Linear Algebra & Geometry - Notes

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Contents

1 Euclidean Plane, Vectors, Cartesian Co-Ordinates & Complex Numbers

1.1 Vectors

Definition 1.01 - *Vectors*
Ordered sets of real numbers.
Denoted by $\boldsymbol{v} = (v_1, v_2, v_3, ...) = \begin{pmatrix} x \\ y \end{pmatrix}$

Definition 1.02 - *Euclidean Plane*
The set of two dimensional vectors, with real componenets, is called the Euclidean Plane.
Denoted by $\mathbb{R}^2$

Definition 1.03 - *Vector Addition*
Let $\boldsymbol{v}, \boldsymbol{w} \in \mathbb{R}^2$ such that $\boldsymbol{v} = (v_1, v_2)$ and $\boldsymbol{w} = (w_1, w_2)$.
Then $\boldsymbol{v} + \boldsymbol{w} = (v_1 + w_1, v_2 + w_2)$.

Definition 1.03 - *Scalar Multiplcation of Vectors*
Let $\boldsymbol{v} \in \mathbb{R}^2$ and $\lambda \in \mathbb{R}$ such that $\boldsymbol{v} = (v_1, v_2)$.
Then $\lambda\boldsymbol{v} = (\lambda v_1, \lambda v_2)$.

Definition 1.04 - *Norm of vectors*
The norm of a vector is its length from the origin.
Denoted by $||\boldsymbol{v}|| = \sqrt{v_1^2 + v_2^2}$ for $\boldsymbol{v} \in \mathbb{R}^2$.

Theorem 1.05
Let $\boldsymbol{v}, \boldsymbol{w} \in \mathbb{R}^2$ and $\lambda \in \mathbb{R}$ such that $\boldsymbol{v} = (v_1, v_2)$ and $\boldsymbol{w} = (w_1, w_2)$.
Then

$$\begin{aligned} ||\boldsymbol{v}|| &= 0 \text{ iff } \boldsymbol{v} = \boldsymbol{0} \\ ||\lambda\boldsymbol{v}|| &= \sqrt{\lambda^2 v_1^2 + \lambda^2 v_2^2} \\ &= |\lambda|.||\boldsymbol{v}|| \\ ||\boldsymbol{v} + \boldsymbol{w}|| &\leq ||\boldsymbol{v}|| + ||\boldsymbol{w}|| \end{aligned}$$

Definition 1.06 - *Unit Vector*
A vector can be described by its length & direction.
Let $\boldsymbol{v} \in \mathbb{R}^2 \backslash \{\boldsymbol{0}\}$.
Then $\boldsymbol{v} = ||\boldsymbol{v}||\boldsymbol{u}$ where $\boldsymbol{u}$ is the unit vector, $\boldsymbol{u} = \begin{pmatrix} cos\theta \\ sin\theta \end{pmatrix}$
Thus $\forall\ \boldsymbol{v} \in \mathbb{R}^2\ \boldsymbol{v} = \begin{pmatrix} \lambda cos\theta \\ \lambda sin\theta \end{pmatrix}$ for some $\lambda \in \mathbb{R}$.

Definition 1.07 - *Dot Product*
Let $\boldsymbol{v} \in \mathbb{R}^2$ and $\lambda \in \mathbb{R}$ such that $\boldsymbol{v} = (v_1, v_2)$.
Then $\boldsymbol{v} \cdot \boldsymbol{w} = v_1.w_1 + v_2.w_2$.

Remark 1.08 - *Positivity of Dot Product*
Let $\boldsymbol{v} \in \mathbb{R}^2$.
Then $\boldsymbol{v} \cdot \boldsymbol{v} = ||\boldsymbol{v}||^2 = v_1^2 + v_2^2 \geq 0$.

Remark 1.09 - *Angle between vectors in Euclidean Plane*
Let $\boldsymbol{v}, \boldsymbol{w} \in \mathbb{R}^2$.
Set θ to be the angle between $\boldsymbol{v}$ & $\boldsymbol{w}$.
Then

$$cos\theta = \frac{\boldsymbol{v} \cdot \boldsymbol{w}}{||\boldsymbol{v}||\ ||\boldsymbol{w}||}$$

.
Theorem 1.10 - *Cauchy-Schwarz Inequality*
Let $\boldsymbol{v}, \boldsymbol{w} \in \mathbb{R}^2$.
Then

$$|\boldsymbol{v} \cdot \boldsymbol{w}| \leq ||\boldsymbol{v}||\ ||\boldsymbol{w}||$$

Proof

$$\begin{aligned}
\frac{v_1 w_1}{||\boldsymbol{v}||\ ||\boldsymbol{w}||} + \frac{v_2 w_2}{||\boldsymbol{v}||\ ||\boldsymbol{w}||} &\leq \frac{1}{2}\left(\frac{v_1^2}{||\boldsymbol{v}||^2} + \frac{w_1^2}{||\boldsymbol{w}||^2}\right) + \frac{1}{2}\left(\frac{v_2^2}{||\boldsymbol{v}||^2} + \frac{w_2^2}{||\boldsymbol{w}||^2}\right) \\
&\leq \frac{1}{2}\left(\frac{v_1^2 + v_2^2}{||\boldsymbol{v}||^2} + \frac{w_1^2 + w_2^2}{||\boldsymbol{w}||^2}\right) \\
&\leq \frac{1}{2}(1+1) \\
&\leq 1 \\
=> |v_1 w_1 + v_2 w_2| &\leq ||\boldsymbol{v}||\ ||\boldsymbol{w}|| \\
|\boldsymbol{v} \cdot \boldsymbol{w}| &\leq ||\boldsymbol{v}||\ ||\boldsymbol{w}||
\end{aligned}$$

1.2 Complex Numbers

Definition 1.11 - i

$$\begin{aligned} i^2 &= -1 \\ i &= \sqrt{-1} \end{aligned}$$

Definition 1.12 - *Complex Number Set*
The set of complex numbers contains all numbers with an imaginary part.

$$\mathbb{C} := \{x + iy; x, y \in \mathbb{R}\}$$

Complex numbers are often denoted by

$$z = x + iy$$

and we say x is the real part of z and y the imaginary part.
Definition 1.13 - *Complex Conjugate*
Let $z \in \mathbb{C}$ st $z = x + iy$.
Then

$$\bar{z} := x - iy$$

Theorem 1.14 - *Operations on Complex Numbers*
Let $z_1, z_2 \in \mathbb{C}$ st $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$.
Then

$$\begin{aligned} z_1 + z_2 &:= (x_1 + x_2) + i(y_1 + y_2) \\ z_1.z_2 &:= (x_1 + iy_1)(x_2 + iy_2) \\ &:= x_1.x_2 - y_1.y_2 + i(x_1.y_2 + x_2.y_1) \end{aligned}$$

N.B. When dividing by a complex number, multiply top and bottom by the complex conjugate.

Definition 1.15 - *Modulus of Complex Numbers*
The modulus of a complex number is the distance of the number, from the origin, on an Argand diagram. Let $z \in \mathbb{C}$ st $z = x + iy$.
Then

$$\begin{aligned} |z| &:= \sqrt{x^2 + y^2} \\ &:= \sqrt{\bar{z}z} \end{aligned}$$

N.B. Amplitude is an alternative name for the modulus

Definition 1.16 - *Phase of Complex Numbers*
The phase of a complex number is the angle between the positive real axis and the line subtended from the origin and the number, on an Argand digram.

$$z = |z|.(cos\theta + i.sin\theta), \quad \theta = \text{Phase}$$

N.B. Phase of $\bar{z}$ = - Phase of z

Theorem 1.17 - *de Moivre's Formula*

$$z^n = (cos\theta + i.sin\theta)^n = cos(n\theta) + i.sin(n\theta)$$

Theorem 1.18 - *Euler's Formula*

$$e^{i\theta} = cos\theta + i.sin\theta$$

Remark 1.19
Using Euler's formula we can express all complex numbers in terms of e. Thus many properties of the exponential remain true:

$$\begin{aligned} z &= \lambda e^{i\theta}, \qquad\qquad \lambda \in \mathbb{R}, \theta \in [0, 2\pi) \\ => z_1 + z_2 &= \lambda_1.\lambda_2.e^{i(\theta_1+\theta_2)} \\ \&, \frac{z_1}{z_2} &= \frac{\lambda_1}{\lambda_2}.e^{i(\theta_1=\theta_2)} \end{aligned}$$

2 Euclidean Space, $\mathbb{R}^n$

Definition 2.01 - *Euclidean Space*
Let $n \in \mathbb{N}$ then $\forall\ \boldsymbol{x} = (x_1, x_2, ..., x_n)$ with $x_1, x_2, ..., x_n \in \mathbb{R}$ we have that $\boldsymbol{x} \in \mathbb{R}^n$.

Theorem 2.02 - *Operations in Euclidean Space*
Let $(x), (y) \in \mathbb{R}^n$ and $\lambda \in \mathbb{R}$. Then

$$(x) + (y) = (x_1 + y_1, ..., x_n + y_n)$$

And

$$(x) + \lambda.(y) = (x_1 + \lambda.y_1, ..., x_n + \lambda.y_n)$$

Definition 2.03 - *Cartesian Product*
Let $A, B \in \mathbb{R}^n$ be non-empty sets.
Then

$$A \times B := \{(a, b); a \in A, b \in B\}$$

2.1 Dot Product

Definition 2.04 - *Dot Product*
Let $\boldsymbol{v}, \boldsymbol{w} \in \mathbb{R}^n$. Then

$$\begin{aligned}\boldsymbol{v} \cdot \boldsymbol{w} &:= v_1.w_1 + ... + v_n.w_n \\ &:= \sum_{j=1}^{n} v_j.w_j\end{aligned}$$

Theorem 2.05 - *Properties of the Dot Product*
Let $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w} \in \mathbb{R}^n$. Linearity:

$$(\boldsymbol{u} + \lambda\boldsymbol{v}) \cdot \boldsymbol{w} = \boldsymbol{u} \cdot \boldsymbol{w} + \lambda(\boldsymbol{v} \cdot \boldsymbol{w})$$

Symmetry:

$$\boldsymbol{v} \cdot \boldsymbol{w} = \boldsymbol{w} \cdot \boldsymbol{v}$$

Positivity:

$$\boldsymbol{v} \cdot \boldsymbol{v} = v_1^2 + v_2^2 + ... + v_n^2 \geq 0$$

Definition 2.06 - *Orthogonality*
Let $\boldsymbol{v}, \boldsymbol{w} \in \mathbb{R}^n$.
It is said that $(v), (w)$ are orthogonal to each other if $\boldsymbol{v} \cdot \boldsymbol{w} = 0$
N.B. Orthogonal vectors are perpendicular to each other.

Definition 2.07 - *The Norm*
Let $\boldsymbol{x} \in \mathbb{R}^n$.
Then

$$||\boldsymbol{x}|| = \sqrt{\boldsymbol{x} \cdot \boldsymbol{x}} = \sqrt{\sum_{i=1}^{n} x_i^2}$$

Theorem 2.08 - *Properties of the Norm*
Let $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$ and $\lambda \in \mathbb{R}$. Then

$$\begin{aligned} ||\boldsymbol{x}|| &\geq 0 \\ ||\boldsymbol{x}|| &= 0 \text{ iff } \boldsymbol{x} = 0 \\ ||\lambda\boldsymbol{x}|| &= |\lambda|||\boldsymbol{x}|| \\ ||\boldsymbol{x}+\boldsymbol{y}|| &\leq ||\boldsymbol{x}|| + ||\boldsymbol{y}|| \end{aligned}$$

Theorem 2.09 - *Dot Product and Norm*
Let $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$.

$$|\boldsymbol{x} \cdot \boldsymbol{y}| \leq ||\boldsymbol{x}||||\boldsymbol{y}||$$

N.B. $|\boldsymbol{x} \cdot \boldsymbol{y}| = ||\boldsymbol{x}||||\boldsymbol{y}||$ iff $\boldsymbol{x}$ & $\boldsymbol{x}$ are orthogonal.

Theorem 2.10 - *Angle between Vectors*
Let $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$. Then

$$cos\theta = \frac{\boldsymbol{x} \cdot \boldsymbol{y}}{||\boldsymbol{x}||||\boldsymbol{y}||}$$

2.2 Linear Subspaces

Definition 2.11 - *Linear Subspace*
Let $V \subset \mathbb{R}^n$. V is a *Linear Subspace* if:
i) $V \neq \emptyset$;
ii) $\forall\ \boldsymbol{v}, \boldsymbol{w} \in V$ then $\boldsymbol{v} + \boldsymbol{w} \in V$;
iii) $\forall\ \lambda \in \mathbb{R}, \boldsymbol{v} \in V$ then $\lambda\boldsymbol{v} \in V$.

Definition 2.12 - *Span*
Let $\boldsymbol{x_1}, ..., \boldsymbol{x_k} \in \mathbb{R}^n; k \in \mathbb{N}$. Then

$$\text{span}\{\boldsymbol{x_1}, ..., \boldsymbol{x_k}\} := \{\lambda_1\boldsymbol{x_1} + ... + \lambda_k\boldsymbol{x_k}; \lambda_i \in \mathbb{R}, 0 \leq i \geq k\}$$

Definition 2.13 - *Spans are Subspaces*
Let $\boldsymbol{x_1}, ..., \boldsymbol{x_k} \in \mathbb{R}^n;\ k \in \mathbb{N}$. Then $\text{span}\{\boldsymbol{x_1}, ..., \boldsymbol{x_k}\}$ is a linear subspace of $\mathbb{R}^n$.

Theorem 2.14

$$W_{\boldsymbol{a}} := \{\boldsymbol{x} \in \mathbb{R}^n; \boldsymbol{x} \cdot \boldsymbol{a} = 0\} \text{ is a subspace.}$$

Definition 2.15 - *Orthogonal Complement*
Let $V \subset \mathbb{R}^n$. Then,

$$V^{\perp} := \{\boldsymbol{x} \in \mathbb{R}^n; \boldsymbol{x} \cdot \boldsymbol{y}\ \forall\ \boldsymbol{y} \in V\}$$

N.B. $V^{\perp} \subset \mathbb{R}^n$
Theorem 2.16 - *Relationship of Subspaces*
Let V, W be subspaces of $\mathbb{R}$. Then

$$V \cap W \text{ is a subspace.}$$

$$V + W := \{\boldsymbol{v} + \boldsymbol{w}; \boldsymbol{v} \in V, \boldsymbol{w} \in W\} \text{ is a subspace.}$$

Definition 2.17 - *Direct Sum*
Let V_1, V_2, W be subspaces of $\mathbb{R}$. Then W is said to be a *direct sum* if
i) $W = V_1 + V_2$;
ii) $V_1 \cap V_2 = \emptyset$.

3 Linear Equations & Matrices

3.1 Linear Equations

Definition 3.01 - *Multi-Variable Linear Equations*
Linear equations produce a straight line and can have multiple variables.
Examples - $x = 3, y = x + 3, z + 5x - 2y$

Defintion 3.02 - *Systems of Linear Equations*
Let $\boldsymbol{a}, \boldsymbol{x} \in \mathbb{R}^n$ & $b \in \mathbb{R}$ such that $\boldsymbol{a} \cdot \boldsymbol{x} = b$.
$\boldsymbol{a} \cdot \boldsymbol{x} = b$ is a linear equation in x with $S = \{\boldsymbol{x}; \boldsymbol{a} \cdot \boldsymbol{x} = b\}$ as the set of solutions.
N.B. If $b = 0$ then $S(\boldsymbol{a}, 0)$ is a subspace.

Definition 3.03 - *Matrix*
Let $m, n \in \mathbb{N}$, then a $m \times n$ grid of numbers form an "m" by "n" matrix. Each element of the matrix can be reference by a_{ij} with $i = 1, ..., m$ and $j = 1, ..., n$.

$$A = \begin{pmatrix} a_{11} & a_{12} & ... & a_{1n} \\ a_{21} & a_{22} & ... & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & ... & a_{mn} \end{pmatrix}$$

N.B. m,i = rows, n,j = columns

Definition 3.04 - *Sets of Matrices*
$M_{m,n}(\mathbb{R})$ is the set of m x n matrices containing only real numbers.
$M_{m,n}(\mathbb{Z})$ is the set of m x n matrices containing only integers.
$M_n(\mathbb{R})$ is the set square matrices, size n, containing only real numbers.

Definition 3.05 - *Transpose Vectors*
Let $\boldsymbol{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$ then $\boldsymbol{x}^t = \begin{pmatrix} x_1 & x_2 & ... & x_n \end{pmatrix}$

Definition 3.06 - *Vector-Matrix Multiplication*
Let $A \in \mathbb{R}_{m,n}$ and $\boldsymbol{x} \in \mathbb{R}^n$ then

$$A\boldsymbol{x} := \begin{pmatrix} a_{11} & a_{12} & ... & a_{1n} \\ a_{21} & a_{22} & ... & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & ... & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} a_{11}x_1 + a_{12}x_2 + ... + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + ... + a_{2n}x_n \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + ... + a_{mn}x_n \end{pmatrix} = \begin{pmatrix} \boldsymbol{a}_1^t \cdot \boldsymbol{x} \\ \boldsymbol{a}_2^t \cdot \boldsymbol{x} \\ \vdots \\ \boldsymbol{a}_m^t \cdot \boldsymbol{x} \end{pmatrix} \in \mathbb{R}^m$$

This can be simplified to

$$\boldsymbol{y} = A\boldsymbol{x} \text{ with } y_i = \sum_{j=1}^{n} a_{ij} x_j$$

Theorem 3.07 - *Operations on Matrices with Vectors*

i) $A(\boldsymbol{x} + \boldsymbol{y}) = A\boldsymbol{x} + A\boldsymbol{y}, \quad \forall\ \boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$.

ii) $A(\lambda \boldsymbol{x}) = \lambda(A\boldsymbol{x}), \quad \forall \boldsymbol{x} \in \mathbb{R}^n, \lambda \in \mathbb{R}$.

Theorem 3.08
Let $A = (a_{ij}) \in M_{m,n}(\mathbb{R})$ and $B = (b_{ij}) \in M_{l,m}(\mathbb{R})$. Then there exists a $C = (c_{ij}) \in M_{l,n}(\mathbb{R})$ such that
$$C\boldsymbol{x} = B(A\boldsymbol{x}), \quad \forall\ \boldsymbol{x} \in \mathbb{R}^n$$

N.B. $c_{ij} = \sum_{k=1}^{m} b_{ik}a_{kj}$

Theorem 3.09 - *Operation between Matrices*
Let $A, B \in M_{m,n}$ and $C \in M_{l,m}$

i) $C(A + B) = CA + CB$.

ii) $(A + B)C = AC + BC$.

iii) Let $D \in M_{m,n}, E \in M_{n,l}$ & $F \in M_{l,k}$ then
$$E(FG) = (EF)G$$

N.B. $AB \neq BA$

Definition 3.10 - *Types of Matrix*

Upper Triangle $\begin{pmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{pmatrix}$, $\quad a_{ij} = 0$ if $i > j$.

Lower Triangle $\begin{pmatrix} 1 & 0 & 0 \\ 2 & 3 & 0 \\ 4 & 5 & 6 \end{pmatrix}$, $\quad a_{ij} = 0$ if $i < j$.

Symmetric Matrix $\begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 0 \\ 3 & 0 & 1 \end{pmatrix}$, $\quad a_{ij} = a_{ji}$.

Anti-Symmetric $\begin{pmatrix} 1 & -2 & -3 \\ 2 & 0 & -4 \\ 3 & 4 & -1 \end{pmatrix}$, $\quad a_{ij} = -a_{ji}$.