Introduction to Group Theory - Problem Sheet 4

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February 17, 2018

1) a) Find the orders of all elements of D_{12}, the dihedral group of order 12.

$$D_{12} = \{e, a, a^2, a^3, a^4, a^5, b, ab, a^2b, a^3b, a^4b, a^5b\}$$

$$(a^ib)^2 = (a^ib)(a^ib) = a^iba^ib \quad = \quad a^ia^{-i}bb = ee = e \ \forall \ i \in \mathbb{N}$$

$$\begin{aligned} ord(e) &= 1, & ord(b) &= 2, \\ ord(a) &= 6, & ord(ab) &= 2, \\ ord(a^2) &= 3, & ord(a^2b) &= 2, \\ ord(a^3) &= 2, & ord(a^3b) &= 2 \\ ord(a^4) &= 3, & ord(a^4b) &= 2 \\ ord(a^5) &= 6, & ord(a^5b) &= 2 \end{aligned}$$

b) Let G be the group of symmetries of a cube. Describe, geometrically, an element of G of order 2, and element of order 3, and an element of order 4.

Rotating π *rads* about the x-axis has order 2;
Rotating $\frac{\pi}{2}$ *rads* about x-axis, then $\frac{\pi}{2}$ *rads* about the y-axis has order 3;
Rotating $\frac{\pi}{2}$ *rads* about x-axis has order 4.

2) a) Let G be an **abelian** (multiplicatively written) group with elements x, y with

$$\text{ord}(x)= m < \infty \text{ and } \text{ord}(y)=n < \infty$$

Show that if m and n both divide k, then $(xy)^k = 2$, and deduce that $\text{ord}(xy)$ divides $\text{lcm}(m, n)$.

As $m|k$ and $n|k$
Then $\exists\ a, b \in \mathbb{N}$ st $am = k = bn$

Since $\text{ord}(x) = m$ and $\text{ord}(y) = n$
Then $x^m = e = y^n$

$$\begin{aligned} (xy)^k &= x^k y^k \\ &= a^{am} b^{bn} \text{ as } G \text{ is abelian} \\ &= (x^m)^a (y^n)^b \\ &= e^a e^b \\ &= \underline{e} \end{aligned}$$

WIP

b) By finding an example, show taht what you were asked to prove in a) is not true for **non-abelian** groups.

$$\begin{aligned} (213), (213) &\in S_3 && \text{which is not abelian.} \\ \text{ord}((213)) &= 2 \\ \text{ord}((132)) &= 2 \\ \text{lcm}((2, 2)) &= 2 \\ \\ (213)(132) &= (231) \\ \text{ord}((231)) &= 3 \\ => \text{ord}((132)) &> \text{lcm}(2, 2) \\ => \text{ord}((132)) &\nmid \text{lcm}(2, 2) \end{aligned}$$

3) Let G be a, multiplicatively written, group and $g, x \in G$. Prove that $\text{ord}(x) = \text{ord}(gxg^-1)$

Let $n \in \mathbb{N}$ st it is the lowest value where $(gxg^{-1})^n = e$

$$\begin{aligned} &=> \quad (gxg^{-1})(gxg^{-1})...(gxg^{-1}) = e \\ &=> \quad gx(g^{-1}g)x(g^{-1}g)...(g^{-1}g)xg^{-1} = e \\ &=> \quad gx^n g^{-1} = e \\ &=> \quad g^{-1}gx^n g^{-1} g = g^{-1} e g \\ &=> \quad ex^n e = g^{-1} g \\ &=> \quad x^n = e \end{aligned}$$

4) Find all the cyclic subgroups of D_{12}

$$D_{12} = \{e, a, a^2, a^3, a^4, a^5, b, ab, a^2b, a^3b, a^4b, a^5b\}$$
$$\langle e\rangle, \langle a\rangle, \langle a^2\rangle, \langle a^3\rangle, \langle a^4\rangle, \langle a^5\rangle, \langle b\rangle, \langle ab\rangle, \langle a^2b\rangle, \langle a^3b\rangle, \langle a^4b\rangle, \langle a^5b\rangle \leq D_{12}$$

5) Let G be the group of permutations of $\{1, 2, 3, 4, 5\}$. Find an element of G with order 6.

$$\begin{aligned}
\text{Let } I &= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 1 & 5 & 4 \end{pmatrix} \\
e &= (12345) \\
I &= (31254) \\
I^2 &= (23145) \\
I^3 &= (12354) \\
I^4 &= (31245) \\
I^5 &= (23154) \\
I^6 &= (12345) = e \\
&=> \underline{\text{ord}(I) = 6}
\end{aligned}$$

6) Show that the group $(\mathbb{Q}, +)$ of rational numbers under addition is not cyclic.

Suppose $x \in \mathbb{Q}$ st $\langle x\rangle = G$

As $x \in \mathbb{Q}$ then $\exists$ st $a \in \mathbb{Z}, b \in \mathbb{N}$ st $x = \dfrac{a}{b}$

Thus $\dfrac{x}{2} = \dfrac{a}{2b} \in \mathbb{Q}$

$\langle x\rangle = \langle \frac{a}{b}\rangle$

$= \left\{..., \frac{a}{b} - 2.\frac{a}{b}, \frac{a}{b} - \frac{a}{b}, \frac{a}{b}, \frac{a}{b} + \frac{a}{b}, ...\right\}$

$= \left\{..., -\frac{a}{b}, 0, \frac{a}{b}, \frac{2a}{b}, ...\right\}$

As $\dfrac{a}{2b} \notin \langle \frac{a}{b}\rangle$

Then $\dfrac{x}{2} \notin \langle x\rangle$

So $\langle x\rangle$ does not contain all rationals thus $\langle x\rangle \neq G \; \forall \; x \in \mathbb{Q}$