Linear Algebra & Geometry - Notes

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Contents

1 Euclidean Plane, Vectors, Cartesian Co-Ordinates & Complex Numbers

1.1 Vectors

Definition 1.01 - *Vectors*
Ordered sets of real numbers.
Denoted by $\boldsymbol{v} = (v_1, v_2, v_3, ...) = \begin{pmatrix} x \\ y \end{pmatrix}$

Definition 1.02 - *Euclidean Plane*
The set of two dimensional vectors, with real componenets, is called the Euclidean Plane.
Denoted by $\mathbb{R}^2$

Definition 1.03 - *Vector Addition*
Let $\boldsymbol{v}, \boldsymbol{w} \in \mathbb{R}^2$ such that $\boldsymbol{v} = (v_1, v_2)$ and $\boldsymbol{w} = (w_1, w_2)$.
Then $\boldsymbol{v} + \boldsymbol{w} = (v_1 + w_1, v_2 + w_2)$.

Definition 1.03 - *Scalar Multiplcation of Vectors*
Let $\boldsymbol{v} \in \mathbb{R}^2$ and $\lambda \in \mathbb{R}$ such that $\boldsymbol{v} = (v_1, v_2)$.
Then $\lambda\boldsymbol{v} = (\lambda v_1, \lambda v_2)$.

Definition 1.04 - *Norm of vectors*
The norm of a vector is its length from the origin.
Denoted by $||\boldsymbol{v}|| = \sqrt{v_1^2 + v_2^2}$ for $\boldsymbol{v} \in \mathbb{R}^2$.

Theorem 1.05
Let $\boldsymbol{v}, \boldsymbol{w} \in \mathbb{R}^2$ and $\lambda \in \mathbb{R}$ such that $\boldsymbol{v} = (v_1, v_2)$ and $\boldsymbol{w} = (w_1, w_2)$.
Then

$$\begin{aligned} ||\boldsymbol{v}|| &= 0 \text{ iff } \boldsymbol{v} = \boldsymbol{0} \\ ||\lambda\boldsymbol{v}|| &= \sqrt{\lambda^2 v_1^2 + \lambda^2 v_2^2} \\ &= |\lambda|.||\boldsymbol{v}|| \\ ||\boldsymbol{v} + \boldsymbol{w}|| &\leq ||\boldsymbol{v}|| + ||\boldsymbol{w}|| \end{aligned}$$

Definition 1.06 - *Unit Vector*
A vector can be described by its length & direction.
Let $\boldsymbol{v} \in \mathbb{R}^2 \backslash \{\boldsymbol{0}\}$.
Then $\boldsymbol{v} = ||\boldsymbol{v}||\boldsymbol{u}$ where $\boldsymbol{u}$ is the unit vector, $\boldsymbol{u} = \begin{pmatrix} cos\theta \\ sin\theta \end{pmatrix}$
Thus $\forall\ \boldsymbol{v} \in \mathbb{R}^2\ \boldsymbol{v} = \begin{pmatrix} \lambda cos\theta \\ \lambda sin\theta \end{pmatrix}$ for some $\lambda \in \mathbb{R}$.

Definition 1.07 - *Dot Product*
Let $\boldsymbol{v} \in \mathbb{R}^2$ and $\lambda \in \mathbb{R}$ such that $\boldsymbol{v} = (v_1, v_2)$.
Then $\boldsymbol{v} \cdot \boldsymbol{w} = v_1.w_1 + v_2.w_2$.

Remark 1.08 - *Positivity of Dot Product*
Let $\boldsymbol{v} \in \mathbb{R}^2$.
Then $\boldsymbol{v} \cdot \boldsymbol{v} = ||\boldsymbol{v}||^2 = v_1^2 + v_2^2 \geq 0$.

Remark 1.09 - *Angle between vectors in Euclidean Plane*
Let $\boldsymbol{v}, \boldsymbol{w} \in \mathbb{R}^2$.
Set θ to be the angle between $\boldsymbol{v}$ & $\boldsymbol{w}$.
Then

$$cos\theta = \frac{\boldsymbol{v} \cdot \boldsymbol{w}}{||\boldsymbol{v}|| \; ||\boldsymbol{w}||}$$

.
Theorem 1.10 - *Cauchy-Schwarz Inequality*
Let $\boldsymbol{v}, \boldsymbol{w} \in \mathbb{R}^2$.
Then

$$|\boldsymbol{v} \cdot \boldsymbol{w}| \leq ||\boldsymbol{v}|| \; ||\boldsymbol{w}||$$

Proof

$$\begin{aligned}
\frac{v_1 w_1}{||\boldsymbol{v}|| \; ||\boldsymbol{w}||} + \frac{v_2 w_2}{||\boldsymbol{v}|| \; ||\boldsymbol{w}||} &\leq \frac{1}{2}\left(\frac{v_1^2}{||\boldsymbol{v}||^2} + \frac{w_1^2}{||\boldsymbol{w}||^2}\right) + \frac{1}{2}\left(\frac{v_2^2}{||\boldsymbol{v}||^2} + \frac{w_2^2}{||\boldsymbol{w}||^2}\right) \\
&\leq \frac{1}{2}\left(\frac{v_1^2 + v_2^2}{||\boldsymbol{v}||^2} + \frac{w_1^2 + w_2^2}{||\boldsymbol{w}||^2}\right) \\
&\leq \frac{1}{2}(1+1) \\
&\leq 1 \\
=> |v_1 w_1 + v_2 w_2| &\leq ||\boldsymbol{v}|| \; ||\boldsymbol{w}|| \\
|\boldsymbol{v} \cdot \boldsymbol{w}| &\leq ||\boldsymbol{v}|| \; ||\boldsymbol{w}||
\end{aligned}$$

1.2 Complex Numbers

Definition 1.11 - i

$$\begin{aligned}
i^2 &= -1 \\
i &= \sqrt{-1}
\end{aligned}$$

Definition 1.12 - *Complex Number Set*
The set of complex numbers contains all numbers with an imaginary part.

$$\mathbb{C} := \{x + iy; x, y \in \mathbb{R}\}$$

Complex numbers are often denoted by

$$z = x + iy$$

and we say x is the real part of z and y the imaginary part.
Definition 1.13 - *Complex Conjugate*
Let $z \in \mathbb{C}$ st $z = x + iy$.
Then

$$\bar{z} := x - iy$$

Theorem 1.14 - *Operations on Complex Numbers*
Let $z_1, z_2 \in \mathbb{C}$ st $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$.
Then

$$\begin{aligned} z_1 + z_2 &:= (x_1 + x_2) + i(y_1 + y_2) \\ z_1.z_2 &:= (x_1 + iy_1)(x_2 + iy_2) \\ &:= x_1.x_2 - y_1.y_2 + i(x_1.y_2 + x_2.y_1) \end{aligned}$$

N.B. When dividing by a complex number, multiply top and bottom by the complex conjugate.

Definition 1.15 - *Modulus of Complex Numbers*
The modulus of a complex number is the distance of the number, from the origin, on an Argand diagram. Let $z \in \mathbb{C}$ st $z = x + iy$.
Then

$$\begin{aligned} |z| &:= \sqrt{x^2 + y^2} \\ &:= \sqrt{\bar{z}z} \end{aligned}$$

N.B. Amplitude is an alternative name for the modulus

Definition 1.16 - *Phase of Complex Numbers*
The phase of a complex number is the angle between the positive real axis and the line subtended from the origin and the number, on an Argand digram.

$$z = |z|.(cos\theta + i.sin\theta), \quad \theta = \text{Phase}$$

N.B. Phase of $\bar{z}$ = - Phase of z

Theorem 1.17 - *de Moivre's Formula*

$$z^n = (cos\theta + i.sin\theta)^n = cos(n\theta) + i.sin(n\theta)$$

Theorem 1.18 - *Euler's Formula*

$$e^{i\theta} = cos\theta + i.sin\theta$$

Remark 1.19
Using Euler's formula we can express all complex numbers in terms of e. Thus many properties of the exponential remain true:

$$\begin{aligned} z &= \lambda e^{i\theta}, \qquad\qquad \lambda \in \mathbb{R}, \theta \in [0, 2\pi) \\ => z_1 + z_2 &= \lambda_1.\lambda_2.e^{i(\theta_1+\theta_2)} \\ \&, \frac{z_1}{z_2} &= \frac{\lambda_1}{\lambda_2}.e^{i(\theta_1=\theta_2)} \end{aligned}$$

2 Euclidean Space, $\mathbb{R}^n$

Definition 2.01 - *Euclidean Space*
Let $n \in \mathbb{N}$ then $\forall$ $\boldsymbol{x} = (x_1, x_2, ..., x_n)$ with $x_1, x_2, ..., x_n \in \mathbb{R}$ we have that $\boldsymbol{x} \in \mathbb{R}^n$.

Theorem 2.02 - *Operations in Euclidean Space*
Let $(x), (y) \in \mathbb{R}^n$ and $\lambda \in \mathbb{R}$. Then

$$(x) + (y) = (x_1 + y_1, ..., x_n + y_n)$$

And

$$(x) + \lambda.(y) = (x_1 + \lambda.y_1, ..., x_n + \lambda.y_n)$$

Definition 2.03 - *Cartesian Product*
Let $A, B \in \mathbb{R}^n$ be non-empty sets.
Then

$$A \times B := \{(a, b); a \in A, b \in B\}$$

2.1 Dot Product

Definition 2.04 - *Dot Product*
Let $\boldsymbol{v}, \boldsymbol{w} \in \mathbb{R}^n$. Then

$$\begin{aligned} \boldsymbol{v} \cdot \boldsymbol{w} &:= v_1.w_1 + ... + v_n.w_n \\ &:= \sum_{j=1}^{n} v_j.w_j \end{aligned}$$

Theorem 2.05 - *Properties of the Dot Product*
Let $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w} \in \mathbb{R}^n$. Linearity:

$$(\boldsymbol{u} + \lambda \boldsymbol{v}) \cdot \boldsymbol{w} = \boldsymbol{u} \cdot \boldsymbol{w} + \lambda(\boldsymbol{v} \cdot \boldsymbol{w})$$

Symmetry:

$$\boldsymbol{v} \cdot \boldsymbol{w} = \boldsymbol{w} \cdot \boldsymbol{v}$$

Positivity:

$$\boldsymbol{v} \cdot \boldsymbol{v} = v_1^2 + v_2^2 + ... + v_n^2 \geq 0$$

Definition 2.06 - *Orthogonality*
Let $\boldsymbol{v}, \boldsymbol{w} \in \mathbb{R}^n$.
It is said that $(v), (w)$ are orthogonal to each other if $\boldsymbol{v} \cdot \boldsymbol{w} = 0$
N.B. Orthogonal vectors are perpendicular to each other.

Definition 2.07 - *The Norm*
Let $\boldsymbol{x} \in \mathbb{R}^n$.
Then

$$||\boldsymbol{x}|| = \sqrt{\boldsymbol{x} \cdot \boldsymbol{x}} = \sqrt{\sum_{i=1}^{n} x_i^2}$$

Theorem 2.08 - *Properties of the Norm*
Let $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$ and $\lambda \in \mathbb{R}$. Then

$$\begin{aligned} ||\boldsymbol{x}|| &\geq 0 \\ ||\boldsymbol{x}|| &= 0 \text{ iff } \boldsymbol{x} = 0 \\ ||\lambda\boldsymbol{x}|| &= |\lambda|||\boldsymbol{x}|| \\ ||\boldsymbol{x}+\boldsymbol{y}|| &\leq ||\boldsymbol{x}|| + ||\boldsymbol{y}|| \end{aligned}$$

Theorem 2.09 - *Dot Product and Norm*
Let $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$.

$$|\boldsymbol{x} \cdot \boldsymbol{y}| \leq ||\boldsymbol{x}||||\boldsymbol{y}||$$

N.B. $|\boldsymbol{x} \cdot \boldsymbol{y}| = ||\boldsymbol{x}||||\boldsymbol{y}||$ iff $\boldsymbol{x}$ & $\boldsymbol{x}$ are orthogonal.

Theorem 2.10 - *Angle between Vectors*
Let $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$. Then

$$cos\theta = \frac{\boldsymbol{x} \cdot \boldsymbol{y}}{||\boldsymbol{x}||||\boldsymbol{y}||}$$

2.2 Linear Subspaces

Definition 2.11 - *Linear Subspace*
Let $V \subset \mathbb{R}^n$. V is a *Linear Subspace* if:
i) $V \neq \emptyset$;
ii) $\forall\ \boldsymbol{v}, \boldsymbol{w} \in V$ then $\boldsymbol{v} + \boldsymbol{w} \in V$;
iii) $\forall\ \lambda \in \mathbb{R}, \boldsymbol{v} \in V$ then $\lambda\boldsymbol{v} \in V$.

Definition 2.12 - *Span*
Let $\boldsymbol{x_1}, ..., \boldsymbol{x_k} \in \mathbb{R}^n; k \in \mathbb{N}$. Then

$$\text{span}\{\boldsymbol{x_1}, ..., \boldsymbol{x_k}\} := \{\lambda_1\boldsymbol{x_1} + ... + \lambda_k\boldsymbol{x_k}; \lambda_i \in \mathbb{R}, 0 \leq i \geq k\}$$

Definition 2.13 - *Spans are Subspaces*
Let $\boldsymbol{x_1}, ..., \boldsymbol{x_k} \in \mathbb{R}^n;\ k \in \mathbb{N}$. Then $\text{span}\{\boldsymbol{x_1}, ..., \boldsymbol{x_k}\}$ is a linear subspace of $\mathbb{R}^n$.

Theorem 2.14

$$W_{\boldsymbol{a}} := \{\boldsymbol{x} \in \mathbb{R}^n; \boldsymbol{x} \cdot \boldsymbol{a} = 0\} \text{ is a subspace.}$$

Definition 2.15 - *Orthogonal Complement*
Let $V \subset \mathbb{R}^n$. Then,

$$V^\perp := \{\boldsymbol{x} \in \mathbb{R}^n; \boldsymbol{x} \cdot \boldsymbol{y}\ \forall\ \boldsymbol{y} \in V\}$$

N.B. $V^\perp \subset \mathbb{R}^n$
Theorem 2.16 - *Relationship of Subspaces*
Let V, W be subspaces of $\mathbb{R}$. Then

$$V \cap W \text{ is a subspace.}$$

$$V + W := \{\boldsymbol{v} + \boldsymbol{w}; \boldsymbol{v} \in V, \boldsymbol{w} \in W\} \text{ is a subspace.}$$

Definition 2.17 - *Direct Sum*
Let V_1, V_2, W be subspaces of $\mathbb{R}$. Then W is said to be a *direct sum* if
i) $W = V_1 + V_2$;
ii) $V_1 \cap V_2 = \emptyset$.

3 Linear Equations & Matrices

3.1 Linear Equations

Definition 3.01 - *Multi-Variable Linear Equations*
Linear equations produce a straight line and can have multiple variables.
Examples - $x = 3, y = x + 3, z + 5x - 2y$

Defintion 3.02 - *Systems of Linear Equations*
Let $\boldsymbol{a}, \boldsymbol{x} \in \mathbb{R}^n$ & $b \in \mathbb{R}$ such that $\boldsymbol{a} \cdot \boldsymbol{x} = b$.
$\boldsymbol{a} \cdot \boldsymbol{x} = b$ is a linear equation in x with $S = \{\boldsymbol{x}; \boldsymbol{a} \cdot \boldsymbol{x} = b\}$ as the set of solutions.
N.B. If $b = 0$ then $S(\boldsymbol{a}, 0)$ is a subspace.

3.2 Matrices

Definition 3.03 - *Matrix*
Let $m, n \in \mathbb{N}$, then a $m \times n$ grid of numbers form an "m" by "n" matrix. Each element of the matrix can be reference by a_{ij} with $i = 1, ..., m$ and $j = 1, ..., n$.

$$A = \begin{pmatrix} a_{11} & a_{12} & ... & a_{1n} \\ a_{21} & a_{22} & ... & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & ... & a_{mn} \end{pmatrix}$$

N.B. m,i = rows, n,j = columns

Definition 3.04 - *Sets of Matrices*
$M_{m,n}(\mathbb{R})$ is the set of m x n matrices containing only real numbers.
$M_{m,n}(\mathbb{Z})$ is the set of m x n matrices containing only integers.
$M_n(\mathbb{R})$ is the set square matrices, size n, containing only real numbers.

Definition 3.05 - *Transpose Vectors*
Let $\boldsymbol{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$ then $\boldsymbol{x}^t = \begin{pmatrix} x_1 & x_2 & ... & x_n \end{pmatrix}$

Definition 3.06 - *Vector-Matrix Multiplication*
Let $A \in \mathbb{R}_{m,n}$ and $\boldsymbol{x} \in \mathbb{R}^n$ then

$$A\boldsymbol{x} := \begin{pmatrix} a_{11} & a_{12} & ... & a_{1n} \\ a_{21} & a_{22} & ... & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & ... & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} a_{11}x_1 + a_{12}x_2 + ... + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + ... + a_{2n}x_n \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + ... + a_{mn}x_n \end{pmatrix} = \begin{pmatrix} \boldsymbol{a}_1^t \cdot \boldsymbol{x} \\ \boldsymbol{a}_2^t \cdot \boldsymbol{x} \\ \vdots \\ \boldsymbol{a}_m^t \cdot \boldsymbol{x} \end{pmatrix} \in \mathbb{R}^m$$

This can be simplified to

$$\boldsymbol{y} = A\boldsymbol{x} \text{ with } y_i = \sum_{j=1}^{n} a_{ij}x_j$$

Theorem 3.07 - *Operations on Matrices with Vectors*

i) $A(\boldsymbol{x} + \boldsymbol{y}) = A\boldsymbol{x} + A\boldsymbol{y}, \quad \forall\ \boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$.

ii) $A(\lambda\boldsymbol{x}) = \lambda(A\boldsymbol{x}), \quad \forall \boldsymbol{x} \in \mathbb{R}^n, \lambda \in \mathbb{R}.$

Theorem 3.08
Let $A = (a_{ij}) \in M_{m,n}(\mathbb{R})$ and $B = (b_{ij}) \in M_{l,m}(\mathbb{R})$. Then there exists a $C = (c_{ij}) \in M_{l,n}(\mathbb{R})$ such that

$$C\boldsymbol{x} = B(A\boldsymbol{x}), \quad \forall\ \boldsymbol{x} \in \mathbb{R}^n$$

N.B. $c_{ij} = \sum_{k=1}^{m} b_{ik}a_{kj}$

Theorem 3.09 - *Operation between Matrices*
Let $A, B \in M_{m,n}$ and $C \in M_{l,m}$

i) $C(A+B) = CA + CB$.

ii) $(A+B)C = AC + BC$.

iii) Let $D \in M_{m,n}, E \in M_{n,l}$ & $F \in M_{l,k}$ then

$$E(FG) = (EF)G$$

N.B. $AB \neq BA$

Definition 3.10 - *Types of Matrix*

Upper Triangle $\begin{pmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{pmatrix}, \quad a_{ij} = 0$ if $i > j$.

Lower Triangle $\begin{pmatrix} 1 & 0 & 0 \\ 2 & 3 & 0 \\ 4 & 5 & 6 \end{pmatrix}, \quad a_{ij} = 0$ if $i < j$.

Symmetric Matrix $\begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 0 \\ 3 & 0 & 1 \end{pmatrix}, \quad a_{ij} = a_{ji}$.

Anti-Symmetric $\begin{pmatrix} 1 & -2 & -3 \\ 2 & 0 & -4 \\ 3 & 4 & -1 \end{pmatrix}, \quad a_{ij} = -a_{ji}$.

Definition 3.11 - *Transposed Matrices*
Let $A = (a_{ij}) \in M_{m,n}(\mathbb{R})$ then the transponse of A, A^t, is an element of $M_{n,m}(\mathbb{R})$.

$$A^t := (aji)$$

Theorem 3.12 - *Transpose Matrix Multiplication*
Let $A \in M_{m,n}(\mathbb{R}), \boldsymbol{x} \in \mathbb{R}^n, \boldsymbol{y} \in \mathbb{R}^m$. Then

$$\boldsymbol{y} \cdot A\boldsymbol{x} = (A_t\boldsymbol{y}) \cdot \boldsymbol{x}$$

Theorem 3.10 - *Transposing Multiplied Matrices*

$$(AB)^t = B^t A^t$$

3.3 Structure of Set of Solutions

Definition 3.13 - *Set of Solutions*
Let $A \in M_{m,n}(\mathbb{R})$ and $\boldsymbol{b} \in \mathbb{R}^m$. Then

$$S(A, \boldsymbol{b}) := \boldsymbol{x} \in \mathbb{R}^n; A\boldsymbol{x} = b$$

Definition 3.14 - *Homogenous Solutions*
The system of $S(A, \mathbf{0})$ is called said to be *homogenous*. All other systems are *inhomogenous*.
N.B. - $S(A, \mathbf{0})$ is a linear subspace.

Theorem 3.15 - *Using Homogenous Solutions*
Let $A \in M_{m,n}(\mathbb{R})$ and $\boldsymbol{b} \in \mathbb{R}^n$. Let $\boldsymbol{x}_0 \in \mathbb{R}^n$ such that $A\boldsymbol{x}_0 = \boldsymbol{b}$, then

$$S(A, \boldsymbol{b}) = \boldsymbol{x}_0 + S(A, \mathbf{0})$$

Remark 3.16 - *Systems of Linear Equations as Matrices*
The system of linear equations $3x + z = 0, y - z = 1, 3x + y = 1$ can be represented by a matrix and a vector.

$$A = \begin{pmatrix} 3 & 0 & 1 \\ 0 & 1 & -1 \\ 3 & 1 & 0 \end{pmatrix}, \ \boldsymbol{b} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

3.4 Solving Systems of Linear Equations

Systems of linear equations can be displayed as matrices which can be reduced and solved by a technique called *Gaussian Elimination.*

Theorem 3.17
There are certain operations that can be performed on a system of linear equations without changing the result:

i) Multiply an equaion by a non-zero constant;

ii) Add a multiple of any equation to another equation;

iii) Swap any two equations.

Definition 3.18 - *Augmented Matrices*
Let $A\boldsymbol{x} = \boldsymbol{b}$ be a system of linear equations. The associated *Augmented Matrix* is

$$(A \ \boldsymbol{b}) \in M_{m,n+1}(\mathbb{R})$$

Theorem 3.19 - *Elementary Row Operations*
From *Theorem 3.17* we can deduce ceratin operations that can be performed on an *Augmented Matrix* which do not alter the solutions:

i) Multiply a row by a non-zero constant, $row\ i \to \lambda(row\ i)$;

ii) Add a multiple of any row to another row, $row\ i \to row\ i + \lambda(row\ j)$;

iii) Swap two rows, $row\ i \leftrightarrow row\ j$.

Definition 3.20 - *Row Echelon Form*
A matrix is in *Row Echelon Form* if:

i) The left-most non-zero value in each row is 1; And,

ii) The leading 1 in each row is one place to the right of the leading 1 in the row below.

Example

$$\begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix}$$

Definition 3.20 - *Reduced Row Echelon Form*
A matrix is in *Reduced Row Echelon Form* if:

i) The matrix is in *row echelon form*; And,

ii) All values in a row, except the leading 1, are 0.

Example

$$\begin{pmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{pmatrix}$$

Theorem 3.21 - *Gaussian Elimination*
Gaussian Elimination is a technique used to solve systems of linear equations. *Example*
Solve $x + y + 2z = 9, 2x + 4y - 3z = 1, 3x + 6y - 5z = 0$.

$$\text{Augmented Matrix} - \begin{pmatrix}1 & 1 & 2 & 9\\2 & 4 & -3 & 1\\3 & 6 & -5 & 0\end{pmatrix}$$

$$\begin{aligned}\text{By EROS} - \begin{pmatrix}1 & 1 & 2 & 9\\2 & 4 & -3 & 1\\3 & 6 & -5 & 0\end{pmatrix} &= \begin{pmatrix}1 & 1 & 2 & 9\\0 & 2 & -7 & -17\\0 & 3 & -11 & 27\end{pmatrix}\\
&= \begin{pmatrix}1 & 1 & 2 & 9\\0 & 2 & -7 & -17\\0 & 1 & -4 & -10\end{pmatrix}\\
&= \begin{pmatrix}1 & 1 & 2 & 9\\0 & 1 & -4 & -10\\0 & 2 & -7 & -17\end{pmatrix}\\
&= \begin{pmatrix}1 & 1 & 2 & 9\\0 & 1 & -4 & -10\\0 & 0 & 1 & 3\end{pmatrix}\\
&= \begin{pmatrix}1 & 0 & 6 & 19\\0 & 1 & -4 & -10\\0 & 0 & 1 & 3\end{pmatrix}\\
&= \begin{pmatrix}1 & 0 & 0 & 1\\0 & 1 & 0 & 2\\0 & 0 & 1 & 3\end{pmatrix}\end{aligned}$$

$$=> \underline{x = 1, y = 2, z = 3}$$

3.5 Elementary Matrices & Inverting Matrices

Definition 3.22 - *Invertible Matrices*
A matrix, $A \in M_{m,n}(\mathbb{R})$, is said to be *Invertible* if there exists $A^{-1} \in M_{n,m}(\mathbb{R})$ such that

$$AA^{-1} = I$$

N.B. - If a matrix is not invertible then it is *Singular*.

Definition 3.23 - *Elementary Matrices*
A matrix, $E \in M_{m,n}(\mathbb{R})$, is said to be an *Elementary Matrix* if it can be obtained by performing Elementary Row Operations on a square identity matrix.
Examples $\begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}, \begin{pmatrix}0 & \lambda\\\mu & 0\end{pmatrix}$

Remark 3.24
All elementary matrices are invertible.

Remark 3.25
Let A be a matrix, and B be a matrix which can be obtained from A by elementary row operations. Then there exists an elementary matrix E such that

$$B = EA$$

.

Theorem 3.26 - *Finding A^{-1}*
Let $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$, $B = \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix}$, $I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.
Then by using EROS to change $(A\ I) \to (I\ B)$, B is the inverse of A.

Theorem 3.27 - *Inverse of a 2x2 Matrix*
Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ then

$$A^{-1} = \frac{1}{ad - bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

4 Linear Independence, Bases & Dimensions

4.1 Linear Independence & Dependence

Definition 4.01 - *Linear Independence & Dependence*
Vectors, $\boldsymbol{x}_1, ..., \boldsymbol{x}_n \in \mathbb{R}^k$, are said to be *linearly dependent* if there exists non-zero real numbers, $\lambda_1, ..., \lambda_n$ such that

$$\lambda_1.\boldsymbol{x}_1 + ... + \lambda_n.\boldsymbol{x}_n = \boldsymbol{0}$$

N.B. - If this is only true if $\lambda_1 = ... = \lambda_n = 0$ then the vectors are said to be *linearly independent.*

Remark 4.02
Vectors are only *linearly dependent* if one of them lies in the span of the rest.

4.2 Bases & Dimensions

Definition 4.03 - *Basis*
A *basis* is a set of vectors, $\boldsymbol{v}_1, ..., \boldsymbol{v}_n \in V$ such that

i) $V = \text{span}\{\boldsymbol{v}_1, ..., \boldsymbol{v}_n\}$;

ii) $\boldsymbol{v}_1, ..., \boldsymbol{v}_n$ are linearly independent.

Definition 4.04 - *Standard Basis*
The *standard basis* for a vector space is the set fewest unit vectors which span it.
Example - $\{\boldsymbol{v}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\}$ are the standard basis for $\mathbb{R}^3$.

Theorem 4.05 - *Basis of a Linear Subspace*
For all elements, $\boldsymbol{v}$, of a linear subspace, $V \subset \mathbb{R}^n$, there exists a unique set of numbers, $\lambda_1, ..., \lambda_n$, such that

$$\boldsymbol{v} = \lambda_1.\boldsymbol{v}_1 + ... + \lambda_n.\boldsymbol{v}_n$$

Theorem 4.06 - *Linear Independence and Bases*
Let $V \subset \mathbb{R}^n$ be a linear subspace with basis $\boldsymbol{v}_1, ..., \boldsymbol{v}_n$. Suppose $\boldsymbol{w}_1, ..., \boldsymbol{w}_k \in V$ are linearly independent, then $k \leq n$.

Definition 4.07 - *Dimension*
Let $V \subset \mathbb{R}^n$ be a linear subspace then the *dimension* of V, $dim(V)$, is the fewest number vectors required to form a basis for V.

4.3 Orthogonal Bases

Definition 4.08 - *Orthogonal*
Let $V \subset \mathbb{R}^n$ be a linear subspace with $\{\boldsymbol{v}_1, ..., \boldsymbol{v}_k\}$ as its basis. This basis is an *orthogonal basis* if it statisfies:

i) $\boldsymbol{v}_i \cdot \boldsymbol{v}_j = 0$ if $i \neq j$;

ii) $\boldsymbol{v}_i \cdot \boldsymbol{v}_i = 1,\ i = 1, ..., k$.

N.B. - This can be generalised to $\boldsymbol{v}_i \cdot \boldsymbol{v}_k = \delta_{ij}$ with $\delta_{ij} := \begin{cases} 1, & i = j \\ 0, & \text{otherwise} \end{cases}$

Theorem 4.09
Let $V \subset \mathbb{R}^n$ be a linear subspace with an orthogonal basis $\{\boldsymbol{v}_1, ..., \boldsymbol{v}_k\}$.
Then for all $\boldsymbol{u} \in V$

$$\boldsymbol{u} = (\boldsymbol{v}_1 \cdot \boldsymbol{u})\boldsymbol{v}_1, ..., (\boldsymbol{v}_k \cdot \boldsymbol{u})\boldsymbol{v}_k$$

5 Linear Maps

Definition 5.01 - *Linear Map*
A map, $T : \mathbb{R}^n \to \mathbb{R}^m$ is a *linear map* if

i) $T(\boldsymbol{x} + \boldsymbol{y}) = T(\boldsymbol{x}) + T(\boldsymbol{y}), \quad \forall\ \boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$;

ii) $T(\lambda \boldsymbol{x}) = \lambda T(\boldsymbol{x}), \quad \forall\ \boldsymbol{x} \in \mathbb{R}^n, \lambda \in \mathbb{R}$.

N.B. - If $m = n$ then T is referred to as a *linear operator*.

Theorem 5.02 - *Properties of Linear Maps*
Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear map. Then $T(\boldsymbol{0}) = \boldsymbol{0}$.

Definiton 5.03 - *Linear Maps as Matrices*
Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear map. Then the associated Matrix is defined as

$$M_T = (t_{ij}) \in M_{m,n}(\mathbb{R})$$

with the elements of M_T defined by

$$t_{ij} = \boldsymbol{e}_i \cdot T(\boldsymbol{e}_j)$$

Theorem 5.04 - *Solutions to Linear Maps from Matrices*
Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear map and M_T be the associated matrix. Then

$$T(\boldsymbol{x}) = M_T \boldsymbol{x}, \quad \forall\ \boldsymbol{x} \in \mathbb{R}^n$$

5.1 Abstract Properties of Linear Maps

Theorem 5.05 - *Relationship between Linear Maps*
Let $S : \mathbb{R}^n \to \mathbb{R}^m$, $T : \mathbb{R}^n \to \mathbb{R}^m$ & $U : \mathbb{R}^m \to \mathbb{R}^k$ be a linear maps and $\lambda \in \mathbb{R}$. Then

i) $(\lambda T)(\boldsymbol{x}) := \lambda T(\boldsymbol{x})$;

ii) $(S+T)(\boldsymbol{x}) = S(\boldsymbol{x}) + T(\boldsymbol{x})$;

iii) $(U \circ S)(\boldsymbol{x}) = U(S(\boldsymbol{x}))$.

Definition 5.06 - *Image & Kernel*
Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear map. Then

i) The *image* of T is defined to be

$$Im(T) := \{\boldsymbol{y} \in \mathbb{R}^m : \exists\ \boldsymbol{x} \in \mathbb{R}^n st\ T(\boldsymbol{x}) = \boldsymbol{y}\}$$

ii) The *kernel* of T is defined to be

$$Ket(T) := \{\boldsymbol{x} \in \mathbb{R}^n : T(\boldsymbol{x}) = \boldsymbol{0}\}$$

Theorem 5.07
Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear map then $Im(T)$ is a linear subsapce of $\mathbb{R}^m$ and $Ker(T)$ is a linear subspaces of $\mathbb{R}^n$

Remark 5.08
Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear map. Then

i) T is surjective if $Im(T) = \mathbb{R}^m$;

ii) T is injective if $Ker(T) = \{0\}$.

5.2 Matrices

Definition 5.09 - *Linear Maps as Matrices*
Let $S : \mathbb{R}^n \to \mathbb{R}^m$, $T : \mathbb{R}^n \to \mathbb{R}^m$ & $U : \mathbb{R}^m \to \mathbb{R}^k$ be a linear maps and $\lambda \in \mathbb{R}$ with M_S, M_T & M_U as the corresponding matrices. Then

i) $M_{\lambda T} = \lambda M_T = (\lambda t_{ij})$;

ii) $M_{S+T} = (s_{ij} + t_{ij}) = M_S + M_T$;

iii) $M_{U \circ S} = (r_{ij})$ where $r_{ik} = \sum_{k=1}^{m} s_{ik} t_{jk}$.

5.3 Rank & Nullity

Defintion 5.10 - *Rank & Nullity*
Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear map. Then we define *Rank* of T by

$$rank(T) := dim(Im(T))$$

and we define *Nullity* of T by

$$nullity(T) := dim(Im(T))$$

N.B. - For all linear maps, $T : \mathbb{R}^n \to \mathbb{R}^m$,

$$nullity(T) + rank(T) = n$$

Remark 5.11
Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear map. Then T is invertible if

i) $rank(T) = n$, or

ii) $nullity(T) = 0$.

Theorem 5.12 - *Relationship of Rank & Nullity between Linear Maps*
Let $S : \mathbb{R}^n \to \mathbb{R}^m$ & $T : \mathbb{R}^k \to \mathbb{R}^n$ be linear maps. Then

i) $S \circ T = 0$ iff $Im(T) \subset Ker(S)$;

ii) $rank(S \circ T) \leq rank(T)$ and $rank(S \circ T) \leq rank(S)$;

iii) $nullity(S \circ T) \geq nullity(T)$ and $nullity(S \circ T) \geq nullity(S) + k - n$;

iv) S is invertible then $rank(S \circ T) = rank(T)$ and $nullity(S \circ T) = nullity(T)$.

6 Determinants

6.1 Definition & Basic Properties

Definition 6.01 - *Determinant Function*
A *determinant function* $d_n : \mathbb{R}^n \times ... \times \mathbb{R}^n \to \mathbb{R}$ is a function which statisfies three conditions:

i) *Multilinear* - $d_2(\lambda \boldsymbol{a}_1 + \mu \boldsymbol{b}, \boldsymbol{a}_2) = \lambda d_2(\boldsymbol{a}_1, \boldsymbol{a}_2) + \mu(\boldsymbol{b}, \boldsymbol{a}_2)$;

ii) *Antisymmetric* - $d_2(\boldsymbol{a}_1, \boldsymbol{a}_2) = -d_2(\boldsymbol{a}_2, \boldsymbol{a}_1)$;

iii) *Normalisation* - $d_2(\boldsymbol{e}_1, \boldsymbol{e}_2) = 1$.

N.B. - Determinant functions only exists for *square matrices.*

Theorem 6.02 - *Properties of Determinant*

i) $det[..., \boldsymbol{a}_i + \lambda \boldsymbol{a}_i, ...] = det[..., \boldsymbol{a}_i, ...] + \lambda det[..., \boldsymbol{a}_i, ...]$;

ii) If A has two identical columns then $det(A) = 0$;

iii) If A has an all zero column then $det(A) = 0$;

iv) $det[...\boldsymbol{a}_i...\boldsymbol{a}_j...] = det[...(\boldsymbol{a}_i + \lambda \boldsymbol{a}_j)...\boldsymbol{a}_j...]$

Theorem 6.03
Let $f_n : \mathbb{R}^n \times ... \times \mathbb{R}^n \to \mathbb{R}$ be a function which is *multilinear* & *Antisymmetric* then

$$f_n(A) = C.det(A)$$

where C is a constant such that $C = f_n(\boldsymbol{e}_1, ..., \boldsymbol{e}_n)$.

Theorem 6.04 - *Determinant of a Triangle Matrix*
Let $A = (a_{ij}) \in M_n(\mathbb{R})$ be a upper triangle matrix, so $a_{ij} = 0$ if $i > j$. Then

$$det(A) = a_{11}.a_{22}. \; ... \; .a_{nn}$$

N.B. - The same is true for lower triangle matrices.

Theorem 6.05 - *Relationship between Determinants*
Let $A, B \in M_n(\mathbb{R})$ then

$$det(AB) = det(A).det(B)$$

but usually

$$det(A + B) \neq det(A) + det(B)$$

Theorem 6.06 - *Determinant & Inverses*
If $det(A) = 0$ then A^{-1} does not exist.

Theorem 6.07 - *Leibniz Formula*
Let $A = (a_{ij}) \in M_n(\mathbb{R})$ then the *Leibniz Formula* states that

$$det(A) := \sum_{\sigma \in S_n} \text{sign}(\sigma) \prod_{j=1}^{n} a_{\sigma(j),j}$$

Where:

- S_n is the group of symmetries for a regular n-sided polygons;
- sign(σ) is the sign function which returns $+1$ for even permuations and -1 for odd permutations.
  A permutation is even if a even number of permutations (swaps) are required to change the identity permutation to the given permutation, σ.

Remark 6.08 - *Determinant of Transpose*
Let A be a square matrix, then

$$det(A) = det(A^t)$$

6.2 Computing Determinant

Theorem 6.09 - *Laplace's Rule*
Let $A \in M_n$ then

$$det(A) = \sum_{j=1}^{n} a_{ij}.(-1)^{i+j}.det(A_{ij})$$

where A_{ij} is the $(n-1) \times (n-1)$ matrix formed when row i and column j are removed from A.
Example Let $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ then $A_{11} = \begin{pmatrix} 4 \end{pmatrix}$ and $A_{12} = \begin{pmatrix} 2 \end{pmatrix}$

Definition 6.10 - *Adjunct Matrices*
Let $A, B \in M_n$ be defined such that $b_{ij} = (-1)^{i+j}.det(A_i j)$ then B is said to be *adjunt* to A.
This means

$$AB = \begin{pmatrix} det(A) & 0 & \dots & 0 \\ 0 & det(A) & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & det(A) \end{pmatrix} = det(A)I$$

Remark 6.11 - *Determinant of Triangle Matrices*
If A is an upper triangle matrix ($a_{ij} = 0$ if $i > j$) then

$$det(A) = a_{11} \times \dots \times a_{nn}$$

If A is a lower triangle matrix ($a_{ij} = 0$ if $i < j$) then

$$det(A) = a_{11} \times \dots \times a_{nn}$$

6.3 Applications of Determinant

Theorem 6.12 - *Linear Equations as Matrices*
A system of m linear equations, each with n variables, can be written as

$$A\boldsymbol{x} = \boldsymbol{b}, \quad A \in M_{mn}(\mathbb{R}), \boldsymbol{x} \in \mathbb{R}^n, \boldsymbol{b} \in \mathbb{R}^m$$

If $det(A) \neq 0$ then we can find an $A^{-1} \in M_{n,m}$ such that

$$\boldsymbol{x} = A^{-1}\boldsymbol{b}$$

Theorem 6.13
Let $A \in M_n(\mathbb{R})$ where $det(A) \neq 0$ then

$$A^{-1} = \frac{1}{det(A)} adj\ A$$

Theorem 6.14 - *Cramer's Rule*
Consider $A\boldsymbol{x} = \boldsymbol{b}$ then

$$x_j = \frac{det(A_j)}{det(A)}$$

where A_j is the matrix A, but the j^{th} column has been replaced by $\boldsymbol{b}$.

Definition 6.15 - *Cross Product*
Let $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^3$ be in the <u>same plane</u> then we define the cross product by

$$\boldsymbol{x} \times \boldsymbol{y} := \begin{vmatrix} \boldsymbol{e}_1 & \boldsymbol{e}_2 & \boldsymbol{e}_3 \\ \boldsymbol{x}_1 & \boldsymbol{x}_2 & \boldsymbol{x}_3 \\ \boldsymbol{y}_1 & \boldsymbol{y}_2 & \boldsymbol{y}_3 \end{vmatrix} = \begin{pmatrix} \boldsymbol{x}_2\boldsymbol{y}_3 - \boldsymbol{x}_3\boldsymbol{y}_2 \\ \boldsymbol{x}_3\boldsymbol{y}_1 - \boldsymbol{x}_1\boldsymbol{y}_3 \\ \boldsymbol{x}_1\boldsymbol{y}_2 - \boldsymbol{x}_2\boldsymbol{y}_1 \end{pmatrix}$$

Theorem 6.16 - *Properties of Cross Product*

i) $\boldsymbol{x} \cdot (\boldsymbol{y} \times \boldsymbol{z}) = \boldsymbol{z} \cdot (\boldsymbol{x} \times \boldsymbol{y}) = \boldsymbol{y} \cdot (\boldsymbol{z} \times \boldsymbol{x})$

ii) $\boldsymbol{x} \times \boldsymbol{y} = -\boldsymbol{y} \times \boldsymbol{x}$

iii) $\boldsymbol{x} \times \boldsymbol{x} = 0$

iv) $(\boldsymbol{x} + \lambda\boldsymbol{y}) \times \boldsymbol{z} = (\boldsymbol{x} \times \boldsymbol{z}) + (\lambda\boldsymbol{y} \times \boldsymbol{z})$

v) $||\boldsymbol{x} \times \boldsymbol{y}||^2 = ||\boldsymbol{x}||^2||\boldsymbol{y}||^2 - (\boldsymbol{x} \cdot \boldsymbol{y})^2$

Theorem 6.17 - *Cross Product and Angle between vectors*
Let θ be the angle between two vectors then

$$||\boldsymbol{x} \times \boldsymbol{y}||^2 = ||\boldsymbol{x}||^2||\boldsymbol{y}||^2 sin^2(\theta)$$

Theorem 6.18 - *Cross Product with Matrices*
Let $A \in M_n(\mathbb{R})$ where $det(A) \neq 0$ then

$$(A\boldsymbol{x}) \times (A\boldsymbol{y}) = [det(A)](A^t)^{-1}(\boldsymbol{x} \cdot \boldsymbol{y})$$

7 Vector Spaces

7.1 Groups & Fields

Definition 7.1 - *Group*
A group, G, is a combination of a set and a map from $G \times G \to G$. The map must obey the following rules:

i) *Associativity* - $f * (g * h) = (f * g) * h$

ii) *Identity Element* - $\exists\ e \in G\ st\ \forall\ g \in G, eg = ge = g$

iii) *Inverse* - $\forall\ g \in G\ \exists\ g^{-1} \in G\ st\ gg^{-1} = e = g^{-1}g$

Definition 7.2 - *Matrix Groups*
The *General Linear Group*, $GL(n, \mathbb{R})$, is a group defined by

$$GL(n, \mathbb{R}) = \{A \in M_n(\mathbb{R}) : det(A) \neq 0\}$$

The identity element is $I \in M_n$ and inverse is A^{-1}.

The *Special Linear Group*, $SL(n, \mathbb{R})$, is a group defined by

$$SL(n, \mathbb{R}) = \{A \in M_n(\mathbb{R}) : det(A) = 1\}$$

The *Orthogonal Group*, $O(n, \mathbb{R})$, is a group defined by

$$O(n, \mathbb{R}) = \{A \in M_n(\mathbb{R}) : A^t = A^{-1}\}$$

The *Special Orthogonal Group*, $SO(n, \mathbb{R})$, is a group defined by

$$SO(n, \mathbb{R}) = \{A \in O(n, \mathbb{R}) : det(A) = \pm 1\}$$

The *Borel Matrix*, $B(n, \mathbb{R})$, is the group of upper triangle matrices with non-zero values on the main diagonal.

The *Permutations Group*, $S(n, \mathbb{R})$, is a group of permutations of $\{1, 2, \dots, n\}$ defined my $n \times n$ matrix

$$\text{e.g.}\ \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}$$

Theorem 7.3 - *Abelian Groups*
Let G be a group. If $\forall\ g, h \in G, gh = hg$ then G is commutative and is called an *Abelian Group.*
N.B. $e = 0$ is the identity element of all Albelian groups.

Definition 7.4 - *Direct/Cartesian Product of a Group*
Let G, H be groups with the same map. Then $G \times H = \{(g, h) : g \in G, h \in H\}$.

Definition 7.5 - *Fields,* $\mathbb{F}$
A field, $\mathbb{F}$, is a set with two binary operations: addition & multiplication.

Theorem 7.6 - *Properties of Fields*

i) $\mathbb{F}$ is an abelian group w.r.t addition;

ii) $\mathbb{F}\backslash\{0\}$ is an abeelian group w.r.t multiplication;

iii) $(x + y).z = x.z + y.z$;

iv) A field always contains 0 & 1.

7.2 Vector Spaces

Definition 7.7 - *Vector Space*
$\mathbb{V}$ is a (linear) vector space over a field, $\mathbb{F}$ if:

i) $\mathbb{V}$ is an abelian group w.r.t addition;

ii) $\forall\ \boldsymbol{v} \in \mathbb{V}\ \&\ \lambda \in \mathbb{F}, \lambda\boldsymbol{v} \in \mathbb{V}$;

iii) $\lambda(\boldsymbol{u}+\boldsymbol{v}) = \lambda\boldsymbol{u} + \lambda\boldsymbol{v}$;

iv) $\lambda(\mu\boldsymbol{v}) = (\lambda\mu)\boldsymbol{v}$;

v) $1.\boldsymbol{v} = \boldsymbol{v}$.

Theorem 7.8 - *Vector Spaces over Fields*
Let W be a vector space over a field, $\mathbb{F}$, and U be a set. Then define

$$F(U,W) := f : U \to W$$

Then $F(U,W)$ is a vector space over $\mathbb{F}$.
This means $F(U,W)$ is linear so for all $\lambda \in \mathbb{F}$ & $f,g \in F(U,W)$ then

$$(f+g)(u) = f(u) + g(u), \quad (\lambda f)(u) = \lambda f(u)$$

7.3 Subspace, Linear Combinations & Span

Definition 7.9 - *Subspace*
Let $\mathbb{V}$ be a vector space over a field $\mathbb{F}$ and $W \subset \mathbb{V}$, W is a subspace if it is a vector space for the operations inherited from $\mathbb{V}$.

Theorem 7.10 - *Properties of Subspaces*
Let $\mathbb{V}$ be a vector space and $U \subset \mathbb{V}$ be a subspace, then U has the following properties:

i) Not empty - $U \neq \emptyset$;

ii) Closed under addition - $\forall\ u,v \in U; (u+v) \in U$;

iii) Closed under multiplication - $\forall\ \lambda \in \mathbb{F}, u \in U; \lambda u \in U$.

Theorem 7.11 - *Subsets of Subspaces*
Let $\mathbb{V}$ be a vector space over $\mathbb{F}$ and $U,W \subset \mathbb{V}$ be subspaces. Then $U \cap W$ is a subspace of $\mathbb{V}$.

Remark 7.12 - *Linear Independence and Span*
Let $\mathbb{V}$ be a vector space over field, $\mathbb{F}$, and $S \subset \mathbb{V}$.
S is linearly dependent if there exists $v \in \mathbb{V}$ such that $span(S) = span(S\backslash\{v\})$.

Definition 7.13 - *Finite Dimensional*
Let $\mathbb{V}$ be a vector space over $\mathbb{F}$.
$\mathbb{V}$ is finitely dimensional if it is a span of a finite set, $S \subset \mathbb{V}$, of vectors.
N.B. - If a vector space is not finite dimensional, then it is *infinitely dimensional*.

Theorem 7.14
Let $\mathbb{V}$ be a vector space over $\mathbb{F}$ with $\mathbb{B}, U \subset \mathbb{V}$.
If $\mathbb{B}$ is a basis for $\mathbb{V}$, with $|\mathbb{B}| < \infty$, and U is linearly independent then

$$|U| \leq |\mathbb{B}|$$

Theorem 7.15 - *Linearly Independent Sets as Bases*
Let $\mathbb{V}$ be a vector space over $\mathbb{F}$ with $U \subset \mathbb{V}$ as a linearly independent set.
Then U can be extended to form a basis of $\mathbb{V}$.

7.4 Direct Sums

Definition 7.16 - *Direct Sum*
Let $\mathbb{V}$ be a vector space over $\mathbb{F}$ and $U, W \subset V$ be subspaces with $U \cap W = \emptyset$ then

$$U \oplus W := U + W$$

This is the direct sum of U and W.

Theorem 7.17 - *Dimension of Direct Sum*
Let $\mathbb{V}$ be a vector space over $\mathbb{F}$ and $U, W \subset V$ be subspaces with $U \cap W = \emptyset$ then

$$dim(U \oplus W) = dim(U) + dim(W)$$

Theorem 7.18 - *Complement*
Let $\mathbb{V}$ be a vector space over $\mathbb{F}$ and $U, W \subset V$ be subspaces with $U \cap W = \emptyset$ if

$$U \oplus W = V$$

then W is said to be the complement of U in V.

7.5 Rank-Nullity Theorem

Definition 7.19 - *Rank & Nullity*
Let $\mathbb{V}, \mathbb{W}$ be vector spaces over $\mathbb{F}$ and $T : \mathbb{V} \to \mathbb{W}$ be a linear map. Then

$$rank(T) := Dim(Im(T)), \quad nullity(T) := Dim(Ker(T))$$

Theorem 7.20 - *Rank-Nullity Theorem*
Let $\mathbb{V}, \mathbb{W}$ be vector spaces over $\mathbb{F}$ and $T : \mathbb{V} \to \mathbb{W}$ be a linear map, with $dim(\mathbb{V}) < \infty$ then

$$Rank(T) + Im(T) = Dim(V)$$

7.6 Projection

Defintion 7.21 - *Projection*
A linear map $P : V \to V$ is called a projection if $P^2 = P$.

Theorem 7.22 - *Image of Projection*
Let $P : V \to V$ be a projection then $v \in Im(P)$ iff $P(v) = v$.

Theorem 7.23 - *Direct Sum of Projection*
Let $P : V \to V$ be a projection then

$$V = Ker(P) \oplus Im(P)$$

7.7 Isomorphisms

Definition 7.24 -*Isomorphisms*
Let V, W be vector spaces over $\mathbb{F}$.
We say that the map $T : V \to W$ is an isomorphism between V & W if

i) T is linear; and

ii) T is bijective.

N.B. - If an isomorphism exists between V & W, then they are said to be isomorphic.

Theorem 7.25 - *Dimension of Isomorphic Spaces*
Let V be a finitely dimensional vector space over $\mathbb{F}$.
If W is isomorphic to V then

$$dim(V) = dim(W)$$

This definition can be extended to say

If two vector spaces have the same dimension, then they are isomorphic.

Proposition 7.26 - *Multiple Bases*
Let $A = \{\boldsymbol{a}_1, \dots, \boldsymbol{a}_n\}$ and $B = \{\boldsymbol{b}_1, \dots, \boldsymbol{b}_n\}$ be different bases for V.
Define $T_A : \mathbb{F}^n \to V$ and $T_B : \mathbb{F}^n \to V$ such that

$$T_A(x_1, \dots, x_n) = x_1.\boldsymbol{a}_1 + \dots + x_n.\boldsymbol{a}_n; \quad T_B(x_1, \dots, x_n) = x_1.\boldsymbol{b}_1 + \dots + x_n.\boldsymbol{b}_n$$

Then for all $\boldsymbol{v} \in V$ there are two ways of expressing $\boldsymbol{v}$.

$$x_1.\boldsymbol{a}_1 + \dots + x_n.\boldsymbol{a}_n = \boldsymbol{v} = x_1.\boldsymbol{b}_1 + \dots + x_n.\boldsymbol{b}_n$$

Unless $A = B$ then $x_i \neq y_i$ for at least one $i \in \mathbb{N}, i \leq n$.

Theorem 7.27 - *Conversion Matrices*
Let A, B be different bases for vector space V, with $dim(V) = n$.
Then an $n \times n$ matrix, C_{AB} can be used to convert elements given in basis A to now be givin in basis B.

Let $\boldsymbol{v} \in V$ and $\boldsymbol{x} = T_A(\boldsymbol{x})$ &$\boldsymbol{u} = T_B(\boldsymbol{x})$ then

$$\boldsymbol{y} = C_{AB}\boldsymbol{x}$$

Theorem 7.28 - *General Relationship between Bases*
Let V be a vector space over $\mathbb{F}$, with $dim(V) = n$.
Let A, B be different bases for V with $A = \{\boldsymbol{a}_1, \dots, \boldsymbol{a}_n\}$ & $B = \{\boldsymbol{b}_1, \dots, \boldsymbol{b}_n\}$
Then for all $\boldsymbol{v} \in V$ we have that

$$\boldsymbol{v} = \sum_{i=1}^{n} v_i.\boldsymbol{a}_i = \sum_{i=1}^{n} v_i.\boldsymbol{b}_i$$

Let $C_{AB} = (c_{ij})$ be the conversion matrix from A to B then

$$v_j = \sum_{i=1}^{n} c_{ij}\boldsymbol{b_i}$$

Theorem 7.29 - *Properties of Transition Matrices*
Let $A, B, C \subset V$ all be different bases for V then

i) $C_{AA} = I$;

ii) $C_{AB}C_{BA} = I$;

iii) $C_{CA}C_{AB} = C_{CB}$.

Theorem 7.30 - *Linear Maps between Vector Spaces as Matrices*
Let V, W both be vector spaces over $\mathbb{F}$, with $dim(V) = n$ and $dim(W) = m$, and $T : V \to W$ be a linear map.
Let $A = \{\boldsymbol{a}_1, \dots, \boldsymbol{a}_n\} \subset V$ and $B = \{\boldsymbol{b}_1, \dots, \boldsymbol{b}_n\} \subset W$ be bases for V & W respectively. Then we can define an $n \times m$ matrix

$$M_{AB}(T) = (m_{ij}) \in M_{n,m}(\mathbb{F})$$

Where m_{ij} are defined to satisfy

$$T(a_j) = \sum_{i=1}^{m} m_{ij} b_i$$

Then

$$\boldsymbol{w} = M_{AB}(T)\boldsymbol{v}$$

With $\boldsymbol{v} \in V, \boldsymbol{w} \in W$
Theorem 7.31 - *Change Basis of Linear Map*
Let V be a vector space over F and $U, W \subset V$ be different bases for V.
Define $T : V \to V$ be a linear map and C to be the transition matrix from basis $U \to W$.
Then C^{-1} is the transition matrix from $W \to U$.
Set A to be the matrix representation of T in basis U. Then

$$A' = C^{-1}AC$$

Where A' is the matrix representation of T in basis W.

8 Eigenvalues & Eigenvectors

8.1 Characteristic Polynomial

Definition 8.1 - *Eigenvectors & Eigenvalues*
Let $\boldsymbol{v} \in V \backslash \{\boldsymbol{0}\}$ and $T : V \to V$ be a linear operator.
$\boldsymbol{v}$ is called an *eigenvector* of T if

$$T(\boldsymbol{v}) = \lambda \boldsymbol{v}, \quad \lambda \in \mathbb{F}$$

This λ is the associatiated *eigenvalue* for $\boldsymbol{v}$.

Definition 8.2 - *Spectrum*
The set of eigenvectors of a linear operator $T : V \to V$ is called the spectrum of T, generally denoted as

$$Spec(T) := \{\boldsymbol{v} \in V : T(\boldsymbol{v}) = \lambda \boldsymbol{v}, \lambda \in \mathbb{F}\}$$

Defintion 8.3 - *Diagonisable*
A linear operator is *diagonisable* if there exists a basis of eigenvectors for it.

Remark 8.4 - *Finding Eigenvalues*
Let A be the matrix which represents a linear operator T, and $\boldsymbol{X}$ be a general eigenvector for T

$$T(\boldsymbol{x}) = A\boldsymbol{x} = \lambda \boldsymbol{x} => (A - \lambda.I)\boldsymbol{x} = \boldsymbol{0}$$

Then λ is an eigenvalue if it satisfies

$$det(A - \lambda.I) = 0$$

Definition 8.5 - *Characteristic Polynomial*
The polynomial which is equivalent to $det(A-\lambda.I)$ is called the *characteristic polynomial* of A.

$$p_A(\lambda) := det(A-\lambda.I)$$

N.B. - λ is an eigenvalue for A if $p_A(\lambda)=0$

Definition 8.6 - *Eigenspace*
let $\lambda \in \mathbb{F}$ be an eigenvalue of T, then the corresponding *eigenspace* is defined as

$$V_\lambda := ker(T-\lambda.I)$$

Remark 8.7 - *Finding Eigenvectors*
Once we have found all $\lambda_1,\dots,\lambda_k$ that satisfy $p_A(\lambda_i)=0$ then we can find the eigenvectors, $\boldsymbol{x}_i$, of A

$$(A-\lambda.I)\boldsymbol{x}_i = \mathbf{0}$$

A good way to start is to produce the linear equations

$$\sum_{j=1}^{n}(A-\lambda.I)_{ij}.x_j = 0$$

For all $i \leq n$. Then solve these, as a series of simultaneous equations, to find the values x_j which produce the eigenvector $\boldsymbol{x}$.
Repeat this process for all $\lambda_1,\dots,\lambda_k$ to find all eigenvectors for A.

Theorem 8.8 - *Similar Characteristic Polynomial*
Let C be an invertible matrix.
Define $A' = C^{-1}AV$ where A & A' are conjugate or similar.
Then $p_A(\lambda) = p_{A'}(\lambda)$.

Theorem 8.9 - *Characteristic Polynomial & Basis*
The characteristic polynomial for T is the same, regardless of the basis of T.
Definition 8.10 - *Trace*
Let $A \in M_n(\mathbb{F})$.
Then the trace of A is defined as

$$Tr(A) := \sum_{i=1}^{n} a_{ii}$$

N.B. - Trace is sometimes called *Spur*.
Remark 8.11
As the terms after the first term of the determinat of a matrix do not contribute to the powers of λ in the characteristic equation then we can ignore them. This means we can deduce that

$$p_T(\lambda) = (-1)^n\lambda^n + (-1)^{n-1}(Tr(A)) + \dots + det(A)$$

Theorem 8.12 - *Diagonalised Matrix*
Let T be a diagonisable matrix with eigenvalues $\{\lambda_1,\lambda_2,\dots,\lambda_n\}$.
Then T can be represented by

$$\Delta = \begin{pmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & & \vdots \\ \vdots & & \ddots & \vdots \\ 0 & \dots & \dots & \lambda_n \end{pmatrix}$$

N.B. - T can also be represented in any basis with, C as the transition matrix, by $C^{-1}\Delta C$.

Theorem 8.13 - *Relationship between Matrix and its Diagonalised Form*
Let T be a matrix and Δ be its diagonalised form, then

$$Det(T) = Det(\Delta) = \prod_{j=1}^{n} \lambda_j$$

And

$$Tr(T) = Tr(\Delta) = \sum_{j=1}^{n} \lambda_j$$

Theorem 8.14 - *Distinct Eigenvectors and Diagonisability*
Eigenvectors, which correspond to distinct eigen values, are linearly independent.
Thus if a matrix, A, has only distinct eigenvalues then it is diagonisable.

8.2 Roots of Characteristic Polynomial

Remark 8.15 - *Degree of Characteristic Equation*
Eigenvalues are roots of $p_A(\lambda) = 0$ where p_A is an equation of degree $dim(A)$.

Remark 8.16 - *Non-Distinct Roots of Characteristic Equation*
If the roots of $P_A(\lambda)$ are not distinct then A may be diagonisable depending on how many eigenvectors are found.

Theorem 8.17 - *Vieta's Theorem*
If $\lambda_1, \dots, \lambda_n$ are roots of the Polynomial

$$\lambda_n + a_1\lambda^{n-1} + \dots + a_n = 0 \equiv p(\lambda) = 0$$

Then $p(\lambda) = (\lambda - \lambda_1)(\lambda - \lambda_2)\dots(\lambda - \lambda_n)$.
So $p(\lambda)$ factorises in the product $\prod_{i=1}^{n}(\lambda - \lambda_i)$ but the λ_is are not necessarily distinct.

Definition 8.18 - *Multiplicity of Roots*
Let $\lambda_1 \in \mathbb{C}$ of characteristic polynomial, $p(\lambda)$.
λ_1 has multiplicity $m_1 \in \mathbb{N}$ if

$$p(\lambda_1) = \frac{dp}{d\lambda}(\lambda_1) = \dots = \frac{d^{m_1-1}p}{d\lambda^{m_1-1}}(\lambda_1) = 0$$

This means that $(\lambda - \lambda_1)^{m_1}$ is a factor of $p(\lambda)$.

Definition 8.19 - *Geometric & Algebraic Multiplicity*
Let $\lambda \in spec(T)$ and V_λ be the corresponding eigenspace.

i) λ has *geometric multiplicity*, $m_g(\lambda) \in \mathbb{N}$, if $dim(V_\lambda) = m_g(\lambda)$;

ii) λ has *algebraic multiplicity*, $m_a(\lambda) \in \mathbb{N}$, if λ has multiplicity m_a of $p_T(\lambda)$

Theorem 8.20 - *Relationship between Geometric & Algebraic Multiplicity*
Let $\lambda \in spec(T)$ then

$$m_g(\lambda) \leq m_a(\lambda)$$

Theorem 8.21 -
Let T be a linear operator on an n dimensional space over $\mathbb{C}$ or $\mathbb{R}$, with eigenvalues $\lambda_1, \dots, \lambda_n$, which are not necessarily distinct. Then

$$det(T) = \prod_{i=1}^{n} \lambda_i \quad \& \quad tr(T) = \sum_{i=1}^{n} \lambda_i$$

9 Inner Product Spaces

9.1 Inner Product, Norm & Orthogonality

Definition 9.01 - *Inner Product (Complex)*
Let V be a vector space over $\mathbb{C}$.
An *inner product* on V is a map, $\langle V, V \rangle : V \times V \to \mathbb{C}$, with the following properties:

i) $\langle v, v \rangle \geq 0$;

ii) $\langle v, w \rangle = \overline{\langle v, w \rangle}$;

iii) $\langle u, v + w \rangle = \langle u, v \rangle + \langle u, w \rangle$;

iv) $\langle \lambda . u, v \rangle = \lambda \langle u, v \rangle$.

Where $u, v, w \in V$ and $\lambda \in \mathbb{C}$.

Definition 9.02 - *Inner Product (Real)*
Let V be a vector space over $\mathbb{R}$.
An *inner product* on V is a map, $\langle , \rangle : V \times V \to \mathbb{C}$, with the following properties:

i) $\langle v, v \rangle \geq 0$;

ii) $\langle v, w \rangle = \langle w, v \rangle$;

iii) $\langle u, v + w \rangle = \langle u, v \rangle + \langle u, w \rangle$;

iv) $\langle \lambda . u, v \rangle = \lambda \langle u, v \rangle$.

Where $u, v, w \in V$ and $\lambda \in \mathbb{C}$.

Definition 9.03 - *Inner Product Space*
A V be a vector space with $\langle , \rangle$ as a defined inner product are called an *inner product space*, denoted by

$$(V, \langle , \rangle)$$

N.B. - If V is over $\mathbb{C}$ then this is called a *complex inner product space.* If V is over $\mathbb{R}$ then this is called a *real inner product space.*

Definition 9.04 - *Norm*
Let $(V, \langle , \rangle)$ be an inner product space, then we define the associated norm as

$$||v|| := \sqrt{\langle v, v \rangle}, \quad v \in V$$

Definition 9.05 - *Orthogonal*
Let $(V, \langle , \rangle)$ be an inner product space, then

i) $v, w \in V$ are *orthogonal*, $v \perp w$, if $\langle v, w \rangle = 0$;

ii) $U, W \subset V$ are *orthogonal*, $U \perp W$, if $u \perp w \ \forall \ u \in U \ \& \ v \in V$.

Definition 9.06 - *Orthogonal Complement*
Let $(V, \langle , \rangle)$ be an inner product space and $W \subset V$.
The *orthogonal complement* is defined as

$$W^{\perp} := \{v \in V : v \perp w \ \forall \ w \in W\}$$

Theorem 9.07 - *Norm of Orthogonal Elements*
Let $(V, \langle , \rangle)$ be an inner product space and $v, w \in V$ with $v \perp w$, then

$$||v + w||^2 = ||v||^2 + ||w||^2$$

Definition 9.08 - *Orthonormal Basis*
Let $(V, \langle , \rangle)$ be an inner product space.
A basis, $\mathbb{B} = \{v_1, \dots, v_n\}$, is called an *orthonormal basis* if

$$\langle v_i, v_j \rangle = \delta_{ij} := \begin{cases} 1 & i = j; \\ 0 & i \neq j. \end{cases}$$

Theorem 9.09 - *Properties of Orthogonal Basis*
Let $(V, \langle , \rangle)$ be an inner product space and $\mathbb{B} = \{v_1, \dots, v_n\}$ an orthonormal basis.
Then $\forall v, w \in V$,

i) $v = \sum_{i=1}^{n} \langle v_i, v \rangle v_i$;

ii) $\langle v, w \rangle = \sum_{i=1}^{n} \overline{\langle v_i, v \rangle} \langle v_i, w \rangle$;

iii) $||v|| = \left[\sum_{i=1}^{n} |\langle v_i, v \rangle|^2\right]^{1/2}$.

9.2 Construction of Orthonormal Basis