

# Linear Algebra & Geometry - Notes

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## Contents

<b>1</b>	<b>Euclidean Plane, Vectors, Cartesian Co-Ordinates &amp; Complex Numbers</b>	<b>3</b>
1.1	Vectors . . . . .	3
1.2	Complex Numbers . . . . .	4
<b>2</b>	<b>Euclidean Space, <math>\mathbb{R}^n</math></b>	<b>6</b>
2.1	Dot Product . . . . .	6
2.2	Linear Subspaces . . . . .	7
<b>3</b>	<b>Linear Equations &amp; Matrices</b>	<b>8</b>
3.1	Linear Equations . . . . .	8
3.2	Matrices . . . . .	8
3.3	Structure of Set of Solutions . . . . .	9
3.4	Solving Systems of Linear Equations . . . . .	10
3.5	Elementary Matrices & Inverting Matrices . . . . .	11
<b>4</b>	<b>Linear Independence, Bases &amp; Dimensions</b>	<b>12</b>
4.1	Linear Independence & Dependence . . . . .	12
4.2	Bases & Dimensions . . . . .	12
4.3	Orthogonal Bases . . . . .	13
<b>5</b>	<b>Linear Maps</b>	<b>13</b>
5.1	Abstract Properties of Linear Maps . . . . .	13
5.2	Matrices . . . . .	14
5.3	Rank & Nullity . . . . .	14
<b>6</b>	<b>Determinants</b>	<b>15</b>
6.1	Definition & Basic Properties . . . . .	15
6.2	Computing Determinant . . . . .	16
6.3	Applications of Determinant . . . . .	16
<b>7</b>	<b>Vector Spaces</b>	<b>17</b>
7.1	Groups & Fields . . . . .	17
7.2	Vector Spaces . . . . .	19
7.3	Subspace, Linear Combinations & Span . . . . .	19
7.4	Direct Sums . . . . .	20
7.5	Rank-Nullity Theorem . . . . .	20
7.6	Projection . . . . .	20
7.7	Isomorphisms . . . . .	20

<b>8 Eigenvalues &amp; Eigenvectors</b>	<b>22</b>
8.1 Characteristic Polynomial . . . . .	22

# 1 Euclidean Plane, Vectors, Cartesian Co-Ordinates & Complex Numbers

## 1.1 Vectors

### Definition 1.01 - Vectors

Ordered sets of real numbers.

Denoted by  $\mathbf{v} = (v_1, v_2, v_3, \dots) = \begin{pmatrix} x \\ y \end{pmatrix}$

### Definition 1.02 - Euclidean Plane

The set of two dimensional vectors, with real componenets, is called the Euclidean Plane.

Denoted by  $\mathbb{R}^2$

### Definition 1.03 - Vector Addition

Let  $\mathbf{v}, \mathbf{w} \in \mathbb{R}^2$  such that  $\mathbf{v} = (v_1, v_2)$  and  $\mathbf{w} = (w_1, w_2)$ .

Then  $\mathbf{v} + \mathbf{w} = (v_1 + w_1, v_2 + w_2)$ .

### Definition 1.03 - Scalar Multiplication of Vectors

Let  $\mathbf{v} \in \mathbb{R}^2$  and  $\lambda \in \mathbb{R}$  such that  $\mathbf{v} = (v_1, v_2)$ .

Then  $\lambda\mathbf{v} = (\lambda v_1, \lambda v_2)$ .

### Definition 1.04 - Norm of vectors

The norm of a vector is its length from the origin.

Denoted by  $\|\mathbf{v}\| = \sqrt{v_1^2 + v_2^2}$  for  $\mathbf{v} \in \mathbb{R}^2$ .

### Theorem 1.05

Let  $\mathbf{v}, \mathbf{w} \in \mathbb{R}^2$  and  $\lambda \in \mathbb{R}$  such that  $\mathbf{v} = (v_1, v_2)$  and  $\mathbf{w} = (w_1, w_2)$ .

Then

$$\begin{aligned} \|\mathbf{v}\| &= 0 \text{ iff } \mathbf{v} = \mathbf{0} \\ \|\lambda\mathbf{v}\| &= \sqrt{\lambda^2 v_1^2 + \lambda^2 v_2^2} \\ &= |\lambda| \cdot \|\mathbf{v}\| \\ \|\mathbf{v} + \mathbf{w}\| &\leq \|\mathbf{v}\| + \|\mathbf{w}\| \end{aligned}$$

### Definition 1.06 - Unit Vector

A vector can be described by its length & direction.

Let  $\mathbf{v} \in \mathbb{R}^2 \setminus \{\mathbf{0}\}$ .

Then  $\mathbf{v} = \|\mathbf{v}\|\mathbf{u}$  where  $\mathbf{u}$  is the unit vector,  $\mathbf{u} = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$

Thus  $\forall \mathbf{v} \in \mathbb{R}^2 \mathbf{v} = \begin{pmatrix} \lambda \cos\theta \\ \lambda \sin\theta \end{pmatrix}$  for some  $\lambda \in \mathbb{R}$ .

### Definition 1.07 - Dot Product

Let  $\mathbf{v} \in \mathbb{R}^2$  and  $\lambda \in \mathbb{R}$  such that  $\mathbf{v} = (v_1, v_2)$ .

Then  $\mathbf{v} \cdot \mathbf{w} = v_1.w_1 + v_2.w_2$ .

### Remark 1.08 - Positivity of Dot Product

Let  $\mathbf{v} \in \mathbb{R}^2$ .

Then  $\mathbf{v} \cdot \mathbf{v} = \|\mathbf{v}\|^2 = v_1^2 + v_2^2 \geq 0$ .

**Remark 1.09** - *Angle between vectors in Euclidean Plane*

Let  $\mathbf{v}, \mathbf{w} \in \mathbb{R}^2$ .

Set  $\theta$  to be the angle between  $\mathbf{v}$  &  $\mathbf{w}$ .

Then

$$\cos\theta = \frac{\mathbf{v} \cdot \mathbf{w}}{\|\mathbf{v}\| \|\mathbf{w}\|}$$

.

**Theorem 1.10** - *Cauchy-Schwarz Inequality*

Let  $\mathbf{v}, \mathbf{w} \in \mathbb{R}^2$ .

Then

$$|\mathbf{v} \cdot \mathbf{w}| \leq \|\mathbf{v}\| \|\mathbf{w}\|$$

*Proof*

$$\begin{aligned} \frac{v_1 w_1}{\|\mathbf{v}\| \|\mathbf{w}\|} + \frac{v_2 w_2}{\|\mathbf{v}\| \|\mathbf{w}\|} &\leq \frac{1}{2} \left( \frac{v_1^2}{\|\mathbf{v}\|^2} + \frac{w_1^2}{\|\mathbf{w}\|^2} \right) + \frac{1}{2} \left( \frac{v_2^2}{\|\mathbf{v}\|^2} + \frac{w_2^2}{\|\mathbf{w}\|^2} \right) \\ &\leq \frac{1}{2} \left( \frac{v_1^2 + v_2^2}{\|\mathbf{v}\|^2} + \frac{w_1^2 + w_2^2}{\|\mathbf{w}\|^2} \right) \\ &\leq \frac{1}{2} (1 + 1) \\ &\leq 1 \\ \Rightarrow |v_1 w_1 + v_2 w_2| &\leq \|\mathbf{v}\| \|\mathbf{w}\| \\ |\mathbf{v} \cdot \mathbf{w}| &\leq \|\mathbf{v}\| \|\mathbf{w}\| \end{aligned}$$

**1.2 Complex Numbers****Definition 1.11** -  $i$ 

$$\begin{aligned} i^2 &= -1 \\ i &= \sqrt{-1} \end{aligned}$$

**Definition 1.12** - *Complex Number Set*

The set of complex numbers contains all numbers with an imaginary part.

$$\mathbb{C} := \{x + iy; x, y \in \mathbb{R}\}$$

Complex numbers are often denoted by

$$z = x + iy$$

and we say  $x$  is the real part of  $z$  and  $y$  the imaginary part.

**Definition 1.13** - *Complex Conjugate*

Let  $z \in \mathbb{C}$  st  $z = x + iy$ .

Then

$$\bar{z} := x - iy$$

**Theorem 1.14 - Operations on Complex Numbers**

Let  $z_1, z_2 \in \mathbb{C}$  st  $z_1 = x_1 + iy_1$  and  $z_2 = x_2 + iy_2$ .

Then

$$\begin{aligned} z_1 + z_2 &:= (x_1 + x_2) + i(y_1 + y_2) \\ z_1 \cdot z_2 &:= (x_1 + iy_1)(x_2 + iy_2) \\ &:= x_1 \cdot x_2 - y_1 \cdot y_2 + i(x_1 \cdot y_2 + x_2 \cdot y_1) \end{aligned}$$

N.B. When dividing by a complex number, multiply top and bottom by the complex conjugate.

**Definition 1.15 - Modulus of Complex Numbers**

The modulus of a complex number is the distance of the number, from the origin, on an Argand diagram. Let  $z \in \mathbb{C}$  st  $z = x + iy$ .

Then

$$\begin{aligned} |z| &:= \sqrt{x^2 + y^2} \\ &:= \sqrt{\bar{z}z} \end{aligned}$$

N.B. Amplitude is an alternative name for the modulus

**Definition 1.16 - Phase of Complex Numbers**

The phase of a complex number is the angle between the positive real axis and the line subtended from the origin and the number, on an Argand diagram.

$$z = |z| \cdot (\cos\theta + i \cdot \sin\theta), \quad \theta = \text{Phase}$$

N.B. Phase of  $\bar{z} = -\text{Phase of } z$

**Theorem 1.17 - de Moivre's Formula**

$$z^n = (\cos\theta + i \cdot \sin\theta)^n = \cos(n\theta) + i \cdot \sin(n\theta)$$

**Theorem 1.18 - Euler's Formula**

$$e^{i\theta} = \cos\theta + i \cdot \sin\theta$$

**Remark 1.19**

Using Euler's formula we can express all complex numbers in terms of  $e$ . Thus many properties of the exponential remain true:

$$\begin{aligned} z &= \lambda e^{i\theta}, & \lambda \in \mathbb{R}, \theta \in [0, 2\pi) \\ \Rightarrow z_1 + z_2 &= \lambda_1 \cdot \lambda_2 \cdot e^{i(\theta_1 + \theta_2)} \\ \&, \frac{z_1}{z_2} &= \frac{\lambda_1}{\lambda_2} \cdot e^{i(\theta_1 - \theta_2)} \end{aligned}$$

## 2 Euclidean Space, $\mathbb{R}^n$

### Definition 2.01 - Euclidean Space

Let  $n \in \mathbb{N}$  then  $\forall \mathbf{x} = (x_1, x_2, \dots, x_n)$  with  $x_1, x_2, \dots, x_n \in \mathbb{R}$  we have that  $\mathbf{x} \in \mathbb{R}^n$ .

### Theorem 2.02 - Operations in Euclidean Space

Let  $(x), (y) \in \mathbb{R}^n$  and  $\lambda \in \mathbb{R}$ . Then

$$(x) + (y) = (x_1 + y_1, \dots, x_n + y_n)$$

And

$$(x) + \lambda.(y) = (x_1 + \lambda.y_1, \dots, x_n + \lambda.y_n)$$

### Definition 2.03 - Cartesian Product

Let  $A, B \in \mathbb{R}^n$  be non-empty sets.

Then

$$A \times B := \{(a, b); a \in A, b \in B\}$$

## 2.1 Dot Product

### Definition 2.04 - Dot Product

Let  $\mathbf{v}, \mathbf{w} \in \mathbb{R}^n$ . Then

$$\begin{aligned} \mathbf{v} \cdot \mathbf{w} &:= v_1.w_1 + \dots + v_n.w_n \\ &:= \sum_{j=1}^n v_j.w_j \end{aligned}$$

### Theorem 2.05 - Properties of the Dot Product

Let  $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^n$ . Linearity:

$$(\mathbf{u} + \lambda\mathbf{v}) \cdot \mathbf{w} = \mathbf{u} \cdot \mathbf{w} + \lambda(\mathbf{v} \cdot \mathbf{w})$$

Symmetry:

$$\mathbf{v} \cdot \mathbf{w} = \mathbf{w} \cdot \mathbf{v}$$

Positivity:

$$\mathbf{v} \cdot \mathbf{v} = v_1^2 + v_2^2 + \dots + v_n^2 \geq 0$$

### Definition 2.06 - Orthogonality

Let  $\mathbf{v}, \mathbf{w} \in \mathbb{R}^n$ .

It is said that  $(\mathbf{v}), (\mathbf{w})$  are orthogonal to each other if  $\mathbf{v} \cdot \mathbf{w} = 0$

N.B. Orthogonal vectors are perpendicular to each other.

### Definition 2.07 - The Norm

Let  $\mathbf{x} \in \mathbb{R}^n$ .

Then

$$\|\mathbf{x}\| = \sqrt{\mathbf{x} \cdot \mathbf{x}} = \sqrt{\sum_{i=1}^n x_i^2}$$

**Theorem 2.08 - Properties of the Norm**

Let  $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$  and  $\lambda \in \mathbb{R}$ . Then

$$\begin{aligned} \|\mathbf{x}\| &\geq 0 \\ \|\mathbf{x}\| &= 0 \text{ iff } \mathbf{x} = \mathbf{0} \\ \|\lambda\mathbf{x}\| &= |\lambda|\|\mathbf{x}\| \\ \|\mathbf{x} + \mathbf{y}\| &\leq \|\mathbf{x}\| + \|\mathbf{y}\| \end{aligned}$$

**Theorem 2.09 - Dot Product and Norm**

Let  $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ .

$$|\mathbf{x} \cdot \mathbf{y}| \leq \|\mathbf{x}\| \|\mathbf{y}\|$$

N.B.  $|\mathbf{x} \cdot \mathbf{y}| = \|\mathbf{x}\| \|\mathbf{y}\|$  iff  $\mathbf{x}$  &  $\mathbf{y}$  are orthogonal.

**Theorem 2.10 - Angle between Vectors**

Let  $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ . Then

$$\cos\theta = \frac{\mathbf{x} \cdot \mathbf{y}}{\|\mathbf{x}\| \|\mathbf{y}\|}$$

**2.2 Linear Subspaces****Definition 2.11 - Linear Subspace**

Let  $V \subset \mathbb{R}^n$ .  $V$  is a *Linear Subspace* if:

- i)  $V \neq \emptyset$ ;
- ii)  $\forall \mathbf{v}, \mathbf{w} \in V$  then  $\mathbf{v} + \mathbf{w} \in V$ ;
- iii)  $\forall \lambda \in \mathbb{R}, \mathbf{v} \in V$  then  $\lambda\mathbf{v} \in V$ .

**Definition 2.12 - Span**

Let  $\mathbf{x}_1, \dots, \mathbf{x}_k \in \mathbb{R}^n; k \in \mathbb{N}$ . Then

$$\text{span}\{\mathbf{x}_1, \dots, \mathbf{x}_k\} := \{\lambda_1\mathbf{x}_1 + \dots + \lambda_k\mathbf{x}_k; \lambda_i \in \mathbb{R}, 0 \leq i \leq k\}$$

**Definition 2.13 - Spans are Subspaces**

Let  $\mathbf{x}_1, \dots, \mathbf{x}_k \in \mathbb{R}^n; k \in \mathbb{N}$ . Then  $\text{span}\{\mathbf{x}_1, \dots, \mathbf{x}_k\}$  is a linear subspace of  $\mathbb{R}^n$ .

**Theorem 2.14**

$$W_{\mathbf{a}} := \{\mathbf{x} \in \mathbb{R}^n; \mathbf{x} \cdot \mathbf{a} = 0\} \text{ is a subspace.}$$

**Definition 2.15 - Orthogonal Complement**

Let  $V \subset \mathbb{R}^n$ . Then,

$$V^\perp := \{\mathbf{x} \in \mathbb{R}^n; \mathbf{x} \cdot \mathbf{y} = 0 \forall \mathbf{y} \in V\}$$

N.B.  $V^\perp \subset \mathbb{R}^n$

**Theorem 2.16 - Relationship of Subspaces**

Let  $V, W$  be subspaces of  $\mathbb{R}^n$ . Then

$$V \cap W \text{ is a subspace.}$$

$$V + W := \{\mathbf{v} + \mathbf{w}; \mathbf{v} \in V, \mathbf{w} \in W\} \text{ is a subspace.}$$

**Definition 2.17 - Direct Sum**

Let  $V_1, V_2, W$  be subspaces of  $\mathbb{R}^n$ . Then  $W$  is said to be a *direct sum* if

- i)  $W = V_1 + V_2$ ;
- ii)  $V_1 \cap V_2 = \emptyset$ .

### 3 Linear Equations & Matrices

#### 3.1 Linear Equations

**Definition 3.01** - *Multi-Variable Linear Equations*

Linear equations produce a straight line and can have multiple variables.

*Examples* -  $x = 3, y = x + 3, z + 5x - 2y$

**Definition 3.02** - *Systems of Linear Equations*

Let  $\mathbf{a}, \mathbf{x} \in \mathbb{R}^n$  &  $b \in \mathbb{R}$  such that  $\mathbf{a} \cdot \mathbf{x} = b$ .

$\mathbf{a} \cdot \mathbf{x} = b$  is a linear equation in  $\mathbf{x}$  with  $S = \{\mathbf{x}; \mathbf{a} \cdot \mathbf{x} = b\}$  as the set of solutions.

N.B. If  $b = 0$  then  $S(\mathbf{a}, 0)$  is a subspace.

#### 3.2 Matrices

**Definition 3.03** - *Matrix*

Let  $m, n \in \mathbb{N}$ , then a  $m \times n$  grid of numbers form an "m" by "n" matrix. Each element of the matrix can be reference by  $a_{ij}$  with  $i = 1, \dots, m$  and  $j = 1, \dots, n$ .

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}$$

N.B. m,i = rows, n,j = columns

**Definition 3.04** - *Sets of Matrices*

$M_{m,n}(\mathbb{R})$  is the set of m x n matrices containing only real numbers.

$M_{m,n}(\mathbb{Z})$  is the set of m x n matrices containing only integers.

$M_n(\mathbb{R})$  is the set square matrices, size n, containing only real numbers.

**Definition 3.05** - *Transpose Vectors*

Let  $\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$  then  $\mathbf{x}^t = (x_1 \ x_2 \ \dots \ x_n)$

**Definition 3.06** - *Vector-Matrix Multiplication*

Let  $A \in \mathbb{R}_{m,n}$  and  $\mathbf{x} \in \mathbb{R}^n$  then

$$A\mathbf{x} := \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n \end{pmatrix} = \begin{pmatrix} \mathbf{a}_1^t \cdot \mathbf{x} \\ \mathbf{a}_2^t \cdot \mathbf{x} \\ \vdots \\ \mathbf{a}_m^t \cdot \mathbf{x} \end{pmatrix} \in \mathbb{R}^m$$

This can be simplified to

$$\mathbf{y} = A\mathbf{x} \text{ with } y_i = \sum_{j=1}^n a_{ij}x_j$$

**Theorem 3.07** - *Operations on Matrices with Vectors*

$$\text{i) } A(\mathbf{x} + \mathbf{y}) = A\mathbf{x} + A\mathbf{y}, \quad \forall \mathbf{x}, \mathbf{y} \in \mathbb{R}^n.$$



$$\text{ii) } A(\lambda \mathbf{x}) = \lambda(A\mathbf{x}), \quad \forall \mathbf{x} \in \mathbb{R}^n, \lambda \in \mathbb{R}.$$

**Theorem 3.08**

Let  $A = (a_{ij}) \in M_{m,n}(\mathbb{R})$  and  $B = (b_{ij}) \in M_{l,m}(\mathbb{R})$ . Then there exists a  $C = (c_{ij}) \in M_{l,n}(\mathbb{R})$  such that

$$C\mathbf{x} = B(A\mathbf{x}), \quad \forall \mathbf{x} \in \mathbb{R}^n$$

N.B.  $c_{ij} = \sum_{k=1}^m b_{ik}a_{kj}$

**Theorem 3.09 - Operation between Matrices**

Let  $A, B \in M_{m,n}$  and  $C \in M_{l,m}$

$$\text{i) } C(A + B) = CA + CB.$$

$$\text{ii) } (A + B)C = AC + BC.$$

$$\text{iii) Let } D \in M_{m,n}, E \in M_{n,l} \text{ \& } F \in M_{l,k} \text{ then}$$

$$E(FG) = (EF)G$$

N.B.  $AB \neq BA$

**Definition 3.10 - Types of Matrix**

$$\text{Upper Triangle } \begin{pmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{pmatrix}, \quad a_{ij} = 0 \text{ if } i > j.$$

$$\text{Lower Triangle } \begin{pmatrix} 1 & 0 & 0 \\ 2 & 3 & 0 \\ 4 & 5 & 6 \end{pmatrix}, \quad a_{ij} = 0 \text{ if } i < j.$$

$$\text{Symmetric Matrix } \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 0 \\ 3 & 0 & 1 \end{pmatrix}, \quad a_{ij} = a_{ji}.$$

$$\text{Anti-Symmetric } \begin{pmatrix} 1 & -2 & -3 \\ 2 & 0 & -4 \\ 3 & 4 & -1 \end{pmatrix}, \quad a_{ij} = -a_{ji}.$$

**Definition 3.11 - Transposed Matrices**

Let  $A = (a_{ij}) \in M_{m,n}(\mathbb{R})$  then the transpose of  $A$ ,  $A^t$ , is an element of  $M_{n,m}(\mathbb{R})$ .

$$A^t := (a_{ji})$$

**Theorem 3.12 - Transpose Matrix Multiplication**

Let  $A \in M_{m,n}(\mathbb{R})$ ,  $\mathbf{x} \in \mathbb{R}^n$ ,  $\mathbf{y} \in \mathbb{R}^m$ . Then

$$\mathbf{y} \cdot A\mathbf{x} = (A^t\mathbf{y}) \cdot \mathbf{x}$$

**Theorem 3.10 - Transposing Multiplied Matrices**

$$(AB)^t = B^t A^t$$

**3.3 Structure of Set of Solutions****Definition 3.13 - Set of Solutions**

Let  $A \in M_{m,n}(\mathbb{R})$  and  $\mathbf{b} \in \mathbb{R}^m$ . Then

$$S(A, \mathbf{b}) := \{\mathbf{x} \in \mathbb{R}^n; A\mathbf{x} = \mathbf{b}\}$$

**Definition 3.14 - Homogenous Solutions**

The system of  $S(A, \mathbf{0})$  is called said to be *homogenous*. All other systems are *inhomogenous*.  
N.B. -  $S(A, \mathbf{0})$  is a linear subspace.

**Theorem 3.15 - Using Homogenous Solutions**

Let  $A \in M_{m,n}(\mathbb{R})$  and  $\mathbf{b} \in \mathbb{R}^n$ . Let  $\mathbf{x}_0 \in \mathbb{R}^n$  such that  $A\mathbf{x}_0 = \mathbf{b}$ , then

$$S(A, \mathbf{b}) = \mathbf{x}_0 + S(A, \mathbf{0})$$

**Remark 3.16 - Systems of Linear Equations as Matrices**

The system of linear equations  $3x + z = 0, y - z = 1, 3x + y = 1$  can be represented by a matrix and a vector.

$$A = \begin{pmatrix} 3 & 0 & 1 \\ 0 & 1 & -1 \\ 3 & 1 & 0 \end{pmatrix}, \mathbf{b} = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}$$

**3.4 Solving Systems of Linear Equations**

Systems of linear equations can be displayed as matrices which can be reduced and solved by a technique called *Gaussian Elimination*.

**Theorem 3.17**

There are certain operations that can be performed on a system of linear equations without changing the result:

- i) Multiply an equation by a non-zero constant;
- ii) Add a multiple of any equation to another equation;
- iii) Swap any two equations.

**Definition 3.18 - Augmented Matrices**

Let  $A\mathbf{x} = \mathbf{b}$  be a system of linear equations. The associated *Augmented Matrix* is

$$(A \ \mathbf{b}) \in M_{m,n+1}(\mathbb{R})$$

**Theorem 3.19 - Elementary Row Operations**

From *Theorem 3.17* we can deduce certain operations that can be performed on an *Augmented Matrix* which do not alter the solutions:

- i) Multiply a row by a non-zero constant,  $\text{row } i \rightarrow \lambda(\text{row } i)$ ;
- ii) Add a multiple of any row to another row,  $\text{row } i \rightarrow \text{row } i + \lambda(\text{row } j)$ ;
- iii) Swap two rows,  $\text{row } i \leftrightarrow \text{row } j$ .

**Definition 3.20 - Row Echelon Form**

A matrix is in *Row Echelon Form* if:

- i) The left-most non-zero value in each row is 1; And,
- ii) The leading 1 in each row is one place to the right of the leading 1 in the row below.

*Example*

$$\begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix}$$

**Definition 3.20 - Reduced Row Echelon Form**

A matrix is in *Reduced Row Echelon Form* if:

- i) The matrix is in *row echelon form*; And,
- ii) All values in a row, except the leading 1, are 0.

*Example*

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

**Theorem 3.21 - Gaussian Elimination**

*Gaussian Elimination* is a technique used to solve systems of linear equations. *Example*  
Solve  $x + y + 2z = 9, 2x + 4y - 3z = 1, 3x + 6y - 5z = 0$ .

$$\begin{aligned} \text{Augmented Matrix} &= \begin{pmatrix} 1 & 1 & 2 & 9 \\ 2 & 4 & -3 & 1 \\ 3 & 6 & -5 & 0 \end{pmatrix} \\ \text{By EROS} &= \begin{pmatrix} 1 & 1 & 2 & 9 \\ 2 & 4 & -3 & 1 \\ 3 & 6 & -5 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 2 & 9 \\ 0 & 2 & -7 & -17 \\ 0 & 3 & -11 & 27 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 1 & 2 & 9 \\ 0 & 2 & -7 & -17 \\ 0 & 1 & -4 & -10 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 1 & 2 & 9 \\ 0 & 1 & -4 & -10 \\ 0 & 2 & -7 & -17 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 1 & 2 & 9 \\ 0 & 1 & -4 & -10 \\ 0 & 0 & 1 & 3 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 & 6 & 19 \\ 0 & 1 & -4 & -10 \\ 0 & 0 & 1 & 3 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 3 \end{pmatrix} \\ &=> \underline{x = 1, y = 2, z = 3} \end{aligned}$$

### 3.5 Elementary Matrices & Inverting Matrices

**Definition 3.22 - Invertible Matrices**

A matrix,  $A \in M_{m,n}(\mathbb{R})$ , is said to be *Invertible* if there exists  $A^{-1} \in M_{n,m}(\mathbb{R})$  such that

$$AA^{-1} = I$$

N.B. - If a matrix is not invertible then it is *Singular*.

**Definition 3.23 - Elementary Matrices**

A matrix,  $E \in M_{m,n}(\mathbb{R})$ , is said to be an *Elementary Matrix* if it can be obtained by performing Elementary Row Operations on a square identity matrix.

*Examples*  $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & \lambda \\ \mu & 0 \end{pmatrix}$

**Remark 3.24**

All elementary matrices are invertible.

**Remark 3.25**

Let  $A$  be a matrix, and  $B$  be a matrix which can be obtained from  $A$  by elementary row operations. Then there exists an elementary matrix  $E$  such that

$$B = EA$$

**Theorem 3.26 - Finding  $A^{-1}$** 

Let  $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$ ,  $B = \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix}$ ,  $I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ .

Then by using EROS to change  $(A \ I) \rightarrow (I \ B)$ ,  $B$  is the inverse of  $A$ .

**Theorem 3.27 - Inverse of a  $2 \times 2$  Matrix**

Let  $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$  then

$$A^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

## 4 Linear Independence, Bases & Dimensions

### 4.1 Linear Independence & Dependence

**Definition 4.01 - Linear Independence & Dependence**

Vectors,  $\mathbf{x}_1, \dots, \mathbf{x}_n \in \mathbb{R}^k$ , are said to be *linearly dependent* if there exists non-zero real numbers,  $\lambda_1, \dots, \lambda_n$  such that

$$\lambda_1 \cdot \mathbf{x}_1 + \dots + \lambda_n \cdot \mathbf{x}_n = \mathbf{0}$$

N.B. - If this is only true if  $\lambda_1 = \dots = \lambda_n = 0$  then the vectors are said to be *linearly independent*.

**Remark 4.02**

Vectors are only *linearly dependent* if one of them lies in the span of the rest.

### 4.2 Bases & Dimensions

**Definition 4.03 - Basis**

A *basis* is a set of vectors,  $\mathbf{v}_1, \dots, \mathbf{v}_n \in V$  such that

- i)  $V = \text{span}\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ ;
- ii)  $\mathbf{v}_1, \dots, \mathbf{v}_n$  are linearly independent.

**Definition 4.04 - Standard Basis**

The *standard basis* for a vector space is the set fewest unit vectors which span it.

*Example* -  $\{\mathbf{v}_1, \mathbf{e}_2, \mathbf{e}_3\}$  are the standard basis for  $\mathbb{R}^3$ .

**Theorem 4.05 - Basis of a Linear Subspace**

For all elements,  $\mathbf{v}$ , of a linear subspace,  $V \subset \mathbb{R}^n$ , there exists a unique set of numbers,  $\lambda_1, \dots, \lambda_n$ , such that

$$\mathbf{v} = \lambda_1 \cdot \mathbf{v}_1 + \dots + \lambda_n \cdot \mathbf{v}_n$$

**Theorem 4.06 - Linear Independence and Bases**

Let  $V \subset \mathbb{R}^n$  be a linear subspace with basis  $\mathbf{v}_1, \dots, \mathbf{v}_n$ . Suppose  $\mathbf{w}_1, \dots, \mathbf{w}_k \in V$  are linearly independent, then  $k \leq n$ .

**Definition 4.07 - Dimension**

Let  $V \subset \mathbb{R}^n$  be a linear subspace then the *dimension* of  $V$ ,  $\dim(V)$ , is the fewest number vectors required to form a basis for  $V$ .

**4.3 Orthogonal Bases****Definition 4.08 - Orthogonal**

Let  $V \subset \mathbb{R}^n$  be a linear subspace with  $\{\mathbf{v}_1, \dots, \mathbf{v}_k\}$  as its basis. This basis is an *orthogonal basis* if it satisfies:

- i)  $\mathbf{v}_i \cdot \mathbf{v}_j = 0$  if  $i \neq j$ ;
- ii)  $\mathbf{v}_i \cdot \mathbf{v}_i = 1$ ,  $i = 1, \dots, k$ .

N.B. - This can be generalised to  $\mathbf{v}_i \cdot \mathbf{v}_k = \delta_{ij}$  with  $\delta_{ij} := \begin{cases} 1, & i = j \\ 0, & \text{otherwise} \end{cases}$

**Theorem 4.09**

Let  $V \subset \mathbb{R}^n$  be a linear subspace with an orthogonal basis  $\{\mathbf{v}_1, \dots, \mathbf{v}_k\}$ . Then for all  $\mathbf{u} \in V$

$$\mathbf{u} = (\mathbf{v}_1 \cdot \mathbf{u})\mathbf{v}_1, \dots, (\mathbf{v}_k \cdot \mathbf{u})\mathbf{v}_k$$

**5 Linear Maps****Definition 5.01 - Linear Map**

A map,  $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$  is a *linear map* if

- i)  $T(\mathbf{x} + \mathbf{y}) = T(\mathbf{x}) + T(\mathbf{y})$ ,  $\forall \mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ ;
- ii)  $T(\lambda\mathbf{x}) = \lambda T(\mathbf{x})$ ,  $\forall \mathbf{x} \in \mathbb{R}^n, \lambda \in \mathbb{R}$ .

N.B. - If  $m = n$  then  $T$  is referred to as a *linear operator*.

**Theorem 5.02 - Properties of Linear Maps**

Let  $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$  be a linear map. Then  $T(\mathbf{0}) = \mathbf{0}$ .

**Definition 5.03 - Linear Maps as Matrices**

Let  $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$  be a linear map. Then the associated Matrix is defined as

$$M_T = (t_{ij}) \in M_{m,n}(\mathbb{R})$$

with the elements of  $M_T$  defined by

$$t_{ij} = \mathbf{e}_i \cdot T(\mathbf{e}_j)$$

**Theorem 5.04 - Solutions to Linear Maps from Matrices**

Let  $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$  be a linear map and  $M_T$  be the associated matrix. Then

$$T(\mathbf{x}) = M_T \mathbf{x}, \quad \forall \mathbf{x} \in \mathbb{R}^n$$

**5.1 Abstract Properties of Linear Maps****Theorem 5.05 - Relationship between Linear Maps**

Let  $S : \mathbb{R}^n \rightarrow \mathbb{R}^m$ ,  $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$  &  $U : \mathbb{R}^m \rightarrow \mathbb{R}^k$  be a linear maps and  $\lambda \in \mathbb{R}$ . Then

- i)  $(\lambda T)(\mathbf{x}) := \lambda T(\mathbf{x})$ ;

$$\text{ii) } (S + T)(\mathbf{x}) = S(\mathbf{x}) + T(\mathbf{x});$$

$$\text{iii) } (U \circ S)(\mathbf{x}) = U(S(\mathbf{x})).$$

**Definition 5.06 - Image & Kernel**

Let  $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$  be a linear map. Then

i) The *image* of  $T$  is defined to be

$$\text{Im}(T) := \{\mathbf{y} \in \mathbb{R}^m : \exists \mathbf{x} \in \mathbb{R}^n \text{ st } T(\mathbf{x}) = \mathbf{y}\}$$

ii) The *kernel* of  $T$  is defined to be

$$\text{Ket}(T) := \{\mathbf{x} \in \mathbb{R}^n : T(\mathbf{x}) = \mathbf{0}\}$$

**Theorem 5.07**

Let  $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$  be a linear map then  $\text{Im}(T)$  is a linear subspace of  $\mathbb{R}^m$  and  $\text{Ket}(T)$  is a linear subspace of  $\mathbb{R}^n$

**Remark 5.08**

Let  $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$  be a linear map. Then

i)  $T$  is surjective if  $\text{Im}(T) = \mathbb{R}^m$ ;

ii)  $T$  is injective if  $\text{Ket}(T) = \{0\}$ .

**5.2 Matrices****Definition 5.09 - Linear Maps as Matrices**

Let  $S : \mathbb{R}^n \rightarrow \mathbb{R}^m$ ,  $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$  &  $U : \mathbb{R}^m \rightarrow \mathbb{R}^k$  be linear maps and  $\lambda \in \mathbb{R}$  with  $M_S$ ,  $M_T$  &  $M_U$  as the corresponding matrices. Then

$$\text{i) } M_{\lambda T} = \lambda M_T = (\lambda t_{ij});$$

$$\text{ii) } M_{S+T} = (s_{ij} + t_{ij}) = M_S + M_T;$$

$$\text{iii) } M_{U \circ S} = (r_{ij}) \text{ where } r_{ik} = \sum_{j=1}^m s_{ij} t_{jk}.$$

**5.3 Rank & Nullity****Definition 5.10 - Rank & Nullity**

Let  $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$  be a linear map. Then we define *Rank* of  $T$  by

$$\text{rank}(T) := \dim(\text{Im}(T))$$

and we define *Nullity* of  $T$  by

$$\text{nullity}(T) := \dim(\text{Ket}(T))$$

N.B. - For all linear maps,  $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ ,

$$\text{nullity}(T) + \text{rank}(T) = n$$

**Remark 5.11**

Let  $T : \mathbb{R}^n \rightarrow \mathbb{R}^n$  be a linear map. Then  $T$  is invertible if

i)  $\text{rank}(T) = n$ , or

ii)  $\text{nullity}(T) = 0$ .

**Theorem 5.12 - Relationship of Rank & Nullity between Linear Maps**

Let  $S : \mathbb{R}^n \rightarrow \mathbb{R}^m$  &  $T : \mathbb{R}^k \rightarrow \mathbb{R}^n$  be linear maps. Then

- i)  $S \circ T = 0$  iff  $\text{Im}(T) \subset \text{Ker}(S)$ ;
- ii)  $\text{rank}(S \circ T) \leq \text{rank}(T)$  and  $\text{rank}(S \circ T) \leq \text{rank}(S)$ ;
- iii)  $\text{nullity}(S \circ T) \geq \text{nullity}(T)$  and  $\text{nullity}(S \circ T) \geq \text{nullity}(S) + k - n$ ;
- iv)  $S$  is invertible then  $\text{rank}(S \circ T) = \text{rank}(T)$  and  $\text{nullity}(S \circ T) = \text{nullity}(T)$ .

## 6 Determinants

### 6.1 Definition & Basic Properties

**Definition 6.01 - Determinant Function**

A *determinant function*  $d_n : \mathbb{R}^n \times \dots \times \mathbb{R}^n \rightarrow \mathbb{R}$  is a function which satisfies three conditions:

- i) *Multilinear* -  $d_2(\lambda \mathbf{a}_1 + \mu \mathbf{b}, \mathbf{a}_2) = \lambda d_2(\mathbf{a}_1, \mathbf{a}_2) + \mu d_2(\mathbf{b}, \mathbf{a}_2)$ ;
- ii) *Antisymmetric* -  $d_2(\mathbf{a}_1, \mathbf{a}_2) = -d_2(\mathbf{a}_2, \mathbf{a}_1)$ ;
- iii) *Normalisation* -  $d_2(\mathbf{e}_1, \mathbf{e}_2) = 1$ .

N.B. - Determinant functions only exists for *square matrices*.

**Theorem 6.02 - Properties of Determinant**

- i)  $\det[\dots, \mathbf{a}_i + \lambda \mathbf{a}_i, \dots] = \det[\dots, \mathbf{a}_i, \dots] + \lambda \det[\dots, \mathbf{a}_i, \dots]$ ;
- ii) If  $A$  has two identical columns then  $\det(A) = 0$ ;
- iii) If  $A$  has an all zero column then  $\det(A) = 0$ ;
- iv)  $\det[\dots \mathbf{a}_i \dots \mathbf{a}_j \dots] = \det[\dots (\mathbf{a}_i + \lambda \mathbf{a}_j) \dots \mathbf{a}_j \dots]$

**Theorem 6.03**

Let  $f_n : \mathbb{R}^n \times \dots \times \mathbb{R}^n \rightarrow \mathbb{R}$  be a function which is *multilinear* & *Antisymmetric* then

$$f_n(A) = C \cdot \det(A)$$

where  $C$  is a constant such that  $C = f_n(\mathbf{e}_1, \dots, \mathbf{e}_n)$ .

**Theorem 6.04 - Determinant of a Triangle Matrix**

Let  $A = (a_{ij}) \in M_n(\mathbb{R})$  be an upper triangle matrix, so  $a_{ij} = 0$  if  $i > j$ . Then

$$\det(A) = a_{11} \cdot a_{22} \cdot \dots \cdot a_{nn}$$

N.B. - The same is true for lower triangle matrices.

**Theorem 6.05 - Relationship between Determinants**

Let  $A, B \in M_n(\mathbb{R})$  then

$$\det(AB) = \det(A) \cdot \det(B)$$

but usually

$$\det(A + B) \neq \det(A) + \det(B)$$

**Theorem 6.06 - Determinant & Inverses**

If  $\det(A) = 0$  then  $A^{-1}$  does not exist.

**Theorem 6.07 - Leibniz Formula**

Let  $A = (a_{ij}) \in M_n(\mathbb{R})$  then the *Leibniz Formula* states that

$$\det(A) := \sum_{\sigma \in S_n} \text{sign}(\sigma) \prod_{j=1}^n a_{\sigma(j),j}$$

Where:

- $S_n$  is the group of symmetries for a regular n-sided polygons;
- $\text{sign}(\sigma)$  is the sign function which returns +1 for even permutations and -1 for odd permutations.  
A permutation is even if a even number of permutations (swaps) are required to change the identity permutation to the given permutation,  $\sigma$ .

**Remark 6.08 - Determinant of Transpose**

Let  $A$  be a square matrix, then

$$\det(A) = \det(A^t)$$

**6.2 Computing Determinant****Theorem 6.09 - Laplace's Rule**

Let  $A \in M_n$  then

$$\det(A) = \sum_{j=1}^n a_{ij} \cdot (-1)^{i+j} \cdot \det(A_{ij})$$

where  $A_{ij}$  is the  $(n-1) \times (n-1)$  matrix formed when row  $i$  and column  $j$  are removed from  $A$ .

*Example* Let  $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$  then  $A_{11} = (4)$  and  $A_{12} = (2)$

**Definition 6.10 - Adjunct Matrices**

Let  $A, B \in M_n$  be defined such that  $b_{ij} = (-1)^{i+j} \cdot \det(A_{ij})$  then  $B$  is said to be *adjunct* to  $A$ . This means

$$AB = \begin{pmatrix} \det(A) & 0 & \dots & 0 \\ 0 & \det(A) & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \det(A) \end{pmatrix} = \det(A)I$$

**Remark 6.11 - Determinant of Triangle Matrices**

If  $A$  is an upper triangle matrix ( $a_{ij} = 0$  if  $i > j$ ) then

$$\det(A) = a_{11} \times \dots \times a_{nn}$$

If  $A$  is a lower triangle matrix ( $a_{ij} = 0$  if  $i < j$ ) then

$$\det(A) = a_{11} \times \dots \times a_{nn}$$

**6.3 Applications of Determinant****Theorem 6.12 - Linear Equations as Matrices**

A system of  $m$  linear equations, each with  $n$  variables, can be written as

$$A\mathbf{x} = \mathbf{b}, \quad A \in M_{mn}(\mathbb{R}), \mathbf{x} \in \mathbb{R}^n, \mathbf{b} \in \mathbb{R}^m$$



If  $\det(A) \neq 0$  then we can find an  $A^{-1} \in M_{n,m}$  such that

$$\mathbf{x} = A^{-1}\mathbf{b}$$

**Theorem 6.13**

Let  $A \in M_n(\mathbb{R})$  where  $\det(A) \neq 0$  then

$$A^{-1} = \frac{1}{\det(A)} \text{adj } A$$

**Theorem 6.14 - Cramer's Rule**

Consider  $A\mathbf{x} = \mathbf{b}$  then

$$x_j = \frac{\det(A_j)}{\det(A)}$$

where  $A_j$  is the matrix  $A$ , but the  $j^{\text{th}}$  column has been replaced by  $\mathbf{b}$ .

**Definition 6.15 - Cross Product**

Let  $\mathbf{x}, \mathbf{y} \in \mathbb{R}^3$  be in the same plane then we define the cross product by

$$\mathbf{x} \times \mathbf{y} := \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix} = \begin{pmatrix} x_2 y_3 - x_3 y_2 \\ x_3 y_1 - x_1 y_3 \\ x_1 y_2 - x_2 y_1 \end{pmatrix}$$

**Theorem 6.16 - Properties of Cross Product**

- i)  $\mathbf{x} \cdot (\mathbf{y} \times \mathbf{z}) = \mathbf{z} \cdot (\mathbf{x} \times \mathbf{y}) = \mathbf{y} \cdot (\mathbf{z} \times \mathbf{x})$
- ii)  $\mathbf{x} \times \mathbf{y} = -\mathbf{y} \times \mathbf{x}$
- iii)  $\mathbf{x} \times \mathbf{x} = \mathbf{0}$
- iv)  $(\mathbf{x} + \lambda \mathbf{y}) \times \mathbf{z} = (\mathbf{x} \times \mathbf{z}) + (\lambda \mathbf{y} \times \mathbf{z})$
- v)  $\|\mathbf{x} \times \mathbf{y}\|^2 = \|\mathbf{x}\|^2 \|\mathbf{y}\|^2 - (\mathbf{x} \cdot \mathbf{y})^2$

**Theorem 6.17 - Cross Product and Angle between vectors**

Let  $\theta$  be the angle between two vectors then

$$\|\mathbf{x} \times \mathbf{y}\|^2 = \|\mathbf{x}\|^2 \|\mathbf{y}\|^2 \sin^2(\theta)$$

**Theorem 6.18 - Cross Product with Matrices**

Let  $A \in M_n(\mathbb{R})$  where  $\det(A) \neq 0$  then

$$(A\mathbf{x}) \times (A\mathbf{y}) = [\det(A)](A^t)^{-1}(\mathbf{x} \cdot \mathbf{y})$$

## 7 Vector Spaces

### 7.1 Groups & Fields

**Definition 7.1 - Group**

A group,  $G$ , is a combination of a set and a map from  $G \times G \rightarrow G$ . The map must obey the following rules:

- i) *Associativity* -  $f * (g * h) = (f * g) * h$
- ii) *Identity Element* -  $\exists e \in G \text{ st } \forall g \in G, eg = ge = g$

iii) *Inverse* -  $\forall g \in G \exists g^{-1} \in G$  st  $gg^{-1} = e = g^{-1}g$

**Definition 7.2 - Matrix Groups**

The *General Linear Group*,  $GL(n, \mathbb{R})$ , is a group defined by

$$GL(n, \mathbb{R}) = \{A \in M_n(\mathbb{R}) : \det(A) \neq 0\}$$

The identity element is  $I \in M_n$  and inverse is  $A^{-1}$ .

The *Special Linear Group*,  $SL(n, \mathbb{R})$ , is a group defined by

$$SL(n, \mathbb{R}) = \{A \in M_n(\mathbb{R}) : \det(A) = 1\}$$

The *Orthogonal Group*,  $O(n, \mathbb{R})$ , is a group defined by

$$O(n, \mathbb{R}) = \{A \in M_n(\mathbb{R}) : A^t = A^{-1}\}$$

The *Special Orthogonal Group*,  $SO(n, \mathbb{R})$ , is a group defined by

$$SO(n, \mathbb{R}) = \{A \in O(n, \mathbb{R}) : \det(A) = \pm 1\}$$

The *Borel Matrix*,  $B(n, \mathbb{R})$ , is the group of upper triangle matrices with non-zero values on the main diagonal.

The *Permutations Group*,  $S(n, \mathbb{R})$ , is a group of permutations of  $\{1, 2, \dots, n\}$  defined by  $n \times n$  matrix

$$\text{e.g. } \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix}$$

**Theorem 7.3 - Abelian Groups**

Let  $G$  be a group. If  $\forall g, h \in G, gh = hg$  then  $G$  is commutative and is called an *Abelian Group*.  
N.B.  $e = 0$  is the identity element of all Abelian groups.

**Definition 7.4 - Direct/Cartesian Product of a Group**

Let  $G, H$  be groups with the same map. Then  $G \times H = \{(g, h) : g \in G, h \in H\}$ .

**Definition 7.5 - Fields,  $\mathbb{F}$**

A field,  $\mathbb{F}$ , is a set with two binary operations: addition & multiplication.

**Theorem 7.6 - Properties of Fields**

- i)  $\mathbb{F}$  is an abelian group w.r.t addition;
- ii)  $\mathbb{F} \setminus \{0\}$  is an abelian group w.r.t multiplication;
- iii)  $(x + y).z = x.z + y.z$ ;
- iv) A field always contains 0 & 1.

## 7.2 Vector Spaces

### Definition 7.7 - Vector Space

$\mathbb{V}$  is a (linear) vector space over a field,  $\mathbb{F}$  if:

- i)  $\mathbb{V}$  is an abelian group w.r.t addition;
- ii)  $\forall \mathbf{v} \in \mathbb{V} \ \& \ \lambda \in \mathbb{F}, \lambda \mathbf{v} \in \mathbb{V}$ ;
- iii)  $\lambda(\mathbf{u} + \mathbf{v}) = \lambda \mathbf{u} + \lambda \mathbf{v}$ ;
- iv)  $\lambda(\mu \mathbf{v}) = (\lambda \mu) \mathbf{v}$ ;
- v)  $1 \cdot \mathbf{v} = \mathbf{v}$ .

### Theorem 7.8 - Vector Spaces over Fields

Let  $W$  be a vector space over a field,  $\mathbb{F}$ , and  $U$  be a set. Then define

$$F(U, W) := f : U \rightarrow W$$

Then  $F(U, W)$  is a vector space over  $\mathbb{F}$ .

This means  $F(U, W)$  is linear so for all  $\lambda \in \mathbb{F} \ \& \ f, g \in F(U, W)$  then

$$(f + g)(u) = f(u) + g(u), \quad (\lambda f)(u) = \lambda f(u)$$

## 7.3 Subspace, Linear Combinations & Span

### Definition 7.9 - Subspace

Let  $\mathbb{V}$  be a vector space over a field  $\mathbb{F}$  and  $W \subset \mathbb{V}$ ,  $W$  is a subspace if it is a vector space for the operations inherited from  $\mathbb{V}$ .

### Theorem 7.10 - Properties of Subspaces

Let  $\mathbb{V}$  be a vector space and  $U \subset \mathbb{V}$  be a subspace, then  $U$  has the following properties:

- i) Not empty -  $U \neq \emptyset$ ;
- ii) Closed under addition -  $\forall u, v \in U; (u + v) \in U$ ;
- iii) Closed under multiplication -  $\forall \lambda \in \mathbb{F}, u \in U; \lambda u \in U$ .

### Theorem 7.11 - Subsets of Subspaces

Let  $\mathbb{V}$  be a vector space over  $\mathbb{F}$  and  $U, W \subset \mathbb{V}$  be subspaces. Then  $U \cap W$  is a subspace of  $\mathbb{V}$ .

### Remark 7.12 - Linear Independence and Span

Let  $\mathbb{V}$  be a vector space over field,  $\mathbb{F}$ , and  $S \subset \mathbb{V}$ .

$S$  is linearly dependent if there exists  $v \in \mathbb{V}$  such that  $\text{span}(S) = \text{span}(S \setminus \{v\})$ .

### Definition 7.13 - Finite Dimensional

Let  $\mathbb{V}$  be a vector space over  $\mathbb{F}$ .

$\mathbb{V}$  is finitely dimensional if it is a span of a finite set,  $S \subset \mathbb{V}$ , of vectors.

N.B. - If a vector space is not finite dimensional, then it is *infinitely dimensional*.

### Theorem 7.14

Let  $\mathbb{V}$  be a vector space over  $\mathbb{F}$  with  $\mathbb{B}, U \subset \mathbb{V}$ .

If  $\mathbb{B}$  is a basis for  $\mathbb{V}$ , with  $|\mathbb{B}| < \infty$ , and  $U$  is linearly independent then

$$|U| \leq |\mathbb{B}|$$

### Theorem 7.15 - Linearly Independent Sets as Bases

Let  $\mathbb{V}$  be a vector space over  $\mathbb{F}$  with  $U \subset \mathbb{V}$  as a linearly independent set.

Then  $U$  can be extended to form a basis of  $\mathbb{V}$ .

## 7.4 Direct Sums

### Definition 7.16 - Direct Sum

Let  $\mathbb{V}$  be a vector space over  $\mathbb{F}$  and  $U, W \subset V$  be subspaces with  $U \cap W = \emptyset$  then

$$U \oplus W := U + W$$

This is the direct sum of  $U$  and  $W$ .

### Theorem 7.17 - Dimension of Direct Sum

Let  $\mathbb{V}$  be a vector space over  $\mathbb{F}$  and  $U, W \subset V$  be subspaces with  $U \cap W = \emptyset$  then

$$\dim(U \oplus W) = \dim(U) + \dim(W)$$

### Theorem 7.18 - Complement

Let  $\mathbb{V}$  be a vector space over  $\mathbb{F}$  and  $U, W \subset V$  be subspaces with  $U \cap W = \emptyset$  if

$$U \oplus W = V$$

then  $W$  is said to be the complement of  $U$  in  $V$ .

## 7.5 Rank-Nullity Theorem

### Definition 7.19 - Rank & Nullity

Let  $\mathbb{V}, \mathbb{W}$  be vector spaces over  $\mathbb{F}$  and  $T : \mathbb{V} \rightarrow \mathbb{W}$  be a linear map. Then

$$\text{rank}(T) := \dim(\text{Im}(T)), \quad \text{nullity}(T) := \dim(\text{Ker}(T))$$

### Theorem 7.20 - Rank-Nullity Theorem

Let  $\mathbb{V}, \mathbb{W}$  be vector spaces over  $\mathbb{F}$  and  $T : \mathbb{V} \rightarrow \mathbb{W}$  be a linear map, with  $\dim(\mathbb{V}) < \infty$  then

$$\text{Rank}(T) + \dim(\text{Ker}(T)) = \dim(\mathbb{V})$$

## 7.6 Projection

### Definition 7.21 - Projection

A linear map  $P : V \rightarrow V$  is called a projection if  $P^2 = P$ .

### Theorem 7.22 - Image of Projection

Let  $P : V \rightarrow V$  be a projection then  $v \in \text{Im}(P)$  iff  $P(v) = v$ .

### Theorem 7.23 - Direct Sum of Projection

Let  $P : V \rightarrow V$  be a projection then

$$V = \text{Ker}(P) \oplus \text{Im}(P)$$

## 7.7 Isomorphisms

### Definition 7.24 - Isomorphisms

Let  $V, W$  be vector spaces over  $\mathbb{F}$ .

We say that the map  $T : V \rightarrow W$  is an isomorphism between  $V$  &  $W$  if

- i)  $T$  is linear; and
- ii)  $T$  is bijective.

N.B. - If an isomorphism exists between  $V$  &  $W$ , then they are said to be isomorphic.

**Theorem 7.25 - Dimension of Isomorphic Spaces**

Let  $V$  be a finitely dimensional vector space over  $\mathbb{F}$ .

If  $W$  is isomorphic to  $V$  then

$$\dim(V) = \dim(W)$$

This definition can be extended to say

*If two vector spaces have the same dimension, then they are isomorphic.*

**Proposition 7.26 - Multiple Bases**

Let  $A = \{\mathbf{a}_1, \dots, \mathbf{a}_n\}$  and  $B = \{\mathbf{b}_1, \dots, \mathbf{b}_n\}$  be different bases for  $V$ .

Define  $T_A : \mathbb{F}^n \rightarrow V$  and  $T_B : \mathbb{F}^n \rightarrow V$  such that

$$T_A(x_1, \dots, x_n) = x_1 \cdot \mathbf{a}_1 + \dots + x_n \cdot \mathbf{a}_n; \quad T_B(x_1, \dots, x_n) = x_1 \cdot \mathbf{b}_1 + \dots + x_n \cdot \mathbf{b}_n$$

Then for all  $\mathbf{v} \in V$  there are two ways of expressing  $\mathbf{v}$ .

$$x_1 \cdot \mathbf{a}_1 + \dots + x_n \cdot \mathbf{a}_n = \mathbf{v} = x_1 \cdot \mathbf{b}_1 + \dots + x_n \cdot \mathbf{b}_n$$

Unless  $A = B$  then  $x_i \neq y_i$  for at least one  $i \in \mathbb{N}, i \leq n$ .

**Theorem 7.27 - Conversion Matrices**

Let  $A, B$  be different bases for vector space  $V$ , with  $\dim(V) = n$ .

Then an  $n \times n$  matrix,  $C_{AB}$  can be used to convert elements given in basis  $A$  to now be given in basis  $B$ .

Let  $\mathbf{v} \in V$  and  $\mathbf{x} = T_A(\mathbf{x})$  &  $\mathbf{u} = T_B(\mathbf{x})$  then

$$\mathbf{y} = C_{AB}\mathbf{x}$$

**Theorem 7.28 - General Relationship between Bases**

Let  $V$  be a vector space over  $\mathbb{F}$ , with  $\dim(V) = n$ .

Let  $A, B$  be different bases for  $V$  with  $A = \{\mathbf{a}_1, \dots, \mathbf{a}_n\}$  &  $B = \{\mathbf{b}_1, \dots, \mathbf{b}_n\}$

Then for all  $\mathbf{v} \in V$  we have that

$$\mathbf{v} = \sum_{i=1}^n v_i \cdot \mathbf{a}_i = \sum_{i=1}^n v_i \cdot \mathbf{b}_i$$

Let  $C_{AB} = (c_{ij})$  be the conversion matrix from  $A$  to  $B$  then

$$v_j = \sum_{i=1}^n c_{ij} \mathbf{b}_i$$

**Theorem 7.29 - Properties of Transition Matrices**

Let  $A, B, C \subset V$  all be different bases for  $V$  then

- i)  $C_{AA} = I$ ;
- ii)  $C_{AB}C_{BA} = I$ ;
- iii)  $C_{CA}C_{AB} = C_{CB}$ .

**Theorem 7.30** - *Linear Maps between Vector Spaces as Matrices*

Let  $V, W$  both be vector spaces over  $\mathbb{F}$ , with  $\dim(V) = n$  and  $\dim(W) = m$ , and  $T : V \rightarrow W$  be a linear map.

Let  $A = \{\mathbf{a}_1, \dots, \mathbf{a}_n\} \subset V$  and  $B = \{\mathbf{b}_1, \dots, \mathbf{b}_m\} \subset W$  be bases for  $V$  &  $W$  respectively. Then we can define an  $n \times m$  matrix

$$M_{AB}(T) = (m_{ij}) \in M_{n,m}(\mathbb{F})$$

Where  $m_{ij}$  are defined to satisfy

$$T(\mathbf{a}_j) = \sum_{i=1}^m m_{ij} \mathbf{b}_i$$

Then

$$\mathbf{w} = M_{AB}(T)\mathbf{v}$$

With  $\mathbf{v} \in V, \mathbf{w} \in W$

**Theorem 7.31** - *Change Basis of Linear Map*

Let  $V$  be a vector space over  $F$  and  $U, W \subset V$  be different bases for  $V$ .

Define  $T : V \rightarrow V$  be a linear map and  $C$  to be the transition matrix from basis  $U \rightarrow W$ .

Then  $C^{-1}$  is the transition matrix from  $W \rightarrow U$ .

Set  $A$  to be the matrix representation of  $T$  in basis  $U$ . Then

$$A' = C^{-1}AC$$

Where  $A'$  is the matrix representation of  $T$  in basis  $W$ .

## 8 Eigenvalues & Eigenvectors

### 8.1 Characteristic Polynomial

**Definition 8.1** - *Eigenvectors & Eigenvalues*

Let  $\mathbf{v} \in V \setminus \{\mathbf{0}\}$  and  $T : V \rightarrow V$  be a linear operator.

$\mathbf{v}$  is called an *eigenvector* of  $T$  if

$$T(\mathbf{v}) = \lambda \mathbf{v}, \quad \lambda \in \mathbb{F}$$

This  $\lambda$  is the associated *eigenvalue* for  $\mathbf{v}$ .

**Definition 8.2** - *Spectrum*

The set of eigenvalues of a linear operator  $T : V \rightarrow V$  is called the spectrum of  $T$ , generally denoted as

$$\text{Spec}(T) := \{\mathbf{v} \in V : T(\mathbf{v}) = \lambda \mathbf{v}, \lambda \in \mathbb{F}\}$$

**Definition 8.3** - *Diagonalisable*

A linear operator is *diagonalisable* if there exists a basis of eigenvectors for it.

**Remark 8.4** - *Finding Eigenvalues*

Let  $A$  be the matrix which represents a linear operator  $T$ , and  $\mathbf{X}$  be a general eigenvector for  $T$

$$T(\mathbf{x}) = A\mathbf{x} = \lambda \mathbf{x} \Rightarrow (A - \lambda I)\mathbf{x} = \mathbf{0}$$

Then  $\lambda$  is an eigenvalue if it satisfies

$$\det(A - \lambda I) = 0$$

**Definition 8.5 - Characteristic Polynomial**

The polynomial which is equivalent to  $\det(A - \lambda.I)$  is called the *characteristic polynomial* of  $A$ .

$$p_A(\lambda) := \det(A - \lambda.I)$$

N.B. -  $\lambda$  is an eigenvalue for  $A$  if  $p_A(\lambda) = 0$

**Remark 8.6 - Finding Eigenvectors**

Once we have found all  $\lambda_1, \dots, \lambda_k$  that satisfy  $p_A(\lambda_i) = 0$  then we can find the eigenvectors,  $\mathbf{x}_i$ , of  $A$

$$(A - \lambda.I)\mathbf{x}_i = \mathbf{0}$$

A good way to start is to produce the linear equations

$$\sum_{j=1}^n (A - \lambda.I)_{ij} \cdot x_j = 0$$

For all  $i \leq n$ . Then solve these, as a series of simultaneous equations, to find the values  $x_j$  which produce the eigenvector  $\mathbf{x}$ .

Repeat this process for all  $\lambda_1, \dots, \lambda_k$  to find all eigenvectors for  $A$ .