Linear Algebra & Geometry - Application Notes

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How to convert a series of linear equations to an augmented matrix.

Process
When given a series of n linear equations, each with the same m, then an augmented $(n+1) \times m$ matrix can be formed.

i) Set the first n elements of each *row* of the matrix to be the coefficients for one of the linear equations.
ii) Set the *last* element of the row to be the value of this linear equation.

Example
Convert the following linear equations to an augment matrix

$$x + z = 1, 2x - 2y = 1 \ \& \ -3y - 3x = -1$$

$$\begin{pmatrix} 1 & 0 & 1 & 1 \\ 2 & -2 & 0 & 1 \\ 0 & -3 & -3 & -1 \end{pmatrix}$$

How to solve a series of linear equations using an augmented matrix.

Theory
The following elementary row operations can be carried out without altering the solution of an augmented matrix.

i) row $i \mapsto \lambda$row i for $\lambda \in \mathbb{R}\backslash\{0\}$;
ii) row $i \mapsto$ row $i + \lambda$ row j for $\lambda \in \mathbb{R}\backslash\{0\}$; and,
iii) row $j \mapsto$ row i & row $i \mapsto$ row j.

Process

i) Convert the linear equations into an augmented matrix;
ii) Perform elementary rows operations until the *first* n rows and columns form an identity matrix; and,
iii) The values in the *last* column are the solutions.

Example
Solve the following system of linear equations

$$x+y+2z=9, 2x+4y-3z=1 \ \& \ 3x+6y-5z=0$$

$$\begin{aligned}
\begin{pmatrix}1&1&2&9\\2&4&-3&1\\3&6&-5&0\end{pmatrix} &= \begin{pmatrix}1&1&2&9\\0&2&-7&-17\\0&3&-11&-27\end{pmatrix} = \begin{pmatrix}1&1&2&9\\0&2&-7&-17\\0&1&-4&-10\end{pmatrix}\\
&= \begin{pmatrix}1&1&2&9\\0&0&1&3\\0&1&-4&-10\end{pmatrix} = \begin{pmatrix}1&1&2&9\\0&1&-4&-10\\0&0&1&3\end{pmatrix}\\
&= \begin{pmatrix}1&1&0&3\\0&1&0&2\\0&0&1&3\end{pmatrix} = \underline{\begin{pmatrix}1&0&0&1\\0&1&0&2\\0&0&1&3\end{pmatrix}}
\end{aligned}$$

So the only solution to this system of linear equations is $x=1, y=2 \ \& \ z=3$.

How to find the Determinant

Process
The determinant can be found using *Laplace's Rule.*

$$det(A)=\sum_{j=1}^{n} a_{ij}.(-1)^{i+j}.det(A_{ij}) \text{ for a fixed } i$$

Where A_{ij} is the matrix A without its $\underline{i^{th} \text{ row and } j^{th} \text{ column.}}$

Example
Find the determinant of $\begin{pmatrix}1&2&3\\4&5&6\\7&8&9\end{pmatrix}$.

$$\begin{aligned}
\begin{vmatrix}1&2&3\\4&5&6\\7&8&9\end{vmatrix} &= 1\begin{vmatrix}5&6\\8&9\end{vmatrix} - 2\begin{vmatrix}4&6\\7&9\end{vmatrix} + 3\begin{vmatrix}4&5\\7&8\end{vmatrix}\\
&= 1[5\times 9-6\times 8]-2[4\times 9-6\times 7]+3[4\times 8-5\times 7]\\
&= 1(-3)-2(-6)+3(-3)\\
&= -3+12-9\\
&= \underline{0}
\end{aligned}$$

How to Invert a Matrix

Theory
A matrix is *invertible* iff $det(A)\neq 0$.
An inverted matrix, A^{-1}, obeys $AA^{-1}=I$.
The *adjoint* of a matrix can be found using

$$(adj(A))_{ij}=(-1)^{i+j}det(A^t_{ij})$$

Where A^t_{ij} is the transpose of matrix A without its $\underline{i^{th} \text{ row and } j^{th} \text{ column.}}$

<u>*Process*</u>
The inverse of a matrix A can be found by

$$A^{-1} = \frac{adj(A)}{det(A)}$$

<u>*Example of Adjoint Matrix*</u>

Find the adjoint of $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$.

$$\begin{aligned}
A^t &= \begin{pmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{pmatrix} \\
adj(A) &= \begin{pmatrix} +\begin{vmatrix} 5 & 8 \\ 6 & 9 \end{vmatrix} & -\begin{vmatrix} 2 & 8 \\ 3 & 9 \end{vmatrix} & +\begin{vmatrix} 2 & 5 \\ 3 & 6 \end{vmatrix} \\ -\begin{vmatrix} 4 & 7 \\ 6 & 9 \end{vmatrix} & +\begin{vmatrix} 1 & 7 \\ 3 & 9 \end{vmatrix} & -\begin{vmatrix} 1 & 4 \\ 3 & 6 \end{vmatrix} \\ +\begin{vmatrix} 4 & 7 \\ 5 & 8 \end{vmatrix} & -\begin{vmatrix} 1 & 7 \\ 2 & 8 \end{vmatrix} & +\begin{vmatrix} 1 & 4 \\ 2 & 5 \end{vmatrix} \end{pmatrix} \\
&= \underline{\begin{pmatrix} -3 & 6 & -3 \\ 6 & -12 & 6 \\ -3 & 6 & -3 \end{pmatrix}}
\end{aligned}$$

Example of Inverse Matrix

Find the inverse of $A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 5 & 6 & 0 \end{pmatrix}$

$$\begin{aligned}
A^t &= \begin{pmatrix} 1 & 0 & 5 \\ 2 & 1 & 6 \\ 3 & 4 & 0 \end{pmatrix} \\
adj(A) &= \begin{pmatrix} +\begin{vmatrix} 1 & 6 \\ 4 & 0 \end{vmatrix} & -\begin{vmatrix} 2 & 6 \\ 3 & 0 \end{vmatrix} & +\begin{vmatrix} 2 & 1 \\ 3 & 4 \end{vmatrix} \\ -\begin{vmatrix} 0 & 5 \\ 4 & 0 \end{vmatrix} & +\begin{vmatrix} 1 & 5 \\ 3 & 0 \end{vmatrix} & -\begin{vmatrix} 1 & 0 \\ 3 & 4 \end{vmatrix} \\ +\begin{vmatrix} 0 & 5 \\ 1 & 6 \end{vmatrix} & -\begin{vmatrix} 0 & 5 \\ 2 & 6 \end{vmatrix} & +\begin{vmatrix} 1 & 0 \\ 2 & 1 \end{vmatrix} \end{pmatrix} \\
&= \begin{pmatrix} -24 & 18 & 5 \\ 20 & -15 & -4 \\ -5 & 4 & -1 \end{pmatrix} \\
det(A) &= 1\begin{vmatrix} 1 & 4 \\ 6 & 0 \end{vmatrix} - 2\begin{vmatrix} 0 & 4 \\ 5 & 0 \end{vmatrix} + 3\begin{vmatrix} 0 & 1 \\ 5 & 6 \end{vmatrix} \\
&= -24 + 40 - 15 \\
&= 1 \\
A^{-1} &= \frac{adj(A)}{det(A)} \\
&= \frac{1}{1}\begin{pmatrix} -24 & 18 & 5 \\ 20 & -15 & -4 \\ -5 & 4 & -1 \end{pmatrix} \\
&= \underline{\begin{pmatrix} -24 & 18 & 5 \\ 20 & -15 & -4 \\ -5 & 4 & -1 \end{pmatrix}}
\end{aligned}$$

How to Form a Basis

Theory

Let V be a vector space.
Then $\{v_1, \dots, v_n\}$ are a basis for V if they are all linearly independent
and $V = span(\{v_1, \dots, v_n\})$.

Process

Let $v_1, \dots, v_n \in V$ and $\{w_1, \dots, w_n\}$ be a basis, to be formed, for V.

i) Construct the augmented matrix $\begin{pmatrix} v_1 & \dots & v_n & 0 \end{pmatrix}$;

ii) Reduce this matrix to row echeldon form;

iii) Identify the vectors which are associated to columns with the leading 1s. These vectors form a basis for the rest.

N.B. - If the reduced matrix has an all zero row then the given vectors are **not** linearly independent.

Example

Find a basis for $\left\{ \begin{pmatrix}1\\2\\1\end{pmatrix}, \begin{pmatrix}-1\\-1\\1\end{pmatrix}, \begin{pmatrix}1\\4\\5\end{pmatrix}, \begin{pmatrix}3\\4\\-1\end{pmatrix}, \begin{pmatrix}0\\1\\2\end{pmatrix}, \begin{pmatrix}4\\9\\6\end{pmatrix} \right\}$.

$$\begin{aligned}
\text{Augmented matrix} \quad & \begin{pmatrix}1 & -1 & 1 & 3 & 0 & 4 & 0\\ 2 & -1 & 4 & 4 & 1 & 9 & 0\\ 1 & 1 & 5 & -1 & 2 & 6 & 0\end{pmatrix}\\
= & \begin{pmatrix}1 & -1 & 1 & 3 & 0 & 4 & 0\\ 0 & 1 & 2 & -2 & 1 & 1 & 0\\ 0 & 2 & 4 & -4 & 2 & 2 & 0\end{pmatrix}\\
= & \begin{pmatrix}1 & -1 & 1 & 3 & 0 & 4 & 0\\ 0 & 1 & 2 & -2 & 1 & 1 & 0\\ 0 & 0 & 0 & 0 & 0 & 0 & 0\end{pmatrix}
\end{aligned}$$

So the first & second vectors, $\begin{pmatrix}1\\2\\1\end{pmatrix}$ & $\begin{pmatrix}-1\\-1\\1\end{pmatrix}$, form a basis for the rest of the vectors.

How to Check is a Basis is Orthogonal

Theory

Let V be a vector space and $v, w \in V$.
Then v & w are orthogonal if

$$v \cdot w = 0$$

This means v & w are at $\frac{\pi}{2} rads$ to each other.
A basis, $\{v_1, \dots, v_n\}$, is orthogonal if

$$v_i \cdot v_j = 0 \text{ iff } i \neq j$$

Process

Say we are given n vectors.
Then we need to check the *dot product* of all possible combinations obey the conditions.
As $v \cdot w = w \cdot v \ \forall \ v, w$ then there are $\frac{1}{2}n(n+1)$ such combinations.

Example

Are $\boldsymbol{v}_1 = \begin{pmatrix}-1\\2\\1\end{pmatrix}$, $\boldsymbol{v}_2 = \begin{pmatrix}1\\0\\1\end{pmatrix}$ & $\boldsymbol{v}_3 = \begin{pmatrix}-1\\-1\\1\end{pmatrix}$ orthogonal.

$$\begin{array}{lclcl}
\boldsymbol{v}_1 \cdot \boldsymbol{v}_1 & = & 1+4+1 & = & 6\\
\boldsymbol{v}_1 \cdot \boldsymbol{v}_2 & = & -1+0+1 & = & 0\\
\boldsymbol{v}_1 \cdot \boldsymbol{v}_3 & = & 1-2+1 & = & 0\\
\boldsymbol{v}_2 \cdot \boldsymbol{v}_2 & = & 1+0+1 & = & 2\\
\boldsymbol{v}_2 \cdot \boldsymbol{v}_3 & = & -1+0+1 & = & 0\\
\boldsymbol{v}_3 \cdot \boldsymbol{v}_3 & = & 1+1+1 & = & 3
\end{array}$$

Yes they are.

How to Prove a Map is a Linear Map

Theory

A map, T, is a linear map if

i) $T(x+y) = T(x) + T(y)\ \forall\ x, y$; and,

ii) $T(\lambda x) = \lambda T(x)$.

Process

Need to prove both of these are conditions are true in all cases.

Example

Prove that $T : \mathbb{R}^2 \to \mathbb{R}^3$ defined as $T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \\ x+2y \end{pmatrix}$ is a linear map.

Let $\boldsymbol{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$, $\boldsymbol{y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$ and $\lambda \in \mathbb{R}$.

$$\begin{aligned}
T(\boldsymbol{x}+\boldsymbol{y}) &= T\left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}\right) \\
&= T\left(\begin{pmatrix} x_1 + x_1 \\ x_2 + y_2 \end{pmatrix}\right) \\
&= \begin{pmatrix} x_1 + y_1 \\ x_2 + y_2 \\ x_1 + y_1 + 2x_2 + 2y_2 \end{pmatrix} \\
&= \begin{pmatrix} x_1 + y_1 \\ x_2 + y_2 \\ (x_1 + 2x_2) + (y_1 + 2y_2) \end{pmatrix} \\
&= \begin{pmatrix} x_1 \\ x_2 \\ x_1 + 2x_2 \end{pmatrix} + \begin{pmatrix} y_1 \\ y_2 \\ y_1 + 2y_2 \end{pmatrix} \\
&= T(\boldsymbol{x}) + T(\boldsymbol{y})
\end{aligned}$$

$$\begin{aligned}
T(\lambda\boldsymbol{x}) &= T\left(\begin{pmatrix} \lambda x_1 \\ \lambda x_2 \end{pmatrix}\right) \\
&= \begin{pmatrix} \lambda x_1 \\ \lambda x_2 \\ \lambda x_1 + 2\lambda x_2 \end{pmatrix} \\
&= \begin{pmatrix} \lambda x_1 \\ \lambda x_2 \\ \lambda(x_1 + 2x_2) \end{pmatrix} \\
&= \lambda \begin{pmatrix} x_1 \\ x_2 \\ x_1 + 2x_2 \end{pmatrix} \\
&= \lambda T(\boldsymbol{x})
\end{aligned}$$

T is a linear map.

How to Represent a Linear Map as a Matrix

Process
Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear map.
Then T can be represented by the matrix $M_T \in M_{m,n}(\mathbb{R})$ where $M_T x = T(x)$.

$$t_{ij} = e_i \cdot T(e_j), \quad e_j \in \mathbb{R}^n \ \& \ e_i \in \mathbb{R}^m$$

Perform this for all combinations of $i \in [1, m]$ & $j \in [1, n]$

Example
Write $T : \mathbb{R}^2 \to \mathbb{R}^3$ defined as $T\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \\ x + 2y \end{pmatrix}$ as a matrix.

$$T(\boldsymbol{e}_1) = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \ \& \ T(\boldsymbol{e}_2) = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$$

$$\begin{aligned}
t_{11} &= \boldsymbol{e}_1 \cdot T(\boldsymbol{e}_1) &&= (1,0,0) \cdot (1,0,1) &&= 1+0+0 = 1 \\
t_{12} &= \boldsymbol{e}_2 \cdot T(\boldsymbol{e}_1) &&= (1,0,0) \cdot (0,1,2) &&= 0+0+0 = 0 \\
t_{21} &= \boldsymbol{e}_2 \cdot T(\boldsymbol{e}_1) &&= (0,1,0) \cdot (1,0,1) &&= 0+0+0 = 0 \\
t_{22} &= \boldsymbol{e}_2 \cdot T(\boldsymbol{e}_2) &&= (0,1,0) \cdot (0,1,2) &&= 0+1+0 = 1 \\
t_{31} &= \boldsymbol{e}_3 \cdot T(\boldsymbol{e}_1) &&= (0,0,1) \cdot (1,0,1) &&= 0+0+1 = 1 \\
t_{32} &= \boldsymbol{e}_3 \cdot T(\boldsymbol{e}_2) &&= (0,0,1) \cdot (0,1,2) &&= 0+0+2 = 2 \\
M_T &= \underline{\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 2 \end{pmatrix}}
\end{aligned}$$

How to Find the Cross Product

Theory
The cross product is a vector which is perpendicular to the plane formed by two other vectors in $\mathbb{R}^3$.
The magnitude of this vector is equal to the area parallelogram formed with these two vectors as the sides.
The cross product is denoted as $\boldsymbol{x} \times \boldsymbol{y}$.

Process

$$\boldsymbol{x} \times \boldsymbol{y} = \begin{vmatrix} \boldsymbol{e}_1 & \boldsymbol{e}_2 & \boldsymbol{e}_3 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix} = \boldsymbol{e}_1 \begin{vmatrix} x_2 & x_3 \\ y_2 & y_3 \end{vmatrix} - \boldsymbol{e}_2 \begin{vmatrix} x_1 & x_3 \\ y_1 & y_3 \end{vmatrix} + \boldsymbol{e}_3 \begin{vmatrix} x_1 & x_2 \\ y_1 & y_2 \end{vmatrix}$$

Example
Let $\boldsymbol{v} = \begin{pmatrix} 3 \\ -1 \\ 5 \end{pmatrix}$ and $\boldsymbol{w} = \begin{pmatrix} 0 \\ 4 \\ -2 \end{pmatrix}$. Compute $\boldsymbol{v} \times \boldsymbol{w}$.

$$\begin{aligned}
\boldsymbol{v} \times \boldsymbol{w} &= \begin{vmatrix} \boldsymbol{e}_1 & \boldsymbol{e}_2 & \boldsymbol{e}_3 \\ 3 & -1 & 5 \\ 0 & 4 & -2 \end{vmatrix} \\
&= \boldsymbol{e}_1 \begin{vmatrix} -1 & 5 \\ 4 & -2 \end{vmatrix} - \boldsymbol{e}_2 \begin{vmatrix} 3 & 5 \\ 0 & -2 \end{vmatrix} + \boldsymbol{e}_3 \begin{vmatrix} 3 & -1 \\ 0 & 4 \end{vmatrix} \\
&= \boldsymbol{e}_1(-18) - \boldsymbol{e}_2(-6) + \boldsymbol{e}_3(12) \\
&= \underline{\begin{pmatrix} -18 \\ 6 \\ 12 \end{pmatrix}}
\end{aligned}$$

How to prove a Set is a Vector Space

Theory
A *field* is a set of numbers for which addition & multiplication are defined.
A *vector space* is aset of vectors which are part of a field and obey the following conditions.

i) V is abelian wrt addition, $v + w = w + v \ \forall \ v, w \in V$;
ii) $\lambda v \in V \ \forall \ \lambda \in \mathbb{F}, v \in V$;
iii) $\lambda(v + w) = \lambda v + \lambda w$;
iv) $\lambda(\mu v) = (\lambda\mu)v$; and,
v) $1.v = v$.

Process
Need to prove these are conditions are true in all cases.

Example
Let $V \subset \mathbb{R}^2$ be defined by $V = \{(v, 2v) : v \in \mathbb{R}\}$
Show V is a vector space over $\mathbb{R}$.

Let $\boldsymbol{x}, \boldsymbol{y} \in V$ such that $\boldsymbol{x} = (x, 2x)$ & $\boldsymbol{y} = (y, 2y)$.

$$\boldsymbol{x} + \boldsymbol{y} = (x + y, 2x + 2y) = (y + x, 2y + 2x) = \boldsymbol{y} + \boldsymbol{x}.$$

So V is abelian.

Let $\lambda \in \mathbb{R}$.
$\lambda\boldsymbol{x} = (\lambda x, 2\lambda\boldsymbol{x})$
$\lambda\boldsymbol{x} \in \mathbb{R}$ so $\lambda\boldsymbol{x} \in V$

$$\begin{aligned}
\lambda(\boldsymbol{x} + \boldsymbol{y}) &= \lambda(x + y, 2x + 2y) \\
&= (\lambda(x + y), \lambda(2x + 2y)) \\
&= (\lambda x + \lambda y, 2\lambda x + 2\lambda y) \\
&= (\lambda x, 2\lambda x) + (\lambda y, 2\lambda y) \\
&= \lambda\boldsymbol{x} + \lambda\boldsymbol{y}
\end{aligned}$$

Let $\mu \in \mathbb{R}$.

$$\begin{array}{rcl} \lambda(\mu\boldsymbol{x}) & = & \lambda(\mu x, 2\mu x) \\ & = & (\lambda\mu x, 2\lambda\mu x) \\ & = & ((\lambda\mu)x, 2(\lambda\mu)x) \\ & = & (\lambda\mu)\boldsymbol{x} \end{array}$$

$$\begin{array}{rcl} 1.\boldsymbol{x} & = & (1.x, 1.2.x) \\ & = & (x, 2x) \\ & = & \boldsymbol{x} \end{array}$$

$\underline{V \text{ is a vector space.}}$

How to Change Basis
Let $A = \{a_1, \dots, a_n\}$ and $B = \{b_1, \dots, b_n\}$ be different bases of a vector space V.

Process i) - Conversion Function
Define $T_A : \mathbb{F}^n \to V$ such that $T_A(x_1, \dots, x_n) = x_1\boldsymbol{a}_1 + \dots + x_n\boldsymbol{a}_n$ This will convert $\boldsymbol{x}$ from the standard basis to A.

Process ii) - Transition Matrix
A matrix, $C_{AB} \in M_n$, can be used to convert from basis A to B such that

$$T_B(\boldsymbol{x}) = C_{AB}T_A(\boldsymbol{x})$$

To do this follow

i) Determine each vector of B interms of the vectors of A;
ii) Fill C_{AB} by placing the values from each vector in a seperate row.

Example

Find a transition matrix between $A = \left\{\begin{pmatrix}-3\\0\\-3\end{pmatrix}, \begin{pmatrix}-3\\2\\1\end{pmatrix}, \begin{pmatrix}1\\6\\1\end{pmatrix}\right\}$ and $B = \left\{\begin{pmatrix}-6\\-6\\0\end{pmatrix}, \begin{pmatrix}-2\\-6\\4\end{pmatrix}, \begin{pmatrix}-2\\-3\\7\end{pmatrix}\right\}$

Set $\begin{pmatrix}-6\\-6\\0\end{pmatrix} = \begin{pmatrix}-3a_1 - 3b_1 + c_1\\2b_1 + 6c_1\\-3a_1 + b_1 + c_1\end{pmatrix}$
Then

$$\begin{array}{rrcrcrcl} & 2b_1 + 6c_1 & = & -6 & \implies & b_1 & = & -3 - 3c_1 \\ \implies & -3a_1 + (-3 - 3c_1) + c_1 & = & 0 & \implies & 3a_1 & = & -2c_1 - 3 \\ \implies & -(-2c_1 - 3) - 3(-3 - 3c_1) + c_1 & = & -6 & \implies & 12c_1 & = & -18 \\ \implies & \multicolumn{7}{l}{c_1 = \underline{-3/2}, b_1 = \underline{3/2} \text{ \& } a_1 = \underline{0}} \end{array}$$

Set $\begin{pmatrix}-2\\-6\\4\end{pmatrix} = \begin{pmatrix}-3a_2 - 3b_2 + c_2\\2b_2 + 6c_2\\-3a_2 + b_2 + c_2\end{pmatrix}$
Then

$$\begin{array}{rrcrcrcl} & 2b_2 + 6c_2 & = & -6 & \implies & b_2 & = & -3 - 3c_2 \\ \implies & -3a_2 + (-3 - 3c_2) + c_2 & = & 4 & \implies & 3a_2 & = & -2c_2 - 7 \\ \implies & -(-2c_2 - 7) - 3(-3 - 3c_2) + c_2 & = & -2 & \implies & 12c_2 & = & -18 \\ \implies & \multicolumn{7}{l}{c_2 = \underline{-3/2}, b_2 = \underline{3/2} \text{ \& } a_2 = \underline{-4/3}} \end{array}$$

Set $\begin{pmatrix}-2\\-3\\7\end{pmatrix} = \begin{pmatrix}-3a_3-3b_3+c_3\\2b_3+6c_3\\-3a_3+b_3+c_3\end{pmatrix}$
Then

$$\begin{array}{rrcrcrcl} & 2b_3+6c_3 & = & -3 & \implies & b_3 & = & -\frac{3}{2}-3c_3 \\ \implies & -3a_3+(-\frac{3}{2}-3c_3)+c_3 & = & 7 & \implies & 3a_3 & = & -2c_3-\frac{17}{2} \\ \implies & -(-2c_3-\frac{17}{2})-3(-\frac{3}{2}-3c_3)+c_3 & = & -2 & \implies & 12c_3 & = & -15 \\ \implies & c_3=\underline{-5/2}, b_3=\underline{-2} \;\&\; a_3=\underline{9/4} & & & & & & \end{array}$$

. So $\underline{M_T = \begin{pmatrix}0 & \frac{3}{2} & -\frac{3}{2}\\ -\frac{4}{3} & \frac{3}{2} & -\frac{3}{2}\\ -2 & \frac{9}{4} & -\frac{5}{4}\end{pmatrix}}$

How to Find Eigenvalues of a Matrix

Theory
Let V be a vector space & $T: V \to V$ be a linear operator.
If $\exists\ \boldsymbol{v} \in V\backslash\{0\}$ st $T(\boldsymbol{v}) = \lambda\boldsymbol{v}$ for $\lambda \in \mathbb{F}$ then λ is a *eigenvalue* of T.

Process

i) Solve $det(A-\lambda I)=0$;
ii) The $\lambda \in \mathbb{F}$ for which this is true are *eigenvalues of* A.

Example

Find the eigenvalues for $A = \begin{pmatrix}1 & 1 & 1\\0 & -2 & 0\\1 & 0 & 1\end{pmatrix}$.
Set $det(A-\lambda I)=0$

$$\begin{array}{rrcl} \implies & \begin{vmatrix}1-\lambda & 1 & 1\\0 & -2-\lambda & 0\\1 & 0 & 1-\lambda\end{vmatrix} & = & 0 \\ \implies & (1-\lambda)\begin{vmatrix}-2-\lambda & 0\\0 & 1-\lambda\end{vmatrix} - 1\begin{vmatrix}0 & 0\\1 & 1-\lambda\end{vmatrix} + 1\begin{vmatrix}0 & -2-\lambda\\1 & 0\end{vmatrix} & = & (1-\lambda)(1-\lambda)(-2-\lambda)-0-(-2-\lambda) \\ & & = & -(2+\lambda)\left[(1-\lambda)^2-1\right] \\ & & = & -(2+\lambda)\left[\lambda^2-2\lambda\right] \\ & & = & -\lambda(2+\lambda)(\lambda-2) \end{array}$$

$\lambda = 0, -2, 2$ are solutions to this and thus are the eigenvalues of A.

How to Find Eigenvectors of a Matrix

Theory
Let V be a vector space & $T: V \to V$ be a linear operator.
If $\exists\ \lambda \in \mathbb{F}$ st $T(\boldsymbol{v}) = \lambda(\boldsymbol{v})$ for $\boldsymbol{v} \in V\backslash\{0\}$ then $\boldsymbol{v}$ is a *eigenvector* of T.

Process

i) Find the eigenvalues, $\{\lambda_1, \dots, \lambda_n\}$, of A;
ii) Find the $\boldsymbol{x}$ that satisfy $(A-\lambda_i I)\boldsymbol{x} = \mathbf{0}$ for atleast one $\lambda_i \in \{\lambda_1, \dots, \lambda_n\}$;
iii) These $\boldsymbol{x}$ are the *eigenvectors* of A.

Example

Find the eigenvectors for $A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & 1 \end{pmatrix}$.

The eigenvalues for A are $\lambda_1 = 0, \lambda_2 = -2$ & $\lambda_3 = 2$.

$$\begin{array}{rlcl}
 & \text{Set } (A - \lambda_1 I)\boldsymbol{x} & = & \mathbf{0} \\
\implies & A\boldsymbol{x} & = & \mathbf{0} \\
\implies & \begin{pmatrix} 1 & 1 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} & = & \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \\
\implies & x_2 = 0 \text{ \& } x_1 + x_3 = 0 & & \\
\implies & x_3 = -x_1 & & \\
 & \text{So } \boldsymbol{x} = c \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} & &
\end{array}$$

$$\begin{array}{rlcl}
 & \text{Set } (A - \lambda_2 I)\boldsymbol{x} & = & \mathbf{0} \\
\implies & (A + 2I)\boldsymbol{x} & = & \mathbf{0} \\
\implies & \begin{pmatrix} 3 & 1 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} & = & \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \\
\implies & x_1 + 3x_3 = 0 & & \\
\implies & x_1 = -3x_3 & & \\
 & 3x_1 + x_2 + x_3 = 0 & \implies & -9x_3 + x_2 + x_3 = 0 \\
\implies & x_2 = 8x_3 & & \\
 & \text{So } \boldsymbol{x} = c \begin{pmatrix} -3 \\ 8 \\ 1 \end{pmatrix} & &
\end{array}$$

$$\begin{array}{rlcl}
 & \text{Set } (A - \lambda_3 I)\boldsymbol{x} & = & \mathbf{0} \\
\implies & (A - 2I)\boldsymbol{x} & = & \mathbf{0} \\
\implies & \begin{pmatrix} 1- & 1 & 1 \\ 0 & -4 & 0 \\ 1 & 0 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} & = & \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \\
\implies & x_2 = 0 \text{ \& } x_1 - x_3 = 0 & & \\
\implies & x_3 = x_1 & & \\
 & \text{So } \boldsymbol{x} = c \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} & &
\end{array}$$

Then $\underline{\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} -3 \\ 8 \\ 1 \end{pmatrix} \text{ \& } \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}}$ are eigenvectors of A.

Inner Product

Theory

The *inner product* is a generalised form of the dot product which accounts for the complex plane.

<u>*General Properties*</u>

i) $\langle v, v\rangle = \|v\|^2 \geq 0$;

ii) $\langle u, v+w\rangle = \langle u, v\rangle + \langle u, w\rangle$; and,

iii) $\langle \lambda u, v\rangle = \lambda\langle u, v\rangle = \langle u, \lambda v\rangle$.

<u>*Difference in Properties between Real & Complex Plane*</u>
Let $r_1, r_2 \in V \subset \mathbb{R}^n$ & $c_1, c_2 \in W \subset \mathbb{C}^m$.
Then $\langle r_1, r_2\rangle = \langle r_2, r_1\rangle$ but $\langle c_1, c_2\rangle = \overline{\langle c_2, c_1\rangle}$ only.

How to construct an Orthonormal Basis

<u>*Theory*</u>
An *orthonormal basis* is a basis where all the vectors are *orthogonal* to each other, and are of *unit length.*
So $\langle v_i, v_j\rangle = 0$ if $i \neq j$ and $\langle v_i, v_i\rangle = 1$.

<u>*Process*</u>

i) Take a set of n linearly independent vecotrs, $\{v_1, \dots, v_n\} \subset V \subset \mathbb{F}^n$;

ii) Define $x_j = \dfrac{v_j - \sum_{i=1}^{j-1}\langle x_i, v_j\rangle x_i}{\|v_j - \sum_{i=1}^{j-1}\langle x_i, v_j\rangle x_i\|}$;

iii) Repeat this $\forall\ j \in [1, n]$;

iv) Then $\{x_1, \dots, x_n\}$ is an orthonormal basis of V.

<u>*Example*</u>
Let $\boldsymbol{v}_1 = \begin{pmatrix}0\\0\\1\\1\end{pmatrix}, \boldsymbol{v}_2 = \begin{pmatrix}0\\1\\1\\0\end{pmatrix}$ and $\boldsymbol{v}_3 = \begin{pmatrix}1\\1\\0\\0\end{pmatrix}$.
Find an orthonormal basis for V where $V = span(\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\})$.

Set $\boldsymbol{x}_1 = \dfrac{\boldsymbol{v}_1}{\|\boldsymbol{v}_1\|} = \dfrac{1}{\sqrt{2}}\begin{pmatrix}0\\0\\1\\1\end{pmatrix}$

Then $\langle \boldsymbol{v}_2, \boldsymbol{x}_1\rangle = \overline{\frac{1}{\sqrt{2}}(1) = \frac{1}{\sqrt{2}}}$

and $\boldsymbol{v}_2 - \langle \boldsymbol{v}_2, \boldsymbol{x}_1\rangle\boldsymbol{x}_1 = \begin{pmatrix}0\\1\\1\\0\end{pmatrix} - \begin{pmatrix}0\\0\\1/2\\1/2\end{pmatrix} = \begin{pmatrix}0\\1\\1/2\\-1/2\end{pmatrix}$

Set $\boldsymbol{x}_2 = \dfrac{\boldsymbol{v}_2 - \langle \boldsymbol{v}_2, \boldsymbol{x}_1\rangle\boldsymbol{x}_1}{\|\boldsymbol{v}_2 - \langle \boldsymbol{v}_2, \boldsymbol{x}_1\rangle\boldsymbol{x}_1\|} = \sqrt{\dfrac{2}{3}}\begin{pmatrix}0\\1\\1/2\\-1/2\end{pmatrix}$

Then $\langle \boldsymbol{v}_3, \boldsymbol{x}_1\rangle = \frac{1}{\sqrt{2}}(0) = 0$ and $\overline{\langle \boldsymbol{v}_3, \boldsymbol{x}_2\rangle = \sqrt{\tfrac{2}{3}}(1) = \sqrt{\tfrac{2}{3}}}$

So $\boldsymbol{v}_3 - \langle \boldsymbol{v}_3, \boldsymbol{x}_1\rangle\boldsymbol{x}_1 - \langle \boldsymbol{v}_3, \boldsymbol{x}_2\rangle\boldsymbol{x}_2 = \boldsymbol{v}_3 - \sqrt{\frac{2}{3}}\boldsymbol{x}_2 = \begin{pmatrix}1\\1\\0\\0\end{pmatrix} - \frac{2}{3}\begin{pmatrix}0\\1\\1/2\\-1/2\end{pmatrix} = \begin{pmatrix}1\\1/3\\-1/3\\1/3\end{pmatrix}$

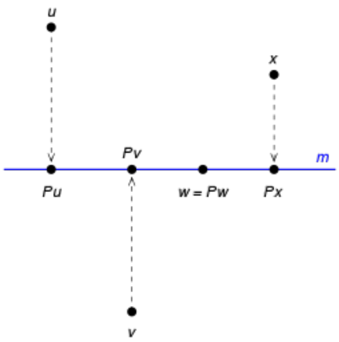
Set $\boldsymbol{x}_3 = \dfrac{\boldsymbol{v}_3 - \langle \boldsymbol{v}_3, \boldsymbol{x}_1\rangle \boldsymbol{x}_1 - \langle \boldsymbol{v}_3, \boldsymbol{x}_2\rangle \boldsymbol{x}_2}{\|\boldsymbol{v}_3 - \langle \boldsymbol{v}_3, \boldsymbol{x}_1\rangle \boldsymbol{x}_1 - \langle \boldsymbol{v}_3, \boldsymbol{x}_2\rangle \boldsymbol{x}_2\|} = \underline{\sqrt{\dfrac{3}{4}}\begin{pmatrix}1\\1/3\\-1/3\\1/3\end{pmatrix}}$

Thus $\underline{\dfrac{1}{\sqrt{2}}\begin{pmatrix}0\\0\\1\\1\end{pmatrix}, \sqrt{\dfrac{2}{3}}\begin{pmatrix}0\\1\\1/2\\-1/2\end{pmatrix} \;\&\; \sqrt{\dfrac{3}{4}}\begin{pmatrix}1\\1/3\\-1/3\\1/3\end{pmatrix}}$ form an orthonormal basis of V.

How to Find the Distance of a Point from a Plane

Theory

A *projection* is a linear map where $P^2 = P$.
So repeating a projection does not change its value.
This is clear when you consider that a projection projects points onto a single plane, so when a point on this plane is projected it doesn't move.



Process

Let $W \subset \mathbb{R}^n$ be a plane and $v \in \mathbb{R}^{n+1}$ be a point.

i) Find an orthonormal basis for W, $\{x_1, \dots, x_n\}$;
ii) Define the projection $P(v) = \langle x_1, v\rangle x_1 + \cdots + \langle x_n, v\rangle x_n$;
iii) Then $P(v)$ is the vector from v to the closest point on W;
iv) So $\|P(v)\|$ is the distance of v from W.

Example

Let $W = span\left(\{(1,1,1,1), (-1,4,4,1), (4,-2,2,0)\}\right) \subset \mathbb{R}^4$. Find the distance of $\boldsymbol{v} = (1,2,3,4)$ from W.

W has an orthonormal basis of $\begin{pmatrix}1/2\\1/2\\1/2\\1/2\end{pmatrix}, \begin{pmatrix}-1/\sqrt{2}\\\sqrt{2}/3\\\sqrt{2}/3\\-1/3\sqrt{2}\end{pmatrix} \;\&\; \begin{pmatrix}1/2\sqrt{3}\\-5/6\sqrt{3}\\7/6\sqrt{3}\\-5/6\sqrt{3}\end{pmatrix}$.

Then

$$\begin{aligned}
\langle \boldsymbol{x}_1, \boldsymbol{v}\rangle &= \tfrac{1}{2}(1+2+3+4) &&= 5\\
\langle \boldsymbol{x}_2, \boldsymbol{v}\rangle &= \tfrac{1}{3\sqrt{2}}(-3+4+6-4) &&= \tfrac{1}{\sqrt{2}}\\
\langle \boldsymbol{x}_3, \boldsymbol{v}\rangle &= \tfrac{1}{2\sqrt{3}}(1+\tfrac{10}{3}+7-\tfrac{20}{3}) &&= \tfrac{-1}{\sqrt{3}}
\end{aligned}$$

So $P(\boldsymbol{v}) = 5\boldsymbol{x}_1 + \dfrac{1}{\sqrt{2}}\boldsymbol{x}_2 - \dfrac{1}{\sqrt{3}}\boldsymbol{x}_3 = \left(\dfrac{11}{6}, \dfrac{28}{9}, \dfrac{22}{9}, \dfrac{47}{18}\right)$

$\|\boldsymbol{v} - P(\boldsymbol{v})\| = \left\|\left(\dfrac{-5}{6}, \dfrac{-10}{9}, \dfrac{5}{9}, \dfrac{25}{18}\right)\right\| = \underline{\dfrac{5}{\sqrt{6}}}$

Hermitian

Theory
A square matrix, $A \in M_n(\mathbb{F})$, is a *hermitian* if $A = \bar{A}^t = A^*$.
i.e. A equals the complex conjugate of its transpose.
The eigenvectors of a hermitian form an orthonormal basis.
A is *unitary* if $AA^* = I$.

Adjoint Operator
Let T be a linear operator.
The adjoint operator, T^*, of T is defined such that $\langle v, T(w)\rangle = \langle T^*v, w\rangle$.
This maintains the properties of the inner product.

Normal Matrix
A matrix, A, is normal if $AA^* = A^*A$.
This means $A^* = A^t$ or $A^* = A^{-1}$.