Calculus 1 - Notes

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Contents

1   Before Calculus

1.1   Fundamental Theorem of Calculus

Definition 1.01 - *Fundamental Theorem of Calculus*
The Fundamental Theorem of Calculus states

$$\frac{d}{dx} \int_a^x f(t)dt = f(x)$$

Definition 1.02 - *Common Sets of Numbers*
Natural Numbers, set of positive integers - $\mathbb{N} := \{1, 2, 3, ...\}$.
Whole Numbers, set of all integers - $\mathbb{Z} := \{..., -2, -1, 0, 1, 2, ...\}$.
Rational Numbers, set of fractions - $\mathbb{Q} := \left\{\frac{p}{q} : p \in \mathbb{Z}, q \in \mathbb{N}\right\}$.
Real Numbers, set of all rational & irrational numbers - $\mathbb{R}$.

1.2   Intervals

Definition 1.03 - *Intervals*
Sets of real numbers that fulfil in given ranges.
<u>Notation</u>

$$[a, b] := \{x \in \mathbb{R} : a \leq x \leq b\}$$
$$(a, b] := \{x \in \mathbb{R} : a < x \leq b\}$$
$$[a, b) := \{x \in \mathbb{R} : a \leq x < b\}$$
$$(a, b) := \{x \in \mathbb{R} : a < x < b\}$$

<u>Example</u>
In what interval does x lie such that:

$$|3x + 4| < |2x - 1|$$

Solution

$$\text{Case } 1 : x \geq \frac{1}{2}$$
$$=> 1 - 2x < 3x + 4 < 2x - 1$$
$$=> 1 - 2x < 3x + 4$$
$$=> x > \frac{-3}{5}$$

$$\text{And,} => 3x + 4 < 2x - 1$$
$$=> x < -5$$

There are no real solutions in this range.

$$\text{Case } 2 : x < \frac{1}{2}$$
$$=> 2x - 1 < 3x + 4 < 1 - 2x$$
$$=> 2x - 1 < 3x + 4$$
$$=> -5 < x$$

$$\text{And,} => 3x + 4 < 1 - 2x$$
$$=> 5x < -3$$
$$=> x < \frac{-3}{5}$$

$$=> -5 < x < \frac{-3}{5}, \; x \in \left(-5, \frac{-3}{5}\right)$$

Definition 1.04 - *Functions*
Functions map values between fields of numbers. The signature of a function is defined by

$$f : A \to B$$

Where f is the name of the function, A is the domain and B is the co-domain.
The *Domain* of a function is the set of numbers it can take as an input.
The *Co-Domain* is the set of numbers that the domain is mapped to.

<u>N.B.</u> - A function is valid iff it maps each value in the domain to a single value in the co-domain.

Definition 1.05 - *Maximal Domain*
The *Maximal Domain* of a function is the largest set of values which can serve as the domain of a function.

Remark 1.06 - *Types of Function*
Let $f : A \to B$
Polynomials

$$f(x) = a_0 + a_1 x + ... + a_n x^n$$

Rational

$$f(x) = \frac{p(x)}{q(x)}, \quad q(x) \neq 0 \; \forall \; x \in A$$

Trigonometric

$$sin(x), \; cos(x), \; tan(x) \text{ etc.}$$

2   Limits

2.1   Limits

Definition 2.01 - *Limits*
A limit is the value a function tends to, for a given x.
i.e. The value f(x) has at it gets very close to x.

Formally We say L is the limit of $f(x)$ as x tends to x_0 if

$$\forall \; \varepsilon > 0, \; \exists \; \delta > 0 \text{ st if } x \in A \text{ and } |x - x_0| < \delta => |f(x) - L| < \varepsilon$$

Notation

$$\lim_{x \to x_0} f(x) = L$$

Definition 2.02 - *Directional Limits*
Sometimes the value of a limit depends on which direction you approach it from.
$\lim_{x \to x_0+}$ is used when approaching from values greater than x_0.
$\lim_{x \to x_0-}$ is used when approaching from values less than x_0.

Theorem 2.03 - *Operations with limits*
Let $\lim_{x \to x_0} f(x) = L_f$ and $\lim_{x \to x_0} g(x) = L_g$ Then

$$\lim_{x \to x_0} [f(x) + g(x)] = L_f + L_g$$
$$\lim_{x \to x_0} f(x).g(x) \quad = L_f.L_g$$
$$\lim_{x \to x_0} \frac{f(x)}{g(x)} \quad = \frac{L_f}{L_g} \quad L_g \neq 0$$

2.2   Exponential Function

Definition 2.04 - *Exponential Function*

$$e := \lim_{x \to \infty} \left(1 + \frac{1}{n}\right)^n \simeq 2.7182818...$$

Theorem 2.05 - *Binomial Expansion*
A techique for expanding binomial expressions

$$\left(1 + \frac{x}{n}\right)^n = \sum_{i=0}^{n} \binom{i}{n}.1^{(n-i)}.\left(\frac{x}{n}\right)^i$$
$$= 1 + x + \frac{n-1}{2n}.x^2 + ... + \frac{x^n}{n^n}$$

3   The Derivative

Definition 3.01 - *Differentiable Equations*
Let $f : A \to B$ and $x_0 \in A$.
f is differentiable at x_0 if $\exists \ L \in B$ such that

$$L = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

If this limit exists $\forall \ x \in A$ then we can define the derivative of $f(x)$

$$f'(x) := \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}$$

Definition 3.02 - *Notation for Differentiation*
There are two ways to denote the derivative of an equation

$$f'(x) \iff \frac{df}{dx}, f''(x) \iff \frac{d^2 f}{dx^2}, ..., f^{(n)}(x) \iff \frac{d^n f}{dx^n}$$

<u>N.B.</u> - Using $\frac{df}{dx}$ is more informative, especially for equations with multiple variables.

3.1　Techniques for finding derivative

Theorem 3.03 - *Sum Rule*
Let f, g be differentiable with respect to x.

$$(f + g)' = f' + g'$$

Theorem 3.04 - *Product Rule*
Let f, g be differentiable with respect to x.

$$(fg)' = f'g + fg'$$

Theorem 3.05 - *Quotient Rule*
Let f, g be differentiable with respect to x.

$$\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$$

Definition 3.06 - *Composite Functions*
Let $f : B \to C$ and $g : A \to B$ Then

$$(f \circ g)(x) = f(g(x))$$

Theorem 3.07 - *Chain Rule*
Let f, g be differentiable with respect to x.

$$\frac{d}{dx}f(g(x)) = f'(g(x)).g'(x)$$

3.2　Implicit Differentiation

Definition 3.08 - *Implicit Differentiation*
Sometimes it is hard to isolate variables in multi-variable equations, in these cases differentiate both sides with respect to the same variable.
Remembering

$$\frac{d}{dx}(x) = 1 \text{ and } \frac{d}{dx}(y) = \frac{dy}{dx} = y'$$

Example
Find y if $x^3 + y^3 = 6xy$

$$\frac{d}{dx}\left(x^3 + y^3\right) = \frac{d}{dx}\left(6xy\right)$$
$$=> 3x^2 + 3y^2.y' = 6y + 6x.y'$$
$$=> y'(3y^2 - 6x) = 6y - 3x^2$$
$$=> y' = \frac{2y - x^2}{y^2 - 2x}$$

3.3   Applications of The Derivative

Thoerem 3.09 - *Netwon's Method*
Let f be differentiable. Using *Newton's Method* we can approximate a solution to $f(x) = 0$.

   i) Take an inital guess, x_0;

   ii) Find the value of x where the tangent to x_0 on $f(x)$ intercepts the x-axis;

   iii) Use this value as the next guess;

   iv) Repeat until the value of x reduces little.

The equation for the tangent is

$$y = f(x_0) + (x - x_0)f'(x_0)$$

so a simplified equation for the process can be deduced

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

Theorem 3.10 - *Angle between Intersecting Curves*
Let $y = f_1(x)$ and $y = f_2(x)$ be two curves which intersect at (x_0, y_0).
Then $y_0 = f_1(x_0) = f_2(x_0)$
Let m_1, m_2 be the gradient of the tangents to f_1 & f_2 at x_0.
Then $\theta_i := tan^{-1}(m_i)$ for $i = 1, 2$.
Let $\phi = |\theta_1 - \theta_2|$, then

$$\phi = \left| \frac{m_1 - m_2}{1 + m_1 m_2} \right|$$

Theorem 3.11 - *L'Hospital's Rule*
For two equations equations, f, g with limit of $-\infty, 0$ or ∞ as x tends to a, it is hard to solve

$$\lim_{x \to a} \frac{f(x)}{g(x)}$$

Provided the limit exists, L'Hospital's Rule states that

$$\lim_{x \to a} \frac{f(x)}{g(x)} \iff \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

3.4   Sketching Curves

Remark 3.11 - *Sketching Curves*
Evaluating the derivative of a curve can make it easier to sketch:

   i) When $f'(x) > 0$ the curve is heading upwards;

   ii) When $f'(x) < 0$ the curve is heading downwards;

   iii) When $f'(x) = 0$ the curve is flat;

   iv) When $f'(x) = \infty, -\infty$ there are assymptotes.

Definition 3.12 - *Even Functions*
If $f(x) = f(-x)$ then the function is symmetrical and said to be *even*.
Examples - $x^2, cos(x), |x|$

Definition 3.13 - *Odd Functions*
If $f(x) = -f(-x)$ then the function is said to be *odd*.
Examples - $x, sin(x), x.cos(x)$

Remark 3.14
Some functions are neither *odd* nor *even*.
Example - $x + x^2$

4   Inegration

4.1   The Primitive

Definition 4.01 - *The Primitive*
A function, $F : A \to \mathbb{R}$, is a primative for the function $f : A \to \mathbb{R}$ if F is differentiable and

$$\frac{d}{dx}F = f$$

<u>N.B.</u> - Primitives are also called *Indefinite Integral* or *Anti-Derivative*.

Remark 4.02 - *Area Under a Curve*
Let $f : [a, b] \to \mathbb{R}$ be a continuous function. Then the area between the curve and the x-axis is found by integration.

$$A := \int_a^b f(x)dx$$

Definition 4.03 - *Convergent Improper Integrals*
Let $b > a$ and define a function, $f : [a, \infty) \to \mathbb{R}$, which is continuous in $[a, b]$ Then

$$\int_0^\infty f(x)dx := \lim_{b \to \infty} \int_a^b f(x)dx$$

If this limit exists then the improper integral is *convergent*, otherwise it is *divergent*.

Definition 4.04 - *Definite Integral*
Let F be the primative for the function f. Then

$$\int_b^a f(x)dx = F(a) - F(b)$$

<u>Notation</u> - $F(x)\big|_a^b = F(b) - F(a)$

Remark 4.05 - *Summing Definite Inegrals*
For all $a < c < b$

$$\int_a^b f(x)dx = \int_c^a f(x)dx + \int_b^c f(x)dx$$

$$\int_b^a f(x)d := -\int_a^b f(x)dx$$

Theorem 4.06 - *Taylor Series*
Functions can be expanded into polynomial form with degree n, T_n, and remainder R_n such that $f(x) = T_n(x) + R_n(x)$.

$$T_n(x) = f(a) + (x-a)f'(a) + ... + \frac{1}{n}.(x-a)^n.f^n(a)$$

$$R_n(x) = \frac{1}{n} \int_a^x (x-t)^n.f^{(n+1)}(t)dt$$

5 Parametric Curves & Arc-Length

5.1 Parametric Curves

Definition 5.01 - *Parametric Curves*
Parametric equations are an alternative to Cartesian equations, for representing curves. They can also represent a point in 3D space.

$$\boldsymbol{p} = \begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix}$$

Theorem 5.02 - *Parametric to Cartesian Equations*
As all equations in a Parametric series have a common variable, substition can be used to form a single equation.
Example Let $\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} t-2 \\ \frac{t}{t-2} \end{pmatrix}$.

$$x = t - 2$$
$$=> t = x + 2$$
$$=> y = \frac{x+2}{(x+2)-2}$$
$$= \frac{x+2}{x}$$
$$y = 1 + \frac{2}{x}$$

5.2 Tangent of a Curve

Theorem 5.02 - *Tangent to a Parametric Curve*
Let $(x(t), y(t))$ be a parametric equation. If we want to find the tangent at a point on the line, (a, b), we need to find the value t_0 such that $x(t_0) = a$ & $y(t_0) = b$.
Then by using the chain rule we can deduce the following equation for the tangent when $t = t_0$:

$$\frac{dy(t_0)}{dx(t_0)} = \frac{y - y(t_0)}{x - x(t_0)}$$

. Similarly we can deduce the equation for the normal when $t = t_0$:

$$-\frac{dx(t_0)}{dy(t_0)} = \frac{y - y(t_0)}{x - x(t_0)}$$

5.3　Arc-Length

Theorem 5.03 - *Arc-Length*
Arc-Length is the length of a curve, following a function, between two points. For a cartesian equation, $y = f(x)$, between the points x and $x + dx$ is

$$ds = \sqrt{dx^2 + dy^2}$$

So for a set of parametric equations, $(x(t), y(t))$, $a \leq t \leq b$,

$$ds = \sqrt{\frac{dx^2}{dt} + \frac{dy^2}{dt}}$$

To find the length of a curve between points a and b

$$s = \int_a^b \sqrt{\frac{dx^2}{dt} + \frac{dy^2}{dt}} dt$$

Definition 5.04 - *Curvature*
Curvature measures how fast the unit tangent vector to a curve rotates. Curvature of a curve, $y = f(x)$, can be found using the equation:

$$K(x) = \frac{|y''(x)|}{[1 + (y'(x))^2]^{\frac{3}{2}}}$$

For a set of parametric equations, $(x(t), y(t))$, it can be found using:

$$K(t_0) = \frac{y''(t_0).x'(t_0) - y'(t_0).x''(t_0)}{[(x'(t_0))^2 + (y'(t_0))^2]^{\frac{3}{2}}}$$

5.4　Level Curves

Definition 5.05 - *Level Curves*
Let $f : \mathbb{R}^d \to \mathbb{R}$ be a function with $d \geq 2$, $d \in \mathbb{N}$. A level curve for f is the set of real solutions for $f(\boldsymbol{x}) = c, c \in \mathbb{R}$.
<u>N.B</u> - $f(\boldsymbol{x}) = c$ is often written as $f = c$.

6　Differential Equations

Definition 6.01 - *Differential Equations*
Differential equations take the form

$$f(x, y, \frac{dx}{dy}, ..., \frac{d^{(n)}y}{dx^{(n)}}) = 0, \ x \in I$$

6.1　First Order Differential Equations

Definition 6.02 - *First Order*
First order differential equations are equations of form $f(x, y, \frac{dx}{dy}) = 0$.

Definition 6.03 - *Seperable Equations*
An equation, f, is said to be seperable if there exists two equations, $M(x)$, $N(y)$, such that

$$f(x, y, y') = y' - M(x).N(y)$$

Thus

$$y' = M(x).N(y)$$

$$=> \frac{y'}{N(y)} = M(x)$$

$$=> \int \frac{1}{N(y)} dy = \int M(x) dx$$

After integration, the equation can be rearranged to be in terms of y.

6.2 Integrating Factor

Theorem 6.04 - *Integrating Factor*
Consider the equation $y' + f(x)y + g(x)$. Let $F(x) = \int f(x)dx$. Thus

$$e^{F(x)}.y' + e^{F(x)}.y = e^{F(x)}.g(x)$$

$$=> \frac{d}{dx}\left(e^{F(x)}.y\right) = e^{F(x)}.g(x)$$

$$=> e^{F(x)}.y = \int e^{F(x)}.g(x) \ dx$$

$$=> y = e^{-F(x)} \int e^{F(x)}.g(x) \ dx$$

6.3 Second Order Differential Equations

Definition 6.05 - *Linear Differential Equations*
A differential equation is said to be *linear* if it can be written in the form

$$Ay(x) := a_n(x).y^{(n)}(x) + ... + a_1(x).y'(x) + a_0(x).y(x) = b(x)$$

We define the set of solutions as

$$S(A, b) := y : I \to \mathbb{R}; Ay = b$$

If the only solution is $b = 0$ then the system is homogenous, otherwise it is inhomogenous.

Definition 6.06 - *Particular & Complimentary Solutions*
When solving a differential equation, $Ay(x) = b(x)$, we need to find two functions in order to find the final solution.

  i) Complementary Function, y_c - The homogenous case of the equation, $Ay(x) = 0$;

  ii) Particular Function, y_p - The inhomogenous case of the equation, $Ay(x) = b(x)$ for a given $b(x)$.

Then $y = y_c + y_p$ is the final solution for $Ay(x) = b(x)$.

Theorem 6.07 - *Complementary Functon of LDEs with Constant Coefficients*
Take a linear differential equation

$$a_n.y^{(n)}(x) + ... + a_1.y'(x) + a_0.y(x) = b(x)$$

where $a_n, ..., a_1, a_0 \in \mathbb{R}$ & $b(x) : \mathbb{R} \to \mathbb{R}$ are all constant.
To find the *Complementary Function* we solve the equation

$$a_n.\lambda^n + ... + a_1.\lambda^n + a_0 = 0$$

. to get solutions $\lambda_1, ..., \lambda_k$ and then produce the complimentary function

$$y_c(x) = \mu_1 e^{\lambda_1 x} + ... + \mu_k e^{\lambda_k x}$$

Where $\mu_1, ..., \mu_k$ are constants to be found later, by comparing with $b(x)$.
Remark 6.08 - *Complementary Function*
The complementary function, y_c, for differential equations with constant coefficients depends
upon the $\lambda_1, ..., \lambda_k$ we find, due to Euler's Formula.

   i) $\lambda_i = c, \quad y_{c_i} = \mu_i e^{\lambda_i x}$;

   ii) $\lambda_i = \pm ib, \quad y_{c_i} = \mu_{i_1} cos(bx) + \mu_{i_2} sin(bx)$;

   iii) $\lambda_i = a \pm ib, \quad y_{c_i} = e^{ax}[\mu_{i_1} cos(bx) + \mu_{i_2} sin(bx)]$.

Then $y_c = \sum_{j=1}^{k} y_{c_j}$.

Remark 6.09 - *Particular Function*
The particular function, y_p, for a differential equation with constand coefficients, $Ay(x) = b(x)$,
depends on the form of $b(x)$.

   i) $b(x) = a_n x^n + ... + a_1 x + a_0, \quad y_p = b_n x^n + ... + b_1 x + b_0$;

   ii) $b(x) = ae^{bx}, \quad y_p = \alpha e^{\beta x}$;

   iii) $b(x) = a.sin(bx) + c.cos(dx), \quad y_p = \alpha sin(\beta x) + \gamma cos(\delta x)$.

Where the constants of y_p are values to be found, when given certain conditions.
Theorem 6.10 - *Particular Functon of LDEs with Constant Coefficients*
Take a linear differential equation

$$a_n.y^{(n)}(x) + ... + a_1.y'(x) + a_0.y(x) = b(x)$$

where $a_n, ..., a_1, a_0 \in \mathbb{R}$ & $b(x) : \mathbb{R} \to \mathbb{R}$ are all constant.
Deduce the particular function for the differential equation, given $b(x)$, and then differentiate
y_p n times.
Substitute in these values, in place of the ys, into the original equation and solve to find values
for the constants in y_p.

Example
Solve $y'' - y' + y = x^2$.

<div align="center">

2Complementary Function

Let $\lambda^2 - \lambda + 1 = 0$

$$=> \lambda = \frac{1 \pm \sqrt{1 - 4}}{2}$$

</div>

$$= \frac{1}{2} \pm i\frac{\sqrt{3}}{2}$$

$$\Rightarrow y_c = e^{\frac{x}{2}}[A\cos(x\frac{\sqrt{3}}{2}) + B\sin(\frac{\sqrt{3}}{2})]$$

Particular Function

$$\text{Let } y_p = \alpha x^2 + \beta x + \gamma$$
$$\Rightarrow y_p'(x) = 2\alpha x + \beta, \&$$
$$\Rightarrow y_p''(x) = 2\alpha$$
$$\Rightarrow (2\alpha) - (2\alpha x + \beta) + (\alpha x^2 + \beta x + \gamma) = x^2$$
$$\Rightarrow x^2[\alpha] + x[\beta - 2\alpha] + [2\alpha - \beta + \gamma] = x^2$$

$$[x^2] : \alpha = 1$$
$$[x] : \beta - 2\alpha = 0$$
$$\Rightarrow \beta = 2$$
$$[x^0] : 2\alpha + \gamma - \beta = 0$$
$$\Rightarrow \gamma = 0$$
$$\Rightarrow y_p = x^2 + 2x$$

$$\Rightarrow y = x^2 + 2x + e^{\frac{x}{2}}[A\cos(x\frac{\sqrt{3}}{2}) + B\sin(\frac{\sqrt{3}}{2})]$$

6.4   Wronskian

Definition 6.11 - *The Wronskian*
The *Wronskian*, $W[y_1, y_2]$, of two differentiable functions is defined by

$$W[y_1, y_2](x) = y_1(x).y_2'(x) - y_1'(x).y_2(x)$$

The notation for this is

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} := ad - bc$$

So

$$W[y_1, y_2] = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$

Remark 6.12
If $W[y_1, y_2] \neq 0$ then y_1, y_2 are linearly independent.

6.5   Variation of Constants

Theorem 6.13
This is a technique for solving all differential equations, not just ones with constant coefficients, and assumes we know the complementary function, y_c.
Consider the equation

$$Ay(x) := y''(x) + \beta(x)y'(x) + \gamma(x)y(x) = b(x), \quad \text{for a known } b(x)$$

Suppose we have a complementary function in the form

$$y_c = \lambda_1(x)y_1(x) + \lambda_2(x)y_2(x)$$

where y_1 & y_2 are linearly independent, thus $W[y_1, y_2] \neq 0$.
Then

$$y_p' = \lambda_1' y_1 + \lambda_1 y_1' + \lambda_2' y_2 + \lambda_2 y_2'$$

As λ_1, λ_2 are constant then $\lambda_1' = \lambda_2' = 0$ so

$$y_p' = \lambda_1 y_1' + \lambda_2 y_2'$$

By differentiating and then substituting back into the original equation we see y_p is a solution
iff

$$\lambda_1' y_1' + \lambda_2' y_2' = f$$

In matrix form we have

$$\begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix} \begin{pmatrix} \lambda_1' \\ \lambda_2' \end{pmatrix} = \begin{pmatrix} 0 \\ f \end{pmatrix}$$

Then by Cramer's rule we have

$$\lambda_1' = \frac{\begin{vmatrix} 0 & y_2 \\ f & y_2' \end{vmatrix}}{W[y_1 \ y_2]} = \frac{-y_2 f}{W[y_1 \ y_2]}$$

and

$$\lambda_2' = \frac{\begin{vmatrix} y_1 & 0 \\ y_1' & f \end{vmatrix}}{W[y_1 \ y_2]} = \frac{y_1 f}{W[y_1 \ y_2]}$$

Giving use a solution for $y_p' = \lambda_1' y_1 + \lambda_1 y_1' + \lambda_2' y_2 + \lambda_2 y_2'$.

7   Applied Differential Equations

Definition 7.01 - *Denoting Limit Relationships*
We use

$$F(x) \sim G(x) \text{ as } x \to a$$

to denote

$$\lim_{x \to a} \frac{F(x)}{G(x)} = 1$$

Theorem 7.02 - *Vibrating String*
If we are given a string which is L long then we can define an equation, $y(x, t)$, which describe
the displacement of a point x along the string, at time t.

$$y(x, t) = u(x)e^{i\omega t}$$

Where $\frac{\omega}{2\pi}$ is the frequency of the string and $u(x) = A\cos(\omega x) + B\sin(\omega x)$.
We can generalise this for strings with n anti-nodes.

$$\omega_n := \frac{n\pi}{L}, \quad u_n := \sin(\omega_n x)$$

8   Liner Difference Equations

Definition 8.01 - *Difference Equations*
A difference equation is an equation of the form

$$F(n, y_n, ..., y_{n+d}) = 0, \quad n, d \in \mathbb{N}$$

8.1   First-Order Linear Difference Equation

Definition 8.02 - *Liner First-Order Difference Equations*
A *Linear First-Order Difference Equation* is an equation, F, which can be described by

$$F(n, y_n, y_{n+1}) = a_n y_{n+1} + b_n y_n - f_n$$

where a_n, b_n, f_n are all known sequences.

Example 8.03
By taking a simple equation

$$y_{n+1} - y_n = f_n$$

we can see that

$$y_{n+1} = y_n + (y_{n+1} - y_n) = y_n + f_n = ... = y_{n_0} + f_{n_0} + ... + f_{n-1} + f_n$$

So

$$y_n = y_{n_0} + \sum_{j=n_0}^{n-1} f_j$$

Theorem 8.04 - *Solving First-Order Liner Difference Equations*
From *Definition 8.02* we can generalise the equation to show that

$$y_{n+1} + b_n y_n = f_n$$

Then

$$\frac{-1}{b_n} y_{n+1} - y_n = \frac{-1}{b_n} f_n$$

We now define the *Summing Factor*, S_n, as

$$S_n := \prod_{j=n_0}^{n-1} \frac{-1}{b_n}$$

. We multiply both sides of the original equation by the summing factor and as $S_n(\frac{-1}{b_n}) = S_{n+1}$
we get

$$S_{n+1} y_{n+1} - S_n y_n = S_{n+1} f_n$$

As this has the same form as the example in *8.03* we can now deduce

$$S_n y_n = y_{n_0} + \sum_{j=n_0}^{n-1} S_{j+1} . f_j$$

8.2   Second-Order Linear Difference Equation

Definition 8.05 - *Second-Order Linear Difference Equation*
A *Linear Second-Order Difference Equation* is an equation, F, which can be described by

$$a_n y_{n+2} + b_n y_{n+1} + c_n y_n = f_n$$

where a_n, b_n, c_n, f_n are know sequences.

Remark 8.06 - *Solving Second-Order Linear Difference Equations*
Similar to solving second-order differential equations we need to consider to cases. The *homogenous* & *inhomogenous* cases. So two sequences will be found the complementary sequence, y_n^c, and the particular sequence, y_n^p. The final solution for y_n is given by

$$y_n = y_n^c + y_n^p$$

Definition 8.07 - *Wronskian of Sequences*
For two sequences $u_n \& v_n$ we define the Wronskian to be

$$W_n := \begin{vmatrix} u_n & v_n \\ u_{n+1} & v_{n+1} \end{vmatrix} = u_n.v_{n+1} - v_n.u_{n+1}$$

Theorem 8.08 - *Homogenous Case with Constant Cofficients*
Take the equation

$$ay_{n+2} + by_{n+1} + cy_n = 0$$

where a, b, c are known constants. We look for solutions of the form

$$y_n = \lambda^n$$

By substition we get the equation $a\lambda^2 + b\lambda + c = 0$. By solving for λ we find a solution

   i) λ has two real solutions - $y_n = A\lambda_1^n + B\lambda_2^n$;

   ii) λ has one real solution - $y_n = (An + B)\lambda^n$;

   iii) λ has only an imaginamry solution - $y_n = \Lambda e^{i\theta}, \quad \Lambda^2 := \frac{c}{a}, \theta := tan^{-1}(\frac{4ac - b^2}{-b})$.

Theorem 8.09 - *Homogenous Case Second-Order Linear Difference Equations*
The homogenous case finds solutions for

$$a_n y_{n+2} + b_n y_{n+1} + c_n y_n = 0$$

Suppose $y_n = u_n$ and $y_n = v_n$ are solutions to this homogenous equation. Then

$$W_n[u_n \ v_n] = W_{n_0} \prod_{j=n_0}^{n-1} \frac{c_j}{a_j}$$

So

$$u_n.v_{n+1} - v_n.u_{n+1} = W_{n_0} \prod_{j=n_0}^{n-1} \frac{c_j}{a_j}$$

Which can be rearranged to be in the form of a first order difference equation, such as *Example 8.03*

$$\frac{u_n}{u_{n+1}} v_{n+1} - v_n = W_{n_0} \prod_{j=n_0}^{n-1} \frac{c_j}{a_j}$$

Which has a summing factor

$$S_n = \frac{1}{u_n}$$

By multiplying both sides by S_n and simplifying we get

$$v_n = u_n \sum_{j=n_0}^{n-1} \frac{1}{u_j.u_{j+1}} \prod_{k=n_0}^{n-1} \frac{c_k}{a_k}$$

Typically you need to solve the product part of the equation to get a result for the sequence u_n.

Remark 8.10 - *Inomogenous Case Second-Order Linear Difference Equations*
Generally the best way to do this is to make an educated guess based on the right hand side of the equation. So if the RHS is a polynomial, guess a polynomial, etc. Similar to solving differential equations.

9   Several Variables - Differentiability

Definition 9.1 - *Several Variable Function*
Let $d \in \mathbb{N}, A \subset \mathbb{R}^d$ & $B \subset \mathbb{R}^d$.
A function $\boldsymbol{f} : A \to B$ is a map which, for all $\boldsymbol{x} \in A$, assigns a unique value $\boldsymbol{f}(\boldsymbol{x}) \in B$.

Defintion 9.2 - *Linear Functions*
A function $\boldsymbol{f} : \mathbb{R}^d \to \mathbb{R}^n$ is *linear* if it can be given in terms of a matrix $A \in M_{n,d}(\mathbb{R})$ where

$$\boldsymbol{f}(\boldsymbol{x}) = A\boldsymbol{x}$$

Theorem 9.3 - *Properties of Liner Functions*
If a function is linear then the following are true:

   i) $\boldsymbol{f}(\lambda\boldsymbol{x}) = \lambda\boldsymbol{f}(\boldsymbol{x})$;

   ii) $\boldsymbol{f}(\boldsymbol{x} + \boldsymbol{y}) = \boldsymbol{f}(\boldsymbol{x}) + \boldsymbol{f}(\boldsymbol{y})$.

Definition 9.4 - *Continuous Several Variable Function*
Let $\boldsymbol{f} : A \to B$ and $\boldsymbol{a} \in A$. Then $\boldsymbol{f}$ is continuous if

$$\lim_{\boldsymbol{x} \to \boldsymbol{a}} \boldsymbol{f}(\boldsymbol{x}) = \boldsymbol{f}(\boldsymbol{a})$$

9.1   The Derivative

Defintion 9.5 - *Norm of a Vector*
The norm of a vector, $\boldsymbol{x} \in \mathbb{R}^d$ is

$$||\boldsymbol{x}|| := \left(\sum_{j=1}^{d} x_j\right)^{1/2}$$

Definition 9.6 - *Derivative of Several Variable Function*
A function, $\boldsymbol{f} : \mathbb{R}^d \to \mathbb{R}^n$, is said to be differentiable at the point $\boldsymbol{x} \in \mathbb{R}^d$ if the exists an $A \in M_{n,d}(\mathbb{R})$ such that
$$\lim_{\boldsymbol{h} \to \boldsymbol{0}} \frac{||\boldsymbol{f}(\boldsymbol{x} + \boldsymbol{h}) - \boldsymbol{f}(\boldsymbol{x}) - A\boldsymbol{h}||}{||\boldsymbol{h}||} = 0$$
This A is the derivative of $\boldsymbol{f}(\boldsymbol{x})$.
$$\boldsymbol{f}'(\boldsymbol{x}) := A$$

Remark 9.7
If consider the following several variable function

$$\boldsymbol{f} = \begin{pmatrix} f_1 \\ \vdots \\ f_n \end{pmatrix}$$

Then

$$\boldsymbol{f}' = \begin{pmatrix} f_1' \\ \vdots \\ f_n' \end{pmatrix}$$

10   Directional & Partial Derivatives

10.1   Directional Derivative

Definition 10.1 - *Direction*
A direction in $\mathbb{R}^d$ is a vector of unit length.
In $\mathbb{R}^2$ every direction can be given by $\boldsymbol{u} = \begin{pmatrix} cos(\theta) \\ sin(\theta) \end{pmatrix}$, where θ is the angle from positive x axis

In $\mathbb{R}^3$ every direction can be given by $\boldsymbol{u} = \begin{pmatrix} sin(\phi)cos(\theta) \\ sin(\phi)sin(\theta) \\ cos(\phi) \end{pmatrix}$, where ϕ is the angle from positive

z axis and θ is the angle from the positive x axis.
Definition 10.2 - *Spherical Co-ordinates*
The spherical co-ordinates describe points in three dimension space.
The distance of a point from the origin is

$$r = \rho sin(\phi)$$

Then

$$\boldsymbol{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} rcos(\theta) \\ rsin(\theta) \\ \rho cos(\phi) \end{pmatrix} = \begin{pmatrix} \rho sin(\phi)cos(\theta) \\ \rho sin(\phi)sin(\theta) \\ \rho cos(\phi) \end{pmatrix}$$

Defintion 10.3 - *Direction Derivative*
The direction derivative of $\boldsymbol{f}$ in the direction of $\boldsymbol{u}$ at the point $\boldsymbol{x}_0$ is the vector:

$$D_{\boldsymbol{u}}\boldsymbol{f}(\boldsymbol{x}_0) := \frac{d}{dt}\boldsymbol{f}(\boldsymbol{x}_0 + t\boldsymbol{u}) \mid_{t=0}$$

Theorem 10.4
For all $\boldsymbol{f} : \mathbb{R}^d \to \mathbb{R}^n$ the directional derivative at $\boldsymbol{u}$ in $\mathbb{R}^d$ we have

$$D_{\boldsymbol{u}}\boldsymbol{f}(\boldsymbol{x}) = \boldsymbol{f}'(\boldsymbol{x}).\boldsymbol{u}$$

10.2   Partial Derivative

Defintion 10.5 - *Partial Derivative*
Let $f : \mathbb{R}^d \to \mathbb{R}$. Then the direction derivative, $D_{\boldsymbol{e}_j}f(\boldsymbol{x})$, if it exists, is called the *partial derivative* of f with respect to x_j at $\boldsymbol{x}$.
This is denoted by

$$\frac{\partial f}{\partial x_j}(\boldsymbol{x}) \text{ or } f_{x_j}(\boldsymbol{x})$$

Proposition 10.6 - *Partial Derivative as a Matrix*
If $\boldsymbol{f} : \mathbb{R}^d \to \mathbb{R}^n$ is differentiable then

$$\boldsymbol{f}'(\boldsymbol{x}) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_d} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_d} \end{pmatrix}$$

Remark 10.7 - *Second Order Partial Derivatives*

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x} \cdot \frac{\partial f}{\partial x}$$
$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x} \cdot \frac{\partial f}{\partial y}$$
$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y} \cdot \frac{\partial f}{\partial y}$$

11   Gradient & Chain Rule in Several Variables

11.1   Chain Rule

Theorem 11.1 - *Chain rule in Several Variables*
Let $\boldsymbol{f} : \mathbb{R}^d \to \mathbb{R}^n$ & $\boldsymbol{g} : \mathbb{R}^m \to \mathbb{R}^d$ be differentiable. Then

$$[\boldsymbol{f} \cdot \boldsymbol{g}]' = \boldsymbol{f}'(\boldsymbol{g}(\boldsymbol{x})).\boldsymbol{g}'(\boldsymbol{x})$$

11.2   Invertible Maps & Implicit Differentiation

Definition 11.2 - *Implicit Differentiation*
Let $A, B \subset \mathbb{R}^d$ and $\boldsymbol{f} : A \to B$.
If $\boldsymbol{f}$ is invertible then be cannot denote the inverse by $\boldsymbol{f}^{-1} : B \to A$ and

$$[\boldsymbol{f}^{-1} \cdot \boldsymbol{f}](\boldsymbol{x}) = \boldsymbol{x}$$

If $\boldsymbol{f}$ and $\boldsymbol{f}^{-1}$ are differentiable then by the chain run

$$\boldsymbol{f}'(\boldsymbol{f}^{-1}(\boldsymbol{x}))\boldsymbol{f}^{-1}\,'(\boldsymbol{x}) = I$$

Hence

$$\boldsymbol{f}^{-1}\,'(\boldsymbol{x}) = [\boldsymbol{f}'(\boldsymbol{f}(\boldsymbol{x}))]^{-1}$$

11.3   The Gradient

Defintion 11.3 - *The Gradient*
Let $f : \mathbb{R}^d \to \mathbb{R}$ be a map with first-order partial derivatives at every point of $\mathbb{R}^d$.
Then the gradient at a point $\boldsymbol{x} \in \mathbb{R}^d$, $\nabla f : \mathbb{R}^d \to \mathbb{R}^d$, is defined as

$$\nabla f(\boldsymbol{x}) := (f_{x_1}(\boldsymbol{x}), \ldots, f_{x_d}(\boldsymbol{x}))$$

Definition 11.4 - *Direction of Greatest Change*
By considering the case in two dimensions then

$$(D_{\boldsymbol{u}}f)(x, y) = \nabla f(x, y) \cdot \boldsymbol{u}$$

Where $\boldsymbol{u} = (cos\theta, sin\theta)$.
Then

$$(D_{\boldsymbol{u}}f)(x_0, y_0) = cos\theta.\frac{\partial f}{\partial x}(x_0, y_0) + sin\theta.\frac{\partial f}{\partial y}(x_0, y_0)$$

The greatest change occurs when

$$\frac{d}{d\theta}[cos\theta.\frac{\partial f}{\partial x}(x_0, y_0) + sin\theta.\frac{\partial f}{\partial y}(x_0, y_0)] = 0$$

$$=> -sin\theta.\frac{\partial f}{\partial x}(x_0, y_0) + cos\theta.\frac{\partial f}{\partial y}(x_0, y_0)] \quad =$$

$$=> \nabla f(x_0, y_0) \cdot (-sin\theta, cos\theta) \qquad\qquad =$$

We can thus deduce that

$$\nabla f(x_0, y_0) \perp (-sin\theta, cos\theta)$$

So

$$\boldsymbol{u} \parallel \nabla f(x_0, y_0)$$

We can then establish the direction, $\boldsymbol{u}$ occurs when $\boldsymbol{u}$ satisfies

$$\boldsymbol{u} = \frac{\nabla f(x_0, y_0)}{||\nabla f(x_0, y_0)||}$$

12   Integration over Two-Dimensional Domains

Remark 12.1 - *Notation*
Let $D \subset \mathbb{R}^2$.
When integrating over D the following notation is used

$$\iint_D f(x,y) dx dy$$

When D is specified e.g. $D = \{(x,y) : g(y) < x < h(y); a < y < b\}$ then we denote it by

$$\int_a^b \left\{ \int_{g(y)}^{h(y)} f(x,y) dx \right\} dy$$

Theorem 12.2 - *Rectangular Domain*
in terms of distance from origin If we are given a domain between constant values, $D = \{(x,y) : c < y < d; a < x < b\}$ then we perform

$$V = \int_c^d \left\{ \int_a^b f(x,y) dx \right\} dy$$

And can perform the integrations are seperate calculations.
Theorem 12.3 - *Triangle Domain*
In triangular domains the value of x depends on y, so can be expressed as a function of y. If we take $0 < x < a$ and $0 < y < b$ then $x = \frac{ya}{b}$. This gives us a domain of

$$D = \{(x,y) : \frac{ya}{b} < x < a; 0 < y < b\}$$

We can solve the intregral by performing:

$$V = \int_0^b \left\{ \int_{\frac{ay}{b}}^a f(x,y) dx \right\} dy$$

Theorem 12.4 - *Genreal Domains*
If we have a domain, D, define by $D = \{(x,y) \in \mathbb{R}^2 : L(x) \leq y \leq U(x), a \leq x \leq b\}$. Then the integral over D can be found by

$$V = \int_a^b \left\{ \int_{L(x)}^{U(x)} f(x,y) dy \right\} dx$$

13   Polar Co-ordinates

Defintion 13.1 - *Cartesian to Polar Co-ordinates*
Polar co-ordinates are an alternative way to denoting a point in two dimensional space, using distance from origin and angle from positive x-axis.
Polar co-ordiantes are given in the form (r,θ). Using pythagoras' theorem and trigonometry we can see that
$$r = \sqrt{x^2 + y^2}, \ \theta = tan^{-1}\left(\frac{y}{x}\right)$$

Theorem 13.2 - *Polar to Cartesian Co-ordinates*
We can convert a polar co-ordinate, (r,θ), to cartesian co-ordinates, (x,y) using the following formula
$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} rcos(\theta) \\ rsin(\theta) \end{pmatrix}$$

Remark 13.3 - *Plotting Curves expressed as polar co-ordiantes*
There are two common ways to do this:

   i) Convert back to cartesian form;

   ii) Interpolation, plot a few points and link them.

Theorem 13.4 - *Computing Curvature of Polar Curve*
Suppose a curve is defined by $r = R(\theta)$. This means that

$$x = rcos(\theta) = R(\theta)cos(\theta)$$
$$y = rsin(\theta) = R(\theta)sin(\theta)$$

By substituting into the equation for curvature of a cartesian curve we get

$$k(\theta) = \frac{|R(\theta)^2 + 2[R'(\theta)]^2 - R(\theta)R''(\theta)|}{\{R(\theta)^2 + [R'(\theta)]^2\}^{3/2}}$$

Theorem 13.5 - *Gradient of a Polar Curve*
Consider the equation $f(x, y) = u(r, \theta)$ then by using substitution and chain rule we can deduce the following equations:

$$\frac{\partial f}{\partial x} = cos(\theta)\frac{\partial u}{\partial r} - \frac{sin(\theta)}{r}\frac{\partial u}{\partial \theta}$$
$$\frac{\partial f}{\partial y} = sin(\theta)\frac{\partial u}{\partial r} + \frac{cos(\theta)}{r}\frac{\partial u}{\partial \theta}$$

13.1   Change of Variable Formula

Theorem 13.6 - *Single Variable Substitution*
By making the substitution of $x = g(u)$ we can produce the following formula

$$\int_{g(a)}^{g(b)} f(x)dx = \int_a^b (f \cdot g)(u)g'(u)du$$

Theorem 13.6 - *Double Variable Substitution*
Suppose we make the following substitution

$$\begin{pmatrix} x \\ y \end{pmatrix} = \boldsymbol{g}(u, v) := \begin{pmatrix} g_1(u, v) \\ g_2(u, v) \end{pmatrix}$$

Then we can produce the following equation

$$\iint_{\boldsymbol{g}(D)} f(x, y)dxdy = \iint_D (f \cdot \boldsymbol{g})(u, v)|det\{\boldsymbol{g}'(u, v)\}dudv$$

Remark 13.7 - *Cartesian to Polar Double Integration*
By using the formula in 13.6 we can change produce the following equation, by using the substitution $(x, y) = (rcos(\theta), rsin(\theta))$

$$\boldsymbol{g}(r, \theta) = (rcos(\theta), rsin(\theta))$$

$$\iint_{\boldsymbol{g}(D)} f(x, y)dxdy = \iint_D f(rcos(\theta), rsin(\theta)).r.drd\theta$$

14   Triple Integration

Remark 14.1 - *Triple Integration*
If we let D be a three dimensional region, so $D \subset \mathbb{R}^3$. Then by doing

$$\iiint_D f(x,y,z)dxdydz$$

we produce a *hypervolume*.
By setting $f(x,y,z) = 1$ we can find the volume of D. If we set it to be a variable equation then we can determine other properties such as mass, center of mass etc.

Theorem 14.2 - *Box-Like region*
If we define D as $D = \{(x,y,z) \in \mathbb{R}^3 : a \le x \le A; b \le y \le B; c \le z \le C\}$. We can integrate over D by

$$\iiint_D f(x,y,z)dxdydz = \int_c^C \left\{ \int_b^B \left\{ \int_a^A f(x,y,z)dx \right\} dy \right\} dz$$

Theorem 14.3 - *General Domains*
If we define D as $D = \{(x,y,z) \in \mathbb{R}^3 : a \le x \le A(y,z); b \le y \le B(z); c \le z \le C\}$. We can integrate over D by

$$\iiint_D f(x,y,z)dxdydz = \int_c^C \left\{ \int_b^{B(z)} \left\{ \int_a^{A(x,y)} f(x,y,z)dx \right\} dy \right\} dz$$

14.1   Substitution with Triple Integrals

Theorem 14.4 - *Substitution with Triple Integration*
If we want to perform a substitution define by

$$x = g_1(u,v,w),$$
$$y = g_2(u,v,w),$$
$$z = g_3(u,v,w)$$

Then we can produce the following equation

$$\iiint_{\boldsymbol{g}(D)} f(x,y,z)dxdydz = \iiint_D (f \cdot \boldsymbol{g})(u,v,w).|det\{\boldsymbol{g}'(u,v,w)\}|dudvdw$$

Remark 14.5 - *Cartesian to Spherical Triple Integration*
If we can to covnert a cartesian region to a spherical region we use the following substitution

$$x = \rho sin(\phi)cos(\theta) = g_1(\rho,\phi,\theta),$$
$$x = \rho sin(\phi)sin(\theta) = g_2(\rho,\phi,\theta),$$
$$x = \rho sin(\phi) \qquad = g_3(\rho,\phi,\theta)$$

This means $det\{\boldsymbol{g}'(\rho,\phi,\theta)\} = \rho^2 sin(\phi)$. Then

$$\iiint_{\boldsymbol{g}(D)} f(x,y,z)dxdydz = \iiint_D (f \cdot \boldsymbol{g})(\rho,\phi,\theta).(\rho^2 sin(\phi)).d\rho d\phi d\theta$$

14.2   Applications of Multiple Integration

Definition 14.6 - *Multiple Integrations*
When integrating over multiple dimensions we dont need to write $\int$ every time, instead we can denoted it as

$$\int_R f(\boldsymbol{x})d\boldsymbol{x}$$

or

$$\int \cdots \int_R f(\boldsymbol{x})dx_1 \ldots dx_d$$

Theorem 14.7 - *Centre of Mass of a Volume*
By considering a region $R \subset \mathbb{R}^d$ whose density is describe by $f(\boldsymbol{x})$. Then its mass can be found by

$$m = \int_R f(\boldsymbol{x})d\boldsymbol{x}$$

Then its centre of mass, $\bar{x} \in \mathbb{R}^d$, can be found by

$$\bar{x} := \frac{1}{m}\int_R \boldsymbol{x}f(\boldsymbol{x})d\boldsymbol{x}$$

15   Local Extrema & Taylor's Theorem in Several Variables

15.1   Taylor's Theorem

Theorem 15.1 - *Taylor's Theorem for a Single Variable*
If $f : \mathbb{R} \to \mathbb{R}$ is smooth and $a \in \mathbb{R}$, then

$$f(x) = f(a) + (x-a)f'(a) + \frac{1}{2!}(x-a)^2 f''(a) + \ldots$$

$$f(x) = \sum_{j=1}^{\infty} \frac{1}{j!} \cdot (x-a)^j \cdot \left[\frac{d^j f}{dx^j}(a)\right]$$

Theorem 15.2 - *Taylor's Theorem for a Two Variables*
If $f : \mathbb{R}^2 \to \mathbb{R}$ is smooth and $a, b \in \mathbb{R}$, then

$$f(x, y) = f(a, b) + f'(a, b)\begin{pmatrix} x-a \\ y-b \end{pmatrix} + \frac{1}{2}\begin{pmatrix} x-a \\ y-b \end{pmatrix}^t f''(a, b)\begin{pmatrix} x-a \\ y-b \end{pmatrix} + \ldots$$

<u>N.B</u> - $f'(x, y) \in M_{1,2}(\mathbb{F})$ & $f''(x, y) \in M_2(\mathbb{F})$

15.2   Local Extrema

Definition 15.3 - *Local Minima*
Let $f : \mathbb{R}^d \to \mathbb{R}$.
We say the point $\boldsymbol{x}_0 \in \mathbb{R}^d$ is a local minimum if

$$\exists\, \delta > 0 \; st \; \forall\, \boldsymbol{x} \in \mathbb{R}^d, ||\boldsymbol{x} - \boldsymbol{x}_0|| < d \text{ then } f(\boldsymbol{x_0}) \leq f(\boldsymbol{x})$$

Definition 15.4 - *Local Maxima*
Let $f : \mathbb{R}^d \to \mathbb{R}$.
We say the point $\boldsymbol{x}_0 \in \mathbb{R}^d$ is a local maxima if

$$\exists\, \delta > 0 \; st \; \forall\, \boldsymbol{x} \in \mathbb{R}^d, ||\boldsymbol{x} - \boldsymbol{x}_0|| < d \text{ then } f(\boldsymbol{x_0}) \geq f(\boldsymbol{x})$$

Definition 15.5 - *Local Extremum*
A point is a *Local Extremum* if it is either a local minimum or maximum.

Definition 15.6 - *Critical point*
A point, $\boldsymbol{x}_0$, is a critical point of f if
$$f'(\boldsymbol{x}_0) = \boldsymbol{0}$$

Proposition 15.7 - *Extremum and Critical Point*
If $\boldsymbol{x}_0$ is a local extremum of f then it is a critical point of f.

Remark 15.8 - *Classifing Critical Points*
Let x_0 be a critical point of f.
Then Taylor's theorem says that

$$f(\boldsymbol{x}) = f(\boldsymbol{x}_0) + \frac{1}{2}(\boldsymbol{x} - \boldsymbol{x}_0)^t . f''(\boldsymbol{x}_0)(\boldsymbol{x} - \boldsymbol{x}_0) + ...$$

Definition 15.9 - *Define & Indefine Matrices*
Let $A \in M_d$, then A is

   i) Positive definite if $\forall\ \boldsymbol{x} \in \mathbb{R}^d, \boldsymbol{x} \neq \boldsymbol{0} => \boldsymbol{x}^t A \boldsymbol{x} > 0$;

   ii) Negative definite if $\forall\ \boldsymbol{x} \in \mathbb{R}^d, \boldsymbol{x} \neq \boldsymbol{0} => \boldsymbol{x}^t A \boldsymbol{x} < 0$;

   iii) Indefinite if $\exists\ \boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^d\ st\ \boldsymbol{x}^t A \boldsymbol{x} > 0$& $\boldsymbol{y}^t A \boldsymbol{y} < 0$.

Theorem 15.10 *Extrema and Definite Matrices*
The critical point $\boldsymbol{x}_0 \in \mathbb{R}^d$ is a

   i) Local minimum iff $f''(\boldsymbol{x}_0)$ is a positive-definite;

   ii) Local maximum iff $f''(\boldsymbol{x}_0)$ is a negative-definite;

If $f''(\boldsymbol{x}_0)$ is indefinite then we say that $\boldsymbol{x}_0$ is a *saddle*.

Theorem 15.11 - *Definite Matrices and Eigenvalues*
We say $f''(\boldsymbol{x}_0)$ is

   i) Positive definite iff all its eigenvalues are strictly positive;

   ii) Negative definite iff all its eigenvalues are strictly negative;

   iii) Indefinite if it has both positive and negative eigenvalues.