Calculus 1 - Notes

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February 21, 2018

Contents

1 Before Calculus

1.1 Fundamental Theorem of Calculus

Definition 1.01 - *Fundamental Theorem of Calculus*
The Fundamental Theorem of Calculus states

$$\frac{d}{dx}\int_a^x f(t)dt = f(x)$$

Definition 1.02 - *Common Sets of Numbers*
Natural Numbers, set of positive integers - $\mathbb{N} := \{1, 2, 3, ...\}$.
Whole Numbers, set of all integers - $\mathbb{Z} := \{..., -2, -1, 0, 1, 2, ...\}$.
Rational Numbers, set of fractions - $\mathbb{Q} := \left\{\frac{p}{q} : p \in \mathbb{Z}, q \in \mathbb{N}\right\}$.
Real Numbers, set of all rational & irrational numbers - $\mathbb{R}$.

1.2 Intervals

Definition 1.03 - *Intervals*
Sets of real numbers that fulfil in given ranges.
Notation

$$\begin{aligned}
[a, b] &:= \{x \in \mathbb{R} : a \leq x \leq b\} \\
(a, b] &:= \{x \in \mathbb{R} : a < x \leq b\} \\
[a, b) &:= \{x \in \mathbb{R} : a \leq x < b\} \\
(a, b) &:= \{x \in \mathbb{R} : a < x < b\}
\end{aligned}$$

Example
In what interval does x lie such that:

$$|3x + 4| < |2x - 1|$$

Solution

$$\begin{aligned}
\text{Case 1} : x \geq \frac{1}{2}& \\
&=> 1 - 2x < 3x + 4 < 2x - 1 \\
&=> 1 - 2x < 3x + 4 \\
&=> x > \frac{-3}{5} \\
\\
\text{And,} &=> 3x + 4 < 2x - 1 \\
&=> x < -5 \\
\text{There are no real solutions in this range.}& \\
\text{Case 2} : x < \frac{1}{2}& \\
&=> 2x - 1 < 3x + 4 < 1 - 2x \\
&=> 2x - 1 < 3x + 4 \\
&=> -5 < x
\end{aligned}$$

$$\begin{aligned}\text{And,} &=> 3x+4 < 1-2x \\ &=> 5x < -3 \\ &=> x < \frac{-3}{5} \\ \\ &=> -5 < x < \frac{-3}{5},\ \underline{x \in \left(-5, \frac{-3}{5}\right)}\end{aligned}$$

Definition 1.04 - *Functions*
Functions map values between fields of numbers. The signature of a function is defined by

$$f : A \to B$$

Where f is the name of the function, A is the domain and B is the co-domain.
The *Domain* of a function is the set of numbers it can take as an input.
The *Co-Domain* is the set of numbers that the domain is mapped to.

N.B. - A function is valid iff it maps each value in the domain to a single value in the co-domain.

Definition 1.05 - *Maximal Domain*
The *Maximal Domain* of a function is the largest set of values which can serve as the domain of a function.

Remark 1.06 - *Types of Function*
Let $f : A \to B$
Polynomials

$$f(x) = a_0 + a_1x + ... + a_nx^n$$

Rational

$$f(x) = \frac{p(x)}{q(x)}, \quad q(x) \neq 0\ \forall\ x \in A$$

Trigonometric

$$sin(x),\ cos(x),\ tan(x) \text{ etc.}$$

2 Limits

2.1 Limits

Definition 2.01 - *Limits*
A limit is the value a function tends to, for a given x.
i.e. The value f(x) has at it gets very close to x.

Formally We say L is the limit of $f(x)$ as x tends to x_0 if

$$\forall\ \varepsilon > 0,\ \exists\ \delta > 0 \text{ st if } x \in A \text{ and } |x - x_0| < \delta => |f(x) - L| < \varepsilon$$

Notation

$$\lim_{x \to x_0} f(x) = L$$

Definition 2.02 - *Directional Limits*
Sometimes the value of a limit depends on which direction you approach it from.
$\lim_{x \to x_0+}$ is used when approaching from values greater than x_0.
$\lim_{x \to x_0-}$ is used when approaching from values less than x_0.

Theorem 2.03 - *Operations with limits*
Let $\lim_{x \to x_0} f(x) = L_f$ and $\lim_{x \to x_0} g(x) = L_g$ Then

$$\begin{aligned}
\lim_{x \to x_0} [f(x) + g(x)] &= L_f + L_g \\
\lim_{x \to x_0} f(x).g(x) &= L_f.L_g \\
\lim_{x \to x_0} \frac{f(x)}{g(x)} &= \frac{L_f}{L_g} \quad L_g \neq 0
\end{aligned}$$

2.2 Exponential Function

Definition 2.04 - *Exponential Function*

$$e := \lim_{x \to \infty} \left(1 + \frac{1}{n}\right)^n \simeq 2.7182818...$$

Theorem 2.05 - *Binomial Expansion*
A techique for expanding binomial expressions

$$\begin{aligned}
\left(1 + \frac{x}{n}\right)^n &= \sum_{i=0}^{n} \binom{i}{n} .1^{(n-i)}. \left(\frac{x}{n}\right)^i \\
&= 1 + x + \frac{n-1}{2n}.x^2 + ... + \frac{x^n}{n^n}
\end{aligned}$$

3 The Derivative

Definition 3.01 - *Differentiable Equations*
Let $f : A \to B$ and $x_0 \in A$.
f is differentiable at x_0 if $\exists\ L \in B$ such that

$$L = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

If this limit exists $\forall\ x \in A$ then we can define the derivative of $f(x)$

$$f'(x) := \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}$$

Definition 3.02 - *Notation for Differentiation*
There are two ways to denote the derivative of an equation

$$f'(x) \iff \frac{df}{dx}, f''(x) \iff \frac{d^2 f}{dx^2}, ..., f^{(n)}(x) \iff \frac{d^n f}{dx^n}$$

N.B. - Using $\frac{df}{dx}$ is more informative, especially for equations with multiple variables.

3.1 Techniques for finding derivative

Theorem 3.03 - *Sum Rule*
Let f, g be differentiable with respect to x.

$$(f+g)' = f' + g'$$

Theorem 3.04 - *Product Rule*
Let f, g be differentiable with respect to x.

$$(fg)' = f'g + fg'$$

Theorem 3.05 - *Quotient Rule*
Let f, g be differentiable with respect to x.

$$\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$$

Definition 3.06 - *Composite Functions*
Let $f : B \to C$ and $g : A \to B$ Then

$$(f \circ g)(x) = f(g(x))$$

Theorem 3.07 - *Chain Rule*
Let f, g be differentiable with respect to x.

$$\frac{d}{dx} f(g(x)) = f'(g(x)).g'(x)$$

3.2 Implicit Differentiation

Definition 3.08 - *Implicit Differentiation*
Sometimes it is hard to isolate variables in multi-variable equations, in these cases differentiate both sides with respect to the same variable.
Remembering

$$\frac{d}{dx}(x) = 1 \text{ and } \frac{d}{dx}(y) = \frac{dy}{dx} = y'$$

Example
Find y if $x^3 + y^3 = 6xy$

$$\begin{aligned}
\frac{d}{dx}\left(x^3 + y^3\right) &= \frac{d}{dx}(6xy) \\
=> 3x^2 + 3y^2.y' &= 6y + 6x.y' \\
=> y'(3y^2 - 6x) &= 6y - 3x^2 \\
=> y' &= \frac{2y - x^2}{y^2 - 2x}
\end{aligned}$$