# Calculus 1 - Notes

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## Contents

# 1 Before Calculus

## 1.1 Fundamental Theorem of Calculus

**Definition 1.01 -** *Fundamental Theorem of Calculus*
The Fundamental Theorem of Calculus states

$$\frac{d}{dx}\int_a^x f(t)dt = f(x)$$

**Definition 1.02 -** *Common Sets of Numbers*
Natural Numbers, set of positive integers - $\mathbb{N} := \{1, 2, 3, ...\}$.
Whole Numbers, set of all integers - $\mathbb{Z} := \{..., -2, -1, 0, 1, 2, ...\}$.
Rational Numbers, set of fractions - $\mathbb{Q} := \left\{\frac{p}{q} : p \in \mathbb{Z}, q \in \mathbb{N}\right\}$.
Real Numbers, set of all rational & irrational numbers - $\mathbb{R}$.

## 1.2 Intervals

**Definition 1.03 -** *Intervals*
Sets of real numbers that fulfil in given ranges.
Notation

$$\begin{aligned}
[a, b] &:= \{x \in \mathbb{R} : a \leq x \leq b\} \\
(a, b] &:= \{x \in \mathbb{R} : a < x \leq b\} \\
[a, b) &:= \{x \in \mathbb{R} : a \leq x < b\} \\
(a, b) &:= \{x \in \mathbb{R} : a < x < b\}
\end{aligned}$$

Example
In what interval does x lie such that:

$$|3x + 4| < |2x - 1|$$

*Solution*

$$\begin{aligned}
\text{Case } 1 : x \geq \frac{1}{2}& \\
&=> 1 - 2x < 3x + 4 < 2x - 1 \\
&=> 1 - 2x < 3x + 4 \\
&=> x > \frac{-3}{5} \\
\\
\text{And, } &=> 3x + 4 < 2x - 1 \\
&=> x < -5
\end{aligned}$$

There are no real solutions in this range.

$$\begin{aligned}
\text{Case } 2 : x < \frac{1}{2}& \\
&=> 2x - 1 < 3x + 4 < 1 - 2x \\
&=> 2x - 1 < 3x + 4 \\
&=> -5 < x
\end{aligned}$$

$$\begin{aligned}\text{And,} => 3x+4 &< 1-2x\\ => 5x &< -3\\ => x &< \frac{-3}{5}\end{aligned}$$

$$=> -5 < x < \frac{-3}{5},\ \underline{x \in \left(-5, \frac{-3}{5}\right)}$$

**Definition 1.04 -** *Functions*
Functions map values between fields of numbers. The signature of a function is defined by

$$f : A \rightarrow B$$

Where $f$ is the name of the function, $A$ is the domain and $B$ is the co-domain.
The *Domain* of a function is the set of numbers it can take as an input.
The *Co-Domain* is the set of numbers that the domain is mapped to.

N.B. - A function is valid iff it maps each value in the domain to a single value in the co-domain.

**Definition 1.05 -** *Maximal Domain*
The *Maximal Domain* of a function is the largest set of values which can serve as the domain of a function.

**Remark 1.06 -** *Types of Function*
Let $f : A \rightarrow B$
*Polynomials*

$$f(x) = a_0 + a_1x + ... + a_nx^n$$

*Rational*

$$f(x) = \frac{p(x)}{q(x)},\quad q(x) \neq 0\ \forall\ x \in A$$

*Trigonometric*

$$sin(x),\ cos(x),\ tan(x)\text{ etc.}$$

# 2 Limits

## 2.1 Limits

**Definition 2.01 -** *Limits*
A limit is the value a function tends to, for a given x.
*i.e.* The value f(x) has at it gets very close to x.

*Formally* We say $L$ is the limit of $f(x)$ as $x$ tends to $x_0$ if

$$\forall\ \varepsilon > 0,\ \exists\ \delta > 0 \text{ st if } x \in A \text{ and } |x - x_0| < \delta => |f(x) - L| < \varepsilon$$

*Notation*

$$\lim_{x \to x_0} f(x) = L$$

**Definition 2.02 -** *Directional Limits*
Sometimes the value of a limit depends on which direction you approach it from.
$\lim_{x\to x_0+}$ is used when approaching from values greater than $x_0$.
$\lim_{x\to x_0-}$ is used when approaching from values less than $x_0$.

**Theorem 2.03 -** *Operations with limits*
Let $\lim_{x\to x_0} f(x) = L_f$ and $\lim_{x\to x_0} g(x) = L_g$ Then

$$\begin{aligned}
\lim_{x\to x_0} [f(x) + g(x)] &= L_f + L_g \\
\lim_{x\to x_0} f(x).g(x) &= L_f.L_g \\
\lim_{x\to x_0} \frac{f(x)}{g(x)} &= \frac{L_f}{L_g} \quad L_g \neq 0
\end{aligned}$$

## 2.2 Exponential Function

**Definition 2.04 -** *Exponential Function*

$$e := \lim_{x\to\infty} \left(1 + \frac{1}{n}\right)^n \simeq 2.7182818...$$

**Theorem 2.05 -** *Binomial Expansion*
A techique for expanding binomial expressions

$$\begin{aligned}
\left(1 + \frac{x}{n}\right)^n &= \sum_{i=0}^{n} \binom{i}{n} .1^{(n-i)}.\left(\frac{x}{n}\right)^i \\
&= 1 + x + \frac{n-1}{2n}.x^2 + ... + \frac{x^n}{n^n}
\end{aligned}$$

# 3 The Derivative

**Definition 3.01 -** *Differentiable Equations*
Let $f : A \to B$ and $x_0 \in A$.
$f$ is differentiable at $x_0$ if $\exists\ L \in B$ such that

$$L = \lim_{h\to 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

If this limit exists $\forall\ x \in A$ then we can define the derivative of $f(x)$

$$f'(x) := \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

**Definition 3.02 -** *Notation for Differentiation*
There are two ways to denote the derivative of an equation

$$f'(x) \iff \frac{df}{dx}, f''(x) \iff \frac{d^2f}{dx^2}, ..., f^{(n)}(x) \iff \frac{d^nf}{dx^n}$$

N.B. - Using $\dfrac{df}{dx}$ is more informative, especially for equations with multiple variables.

### 3.1 Techniques for finding derivative

**Theorem 3.03 -** *Sum Rule*
Let $f, g$ be differentiable with respect to x.

$$(f+g)' = f' + g'$$

**Theorem 3.04 -** *Product Rule*
Let $f, g$ be differentiable with respect to x.

$$(fg)' = f'g + fg'$$

**Theorem 3.05 -** *Quotient Rule*
Let $f, g$ be differentiable with respect to x.

$$\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$$

**Definition 3.06 -** *Composite Functions*
Let $f : B \to C$ and $g : A \to B$ Then

$$(f \circ g)(x) = f(g(x))$$

**Theorem 3.07 -** *Chain Rule*
Let $f, g$ be differentiable with respect to x.

$$\frac{d}{dx} f(g(x)) = f'(g(x)).g'(x)$$

### 3.2 Implicit Differentiation

**Definition 3.08 -** *Implicit Differentiation*
Sometimes it is hard to isolate variables in multi-variable equations, in these cases differentiate both sides with respect to the same variable.
Remembering

$$\frac{d}{dx}(x) = 1 \text{ and } \frac{d}{dx}(y) = \frac{dy}{dx} = y'$$

*Example*
Find $y$ if $x^3 + y^3 = 6xy$

$$\begin{aligned}
\frac{d}{dx}\left(x^3 + y^3\right) &= \frac{d}{dx}(6xy) \\
=> 3x^2 + 3y^2.y' &= 6y + 6x.y' \\
=> y'(3y^2 - 6x) &= 6y - 3x^2 \\
=> y' \qquad &= \frac{2y - x^2}{y^2 - 2x}
\end{aligned}$$

## 3.3 Applications of The Derivative

**Thoerem 3.09 -** *Netwon's Method*
Let $f$ be differentiable. Using *Newton's Method* we can approximate a solution to $f(x) = 0$.

i) Take an inital guess, $x_0$;

ii) Find the value of $x$ where the tangent to $x_0$ on $f(x)$ intercepts the x-axis;

iii) Use this value as the next guess;

iv) Repeat until the value of $x$ reduces little.

The equation for the tangent is

$$y = f(x_0) + (x - x_0)f'(x_0)$$

so a simplified equation for the process can be deduced

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

**Theorem 3.10 -** *Angle between Intersecting Curves*
Let $y = f_1(x)$ and $y = f_2(x)$ be two curves which intersect at $(x_0, y_0)$.
Then $y_0 = f_1(x_0) = f_2(x_0)$
Let $m_1, m_2$ be the gradient of the tangents to $f_1$ & $f_2$ at $x_0$.
Then $\theta_i := tan^{-1}(m_i)$ for $i = 1, 2$.
Let $\phi = |\theta_1 - \theta_2|$, then

$$\phi = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$$

**Theorem 3.11 -** *L'Hospital's Rule*
For two equations equations, $f, g$ with limit of $-\infty, 0$ or $\infty$ as $x$ tends to $a$, it is hard to solve

$$\lim_{x\to a} \frac{f(x)}{g(x)}$$

Provided the limit exists, L'Hospital's Rule states that

$$\lim_{x\to a} \frac{f(x)}{g(x)} \iff \lim_{x\to a} \frac{f'(x)}{g'(x)}$$

## 3.4 Sketching Curves

**Remark 3.11 -** *Sketching Curves*
Evaluating the derivative of a curve can make it easier to sketch:

i) When $f'(x) > 0$ the curve is heading upwards;

ii) When $f'(x) < 0$ the curve is heading downwards;

iii) When $f'(x) = 0$ the curve is flat;

iv) When $f'(x) = \infty, -\infty$ there are assymptotes.

**Definition 3.12 -** *Even Functions*
If $f(x) = f(-x)$ then the function is symmetrical and said to be *even*.
*Examples* - $x^2, cos(x), |x|$

**Definition 3.13 -** *Odd Functions*
If $f(x) = -f(-x)$ then the function is said to be *odd*.
*Examples* - $x, sin(x), x.cos(x)$

**Remark 3.14**
Some functions are neither *odd* nor *even*.
*Example* - $x + x^2$

# 4 Inegration

## 4.1 The Primitive

**Definition 4.01 -** *The Primitive*
A function, $F : A \to \mathbb{R}$, is a primative for the function $f : A \to \mathbb{R}$ if $F$ is differentiable and

$$\frac{d}{dx}F = f$$

N.B. - Primitives are also called *Indefinite Integral* or *Anti-Derivative*.

**Remark 4.02 -** *Area Under a Curve*
Let $f : [a, b] \to \mathbb{R}$ be a continuous function. Then the area between the curve and the x-axis is found by integration.

$$A := \int_a^b f(x)dx$$

**Definition 4.03 -** *Convergent Improper Integrals*
Let $b > a$ and define a function, $f : [a, \infty) \to \mathbb{R}$, which is continuous in $[a, b]$ Then

$$\int_0^\infty f(x)dx := \lim_{b\to\infty} \int_a^b f(x)dx$$

If this limit exists then the improper integral is *convergent*, otherwise it is *divergent*.

**Definition 4.04 -** *Definite Integral*
Let $F$ be the primative for the function $f$. Then

$$\int_b^a f(x)dx = F(a) - F(b)$$

Notation - $F(x)\big|_a^b = F(b) - F(a)$

**Remark 4.05 -** *Summing Definite Inegrals*
For all $a < c < b$

$$\int_a^b f(x)dx = \int_c^a f(x)dx + \int_b^c f(x)dx$$

$$\int_b^a f(x)d := -\int_a^b f(x)dx$$

**Theorem 4.06 -** *Taylor Series*
Functions can be expanded into polynomial form with degree $n$, $T_n$, and remainder $R_n$ such that $f(x) = T_n(x) + R_n(x)$.

$$T_n(x) = f(a) + (x-a)f'(a) + ... + \frac{1}{n}.(x-a)^n.f^n(a)$$

$$R_n(x) = \frac{1}{n}\int_a^x (x-t)^n.f^{(n+1)}(t)dt$$

# 5 Parametric Curves & Arc-Length

## 5.1 Parametric Curves

**Definition 5.01 -** *Parametric Curves*
*Parametric equations* are an alternative to Cartesian equations, for representing curves. They can also represent a point in 3D space.

$$\boldsymbol{p} = \begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix}$$

**Theorem 5.02 -** *Parametric to Cartesian Equations*
As all equations in a Parametric series have a common variable, substition can be used to form a single equation.
*Example* Let $\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} t-2 \\ \frac{t}{t-2} \end{pmatrix}$.

$$\begin{aligned} x &= t-2 \\ => t &= x+2 \\ => y &= \frac{x+2}{(x+2)-2} \\ &= \frac{x+2}{x} \\ y &= \underline{1 + \frac{2}{x}} \end{aligned}$$

## 5.2 Tangent of a Curve

**Theorem 5.02 -** *Tangent to a Parametric Curve*
Let $(x(t), y(t))$ be a parametric equation. If we want to find the tangent at a point on the line, $(a, b)$, we need to find the value $t_0$ such that $x(t_0) = a$ & $y(t_0) = b$.
Then by using the chain rule we can deduce the following equation for the tangent when $t = t_0$:

$$\frac{dy(t_0)}{dx(t_0)} = \frac{y - y(t_0)}{x - x(t_0)}$$

. Similarly we can deduce the equation for the normal when $t = t_0$:

$$-\frac{dx(t_0)}{dy(t_0)} = \frac{y - y(t_0)}{x - x(t_0)}$$

### 5.3 Arc-Length

**Theorem 5.03 -** *Arc-Length*
Arc-Length is the length of a curve, following a function, between two points. For a cartesian equation, $y = f(x)$, between the points $x$ and $x + dx$ is

$$ds = \sqrt{dx^2 + dy^2}$$

So for a set of parametric equations, $(x(t), y(t)),\ a \leq t \leq b$,

$$ds = \sqrt{\frac{dx^2}{dt} + \frac{dy^2}{dt}}$$

To find the length of a curve between points $a$ and $b$

$$s = \int_a^b \sqrt{\frac{dx^2}{dt} + \frac{dy^2}{dt}}dt$$

**Definition 5.04 -** *Curvature*
Curvature measures how fast the unit tangent vector to a curve rotates. Curvature of a curve, $y = f(x)$, can be found using the equation:

$$K(x) = \frac{|y''(x)|}{[1 + (y'(x))^2]^{\frac{3}{2}}}$$

For a set of parametric equations, $(x(t), y(t))$, it can be found using:

$$K(t_0) = \frac{y''(t_0).x'(t_0) - y'(t_0).x''(t_0)}{[(x'(t_0))^2 + (y'(t_0))^2]^{\frac{3}{2}}}$$

### 5.4 Level Curves

**Definition 5.05 -** *Level Curves*
Let $f : \mathbb{R}^d \to \mathbb{R}$ be a function with $d \geq 2,\ d \in \mathbb{N}$. A level curve for $f$ is the set of real solutions for $f(\boldsymbol{x}) = c, c \in \mathbb{R}$.
<u>N.B</u> - $f(\boldsymbol{x}) = c$ is often written as $f = c$.

## 6 Differential Equations

**Definition 6.01 -** *Differential Equations*
Differential equations take the form

$$f(x, y, \frac{dx}{dy}, ..., \frac{d^{(n)}y}{dx^{(n)}}) = 0,\ x \in I$$

### 6.1 First Order Differential Equations

**Definition 6.02 -** *First Order*
First order differential equations are equations of form $f(x, y, \frac{dx}{dy}) = 0$.

**Definition 6.03 -** *Seperable Equations*
An equation, $f$, is said to be seperable if there exists two equations, $M(x),\ N(y)$, such that

$$f(x, y, y') = y' - M(x).N(y)$$

Thus

$$
\begin{aligned}
y' &= M(x).N(y) \\
=> \frac{y'}{N(y)} &= M(x) \\
=> \int \frac{1}{N(y)} dy &= \int M(x) dx
\end{aligned}
$$

After integration, the equation can be rearranged to be in terms of y.

## 6.2 Integrating Factor

**Theorem 6.04 -** *Integrating Factor*
Consider the equation $y' + f(x)y + g(x)$. Let $F(x) = \int f(x)dx$. Thus

$$
\begin{aligned}
e^{F(x)}.y' + e^{F(x)}.y &= e^{F(x)}.g(x) \\
=> \frac{d}{dx}\left(e^{F(x)}.y\right) &= e^{F(x)}.g(x) \\
=> e^{F(x)}.y &= \int e^{F(x)}.g(x)\ dx \\
=> y &= e^{-F(x)} \int e^{F(x)}.g(x)\ dx
\end{aligned}
$$

## 6.3 Second Order Differential Equations

**Definition 6.05 -** *Linear Differential Equations*
A differential equation is said to be *linear* if it can be written in the form

$$Ay(x) := a_n(x).y^{(n)}(x) + ... + a_1(x).y'(x) + a_0(x).y(x) = b(x)$$

We define the set of solutions as

$$S(A, b) := y : I \to \mathbb{R}; Ay = b$$

If the only solution is $b = 0$ then the system is homogenous, otherwise it is inhomogenous.

**Definition 6.06 -** *Particular & Complimentary Solutions*
When solving a differential equation, $Ay(x) = b(x)$, we need to find two functions in order to find the final solution.

i) Complementary Function, $y_c$ - The homogenous case of the equation, $Ay(x) = 0$;

ii) Particular Function, $y_p$ - The inhomogenous case of the equation, $Ay(x) = b(x)$ for a given $b(x)$.

Then $y = y_c + y_p$ is the final solution for $Ay(x) = b(x)$.

**Theorem 6.07 -** *Complementary Functon of LDEs with Constant Coefficients*
Take a linear differential equation

$$a_n.y^{(n)}(x) + ... + a_1.y'(x) + a_0.y(x) = b(x)$$

where $a_n, ..., a_1, a_0 \in \mathbb{R}$ & $b(x) : \mathbb{R} \to \mathbb{R}$ are all constant.
To find the *Complementary Function* we solve the equation

$$a_n.\lambda^n + ... + a_1.\lambda^n + a_0 = 0$$

. to get solutions $\lambda_1, ..., \lambda_k$ and then produce the complimentary function

$$y_c(x) = \mu_1 e^{\lambda_1 x} + ... + \mu_k e^{\lambda_k x}$$

Where $\mu_1, ..., \mu_k$ are constants to be found later, by comparing with $b(x)$.
**Remark 6.08 -** *Complementary Function*
The complementary function, $y_c$, for differential equations with constant coefficients depends upon the $\lambda_1, ..., \lambda_k$ we find, due to Euler's Formula.

i) $\lambda_i = c, \quad y_{c_i} = \mu_i e^{\lambda_i x}$;

ii) $\lambda_i = \pm ib, \quad y_{c_i} = \mu_{i_1} cos(bx) + \mu_{i_2} sin(bx)$;

iii) $\lambda_i = a \pm ib, \quad y_{c_i} = e^{ax}[\mu_{i_1} cos(bx) + \mu_{i_2} sin(bx)]$.

Then $y_c = \sum_{j=1}^{k} y_{c_j}$.

**Remark 6.09 -** *Particular Function*
The particular function, $y_p$, for a differential equation with constand coefficients, $Ay(x) = b(x)$, depends on the form of $b(x)$.

i) $b(x) = a_n x^n + ... + a_1 x + a_0, \quad y_p = b_n x^n + ... + b_1 x + b_0$;

ii) $b(x) = ae^{bx}, \quad y_p = \alpha e^{\beta x}$;

iii) $b(x) = a.sin(bx) + c.cos(dx), \quad y_p = \alpha sin(\beta x) + \gamma cos(\delta x)$.

Where the constants of $y_p$ are values to be found, when given certain conditions.
**Theorem 6.10 -** *Particular Functon of LDEs with Constant Coefficients*
Take a linear differential equation

$$a_n.y^{(n)}(x) + ... + a_1.y'(x) + a_0.y(x) = b(x)$$

where $a_n, ..., a_1, a_0 \in \mathbb{R}$ & $b(x) : \mathbb{R} \to \mathbb{R}$ are all constant.
Deduce the particular function for the differential equation, given $b(x)$, and then differentiate $y_p$ n times.
Substitute in these values, in place of the $y$s, into the original equation and solve to find values for the constants in $y_p$.

<u>Example</u>
Solve $y'' - y' + y = x^2$.

2**Complementary Function**

$$\text{Let } \lambda^2 - \lambda + 1 = 0$$

$$=> \lambda = \frac{1 \pm \sqrt{1-4}}{2}$$

$$= \frac{1}{2} \pm i\frac{\sqrt{3}}{2}$$
$$=> y_c = e^{\frac{x}{2}}[Acos(x\frac{\sqrt{3}}{2}) + Bsin(\frac{\sqrt{3}}{2})]$$

**Particular Function**

$$\text{Let } y_p = \alpha x^2 + \beta x + \gamma$$
$$=> y_p'(x) = 2\alpha x + \beta, \&$$
$$=> y_p''(x) = 2\alpha$$
$$=> (2\alpha) - (2\alpha x + \beta) + (\alpha x^2 + \beta x + \gamma) = x^2$$
$$=> x^2[\alpha] + x[\beta - 2\alpha] + [2\alpha - \beta + \gamma] = x^2$$

$$[x^2] : \alpha = 1$$
$$[x] : \beta - 2\alpha = 0$$
$$=> \beta = 2$$
$$[x^0] : 2\alpha + \gamma - \beta = 0$$
$$=> \gamma = 0$$
$$=> y_p = x^2 + 2x$$
$$=> \underline{y = x^2 + 2x + e^{\frac{x}{2}}[Acos(x\frac{\sqrt{3}}{2}) + Bsin(\frac{\sqrt{3}}{2})]}$$

### 6.4 Wronskian

**Definition 6.11 -** *The Wronskian*
The *Wronskian*, $W[y_1, y_2]$, of two differentiable functions is defined by

$$W[y_1, y_2](x) = y_1(x).y_2'(x) - y_1'(x).y_2(x)$$

The notation for this is

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} := ad - bc$$

So

$$W[y_1, y_2] = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$

**Remark 6.12**
If $W[y_1, y_2] \neq 0$ then $y_1, y_2$ are linearly independent.

### 6.5 Variation of Constants

**Theorem 6.13**
This is a technique for solving all differential equations, not just ones with constant coefficients, and assumes we know the complementary function, $y_c$.
Consider the equation

$$Ay(x) := y''(x) + \beta(x)y'(x) + \gamma(x)y(x) = b(x), \quad \text{for a known } b(x)$$

Suppose we have a complementary function in the form

$$y_c = \lambda_1(x)y_1(x) + \lambda_2(x)y_2(x)$$

where $y_1$ & $y_2$ are linearly independent, thus $W[y_1, y_2] \neq 0$.
Then
$$y_p' = \lambda_1' y_1 + \lambda_1 y_1' + \lambda_2' y_2 + \lambda_2 y_2'$$
As $\lambda_1, \lambda_2$ are constant then $\lambda_1' = \lambda_2' = 0$ so
$$y_p' = \lambda_1 y_1' + \lambda_2 y_2'$$
By differentiating and then substituting back into the original equation we see $y_p$ is a solution iff
$$\lambda_1' y_1' + \lambda_2' y_2' = f$$
In matrix form we have
$$\begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix} \begin{pmatrix} \lambda_1' \\ \lambda_2' \end{pmatrix} = \begin{pmatrix} 0 \\ f \end{pmatrix}$$
Then by Cramer's rule we have
$$\lambda_1' = \frac{\begin{vmatrix} 0 & y_2 \\ f & y_2' \end{vmatrix}}{W[y_1 \ y_2]} = \frac{-y_2 f}{W[y_1 \ y_2]}$$
and
$$\lambda_2' = \frac{\begin{vmatrix} y_1 & 0 \\ y_1' & f \end{vmatrix}}{W[y_1 \ y_2]} = \frac{y_1 f}{W[y_1 \ y_2]}$$
Giving use a solution for $y_p' = \lambda_1' y_1 + \lambda_1 y_1' + \lambda_2' y_2 + \lambda_2 y_2'$.

# 7 Applied Differential Equations

**Definition 7.01 -** *Denoting Limit Relationships*
We use
$$F(x) \sim G(x) \text{ as } x \to a$$
to denote
$$\lim_{x \to a} \frac{F(x)}{G(x)} = 1$$
**Theorem 7.02 -** *Vibrating String*
If we are given a string which is $L$ long then we can define an equation, $y(x,t)$, which describe the displacement of a point $x$ along the string, at time $t$.
$$y(x,t) = u(x)e^{i\omega t}$$
Where $\frac{\omega}{2\pi}$ is the frequency of the string and $u(x) = Acos(\omega x) + Bsin(\omega x)$.
We can generalise this for strings with $n$ anti-nodes.
$$\omega_n := \frac{n\pi}{L}, \quad u_n := sin(\omega_n x)$$

# 8 Liner Difference Equations

**Definition 8.01 -** *Difference Equations*
A difference equation is an equation of the form
$$F(n, y_n, ..., y_{n+d}) = 0, \quad n, d \in \mathbb{N}$$

## 8.1 First-Order Linear Difference Equation

**Definition 8.02 -** *Liner First-Order Difference Equations*
A *Linear First-Order Difference Equation* is an equation, $F$, which can be described by

$$F(n, y_n, y_{n+1}) = a_n y_{n+1} + b_n y_n - f_n$$

where $a_n, b_n, f_n$ are all known sequences.

**Example 8.03**
By taking a simple equation

$$y_{n+1} - y_n = f_n$$

we can see that

$$y_{n+1} = y_n + (y_{n+1} - y_n) = y_n + f_n = ... = y_{n_0} + f_{n_0} + ... + f_{n-1} + f_n$$

So

$$y_n = y_{n_0} + \sum_{j=n_0}^{n-1} f_j$$

**Theorem 8.04 -** *Solving First-Order Liner Difference Equations*
From *Definition 8.02* we can generalise the equation to show that

$$y_{n+1} + b_n y_n = f_n$$

Then

$$\frac{-1}{b_n} y_{n+1} - y_n = \frac{-1}{b_n} f_n$$

We now define the *Summing Factor*, $S_n$, as

$$S_n := \prod_{j=n_0}^{n-1} \frac{-1}{b_n}$$

. We multiply both sides of the original equation by the summing factor and as $S_n(\frac{-1}{b_n}) = S_{n+1}$ we get

$$S_{n+1} y_{n+1} - S_n y_n = S_{n+1} f_n$$

As this has the same form as the example in *8.03* we can now deduce

$$S_n y_n = y_{n_0} + \sum_{j=n_0}^{n-1} S_{j+1}.f_j$$

## 8.2 Second-Order Linear Difference Equation

**Definition 8.05 -** *Second-Order Linear Difference Equation*
A *Linear Second-Order Difference Equation* is an equation, $F$, which can be described by

$$a_n y_{n+2} + b_n y_{n+1} + c_n y_n = f_n$$

where $a_n, b_n, c_n, f_n$ are know sequences.

**Remark 8.06 -** *Solving Second-Order Linear Difference Equations*
Similar to solving second-order differential equations we need to consider to cases. The *homogenous* & *inhomogenous* cases. So two sequences will be found the complementary sequence, $y_n^c$, and the particular sequence, $y_n^p$. The final solution for $y_n$ is given by

$$y_n = y_n^c + y_n^p$$

**Definition 8.07 -** *Wronskian of Sequences*
For two sequences $u_n \& v_n$ we define the Wronskian to be

$$W_n := \begin{vmatrix} u_n & v_n \\ u_{n+1} & v_{n+1} \end{vmatrix} = u_n.v_{n+1} - v_n.u_{n+1}$$

**Theorem 8.08 -** *Homogenous Case with Constant Cofficients*
Take the equation

$$ay_{n+2} + by_{n+1} + cy_n = 0$$

where $a, b, c$ are known constants. We look for solutions of the form

$$y_n = \lambda^n$$

By substition we get the equation $a\lambda^2 + b\lambda + c = 0$. By solving for $\lambda$ we find a solution

i) $\lambda$ has two real solutions - $y_n = A\lambda_1^n + B\lambda_2^n$;

ii) $\lambda$ has one real solution - $y_n = (An + B)\lambda^n$;

iii) $\lambda$ has only an imaginamry solution - $y_n = \Lambda e^{i\theta}, \quad \Lambda^2 := \frac{c}{a}, \theta := tan^{-1}(\frac{4ac-b^2}{-b})$.

**Theorem 8.09 -** *Homogenous Case Second-Order Linear Difference Equations*
The homogenous case finds solutions for

$$a_n y_{n+2} + b_n y_{n+1} + c_n y_n = 0$$

Suppose $y_n = u_n$ and $y_n = v_n$ are solutions to this homogenous equation. Then

$$W_n[u_n \ v_n] = W_{n_0} \prod_{j=n_0}^{n-1} \frac{c_j}{a_j}$$

So

$$u_n.v_{n+1} - v_n.u_{n+1} = W_{n_0} \prod_{j=n_0}^{n-1} \frac{c_j}{a_j}$$

Which can be rearranged to be in the form of a first order difference equation, such as *Example 8.03*

$$\frac{u_n}{u_{n+1}} v_{n+1} - v_n = W_{n_0} \prod_{j=n_0}^{n-1} \frac{c_j}{a_j}$$

Which has a summing factor

$$S_n = \frac{1}{u_n}$$

By multiplying both sides by $S_n$ and simplifying we get

$$v_n = u_n \sum_{j=n_0}^{n-1} \frac{1}{u_j.u_{j+1}} \prod_{k=n_0}^{n-1} \frac{c_k}{a_k}$$

Typically you need to solve the product part of the equation to get a result for the sequence $u_n$.

**Remark 8.10 -** *Inomogenous Case Second-Order Linear Difference Equations*
Generally the best way to do this is to make an educated guess based on the right hand side of the equation. So if the RHS is a polynomial, guess a polynomial, etc. Similar to solving differential equations.

# 9 Several Variables - Differentiability

**Definition 9.1 -** *Several Variable Function*
Let $d \in \mathbb{N}, A \subset \mathbb{R}^d$ & $B \subset \mathbb{R}^d$.
A function $\boldsymbol{f} : A \to B$ is a map which, for all $\boldsymbol{x} \in A$, assigns a unique value $\boldsymbol{f}(\boldsymbol{x}) \in B$.

**Defintion 9.2 -** *Linear Functions*
A function $\boldsymbol{f} : \mathbb{R}^d \to \mathbb{R}^n$ is *linear* if it can be given in terms of a matrix $A \in M_{n,d}(\mathbb{R})$ where

$$\boldsymbol{f}(\boldsymbol{x}) = A\boldsymbol{x}$$

**Theorem 9.3 -** *Properties of Liner Functions*
If a function is linear then the following are true:

i) $\boldsymbol{f}(\lambda\boldsymbol{x}) = \lambda\boldsymbol{f}(\boldsymbol{x})$;

ii) $\boldsymbol{f}(\boldsymbol{x}+\boldsymbol{y}) = \boldsymbol{f}(\boldsymbol{x}) + \boldsymbol{f}(\boldsymbol{y})$.

**Definition 9.4 -** *Continuous Several Variable Function*
Let $\boldsymbol{f} : A \to B$ and $\boldsymbol{a} \in A$. Then $\boldsymbol{f}$ is continuous if

$$\lim_{\boldsymbol{x}\to\boldsymbol{a}} \boldsymbol{f}(\boldsymbol{x}) = \boldsymbol{f}(\boldsymbol{a})$$

## 9.1 The Derivative

**Defintion 9.5 -** *Norm of a Vector*
The norm of a vector, $\boldsymbol{x} \in \mathbb{R}^d$ is

$$||\boldsymbol{x}|| := \left(\sum_{j=1}^{d} x_j\right)^{1/2}$$

**Definition 9.6 -** *Derivative of Several Variable Function*
A function, $\boldsymbol{f} : \mathbb{R}^d \to \mathbb{R}^n$, is said to be differentiable at the point $\boldsymbol{x} \in \mathbb{R}^d$ if the exists an $A \in M_{n,d}(\mathbb{R})$ such that

$$\lim_{\boldsymbol{h}\to\boldsymbol{0}} \frac{||\boldsymbol{f}(\boldsymbol{x}+\boldsymbol{h}) - \boldsymbol{f}(\boldsymbol{x}) - A\boldsymbol{h}||}{||\boldsymbol{h}||} = 0$$

This $A$ is the derivative of $\boldsymbol{f}(\boldsymbol{x})$.

$$\boldsymbol{f}'(\boldsymbol{x}) := A$$

**Remark 9.7**
If consider the following several variable function

$$\boldsymbol{f} = \begin{pmatrix} f_1 \\ \vdots \\ f_n \end{pmatrix}$$

Then

$$\boldsymbol{f}' = \begin{pmatrix} f_1' \\ \vdots \\ f_n' \end{pmatrix}$$

# 10 Directional & Partial Derivatives

## 10.1 Directional Derivative

**Definition 10.1 -** *Direction*
A direction in $\mathbb{R}^d$ is a vector of unit length.
In $\mathbb{R}^2$ every direction can be given by $\boldsymbol{u} = \begin{pmatrix} cos(\theta) \\ sin(\theta) \end{pmatrix}$, where $\theta$ is the angle from positive $x$ axis
In $\mathbb{R}^3$ every direction can be given by $\boldsymbol{u} = \begin{pmatrix} sin(\phi)cos(\theta) \\ sin(\phi)sin(\theta) \\ cos(\phi) \end{pmatrix}$, where $\phi$ is the angle from positive $z$ axis and $\theta$ is the angle from the positive $x$ axis.
**Definition 10.2 -** *Spherical Co-ordinates*
The spherical co-ordinates describe points in three dimension space.
The distance of a point from the origin is

$$r = \rho sin(\phi)$$

Then

$$\boldsymbol{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} rcos(\theta) \\ rsin(\theta) \\ \rho cos(\phi) \end{pmatrix} = \begin{pmatrix} \rho sin(\phi)cos(\theta) \\ \rho sin(\phi)sin(\theta) \\ \rho cos(\phi) \end{pmatrix}$$

**Defintion 10.3 -** *Direction Derivative*
The direction derivative of $\boldsymbol{f}$ in the direction of $\boldsymbol{u}$ at the point $\boldsymbol{x}_0$ is the vector:

$$D_{\boldsymbol{u}}\boldsymbol{f}(\boldsymbol{x}_0) := \frac{d}{dt}\boldsymbol{f}(\boldsymbol{x}_0 + t\boldsymbol{u}) \Big|_{t=0}$$

**Theorem 10.4**
For all $\boldsymbol{f} : \mathbb{R}^d \to \mathbb{R}^n$ the directional derivative at $\boldsymbol{u}$ in $\mathbb{R}^d$ we have

$$D_{\boldsymbol{u}}\boldsymbol{f}(\boldsymbol{x}) = \boldsymbol{f}'(\boldsymbol{x}).\boldsymbol{u}$$

## 10.2 Partial Derivative

**Defintion 10.5 -** *Partial Derivative*
Let $f : \mathbb{R}^d \to \mathbb{R}$. Then the direction derivative, $D_{\boldsymbol{e}_j}f(\boldsymbol{x})$, if it exists, is called the *partial derivative* of $f$ with respect to $x_j$ at $\boldsymbol{x}$.
This is denoted by

$$\frac{\partial f}{\partial x_j}(\boldsymbol{x}) \text{ or } f_{x_j}(\boldsymbol{x})$$

**Proposition 10.6 -** *Partial Derivative as a Matrix*
If $\boldsymbol{f} : \mathbb{R}^d \to \mathbb{R}^n$ is differentiable then

$$\boldsymbol{f}'(\boldsymbol{x}) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_d} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_d} \end{pmatrix}$$

**Remark 10.7 -** *Second Order Partial Derivatives*

$$\begin{aligned} \frac{\partial^2 f}{\partial x^2} &= \frac{\partial}{\partial x} \cdot \frac{\partial f}{\partial x} \\ \frac{\partial^2 f}{\partial x \partial y} &= \frac{\partial}{\partial x} \cdot \frac{\partial f}{\partial y} \\ \frac{\partial^2 f}{\partial y^2} &= \frac{\partial}{\partial y} \cdot \frac{\partial f}{\partial y} \end{aligned}$$