

Calculus 1 - Notes

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1 Before Calculus

1.1 Fundamental Theorem of Calculus

Definition 1.01 - *Fundamental Theorem of Calculus*

The Fundamental Theorem of Calculus states

$$\frac{d}{dx} \int_a^x f(t)dt = f(x)$$

Definition 1.02 - *Common Sets of Numbers*

Natural Numbers, set of positive integers - $\mathbb{N} := \{1, 2, 3, \dots\}$.

Whole Numbers, set of all integers - $\mathbb{Z} := \{\dots, -2, -1, 0, 1, 2, \dots\}$.

Rational Numbers, set of fractions - $\mathbb{Q} := \left\{\frac{p}{q} : p \in \mathbb{Z}, q \in \mathbb{N}\right\}$.

Real Numbers, set of all rational & irrational numbers - \mathbb{R} .

1.2 Intervals

Definition 1.03 - *Intervals*

Sets of real numbers that fulfil in given ranges.

Notation

$$[a, b] := \{x \in \mathbb{R} : a \leq x \leq b\}$$

$$(a, b] := \{x \in \mathbb{R} : a < x \leq b\}$$

$$[a, b) := \{x \in \mathbb{R} : a \leq x < b\}$$

$$(a, b) := \{x \in \mathbb{R} : a < x < b\}$$

Example

In what interval does x lie such that:

$$|3x + 4| < |2x - 1|$$

Solution

$$\text{Case 1 : } x \geq \frac{1}{2}$$

$$\Rightarrow 1 - 2x < 3x + 4 < 2x - 1$$

$$\Rightarrow 1 - 2x < 3x + 4$$

$$\Rightarrow x > \frac{-3}{5}$$

$$\text{And, } \Rightarrow 3x + 4 < 2x - 1$$

$$\Rightarrow x < -5$$

There are no real solutions in this range.

$$\text{Case 2 : } x < \frac{1}{2}$$

$$\Rightarrow 2x - 1 < 3x + 4 < 1 - 2x$$

$$\Rightarrow 2x - 1 < 3x + 4$$

$$\Rightarrow -5 < x$$

$$\text{And, } \Rightarrow 3x + 4 < 1 - 2x$$

$$\Rightarrow 5x < -3$$

$$\Rightarrow x < \frac{-3}{5}$$

$$\Rightarrow -5 < x < \frac{-3}{5}, \quad x \in \left(-5, \frac{-3}{5} \right)$$

Definition 1.04 - Functions

Functions map values between fields of numbers. The signature of a function is defined by

$$f : A \rightarrow B$$

Where f is the name of the function, A is the domain and B is the co-domain.

The *Domain* of a function is the set of numbers it can take as an input.

The *Co-Domain* is the set of numbers that the domain is mapped to.

N.B. - A function is valid iff it maps each value in the domain to a single value in the co-domain.

Definition 1.05 - Maximal Domain

The *Maximal Domain* of a function is the largest set of values which can serve as the domain of a function.

Remark 1.06 - Types of Function

Let $f : A \rightarrow B$

Polynomials

$$f(x) = a_0 + a_1x + \dots + a_nx^n$$

Rational

$$f(x) = \frac{p(x)}{q(x)}, \quad q(x) \neq 0 \quad \forall x \in A$$

Trigonometric

$$\sin(x), \cos(x), \tan(x) \text{ etc.}$$

2 Limits

2.1 Limits

Definition 2.01 - Limits

A limit is the value a function tends to, for a given x .

i.e. The value $f(x)$ has at it gets very close to x .

Formally We say L is the limit of $f(x)$ as x tends to x_0 if

$$\forall \varepsilon > 0, \exists \delta > 0 \text{ st if } x \in A \text{ and } |x - x_0| < \delta \Rightarrow |f(x) - L| < \varepsilon$$

Notation

$$\lim_{x \rightarrow x_0} f(x) = L$$

Definition 2.02 - Directional Limits

Sometimes the value of a limit depends on which direction you approach it from.

$\lim_{x \rightarrow x_0+}$ is used when approaching from values greater than x_0 .

$\lim_{x \rightarrow x_0-}$ is used when approaching from values less than x_0 .

Theorem 2.03 - Operations with limits

Let $\lim_{x \rightarrow x_0} f(x) = L_f$ and $\lim_{x \rightarrow x_0} g(x) = L_g$ Then

$$\begin{aligned}\lim_{x \rightarrow x_0} [f(x) + g(x)] &= L_f + L_g \\ \lim_{x \rightarrow x_0} f(x) \cdot g(x) &= L_f \cdot L_g \\ \lim_{x \rightarrow x_0} \frac{f(x)}{g(x)} &= \frac{L_f}{L_g} \quad L_g \neq 0\end{aligned}$$

2.2 Exponential Function**Definition 2.04 - Exponential Function**

$$e := \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^n \simeq 2.7182818...$$

Theorem 2.05 - Binomial Expansion

A technique for expanding binomial expressions

$$\begin{aligned}\left(1 + \frac{x}{n}\right)^n &= \sum_{i=0}^n \binom{n}{i} \cdot 1^{(n-i)} \cdot \left(\frac{x}{n}\right)^i \\ &= 1 + x + \frac{n-1}{2n} \cdot x^2 + \dots + \frac{x^n}{n^n}\end{aligned}$$

3 The Derivative**Definition 3.01 - Differentiable Equations**

Let $f : A \rightarrow B$ and $x_0 \in A$.

f is differentiable at x_0 if $\exists L \in B$ such that

$$L = \lim_{h \rightarrow 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

If this limit exists $\forall x \in A$ then we can define the derivative of $f(x)$

$$f'(x) := \lim_{h \rightarrow 0} \frac{f(x + h) - f(x)}{h}$$

Definition 3.02 - Notation for Differentiation

There are two ways to denote the derivative of an equation

$$f'(x) \iff \frac{df}{dx}, f''(x) \iff \frac{d^2f}{dx^2}, \dots, f^{(n)}(x) \iff \frac{d^n f}{dx^n}$$

N.B. - Using $\frac{df}{dx}$ is more informative, especially for equations with multiple variables.

3.1 Techniques for finding derivative

Theorem 3.03 - Sum Rule

Let f, g be differentiable with respect to x .

$$(f + g)' = f' + g'$$

Theorem 3.04 - Product Rule

Let f, g be differentiable with respect to x .

$$(fg)' = f'g + fg'$$

Theorem 3.05 - Quotient Rule

Let f, g be differentiable with respect to x .

$$\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$$

Definition 3.06 - Composite Functions

Let $f : B \rightarrow C$ and $g : A \rightarrow B$ Then

$$(f \circ g)(x) = f(g(x))$$

Theorem 3.07 - Chain Rule

Let f, g be differentiable with respect to x .

$$\frac{d}{dx}f(g(x)) = f'(g(x)).g'(x)$$

3.2 Implicit Differentiation

Definition 3.08 - Implicit Differentiation

Sometimes it is hard to isolate variables in multi-variable equations, in these cases differentiate both sides with respect to the same variable.

Remembering

$$\frac{d}{dx}(x) = 1 \text{ and } \frac{d}{dx}(y) = \frac{dy}{dx} = y'$$

Example

Find y if $x^3 + y^3 = 6xy$

$$\begin{aligned} \frac{d}{dx}(x^3 + y^3) &= \frac{d}{dx}(6xy) \\ \Rightarrow 3x^2 + 3y^2.y' &= 6y + 6x.y' \\ \Rightarrow y'(3y^2 - 6x) &= 6y - 3x^2 \\ \Rightarrow y' &= \frac{2y - x^2}{y^2 - 2x} \end{aligned}$$

3.3 Applications of The Derivative

Theorem 3.09 - Newton's Method

Let f be differentiable. Using *Newton's Method* we can approximate a solution to $f(x) = 0$.

- i) Take an initial guess, x_0 ;
- ii) Find the value of x where the tangent to x_0 on $f(x)$ intercepts the x-axis;
- iii) Use this value as the next guess;
- iv) Repeat until the value of x reduces little.

The equation for the tangent is

$$y = f(x_0) + (x - x_0)f'(x_0)$$

so a simplified equation for the process can be deduced

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

Theorem 3.10 - Angle between Intersecting Curves

Let $y = f_1(x)$ and $y = f_2(x)$ be two curves which intersect at (x_0, y_0) .

Then $y_0 = f_1(x_0) = f_2(x_0)$

Let m_1, m_2 be the gradient of the tangents to f_1 & f_2 at x_0 .

Then $\theta_i := \tan^{-1}(m_i)$ for $i = 1, 2$.

Let $\phi = |\theta_1 - \theta_2|$, then

$$\phi = \left| \frac{m_1 - m_2}{1 + m_1 m_2} \right|$$

Theorem 3.11 - L'Hospital's Rule

For two equations f, g with limit of $-\infty, 0$ or ∞ as x tends to a , it is hard to solve

$$\lim_{x \rightarrow a} \frac{f(x)}{g(x)}$$

Provided the limit exists, L'Hospital's Rule states that

$$\lim_{x \rightarrow a} \frac{f(x)}{g(x)} \iff \lim_{x \rightarrow a} \frac{f'(x)}{g'(x)}$$

3.4 Sketching Curves

Remark 3.11 - Sketching Curves

Evaluating the derivative of a curve can make it easier to sketch:

- i) When $f'(x) > 0$ the curve is heading upwards;
- ii) When $f'(x) < 0$ the curve is heading downwards;
- iii) When $f'(x) = 0$ the curve is flat;
- iv) When $f'(x) = \infty, -\infty$ there are asymptotes.

Definition 3.12 - Even Functions

If $f(x) = f(-x)$ then the function is symmetrical and said to be *even*.

Examples - $x^2, \cos(x), |x|$

Definition 3.13 - Odd Functions

If $f(x) = -f(-x)$ then the function is said to be *odd*.

Examples - $x, \sin(x), x \cdot \cos(x)$

Remark 3.14

Some functions are neither *odd* nor *even*.

Example - $x + x^2$