Linear Algebra & Geometry - Notes

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1 Euclidean Plane, Vectors, Cartesian Co-Ordinates & Complex Numbers

1.1 Vectors

Definition 1.01 - *Vectors*
Ordered sets of real numbers.
Denoted by $\boldsymbol{v} = (v_1, v_2, v_3, ...) = \begin{pmatrix} x \\ y \end{pmatrix}$

Definition 1.02 - *Euclidean Plane*
The set of two dimensional vectors, with real componenets, is called the Euclidean Plane.
Denoted by $\mathbb{R}^2$

Definition 1.03 - *Vector Addition*
Let $\boldsymbol{v}, \boldsymbol{w} \in \mathbb{R}^2$ such that $\boldsymbol{v} = (v_1, v_2)$ and $\boldsymbol{w} = (w_1, w_2)$.
Then $\boldsymbol{v} + \boldsymbol{w} = (v_1 + w_1, v_2 + w_2)$.

Definition 1.03 - *Scalar Multiplcation of Vectors*
Let $\boldsymbol{v} \in \mathbb{R}^2$ and $\lambda \in \mathbb{R}$ such that $\boldsymbol{v} = (v_1, v_2)$.
Then $\lambda\boldsymbol{v} = (\lambda v_1, \lambda v_2)$.

Definition 1.04 - *Norm of vectors*
The norm of a vector is its length from the origin.
Denoted by $||\boldsymbol{v}|| = \sqrt{v_1^2 + v_2^2}$ for $\boldsymbol{v} \in \mathbb{R}^2$.

Theorem 1.05
Let $\boldsymbol{v}, \boldsymbol{w} \in \mathbb{R}^2$ and $\lambda \in \mathbb{R}$ such that $\boldsymbol{v} = (v_1, v_2)$ and $\boldsymbol{w} = (w_1, w_2)$.
Then

$$\begin{aligned} ||\boldsymbol{v}|| &= 0 \text{ iff } \boldsymbol{v} = \boldsymbol{0} \\ ||\lambda\boldsymbol{v}|| &= \sqrt{\lambda^2 v_1^2 + \lambda^2 v_2^2} \\ &= |\lambda|.||\boldsymbol{v}|| \\ ||\boldsymbol{v} + \boldsymbol{w}|| &\leq ||\boldsymbol{v}|| + ||\boldsymbol{w}|| \end{aligned}$$

Definition 1.06 - *Unit Vector*
A vector can be described by its length & direction.
Let $\boldsymbol{v} \in \mathbb{R}^2 \backslash \{\boldsymbol{0}\}$.
Then $\boldsymbol{v} = ||\boldsymbol{v}||\boldsymbol{u}$ where $\boldsymbol{u}$ is the unit vector, $\boldsymbol{u} = \begin{pmatrix} cos\theta \\ sin\theta \end{pmatrix}$
Thus $\forall\ \boldsymbol{v} \in \mathbb{R}^2\ \boldsymbol{v} = \begin{pmatrix} \lambda cos\theta \\ \lambda sin\theta \end{pmatrix}$ for some $\lambda \in \mathbb{R}$.

Definition 1.07 - *Dot Product*
Let $\boldsymbol{v} \in \mathbb{R}^2$ and $\lambda \in \mathbb{R}$ such that $\boldsymbol{v} = (v_1, v_2)$.
Then $\boldsymbol{v} \cdot \boldsymbol{w} = v_1.w_1 + v_2.w_2$.

Remark 1.08 - *Positivity of Dot Product*
Let $\boldsymbol{v} \in \mathbb{R}^2$.
Then $\boldsymbol{v} \cdot \boldsymbol{v} = ||\boldsymbol{v}||^2 = v_1^2 + v_2^2 \geq 0$.

Remark 1.09 - *Angle between vectors in Euclidean Plane*
Let $\boldsymbol{v}, \boldsymbol{w} \in \mathbb{R}^2$.
Set θ to be the angle between $\boldsymbol{v}$ & $\boldsymbol{w}$.
Then

$$cos\theta = \frac{\boldsymbol{v} \cdot \boldsymbol{w}}{||\boldsymbol{v}||\ ||\boldsymbol{w}||}$$

.
Theorem 1.10 - *Cauchy-Schwarz Inequality*
Let $\boldsymbol{v}, \boldsymbol{w} \in \mathbb{R}^2$.
Then

$$|\boldsymbol{v} \cdot \boldsymbol{w}| \leq ||\boldsymbol{v}||\ ||\boldsymbol{w}||$$

Proof

$$\begin{aligned}
\frac{v_1 w_1}{||\boldsymbol{v}||\ ||\boldsymbol{w}||} + \frac{v_2 w_2}{||\boldsymbol{v}||\ ||\boldsymbol{w}||} &\leq \frac{1}{2}\left(\frac{v_1^2}{||\boldsymbol{v}||^2} + \frac{w_1^2}{||\boldsymbol{w}||^2}\right) + \frac{1}{2}\left(\frac{v_2^2}{||\boldsymbol{v}||^2} + \frac{w_2^2}{||\boldsymbol{w}||^2}\right) \\
&\leq \frac{1}{2}\left(\frac{v_1^2 + v_2^2}{||\boldsymbol{v}||^2} + \frac{w_1^2 + w_2^2}{||\boldsymbol{w}||^2}\right) \\
&\leq \frac{1}{2}(1+1) \\
&\leq 1 \\
=> |v_1 w_1 + v_2 w_2| &\leq ||\boldsymbol{v}||\ ||\boldsymbol{w}|| \\
|\boldsymbol{v} \cdot \boldsymbol{w}| &\leq ||\boldsymbol{v}||\ ||\boldsymbol{w}||
\end{aligned}$$

1.2 Complex Numbers

Definition 1.11 - i

$$\begin{aligned} i^2 &= -1 \\ i &= \sqrt{-1} \end{aligned}$$

Definition 1.12 - *Complex Number Set*
The set of complex numbers contains all numbers with an imaginary part.

$$\mathbb{C} := \{x + iy; x, y \in \mathbb{R}\}$$

Complex numbers are often denoted by

$$z = x + iy$$

and we say x is the real part of z and y the imaginary part.
Definition 1.13 - *Complex Conjugate*
Let $z \in \mathbb{C}$ st $z = x + iy$.
Then

$$\bar{z} := x - iy$$

Theorem 1.14 - *Operations on Complex Numbers*
Let $z_1, z_2 \in \mathbb{C}$ st $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$.
Then

$$\begin{aligned}
z_1 + z_2 &:= (x_1 + x_2) + i(y_1 + y_2) \\
z_1.z_2 &:= (x_1 + iy_1)(x_2 + iy_2) \\
&:= x_1.x_2 - y_1.y_2 + i(x_1.y_2 + x_2.y_1)
\end{aligned}$$

N.B. When dividing by a complex number, multiply top and bottom by the complex conjugate.

Definition 1.15 - *Modulus of Complex Numbers*
The modulus of a complex number is the distance of the number, from the origin, on an Argand diagram. Let $z \in \mathbb{C}$ st $z = x + iy$.
Then

$$\begin{aligned}
|z| &:= \sqrt{x^2 + y^2} \\
&:= \sqrt{\bar{z}z}
\end{aligned}$$

N.B. Amplitude is an alternative name for the modulus

Definition 1.16 - *Phase of Complex Numbers*
The phase of a complex number is the angle between the positive real axis and the line subtended from the origin and the number, on an Argand digram.

$$z = |z|.(cos\theta + i.sin\theta), \quad \theta = \text{Phase}$$

N.B. Phase of $\bar{z}$ = - Phase of z

Theorem 1.17 - *de Moivre's Formula*

$$z^n = (cos\theta + i.sin\theta)^n = cos(n\theta) + i.sin(n\theta)$$

Theorem 1.18 - *Euler's Formula*

$$e^{i\theta} = cos\theta + i.sin\theta$$

Remark 1.19
Using Euler's formula we can express all complex numbers in terms of e. Thus many properties of the exponential remain true:

$$\begin{aligned}
z &= \lambda e^{i\theta}, \qquad\qquad \lambda \in \mathbb{R}, \theta \in [0, 2\pi) \\
=> z_1 + z_2 &= \lambda_1.\lambda_2.e^{i(\theta_1+\theta_2)} \\
\&, \frac{z_1}{z_2} &= \frac{\lambda_1}{\lambda_2}.e^{i(\theta_1=\theta_2)}
\end{aligned}$$

2 Euclidean Space, $\mathbb{R}^n$

Definition 2.01 - *Euclidean Space*
Let $n \in \mathbb{N}$ then $\forall$ $\boldsymbol{x} = (x_1, x_2, ..., x_n)$ with $x_1, x_2, ..., x_n \in \mathbb{R}$ we have that $\boldsymbol{x} \in \mathbb{R}^n$.

Theorem 2.02 - *Operations in Euclidean Space*
Let $(x), (y) \in \mathbb{R}^n$ and $\lambda \in \mathbb{R}$. Then

$$(x) + (y) = (x_1 + y_1, ..., x_n + y_n)$$

And

$$(x) + \lambda.(y) = (x_1 + \lambda.y_1, ..., x_n + \lambda.y_n)$$

Definition 2.03 - *Cartesian Product*
Let $A, B \in \mathbb{R}^n$ be non-empty sets.
Then

$$A \times B := \{(a, b); a \in A, b \in B\}$$

2.1 Dot Product

Definition 2.04 - *Dot Product*
Let $\boldsymbol{v}, \boldsymbol{w} \in \mathbb{R}^n$. Then

$$\begin{aligned} \boldsymbol{v} \cdot \boldsymbol{w} &:= v_1.w_1 + ... + v_n.w_n \\ &:= \sum_{j=1}^{n} v_j.w_j \end{aligned}$$

Theorem 2.05 - *Properties of the Dot Product*
Let $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w} \in \mathbb{R}^n$. Linearity:

$$(\boldsymbol{u} + \lambda\boldsymbol{v}) \cdot \boldsymbol{w} = \boldsymbol{u} \cdot \boldsymbol{w} + \lambda(\boldsymbol{v} \cdot \boldsymbol{w})$$

Symmetry:

$$\boldsymbol{v} \cdot \boldsymbol{w} = \boldsymbol{w} \cdot \boldsymbol{v}$$

Positivity:

$$\boldsymbol{v} \cdot \boldsymbol{v} = v_1^2 + v_2^2 + ... + v_n^2 \geq 0$$

Definition 2.06 - *Orthogonality*
Let $\boldsymbol{v}, \boldsymbol{w} \in \mathbb{R}^n$.
It is said that $(v), (w)$ are orthogonal to each other if $\boldsymbol{v} \cdot \boldsymbol{w} = 0$
N.B. Orthogonal vectors are perpendicular to each other.

Definition 2.07 - *The Norm*
Let $\boldsymbol{x} \in \mathbb{R}^n$.
Then

$$||\boldsymbol{x}|| = \sqrt{\boldsymbol{x} \cdot \boldsymbol{x}} = \sqrt{\sum_{i=1}^{n} x_i^2}$$

Theorem 2.08 - *Properties of the Norm*
Let $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$ and $\lambda \in \mathbb{R}$. Then

$$\begin{aligned} ||\boldsymbol{x}|| &\geq 0 \\ ||\boldsymbol{x}|| &= 0 \text{ iff } \boldsymbol{x} = 0 \\ ||\lambda \boldsymbol{x}|| &= |\lambda|||\boldsymbol{x}|| \\ ||\boldsymbol{x} + \boldsymbol{y}|| &\leq ||\boldsymbol{x}|| + ||\boldsymbol{y}|| \end{aligned}$$

Theorem 2.09 - *Dot Product and Norm*
Let $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$.

$$|\boldsymbol{x} \cdot \boldsymbol{y}| \leq ||\boldsymbol{x}||||\boldsymbol{y}||$$

N.B. $|\boldsymbol{x} \cdot \boldsymbol{y}| = ||\boldsymbol{x}||||\boldsymbol{y}||$ iff $\boldsymbol{x}$ & $\boldsymbol{x}$ are orthogonal.

Theorem 2.10 - *Angle between Vectors*
Let $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$. Then

$$cos\theta = \frac{\boldsymbol{x} \cdot \boldsymbol{y}}{||\boldsymbol{x}||||\boldsymbol{y}||}$$

2.2 Linear Subspaces

Definition 2.11 - *Linear Subspace*
Let $V \subset \mathbb{R}^n$. V is a *Linear Subspace* if:
i) $V \neq \emptyset$;
ii) $\forall\ \boldsymbol{v}, \boldsymbol{w} \in V$ then $\boldsymbol{v} + \boldsymbol{w} \in V$;
iii) $\forall\ \lambda \in \mathbb{R}, \boldsymbol{v} \in V$ then $\lambda \boldsymbol{v} \in V$.

Definition 2.12 - *Span*
Let $\boldsymbol{x_1}, ..., \boldsymbol{x_k} \in \mathbb{R}^n; k \in \mathbb{N}$. Then

$$\text{span}\{\boldsymbol{x_1}, ..., \boldsymbol{x_k}\} := \{\lambda_1 \boldsymbol{x_1} + ... + \lambda_k \boldsymbol{x_k}; \lambda_i \in \mathbb{R}, 0 \leq i \geq k\}$$

Definition 2.13 - *Spans are Subspaces*
Let $\boldsymbol{x_1}, ..., \boldsymbol{x_k} \in \mathbb{R}^n;\ k \in \mathbb{N}$. Then $\text{span}\{\boldsymbol{x_1}, ..., \boldsymbol{x_k}\}$ is a linear subspace of $\mathbb{R}^n$.

Theorem 2.14

$$W_{\boldsymbol{a}} := \{\boldsymbol{x} \in \mathbb{R}^n; \boldsymbol{x} \cdot \boldsymbol{a} = 0\} \text{ is a subspace.}$$

Definition 2.15 - *Orthogonal Complement*
Let $V \subset \mathbb{R}^n$. Then,

$$V^{\perp} := \{\boldsymbol{x} \in \mathbb{R}^n; \boldsymbol{x} \cdot \boldsymbol{y}\ \forall\ \boldsymbol{y} \in V\}$$

N.B. $V^{\perp} \subset \mathbb{R}^n$

Theorem 2.16 - *Relationship of Subspaces*
Let V, W be subspaces of $\mathbb{R}$. Then

$$V \cap W \text{ is a subspace.}$$

$$V + W := \{\boldsymbol{v} + \boldsymbol{w}; \boldsymbol{v} \in V, \boldsymbol{w} \in W\} \text{ is a subspace.}$$

Definition 2.17 - *Direct Sum*
Let V_1, V_2, W be subspaces of $\mathbb{R}$. Then W is said to be a *direct sum* if
i) $W = V_1 + V_2$;
ii) $V_1 \cap V_2 = \emptyset$.

3 Linear Equations & Matrices

3.1 Linear Equations

Definition 3.01 - *Multi-Variable Linear Equations*
Linear equations produce a straight line and can have multiple variables.
Examples - $x = 3, y = x + 3, z + 5x - 2y$

Defintion 3.02 - *Systems of Linear Equations*
Let $\boldsymbol{a}, \boldsymbol{x} \in \mathbb{R}^n$ & $b \in \mathbb{R}$ such that $\boldsymbol{a} \cdot \boldsymbol{x} = b$.
$\boldsymbol{a} \cdot \boldsymbol{x} = b$ is a linear equation in x with $S = \{\boldsymbol{x}; \boldsymbol{a} \cdot \boldsymbol{x} = b\}$ as the set of solutions.
N.B. If $b = 0$ then $S(\boldsymbol{a}, 0)$ is a subspace.

3.2 Matrices

Definition 3.03 - *Matrix*
Let $m, n \in \mathbb{N}$, then a $m \times n$ grid of numbers form an "m" by "n" matrix. Each element of the matrix can be reference by a_{ij} with $i = 1, ..., m$ and $j = 1, ..., n$.

$$A = \begin{pmatrix} a_{11} & a_{12} & ... & a_{1n} \\ a_{21} & a_{22} & ... & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & ... & a_{mn} \end{pmatrix}$$

N.B. m,i = rows, n,j = columns

Definition 3.04 - *Sets of Matrices*
$M_{m,n}(\mathbb{R})$ is the set of m x n matrices containing only real numbers.
$M_{m,n}(\mathbb{Z})$ is the set of m x n matrices containing only integers.
$M_n(\mathbb{R})$ is the set square matrices, size n, containing only real numbers.

Definition 3.05 - *Transpose Vectors*

Let $\boldsymbol{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$ then $\boldsymbol{x}^t = \begin{pmatrix} x_1 & x_2 & ... & x_n \end{pmatrix}$

Definition 3.06 - *Vector-Matrix Multiplication*
Let $A \in \mathbb{R}_{m,n}$ and $\boldsymbol{x} \in \mathbb{R}^n$ then

$$A\boldsymbol{x} := \begin{pmatrix} a_{11} & a_{12} & ... & a_{1n} \\ a_{21} & a_{22} & ... & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & ... & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} a_{11}x_1 + a_{12}x_2 + ... + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + ... + a_{2n}x_n \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + ... + a_{mn}x_n \end{pmatrix} = \begin{pmatrix} \boldsymbol{a}_1^t \cdot \boldsymbol{x} \\ \boldsymbol{a}_2^t \cdot \boldsymbol{x} \\ \vdots \\ \boldsymbol{a}_m^t \cdot \boldsymbol{x} \end{pmatrix} \in \mathbb{R}^m$$

This can be simplified to

$$\boldsymbol{y} = A\boldsymbol{x} \text{ with } y_i = \sum_{j=1}^{n} a_{ij}x_j$$

Theorem 3.07 - *Operations on Matrices with Vectors*

i) $A(\boldsymbol{x} + \boldsymbol{y}) = A\boldsymbol{x} + A\boldsymbol{y}, \quad \forall\ \boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$.

ii) $A(\lambda \boldsymbol{x}) = \lambda(A\boldsymbol{x}), \quad \forall \boldsymbol{x} \in \mathbb{R}^n, \lambda \in \mathbb{R}$.

Theorem 3.08
Let $A = (a_{ij}) \in M_{m,n}(\mathbb{R})$ and $B = (b_{ij}) \in M_{l,m}(\mathbb{R})$. Then there exists a $C = (c_{ij}) \in M_{l,n}(\mathbb{R})$ such that

$$C\boldsymbol{x} = B(A\boldsymbol{x}), \quad \forall\ \boldsymbol{x} \in \mathbb{R}^n$$

N.B. $c_{ij} = \sum_{k=1}^{m} b_{ik}a_{kj}$

Theorem 3.09 - *Operation between Matrices*
Let $A, B \in M_{m,n}$ and $C \in M_{l,m}$

i) $C(A + B) = CA + CB$.

ii) $(A + B)C = AC + BC$.

iii) Let $D \in M_{m,n}, E \in M_{n,l}$ & $F \in M_{l,k}$ then

$$E(FG) = (EF)G$$

N.B. $AB \neq BA$

Definition 3.10 - *Types of Matrix*

Upper Triangle $\begin{pmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{pmatrix}, \quad a_{ij} = 0$ if $i > j$.

Lower Triangle $\begin{pmatrix} 1 & 0 & 0 \\ 2 & 3 & 0 \\ 4 & 5 & 6 \end{pmatrix}, \quad a_{ij} = 0$ if $i < j$.

Symmetric Matrix $\begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 0 \\ 3 & 0 & 1 \end{pmatrix}, \quad a_{ij} = a_{ji}$.

Anti-Symmetric $\begin{pmatrix} 1 & -2 & -3 \\ 2 & 0 & -4 \\ 3 & 4 & -1 \end{pmatrix}, \quad a_{ij} = -a_{ji}$.

Definition 3.11 - *Transposed Matrices*
Let $A = (a_{ij}) \in M_{m,n}(\mathbb{R})$ then the transponse of A, A^t, is an element of $M_{n,m}(\mathbb{R})$.

$$A^t := (aji)$$

Theorem 3.12 - *Transpose Matrix Multiplication*
Let $A \in M_{m,n}(\mathbb{R}), \boldsymbol{x} \in \mathbb{R}^n, \boldsymbol{y} \in \mathbb{R}^m$. Then

$$\boldsymbol{y} \cdot A\boldsymbol{x} = (A_t\boldsymbol{y}) \cdot \boldsymbol{x}$$

Theorem 3.10 - *Transposing Multiplied Matrices*

$$(AB)^t = B^tA^t$$

3.3 Structure of Set of Solutions

Definition 3.13 - *Set of Solutions*
Let $A \in M_{m,n}(\mathbb{R})$ and $\boldsymbol{b} \in \mathbb{R}^m$. Then

$$S(A, \boldsymbol{b}) := \boldsymbol{x} \in \mathbb{R}^n; A\boldsymbol{x} = b$$

Definition 3.14 - *Homogenous Solutions*
The system of $S(A, \mathbf{0})$ is called said to be *homogenous*. All other systems are *inhomogenous*.
N.B. - $S(A, \mathbf{0})$ is a linear subspace.

Theorem 3.15 - *Using Homogenous Solutions*
Let $A \in M_{m,n}(\mathbb{R})$ and $\boldsymbol{b} \in \mathbb{R}^n$. Let $\boldsymbol{x}_0 \in \mathbb{R}^n$ such that $A\boldsymbol{x}_0 = \boldsymbol{b}$, then

$$S(A, \boldsymbol{b}) = \boldsymbol{x}_0 + S(A, \mathbf{0})$$

Remark 3.16 - *Systems of Linear Equations as Matrices*
The system of linear equations $3x + z = 0, y - z = 1, 3x + y = 1$ can be represented by a matrix and a vector.

$$A = \begin{pmatrix} 3 & 0 & 1 \\ 0 & 1 & -1 \\ 3 & 1 & 0 \end{pmatrix}, \ \boldsymbol{b} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

3.4 Solving Systems of Linear Equations

Systems of linear equations can be displayed as matrices which can be reduced and solved by a technique called *Gaussian Elimination*.

Theorem 3.17
There are certain operations that can be performed on a system of linear equations without changing the result:

i) Multiply an equaion by a non-zero constant;

ii) Add a multiple of any equation to another equation;

iii) Swap any two equations.

Definition 3.18 - *Augmented Matrices*
Let $A\boldsymbol{x} = \boldsymbol{b}$ be a system of linear equations. The associated *Augmented Matrix* is

$$(A\ \boldsymbol{b}) \in M_{m,n+1}(\mathbb{R})$$

Theorem 3.19 - *Elementary Row Operations*
From *Theorem 3.17* we can deduce ceratin operations that can be performed on an *Augmented Matrix* which do not alter the solutions:

i) Multiply a row by a non-zero constant, $row\ i \to \lambda(row\ i)$;

ii) Add a multiple of any row to another row, $row\ i \to row\ i + \lambda(row\ j)$;

iii) Swap two rows, $row\ i \leftrightarrow row\ j$.

Definition 3.20 - *Row Echelon Form*
A matrix is in *Row Echelon Form* if:

i) The left-most non-zero value in each row is 1; And,

ii) The leading 1 in each row is one place to the right of the leading 1 in the row below.

Example

$$\begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix}$$

Definition 3.20 - *Reduced Row Echelon Form*
A matrix is in *Reduced Row Echelon Form* if:

i) The matrix is in *row echelon form*; And,

ii) All values in a row, except the leading 1, are 0.

Example

$$\begin{pmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & 0 & 1\end{pmatrix}$$

Theorem 3.21 - *Gaussian Elimination*
Gaussian Elimination is a technique used to solve systems of linear equations. *Example*
Solve $x + y + 2z = 9, 2x + 4y - 3z = 1, 3x + 6y - 5z = 0$.

$$\text{Augmented Matrix} - \begin{pmatrix}1 & 1 & 2 & 9\\2 & 4 & -3 & 1\\3 & 6 & -5 & 0\end{pmatrix}$$

$$\begin{aligned}\text{By EROS} - \begin{pmatrix}1 & 1 & 2 & 9\\2 & 4 & -3 & 1\\3 & 6 & -5 & 0\end{pmatrix} &= \begin{pmatrix}1 & 1 & 2 & 9\\0 & 2 & -7 & -17\\0 & 3 & -11 & 27\end{pmatrix}\\
&= \begin{pmatrix}1 & 1 & 2 & 9\\0 & 2 & -7 & -17\\0 & 1 & -4 & -10\end{pmatrix}\\
&= \begin{pmatrix}1 & 1 & 2 & 9\\0 & 1 & -4 & -10\\0 & 2 & -7 & -17\end{pmatrix}\\
&= \begin{pmatrix}1 & 1 & 2 & 9\\0 & 1 & -4 & -10\\0 & 0 & 1 & 3\end{pmatrix}\\
&= \begin{pmatrix}1 & 0 & 6 & 19\\0 & 1 & -4 & -10\\0 & 0 & 1 & 3\end{pmatrix}\\
&= \begin{pmatrix}1 & 0 & 0 & 1\\0 & 1 & 0 & 2\\0 & 0 & 1 & 3\end{pmatrix}\end{aligned}$$

$$=> \underline{x = 1, y = 2, z = 3}$$

3.5 Elementary Matrices & Inverting Matrices

Definition 3.22 - *Invertible Matrices*
A matrix, $A \in M_{m,n}(\mathbb{R})$, is said to be *Invertible* if there exists $A^{-1} \in M_{n,m}(\mathbb{R})$ such that

$$AA^{-1} = I$$

<u>N.B.</u> - If a matrix is not invertible then it is *Singular*.

Definition 3.23 - *Elementary Matrices*
A matrix, $E \in M_{m,n}(\mathbb{R})$, is said to be an *Elementary Matrix* if it can be obtained by performing Elementary Row Operations on a square identity matrix.
Examples $\begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}, \begin{pmatrix}0 & \lambda\\\mu & 0\end{pmatrix}$

Remark 3.24
All elementary matrices are invertible.

Remark 3.25
Let A be a matrix, and B be a matrix which can be obtained from A by elementary row operations. Then there exists an elementary matrix E such that

$$B = EA$$

.

Theorem 3.26 - *Finding A^{-1}*
Let $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$, $B = \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix}$, $I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.
Then by using EROS to change $(A\ I) \to (I\ B)$, B is the inverse of A.

Theorem 3.27 - *Inverse of a 2x2 Matrix*
Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ then

$$A^{-1} = \frac{1}{ad - bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

4 Linear Independence, Bases & Dimensions

4.1 Linear Independence & Dependence

Definition 4.01 - *Linear Independence & Dependence*
Vectors, $\boldsymbol{x}_1, ..., \boldsymbol{x}_n \in \mathbb{R}^k$, are said to be *linearly dependent* if there exists non-zero real numbers, $\lambda_1, ..., \lambda_n$ such that

$$\lambda_1.\boldsymbol{x}_1 + ... + \lambda_n.\boldsymbol{x}_n = \mathbf{0}$$

N.B. - If this is only true if $\lambda_1 = ... = \lambda_n = 0$ then the vectors are said to be *linearly independent.*

Remark 4.02
Vectors are only *linearly dependent* if one of them lies in the span of the rest.

4.2 Bases & Dimensions

Definition 4.03 - *Basis*
A *basis* is a set of vectors, $\boldsymbol{v}_1, ..., \boldsymbol{v}_n \in V$ such that

i) $V = \text{span}\{\boldsymbol{v}_1, ..., \boldsymbol{v}_n\}$;

ii) $\boldsymbol{v}_1, ..., \boldsymbol{v}_n$ are linearly independent.

Definition 4.04 - *Standard Basis*
The *standard basis* for a vector space is the set fewest unit vectors which span it.
Example - $\{\boldsymbol{v}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\}$ are the standard basis for $\mathbb{R}^3$.

Theorem 4.05 - *Basis of a Linear Subspace*
For all elements, $\boldsymbol{v}$, of a linear subspace, $V \subset \mathbb{R}^n$, there exists a unique set of numbers, $\lambda_1, ..., \lambda_n$, such that

$$\boldsymbol{v} = \lambda_1.\boldsymbol{v}_1 + ... + \lambda_n.\boldsymbol{v}_n$$

Theorem 4.06 - *Linear Independence and Bases*
Let $V \subset \mathbb{R}^n$ be a linear subspace with basis $\boldsymbol{v}_1, ..., \boldsymbol{v}_n$. Suppose $\boldsymbol{w}_1, ..., \boldsymbol{w}_k \in V$ are linearly independent, then $k \leq n$.

Definition 4.07 - *Dimension*
Let $V \subset \mathbb{R}^n$ be a linear subspace then the *dimension* of V, $dim(V)$, is the fewest number vectors required to form a basis for V.

4.3 Orthogonal Bases

Definition 4.08 - *Orthogonal*
Let $V \subset \mathbb{R}^n$ be a linear subspace with $\{\boldsymbol{v}_1, ..., \boldsymbol{v}_k\}$ as its basis. This basis is an *orthogonal basis* if it statisfies:

i) $\boldsymbol{v}_i \cdot \boldsymbol{v}_j = 0$ if $i \neq j$;

ii) $\boldsymbol{v}_i \cdot \boldsymbol{v}_i = 1,\ i = 1, ..., k$.

N.B. - This can be generalised to $\boldsymbol{v}_i \cdot \boldsymbol{v}_k = \delta_{ij}$ with $\delta_{ij} := \begin{cases} 1, & i = j \\ 0, & \text{otherwise} \end{cases}$

Theorem 4.09
Let $V \subset \mathbb{R}^n$ be a linear subspace with an orthogonal basis $\{\boldsymbol{v}_1, ..., \boldsymbol{v}_k\}$.
Then for all $\boldsymbol{u} \in V$

$$\boldsymbol{u} = (\boldsymbol{v}_1 \cdot \boldsymbol{u})\boldsymbol{v}_1, ..., (\boldsymbol{v}_k \cdot \boldsymbol{u})\boldsymbol{v}_k$$

5 Linear Maps

Definition 5.01 - *Linear Map*
A map, $T : \mathbb{R}^n \to \mathbb{R}^m$ is a *linear map* if

i) $T(\boldsymbol{x} + \boldsymbol{y}) = T(\boldsymbol{x}) + T(\boldsymbol{y}), \quad \forall\ \boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$;

ii) $T(\lambda \boldsymbol{x}) = \lambda T(\boldsymbol{x}), \quad \forall\ \boldsymbol{x} \in \mathbb{R}^n, \lambda \in \mathbb{R}$.

N.B. - If $m = n$ then T is referred to as a *linear operator*.

Theorem 5.02 - *Properties of Linear Maps*
Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear map. Then $T(\boldsymbol{0}) = \boldsymbol{0}$.

Definiton 5.03 - *Linear Maps as Matrices*
Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear map. Then the associated Matrix is defined as

$$M_T = (t_{ij}) \in M_{m,n}(\mathbb{R})$$

with the elements of M_T defined by

$$t_{ij} = \boldsymbol{e}_i \cdot T(\boldsymbol{e}_j)$$

Theorem 5.04 - *Solutions to Linear Maps from Matrices*
Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear map and M_T be the associated matrix. Then

$$T(\boldsymbol{x}) = M_T \boldsymbol{x}, \quad \forall\ \boldsymbol{x} \in \mathbb{R}^n$$

5.1 Abstract Properties of Linear Maps

Theorem 5.05 - *Relationship between Linear Maps*
Let $S : \mathbb{R}^n \to \mathbb{R}^m$, $T : \mathbb{R}^n \to \mathbb{R}^m$ & $U : \mathbb{R}^m \to \mathbb{R}^k$ be a linear maps and $\lambda \in \mathbb{R}$. Then

i) $(\lambda T)(\boldsymbol{x}) := \lambda T(\boldsymbol{x})$;

ii) $(S+T)(\boldsymbol{x}) = S(\boldsymbol{x}) + T(\boldsymbol{x})$;

iii) $(U \circ S)(\boldsymbol{x}) = U(S(\boldsymbol{x}))$.

Definition 5.06 - *Image & Kernel*
Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear map. Then

i) The *image* of T is defined to be

$$Im(T) := \{\boldsymbol{y} \in \mathbb{R}^m : \exists\ \boldsymbol{x} \in \mathbb{R}^n st\ T(\boldsymbol{x}) = \boldsymbol{y}\}$$

ii) The *kernel* of T is defined to be

$$Ket(T) := \{\boldsymbol{x} \in \mathbb{R}^n : T(\boldsymbol{x}) = \boldsymbol{0}\}$$

Theorem 5.07
Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear map then $Im(T)$ is a linear subsapce of $\mathbb{R}^m$ and $Ker(T)$ is a linear subspaces of $\mathbb{R}^n$

Remark 5.08
Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear map. Then

i) T is surjective if $Im(T) = \mathbb{R}^m$;

ii) T is injective if $Ker(T) = \{0\}$.

5.2 Matrices

Definition 5.09 - *Linear Maps as Matrices*
Let $S : \mathbb{R}^n \to \mathbb{R}^m$, $T : \mathbb{R}^n \to \mathbb{R}^m$ & $U : \mathbb{R}^m \to \mathbb{R}^k$ be a linear maps and $\lambda \in \mathbb{R}$ with M_S, M_T & M_U as the corresponding matrices. Then

i) $M_{\lambda T} = \lambda M_T = (\lambda t_{ij})$;

ii) $M_{S+T} = (s_{ij} + t_{ij}) = M_S + M_T$;

iii) $M_{U \circ S} = (r_{ij})$ where $r_{ik} = \sum_{k=1}^{m} s_{ik} t_{jk}$.

5.3 Rank & Nullity

Defintion 5.10 - *Rank & Nullity*
Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear map. Then we define *Rank* of T by

$$rank(T) := dim(Im(T))$$

and we define *Nullity* of T by

$$nullity(T) := dim(Im(T))$$

N.B. - For all linear maps, $T : \mathbb{R}^n \to \mathbb{R}^m$,

$$nullity(T) + rank(T) = n$$

Remark 5.11
Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear map. Then T is invertible if

i) $rank(T) = n$, or

ii) $nullity(T) = 0$.

Theorem 5.12 - *Relationship of Rank & Nullity between Linear Maps*
Let $S:\mathbb{R}^n\to\mathbb{R}^m$ & $T:\mathbb{R}^k\to\mathbb{R}^n$ be linear maps. Then

i) $S\circ T=0$ iff $Im(T)\subset Ker(S)$;

ii) $rank(S\circ T)\leq rank(T)$ and $rank(S\circ T)\leq rank(S)$;

iii) $nullity(S\circ T)\geq nullity(T)$ and $nullity(S\circ T)\geq nullity(S)+k-n$;

iv) S is invertible then $rank(S\circ T)=rank(T)$ and $nullity(S\circ T)=nullity(T)$.