Calculus 1 - Application Notes

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How to Derive the Derivative of a Function.

Theory
The derivative of a function, $f(x)$, is defined to be

$$f'(x) := \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

The derivative gives you a function for the gradient at a given point.

Process
Expanding the numerator will usually cause the h in the denominator to disappear. Any terms which still have an h in them can be discounted as they will tend to 0.
L'Hôpital's Rule is often useful

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$$

Example
Find the derivative of $f(x) = (1 - x^2)^2$

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{(1-(x+h))^2 - (1-x^2)^2}{h} \\
&= \lim_{h\to 0} \frac{1 - 2(x+h)^2 + (x+h)^4 - (1 - 2x^2 + x^4)}{h} \\
&= \lim_{h\to 0} \frac{-2\left[x^2 + 2xh + h^2\right] + \left[x^4 + 4x^3 + 6x^2h + 4xh^3 + h^4\right] + x^2 - x^4}{h} \\
&= \lim_{h\to 0} \frac{-4xh - 2h^2 + 4x^3h + 6x^2h^2 + 4xh^3 + h^4}{h} \\
&= \lim_{h\to 0} -4x - 2h + 4x^3 + 6x^2h + 4xh^2 + h^3 \\
&= -4x + 4x^3 \\
f'(x) &= \underline{4x\left(x^2 - 1\right)}
\end{aligned}$$

Techniques for Finding the Derivative.

Sum Rule

$$(f+g)' = f' + g'$$

Example
Find the derivative of $(2x + x^3)$.

$$\begin{aligned}
(2x + x^3)' &= (2x)' + (x^3)' \\
&= \underline{2 + 3x^2}
\end{aligned}$$

Product Rule

$$(fg)' = f'g + fg'$$

Example
Find the derivative of $2xsin(x)$.

$$\begin{aligned}(2x\sin(x))' &= (2x)'\sin(x) + 2x(\sin(x))' \\ &= \underline{2\sin(x) + 2x\cos(x)}\end{aligned}$$

Quotient Rule

$$\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$$

Example
Find the derivative of $\dfrac{2x}{\sin(x)}$.

$$\begin{aligned}\left(\frac{2x}{\sin(x)}\right)' &= \frac{(2x)'\sin(x) - 2x(\sin(x))'}{\sin^2(x)} \\ &= \frac{2\sin(x) - 2x\cos(x)}{\sin^2(x)} \\ &= 2\left[\frac{1}{\sin(x)} - \frac{x}{\tan(x)\sin(x)}\right] \\ &= \underline{2\text{cosec}(x)\left[1 - x\cot(x)\right]}\end{aligned}$$

Chain Rule

$$f(g(x))' = f'(g(x))g'(x)$$

Example
Find the derivative of $\sin(2x)$.

$$\begin{aligned}f(x) = \sin(x) &\implies f'(x) = \cos(x) \\ g(x) = 2x &\implies g'(x) = 2 \\ \implies sin(2x)' &= \underline{2\cos(2x)}\end{aligned}$$

How to Find the Derivative of an Equation Where Elements Cannot be Easily Seperated.

Theory
Remember that $\frac{d}{dx}(x) = 1$ & $\frac{d}{dx}y = \frac{dy}{dx} = y'$.
By the chain rule $\left(\frac{1}{dx}\right)(dy) = \frac{dy}{dx}$ & $\frac{dx}{dy} = \frac{1}{dy/dx}$.

Process
Differentiate both sides with respect to the same variable.
to form an equation in termins of the given gradient$\left(\frac{dx}{dy}, \frac{dy}{dx}, \text{ etc.}\right)$

Example
Find $\frac{dy}{dx}$ of $x^3 + y^3 = xy$.

$$\begin{array}{rrcl} & \frac{d}{dx}\left(x^3+y^3\right) & = & \frac{d}{dx}(xy) \\ \implies & 3x^2+3y^2\frac{dy}{dx} & = & y+x\frac{dy}{dx} \\ \implies & \frac{dy}{dx}\left(x-3y^2\right) & = & 3x^2-y \\ \implies & \frac{dy}{dx} & = & \underline{\frac{3x^2-y}{x-3y^2}} \end{array}$$

How to Find the Tangent to a Parametric Curve

Process
Use the following formula gives the tangent when $t = t_0$ as a cartesian equation.

$$\frac{dy/dt}{dx/dt}(t_0) = \frac{y - y(t_0)}{x - x(t_0)}$$

Example
Find the tangent of $x = t^4 + 1$ & $y = t^2 + t$ when $t = 1$.

$$\begin{array}{rrcll} & \frac{dy}{dt} & = & 2t+1 & \\ & \frac{dx}{dt} & = & 4t^3 & \\ \implies & \frac{dy/dt}{dx/dt}(1) & = & \frac{2(1)+1}{4(1)} & = \frac{3}{4} \\ \text{Set} & \frac{3}{4} & = & \frac{y-y(1)}{x-x(1)} & \\ & & = & \frac{y-(1^2+1)}{x-(1^4+1)} & \\ & & = & \frac{y-2}{x-2} & \\ \implies & \frac{3x}{4}-\frac{6}{4} & = & y-2 & \\ \implies & y & = & \underline{\frac{3x}{4}-\frac{1}{2}} & \end{array}$$

How to Find the Arc-Length of a Parametric Curve

Process
The legth of $(x(t), y(t))$ for $a \leq t \leq b$ is found with

$$s = \int_a^b \sqrt{\left(\frac{dx}{dt}(t)\right)^2 + \left(\frac{dy}{dt}(t)\right)^2}\, dt$$

Example
Find the arc-legth of $(3\cos(t), 3\sin(t))$ for $t \in \left[0, \frac{\pi}{2}\right]$.

$$\begin{array}{rcl} s & = & \int_0^{\frac{\pi}{2}} \sqrt{(-3\sin(t))^2+(3\cos(t))^2}dt \\ & = & \int_0^{\frac{\pi}{2}} \sqrt{9\left[\sin^2(t)+\cos^2(t)\right]}dt \\ & = & \int_0^{\frac{\pi}{2}} 3dt \\ & = & 3t\Big|_0^{\frac{\pi}{2}} \\ & = & \underline{\frac{3\pi}{2}} \end{array}$$

How to Find the Curvature of A Curve

Theory
Curvature measures how fast the gradient of a curve is changing at a given point.

Process
For a cartesian curve *curvature* is given by

$$K(x) = \frac{|y''(x)|}{\left[1+(y'(x))^2\right]^{3/2}}$$

For a parametric curve it is given by

$$K(t_0) = \frac{y''(t_0)x'(t_0) - y'(t_0)x''(t_0)}{\left[(x'(t_0))^2 + (y'(t_0))^2\right]^{3/2}}$$

How to Solve a First-Order Differential Equation

Theory
A first-order differential equation takes the form

$$q(x) = p(x) + \frac{dy}{dx}$$

An integrating factor for an equation of this form is found by

$$R := e^{\int p(x)dx}$$

Process
Rearrange the differential equation to the form $q(x) = p(x)y + \frac{dy}{dx}$.
Calculate the integrating factor, R.
Then

$$\begin{array}{rrcl} & Rq(x) & = & \frac{d}{dx}(Ry) \\ \implies & y & = & \frac{\int Rq(x)dx}{R} \end{array}$$

Example
Find a $y = f(x)$ such that $xy' + y = e^x$.

$$\begin{array}{rrcl} \implies & R & = & e^{\int \frac{1}{x}dx} \\ & & = & e^{\ln(x)} \\ & & = & x \\ \implies & e^x & = & \frac{d}{dx}(xy) \\ \implies & y & = & \frac{\int e^x dx}{x} \\ & & = & \underline{\frac{e^x + c}{x}} \end{array}$$

How to Solve a Second-Order Linear Differential Equation

Theory
A second-order linear differential equation takes the form

$$ay + b\frac{dy}{dx} + c\frac{d^2y}{dx^2} = d(x)$$

The *complementary function*, y_c, is a solution to $ay + b\frac{dy}{dx} + c\frac{d^2y}{dx^2} = 0$.
The *particular function*, y_p, is a solution to $ay + b\frac{dy}{dx} + c\frac{d^2y}{dx^2} = d(x)$.

Process

Rearrange the differential equation to the form $ay + b\frac{dy}{dx} + c\frac{d^2y}{dx^2} = d(x)$.

Complementary function.

Set $a\lambda^2 + b\lambda + c = 0$

Solve this to find λ_1 & λ_2.

The form of λ_1 & λ_2 defines the form of the complementary function.

If

$\lambda_1 = \lambda_2 \in \mathbb{R} \implies y_c = \mu_1 e^{\lambda_1 x} + \mu_2 x e^{\lambda_1 x}$;

$\lambda_1, \lambda_2 \in \mathbb{R} \implies y_c = \mu_1 e^{\lambda_1 x} + \mu_2 e^{\lambda_2 x}$;

$\lambda_1, \lambda_2 \in i\mathbb{R} \implies y_c = \mu_1 \cos\left(\frac{\lambda_1}{i}\right) + \mu_2 \cos\left(\frac{\lambda_2}{i}\right)$; or

$\lambda_1, \lambda_2 \in \mathbb{C}$ & $Re(\lambda_1) \neq 0 \implies y_c = e^{Re(\lambda_1)x}\left[\mu_1 \cos(Im(\lambda_1) + \mu_2 \cos(Im(\lambda_1))\right]$. N.B. - These are all just $\mu_i e^{\lambda_i x}$ in Euler's Form.

Particular function.

Establishing a general particular solution.

This depends on the form of $d(x)$.

If

$d(x) = a_n x^n + \cdots + a_1 x + a_0$ set $y_p = b_n x^n + \cdots + b_1 x + b_0$;

$d(x) = ae^{bx}$ set $y_p = ce^{dx}$; or

$d(x) = a\sin(bx) + c\cos(dx)$ set $y_p = f\sin(gx) + h\cos(jx)$. Differentiate the general y_p twice to get y_p' and y_p''.

Substitute these into the original equation, in place of the ys.

Solve this to find values for the constants in y_p.

Finally, $y = y_c + y_p$.

Initial conditions are required to find values for the constants in y_c.

Example

Find $y = f(x)$ such that $10y'' - y = e^x$.

$$
\begin{array}{rrcl}
\text{Set} & 10\lambda^2 + 0\lambda - 1 & = & 0 \\
\implies & 10\lambda^2 & = & 1 \\
\implies & \lambda & = & \pm\frac{1}{\sqrt{10}} \\
& \lambda \in \mathbb{R} \text{ so } y_c & = & \mu_1 e^{\frac{x}{\sqrt{10}}} + \mu_2 e^{-\frac{x}{\sqrt{10}}} \\
\text{Set} & y_p & = & ae^x \\
\implies & y_p' & = & ae^x \\
\& & y_p'' & = & ae^x \\
\implies & 10ae^x - ae^x & = & e^x \\
\implies & 9a & = & 1 \\
\implies & a & = & \frac{1}{9} \\
\implies & y & = & \underline{\frac{1}{9}e^x + \mu_1 e^{\frac{x}{\sqrt{10}}} + \mu_2 e^{-\frac{x}{\sqrt{10}}}}
\end{array}
$$

How to Solve Inhomogenous Second-Order Differential Equations using the Wronskian

Theory

A second-order differential equation is *inhomogenous* if $y''(x)+ay'(x)+by(x)=d(x)$ with $d \neq 0$.
A *Wronskian Matrix* is defined as

$$\Phi[y_1,\dots,y_n] := \begin{pmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{pmatrix}$$

The *Wronskian* is defined as

$$W[y_1,\dots,y_n] := det(\Phi[y_1,\dots,y_n]) = \begin{vmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{vmatrix}$$

If $W[y_1,\dots,y_n] \neq 0$ then $y_1,\dots,y_n$ are linearly independent.

Process

Rearrange the differential equation to the form $y + a\frac{dy}{dx} + b\frac{d^2y}{dx^2} = c(x)$.
Find the complementary function, y_c, of this equation, as shown earlier.
y_c will have the form $y_c = \lambda_1 z_1(x) + \lambda_2 z_2(x)$ where $z_1(x)$ & $z_2(x)$ are linearly independent.
Define the particular solution to be $y_p = \mu_1(x)z_1(x) + \mu_2(x)z_2(x)$.
Then $\mu_1'(x) = \frac{\begin{vmatrix} 0 & z_2(x) \\ c(x) & z_2(x)' \end{vmatrix}}{W[z_1(x),z_2(x)]} = \frac{-x_2(x)c(x)}{W[z_1(x),z_2(x)]}$ and $\mu_2'(x) = \frac{\begin{vmatrix} z_1(x) & 0 \\ z_2(x)' & c(x) \end{vmatrix}}{W[z_1(x),z_2(x)]} = \frac{z_1(x)c(x)}{W[z_1(x),z_2(x)]}$.
Use integration to find $\mu_1(x)$ & $\mu_2(x)$.
Finally $y = \mu_1(x)z_1(x) + \mu_2(x)z_2(x)$.

Example
Find $y = f(x)$ such that $y'' - y = x$.

$$\begin{array}{rrcl}
\text{Set} & \lambda^2 - 1 & = & 0 \\
\Longrightarrow & \lambda^2 & = & 1 \\
\Longrightarrow & \lambda_1 = 1 & \& & \lambda_2 = -1 \\
\text{Set} & y_c & = & \mu_1 e^x + \mu_2 e^{-1} \\
 & & & \\
\text{So} & z_1(x) = e^x & \& & z_2(x) = e^{-x} \\
 & W[z_1(x), z_2(x)] & = & \begin{vmatrix} e^x & e^{-x} \\ e^x & -e^{-x} \end{vmatrix} \\
 & & = & -1 - 1 \\
 & & = & -2 \\
\Longrightarrow & \mu_1'(x) & = & \dfrac{\begin{vmatrix} 0 & e^{-x} \\ x & -e^{-x} \end{vmatrix}}{-2} \\
 & & = & \dfrac{-xe^{-x}}{-2} \\
 & & = & \dfrac{xe^{-x}}{2} \\
\& & \mu_2'(x) & = & \dfrac{\begin{vmatrix} e^x & 0 \\ e^x & 0 \end{vmatrix}}{-2} \\
 & & = & \frac{-xe^x}{2} \\
\Longrightarrow & \mu_1 & = & -\frac{1}{2}(1+x)e^{-x} \\
\& & \mu_2 & = & \frac{1}{2}(1-x)e^x \\
\Longrightarrow & y & = & (-\frac{1}{2}(1+x)e^{-x})e^x + (\frac{1}{2}(1-x)e^x)e^{-x} \\
 & & = & -\frac{1}{2}(1+x) + \frac{1}{2}(1-x) \\
 & y & = & \underline{-x}
\end{array}$$

How to Solve a First-Order linear Difference Equations

Theory
A first-order linear difference equation takes the form

$$y_{n+1} = f + ay_n$$

where y_n & y_{n+1} are part of a sequence.
This is a recursive function so requires a stopping condition.
Geometric progressions, $\{a, ar, ar^2, \dots\}$, are a first-order linear difference equation where $f = 0$.
The sum of the first n terms of a geometric progression is $\frac{a(1-r^n)}{1-r}$.

Process
Rearrange the equation to the form $y_{n+1} = ay_n + b$.

$$\begin{array}{rrcl}
\Longrightarrow & y_{n+1} & = & a[ay_{n-1} + b] + b \\
 & & = & b[1+a] + a^2 y_{n-1} \\
\Longrightarrow & y_{n+1} & = & b[1 + a + \dots + a^{n-1}] + a^n y_1
\end{array}$$

Since $1 + a + \dots + a^{n-1}$ is a geometric sequence then
$y_{n+1} = b\left[\frac{1-a^{n-1}}{1-a}\right] + a^n y_1$
This is now solved.

Example
Find $y_n = f(n)$ such that $3y_{n+1} + y_n = 15$.

$$\begin{aligned}
\implies \quad y_{n+1} &= 5 - \tfrac{1}{3}y_n \\
&= 5\left[1 - \tfrac{1}{3}\right] + \left[\tfrac{-1}{3}\right]^2 .y_{n-1} \\
&= 5\left[1 - \tfrac{1}{3}\right] + \cdots + \left(\tfrac{-1}{3}\right)^n\Big] + \left[\tfrac{-1}{3}\right]^n .y_0 \\
&= 5\left[\frac{1-\left(\frac{-1}{3}\right)^n}{1-\frac{-1}{3}}\right] + \left[\tfrac{-1}{3}\right]^n .y_0 \\
&= \tfrac{15}{4}\left[1 - \left(\tfrac{-1}{3}\right)^n\right] + \left[\tfrac{-1}{3}\right]^n .y_0 \\
&= \underline{\left(\tfrac{-1}{3}\right)^n \left[y_0 - \tfrac{15}{4}\right] + \tfrac{15}{4}}
\end{aligned}$$

How to Solve Second-Order Linear Difference Equations

Theory
A second-order linear difference equation takes the form

$$ay_{n+2} + by_{n+1} + cy_n = d(n)$$

The solution for y_n has two parts, a complementary & particular solution. $y_n = y_n^c + y_n^p$.
The complementary equation deals with the homogenous case & the particular equation with the inhomogenous case.

Process
Rearrange the difference equation to form $ay_{n+2} + by_{n+1} + cy_n = d(n)$.
Set $a\lambda^2 + b\lambda + c = 0$." Solve to find λ_1 & λ_2. Let y_n^c be the complementary function. Set it with these conditions.
If

$\lambda_1 = \lambda_2 \in \mathbb{R} \implies y_n^c = \mu_1\lambda_1^n + n\mu_2\lambda_2^n$;
$\lambda_1, \lambda_2 \in \mathbb{R} \implies y_n^c = \mu_1\lambda^n + \mu_2\lambda_2^n$; or,
$\lambda_1, \lambda_2 \in \mathbb{C}\ st\ \lambda_1, \lambda_2 = Re^{\pm i\theta} \implies y_n^c = R^n.[\mu_1 cos(n\theta) + \mu_2 sin(n\theta)]$.

Let y_n^p be the particular function. Set it with these conditions.
If

$d(n) = a \in \mathbb{R}\ \forall\ n \implies y_n^p = A$;
$d(n) = an \implies y_n^p = An + B$;
$d(n) = an^2 \implies y_n^p = An^2 + Bn + C$;
$d(n) = a\sin(bn) + c\cos(bn) \implies y_n^p = A\sin(bn) + B\cos(bt)$; or,
$d(n) = cn^t \implies y_n^p = An^t$.

Expand this to find y_{n+1} & y_{n+2}.
Substitue these into the original equation to find values for the constants, by comparing coefficients.
Finally, $y_n = y_n^c + y_n^p$.

Example
Find $y_n = f(n)$ such that $y_{n+2} - 4y_{n+1} + 4y_n = n$

$$\begin{array}{lrcl}
\text{Set} & \lambda^2 - 4\lambda + 4 & = & 0 \\
\implies & (\lambda - 2)^2 & = & 0 \\
\implies & \lambda_1 = 2 = \lambda_2 & & \\
\text{Set} & y_n^c & = & \mu_1 2^n + n\mu 2^n \\
 & & & \\
\text{Let} & y_n^p & = & An + b \\
\implies & y_{n+1}^p & = & A(n+1) + b \\
\& & y_{n+2}^p & = & A(n+2) + b \\
\implies & [A(n+2) + B] & & \\
 & -4[A(n+1) + B] + 4[An + B] & = & n \\
\implies & An - 2A + B & = & n \\
 & [n^1] : A & = & 1 \\
 & [n^0] : -2A + B & = & 0 \\
\implies & B & = & 2A = 2 \\
\implies & y_n^p & = & n + 2 \\
\implies & y_n & = & \underline{n + 2 + \mu_1 2^n + n\mu 2^n}
\end{array}$$

How to Find a Directional Derivative

Theory
figure A *direction* is a unit length vector.
This takes the form $\begin{pmatrix} \cos(\theta) \\ \sin(\theta) \end{pmatrix}$ in $\mathbb{R}^2$ & $\begin{pmatrix} \sin(\phi)\cos(\theta) \\ \sin(\phi)\sin(\theta) \\ \cos(\phi) \end{pmatrix}$ in $\mathbb{R}^3$.
A *directional derivative* gives the rate of change of a multi-variable function in a particular direction.

Process
$D_{\boldsymbol{u}}\boldsymbol{f}(\boldsymbol{x}) = \boldsymbol{f}'(\boldsymbol{x}) \cdot \boldsymbol{u}$ where $\boldsymbol{u}$ is a direction.

How to Find a Partial Derivative

Theory
A *partial derivative* is a directional derivative where the direction is a standard basis vector.
So $D_{\boldsymbol{e}_j}\boldsymbol{f}(\boldsymbol{x})$ is a partial derivative.
This can be denoted as $D_{\boldsymbol{e}_j}\boldsymbol{f}(\boldsymbol{x}) = \boldsymbol{f}'_j(\boldsymbol{x}) = \frac{\partial f}{\partial x_j}(\boldsymbol{x})$
A matrix of all the partial derivatives can be formed. If $\boldsymbol{f} : \mathbb{R}^n \to \mathbb{R}^m, n \in \mathbb{N}$ then $\boldsymbol{f}' \in M_{m,n}$.

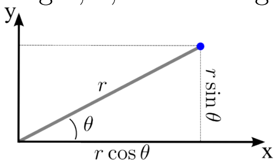
$$\boldsymbol{f}'(\boldsymbol{x}) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1} & \frac{\partial f_m}{\partial x_2} & \cdots & \frac{\partial f_m}{\partial x_n} \end{pmatrix}$$

Example
Find $\boldsymbol{f}'(\boldsymbol{x})$ of $\boldsymbol{f}'(x, y, z) = (x, 2x + y, z^2)$.

$$\boldsymbol{f}'(\boldsymbol{x}) = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2z \end{pmatrix}$$

How to Convert Between Cartesian & Polar Co-ordinates

Theory
Polar co-ordinates describe a point in two-dimensional space in terms of its distance from the origin, r, and its angle from the positive x-axis, θ.



Process - Cartesian to Polar
For a point $\boldsymbol{v} = \begin{pmatrix} r \\ \theta \end{pmatrix}$ in polar co-ordiantes.
$\boldsymbol{v} = \begin{pmatrix} r\cos(\theta) \\ r\sin(\theta) \end{pmatrix}$ in cartesian co-ordinates.

Process - Polar to Cartesian
For a point $\boldsymbol{v} = \begin{pmatrix} x \\ y \end{pmatrix}$ in cartesian co-ordinates.

$$x = r\cos(\theta) \ \& \ y = r\sin(\theta) \implies \tan(\theta) = \frac{y}{x} \ \& \ r = \sqrt{x^2+y^2}$$

So $\boldsymbol{v} = \begin{pmatrix} \sqrt{x^2+y^2} \\ \tan^{-1}\left(\frac{y}{x}\right) \end{pmatrix} = \begin{pmatrix} r \\ \theta \end{pmatrix}$ in polar co-ordinates.

Example
Polar → Cartesian
Let $\boldsymbol{v} = (60, \frac{\pi}{4})$.

$$\boldsymbol{v} = \begin{pmatrix} 60\cos(\frac{\pi}{4}) \\ 60\sin(\frac{\pi}{4}) \end{pmatrix} = \begin{pmatrix} 60(\frac{1}{\sqrt{2}}) \\ 60(\frac{1}{\sqrt{2}}) \end{pmatrix} = \underline{\begin{pmatrix} 30\sqrt{2} \\ 30\sqrt{2} \end{pmatrix}}$$

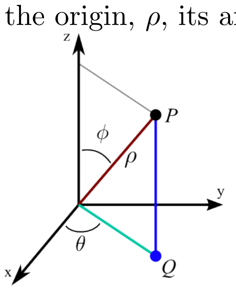
Cartesian → Polar
Let $\boldsymbol{v} = (-\sqrt{3}, 1)$.

$$\theta = \tan^{-1}(\frac{-1}{\sqrt{3}}) = -\frac{\pi}{6} \ \& \ r = \sqrt{\sqrt{3}^2 + 1^2} = \sqrt{4} = 2$$

$\implies \boldsymbol{v} = \underline{(2, \frac{-\pi}{6})}$.

How to Convert Between Cartesian & Sperical Co-Ordinates

Theory
Spherial co-ordinates describe a point in three-dimensional space in terms of its distance from the origin, ρ, its angle from the positive x-axis, θ, & its angel from the positive z-axis, ϕ.



Process - Cartesian to Spherical
Let $\boldsymbol{v} = (\rho, \theta, \phi)$ in spherical co-ordinates.
Then $\boldsymbol{v} = \begin{pmatrix} \rho\sin(\phi)\cos(\theta) \\ \rho\sin(\phi)\sin(\theta) \\ \rho\cos(\phi) \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ in cartesian co-ordiantes.

Process - Spherical to Cartesian
Let $\boldsymbol{v} = (x, y, z)$ in spherical co-ordinates.
Then $\boldsymbol{v} = \begin{pmatrix} \sqrt{x^2+y^2+z^2} \\ \tan^{-1}(\frac{y}{x}) \\ \cos^{-1}\left(\frac{z}{\sqrt{x^2+y^2+z^2}}\right) \end{pmatrix} = \begin{pmatrix} \rho \\ \theta \\ \phi \end{pmatrix}$ in spherical co-ordiantes.

How to Find a Volume of a Region

Theory
Multiple integeration is used to find the volume of a region in three-dimensional space.
Multiple integeration involves integerating over the same region, with a multiple different variables, sequentially.
These can be treated as seperate integrals, processing the innermost first.
They are denoted by $\int_R f(\boldsymbol{x})d\boldsymbol{x}$ where R is the region and $f(\boldsymbol{x})$ is a density function.
This can be expanded as

$$\int_{a_n}^{b_n} \left\{ \cdots \int_{a_2}^{b_2} \left\{ \int_{a_1}^{b_1} f(\boldsymbol{x}) dx_1 \right\} dx_2 \dots \right\} dx_n$$

where $R = \{\boldsymbol{x} \in \mathbb{R}^n : a_n \leq x_n \leq b_n, \dots, a_1 \leq x_1 \leq b_1\}$.

Process
When wanting to find the volume of $R \subset \mathbb{R}^3$ set $f\boldsymbol{x} = 1$ and perform a triple integration.

Example
Find the volume of $R = \{(x, y, z) \in \mathbb{R}^3 : 0 \leq x \leq 1 - y - z, 0 \leq y \leq 1 - z, 0 \leq z \leq 1\}$.

$$\begin{aligned}
\textstyle\int_D 1\ dxdydz &= \int_0^1 \left\{ \int_0^{1-z} \left\{ \int_0^{1-y-z} 1\ dx \right\} dy \right\} dz \\
&= \int_0^1 \left\{ \int_0^{1-z} 1 - y - z\ dy \right\} dz \\
&= \int_0^1 y - \frac{y^2}{2} - yz\Big|_0^{1-z}\ dz \\
&= \int_0^1 1 - z - \frac{(1-z)^2}{2} - z + z^2 dz \\
&= \int_0^1 (1-z)^2 - \frac{(1-z)^2}{2} dz \\
&= \int_0^1 \frac{(1-z)^2}{2} dz \\
&= -\frac{(1-z)^3}{6}\Big|_0^1 \\
&= \left[-\tfrac{1-1}{6}\right] - \left[-\tfrac{1-0}{6}\right] \\
&= \underline{\frac{1}{6}}
\end{aligned}$$

How to Find Centre of Mass of a Region using Integration

Process
Find the volume of a region, as shown before.
Find the mass of the region using the density function, $m = f(V)$.
Find the centre of mass by performing $\bar{x} := \frac{1}{m}\int_R x f(\boldsymbol{x})d\boldsymbol{x}$.
Repeat this for $\bar{y}$ & $\bar{z}$.

Example
Find the centre of mass of $D = \{(x, y, z) \in \mathbb{R}^3 : 0 \leq x \leq a, 0 \leq y \leq a, 0 \leq z \leq a\}$ with density function $f(x, y, z) = x^2 + y^2 + z^2$.

$$\begin{aligned}
m &= \int_0^a\int_0^a\int_0^a f(\boldsymbol{x})\ dxdydz \\
&= \int_0^a\int_0^a\int_0^a x^2+y^2+z^2\ dxdydz \\
&= \int_0^a\int_0^a \frac{a^3}{3}+ay^2+az^2dydz \\
&= \int_0^a \frac{a^4}{3}+\frac{a^4}{3}+a^2z^2dz \\
&= \frac{a^5}{3}+\frac{a^5}{3}+\frac{a^5}{3} \\
&= \underline{a^5} \\
\implies \bar{x} &= \frac{1}{a^5}\int_0^a\int_0^a\int_0^a xf(\boldsymbol{x})\ dxdydz \\
&= \frac{1}{a^5}\int_0^a\int_0^a\int_0^a x^3+xy^2+xz^2\ dxdydz \\
&= \frac{1}{a^5}\int_0^a\int_0^a \frac{a^4}{4}+\frac{a^2}{2}y^2+\frac{a^2}{2}z^2\ dydz \\
&= \frac{1}{a^5}\int_0^a\int_0^a \frac{a^5}{4}+\frac{a^5}{6}+\frac{a^3}{2}z^2\ dz \\
&= \frac{1}{a^5}\left[\frac{a^6}{4}+\frac{a^6}{6}+\frac{a^6}{6}\right] \\
&= \underline{\frac{7a}{12}}
\end{aligned}$$

Since this calculation is the same for both $\bar{y}$ & $\bar{z}$ so $\bar{\boldsymbol{x}} = \left(\frac{7a}{12}, \frac{7a}{12}, \frac{7a}{12}\right)$.

How to Determine the Stability of Equilibria of a System of Linear Differential Equations

Theory
A point, $\boldsymbol{x}$, is an equilibrium of a function if $\boldsymbol{f}(\boldsymbol{x}) = \boldsymbol{0}$.
An equilibrium, $\boldsymbol{e}$, is stable if

$\forall\ \epsilon > 0\ \exists\ \delta > 0\ st\ \forall\ \boldsymbol{f}(\boldsymbol{0}) \in \mathbb{R}^d\ \&\ t \geq 0, \|\boldsymbol{f}(\boldsymbol{0}) - \boldsymbol{e}\| < \delta \implies \|\boldsymbol{f}(\boldsymbol{t}) - \boldsymbol{e}\| \leq \epsilon$

otherwise it is unstable.
Equilibria in $\mathbb{R}^2$ are classified as: *Node* if both eigenvalues of $\boldsymbol{f}'(\boldsymbol{e})$ are real; *Centre* if both eigenvalues of $\boldsymbol{f}'(\boldsymbol{e})$ are purely imaginary; or *spiral* if the eigenvalues of $\boldsymbol{f}'(\boldsymbol{e})$ form a complex conjugate.

Process
An equilibrium ,$\boldsymbol{e}$, is stable if the real parts of all the eigenvalues of $\boldsymbol{f}'(\boldsymbol{e})$ are negative.

Example

Find and determine the stability of the equilibria of $\boldsymbol{f}\begin{pmatrix}x\\y\end{pmatrix} := \begin{pmatrix}x'\\y'\end{pmatrix} = \begin{pmatrix}x(1-\frac{y}{2})\\y(\frac{-3}{4}+\frac{x}{4})\end{pmatrix}$.

$$\begin{array}{lccl}
\text{Set} & x(1-\frac{y}{2})=0 & \& & y(\frac{-3}{4}+\frac{x}{4})=0\\
\implies & (0,0)\ \&\ (3,2) & & \text{are equilibria}\\
 & \boldsymbol{f}'(x,y) & = & \begin{pmatrix}1-\frac{y}{2} & -\frac{x}{2}\\ \frac{y}{4} & \frac{x}{4}-\frac{3}{4}\end{pmatrix}\\
\implies & \boldsymbol{f}'(0,0) & = & \begin{pmatrix}1 & 0\\ 0 & -\frac{3}{4}\end{pmatrix}\\
 & 1\ \&\ \frac{-3}{4} & & \text{are eigenvalues}\\
 & 1>0 & & (0,0)\text{ is unstable}\\
\& & \boldsymbol{f}'(3,2) & = & \begin{pmatrix}0 & \frac{-3}{2}\\ \frac{1}{2} & 0\end{pmatrix}\\
 & \begin{vmatrix}-\lambda & \frac{-3}{2}\\ \frac{1}{2} & -\lambda\end{vmatrix} & = & \lambda^2+\frac{3}{4}\\
\implies & \lambda & = & \pm i\frac{\sqrt{3}}{2}
\end{array}$$

Cannot conclude stability of $(3,2)$.

How to Determine the Stability of Equilibria of a Discrete Dynamic System

Theory

A discrete dynamic system is a recurrence relation where $\boldsymbol{x}_{n+1} = \boldsymbol{f}(\boldsymbol{x}_n)$.
A vector, $\boldsymbol{x}$, is an equilibrium if $\boldsymbol{f}(\boldsymbol{x}) = \boldsymbol{x}$.
An equilibrium, $\boldsymbol{x}$, is stable if

$\forall\ \epsilon>0\ \exists\ \delta>0\ st\ \|\boldsymbol{x}_0-\boldsymbol{x}\| \implies \|\boldsymbol{x}_n-\boldsymbol{x}\|\ \forall\ n\in\mathbb{N}$.

Process

To find equilibria set $\boldsymbol{f}(\boldsymbol{x}) = \boldsymbol{x}$ for a general $\boldsymbol{x}$.
Seperate these into their seperate functions for each dimension, the solve.
An equilibrium is stable if $f'(\boldsymbol{x}) < 1$.

Example

Find and determine the equilibria of $x_{n+1} = \frac{3x_n}{2}(1-x_n)$.

$$\begin{array}{lrcl}
\text{Set} & x & = & \frac{3x}{2}(1-x)\\
\implies & \frac{3x^2}{2}-\frac{x}{2} & = & 0\\
\implies & \frac{x}{2}(3x-1) & = & 0\\
\implies & x=0 & \& & x=\frac{1}{3}\text{ are equilibria.}\\
 & & & \\
 & f(x) & = & \frac{3}{2}-3x\\
\implies & f(0) & = & \frac{3}{2}\implies\text{unstable.}\\
\& & f(\frac{1}{3}) & = & \frac{1}{2}\implies\text{stable.}
\end{array}$$

Find general solution to system of first-order-differential-equations.

Process

i) Express system as matrix $\begin{pmatrix}x_1'\\ \vdots\\ x_n'\end{pmatrix} = A\begin{pmatrix}x_1\\ \vdots\\ x_n\end{pmatrix}, A\in M_n$.

ii) Find the eigenvalues & eigenvectors of A, $\{(\lambda, \boldsymbol{v})\}$.

iii) Express system as $\begin{pmatrix} x \\ y \end{pmatrix} = \mu_1.e^{\lambda_1.t}\boldsymbol{v}_1 + \cdots + \mu_n.e^{\lambda_n.t}\boldsymbol{v}_n$ where $\mu_1, \ldots, \mu_n$ are constants to be found.