDEs with Constant Coefficients

Take a linear differential equation

$$a_n \cdot y^{(n)}(x) + \dots + a_1 \cdot y'(x) + a_0 \cdot y(x) = b(x)$$

where $a_n, \dots, a_1, a_0 \in \mathbb{R}$ & $b(x) : \mathbb{R} \rightarrow \mathbb{R}$ are all constant.

Deduce the particular function for the differential equation, given $b(x)$, and then differentiate y_p n times.

Substitute in these values, in place of the y s, into the original equation and solve to find values for the constants in y_p .

Example

Solve $y'' - y' + y = x^2$.

2Complementary Function

Let $\lambda^2 - \lambda + 1 = 0$

$$\Rightarrow \lambda = \frac{1 \pm \sqrt{1-4}}{2}$$

$$= \frac{1}{2} \pm i \frac{\sqrt{3}}{2}$$

$$\Rightarrow y_c = e^{\frac{x}{2}} [A \cos(x \frac{\sqrt{3}}{2}) + B \sin(\frac{\sqrt{3}}{2})]$$

Particular Function

$$\text{Let } y_p = \alpha x^2 + \beta x + \gamma$$

$$\Rightarrow y_p'(x) = 2\alpha x + \beta, \&$$

$$\Rightarrow y_p''(x) = 2\alpha$$

$$\Rightarrow (2\alpha) - (2\alpha x + \beta) + (\alpha x^2 + \beta x + \gamma) = x^2$$

$$\Rightarrow x^2[\alpha] + x[\beta - 2\alpha] + [2\alpha - \beta + \gamma] = x^2$$

$$[x^2] : \alpha = 1$$

$$[x] : \beta - 2\alpha = 0$$

$$\Rightarrow \beta = 2$$

$$[x^0] : 2\alpha + \gamma - \beta = 0$$

$$\Rightarrow \gamma = 0$$

$$\Rightarrow y_p = x^2 + 2x$$

$$\Rightarrow y = x^2 + 2x + e^{\frac{x}{2}} [A \cos(x \frac{\sqrt{3}}{2}) + B \sin(\frac{\sqrt{3}}{2})]$$

6.4 Wronskian

Definition 6.11 - The Wronskian

The *Wronskian*, $W[y_1, y_2]$, of two differentiable functions is defined by

$$W[y_1, y_2](x) = y_1(x) \cdot y_2'(x) - y_1'(x) \cdot y_2(x)$$

The notation for this is

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} := ad - bc$$

So

$$W[y_1, y_2] = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$

Remark 6.12

If $W[y_1, y_2] \neq 0$ then y_1, y_2 are linearly independent.

6.5 Variation of Constants

Theorem 6.13

This is a technique for solving all differential equations, not just ones with constant coefficients, and assumes we know the complementary function, y_c .

Consider the equation

$$Ay(x) := y''(x) + \beta(x)y'(x) + \gamma(x)y(x) = b(x), \quad \text{for a known } b(x)$$

Suppose we have a complementary function in the form

$$y_c = \lambda_1(x)y_1(x) + \lambda_2(x)y_2(x)$$

where y_1 & y_2 are linearly independent, thus $W[y_1, y_2] \neq 0$.

Then

$$y_p' = \lambda_1' y_1 + \lambda_1 y_1' + \lambda_2' y_2 + \lambda_2 y_2'$$

As λ_1, λ_2 are constant then $\lambda_1' = \lambda_2' = 0$ so

$$y_p' = \lambda_1 y_1' + \lambda_2 y_2'$$

By differentiating and then substituting back into the original equation we see y_p is a solution iff

$$\lambda_1' y_1' + \lambda_2' y_2' = f$$

In matrix form we have

$$\begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix} \begin{pmatrix} \lambda_1' \\ \lambda_2' \end{pmatrix} = \begin{pmatrix} 0 \\ f \end{pmatrix}$$

Then by Cramer's rule we have

$$\lambda_1' = \frac{\begin{vmatrix} 0 & y_2 \\ f & y_2' \end{vmatrix}}{W[y_1 \ y_2]} = \frac{-y_2 f}{W[y_1 \ y_2]}$$

and

$$\lambda_2' = \frac{\begin{vmatrix} y_1 & 0 \\ y_1' & f \end{vmatrix}}{W[y_1 \ y_2]} = \frac{y_1 f}{W[y_1 \ y_2]}$$

Giving use a solution for $y_p' = \lambda_1' y_1 + \lambda_1 y_1' + \lambda_2' y_2 + \lambda_2 y_2'$.

7 Applied Differential Equations

Definition 7.01 - *Denoting Limit Relationships*

We use

$$F(x) \sim G(x) \text{ as } x \rightarrow a$$

to denote

$$\lim_{x \rightarrow a} \frac{F(x)}{G(x)} = 1$$

Theorem 7.02 - *Vibrating String*

If we are given a string which is L long then we can define an equation, $y(x, t)$, which describe the displacement of a point x along the string, at time t .

$$y(x, t) = u(x)e^{i\omega t}$$

Where $\frac{\omega}{2\pi}$ is the frequency of the string and $u(x) = A\cos(\omega x) + B\sin(\omega x)$.

We can generalise this for strings with n anti-nodes.

$$\omega_n := \frac{n\pi}{L}, \quad u_n := \sin(\omega_n x)$$

8 Liner Difference Equations

Definition 8.01 - *Difference Equations*

A difference equation is an equation of the form

$$F(n, y_n, \dots, y_{n+d}) = 0, \quad n, d \in \mathbb{N}$$

8.1 First-Order Linear Difference Equation

Definition 8.02 - Linear First-Order Difference Equations

A *Linear First-Order Difference Equation* is an equation, F , which can be described by

$$F(n, y_n, y_{n+1}) = a_n y_{n+1} + b_n y_n - f_n$$

where a_n, b_n, f_n are all known sequences.

Example 8.03

By taking a simple equation

$$y_{n+1} - y_n = f_n$$

we can see that

$$y_{n+1} = y_n + (y_{n+1} - y_n) = y_n + f_n = \dots = y_{n_0} + f_{n_0} + \dots + f_{n-1} + f_n$$

So

$$y_n = y_{n_0} + \sum_{j=n_0}^{n-1} f_j$$

Theorem 8.04 - Solving First-Order Linear Difference Equations

From *Definition 8.02* we can generalise the equation to show that

$$y_{n+1} + b_n y_n = f_n$$

Then

$$\frac{-1}{b_n} y_{n+1} - y_n = \frac{-1}{b_n} f_n$$

We now define the *Summing Factor*, S_n , as

$$S_n := \prod_{j=n_0}^{n-1} \frac{-1}{b_j}$$

. We multiply both sides of the original equation by the summing factor and as $S_n(\frac{-1}{b_n}) = S_{n+1}$ we get

$$S_{n+1} y_{n+1} - S_n y_n = S_{n+1} f_n$$

As this has the same form as the example in 8.03 we can now deduce

$$S_n y_n = y_{n_0} + \sum_{j=n_0}^{n-1} S_{j+1} \cdot f_j$$

8.2 Second-Order Linear Difference Equation

Definition 8.05 - Second-Order Linear Difference Equation

A *Linear Second-Order Difference Equation* is an equation, F , which can be described by

$$a_n y_{n+2} + b_n y_{n+1} + c_n y_n = f_n$$

where a_n, b_n, c_n, f_n are known sequences.

Remark 8.06 - Solving Second-Order Linear Difference Equations

Similar to solving second-order differential equations we need to consider two cases. The *homogeneous* & *inhomogeneous* cases. So two sequences will be found the complementary sequence, y_n^c , and the particular sequence, y_n^p . The final solution for y_n is given by

$$y_n = y_n^c + y_n^p$$

Definition 8.07 - Wronskian of Sequences

For two sequences u_n, v_n we define the Wronskian to be

$$W_n := \begin{vmatrix} u_n & v_n \\ u_{n+1} & v_{n+1} \end{vmatrix} = u_n \cdot v_{n+1} - v_n \cdot u_{n+1}$$

Theorem 8.08 - Homogenous Case with Constant Coefficients

Take the equation

$$ay_{n+2} + by_{n+1} + cy_n = 0$$

where a, b, c are known constants. We look for solutions of the form

$$y_n = \lambda^n$$

By substitution we get the equation $a\lambda^2 + b\lambda + c = 0$. By solving for λ we find a solution

- i) λ has two real solutions - $y_n = A\lambda_1^n + B\lambda_2^n$;
- ii) λ has one real solution - $y_n = (An + B)\lambda^n$;
- iii) λ has only an imaginary solution - $y_n = \Lambda e^{i\theta}$, $\Lambda^2 := \frac{c}{a}, \theta := \tan^{-1}(\frac{4ac-b^2}{-b})$.

Theorem 8.09 - Homogenous Case Second-Order Linear Difference Equations

The homogenous case finds solutions for

$$a_n y_{n+2} + b_n y_{n+1} + c_n y_n = 0$$

Suppose $y_n = u_n$ and $y_n = v_n$ are solutions to this homogenous equation. Then

$$W_n[u_n, v_n] = W_{n_0} \prod_{j=n_0}^{n-1} \frac{c_j}{a_j}$$

So

$$u_n \cdot v_{n+1} - v_n \cdot u_{n+1} = W_{n_0} \prod_{j=n_0}^{n-1} \frac{c_j}{a_j}$$

Which can be rearranged to be in the form of a first order difference equation, such as *Example 8.03*

$$\frac{u_n}{u_{n+1}} v_{n+1} - v_n = W_{n_0} \prod_{j=n_0}^{n-1} \frac{c_j}{a_j}$$

Which has a summing factor

$$S_n = \frac{1}{u_n}$$

By multiplying both sides by S_n and simplifying we get

$$v_n = u_n \sum_{j=n_0}^{n-1} \frac{1}{u_j \cdot u_{j+1}} \prod_{k=n_0}^{n-1} \frac{c_k}{a_k}$$

Typically you need to solve the product part of the equation to get a result for the sequence u_n .

Remark 8.10 - Inhomogenous Case Second-Order Linear Difference Equations

Generally the best way to do this is to make an educated guess based on the right hand side of the equation. So if the RHS is a polynomial, guess a polynomial, etc. Similar to solving differential equations.

9 Several Variables - Differentiability

Definition 9.1 - Several Variable Function

Let $d \in \mathbb{N}$, $A \subset \mathbb{R}^d$ & $B \subset \mathbb{R}^d$.

A function $\mathbf{f} : A \rightarrow B$ is a map which, for all $\mathbf{x} \in A$, assigns a unique value $\mathbf{f}(\mathbf{x}) \in B$.

Definition 9.2 - Linear Functions

A function $\mathbf{f} : \mathbb{R}^d \rightarrow \mathbb{R}^n$ is *linear* if it can be given in terms of a matrix $A \in M_{n,d}(\mathbb{R})$ where

$$\mathbf{f}(\mathbf{x}) = A\mathbf{x}$$

Theorem 9.3 - Properties of Linear Functions

If a function is linear then the following are true:

- i) $\mathbf{f}(\lambda\mathbf{x}) = \lambda\mathbf{f}(\mathbf{x})$;
- ii) $\mathbf{f}(\mathbf{x} + \mathbf{y}) = \mathbf{f}(\mathbf{x}) + \mathbf{f}(\mathbf{y})$.

Definition 9.4 - Continuous Several Variable Function

Let $\mathbf{f} : A \rightarrow B$ and $\mathbf{a} \in A$. Then \mathbf{f} is continuous if

$$\lim_{\mathbf{x} \rightarrow \mathbf{a}} \mathbf{f}(\mathbf{x}) = \mathbf{f}(\mathbf{a})$$

9.1 The Derivative

Definition 9.5 - Norm of a Vector

The norm of a vector, $\mathbf{x} \in \mathbb{R}^d$ is

$$\|\mathbf{x}\| := \left(\sum_{j=1}^d x_j^2 \right)^{1/2}$$

Definition 9.6 - Derivative of Several Variable Function

A function, $\mathbf{f} : \mathbb{R}^d \rightarrow \mathbb{R}^n$, is said to be differentiable at the point $\mathbf{x} \in \mathbb{R}^d$ if there exists an $A \in M_{n,d}(\mathbb{R})$ such that

$$\lim_{\mathbf{h} \rightarrow \mathbf{0}} \frac{\|\mathbf{f}(\mathbf{x} + \mathbf{h}) - \mathbf{f}(\mathbf{x}) - A\mathbf{h}\|}{\|\mathbf{h}\|} = 0$$

This A is the derivative of $\mathbf{f}(\mathbf{x})$.

$$\mathbf{f}'(\mathbf{x}) := A$$

Remark 9.7

If consider the following several variable function

$$\mathbf{f} = \begin{pmatrix} f_1 \\ \vdots \\ f_n \end{pmatrix}$$

Then

$$\mathbf{f}' = \begin{pmatrix} f'_1 \\ \vdots \\ f'_n \end{pmatrix}$$

10 Directional & Partial Derivatives

10.1 Directional Derivative

Definition 10.1 - Direction

A direction in \mathbb{R}^d is a vector of unit length.

In \mathbb{R}^2 every direction can be given by $\mathbf{u} = \begin{pmatrix} \cos(\theta) \\ \sin(\theta) \end{pmatrix}$, where θ is the angle from positive x axis

In \mathbb{R}^3 every direction can be given by $\mathbf{u} = \begin{pmatrix} \sin(\phi)\cos(\theta) \\ \sin(\phi)\sin(\theta) \\ \cos(\phi) \end{pmatrix}$, where ϕ is the angle from positive z axis and θ is the angle from the positive x axis.

Definition 10.2 - Spherical Co-ordinates

The spherical co-ordinates describe points in three dimension space.

The distance of a point from the origin is

$$r = \rho \sin(\phi)$$

Then

$$\mathbf{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} r\cos(\theta) \\ r\sin(\theta) \\ \rho\cos(\phi) \end{pmatrix} = \begin{pmatrix} \rho\sin(\phi)\cos(\theta) \\ \rho\sin(\phi)\sin(\theta) \\ \rho\cos(\phi) \end{pmatrix}$$

Definition 10.3 - Direction Derivative

The direction derivative of \mathbf{f} in the direction of \mathbf{u} at the point \mathbf{x}_0 is the vector:

$$D_{\mathbf{u}}\mathbf{f}(\mathbf{x}_0) := \frac{d}{dt}\mathbf{f}(\mathbf{x}_0 + t\mathbf{u}) \big|_{t=0}$$

Theorem 10.4

For all $\mathbf{f} : \mathbb{R}^d \rightarrow \mathbb{R}^n$ the directional derivative at \mathbf{u} in \mathbb{R}^d we have

$$D_{\mathbf{u}}\mathbf{f}(\mathbf{x}) = \mathbf{f}'(\mathbf{x}) \cdot \mathbf{u}$$

10.2 Partial Derivative

Definition 10.5 - Partial Derivative

Let $f : \mathbb{R}^d \rightarrow \mathbb{R}$. Then the direction derivative, $D_{\mathbf{e}_j}f(\mathbf{x})$, if it exists, is called the *partial derivative* of f with respect to x_j at \mathbf{x} .

This is denoted by

$$\frac{\partial f}{\partial x_j}(\mathbf{x}) \text{ or } f_{x_j}(\mathbf{x})$$

Proposition 10.6 - Partial Derivative as a Matrix

If $\mathbf{f} : \mathbb{R}^d \rightarrow \mathbb{R}^n$ is differentiable then

$$\mathbf{f}'(\mathbf{x}) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_d} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_d} \end{pmatrix}$$

Remark 10.7 - Second Order Partial Derivatives

$$\begin{aligned} \frac{\partial^2 f}{\partial x^2} &= \frac{\partial}{\partial x} \cdot \frac{\partial f}{\partial x} \\ \frac{\partial^2 f}{\partial x \partial y} &= \frac{\partial}{\partial x} \cdot \frac{\partial f}{\partial y} \\ \frac{\partial^2 f}{\partial y^2} &= \frac{\partial}{\partial y} \cdot \frac{\partial f}{\partial y} \end{aligned}$$

11 Gradient & Chain Rule in Several Variables

11.1 Chain Rule

Theorem 11.1 - Chain rule in Several Variables

Let $\mathbf{f} : \mathbb{R}^d \rightarrow \mathbb{R}^n$ & $\mathbf{g} : \mathbb{R}^m \rightarrow \mathbb{R}^d$ be differentiable. Then

$$[\mathbf{f} \cdot \mathbf{g}]' = \mathbf{f}'(\mathbf{g}(\mathbf{x})) \cdot \mathbf{g}'(\mathbf{x})$$

11.2 Invertible Maps & Implicit Differentiation

Definition 11.2 - Implicit Differentiation

Let $A, B \subset \mathbb{R}^d$ and $\mathbf{f} : A \rightarrow B$.

If \mathbf{f} is invertible then we cannot denote the inverse by $\mathbf{f}^{-1} : B \rightarrow A$ and

$$[\mathbf{f}^{-1} \cdot \mathbf{f}](\mathbf{x}) = \mathbf{x}$$

If \mathbf{f} and \mathbf{f}^{-1} are differentiable then by the chain rule

$$\mathbf{f}'(\mathbf{f}^{-1}(\mathbf{x})) \mathbf{f}^{-1}'(\mathbf{x}) = I$$

Hence

$$\mathbf{f}^{-1}'(\mathbf{x}) = [\mathbf{f}'(\mathbf{f}(\mathbf{x}))]^{-1}$$

11.3 The Gradient

Definition 11.3 - The Gradient

Let $f : \mathbb{R}^d \rightarrow \mathbb{R}$ be a map with first-order partial derivatives at every point of \mathbb{R}^d .

Then the gradient at a point $\mathbf{x} \in \mathbb{R}^d$, $\nabla f : \mathbb{R}^d \rightarrow \mathbb{R}^d$, is defined as

$$\nabla f(\mathbf{x}) := (f_{x_1}(\mathbf{x}), \dots, f_{x_d}(\mathbf{x}))$$

Definition 11.4 - Direction of Greatest Change

By considering the case in two dimensions then

$$(D_{\mathbf{u}}f)(x, y) = \nabla f(x, y) \cdot \mathbf{u}$$

Where $\mathbf{u} = (\cos\theta, \sin\theta)$.

Then

$$(D_{\mathbf{u}}f)(x_0, y_0) = \cos\theta \cdot \frac{\partial f}{\partial x}(x_0, y_0) + \sin\theta \cdot \frac{\partial f}{\partial y}(x_0, y_0)$$

The greatest change occurs when

$$\begin{aligned} \frac{d}{d\theta} [\cos\theta \cdot \frac{\partial f}{\partial x}(x_0, y_0) + \sin\theta \cdot \frac{\partial f}{\partial y}(x_0, y_0)] &= 0 \\ \Rightarrow -\sin\theta \cdot \frac{\partial f}{\partial x}(x_0, y_0) + \cos\theta \cdot \frac{\partial f}{\partial y}(x_0, y_0) &= \\ \Rightarrow \nabla f(x_0, y_0) \cdot (-\sin\theta, \cos\theta) &= \end{aligned}$$

We can thus deduce that

$$\nabla f(x_0, y_0) \perp (-\sin\theta, \cos\theta)$$

So

$$\mathbf{u} \parallel \nabla f(x_0, y_0)$$

We can then establish the direction, \mathbf{u} occurs when \mathbf{u} satisfies

$$\mathbf{u} = \frac{\nabla f(x_0, y_0)}{\|\nabla f(x_0, y_0)\|}$$

12 Integration over Two-Dimensional Domains

Remark 12.1 - Notation

Let $D \subset \mathbb{R}^2$.

When integrating over D the following notation is used

$$\iint_D f(x, y) dx dy$$

When D is specified e.g. $D = \{(x, y) : g(y) < x < h(y); a < y < b\}$ then we denote it by

$$\int_a^b \left\{ \int_{g(y)}^{h(y)} f(x, y) dx \right\} dy$$

Theorem 12.2 - Rectangular Domain

in terms of distance from origin If we are given a domain between constant values, $D = \{(x, y) : c < y < d; a < x < b\}$ then we perform

$$V = \int_c^d \left\{ \int_a^b f(x, y) dx \right\} dy$$

And can perform the integrations are separate calculations.

Theorem 12.3 - Triangle Domain

In triangular domains the value of x depends on y , so can be expressed as a function of y . If we take $0 < x < a$ and $0 < y < b$ then $x = \frac{ya}{b}$. This gives us a domain of

$$D = \{(x, y) : \frac{ya}{b} < x < a; 0 < y < b\}$$

We can solve the integral by performing:

$$V = \int_0^b \left\{ \int_{\frac{ay}{b}}^a f(x, y) dx \right\} dy$$

Theorem 12.4 - Genreal Domains

If we have a domain, D , define by $D = \{(x, y) \in \mathbb{R}^2 : L(x) \leq y \leq U(x), a \leq x \leq b\}$. Then the integral over D can be found by

$$V = \int_a^b \left\{ \int_{L(x)}^{U(x)} f(x, y) dy \right\} dx$$

13 Polar Co-ordinates

Defintion 13.1 - Cartesian to Polar Co-ordinates

Polar co-ordinates are an alternative way to denoting a point in two dimensional space, using distance from origin and angle from positive x-axis.

Polar co-ordiantes are given in the form (r, θ) . Using pythagoras' theorem and trigonometry we can see that

$$r = \sqrt{x^2 + y^2}, \theta = \tan^{-1} \left(\frac{y}{x} \right)$$

Theorem 13.2 - Polar to Cartesian Co-ordinates

We can convert a polar co-ordinate, (r, θ) , to cartesian co-ordinates, (x, y) using the following formula

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} r \cos(\theta) \\ r \sin(\theta) \end{pmatrix}$$

Remark 13.3 - Plotting Curves expressed as polar co-ordiantes

There are two common ways to do this:

- i) Convert back to cartesian form;
- ii) Interpolation, plot a few points and link them.

Theorem 13.4 - Computing Curvature of Polar Curve

Suppose a curve is defined by $r = R(\theta)$. This means that

$$\begin{aligned}x &= r \cos(\theta) = R(\theta) \cos(\theta) \\ y &= r \sin(\theta) = R(\theta) \sin(\theta)\end{aligned}$$

By substituting into the equation for curvature of a cartesian curve we get

$$k(\theta) = \frac{|R(\theta)^2 + 2[R'(\theta)]^2 - R(\theta)R''(\theta)|}{\{R(\theta)^2 + [R'(\theta)]^2\}^{3/2}}$$

Theorem 13.5 - Gradient of a Polar Curve

Consider the equation $f(x, y) = u(r, \theta)$ then by using substitution and chain rule we can deduce the following equations:

$$\begin{aligned}\frac{\partial f}{\partial x} &= \cos(\theta) \frac{\partial u}{\partial r} - \frac{\sin(\theta)}{r} \frac{\partial u}{\partial \theta} \\ \frac{\partial f}{\partial y} &= \sin(\theta) \frac{\partial u}{\partial r} + \frac{\cos(\theta)}{r} \frac{\partial u}{\partial \theta}\end{aligned}$$

13.1 Change of Variable Formula**Theorem 13.6 - Single Variable Substitution**

By making the substitution of $x = g(u)$ we can produce the following formula

$$\int_{g(a)}^{g(b)} f(x) dx = \int_a^b (f \cdot g)(u) g'(u) du$$

Theorem 13.6 - Double Variable Substitution

Suppose we make the following substitution

$$\begin{pmatrix} x \\ y \end{pmatrix} = \mathbf{g}(u, v) := \begin{pmatrix} g_1(u, v) \\ g_2(u, v) \end{pmatrix}$$

Then we can produce the following equation

$$\iint_{\mathbf{g}(D)} f(x, y) dx dy = \iint_D (f \cdot \mathbf{g})(u, v) |\det\{\mathbf{g}'(u, v)\}| du dv$$

Remark 13.7 - Cartesian to Polar Double Integration

By using the formula in 13.6 we can change produce the following equation, by using the substitution $(x, y) = (r \cos(\theta), r \sin(\theta))$

$$\mathbf{g}(r, \theta) = (r \cos(\theta), r \sin(\theta))$$

$$\iint_{\mathbf{g}(D)} f(x, y) dx dy = \iint_D f(r \cos(\theta), r \sin(\theta)) \cdot r \cdot dr d\theta$$

14 Triple Integration

Remark 14.1 - Triple Integration

If we let D be a three dimensional region, so $D \subset \mathbb{R}^3$. Then by doing

$$\iiint_D f(x, y, z) dx dy dz$$

we produce a *hypervolume*.

By setting $f(x, y, z) = 1$ we can find the volume of D . If we set it to be a variable equation then we can determine other properties such as mass, center of mass etc.

Theorem 14.2 - Box-Like region

If we define D as $D = \{(x, y, z) \in \mathbb{R}^3 : a \leq x \leq A; b \leq y \leq B; c \leq z \leq C\}$. We can integrate over D by

$$\iiint_D f(x, y, z) dx dy dz = \int_c^C \left\{ \int_b^B \left\{ \int_a^A f(x, y, z) dx \right\} dy \right\} dz$$

Theorem 14.3 - General Domains

If we define D as $D = \{(x, y, z) \in \mathbb{R}^3 : a \leq x \leq A(y, z); b \leq y \leq B(z); c \leq z \leq C\}$. We can integrate over D by

$$\iiint_D f(x, y, z) dx dy dz = \int_c^C \left\{ \int_b^{B(z)} \left\{ \int_a^{A(y, z)} f(x, y, z) dx \right\} dy \right\} dz$$

14.1 Substitution with Triple Integrals

Theorem 14.4 - Substitution with Triple Integration

If we want to perform a substitution define by

$$\begin{aligned} x &= g_1(u, v, w), \\ y &= g_2(u, v, w), \\ z &= g_3(u, v, w) \end{aligned}$$

Then we can produce the following equation

$$\iiint_{g(D)} f(x, y, z) dx dy dz = \iiint_D (f \cdot \mathbf{g})(u, v, w) \cdot |\det\{\mathbf{g}'(u, v, w)\}| du dv dw$$

Remark 14.5 - Cartesian to Spherical Triple Integration

If we can convert a cartesian region to a spherical region we use the following substitution

$$\begin{aligned} x &= \rho \sin(\phi) \cos(\theta) = g_1(\rho, \phi, \theta), \\ x &= \rho \sin(\phi) \sin(\theta) = g_2(\rho, \phi, \theta), \\ x &= \rho \sin(\phi) &= g_3(\rho, \phi, \theta) \end{aligned}$$

This means $\det\{\mathbf{g}'(\rho, \phi, \theta)\} = \rho^2 \sin(\phi)$. Then

$$\iiint_{g(D)} f(x, y, z) dx dy dz = \iiint_D (f \cdot \mathbf{g})(\rho, \phi, \theta) \cdot (\rho^2 \sin(\phi)) \cdot d\rho d\phi d\theta$$

14.2 Applications of Multiple Integration

Definition 14.6 - Multiple Integrations

When integrating over multiple dimensions we don't need to write \int every time, instead we can denote it as

$$\int_R f(\mathbf{x}) d\mathbf{x}$$

or

$$\int \cdots \int_R f(\mathbf{x}) dx_1 \cdots dx_d$$

Theorem 14.7 - Centre of Mass of a Volume

By considering a region $R \subset \mathbb{R}^d$ whose density is described by $f(\mathbf{x})$. Then its mass can be found by

$$m = \int_R f(\mathbf{x}) d\mathbf{x}$$

Then its centre of mass, $\bar{\mathbf{x}} \in \mathbb{R}^d$, can be found by

$$\bar{\mathbf{x}} := \frac{1}{m} \int_R \mathbf{x} f(\mathbf{x}) d\mathbf{x}$$

15 Local Extrema & Taylor's Theorem in Several Variables

15.1 Taylor's Theorem

Theorem 15.1 - Taylor's Theorem for a Single Variable

If $f : \mathbb{R} \rightarrow \mathbb{R}$ is smooth and $a \in \mathbb{R}$, then

$$f(x) = f(a) + (x-a)f'(a) + \frac{1}{2!}(x-a)^2 f''(a) + \dots$$

$$f(x) = \sum_{j=1}^{\infty} \frac{1}{j!} (x-a)^j \cdot \left[\frac{d^j f}{dx^j}(a) \right]$$

Theorem 15.2 - Taylor's Theorem for a Two Variables

If $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ is smooth and $a, b \in \mathbb{R}$, then

$$f(x, y) = f(a, b) + f'(a, b) \begin{pmatrix} x-a \\ y-b \end{pmatrix} + \frac{1}{2} \begin{pmatrix} x-a \\ y-b \end{pmatrix}^t f''(a, b) \begin{pmatrix} x-a \\ y-b \end{pmatrix} + \dots$$

N.B - $f'(x, y) \in M_{1,2}(\mathbb{F})$ & $f''(x, y) \in M_2(\mathbb{F})$

15.2 Local Extrema

Definition 15.3 - Local Minima

Let $f : \mathbb{R}^d \rightarrow \mathbb{R}$.

We say the point $\mathbf{x}_0 \in \mathbb{R}^d$ is a local minimum if

$$\exists \delta > 0 \text{ st } \forall \mathbf{x} \in \mathbb{R}^d, \|\mathbf{x} - \mathbf{x}_0\| < \delta \text{ then } f(\mathbf{x}_0) \leq f(\mathbf{x})$$

Definition 15.4 - Local Maxima

Let $f : \mathbb{R}^d \rightarrow \mathbb{R}$.

We say the point $\mathbf{x}_0 \in \mathbb{R}^d$ is a local maxima if

$$\exists \delta > 0 \text{ st } \forall \mathbf{x} \in \mathbb{R}^d, \|\mathbf{x} - \mathbf{x}_0\| < \delta \text{ then } f(\mathbf{x}_0) \geq f(\mathbf{x})$$

Definition 15.5 - Local Extremum

A point is a *Local Extremum* if it is either a local minimum or maximum.

Definition 15.6 - Critical point

A point, \mathbf{x}_0 , is a critical point of f if

$$f'(\mathbf{x}_0) = \mathbf{0}$$

Proposition 15.7 - Extremum and Critical Point

If \mathbf{x}_0 is a local extremum of f then it is a critical point of f .

Remark 15.8 - Classifying Critical Points

Let \mathbf{x}_0 be a critical point of f .

Then Taylor's theorem says that

$$f(\mathbf{x}) = f(\mathbf{x}_0) + \frac{1}{2}(\mathbf{x} - \mathbf{x}_0)^t \cdot f''(\mathbf{x}_0)(\mathbf{x} - \mathbf{x}_0) + \dots$$

Definition 15.9 - Define & Indefinite Matrices

Let $A \in M_d$, then A is

- i) Positive definite if $\forall \mathbf{x} \in \mathbb{R}^d, \mathbf{x} \neq \mathbf{0} \Rightarrow \mathbf{x}^t A \mathbf{x} > 0$;
- ii) Negative definite if $\forall \mathbf{x} \in \mathbb{R}^d, \mathbf{x} \neq \mathbf{0} \Rightarrow \mathbf{x}^t A \mathbf{x} < 0$;
- iii) Indefinite if $\exists \mathbf{x}, \mathbf{y} \in \mathbb{R}^d$ st $\mathbf{x}^t A \mathbf{x} > 0$ & $\mathbf{y}^t A \mathbf{y} < 0$.

Theorem 15.10 Extrema and Definite Matrices

The critical point $\mathbf{x}_0 \in \mathbb{R}^d$ is a

- i) Local minimum iff $f''(\mathbf{x}_0)$ is a positive-definite;
- ii) Local maximum iff $f''(\mathbf{x}_0)$ is a negative-definite;

If $f''(\mathbf{x}_0)$ is indefinite then we say that \mathbf{x}_0 is a *saddle*.

Theorem 15.11 - Definite Matrices and Eigenvalues

We say $f''(\mathbf{x}_0)$ is

- i) Positive definite iff all its eigenvalues are strictly positive;
- ii) Negative definite iff all its eigenvalues are strictly negative;
- iii) Indefinite if it has both positive and negative eigenvalues.

16 Systems of Linear Differential Equations

Theorem 16.1 - Linear Case

Define a function, $\mathbf{f} : \mathbb{R}^d \rightarrow \mathbb{R}^d$, such that $\mathbf{f}(\mathbf{x}) = A\mathbf{x}$ where $\mathbf{x} \in \mathbb{R}^d$ and $A \in M_d$ is diagonalisable. Let $\{\mathbf{v}_1, \dots, \mathbf{v}_d\}$ be eigenvectors for A , with $\{\lambda_1, \dots, \lambda_d\}$ as the corresponding eigenvalues. Then the general solution is

$$\mathbf{x}(t) = \sum_{j=1}^d c_j(0) \cdot e^{\lambda_j \cdot t} \cdot \mathbf{v}_j$$

values of c are determined in the particular solution from the initial solutions.

Definition 16.2 - Equilibria

Let $\mathbf{f} : \mathbb{R}^d \rightarrow \mathbb{R}^d$, $\mathbf{e} \in \mathbb{R}^d$ is said to be an equilibrium of \mathbf{f} if

$$\mathbf{f}(\mathbf{e}) = \mathbf{0}$$

Theorem 16.3 - Particular Solutions and Equilibria

Let $t \in \mathbb{R}$, $\mathbf{f} : \mathbb{R}^d \rightarrow \mathbb{R}^d$ & $\mathbf{x} : \mathbb{R} \rightarrow \mathbb{R}^d$. If

$$\mathbf{f}(\mathbf{x}(t)) = \mathbf{0}$$

then t is a particular solution of \mathbf{f} .

Definition 16.4 - Stable/Unstable Equilibria

Let \mathbf{e} be an equilibrium. \mathbf{e} is said to *stable* if

$$\forall \epsilon > 0, \exists \delta > 0 \text{ st } \forall \mathbf{x}(0) \in \mathbb{R}^d \text{ \& } t \geq 0, \|\mathbf{x}(0) - \mathbf{e}\| < \delta \Rightarrow \|\mathbf{x}(t) - \mathbf{e}\| < \epsilon$$

Otherwise the equilibrium point is *unstable*.

Theorem 16.5 - Determining Stability of Equilibria

If for all eigenvalues, λ_j , of $\mathbf{f}'(\mathbf{e})$ is such that

$$\operatorname{Re}(\lambda_j) < 0$$

then the equilibrium, \mathbf{e} , is *stable*.

If there exists one eigenvalue, λ_i , where $\operatorname{Re}(\lambda_i) > 0$ then \mathbf{e} is *unstable*.