

# Calculus 1 - Notes

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# 1 Before Calculus

## 1.1 Fundamental Theorem of Calculus

### Definition 1.01 - Fundamental Theorem of Calculus

The *Fundamental Theorem of Calculus* states

$$\frac{d}{dx} \int_a^x f(t)dt = f(x)$$

### Definition 1.02 - Common Sets of Numbers

Natural Numbers, set of positive integers -  $\mathbb{N} := \{1, 2, 3, \dots\}$ .

Whole Numbers, set of all integers -  $\mathbb{Z} := \{\dots, -2, -1, 0, 1, 2, \dots\}$ .

Rational Numbers, set of fractions -  $\mathbb{Q} := \left\{\frac{p}{q} : p \in \mathbb{Z}, q \in \mathbb{N}\right\}$ .

Real Numbers, set of all rational & irrational numbers -  $\mathbb{R}$ .

## 1.2 Intervals

### Definition 1.03 - Intervals

Sets of real numbers that fulfil in given ranges.

Notation

$$[a, b] := \{x \in \mathbb{R} : a \leq x \leq b\}$$

$$(a, b] := \{x \in \mathbb{R} : a < x \leq b\}$$

$$[a, b) := \{x \in \mathbb{R} : a \leq x < b\}$$

$$(a, b) := \{x \in \mathbb{R} : a < x < b\}$$

### Definition 1.04 - Functions

*Functions* map values between fields of numbers. The signature of a function is defined by

$$f : A \rightarrow B$$

Where  $f$  is the name of the function,  $A$  is the domain and  $B$  is the co-domain.

### Definition 1.05 Domain & Co-Domain

The *Domain* of a function is the set of numbers it can take as an input.

The *Co-Domain* is the set of numbers that the domain is mapped to.

N.B. - A function is valid iff it maps each value in the domain to a single value in the co-domain.

### Definition 1.06 - Maximal Domain

The *Maximal Domain* of a function is the largest set of values which can serve as the domain of a function.

### Remark 1.07 - Types of Function

Let  $f : A \rightarrow B$

*Polynomials*

$$f(x) = a_0 + a_1x + \dots + a_nx^n$$

*Rational*

$$f(x) = \frac{p(x)}{q(x)}, \quad q(x) \neq 0 \quad \forall x \in A$$

*Trigonometric*

$$\sin(x), \cos(x), \tan(x) \text{ etc.}$$

## 2 Limits

### 2.1 Limits

#### Definition 2.01 - Limits

A *limit* is the value a function tends to, as the input converges to a given  $x$ .  
*i.e.* The value  $f(x)$  has at it gets very close to  $x$ .

#### Formal Definition

We say  $L$  is the limit of  $f(x)$  as  $x$  tends to  $x_0$  if

$$\forall \varepsilon > 0, \exists \delta > 0 \text{ st if } x \in A \text{ and } |x - x_0| < \delta \Rightarrow |f(x) - L| < \varepsilon$$

#### Notation

$$\lim_{x \rightarrow x_0} f(x) = L$$

#### Definition 2.02 - Directional Limits

Sometimes the value of a limit depends on which direction you approach it from.

$\lim_{x \rightarrow x_0+}$  is used when approaching from values greater than  $x_0$ .

$\lim_{x \rightarrow x_0-}$  is used when approaching from values less than  $x_0$ .

#### Theorem 2.03 - Operations with limits

Let  $\lim_{x \rightarrow x_0} f(x) = L_f$  and  $\lim_{x \rightarrow x_0} g(x) = L_g$  Then

$$\begin{aligned} \lim_{x \rightarrow x_0} [f(x) + g(x)] &= L_f + L_g \\ \lim_{x \rightarrow x_0} [f(x) \cdot g(x)] &= L_f \cdot L_g \\ \lim_{x \rightarrow x_0} \left[ \frac{f(x)}{g(x)} \right] &= \frac{L_f}{L_g}, \quad L_g \neq 0 \end{aligned}$$

### 2.2 Exponential Function

#### Definition 2.04 - Exponential Function

$$e := \lim_{x \rightarrow \infty} \left( 1 + \frac{1}{n} \right)^n \simeq 2.7182818...$$

#### Theorem 2.05 - Binomial Expansion

A technique used for expanding binomial expressions

$$\begin{aligned} \left( 1 + \frac{x}{n} \right)^n &= \sum_{i=0}^n \binom{n}{i} \cdot 1^{(n-i)} \cdot \left( \frac{x}{n} \right)^i \\ &= 1 + x + \frac{n-1}{2n} \cdot x^2 + \dots + \frac{x^n}{n^n} \end{aligned}$$

## 3 The Derivative

#### Definition 3.01 - Differentiable Equations

Let  $f : A \rightarrow B$  and  $x_0 \in A$ .

$f$  is differentiable at  $x_0$  if  $\exists L \in B$  such that

$$L = \lim_{h \rightarrow 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

If this limit exists  $\forall x \in A$  then we can define the derivative of  $f(x)$  by

$$f'(x) := \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

**Definition 3.02 - Notation for Differentiation**

There are two ways to denote the derivative of an equation

$$f'(x) \iff \frac{df}{dx}, f''(x) \iff \frac{d^2f}{dx^2}, \dots, f^{(n)}(x) \iff \frac{d^n f}{dx^n}$$

N.B. - Using  $\frac{df}{dx}$  is more informative, especially for equations with multiple variables.

### 3.1 Techniques for Finding the Derivative

**Theorem 3.03 - Sum Rule**

Let  $f, g$  be differentiable with respect to  $x$ .

$$(f + g)' = f' + g'$$

**Theorem 3.04 - Product Rule**

Let  $f, g$  be differentiable with respect to  $x$ .

$$(fg)' = f'g + fg'$$

**Theorem 3.05 - Quotient Rule**

Let  $f, g$  be differentiable with respect to  $x$ .

$$\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$$

**Definition 3.06 - Composite Functions**

Let  $f : B \rightarrow C$  and  $g : A \rightarrow B$ . Then the following notation can be used

$$(f \circ g)(x) := f(g(x))$$

**Theorem 3.07 - Chain Rule**

Let  $f, g$  be differentiable with respect to  $x$ .

$$\frac{d}{dx} [f(g(x))] = f'(g(x)) \cdot g'(x)$$

### 3.2 Implicit Differentiation

**Definition 3.08 - Implicit Differentiation**

Sometimes it is hard to isolate variables in multi-variable equations, in these cases differentiate both sides with respect to the same variable.

Remembering

$$\frac{d}{dx}(x) = 1 \text{ and } \frac{d}{dx}(y) = \frac{dy}{dx} = y'$$

### 3.3 Applications of The Derivative

#### **Theorem 3.09** - *Newton's Method*

Let  $f$  be differentiable. Using *Newton's Method* we can approximate a solution to  $f(x) = 0$ .

*Process*

- i) Take an initial guess,  $x_0$ ;
- ii) Find the value of  $x$  where the tangent to  $(x_0, f(x_0))$  on  $f(x)$  intercepts the x-axis;
- iii) Use this value as the next guess;
- iv) Repeat until the value of  $x$  reduces little.

The equation for the tangent is

$$y = f(x_0) + (x - x_0)f'(x_0)$$

so a simplified equation for the process can be deduced

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

#### **Theorem 3.10** - *Angle between Intersecting Curves*

Let  $y = f_1(x)$  and  $y = f_2(x)$  be two curves which intersect at  $(x_0, y_0)$ . So

$$y_0 = f_1(x_0) = f_2(x_0)$$

Let  $m_1, m_2$  be the gradient of the tangents to  $f_1$  &  $f_2$  at  $(x_0, f(x_0))$ .

Define  $\theta_i := \tan^{-1}(m_i)$  for  $i = 1, 2$ .

Set  $\phi = |\theta_1 - \theta_2|$ , then

$$\phi = \left| \frac{m_1 - m_2}{1 + m_1 m_2} \right|$$

#### **Theorem 3.11** - *L'Hôpital's Rule*

For two equations  $f, g$  with limit of  $-\infty, 0$  or  $\infty$  as  $x$  tends to  $a$ , it is hard to solve

$$\lim_{x \rightarrow a} \frac{f(x)}{g(x)}$$

Provided this limit exists, L'Hôpital's Rule states that

$$\lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \lim_{x \rightarrow a} \frac{f'(x)}{g'(x)}$$

### 3.4 Sketching Curves

#### **Remark 3.11** - *Sketching Curves*

Evaluating the derivative of a curve can make it easier to sketch:

- i) When  $f'(x) > 0$  the curve is heading upwards.
- ii) When  $f'(x) < 0$  the curve is heading downwards.
- iii) When  $f'(x) = 0$  the curve is flat.
- iv) When  $f'(x) = \infty, -\infty$  there are asymptotes.

**Definition 3.12 - Even Functions**

If  $f(x) = f(-x)$  then the function is symmetrical across the y-axis and said to be *even*.

Examples -  $x^2$ ,  $\cos(x)$ ,  $|x|$

**Definition 3.13 - Odd Functions**

If  $f(x) = -f(-x)$  then the function is said to be *odd*.

Examples -  $x$ ,  $\sin(x)$ ,  $x.\cos(x)$

**Remark 3.14**

Some functions are neither *odd*, or *even*.

Example -  $x + x^2$

## 4 Integration

### 4.1 The Primitive

**Definition 4.01 - The Primitive**

A function,  $F : A \rightarrow \mathbb{R}$ , is a primitive for the function  $f : A \rightarrow \mathbb{R}$  if  $F$  is differentiable and

$$\frac{d}{dx}F = f$$

N.B. - Primitives are also called *Indefinite Integral* or *Anti-Derivative*.

**Definition 4.02 - Definite Integral**

Let  $F$  be the primitive for the function  $f$ . Then

$$\int_b^a f(x)dx = F(a) - F(b)$$

Notation -  $F(x)|_a^b = F(b) - F(a)$

**Remark 4.03 - Area Under a Curve**

Let  $f : [a, b] \rightarrow \mathbb{R}$  be a continuous function. Then the area between the curve and the x-axis is found by integration.

$$A := \int_a^b f(x)dx$$

**Definition 4.04 - Convergent Improper Integrals**

Let  $b > a$  and define a function,  $f : [a, \infty) \rightarrow \mathbb{R}$ , which is continuous in  $[a, b]$  Then

$$\int_a^\infty f(x)dx := \lim_{b \rightarrow \infty} \int_a^b f(x)dx$$

If this limit exists then the improper integral is *convergent*, otherwise it is *divergent*.

**Remark 4.05 - Summing Definite Integrals**

For all  $a < c < b$

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$$

$$\int_b^a f(x)dx := - \int_a^b f(x)dx$$

**Theorem 4.06 - Taylor Series**

Functions,  $f(x)$ , can be expanded into polynomial form with degree  $n$ ,  $T_n$ , and remainder  $R_n$  such that  $f(x) = T_n(x) + R_n(x)$  where

$$T_n(x) = f(a) + (x-a)f'(a) + \dots + \frac{1}{n!}(x-a)^n \cdot f^n(a)$$

$$R_n(x) = \frac{1}{n!} \int_a^x (x-t)^n \cdot f^{(n+1)}(t) dt$$

## 5 Parametric Curves & Arc-Length

### 5.1 Parametric Curves

**Definition 5.01 - Parametric Curves**

*Parametric equations* are an alternative to Cartesian equations, for representing curves. They can also represent a point in 3D space.

$$\mathbf{p} = \begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix}$$

**Theorem 5.02 - Parametric to Cartesian Equations**

As all equations in a Parametric series have a common variable, substitution can be used to form a single equation.

### 5.2 Tangent to a Curve

**Theorem 5.02 - Tangent to a Parametric Curve**

Let  $(x(t), y(t))$  be a series of parametric equation.

If we want to find the tangent at a point on the line,  $(a, b)$ , we need to find the value  $t_0$  such that  $x(t_0) = a$  &  $y(t_0) = b$ .

Then by using the chain rule we can deduce the following equation for the tangent when  $t = t_0$

$$\frac{dy(t_0)}{dx(t_0)} = \frac{y - y(t_0)}{x - x(t_0)}$$

Similarly we can deduce the equation for the *normal* when  $t = t_0$

$$-\frac{dx(t_0)}{dy(t_0)} = \frac{y - y(t_0)}{x - x(t_0)}$$

### 5.3 Arc-Length

**Theorem 5.03 - Arc-Length**

*Arc-Length* is the length of a curve, following a function, between two points.

For a *cartesian equation*,  $y = f(x)$ , between the points  $x$  and  $x + dx$  is

$$ds = \sqrt{dx^2 + dy^2}$$

So for a set of *parametric equations*,  $(x(t), y(t))$ ,  $a \leq t \leq b$ ,

$$ds = \sqrt{\frac{dx^2}{dt} + \frac{dy^2}{dt}}$$



To find the length of a curve between points  $a$  and  $b$

$$s = \int_a^b \sqrt{\frac{dx^2}{dt} + \frac{dy^2}{dt}} dt$$

**Definition 5.04 - Curvature**

*Curvature* measures how fast the unit tangent vector to a curve rotates. Curvature of a curve,  $y = f(x)$ , can be found using the equation:

$$K(x) = \frac{|y''(x)|}{[1 + (y'(x))^2]^{\frac{3}{2}}}$$

For a set of parametric equations,  $(x(t), y(t))$ , it can be found using:

$$K(t_0) = \frac{y''(t_0).x'(t_0) - y'(t_0).x''(t_0)}{[(x'(t_0))^2 + (y'(t_0))^2]^{\frac{3}{2}}}$$

## 5.4 Level Curves

**Definition 5.05 - Level Curves**

Let  $f : \mathbb{R}^d \rightarrow \mathbb{R}$  be a function with  $d \geq 2$ ,  $d \in \mathbb{N}$ .

A *level curve* for  $f$  is the set of real solutions for  $f(\mathbf{x}) = c$  where  $c \in \mathbb{R}$  is a constant.

N.B -  $f(\mathbf{x}) = c$  is often written as  $f = c$ .

## 6 Differential Equations

**Definition 6.01 - Differential Equations**

Differential equations take the form

$$f(x, y, \frac{dx}{dy}, \dots, \frac{d^{(n)}y}{dx^{(n)}}) = 0, \quad x \in I$$

### 6.1 First Order Differential Equations

**Definition 6.02 - First Order Differential Equations**

First order differential equations are equations of form  $f(x, y, \frac{dx}{dy}) = 0$ .

**Definition 6.03 - Seperable Equations**

An equation,  $f$ , is said to be seperable if there exists two equations,  $M(x)$ ,  $N(y)$ , such that

$$f(x, y, y') = y' - M(x).N(y)$$

Thus

$$\begin{aligned} y' &= M(x).N(y) \\ \implies \frac{y'}{N(y)} &= M(x) \\ \implies \int \frac{1}{N(y)} dy &= \int M(x) dx \end{aligned}$$

After integration, the equation can be rearranged to be in terms of  $y$ .

## 6.2 Integrating Factor

### Theorem 6.04 - Integrating Factor

Consider the equation  $y' + f(x)y + g(x)$ .

Let  $F(x) = \int f(x)dx$ . Thus

$$\begin{aligned} e^{F(x)}.y' + e^{F(x)}.y &= e^{F(x)}.g(x) \\ \Rightarrow \frac{d}{dx} (e^{F(x)}.y) &= e^{F(x)}.g(x) \\ \Rightarrow e^{F(x)}.y &= \int e^{F(x)}.g(x) dx \\ \Rightarrow y &= e^{-F(x)} \int e^{F(x)}.g(x) dx \end{aligned}$$

## 6.3 Second Order Differential Equations

### Definition 6.05 - Linear Differential Equations

A differential equation is said to be *linear* if it can be written in the form

$$Ay(x) := a_n(x).y^{(n)}(x) + \dots + a_1(x).y'(x) + a_0(x).y(x) = b(x)$$

We define the set of solutions as

$$S(A, b) := \{y : I \rightarrow \mathbb{R}; Ay = b\}$$

If the only solution is  $b(x) = 0 \forall x$  then the system is *homogenous*, otherwise it is *inhomogenous*.

### Definition 6.06 - Particular & Complimentary Solutions

When solving a differential equation,  $Ay(x) = b(x)$ , we need to find two functions in order to find the final solution.

- i) Complementary Function,  $y_c$  - The homogenous case of the equation,  $Ay(x) = 0$ ; and,
- ii) Particular Function,  $y_p$  - The inhomogenous case of the equation,  $Ay(x) = b(x)$  for a known  $b(x)$ .

Then  $y = y_c + y_p$  is the final solution for  $Ay(x) = b(x)$ , for the given  $b(x)$ .

### Theorem 6.07 - Finding the Complementary Function

Take a linear differential equation

$$a_n.y^{(n)}(x) + \dots + a_1.y'(x) + a_0.y(x) = b(x)$$

where  $a_n, \dots, a_1, a_0 \in \mathbb{R}$  &  $b(x) : \mathbb{R} \rightarrow \mathbb{R}$  are all constant.

To find the *complementary function* we solve the equation

$$a_n.\lambda^n + \dots + a_1.\lambda + a_0 = 0$$

to get solutions  $\lambda_1, \dots, \lambda_k$  and then produce the complimentary function

$$y_c(x) = \mu_1 e^{\lambda_1 x} + \dots + \mu_k e^{\lambda_k x}$$

Where  $\mu_1, \dots, \mu_k$  are constants to be found later, by comparing with  $b(x)$ .

### Remark 6.08 - Complementary Function

The *complementary function*,  $y_c$ , for differential equations with constant coefficients depends upon the  $\lambda_1, \dots, \lambda_k$  we find, due to Euler's Formula.

- i)  $\lambda_i = c, \quad y_{c_i} = \mu_i e^{\lambda_i x};$
- ii)  $\lambda_i = \pm ib, \quad y_{c_i} = \mu_{i_1} \cos(bx) + \mu_{i_2} \sin(bx);$
- iii)  $\lambda_i = a \pm ib, \quad y_{c_i} = e^{ax} [\mu_{i_1} \cos(bx) + \mu_{i_2} \sin(bx)].$

Then  $y_c = \sum_{j=1}^k y_{c_j}$ .

**Remark 6.09 - Particular Function**

The *particular function*,  $y_p$ , for a differential equation with constant coefficients,  $Ay(x) = b(x)$ , depends on the form of  $b(x)$ .

- i)  $b(x) = a_n x^n + \dots + a_1 x + a_0 \implies y_p = b_n x^n + \dots + b_1 x + b_0;$
- ii)  $b(x) = a e^{bx} \implies y_p = \alpha e^{\beta x};$
- iii)  $b(x) = a \sin(bx) + c \cos(dx) \implies y_p = \alpha \sin(\beta x) + \gamma \cos(\delta x).$

Where the constants of  $y_p$  are values to be found, when given certain conditions.

**Theorem 6.10 - Finding the Particular Function**

Take a linear differential equation

$$a_n y^{(n)}(x) + \dots + a_1 y'(x) + a_0 y(x) = b(x)$$

where  $a_n, \dots, a_1, a_0 \in \mathbb{R}$  &  $b(x) : \mathbb{R} \rightarrow \mathbb{R}$  are all constant.

Deduce the particular function for the differential equation, given  $b(x)$ , and then differentiate  $y_p$   $n$  times.

Substitute in these values, in place of the  $y$ s, into the original equation and solve to find values for the constants in  $y_p$ .

## 6.4 The Wronskian

**Definition 6.11 - The Wronskian**

The *Wronskian*,  $W[y_1, y_2]$ , of two differentiable functions is defined by

$$W[y_1, y_2](x) = y_1(x) \cdot y_2'(x) - y_1'(x) \cdot y_2(x)$$

The notation for this is

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} := ad - bc$$

So

$$W[y_1, y_2] = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$

**Remark 6.12**

If  $W[y_1, y_2] \neq 0$  then  $y_1, y_2$  are linearly independent.

## 6.5 Variation of Constants

**Theorem 6.13 - Variation of Constants**

This is a technique for solving all differential equations, not just ones with constant coefficients, assuming we know the complementary function,  $y_c$ .

Consider the equation

$$Ay(x) = y''(x) + \beta(x)y'(x) + \gamma(x)y(x) = b(x), \quad \text{for a known } b(x)$$

And suppose we have a complementary function in the form

$$y_c = \lambda_1(x)y_1(x) + \lambda_2(x)y_2(x)$$

where  $y_1$  &  $y_2$  are linearly independent, thus  $W[y_1, y_2] \neq 0$ .

Then

$$y'_p = \lambda'_1 y_1 + \lambda_1 y'_1 + \lambda'_2 y_2 + \lambda_2 y'_2$$

As  $\lambda_1, \lambda_2$  are constant then  $\lambda'_1 = \lambda'_2 = 0$  so

$$y'_p = \lambda_1 y'_1 + \lambda_2 y'_2$$

By differentiating and then substituting back into the original equation we see  $y_p$  is a solution iff

$$\lambda'_1 y'_1 + \lambda'_2 y'_2 = f$$

In matrix form we have

$$\begin{pmatrix} y_1 & y_2 \\ y'_1 & y'_2 \end{pmatrix} \begin{pmatrix} \lambda'_1 \\ \lambda'_2 \end{pmatrix} = \begin{pmatrix} 0 \\ f \end{pmatrix}$$

Then by Cramer's rule we have

$$\lambda'_1 = \frac{\begin{vmatrix} 0 & y_2 \\ f & y'_2 \end{vmatrix}}{W[y_1 \ y_2]} = \frac{-y_2 f}{W[y_1 \ y_2]}$$

and

$$\lambda'_2 = \frac{\begin{vmatrix} y_1 & 0 \\ y'_1 & f \end{vmatrix}}{W[y_1 \ y_2]} = \frac{y_1 f}{W[y_1 \ y_2]}$$

Giving use a solution for  $y'_p = \lambda'_1 y_1 + \lambda_1 y'_1 + \lambda'_2 y_2 + \lambda_2 y'_2$ .

## 7 Applied Differential Equations

### Definition 7.01 - Denoting Limit Relationships

We use

$$F(x) \sim G(x) \text{ as } x \rightarrow a$$

to denote

$$\lim_{x \rightarrow a} \frac{F(x)}{G(x)} = 1$$

### Theorem 7.02 - Vibrating String

If we are given a string which is  $L$  long then we can define an equation,  $y(x, t)$ , which describe the displacement of a point  $x$  along the string, at time  $t$ .

$$y(x, t) = u(x)e^{i\omega t}$$

Where  $\frac{\omega}{2\pi}$  is the frequency of the string and  $u(x) = A\cos(\omega x) + B\sin(\omega x)$ .

We can generalise this for strings with  $n$  anti-nodes.

$$\omega_n := \frac{n\pi}{L}, \quad u_n := \sin(\omega_n x)$$

## 8 Liner Difference Equations

### Definition 8.01 - Difference Equations

A difference equation is an equation of the form

$$F(n, y_n, \dots, y_{n+d}) = 0, \quad n, d \in \mathbb{N}$$

where  $y$  is a sequence.

### 8.1 First-Order Linear Difference Equation

#### Definition 8.02 - Liner First-Order Difference Equations

A *Linear First-Order Difference Equation* is an equation,  $F$ , which can be described by

$$F(n, y_n, y_{n+1}) = a_n y_{n+1} + b_n y_n - f_n = 0$$

where  $a_n, b_n, f_n$  are all known sequences.

#### Example 8.03

By taking a simple equation

$$y_{n+1} - y_n = f_n$$

we can see that

$$y_{n+1} = y_n + (y_{n+1} - y_n) = y_n + f_n = \dots = y_{n_0} + f_{n_0} + \dots + f_{n-1} + f_n$$

So

$$y_n = y_{n_0} + \sum_{j=n_0}^{n-1} f_j$$

#### Theorem 8.04 - Solving First-Order Liner Difference Equations

From *Definition 8.02* we can generalise the equation to show that

$$y_{n+1} + b_n y_n = f_n$$

Then

$$\frac{-1}{b_n} y_{n+1} - y_n = \frac{-1}{b_n} f_n$$

We now define the *Summing Factor*,  $S_n$ , as

$$S_n := \prod_{j=n_0}^{n-1} \frac{-1}{b_j}$$

. We multiply both sides of the original equation by the summing factor and as  $S_n(\frac{-1}{b_n}) = S_{n+1}$  we get

$$S_{n+1} y_{n+1} - S_n y_n = S_{n+1} f_n$$

As this has the same form as the example in 8.03 we can now deduce

$$S_n y_n = y_{n_0} + \sum_{j=n_0}^{n-1} S_{j+1} \cdot f_j$$

## 8.2 Second-Order Linear Difference Equation

### Definition 8.05 - Second-Order Linear Difference Equation

A Linear Second-Order Difference Equation is an equation,  $F$ , which can be described by

$$F(n, y_n, y_{n+1}, y_{n+2}) := a_n \cdot y_{n+2} + b_n \cdot y_{n+1} + c_n \cdot y_n = f_n$$

where  $a_n, b_n, c_n, f_n$  are known sequences.

### Remark 8.06 - Solving Second-Order Linear Difference Equations

Similar to solving second-order differential equations we need to consider two cases. The *homogeneous* & *inhomogeneous* cases. So two sequences will be found the complementary sequence,  $y_n^c$ , and the particular sequence,  $y_n^p$ . The final solution for  $y_n$  is given by

$$y_n = y_n^c + y_n^p$$

### Definition 8.07 - Wronskian of Sequences

For two sequences  $u_n$  &  $v_n$  we define the Wronskian to be

$$W_n := \begin{vmatrix} u_n & v_n \\ u_{n+1} & v_{n+1} \end{vmatrix} = u_n \cdot v_{n+1} - v_n \cdot u_{n+1}$$

### Theorem 8.08 - Homogeneous Case with Constant Coefficients

Take the equation

$$a \cdot y_{n+2} + b \cdot y_{n+1} + c \cdot y_n = 0$$

where  $a, b$  &  $c$  are known constants. We look for solutions of the form

$$y_n = \lambda^n$$

By substitution we get the equation  $a\lambda^2 + b\lambda + c = 0$ . By solving for  $\lambda$  we find a solution

- i)  $\lambda$  has two real solutions -  $y_n = A\lambda_1^n + B\lambda_2^n$ ;
- ii)  $\lambda$  has one real solution -  $y_n = (An + B)\lambda^n$ ;
- iii)  $\lambda$  has only an imaginary solution -  $y_n = \Lambda e^{i\theta}$ ,  $\Lambda^2 := \frac{c}{a}, \theta := \tan^{-1}\left(\frac{4ac-b^2}{-b}\right)$ .

### Theorem 8.09 - Homogeneous Case Second-Order Linear Difference Equations

The homogeneous case finds solutions for

$$a_n y_{n+2} + b_n y_{n+1} + c_n y_n = 0$$

Suppose  $y_n = u_n$  and  $y_n = v_n$  are solutions to this homogeneous equation. Then

$$W_n[u_n, v_n] = W_{n_0} \prod_{j=n_0}^{n-1} \frac{c_j}{a_j}$$

So

$$u_n \cdot v_{n+1} - v_n \cdot u_{n+1} = W_{n_0} \prod_{j=n_0}^{n-1} \frac{c_j}{a_j}$$

Which can be rearranged to be in the form of a first order difference equation, such as *Example 8.03*

$$\frac{u_n}{u_{n+1}} v_{n+1} - v_n = W_{n_0} \prod_{j=n_0}^{n-1} \frac{c_j}{a_j}$$

Which has a summing factor

$$S_n = \frac{1}{u_n}$$

By multiplying both sides by  $S_n$  and simplifying we get

$$v_n = u_n \sum_{j=n_0}^{n-1} \frac{1}{u_j \cdot u_{j+1}} \prod_{k=n_0}^{n-1} \frac{c_k}{a_k}$$

Typically you need to solve the product part of the equation to get a result for the sequence  $u_n$ .

### Remark 8.10 - Inhomogenous Case Second-Order Linear Difference Equations

Generally the best way to do this is to make an educated guess based on the right hand side of the equation. So if the RHS is a polynomial, guess a polynomial, etc. Similar to solving differential equations.

## 9 Several Variables - Differentiability

### Definition 9.01 - Several Variable Function

Let  $d \in \mathbb{N}$ ,  $A \subset \mathbb{R}^d$  &  $B \subset \mathbb{R}^d$ .

A function  $\mathbf{f} : A \rightarrow B$  is a map which, for all  $\mathbf{x} \in A$ , assigns a unique value  $\mathbf{f}(\mathbf{x}) \in B$ .

### Definition 9.02 - Linear Functions

A function  $\mathbf{f} : \mathbb{R}^d \rightarrow \mathbb{R}^n$  is *linear* if it can be given in terms of a matrix  $A \in M_{n,d}(\mathbb{R})$  where

$$\mathbf{f}(\mathbf{x}) = A\mathbf{x}$$

### Theorem 9.03 - Properties of Linear Functions

If a function is linear then the following are true:

- i)  $\mathbf{f}(\lambda\mathbf{x}) = \lambda\mathbf{f}(\mathbf{x})$ ; and,
- ii)  $\mathbf{f}(\mathbf{x} + \mathbf{y}) = \mathbf{f}(\mathbf{x}) + \mathbf{f}(\mathbf{y})$ .

### Definition 9.04 - Continuous Several Variable Function

Let  $\mathbf{f} : A \rightarrow B$  and  $\mathbf{a} \in A$ . Then  $\mathbf{f}$  is continuous at  $\mathbf{a}$  if

$$\lim_{\mathbf{x} \rightarrow \mathbf{a}} \mathbf{f}(\mathbf{x}) = \mathbf{f}(\mathbf{a})$$

## 9.1 The Derivative

### Definition 9.05 - Norm of a Vector

The *norm* of a vector,  $\mathbf{x} \in \mathbb{R}^d$  is

$$\|\mathbf{x}\| := \left( \sum_{j=1}^d x_j^2 \right)^{1/2}$$

### Definition 9.06 - Derivative of Several Variable Function

A function,  $\mathbf{f} : \mathbb{R}^d \rightarrow \mathbb{R}^n$ , is said to be differentiable at the point  $\mathbf{x} \in \mathbb{R}^d$  if there exists an  $A \in M_{n,d}(\mathbb{R})$  such that

$$\lim_{\mathbf{h} \rightarrow \mathbf{0}} \frac{\|\mathbf{f}(\mathbf{x} + \mathbf{h}) - \mathbf{f}(\mathbf{x}) - A\mathbf{h}\|}{\|\mathbf{h}\|} = 0$$

This  $A$  is the derivative of  $\mathbf{f}(\mathbf{x})$ .

$$\mathbf{f}'(\mathbf{x}) := A$$

**Remark 9.07**

If consider the following several variable function

$$\mathbf{f} = \begin{pmatrix} f_1 \\ \vdots \\ f_n \end{pmatrix}$$

Then

$$\mathbf{f}' = \begin{pmatrix} f'_1 \\ \vdots \\ f'_n \end{pmatrix}$$

## 10 Directional & Partial Derivatives

### 10.1 Directional Derivative

**Definition 10.01 - Direction**

A *direction* in  $\mathbb{R}^d$  is a vector of unit length.

In  $\mathbb{R}^2$  every direction can be given by  $\mathbf{u} = \begin{pmatrix} \cos(\theta) \\ \sin(\theta) \end{pmatrix}$ , where  $\theta$  is the angle from positive  $x$  axis

In  $\mathbb{R}^3$  every direction can be given by  $\mathbf{u} = \begin{pmatrix} \sin(\phi)\cos(\theta) \\ \sin(\phi)\sin(\theta) \\ \cos(\phi) \end{pmatrix}$ , where  $\phi$  is the angle from positive  $z$  axis and  $\theta$  is the angle from the positive  $x$  axis.

**Definition 10.02 - Spherical Co-ordinates**

The *spherical co-ordinates* describe points in three dimension space.

The distance of a point from the origin is

$$r = \rho \sin(\phi)$$

where  $\rho$  is the length of the line. Then

$$\mathbf{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} r\cos(\theta) \\ r\sin(\theta) \\ \rho\cos(\phi) \end{pmatrix} = \begin{pmatrix} \rho\sin(\phi)\cos(\theta) \\ \rho\sin(\phi)\sin(\theta) \\ \rho\cos(\phi) \end{pmatrix}$$

**Defintion 10.03 - Direction Derivative**

The *direction derivative* of  $\mathbf{f}$  in the direction of  $\mathbf{u}$  at the point  $\mathbf{x}_0$  is the vector:

$$D_{\mathbf{u}}\mathbf{f}(\mathbf{x}_0) := \left. \frac{d}{dt}\mathbf{f}(\mathbf{x}_0 + t\mathbf{u}) \right|_{t=0}$$

**Theorem 10.04**

For all  $\mathbf{f} : \mathbb{R}^d \rightarrow \mathbb{R}^n$  the directional derivative at  $\mathbf{u}$  in  $\mathbb{R}^d$  we have

$$D_{\mathbf{u}}\mathbf{f}(\mathbf{x}) = \mathbf{f}'(\mathbf{x}) \cdot \mathbf{u}$$

### 10.2 Partial Derivative

**Defintion 10.05 - Partial Derivative**

Let  $f : \mathbb{R}^d \rightarrow \mathbb{R}$ .

Then the direction derivative  $D_{\mathbf{e}_j}f(\mathbf{x})$ , if it exists, is called the *partial derivative* of  $f$  with



respect to  $x_j$  at  $\mathbf{x}$ .

This is denoted by

$$\frac{\partial f}{\partial x_j}(\mathbf{x}) \text{ or } f_{x_j}(\mathbf{x})$$

**Proposition 10.06** - *Partial Derivative as a Matrix*

If  $\mathbf{f} : \mathbb{R}^d \rightarrow \mathbb{R}^n$  is differentiable then

$$\mathbf{f}'(\mathbf{x}) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_d} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_d} \end{pmatrix}$$

**Remark 10.07** - *Second Order Partial Derivatives*

$$\begin{aligned} \frac{\partial^2 f}{\partial x^2} &= \frac{\partial}{\partial x} \cdot \frac{\partial f}{\partial x} \\ \frac{\partial^2 f}{\partial x \partial y} &= \frac{\partial}{\partial x} \cdot \frac{\partial f}{\partial y} \\ \frac{\partial^2 f}{\partial y^2} &= \frac{\partial}{\partial y} \cdot \frac{\partial f}{\partial y} \end{aligned}$$

## 11 Gradient & Chain Rule in Several Variables

### 11.1 Chain Rule

**Theorem 11.01** - *Chain rule in Several Variables*

Let  $\mathbf{f} : \mathbb{R}^d \rightarrow \mathbb{R}^n$  &  $\mathbf{g} : \mathbb{R}^m \rightarrow \mathbb{R}^d$  be differentiable. Then

$$[\mathbf{f} \cdot \mathbf{g}]' = \mathbf{f}'(\mathbf{g}(\mathbf{x})) \cdot \mathbf{g}'(\mathbf{x})$$

### 11.2 Invertible Maps & Implicit Differentiation

**Definition 11.02** - *Implicit Differentiation*

Let  $A, B \subset \mathbb{R}^d$  and  $\mathbf{f} : A \rightarrow B$ .

If  $\mathbf{f}$  is invertible then we can denote the inverse by  $\mathbf{f}^{-1} : B \rightarrow A$  and

$$[\mathbf{f}^{-1} \cdot \mathbf{f}](\mathbf{x}) = \mathbf{x}$$

If  $\mathbf{f}$  and  $\mathbf{f}^{-1}$  are differentiable then by the chain rule

$$\mathbf{f}'(\mathbf{f}^{-1}(\mathbf{x})) \mathbf{f}^{-1}{}'(\mathbf{x}) = I$$

Hence

$$\mathbf{f}^{-1}{}'(\mathbf{x}) = [\mathbf{f}'(\mathbf{f}(\mathbf{x}))]^{-1}$$

### 11.3 The Gradient

**Definition 11.03** - *The Gradient*

Let  $f : \mathbb{R}^d \rightarrow \mathbb{R}$  be a map with first-order partial derivatives at every point of  $\mathbb{R}^d$ .

Then the *gradient* at a point  $\mathbf{x} \in \mathbb{R}^d$ ,  $\nabla f : \mathbb{R}^d \rightarrow \mathbb{R}^d$ , is defined as

$$\nabla f(\mathbf{x}) := (f_{x_1}(\mathbf{x}), \dots, f_{x_d}(\mathbf{x}))$$

**Definition 11.04 - Direction of Greatest Change**

By considering the case in two dimensions then

$$(D_{\mathbf{u}}f)(x, y) = \nabla f(x, y) \cdot \mathbf{u}, \quad \mathbf{u} = (\cos\theta, \sin\theta)$$

Then

$$(D_{\mathbf{u}}f)(x_0, y_0) = \cos\theta \cdot \frac{\partial f}{\partial x}(x_0, y_0) + \sin\theta \cdot \frac{\partial f}{\partial y}(x_0, y_0)$$

The greatest change occurs when

$$\begin{aligned} \frac{d}{d\theta} [\cos\theta \cdot \frac{\partial f}{\partial x}(x_0, y_0) + \sin\theta \cdot \frac{\partial f}{\partial y}(x_0, y_0)] &= 0 \\ \Rightarrow -\sin\theta \cdot \frac{\partial f}{\partial x}(x_0, y_0) + \cos\theta \cdot \frac{\partial f}{\partial y}(x_0, y_0) &= 0 \\ \Rightarrow \nabla f(x_0, y_0) \cdot (-\sin\theta, \cos\theta) &= 0 \end{aligned}$$

We can thus deduce that

$$\nabla f(x_0, y_0) \perp (-\sin\theta, \cos\theta)$$

So

$$\mathbf{u} \parallel \nabla f(x_0, y_0)$$

We can then establish the direction,  $\mathbf{u}$ , which occurs when  $\mathbf{u}$  satisfies

$$\mathbf{u} = \frac{\nabla f(x_0, y_0)}{\|\nabla f(x_0, y_0)\|}$$

## 12 Integration over Two-Dimensional Domains

**Remark 12.01 - Notation**

Let  $D \subset \mathbb{R}^2$ .

When integrating over  $D$  the following notation is used

$$\iint_D f(x, y) dx dy$$

When  $D$  is specified e.g.  $D = \{(x, y) : g(y) < x < h(y); a < y < b\}$  then we denote it by

$$\int_a^b \left\{ \int_{g(y)}^{h(y)} f(x, y) dx \right\} dy$$

**Theorem 12.02 - Rectangular Domain**

If we are given a domain between constant values,  $D = \{(x, y) : c < y < d; a < x < b\}$  in terms of distance from origin, then we perform

$$V = \int_c^d \left\{ \int_a^b f(x, y) dx \right\} dy$$

And can perform the integrations are separate calculations.

**Theorem 12.03 - Triangular Domain**

In *triangular domains* the value of  $x$  depends on  $y$ , so can be expressed as a function of  $y$ . If we take  $0 < x < a$  and  $0 < y < b$  then  $x = \frac{ya}{b}$ . This gives us a domain of

$$D = \{(x, y) : \frac{ya}{b} < x < a; 0 < y < b\}$$

We can solve the integral by performing:

$$V = \int_0^b \left\{ \int_{\frac{ay}{b}}^a f(x, y) dx \right\} dy$$

**Theorem 12.04 - General Domains**

If we have a domain,  $D$ , defined by  $D = \{(x, y) \in \mathbb{R}^2 : L(x) \leq y \leq U(x), a \leq x \leq b\}$ . Then the integral over  $D$  can be found by

$$V = \int_a^b \left\{ \int_{L(x)}^{U(x)} f(x, y) dy \right\} dx$$

## 13 Polar Co-ordinates

**Defintion 13.01 - Cartesian to Polar Co-ordinates**

*Polar co-ordinates* are an alternative way of denoting a point in two dimensional space, using distance from the origin and angle from the positive x-axis.

Polar co-ordinates are given in the form  $(r, \theta)$ .

Using pythagoras' theorem and trigonometry we can see that

$$r = \sqrt{x^2 + y^2}, \quad \theta := \tan^{-1} \left( \frac{y}{x} \right)$$

**Theorem 13.02 - Polar to Cartesian Co-ordinates**

We can convert a polar co-ordinate,  $(r, \theta)$ , to cartesian co-ordinates,  $(x, y)$  using the following formula

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} r \cos(\theta) \\ r \sin(\theta) \end{pmatrix}$$

**Remark 13.03 - Plotting Curves expressed as polar co-ordinates**

There are two common ways to do this:

- i) Convert back to cartesian form; or,
- ii) Interpolation, plot a few points and link them.

**Theorem 13.04 - Computing Curvature of Polar Curve**

Suppose a curve is defined by  $r = R(\theta)$ . This means that

$$x = r \cos(\theta) = R(\theta) \cos(\theta)$$

$$y = r \sin(\theta) = R(\theta) \sin(\theta)$$

By substituting into the equation for curvature of a cartesian curve we get

$$K(\theta) = \frac{|R(\theta)^2 + 2[R'(\theta)]^2 - R(\theta)R''(\theta)|}{\{R(\theta)^2 + [R'(\theta)]^2\}^{3/2}}$$

**Theorem 13.05 - Gradient of a Polar Curve**

Consider the equation  $f(x, y) = u(r, \theta)$  then by using substitution and chain rule we can deduce the following equations:

$$\begin{aligned} \frac{\partial f}{\partial x} &= \cos(\theta) \frac{\partial u}{\partial r} - \frac{\sin(\theta)}{r} \frac{\partial u}{\partial \theta} \\ \frac{\partial f}{\partial y} &= \sin(\theta) \frac{\partial u}{\partial r} + \frac{\cos(\theta)}{r} \frac{\partial u}{\partial \theta} \end{aligned}$$

### 13.1 Change of Variable Formula

**Theorem 13.06** - *Single Variable Substitution*

By making the substitution of  $x = g(u)$  we can produce the following formula

$$\int_{g(a)}^{g(b)} f(x)dx = \int_a^b (f \circ g)(u)g'(u)du$$

**Theorem 13.07** - *Double Variable Substitution*

Suppose we make the following substitution

$$\begin{pmatrix} x \\ y \end{pmatrix} = \mathbf{g}(u, v) := \begin{pmatrix} g_1(u, v) \\ g_2(u, v) \end{pmatrix}$$

Then we can produce the following equation

$$\iint_{\mathbf{g}(D)} f(x, y)dxdy = \iint_D (f \circ \mathbf{g})(u, v)|\det\{\mathbf{g}'(u, v)\}|dudv$$

**Remark 13.08** - *Cartesian to Polar Double Integration*

By using the formula in *Theorem 13.6* we can produce the following equation, by using the substitution  $(x, y) = (r\cos(\theta), r\sin(\theta))$

$$\text{Let } \mathbf{g}(r, \theta) = (r\cos(\theta), r\sin(\theta))$$

$$\implies \iint_{\mathbf{g}(D)} f(x, y)dxdy = \iint_D f(r\cos(\theta), r\sin(\theta)).r.dr d\theta$$

## 14 Triple Integration

**Remark 14.01** - *Triple Integration*

If we let  $D$  be a three dimensional region, so  $D \subset \mathbb{R}^3$ . Then by doing

$$\iiint_D f(x, y, z)dxdydz$$

we produce a *hypervolume*.

By setting  $f(x, y, z) = 1$  we can find the volume of  $D$ .

If we set it to be a variable equation then we can determine other properties such as mass, center of mass etc.

**Theorem 14.02** - *Box-Like region*

If we define  $D$  as  $D = \{(x, y, z) \in \mathbb{R}^3 : a \leq x \leq A; b \leq y \leq B; c \leq z \leq C\}$ . We can integrate over  $D$  by

$$\iiint_D f(x, y, z)dxdydz = \int_c^C \left\{ \int_b^B \left\{ \int_a^A f(x, y, z)dx \right\} dy \right\} dz$$

**Theorem 14.03** - *General Domains*

If we define  $D$  as  $D = \{(x, y, z) \in \mathbb{R}^3 : a \leq x \leq A(y, z); b \leq y \leq B(z); c \leq z \leq C\}$ . We can integrate over  $D$  by

$$\iiint_D f(x, y, z)dxdydz = \int_c^C \left\{ \int_b^{B(z)} \left\{ \int_a^{A(y, z)} f(x, y, z)dx \right\} dy \right\} dz$$

## 14.1 Substitution with Triple Integrals

### Theorem 14.04 - Substitution with Triple Integration

If we want to perform a substitution define by

$$\begin{aligned}x &= g_1(u, v, w), \\y &= g_2(u, v, w), \\z &= g_3(u, v, w)\end{aligned}$$

Then we can produce the following equation

$$\iiint_{g(D)} f(x, y, z) dx dy dz = \iiint_D (f \cdot \mathbf{g})(u, v, w) \cdot |\det\{\mathbf{g}'(u, v, w)\}| du dv dw$$

### Remark 14.05 - Cartesian to Spherical Triple Integration

If we can convert a cartesian region to a spherical region we use the following substitution

$$\begin{aligned}x &= \rho \sin(\phi) \cos(\theta) = g_1(\rho, \phi, \theta), \\x &= \rho \sin(\phi) \sin(\theta) = g_2(\rho, \phi, \theta), \\x &= \rho \sin(\phi) &= g_3(\rho, \phi, \theta)\end{aligned}$$

This means  $\det\{\mathbf{g}'(\rho, \phi, \theta)\} = \rho^2 \sin(\phi)$ . Then

$$\iiint_{g(D)} f(x, y, z) dx dy dz = \iiint_D (f \cdot \mathbf{g})(\rho, \phi, \theta) \cdot (\rho^2 \sin(\phi)) \cdot d\rho d\phi d\theta$$

## 14.2 Applications of Multiple Integration

### Definition 14.06 - Multiple Integrations

When integrating over multiple dimensions we don't need to write  $\int$  every time, instead we can denote it as

$$\int_R f(\mathbf{x}) d\mathbf{x}$$

or

$$\int \cdots \int_R f(\mathbf{x}) dx_1 \cdots dx_d$$

### Theorem 14.07 - Centre of Mass

By considering a region  $R \subset \mathbb{R}^d$  whose density is described by  $f(\mathbf{x})$ . Then its mass can be found by

$$m = \int_R f(\mathbf{x}) d\mathbf{x}$$

Then its *centre of mass*,  $\bar{\mathbf{x}} \in \mathbb{R}^d$ , can be found by

$$\bar{\mathbf{x}} := \frac{1}{m} \int_R \mathbf{x} f(\mathbf{x}) d\mathbf{x}$$

## 15 Local Extrema & Taylor's Theorem in Several Variables

### 15.1 Taylor's Theorem

#### Theorem 15.01 - Taylor's Theorem for a Single Variable

If  $f : \mathbb{R} \rightarrow \mathbb{R}$  is smooth and  $a \in \mathbb{R}$ , then

$$f(x) = f(a) + (x - a)f'(a) + \frac{1}{2!}(x - a)^2 f''(a) + \dots$$

$$\implies f(x) = \sum_{j=1}^{\infty} \frac{1}{j!} \cdot (x-a)^j \cdot \left[ \frac{d^j f}{dx^j}(a) \right]$$

**Theorem 15.02 - Taylor's Theorem for a Two Variables**

If  $f : \mathbb{R}^2 \rightarrow \mathbb{R}$  is smooth and  $a, b \in \mathbb{R}$ , then

$$f(x, y) = f(a, b) + f'(a, b) \begin{pmatrix} x-a \\ y-b \end{pmatrix} + \frac{1}{2} \begin{pmatrix} x-a \\ y-b \end{pmatrix}^t f''(a, b) \begin{pmatrix} x-a \\ y-b \end{pmatrix} + \dots$$

N.B -  $f'(x, y) \in M_{1,2}(\mathbb{F})$  &  $f''(x, y) \in M_2(\mathbb{F})$

**15.2 Local Extrema****Definition 15.03 - Local Minima**

Let  $f : \mathbb{R}^d \rightarrow \mathbb{R}$ .

We say the point  $\mathbf{x}_0 \in \mathbb{R}^d$  is a *local minimum* if

$$\exists \delta > 0 \text{ st } \forall \mathbf{x} \in \mathbb{R}^d, \|\mathbf{x} - \mathbf{x}_0\| < \delta \text{ then } f(\mathbf{x}_0) \leq f(\mathbf{x})$$

**Definition 15.04 - Local Maxima**

Let  $f : \mathbb{R}^d \rightarrow \mathbb{R}$ .

We say the point  $\mathbf{x}_0 \in \mathbb{R}^d$  is a *local maximum* if

$$\exists \delta > 0 \text{ st } \forall \mathbf{x} \in \mathbb{R}^d, \|\mathbf{x} - \mathbf{x}_0\| < \delta \text{ then } f(\mathbf{x}_0) \geq f(\mathbf{x})$$

**Definition 15.05 - Local Extremum**

A point is a *local extremum* if it is either a local minimum or maximum.

**Definition 15.06 - Critical point**

A point,  $\mathbf{x}_0$ , is a critical point of  $f$  if

$$f'(\mathbf{x}_0) = \mathbf{0}$$

**Proposition 15.07 - Extremum and Critical Point**

If  $\mathbf{x}_0$  is a local extremum of  $f$  then it is a critical point of  $f$ .

**Remark 15.08 - Classifying Critical Points**

Let  $\mathbf{x}_0$  be a critical point of  $f$ .

Then Taylor's theorem says that

$$f(\mathbf{x}) = f(\mathbf{x}_0) + \frac{1}{2}(\mathbf{x} - \mathbf{x}_0)^t \cdot f''(\mathbf{x}_0)(\mathbf{x} - \mathbf{x}_0) + \dots$$

**Definition 15.09 - Define & Indefine Matrices**

Let  $A \in M_d$ , then  $A$  is

- i) *Positive definite* if  $\forall \mathbf{x} \in \mathbb{R}^d, \mathbf{x} \neq \mathbf{0} \implies \mathbf{x}^t A \mathbf{x} > 0$ ;
- ii) *Negative definite* if  $\forall \mathbf{x} \in \mathbb{R}^d, \mathbf{x} \neq \mathbf{0} \implies \mathbf{x}^t A \mathbf{x} < 0$ ;
- iii) *Indefinite* if  $\exists \mathbf{x}, \mathbf{y} \in \mathbb{R}^d \text{ st } \mathbf{x}^t A \mathbf{x} > 0 \& \mathbf{y}^t A \mathbf{y} < 0$ .

**Theorem 15.10 - Extrema and Definite Matrices**

The critical point  $\mathbf{x}_0 \in \mathbb{R}^d$  is a

- i) Local minimum iff  $f''(\mathbf{x}_0)$  is a positive-definite;
- ii) Local maximum iff  $f''(\mathbf{x}_0)$  is a negative-definite;

If  $f''(\mathbf{x}_0)$  is indefinite then we say that  $\mathbf{x}_0$  is a *saddle*.

**Theorem 15.11 - Definite Matrices and Eigenvalues**

We say  $f''(\mathbf{x}_0)$  is

- i) *Positive definite* iff all its eigenvalues are strictly *positive*;
- ii) *Negative definite* iff all its eigenvalues are strictly *negative*;
- iii) *Indefinite* if it has both positive and negative eigenvalues.

## 16 Systems of Linear Differential Equations

**Theorem 16.01 - Linear Case**

Define a function,  $\mathbf{f} : \mathbb{R}^d \rightarrow \mathbb{R}^d$ , such that  $\mathbf{f}(\mathbf{x}) = A\mathbf{x}$  where  $\mathbf{x} \in \mathbb{R}^d$  and  $A \in M_d$  is diagonalisable. Let  $\{\mathbf{v}_1, \dots, \mathbf{v}_d\}$  be eigenvectors for  $A$ , with  $\{\lambda_1, \dots, \lambda_d\}$  as the corresponding eigenvalues. Then the general solution is

$$\mathbf{x}(t) = \sum_{j=1}^d c_j(0) \cdot e^{\lambda_j \cdot t} \cdot \mathbf{v}_j$$

values of  $c$  are determined in the particular solution from the initial solutions.

**Definition 16.02 - Equilibria**

Let  $\mathbf{f} : \mathbb{R}^d \rightarrow \mathbb{R}^d$ ,  $\mathbf{e} \in \mathbb{R}^d$  is said to be an *equilibrium* of  $\mathbf{f}$  if

$$\mathbf{f}(\mathbf{e}) = \mathbf{0}$$

**Theorem 16.03 - Particular Solutions and Equilibria**

Let  $t \in \mathbb{R}$ ,  $\mathbf{f} : \mathbb{R}^d \rightarrow \mathbb{R}^d$  &  $\mathbf{x} : \mathbb{R} \rightarrow \mathbb{R}^d$ . If

$$\mathbf{f}(\mathbf{x}(t)) = \mathbf{0}$$

then  $t$  is a particular solution of  $\mathbf{f}$ .

**Definition 16.04 - Stable/Unstable Equilibria**

Let  $\mathbf{e}$  be an *equilibrium*.

$\mathbf{e}$  is said to *stable* if

$$\forall \epsilon > 0, \exists \delta > 0 \text{ st } \forall \mathbf{x}(0) \in \mathbb{R}^d \text{ \& } t \geq 0, \|\mathbf{x}(0) - \mathbf{e}\| < \delta \Rightarrow \|\mathbf{x}(t) - \mathbf{e}\| < \epsilon$$

Otherwise the equilibrium point is *unstable*.

**Theorem 16.05 - Determining Stability of Equilibria**

If for all eigenvalues,  $\lambda_j$ , of  $\mathbf{f}'(\mathbf{e})$  is such that

$$\operatorname{Re}(\lambda_j) < 0$$

then the equilibrium,  $\mathbf{e}$ , is *stable*.

If there exists one eigenvalue,  $\lambda_i$ , where  $\operatorname{Re}(\lambda_i) > 0$  then  $\mathbf{e}$  is *unstable*.