Calculus 1 - Notes

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Contents

1 Before Calculus

1.1 Fundamental Theorem of Calculus

Definition 1.01 - *Fundamental Theorem of Calculus*
The *Fundamental Theorem of Calculus* states

$$\frac{d}{dx}\int_a^x f(t)dt = f(x)$$

Definition 1.02 - *Common Sets of Numbers*
Natural Numbers, set of positive integers - $\mathbb{N} := \{1, 2, 3, ...\}$.
Whole Numbers, set of all integers - $\mathbb{Z} := \{..., -2, -1, 0, 1, 2, ...\}$.
Rational Numbers, set of fractions - $\mathbb{Q} := \left\{\frac{p}{q} : p \in \mathbb{Z}, q \in \mathbb{N}\right\}$.
Real Numbers, set of all rational & irrational numbers - $\mathbb{R}$.

1.2 Intervals

Definition 1.03 - *Intervals*
Sets of real numbers that fulfil in given ranges.
Notation

$$\begin{aligned}
[a, b] &:= \{x \in \mathbb{R} : a \leq x \leq b\} \\
(a, b] &:= \{x \in \mathbb{R} : a < x \leq b\} \\
[a, b) &:= \{x \in \mathbb{R} : a \leq x < b\} \\
(a, b) &:= \{x \in \mathbb{R} : a < x < b\}
\end{aligned}$$

Definition 1.04 - *Functions*
Functions map values between fields of numbers. The signature of a function is defined by

$$f : A \to B$$

Where f is the name of the function, A is the domain and B is the co-domain.

Definition 1.05*Domain & Co-Domain*
The *Domain* of a function is the set of numbers it can take as an input.
The *Co-Domain* is the set of numbers that the domain is mapped to.

N.B. - A function is valid iff it maps each value in the domain to a single value in the co-domain.

Definition 1.06 - *Maximal Domain*
The *Maximal Domain* of a function is the largest set of values which can serve as the domain of a function.

Remark 1.07 - *Types of Function*
Let $f : A \to B$
Polynomials

$$f(x) = a_0 + a_1x + ... + a_nx^n$$

Rational

$$f(x) = \frac{p(x)}{q(x)}, \quad q(x) \neq 0 \; \forall \; x \in A$$

Trigonometric

$$sin(x),\ cos(x),\ tan(x) \text{ etc.}$$

2 Limits

2.1 Limits

Definition 2.01 - *Limits*
A *limit* is the value a function tends to, as the input converges to a given x.
i.e. The value f(x) has at it gets very close to x.

Formal Definition
We say L is the limit of $f(x)$ as x tends to x_0 if

$$\forall\ \varepsilon > 0,\ \exists\ \delta > 0 \text{ st if } x \in A \text{ and } |x - x_0| < \delta => |f(x) - L| < \varepsilon$$

Notation

$$\lim_{x \to x_0} f(x) = L$$

Definition 2.02 - *Directional Limits*
Sometimes the value of a limit depends on which direction you approach it from.
$\lim_{x \to x_0+}$ is used when approaching from values greater than x_0.
$\lim_{x \to x_0-}$ is used when approaching from values less than x_0.

Theorem 2.03 - *Operations with limits*
Let $\lim_{x \to x_0} f(x) = L_f$ and $\lim_{x \to x_0} g(x) = L_g$ Then

$$\begin{aligned}
\lim_{x \to x_0} [f(x) + g(x)] &= L_f + L_g \\
\lim_{x \to x_0} [f(x).g(x)] &= L_f.L_g \\
\lim_{x \to x_0} \left[\frac{f(x)}{g(x)}\right] &= \frac{L_f}{L_g}, \quad L_g \neq 0
\end{aligned}$$

2.2 Exponential Function

Definition 2.04 - *Exponential Function*

$$e := \lim_{x \to \infty} \left(1 + \frac{1}{n}\right)^n \simeq 2.7182818...$$

Theorem 2.05 - *Binomial Expansion*
A techique used for expanding binomial expressions

$$\begin{aligned}
\left(1 + \frac{x}{n}\right)^n &= \sum_{i=0}^{n} \binom{i}{n} .1^{(n-i)}. \left(\frac{x}{n}\right)^i \\
&= 1 + x + \frac{n-1}{2n}.x^2 + ... + \frac{x^n}{n^n}
\end{aligned}$$

3 The Derivative

Definition 3.01 - *Differentiable Equations*
Let $f : A \to B$ and $x_0 \in A$.
f is differentiable at x_0 if $\exists\ L \in B$ such that

$$L = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

If this limit exists $\forall\ x \in A$ then we can define the derivative of $f(x)$ by

$$f'(x) := \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Definition 3.02 - *Notation for Differentiation*
There are two ways to denote the derivative of an equation

$$f'(x) \iff \frac{df}{dx}, f''(x) \iff \frac{d^2 f}{dx^2}, ..., f^{(n)}(x) \iff \frac{d^n f}{dx^n}$$

N.B. - Using $\frac{df}{dx}$ is more informative, especially for equations with multiple variables.

3.1 Techniques for Finding the Derivative

Theorem 3.03 - *Sum Rule*
Let f, g be differentiable with respect to x.

$$(f+g)' = f' + g'$$

Theorem 3.04 - *Product Rule*
Let f, g be differentiable with respect to x.

$$(fg)' = f'g + fg'$$

Theorem 3.05 - *Quotient Rule*
Let f, g be differentiable with respect to x.

$$\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$$

Definition 3.06 - *Composite Functions*
Let $f : B \to C$ and $g : A \to B$. Then the following notation can be used

$$(f \circ g)(x) := f(g(x))$$

Theorem 3.07 - *Chain Rule*
Let f, g be differentiable with respect to x.

$$\frac{d}{dx}\left[f(g(x))\right] = f'(g(x)).g'(x)$$

3.2 Implicit Differentiation

Definition 3.08 - *Implicit Differentiation*
Sometimes it is hard to isolate variables in multi-variable equations, in these cases differentiate both sides with respect to the same variable.
Remembering

$$\frac{d}{dx}(x) = 1 \text{ and } \frac{d}{dx}(y) = \frac{dy}{dx} = y'$$

3.3 Applications of The Derivative

Thoerem 3.09 - *Netwon's Method*
Let f be differentiable. Using *Newton's Method* we can approximate a solution to $f(x) = 0$.
Process

i) Take an inital guess, x_0;

ii) Find the value of x where the tangent to $(x_0, f(x_0))$ on $f(x)$ intercepts the x-axis;

iii) Use this value as the next guess;

iv) Repeat until the value of x reduces little.

The equation for the tangent is

$$y = f(x_0) + (x - x_0)f'(x_0)$$

so a simplified equation for the process can be deduced

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

Theorem 3.10 - *Angle between Intersecting Curves*
Let $y = f_1(x)$ and $y = f_2(x)$ be two curves which intersect at (x_0, y_0). So

$$y_0 = f_1(x_0) = f_2(x_0)$$

Let m_1, m_2 be the gradient of the tangents to f_1 & f_2 at $(x_0, f(x_0))$.
Define $\theta_i := tan^{-1}(m_i)$ for $i = 1, 2$.
Set $\phi = |\theta_1 - \theta_2|$, then

$$\phi = \left| \frac{m_1 - m_2}{1 + m_1 m_2} \right|$$

Theorem 3.11 - *L'Hôspital's Rule*
For two equations equations, f, g with limit of $-\infty, 0$ or ∞ as x tends to a, it is hard to solve

$$\lim_{x \to a} \frac{f(x)}{g(x)}$$

Provided this limit exists, L'Hôspital's Rule states that

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

3.4 Sketching Curves

Remark 3.11 - *Sketching Curves*
Evaluating the derivative of a curve can make it easier to sketch:

i) When $f'(x) > 0$ the curve is heading upwards.

ii) When $f'(x) < 0$ the curve is heading downwards.

iii) When $f'(x) = 0$ the curve is flat.

iv) When $f'(x) = \infty, -\infty$ there are assymptotes.

Definition 3.12 - *Even Functions*
If $f(x) = f(-x)$ then the function is symmetrical across the y-axis and said to be *even.*
Examples - $x^2,\ cos(x),\ |x|$

Definition 3.13 - *Odd Functions*
If $f(x) = -f(-x)$ then the function is said to be *odd.*
Examples - $x,\ sin(x),\ x.cos(x)$

Remark 3.14
Some functions are neither *odd*, or *even.*
Example - $x + x^2$

4 Integration

4.1 The Primitive

Definition 4.01 - *The Primitive*
A function, $F : A \to \mathbb{R}$, is a primative for the function $f : A \to \mathbb{R}$ if F is differentiable and

$$\frac{d}{dx}F = f$$

N.B. - Primitives are also called *Indefinite Integral* or *Anti-Derivative.*

Definition 4.02 - *Definite Integral*
Let F be the primative for the function f. Then

$$\int_b^a f(x)dx = F(a) - F(b)$$

Notation - $F(x)\big|_a^b = F(b) - F(a)$

Remark 4.03 - *Area Under a Curve*
Let $f : [a, b] \to \mathbb{R}$ be a continuous function. Then the area between the curve and the x-axis is found by integration.

$$A := \int_a^b f(x)dx$$

Definition 4.04 - *Convergent Improper Integrals*
Let $b > a$ and define a function, $f : [a, \infty) \to \mathbb{R}$, which is continuous in $[a, b]$ Then

$$\int_a^\infty f(x)dx := \lim_{b\to\infty} \int_a^b f(x)dx$$

If this limit exists then the improper integral is *convergent*, otherwise it is *divergent.*

Remark 4.05 - *Summing Definite Inegrals*
For all $a < c < b$

$$\int_a^b f(x)dx = \int_c^a f(x)dx + \int_b^c f(x)dx$$

$$\int_b^a f(x)d := -\int_a^b f(x)dx$$

Theorem 4.06 - *Taylor Series*
Functions, $f(x)$, can be expanded into polynomial form with degree n, T_n, and remainder R_n such that $f(x) = T_n(x) + R_n(x)$ where

$$T_n(x) = f(a) + (x-a)f'(a) + ... + \frac{1}{n}.(x-a)^n.f^n(a)$$

$$R_n(x) = \frac{1}{n}\int_a^x (x-t)^n.f^{(n+1)}(t)dt$$

5 Parametric Curves & Arc-Length

5.1 Parametric Curves

Definition 5.01 - *Parametric Curves*
Parametric equations are an alternative to Cartesian equations, for representing curves. They can also represent a point in 3D space.

$$\boldsymbol{p} = \begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix}$$

Theorem 5.02 - *Parametric to Cartesian Equations*
As all equations in a Parametric series have a common variable, substition can be used to form a single equation.

5.2 Tangent to a Curve

Theorem 5.02 - *Tangent to a Parametric Curve*
Let $(x(t), y(t))$ be a series of parametric equation.
If we want to find the tangent at a point on the line, (a, b), we need to find the value t_0 such that $x(t_0) = a$ & $y(t_0) = b$.
Then by using the chain rule we can deduce the following equation for the tangent when $t = t_0$

$$\frac{dy(t_0)}{dx(t_0)} = \frac{y - y(t_0)}{x - x(t_0)}$$

Similarly we can deduce the equation for the *normal* when $t = t_0$

$$-\frac{dx(t_0)}{dy(t_0)} = \frac{y - y(t_0)}{x - x(t_0)}$$

5.3 Arc-Length

Theorem 5.03 - *Arc-Length*
Arc-Length is the length of a curve, following a function, between two points.
For a *cartesian equation*, $y = f(x)$, between the points x and $x + dx$ is

$$ds = \sqrt{dx^2 + dy^2}$$

So for a set of *parametric equations*, $(x(t), y(t)),\ a \le t \le b$,

$$ds = \sqrt{\frac{dx^2}{dt} + \frac{dy^2}{dt}}$$

To find the length of a curve between points a and b

$$s = \int_a^b \sqrt{\frac{dx^2}{dt} + \frac{dy^2}{dt}} dt$$

Definition 5.04 - *Curvature*
Curvature measures how fast the unit tangent vector to a curve rotates. Curvature of a curve, $y = f(x)$, can be found using the equation:

$$K(x) = \frac{|y''(x)|}{[1 + (y'(x))^2]^{\frac{3}{2}}}$$

For a set of parametric equations, $(x(t), y(t))$, it can be found using:

$$K(t_0) = \frac{y''(t_0).x'(t_0) - y'(t_0).x''(t_0)}{[(x'(t_0))^2 + (y'(t_0))^2]^{\frac{3}{2}}}$$

5.4 Level Curves

Definition 5.05 - *Level Curves*
Let $f : \mathbb{R}^d \to \mathbb{R}$ be a function with $d \geq 2,\ d \in \mathbb{N}$.
A *level curve* for f is the set of real solutions for $f(\boldsymbol{x}) = c$ where $c \in \mathbb{R}$ is a constant.
N.B - $f(\boldsymbol{x}) = c$ is often written as $f = c$.

6 Differential Equations

Definition 6.01 - *Differential Equations*
Differential equations take the form

$$f(x, y, \frac{dx}{dy}, ..., \frac{d^{(n)}y}{dx^{(n)}}) = 0,\ x \in I$$

6.1 First Order Differential Equations

Definition 6.02 - *First Order Differential Equations*
First order differential equations are equations of form $f(x, y, \frac{dx}{dy}) = 0$.

Definition 6.03 - *Seperable Equations*
An equation, f, is said to be seperable if there exists two equations, $M(x),\ N(y)$, such that

$$f(x, y, y') = y' - M(x).N(y)$$

Thus

$$\begin{aligned} y' &= M(x).N(y) \\ \implies \frac{y'}{N(y)} &= M(x) \\ \implies \int \frac{1}{N(y)} dy &= \int M(x) dx \end{aligned}$$

After integration, the equation can be rearranged to be in terms of y.

6.2 Integrating Factor

Theorem 6.04 - *Integrating Factor*
Consider the equation $y' + f(x)y + g(x)$.
Let $F(x) = \int f(x)dx$. Thus

$$\begin{aligned} e^{F(x)}.y' + e^{F(x)}.y &= e^{F(x)}.g(x) \\ => \frac{d}{dx}\left(e^{F(x)}.y\right) &= e^{F(x)}.g(x) \\ => e^{F(x)}.y &= \int e^{F(x)}.g(x)\ dx \\ => y &= e^{-F(x)} \int e^{F(x)}.g(x)\ dx \end{aligned}$$

6.3 Second Order Differential Equations

Definition 6.05 - *Linear Differential Equations*
A differential equation is said to be *linear* if it can be written in the form

$$Ay(x) := a_n(x).y^{(n)}(x) + ... + a_1(x).y'(x) + a_0(x).y(x) = b(x)$$

We define the set of solutions as

$$S(A,b) := y : I \to \mathbb{R}; Ay = b$$

If the only solution is $b(x) = 0\ \forall\ x$ then the system is *homogenous*, otherwise it is *inhomogenous.*
Definition 6.06 - *Particular & Complimentary Solutions*
When solving a differential equation, $Ay(x) = b(x)$, we need to find two functions in order to find the final solution.

i) Complementary Function, y_c - The homogenous case of the equation, $Ay(x) = 0$; and,

ii) Particular Function, y_p - The inhomogenous case of the equation, $Ay(x) = b(x)$ for a known $b(x)$.

Then $y = y_c + y_p$ is the final solution for $Ay(x) = b(x)$, for the given $b(x)$.

Theorem 6.07 - *Finding the Complementary Function*
Take a linear differential equation

$$a_n.y^{(n)}(x) + ... + a_1.y'(x) + a_0.y(x) = b(x)$$

where $a_n, ..., a_1, a_0 \in \mathbb{R}$ & $b(x) : \mathbb{R} \to \mathbb{R}$ are all constant.
To find the *complementary function* we solve the equation

$$a_n.\lambda^n + ... + a_1.\lambda^n + a_0 = 0$$

to get solutions $\lambda_1, ..., \lambda_k$ and then produce the complimentary function

$$y_c(x) = \mu_1 e^{\lambda_1 x} + ... + \mu_k e^{\lambda_k x}$$

Where $\mu_1, ..., \mu_k$ are constants to be found later, by comparing with $b(x)$.

Remark 6.08 - *Complementary Function*
The *complementary function*, y_c, for differential equations with constant coefficients depends upon the $\lambda_1, ..., \lambda_k$ we find, due to Euler's Formula.

i) $\lambda_i = c, \quad y_{c_i} = \mu_i e^{\lambda_i x}$;

ii) $\lambda_i = \pm ib, \quad y_{c_i} = \mu_{i_1} cos(bx) + \mu_{i_2} sin(bx)$;

iii) $\lambda_i = a \pm ib, \quad y_{c_i} = e^{ax}[\mu_{i_1} cos(bx) + \mu_{i_2} sin(bx)]$.

Then $y_c = \sum_{j=1}^{k} y_{c_j}$.

Remark 6.09 - *Particular Function*
The *particular function*, y_p, for a differential equation with constand coefficients, $Ay(x) = b(x)$, depends on the form of $b(x)$.

i) $b(x) = a_n x^n + ... + a_1 x + a_0 \implies y_p = b_n x^n + ... + b_1 x + b_0$;

ii) $b(x) = ae^{bx} \implies y_p = \alpha e^{\beta x}$;

iii) $b(x) = a.sin(bx) + c.cos(dx) \implies y_p = \alpha sin(\beta x) + \gamma cos(\delta x)$.

Where the constants of y_p are values to be found, when given certain conditions.

Theorem 6.10 - *Finding the Particular Function*
Take a linear differential equation

$$a_n.y^{(n)}(x) + ... + a_1.y'(x) + a_0.y(x) = b(x)$$

where $a_n, ..., a_1, a_0 \in \mathbb{R}$ & $b(x) : \mathbb{R} \to \mathbb{R}$ are all constant.
Deduce the particular function for the differential equation, given $b(x)$, and then differentiate y_p n times.
Substitute in these values, in place of the ys, into the original equation and solve to find values for the constants in y_p.

6.4 The Wronskian

Definition 6.11 - *The Wronskian*
The *Wronskian*, $W[y_1, y_2]$, of two differentiable functions is defined by

$$W[y_1, y_2](x) = y_1(x).y_2'(x) - y_1'(x).y_2(x)$$

The notation for this is

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} := ad - bc$$

So

$$W[y_1, y_2] = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$$

Remark 6.12
If $W[y_1, y_2] \neq 0$ then y_1, y_2 are linearly independent.

6.5 Variation of Constants

Theorem 6.13 - *Variation of Constants*
This is a technique for solving all differential equations, not just ones with constant coefficients, assuming we know the complementary function, y_c.
Consider the equation

$$Ay(x) = y''(x) + \beta(x)y'(x) + \gamma(x)y(x) = b(x), \quad \text{for a known } b(x)$$

And suppose we have a complementary function in the form

$$y_c = \lambda_1(x)y_1(x) + \lambda_2(x)y_2(x)$$

where y_1 & y_2 are linearly independent, thus $W[y_1, y_2] \neq 0$.
Then

$$y_p' = \lambda_1' y_1 + \lambda_1 y_1' + \lambda_2' y_2 + \lambda_2 y_2'$$

As λ_1, λ_2 are constant then $\lambda_1' = \lambda_2' = 0$ so

$$y_p' = \lambda_1 y_1' + \lambda_2 y_2'$$

By differentiating and then substituting back into the original equation we see y_p is a solution iff

$$\lambda_1' y_1' + \lambda_2' y_2' = f$$

In matrix form we have

$$\begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix} \begin{pmatrix} \lambda_1' \\ \lambda_2' \end{pmatrix} = \begin{pmatrix} 0 \\ f \end{pmatrix}$$

Then by Cramer's rule (*LAG - Theorem 6.14*) we have

$$\lambda_1' = \frac{\begin{vmatrix} 0 & y_2 \\ f & y_2' \end{vmatrix}}{W[y_1\ y_2]} = \frac{-y_2 f}{W[y_1\ y_2]}$$

and

$$\lambda_2' = \frac{\begin{vmatrix} y_1 & 0 \\ y_1' & f \end{vmatrix}}{W[y_1\ y_2]} = \frac{y_1 f}{W[y_1\ y_2]}$$

Giving use a solution for $y_p' = \lambda_1' y_1 + \lambda_1 y_1' + \lambda_2' y_2 + \lambda_2 y_2'$.

7 Applied Differential Equations

Definition 7.01 - *Denoting Limit Relationships*
We use

$$F(x) \sim G(x) \text{ as } x \to a$$

to denote

$$\lim_{x \to a} \frac{F(x)}{G(x)} = 1$$

Theorem 7.02 - *Vibrating String*
If we are given a string which is L long then we can define an equation, $y(x,t)$, which describe the displacement of a point x along the string, at time t.

$$y(x,t) = u(x)e^{i\omega t}$$

Where $\frac{\omega}{2\pi}$ is the frequency of the string and $u(x) = Acos(\omega x) + Bsin(\omega x)$.
We can generalise this for strings with n anti-nodes.

$$\omega_n := \frac{n\pi}{L}, \quad u_n := sin(\omega_n x)$$

8 Linear Difference Equations

Definition 8.01 - *Difference Equations*
A difference equation is an equation of the form

$$F(n, y_n, ..., y_{n+d}) = 0, \quad n, d \in \mathbb{N}$$

where y is a sequence.

8.1 First-Order Linear Difference Equation

Definition 8.02 - *Linear First-Order Difference Equations*
A *Linear First-Order Difference Equation* is an equation, F, which can be described by

$$F(n, y_n, y_{n+1}) = a_n y_{n+1} + b_n y_n - f_n = 0$$

where a_n, b_n, f_n are all known sequences.

Example 8.03
By taking a simple equation

$$y_{n+1} - y_n = f_n$$

we can see that

$$y_{n+1} = y_n + (y_{n+1} - y_n) = y_n + f_n = ... = y_{n_0} + f_{n_0} + ... + f_{n-1} + f_n$$

So

$$y_n = y_{n_0} + \sum_{j=n_0}^{n-1} f_j$$

Theorem 8.04 - *Solving First-Order Linear Difference Equations*
From *Definition 8.02* we can generalise the equation to show that

$$y_{n+1} + b_n y_n = f_n$$

Then

$$\frac{-1}{b_n} y_{n+1} - y_n = \frac{-1}{b_n} f_n$$

We now define the *Summing Factor*, S_n, as

$$S_n := \prod_{j=n_0}^{n-1} \frac{-1}{b_n}$$

. We multiply both sides of the original equation by the summing factor and as $S_n(\frac{-1}{b_n}) = S_{n+1}$ we get

$$S_{n+1} y_{n+1} - S_n y_n = S_{n+1} f_n$$

As this has the same form as the example in *8.03* we can now deduce

$$S_n y_n = y_{n_0} + \sum_{j=n_0}^{n-1} S_{j+1}.f_j$$

8.2 Second-Order Linear Difference Equation

Definition 8.05 - *Second-Order Linear Difference Equation*
A *Linear Second-Order Difference Equation* is an equation, F, which can be described by

$$F(n, y_n, y_{n+1}, y_{n+2}) := a_n.y_{n+2} + b_n.y_{n+1} + c_n.y_n = f_n$$

where a_n, b_n, c_n, f_n are know sequences.

Remark 8.06 - *Solving Second-Order Linear Difference Equations*
Similar to solving second-order differential equations we need to consider to cases. The *homogenous* & *inhomogenous* cases. So two sequences will be found the complementary sequence, y_n^c, and the particular sequence, y_n^p. The final solution for y_n is given by

$$y_n = y_n^c + y_n^p$$

Definition 8.07 - *Wronskian of Sequences*
For two sequences u_n & v_n we define the Wronskian to be

$$W_n := \begin{vmatrix} u_n & v_n \\ u_{n+1} & v_{n+1} \end{vmatrix} = u_n.v_{n+1} - v_n.u_{n+1}$$

Theorem 8.08 - *Homogenous Case with Constant Cofficients*
Take the equation

$$a.y_{n+2} + b.y_{n+1} + c.y_n = 0$$

where a, b & c are known constants. We look for solutions of the form

$$y_n = \lambda^n$$

By substition we get the equation $a\lambda^2 + b\lambda + c = 0$. By solving for λ we find a solution

i) λ has two real solutions - $y_n = A\lambda_1^n + B\lambda_2^n$;

ii) λ has one real solution - $y_n = (An + B)\lambda^n$;

iii) λ has only an imaginamry solution - $y_n = \Lambda e^{i\theta}, \quad \Lambda^2 := \frac{c}{a}, \theta := tan^{-1}(\frac{4ac-b^2}{-b})$.

Theorem 8.09 - *Homogenous Case Second-Order Linear Difference Equations*
The homogenous case finds solutions for

$$a_n y_{n+2} + b_n y_{n+1} + c_n y_n = 0$$

Suppose $y_n = u_n$ and $y_n = v_n$ are solutions to this homogenous equation. Then

$$W_n[u_n \ v_n] = W_{n_0} \prod_{j=n_0}^{n-1} \frac{c_j}{a_j}$$

So

$$u_n.v_{n+1} - v_n.u_{n+1} = W_{n_0} \prod_{j=n_0}^{n-1} \frac{c_j}{a_j}$$

Which can be rearranged to be in the form of a first order difference equation, such as *Example 8.03*

$$\frac{u_n}{u_{n+1}} v_{n+1} - v_n = W_{n_0} \prod_{j=n_0}^{n-1} \frac{c_j}{a_j}$$

Which has a summing factor

$$S_n = \frac{1}{u_n}$$

By multiplying both sides by S_n and simplifying we get

$$v_n = u_n \sum_{j=n_0}^{n-1} \frac{1}{u_j.u_{j+1}} \prod_{k=n_0}^{n-1} \frac{c_k}{a_k}$$

Typically you need to solve the product part of the equation to get a result for the sequence u_n.

Remark 8.10 - *Inhomogenous Case Second-Order Linear Difference Equations*
Generally the best way to do this is to make an educated guess based on the right hand side of the equation. So if the RHS is a polynomial, guess a polynomial, etc. Similar to solving differential equations.

9 Several Variables - Differentiability

Definition 9.01 - *Several Variable Function*
Let $d \in \mathbb{N}$, $A \subset \mathbb{R}^d$ & $B \subset \mathbb{R}^d$.
A function $\boldsymbol{f} : A \to B$ is a map which, for all $\boldsymbol{x} \in A$, assigns a unique value $\boldsymbol{f}(\boldsymbol{x}) \in B$.

Defintion 9.02 - *Linear Functions*
A function $\boldsymbol{f} : \mathbb{R}^d \to \mathbb{R}^n$ is *linear* if it can be given in terms of a matrix $A \in M_{n,d}(\mathbb{R})$ where

$$\boldsymbol{f}(\boldsymbol{x}) = A\boldsymbol{x}$$

Theorem 9.03 - *Properties of Linear Functions*
If a function is linear then the following are true:

i) $\boldsymbol{f}(\lambda\boldsymbol{x}) = \lambda\boldsymbol{f}(\boldsymbol{x})$; and,

ii) $\boldsymbol{f}(\boldsymbol{x}+\boldsymbol{y}) = \boldsymbol{f}(\boldsymbol{x}) + \boldsymbol{f}(\boldsymbol{y})$.

Definition 9.04 - *Continuous Several Variable Function*
Let $\boldsymbol{f} : A \to B$ and $\boldsymbol{a} \in A$. Then $\boldsymbol{f}$ is continuous at $\boldsymbol{a}$ if

$$\lim_{\boldsymbol{x}\to\boldsymbol{a}} \boldsymbol{f}(\boldsymbol{x}) = \boldsymbol{f}(\boldsymbol{a})$$

9.1 The Derivative

Defintion 9.05 - *Norm of a Vector*
The *norm* of a vector, $\boldsymbol{x} \in \mathbb{R}^d$ is

$$\|\boldsymbol{x}\| := \left(\sum_{j=1}^{d} x_j\right)^{1/2}$$

Definition 9.06 - *Derivative of Several Variable Function*
A function, $\boldsymbol{f} : \mathbb{R}^d \to \mathbb{R}^n$, is said to be differentiable at the point $\boldsymbol{x} \in \mathbb{R}^d$ if the exists an $A \in M_{n,d}(\mathbb{R})$ such that

$$\lim_{\boldsymbol{h}\to\boldsymbol{0}} \frac{\|\boldsymbol{f}(\boldsymbol{x}+\boldsymbol{h}) - \boldsymbol{f}(\boldsymbol{x}) - A\boldsymbol{h}\|}{\|\boldsymbol{h}\|} = 0$$

This A is the derivative of $\boldsymbol{f}(\boldsymbol{x})$.

$$\boldsymbol{f}'(\boldsymbol{x}) := A$$

Remark 9.07
If consider the following several variable function

$$\boldsymbol{f} = \begin{pmatrix} f_1 \\ \vdots \\ f_n \end{pmatrix}$$

Then

$$\boldsymbol{f}' = \begin{pmatrix} f_1' \\ \vdots \\ f_n' \end{pmatrix}$$

10 Directional & Partial Derivatives

10.1 Directional Derivative

Definition 10.01 - *Direction*
A *direction* in $\mathbb{R}^d$ is a vector of unit length.
In $\mathbb{R}^2$ every direction can be given by $\boldsymbol{u} = \begin{pmatrix} cos(\theta) \\ sin(\theta) \end{pmatrix}$, where θ is the angle from positive x axis
In $\mathbb{R}^3$ every direction can be given by $\boldsymbol{u} = \begin{pmatrix} sin(\phi)cos(\theta) \\ sin(\phi)sin(\theta) \\ cos(\phi) \end{pmatrix}$, where ϕ is the angle from positive z axis and θ is the angle from the positive x axis.

Definition 10.02 - *Spherical Co-ordinates*
The *spherical co-ordinates* describe points in three dimension space.
The distance of a point from the origin is

$$r = \rho sin(\phi)$$

where ρ is the length of the line. Then

$$\boldsymbol{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} rcos(\theta) \\ rsin(\theta) \\ \rho cos(\phi) \end{pmatrix} = \begin{pmatrix} \rho sin(\phi)cos(\theta) \\ \rho sin(\phi)sin(\theta) \\ \rho cos(\phi) \end{pmatrix}$$

Defintion 10.03 - *Direction Derivative*
The *direction derivative* of $\boldsymbol{f}$ in the direction of $\boldsymbol{u}$ at the point $\boldsymbol{x}_0$ is the vector:

$$D_{\boldsymbol{u}}\boldsymbol{f}(\boldsymbol{x}_0) := \frac{d}{dt}\boldsymbol{f}(\boldsymbol{x}_0 + t\boldsymbol{u}) \Big|_{t=0}$$

Theorem 10.04
For all $\boldsymbol{f} : \mathbb{R}^d \to \mathbb{R}^n$ the directional derivative at $\boldsymbol{u}$ in $\mathbb{R}^d$ we have

$$D_{\boldsymbol{u}}\boldsymbol{f}(\boldsymbol{x}) = \boldsymbol{f}'(\boldsymbol{x}).\boldsymbol{u}$$

10.2 Partial Derivative

Defintion 10.05 - *Partial Derivative*
Let $f : \mathbb{R}^d \to \mathbb{R}$.
Then the direction derivative $D_{\boldsymbol{e}_j}f(\boldsymbol{x})$, if it exists, is called the *partial derivative* of f with

respect to x_j at $\boldsymbol{x}$.
This is denoted by

$$\frac{\partial f}{\partial x_j}(\boldsymbol{x}) \text{ or } f_{x_j}(\boldsymbol{x})$$

Proposition 10.06 - *Partial Derivative as a Matrix*
If $\boldsymbol{f} : \mathbb{R}^d \to \mathbb{R}^n$ is differentiable then

$$\boldsymbol{f}'(\boldsymbol{x}) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \dots & \frac{\partial f_1}{\partial x_d} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \dots & \frac{\partial f_n}{\partial x_d} \end{pmatrix}$$

Remark 10.07 - *Second Order Partial Derivatives*

$$\begin{aligned} \frac{\partial^2 f}{\partial x^2} &= \frac{\partial}{\partial x} \cdot \frac{\partial f}{\partial x} \\ \frac{\partial^2 f}{\partial x \partial y} &= \frac{\partial}{\partial x} \cdot \frac{\partial f}{\partial y} \\ \frac{\partial^2 f}{\partial y^2} &= \frac{\partial}{\partial y} \cdot \frac{\partial f}{\partial y} \end{aligned}$$

11 Gradient & Chain Rule in Several Variables

11.1 Chain Rule

Theorem 11.01 - *Chain rule in Several Variables*
Let $\boldsymbol{f} : \mathbb{R}^d \to \mathbb{R}^n$ & $\boldsymbol{g} : \mathbb{R}^m \to \mathbb{R}^d$ be differentiable. Then

$$[\boldsymbol{f} \cdot \boldsymbol{g}]' = \boldsymbol{f}'(\boldsymbol{g}(\boldsymbol{x})).\boldsymbol{g}'(\boldsymbol{x})$$

11.2 Invertible Maps & Implicit Differentiation

Definition 11.02 - *Implicit Differentiation*
Let $A, B \subset \mathbb{R}^d$ and $\boldsymbol{f} : A \to B$.
If $\boldsymbol{f}$ is invertible then we can denote the inverse by $\boldsymbol{f}^{-1} : B \to A$ and

$$[\boldsymbol{f}^{-1} \cdot \boldsymbol{f}](\boldsymbol{x}) = \boldsymbol{x}$$

If $\boldsymbol{f}$ and $\boldsymbol{f}^{-1}$ are differentiable then by the chain run

$$\boldsymbol{f}'(\boldsymbol{f}^{-1}(\boldsymbol{x}))\boldsymbol{f}^{-1\,\prime}(\boldsymbol{x}) = I$$

Hence

$$\boldsymbol{f}^{-1\,\prime}(\boldsymbol{x}) = [\boldsymbol{f}'(\boldsymbol{f}(\boldsymbol{x}))]^{-1}$$

11.3 The Gradient

Defintion 11.03 - *The Gradient*
Let $f : \mathbb{R}^d \to \mathbb{R}$ be a map with first-order partial derivatives at every point of $\mathbb{R}^d$.
Then the *gradient* at a point $\boldsymbol{x} \in \mathbb{R}^d$, $\nabla f : \mathbb{R}^d \to \mathbb{R}^d$, is defined as

$$\nabla f(\boldsymbol{x}) := (f_{x_1}(\boldsymbol{x}), \dots, f_{x_d}(\boldsymbol{x}))$$

Definition 11.04 - *Direction of Greatest Change*
By considering the case in two dimensions then

$$(D_{\boldsymbol{u}}f)(x,y) = \nabla f(x,y) \cdot \boldsymbol{u}, \quad \boldsymbol{u} = (cos\theta, sin\theta)$$

Then

$$(D_{\boldsymbol{u}}f)(x_0,y_0) = cos\theta.\frac{\partial f}{\partial x}(x_0,y_0) + sin\theta.\frac{\partial f}{\partial y}(x_0,y_0)$$

The greatest change occurs when

$$\begin{aligned} \frac{d}{d\theta}[cos\theta.\frac{\partial f}{\partial x}(x_0,y_0) + sin\theta.\frac{\partial f}{\partial y}(x_0,y_0)] &= 0 \\ => -sin\theta.\frac{\partial f}{\partial x}(x_0,y_0) + cos\theta.\frac{\partial f}{\partial y}(x_0,y_0)] &= \\ => \nabla f(x_0,y_0) \cdot (-sin\theta, cos\theta) &= \end{aligned}$$

We can thus deduce that

$$\nabla f(x_0,y_0) \perp (-sin\theta, cos\theta)$$

So

$$\boldsymbol{u} \parallel \nabla f(x_0,y_0)$$

We can then establish the direction, $\boldsymbol{u}$, which occurs when $\boldsymbol{u}$ satisfies

$$\boldsymbol{u} = \frac{\nabla f(x_0,y_0)}{\|\nabla f(x_0,y_0)\|}$$

12 Integration over Two-Dimensional Domains

Remark 12.01 - *Notation*
Let $D \subset \mathbb{R}^2$.
When integrating over D the following notation is used

$$\iint_D f(x,y)dxdy$$

When D is specified e.g. $D = \{(x,y) : g(y) < x < h(y); a < y < b\}$ then we denote it by

$$\int_a^b \left\{ \int_{g(y)}^{h(y)} f(x,y)dx \right\} dy$$

Theorem 12.02 - *Rectangular Domain*
If we are given a domain between constant values, $D = \{(x,y) : c < y < d; a < x < b\}$ in terms of distance from origin, then we perform

$$V = \int_c^d \left\{ \int_a^b f(x,y)dx \right\} dy$$

And can perform the integrations are seperate calculations.

Theorem 12.03 - *Trianglular Domain*
In *triangular domains* the value of x depends on y, so can be expressed as a function of y. If we take $0 < x < a$ and $0 < y < b$ then $x = \frac{ya}{b}$. This gives us a domain of

$$D = \{(x,y) : \frac{ya}{b} < x < a; 0 < y < b\}$$

We can solve the intregral by performing:

$$V = \int_0^b \left\{ \int_{\frac{ay}{b}}^a f(x,y)dx \right\} dy$$

Theorem 12.04 - *General Domains*
If we have a domain, D, defined by $D = \{(x, y) \in \mathbb{R}^2 : L(x) \leq y \leq U(x), a \leq x \leq b\}$. Then the integral over D can be found by

$$V = \int_a^b \left\{ \int_{L(x)}^{U(x)} f(x,y)dy \right\} dx$$

13 Polar Co-ordinates

Defintion 13.01 - *Cartesian to Polar Co-ordinates*
Polar co-ordinates are an alternative way of denoting a point in two dimensional space, using distance from the origin and angle from the positive x-axis.
Polar co-ordiantes are given in the form (r, θ).
Using pythagoras' theorem and trigonometry we can see that

$$r = \sqrt{x^2 + y^2}, \quad \theta := tan^{-1}\left(\frac{y}{x}\right)$$

Theorem 13.02 - *Polar to Cartesian Co-ordinates*
We can convert a polar co-ordinate, (r, θ), to cartesian co-ordinates, (x, y) using the following formula

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} rcos(\theta) \\ rsin(\theta) \end{pmatrix}$$

Remark 13.03 - *Plotting Curves expressed as polar co-ordiantes*
There are two common ways to do this:

i) Convert back to cartesian form; or,

ii) Interpolation, plot a few points and link them.

Theorem 13.04 - *Computing Curvature of Polar Curve*
Suppose a curve is defined by $r = R(\theta)$. This means that

$$\begin{aligned} x =& rcos(\theta) = R(\theta)cos(\theta) \\ y =& rsin(\theta) = R(\theta)sin(\theta) \end{aligned}$$

By substituting into the equation for curvature of a cartesian curve we get

$$K(\theta) = \frac{|R(\theta)^2 + 2[R'(\theta)]^2 - R(\theta)R''(\theta)|}{\{R(\theta)^2 + [R'(\theta)]^2\}^{3/2}}$$

Theorem 13.05 - *Gradient of a Polar Curve*
Consider the equation $f(x, y) = u(r, \theta)$ then by using substitution and chain rule we can deduce the following equations:

$$\begin{aligned} \frac{\partial f}{\partial x} &= cos(\theta)\frac{\partial u}{\partial r} - \frac{sin(\theta)}{r}\frac{\partial u}{\partial \theta} \\ \frac{\partial f}{\partial y} &= sin(\theta)\frac{\partial u}{\partial r} + \frac{cos(\theta)}{r}\frac{\partial u}{\partial \theta} \end{aligned}$$

13.1 Change of Variable Formula

Theorem 13.06 - *Single Variable Substitution*
By making the substitution of $x = g(u)$ we can produce the following formula

$$\int_{g(a)}^{g(b)} f(x)dx = \int_a^b (f \cdot g)(u)g'(u)du$$

Theorem 13.07 - *Double Variable Substitution*
Suppose we make the following substitution

$$\begin{pmatrix} x \\ y \end{pmatrix} = \boldsymbol{g}(u,v) := \begin{pmatrix} g_1(u,v) \\ g_2(u,v) \end{pmatrix}$$

Then we can produce the following equation

$$\iint_{\boldsymbol{g}(D)} f(x,y)dxdy = \iint_D (f \cdot \boldsymbol{g})(u,v)|det\{\boldsymbol{g}'(u,v)\}dudv$$

Remark 13.08 - *Cartesian to Polar Double Integration*
By using the formula in *Theorem 13.6* we can produce the following equation, by using the substitution $(x,y) = (rcos(\theta), rsin(\theta))$

$$\text{Let } \boldsymbol{g}(r,\theta) = (rcos(\theta), rsin(\theta))$$

$$\implies \iint_{\boldsymbol{g}(D)} f(x,y)dxdy = \iint_D f(rcos(\theta), rsin(\theta)).r.drd\theta$$

14 Triple Integration

Remark 14.01 - *Triple Integration*
If we let D be a three dimensional region, so $D \subset \mathbb{R}^3$. Then by doing

$$\iiint_D f(x,y,z)dxdydz$$

we produce a *hypervolume.*
By setting $f(x,y,z) = 1$ we can find the volume of D.
If we set it to be a variable equation then we can determine other properties such as mass, center of mass etc.

Theorem 14.02 - *Box-Like region*
If we define D as $D = \{(x,y,z) \in \mathbb{R}^3 : a \leq x \leq A; b \leq y \leq B; c \leq z \leq C\}$. We can integrate over D by

$$\iiint_D f(x,y,z)dxdydz = \int_c^C \left\{ \int_b^B \left\{ \int_a^A f(x,y,z)dx \right\} dy \right\} dz$$

Theorem 14.03 - *General Domains*
If we define D as $D = \{(x,y,z) \in \mathbb{R}^3 : a \leq x \leq A(y,z); b \leq y \leq B(z); c \leq z \leq C\}$. We can integrate over D by

$$\iiint_D f(x,y,z)dxdydz = \int_c^C \left\{ \int_b^{B(z)} \left\{ \int_a^{A(x,y)} f(x,y,z)dx \right\} dy \right\} dz$$

14.1 Substitution with Triple Integrals

Theorem 14.04 - *Substitution with Triple Integration*
If we want to perform a substitution define by

$$x = g_1(u, v, w),$$
$$y = g_2(u, v, w),$$
$$z = g_3(u, v, w)$$

Then we can produce the following equation

$$\iiint_{\boldsymbol{g}(D)} f(x, y, z)dxdydz = \iiint_{D} (f \cdot \boldsymbol{g})(u, v, w).|det\{\boldsymbol{g}'(u, v, w)\}|dudvdw$$

Remark 14.05 - *Cartesian to Spherical Triple Integration*
If we can to covnert a cartesian region to a spherical region we use the following substitution

$$x = \rho sin(\phi)cos(\theta) = g_1(\rho, \phi, \theta),$$
$$x = \rho sin(\phi)sin(\theta) = g_2(\rho, \phi, \theta),$$
$$x = \rho sin(\phi) \qquad = g_3(\rho, \phi, \theta)$$

This means $det\{\boldsymbol{g}'(\rho, \phi, \theta)\} = \rho^2 sin(\phi)$. Then

$$\iiint_{\boldsymbol{g}(D)} f(x, y, z)dxdydz = \iiint_{D} (f \cdot \boldsymbol{g})(\rho, \phi, \theta).(\rho^2 sin(\phi)).d\rho d\phi d\theta$$

14.2 Applications of Multiple Integration

Definition 14.06 - *Multiple Integrations*
When integrating over multiple dimensions we dont need to write $\int$ every time, instead we can denoted it as

$$\int_R f(\boldsymbol{x})d\boldsymbol{x}$$

or

$$\int \cdots \int_R f(\boldsymbol{x})dx_1 \dots dx_d$$

Theorem 14.07 - *Centre of Mass*
By considering a region $R \subset \mathbb{R}^d$ whose density is describe by $f(\boldsymbol{x})$. Then its mass can be found by

$$m = \int_R f(\boldsymbol{x})d\boldsymbol{x}$$

Then its *centre of mass*, $\bar{x} \in \mathbb{R}^d$, can be found by

$$\bar{x} := \frac{1}{m} \int_R \boldsymbol{x} f(\boldsymbol{x})d\boldsymbol{x}$$

15 Local Extrema & Taylor's Theorem in Several Variables

15.1 Taylor's Theorem

Theorem 15.01 - *Taylor's Theorem for a Single Variable*
If $f : \mathbb{R} \to \mathbb{R}$ is smooth and $a \in \mathbb{R}$, then

$$f(x) = f(a) + (x - a)f'(a) + \frac{1}{2!}(x - a)^2 f''(a) + \dots$$

$$\implies f(x) = \sum_{j=1}^{\infty} \frac{1}{j!}.(x-a)^j. \left[\frac{d^j f}{dx^j}(a)\right]$$

Theorem 15.02 - *Taylor's Theorem for a Two Variables*
If $f : \mathbb{R}^2 \to \mathbb{R}$ is smooth and $a, b \in \mathbb{R}$, then

$$f(x,y) = f(a,b) + f'(a,b)\begin{pmatrix} x-a \\ y-b \end{pmatrix} + \frac{1}{2}\begin{pmatrix} x-a \\ y-b \end{pmatrix}^t f''(a,b)\begin{pmatrix} x-a \\ y-b \end{pmatrix} + \dots$$

<u>N.B</u> - $f'(x,y) \in M_{1,2}(\mathbb{F})$ & $f''(x,y) \in M_2(\mathbb{F})$

15.2 Local Extrema

Definition 15.03 - *Local Minima*
Let $f : \mathbb{R}^d \to \mathbb{R}$.
We say the point $\boldsymbol{x}_0 \in \mathbb{R}^d$ is a *local minimum* if

$$\exists\ \delta > 0\ st\ \forall\ \boldsymbol{x} \in \mathbb{R}^d, \|\boldsymbol{x} - \boldsymbol{x}_0\| < d \text{ then } f(\boldsymbol{x_0}) \leq f(\boldsymbol{x})$$

Definition 15.04 - *Local Maxima*
Let $f : \mathbb{R}^d \to \mathbb{R}$.
We say the point $\boldsymbol{x}_0 \in \mathbb{R}^d$ is a *local maximum* if

$$\exists\ \delta > 0\ st\ \forall\ \boldsymbol{x} \in \mathbb{R}^d, \|\boldsymbol{x} - \boldsymbol{x}_0\| < d \text{ then } f(\boldsymbol{x_0}) \geq f(\boldsymbol{x})$$

Definition 15.05 - *Local Extremum*
A point is a *local extremum* if it is either a local minimum or maximum.

Definition 15.06 - *Critical point*
A point, $\boldsymbol{x}_0$, is a critical point of f if

$$f'(\boldsymbol{x}_0) = \boldsymbol{0}$$

Proposition 15.07 - *Extremum and Critical Point*
If $\boldsymbol{x}_0$ is a local extremum of f then it is a critical point of f.

Remark 15.08 - *Classifing Critical Points*
Let $\boldsymbol{x}_0$ be a critical point of f.
Then Taylor's theorem says that

$$f(\boldsymbol{x}) = f(\boldsymbol{x}_0) + \frac{1}{2}(\boldsymbol{x} - \boldsymbol{x}_0)^t.f''(\boldsymbol{x}_0)(\boldsymbol{x} - \boldsymbol{x}_0) + ...$$

Definition 15.09 - *Define & Indefine Matrices*
Let $A \in M_d$, then A is

i) *Positive definite* if $\forall\ \boldsymbol{x} \in \mathbb{R}^d, \boldsymbol{x} \neq \boldsymbol{0} => \boldsymbol{x}^t A\boldsymbol{x} > 0$;

ii) *Negative definite* if $\forall\ \boldsymbol{x} \in \mathbb{R}^d, \boldsymbol{x} \neq \boldsymbol{0} => \boldsymbol{x}^t A\boldsymbol{x} < 0$;

iii) *Indefinite* if $\exists\ \boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^d\ st\ \boldsymbol{x}^t A\boldsymbol{x} > 0 \&\ \boldsymbol{y}^t A\boldsymbol{y} < 0$.

Theorem 15.10 - *Extrema and Definite Matrices*
The critical point $\boldsymbol{x}_0 \in \mathbb{R}^d$ is a

i) Local minimum iff $f''(\boldsymbol{x}_0)$ is a positive-definite;

ii) Local maximum iff $f''(\boldsymbol{x}_0)$ is a negative-definite;

If $f''(\boldsymbol{x}_0)$ is indefinite then we say that $\boldsymbol{x}_0$ is a *saddle*.

Theorem 15.11 - *Definite Matrices and Eigenvalues*
We say $f''(\boldsymbol{x}_0)$ is

i) *Positive definite* iff all its eigenvalues are strictly *positive*;

ii) *Negative definite* iff all its eigenvalues are strictly *negative*;

iii) *Indefinite* if it has both positive and negative eigenvalues.

16 Systems of Linear Differential Equations

Theorem 16.01 - *Linear Case*
Define a function, $\boldsymbol{f}: \mathbb{R}^d \to \mathbb{R}^d$, such that $\boldsymbol{f}(\boldsymbol{x}) = A\boldsymbol{x}$ where $\boldsymbol{x} \in \mathbb{R}^d$ and $A \in M_d$ is diagonisable.
Let $\{\boldsymbol{v}_1, \dots, \boldsymbol{v}_d\}$ be eigenvectors for A, with $\{\lambda_1, \dots, \lambda_d\}$ as the corresponding eigenvalues.
Then the general solution is

$$\boldsymbol{x}(t) = \sum_{j=1}^{d} c_j(0).e^{\lambda_j.t}.\boldsymbol{v}_j$$

values of c are determined in the particular solution from the initial solutions.

Definition 16.02 - *Equilibria*
Let $\boldsymbol{f}: \mathbb{R}^d \to \mathbb{R}^d$, $\boldsymbol{e} \in \mathbb{R}^d$ is said to be an *equilibrium* of $\boldsymbol{f}$ if

$$\boldsymbol{f}(\boldsymbol{e}) = \boldsymbol{0}$$

Theorem 16.03 - *Particular Solutions and Equilibria*
Let $t \in \mathbb{R}$, $\boldsymbol{f}: \mathbb{R}^d \to \mathbb{R}^d$ & $\boldsymbol{x}: \mathbb{R} \to \mathbb{R}^d$. If

$$\boldsymbol{f}(\boldsymbol{x}(t)) = \boldsymbol{0}$$

then t is a particular solution of $\boldsymbol{f}$.

16.1 Stability of Equilibria

Definition 16.04 - *Stable/Unstable Equilibria*
Let $\boldsymbol{e}$ be an *equilibrium*.
$\boldsymbol{e}$ is said to *stable* if

$$\forall\ \epsilon > 0, \exists\ \delta > 0\ st\ \forall\ \boldsymbol{x}(0) \in \mathbb{R}^d\ \&\ t \geq 0, \|\boldsymbol{x}(0) - \boldsymbol{e}\| < \delta => \|\boldsymbol{x}(t) - \boldsymbol{e}\| < \epsilon$$

Otherwise the equilibrium point is *unstable*.

Theorem 16.05 - *Determining Stability of Equilibria*
If for all eigenvalues, λ_j, of $\boldsymbol{f}'(\boldsymbol{e})$ is such that

$$Re(\lambda_j) < 0$$

then the equilibrium, $\boldsymbol{e}$, is *stable*.
If there exists one eigenvalue, λ_i, where $Re(\lambda_i) > 0$ then $\boldsymbol{e}$ is *unstable*.

Definition 16.06 - *Classification of Equilibria*
In dimension 2.

i) If both eigenvalues of $\boldsymbol{f}'(\boldsymbol{e})$ are real, $\boldsymbol{e}$ is a *node*;

ii) If both eigenvalues of $\boldsymbol{f}'(\boldsymbol{e})$ are purely imaginary, $\boldsymbol{e}$ is a *centre*; and,

iii) If both eigenvalues of $\boldsymbol{f}'(\boldsymbol{e})$ from a complex conjugate pair, with non-zero real parts, $\boldsymbol{e}$ is a *spiral.*

17 Discrete Dynamic Systems

Definition 17.01 - *Discrete Dynamic System*
A *discrete dynamic system* is a recurrence relation of the form

$$\boldsymbol{x}_{n+1} = \boldsymbol{f}(\boldsymbol{x}_n), n \in \mathbb{N}_0$$

Where $\boldsymbol{X}_n$ is the general term of a sequence $\mathbb{R}^d$, and $\boldsymbol{f} : \mathbb{R}^d \to \mathbb{R}^d$.

N.B. - We use the notation $\boldsymbol{x}_n = \begin{pmatrix} x_{1,n} \\ \vdots \\ x_{d,n} \end{pmatrix}$.

Theorem 17.02 - *Linear Case*
The simplest case of a discrete dynamic system is the *linear case* where

$$\boldsymbol{f}(\boldsymbol{x}) = A\boldsymbol{x}$$

with A as a diagonisable $d \times d$ matrix.
Let $\{\boldsymbol{v}_1, \dots, \boldsymbol{v}_d\}$ be a basis for $\mathbb{R}^d$ consisting of the eigenvectors for A.
Then a general solution is

$$\boldsymbol{x}_n = \sum_{j=1}^{d} c_j.\lambda_j^n.\boldsymbol{v}_j$$

where λ_j is the associated eigenvalue for the eigenvector $\boldsymbol{v}_j$.

Definition 17.03 - *Equilibria of Discrete Dynamic Systems*
Let $\boldsymbol{x} \in \mathbb{R}^d$ such that

$$\boldsymbol{F}(\boldsymbol{x}) = \boldsymbol{x}$$

then $\boldsymbol{x}$ is an *equilibrium* point of $\boldsymbol{f}$.

Definition 17.04 - *Periodic*
We say the solution $\boldsymbol{x}_n$ of a discrete dynamic system is *periodic* if

$$\exists\ p \in \mathbb{N}\ st\ \forall\ n \in \mathbb{N} \quad \boldsymbol{x}_{n+p} = \boldsymbol{x}_n$$

The smallest such p is called the period of the solution.

Definition 17.05 - *P-Cycle*
Let $\boldsymbol{x}_n$ be a periodic solution with period p and $k \in \mathbb{N}$.
Then the ordered list $\boldsymbol{x}_k, \boldsymbol{x}_{k+1}, \dots, \boldsymbol{x}_{k+p-1}$ is called a *p-cycle.*

Definition 17.06 - *Stable Equilibria of Discrete Dynamic Systems*
We say an equilibrium, $\boldsymbol{x}$ of a discrete dynamic system is *stable* if

$$\forall\ \epsilon > 0, \exists\ \delta > 0\ such\ that\ \|\boldsymbol{x}_0 - \boldsymbol{x}\| < \delta \implies \|\boldsymbol{x}_n - \boldsymbol{x}\| < \epsilon\ \forall\ n \in \mathbb{N}$$

Proposition 17.07 - *Stability and Eigenvalues*
The equilibrium $\boldsymbol{x}$ is stable if the modulus of every eigenvalue of $\boldsymbol{f}'(\boldsymbol{X})$ is less than 1.
Otherwise, it is unstable.