Multi-Variable Calculus - Notes

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Contents

1 Review of Differential Calculus with Multi-Variable Functions

1.1 General Maps, $\mathbb{R}^m \to \mathbb{R}^n$

Definition 1.1 - *Scalar Map*
A *scalar map* maps a real-valued vector to a single real value.
They can be represented by the signature $f : \mathbb{R}^n \to \mathbb{R}, n \in \mathbb{N}$.

Definition 1.2 - *Vector Map*
A *vector map* maps a real-valued vector to another real-valued vector.
They can be represented by the signature $\mathbf{F} : \mathbb{R}^m \to \mathbb{R}^n,\ m, n \in \mathbb{R}$.
Vector maps can be considered as a collection of linear maps, so can be considered as

$$\mathbf{F}(\mathbf{x}) = \begin{pmatrix} F_1(\mathbf{x}) \\ \vdots \\ F_n(\mathbf{x}) \end{pmatrix} \quad F_i : \mathbb{R}^m \to \mathbb{R}, i = \{1, \dots, n\}, \mathbf{x} \in \mathbb{R}^m$$

.

Definition 1.3 - *Linear Map*
A general map, $\mathbf{F} : \mathbb{R}^m \to \mathbb{R}^n$, is *linear* if

$$\forall\ \mathbf{x}, \mathbf{y} \in \mathbb{R}^m\ \&\ \lambda,\ \mu \in \mathbb{R},\ \mathbf{F}(\lambda\mathbf{x} + \mu\mathbf{y}) = \lambda\mathbf{F}(\mathbf{x}) + \mu\mathbf{F}(\mathbf{y})$$

Proposition 1.1 - *Linear Maps as Matrices*
$\forall$ *linear maps*, $\mathbf{F} : \mathbb{R}^m \to \mathbb{R}^n\ \exists\ A \in M_{n,m}(\mathbb{R})$ *st* $\mathbf{F}(\mathbf{x}) = A\mathbf{x}$.

1.2 Derivative of a Map

Definition 1.4 - *Derivative*
The *derivative* of a general map, $\mathbf{F} : \mathbb{R}^m \to \mathbb{R}^n$, is an $n \times m$ matrix, $\mathbf{F}'$ st $\{\mathbf{F}'(\mathbf{x})\}_{ij} = \frac{\partial F_i}{\partial x_j}$.

Definition 1.5 - *Derivative of Single Variable Linear Maps*
The *derivative* of a single variable linear map, $f : \mathbb{R} \to \mathbb{R}$, is the map, $f' : \mathbb{R} \to \mathbb{R}$, st the line formed by

$$y = f(x_0) + (x - x_0)f'(x_0)$$

is the tangent to f at $x - x_0$.

Definition 1.6 - *Derivative of a Single Variable Map*
The *derivative*, f', of a single variable map, $f : \mathbb{R} \to \mathbb{R}$, can be defined as a limit

$$f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}$$

Definition 1.7 - *Derivative of Multi-Variable Maps*
The *derivative* of a multi-variable map, $\mathbf{F} : \mathbb{R}^m \to \mathbb{R}^n$, is a matrix, $\mathbf{F}' \in M_{n,m}$, st $\mathbf{F}(\mathbf{x}_0) + \mathbf{F}'(\mathbf{x}_0)(\mathbf{x} - \mathbf{x}_0)$ is a tangent plane at $\mathbf{x} = \mathbf{x}_0$ to the *hyper surface* of $\mathbf{F}(\mathbf{x})$.

Proposition 1.2 - *Derivative of a Linear Map*
If a map $\mathbf{F} : \mathbb{R}^m \to \mathbb{R}^n$ is linear st $\mathbf{F}(\mathbf{x}) = A\mathbf{x},\ A \in M_{n,m}$.
Then $\mathbf{F}'(\mathbf{x}) = A$.
Proof

$$\begin{array}{rrcl} \text{We know} & \mathbf{F}_i & = & \sum_{k=1}^m A_{ik}x_k \\ \implies & \frac{\partial F_i}{\partial x_j} & = & \{\mathbf{F}'\}_{ij} \\ & & = & \sum_{k=1}^m A_{ik}\frac{\partial x_k}{\partial x_j} \\ & & = & \sum_{k=1}^m A_{ik}\delta_{kj} \\ & & = & A_{ij} \end{array}$$

Definition 1.8 - *Jacobian Matrix*
The matrix, A, produced by the derivative of a vector map $\mathbf{F} : \mathbb{R}^m \to \mathbb{R}^n$, with elements $A_{ij} = \frac{\partial F_i}{\partial x_j}$, is called the *Jacobian Matrix.*

1.3 The Gradient of a Function

Definition 1.9 - *Gradient of a Scalar Function*
The *gradient* of a scalar function, $f : \mathbb{R}^m \to \mathbb{R}$, is defined as

$$\nabla f = \begin{pmatrix} \frac{\partial f}{\partial x_1} \\ \vdots \\ \frac{\partial f}{\partial x_m} \end{pmatrix}$$

Remark 1.1 - *Jacobian Matrix as Vector of Gradients*
Each row of a *Jacobian Matrix* can be considered as the *gradient* of that component of the map, as each component is scalar.

$$\mathbf{F}' = \begin{pmatrix} \nabla F_1^T \\ \vdots \\ \nabla F_n^T \end{pmatrix}$$

1.4 Directional Derivative

Definition 1.10 - *Directional Derivative*
The *directional derivative* of a map gives the rate of change in a given direction.
For a map, $\mathbf{F} : \mathbb{R}^m \to \mathbb{R}^n$, at a given point, $\mathbf{x}_0 \in \mathbb{R}^m$, in the given direction, $\hat{\mathbf{v}} \in \mathbb{R}^m$, the *directional derivative* is a vector in $\mathbb{R}^n$ given by the formula

$$D_{\hat{\mathbf{v}}}\mathbf{F}(\mathbf{x}_0) = \begin{pmatrix} \frac{d}{dt}F_1(\mathbf{x}_0 + t\hat{\mathbf{v}})|_{t=0} \\ \vdots \\ \frac{d}{dt}F_n(\mathbf{x}_0 + t\hat{\mathbf{v}})|_{t=0} \end{pmatrix}$$

This can be simplified to

$$D_{\hat{\mathbf{v}}}\mathbf{F}(\mathbf{x}_0) = \mathbf{F}'(\mathbf{x}_0)\hat{\mathbf{v}}$$

Proof

$$\begin{array}{rcl} D_{\hat{\mathbf{v}}}\mathbf{F}(\mathbf{x}_0) & = & \frac{d}{dt}\mathbf{F}(\mathbf{x}_0 + t\hat{\mathbf{v}})\Big|_{t=0} \\ & = & \lim_{h\to 0}\frac{\mathbf{F}(\mathbf{x}_0+(t+h)\hat{\mathbf{v}})-\mathbf{F}(\mathbf{x}_0+t\hat{\mathbf{v}})}{h}\Big|_{t=0} \\ & = & \lim_{h\to 0}\frac{\mathbf{F}(\mathbf{x}_0+h\hat{\mathbf{v}})-\mathbf{F}(\mathbf{x}_0)}{h} \\ & = & \lim_{h\to 0}\frac{\mathbf{F}(\mathbf{x}_0)+\mathbf{F}'(\mathbf{x}_0)(\mathbf{x}_0+h\hat{\mathbf{v}}-\mathbf{x}_0)-\mathbf{F}(\mathbf{x}_0)}{h} \\ & = & \lim_{h\to 0}\mathbf{F}'(\mathbf{x}_0)\hat{\mathbf{v}} \\ & = & \mathbf{F}'(\mathbf{x}_0)\hat{\mathbf{v}} \end{array}$$

Remark 1.2 - *Directional Derivative of a Scalar Map*
For a scalar map, $f : \mathbb{R}^m \to \mathbb{R}$, the *directional derivative* is simplified to

$$D_{\hat{\mathbf{v}}} f(\mathbf{x}_0) = \nabla f \cdot \hat{\mathbf{v}} \in \mathbb{R}$$

1.5 Operations on Maps & their Derivatives

Theorem 1.1 - *Derivative of Map Sums*
Let $\mathbf{F}, \mathbf{G}, \mathbf{H} : \mathbb{R}^m \to \mathbb{R}^n$ st $\mathbf{H} = \mathbf{F} + \mathbf{G}$.
Then $\mathbf{H}' = \mathbf{F}' + \mathbf{G}'$.
Proof

$$\begin{aligned} \mathbf{H}' &= \frac{\partial H_i}{\partial x_j} \\ &= \frac{\partial}{\partial x_j}(F_i + G_i) \\ &= \frac{\partial F_i}{\partial x_j} + \frac{\partial G_i}{\partial x_j} \\ &= \mathbf{F}' + \mathbf{G}' \end{aligned}$$

Theorem 1.2 - *Derivative of Scalar & Vector Map Compositions*
Let $\mathbf{F}, \mathbf{H} : \mathbb{R}^m \to \mathbb{R}^n$ & $f : \mathbb{R}^m \to \mathbb{R}$ st $\mathbf{H}(\mathbf{x}) = f(\mathbf{x})\mathbf{F}(\mathbf{x})$.
Then

$$\begin{aligned} \{\mathbf{H}'(\mathbf{x})\}_{ij} &= \frac{\partial}{\partial x_j}\left[f(\mathbf{x})F_i((\mathbf{x}))\right] \\ &= \frac{\partial f}{\partial x_j}(\mathbf{x})F_i(\mathbf{x}) + f(\mathbf{x})\frac{\partial F_i}{\partial x_j}(\mathbf{x}) \end{aligned}$$

Theorem 1.3 - *Chain Rule for Multi-Variable Functions*
Let $\mathbf{F} : \mathbb{R}^m \to \mathbb{R}^n, \mathbf{G} : \mathbb{R}^n \to \mathbb{R}^p$ & $H : \mathbb{R}^m \to \mathbb{R}^p$.
Define $\mathbf{H} = \mathbf{G} \cdot \mathbf{F}$. Then

$$\mathbf{H}'(\mathbf{x}) = (\mathbf{G}' \cdot \mathbf{F})(\mathbf{x})\mathbf{F}'(\mathbf{x})$$

Proof

Note that $H_i = G_i \begin{pmatrix} F_1(x_1, \dots, x_m) \\ \vdots \\ F_n(x_1, \dots, x_m) \end{pmatrix}$.

$$\begin{aligned} \implies \quad \mathbf{H}'_{ij} &= \frac{\partial H_i}{\partial x_j} \\ &= \frac{\partial G_i}{\partial x_1}\frac{\partial F_1}{\partial x_j} + \dots + \frac{\partial G_i}{\partial x_n}\frac{\partial F_n}{\partial x_j} \\ &= \sum_{k=1}^n \frac{\partial G_i}{\partial x_k} \cdot \frac{\partial F_k}{\partial x_j} \end{aligned}$$

1.6 Inverse Maps

Definition 1.11 - *Inverse Map*
Let $\mathbf{F}, \mathbf{G} : \mathbb{R}^n \to \mathbb{R}^n$.
$\mathbf{G}$ is the *inverse map* of $\mathbf{F}$ if $(\mathbf{G} \cdot \mathbf{F})(\mathbf{x}) = \mathbf{x}$.
Then, $\mathbf{G}$ can be written as $\mathbf{G} = \mathbf{F}^{-1}$.

Remark 1.3 - *Inverse Derivative*
Since $(\mathbf{F}^{-1} \cdot \mathbf{F})(\mathbf{x}) = \mathbf{x} = I\mathbf{x}$, where I is the identity matrix.
By differentiating we find $\mathbf{F}^{-1} \cdot \mathbf{F} = (\mathbf{F}^{-1})'(\mathbf{F}(\mathbf{x}))\ \mathbf{F}'(\mathbf{x}) = I$.
Thus $(\mathbf{F}^{-1})'(\mathbf{F}(\mathbf{x})) = (\mathbf{F}'(\mathbf{x}))^{-1}$.

The derivative of the inverse = Inverse of the derivative.

$$g'(f) = (f'(g))^{-1}$$

Remark 1.4 - *Solving Inverse Derivatives*
Let $\mathbf{F} : \mathbb{R}^m \to \mathbb{R}^n$ be a general map and $\mathbf{x} \in \mathbb{R}^m$. Then

$$(\mathbf{F}^{-1}(\mathbf{x}))' = \frac{1}{\mathbf{F}'(\mathbf{F}^{-1}(\mathbf{x}))}$$

Remark 1.5 - *When is a Matrix Invertible?*
A matrix A is *invertible* if $det(A) \neq 0$.

1.7 Solving Equations

Definition 1.12 - *Jacobian Determinant*
Let $\mathbf{F} : \mathbb{R}^n \to \mathbb{R}^n$ be a general map.
The *Jacobian Determinant* is the determinant of the Jacobian Matrix of $\mathbf{F}$.

$$J_{\mathbf{F}}(\mathbf{x}_0) := \left.\frac{\partial(F_1, \dots, F_n)}{\partial(x_1, \dots, x_n)}\right|_{\mathbf{x}=\mathbf{x}_0}$$

Proof (*Informal*)
For $\mathbf{x} \in \mathbb{R}^m$ close to $\mathbf{x}_0$ we can use $\mathbf{F}(\mathbf{x}_0) + \mathbf{F}'(\mathbf{x}_0)(\mathbf{x} - \mathbf{x}_0)$ to approximate $\mathbf{F}(\mathbf{x})$.
So

$$\begin{array}{rrcl} & y & = & \mathbf{F}(\mathbf{x}) \\ & & \approx & \mathbf{F}(\mathbf{x}_0) + \mathbf{F}'(\mathbf{x}_0)(\mathbf{x} - \mathbf{x}_0) \\ \implies & \mathbf{x} & \approx & \mathbf{x}_0 + [\mathbf{F}'(\mathbf{x}_0)]^{-1}(\mathbf{y} - \mathbf{y}_0) \end{array}$$

Since $\mathbf{y}_0 = \mathbf{F}(\mathbf{x}_0)$ but relies on the existence of the inverse of the Jacobian.

Theorem 1.4 - *Inverse Function Theorem*
Let $\mathbf{F} : \mathbb{R}^n \to \mathbb{R}^n$ & $\mathbf{x}_0, \mathbf{y}_0 \in \mathbb{R}^n$ st $\mathbf{y}_0 = \mathbf{F}(\mathbf{x}_0)$.
The *Inverse Function Theorem* states

If $J_{\mathbf{F}}(\mathbf{x}_0)) \neq 0$ then $\mathbf{y} = \mathbf{F}(\mathbf{x})$ can be solved uniquely as $\mathbf{x} = \mathbf{F}^{-1}(\mathbf{y})$ for $\mathbf{y}$ in the neighbourhood of $\mathbf{y}_0$.

Remark 1.6 - *Inverse Function Theorem & Lack of Inverse*
The *Inverse Function Theorem* does not say anything about the case where the inverse does not exists.

Definition 1.13 - *Implicit Function*
An *implicit function* is a function in which one variable cannot be explicitly expressed in terms of another.
Example - $x^2 + y^2 = 1$.

Theorem 1.5 - *Implicit Function Theorem*
Consider $\mathbf{x} \in \mathbb{R}^m$, $\mathbf{y} \in \mathbb{R}^n$ & $\mathbf{F} : \mathbb{R}^{m+n} \to \mathbb{R}^n$ where $\mathbf{F}$ a system of non-linear equations.
Suppose $\mathbf{F}(\mathbf{x}, \mathbf{y}) = \mathbf{0}$ is satisfied by $(\mathbf{x}_0, \mathbf{y}_0)$.
The *Implicit Function Theorem* states

If $J_{\mathbf{F}}(\mathbf{x}_0, \mathbf{y}_0) \neq 0$ we can express $\mathbf{F}(\mathbf{x}, \mathbf{y}) = \mathbf{0}$ in the form $\mathbf{y} = \mathbf{y}(\mathbf{x})$ for $\mathbf{y} : \mathbb{R}^m \to \mathbb{R}^n$ in the neighbourhood of $\mathbf{y}_0$.

Proof *(Informal)*
The i^{th} component of $\mathbf{F}(\mathbf{x}_0, \mathbf{y}_0) = \mathbf{0}$ is $F_i(x_1, \dots, x_n, y_1, \dots, y_n) = 0$
but $y_i = y_i(x_1, \dots, x_n) \forall\ i \in [1, n]$
So by taking $\frac{\partial F_i}{\partial x_j}$ and, using the chain rule, we get

$$\frac{\partial F_i}{\partial x_j} + \frac{\partial F_i}{\partial y_1}.\frac{\partial y_1}{\partial x_j} + \dots + \frac{\partial F_i}{\partial y_n}.\frac{\partial y_n}{\partial x_j} = 0$$

This can be expressed as

$$\begin{aligned}
&\begin{pmatrix} \frac{\partial F_1}{\partial x_1} & \cdots & \frac{\partial F_1}{\partial x_m} \\ \vdots & \ddots & \vdots \\ \frac{\partial F_n}{\partial x_1} & \cdots & \frac{\partial F_n}{\partial x_m} \end{pmatrix} + \begin{pmatrix} \frac{\partial F_1}{\partial y_1} & \cdots & \frac{\partial F_1}{\partial y_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial F_n}{\partial y_1} & \cdots & \frac{\partial F_n}{\partial y_n} \end{pmatrix} \begin{pmatrix} \frac{\partial y_1}{\partial x_1} & \cdots & \frac{\partial y_1}{\partial x_m} \\ \vdots & \ddots & \vdots \\ \frac{\partial y_n}{\partial x_1} & \cdots & \frac{\partial y_n}{\partial x_m} \end{pmatrix} = 0 \\
\implies &\begin{pmatrix} \frac{\partial y_1}{\partial x_1} & \cdots & \frac{\partial y_1}{\partial x_m} \\ \vdots & \ddots & \vdots \\ \frac{\partial y_n}{\partial x_1} & \cdots & \frac{\partial y_n}{\partial x_m} \end{pmatrix} = - \begin{pmatrix} \frac{\partial F_1}{\partial y_1} & \cdots & \frac{\partial F_1}{\partial y_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial F_n}{\partial y_1} & \cdots & \frac{\partial F_n}{\partial y_n} \end{pmatrix}^{-1} \begin{pmatrix} \frac{\partial F_1}{\partial x_1} & \cdots & \frac{\partial F_1}{\partial x_m} \\ \vdots & \ddots & \vdots \\ \frac{\partial F_n}{\partial x_1} & \cdots & \frac{\partial F_n}{\partial x_m} \end{pmatrix}
\end{aligned}$$

By **Definition 1.6** we have that, for $\mathbf{x}$ near $\mathbf{x}_0$,

$$\begin{aligned}
\mathbf{y} &\approx \mathbf{y}(\mathbf{x}_0) + \mathbf{y}'(\mathbf{x}_0)(\mathbf{x} - \mathbf{x}_0) \\
&= \mathbf{y}_0 + \mathbf{y}'(\mathbf{x}_0)(\mathbf{x} - \mathbf{x}_0)
\end{aligned}$$

So $\mathbf{y}$ is near $\mathbf{y}_0$ provided $\mathbf{y}'(\mathbf{x}_0)$ exists. (*i.e.* $J_{\mathbf{F}}(\mathbf{x}_0, \mathbf{y}_0) \neq 0$).

Remark 1.7 - *Linear Maps & Implicit Function Theorem*
If $\mathbf{F} : \mathbb{R}^{m+n} \to \mathbb{R}^n$ is linear in $\mathbf{y}$ then $\mathbf{F}(\mathbf{x}, \mathbf{y}) = \mathbf{0}$ can be written in the form

$$\mathbf{y} = \mathbf{G}(\mathbf{x})$$

Thus it can use the *inverse function theorem.*

1.8 Higher-Order Derivatives

Remark 1.8 - *Second Order Derivatives*

$$\frac{\partial^2 F_i}{\partial x_j \partial x_k} = \frac{\partial^2 F_i}{\partial x_k \partial x_j}$$

Theorem 1.6 - *Taylor's Theorem for Single Variable Scalar Functions*
For scalar functions $f : \mathbb{R} \to \mathbb{R}$ *Taylor's Theorem* states

$$\begin{aligned}
f(x) &= f(x_0) \\
&+ (x - x_0) f'(x_0) \\
&+ \tfrac{1}{2!}(x - x_0)^2 f''(x_0) \\
&\vdots \\
&+ \tfrac{1}{n!}(x - x_0)^n f^{(n)}(x_0)
\end{aligned}$$

Theorem 1.7 - *Taylor's Theorem for Double Variable Scalar Functions*
For scalar functions $f : \mathbb{R}^2 \to \mathbb{R}$ *Taylor's the*

$$\begin{aligned}
f(x, y) &= f(x_0, y_0) \\
&+ \begin{pmatrix} f_x(x_0, y_0) & f_y(x_0, y_0) \end{pmatrix} \begin{pmatrix} x - x_0 \\ y - y_0 \end{pmatrix} \\
&+ \tfrac{1}{2!} \begin{pmatrix} x - x_0 & y - y_0 \end{pmatrix} \begin{pmatrix} f_{xx}(x_0, y_0) & f_{xy}(x_0, y_0) \\ f_{yx}(x_0, y_0) & f_{yy}(x_0, y_0) \end{pmatrix} \begin{pmatrix} x - x_0 \\ y - y_0 \end{pmatrix} \\
&+ \text{higher order terms}
\end{aligned}$$

Remark 1.9 - *Taylor's Theorem - Vector Functions*
For vector functions, $\mathbf{F} : \mathbb{R}^n \to \mathbb{R}^m$, each component is a scalar function so by *Taylor's Theorem*

$$F_i(\mathbf{x}) = f_i(\mathbf{x}_0) + (\nabla f_i)^T (\mathbf{x} - \mathbf{x}_0) + \text{higher order terms}, \ \mathbf{x} \in \mathbb{R}^m$$

Consider the whole function we get

$$\mathbf{F}(\mathbf{x}) = \mathbf{F}(\mathbf{x}_0) + \mathbf{F}'(\mathbf{x}_0)(\mathbf{x} - \mathbf{x}_0) + \text{higher order terms}$$

If $\mathbf{x} \approx \mathbf{x}_0$ & $|\mathbf{x} - \mathbf{x}_0|^2 < |\mathbf{x} - \mathbf{x}_0|$ then

$$\mathbf{F}(\mathbf{x}) \approx \mathbf{F}(\mathbf{x}_0) + \mathbf{F}'(\mathbf{x}_0)(\mathbf{x} - \mathbf{x}_0)$$

Since the higher order terms tend to size $|\mathbf{x} - \mathbf{x}_0|^2$.

2 Differential Vector Calculus

2.1 Linear Algebra

Theorem 2.1 - *Sampling Property*
Using the *Einstein summation convention*

$$x_i = \delta_{ij} x_j$$

Remark 2.1 - *Cross Product is Anti-Commutative*

$$\mathbf{u} \times \mathbf{v} = -\mathbf{v} \times \mathbf{u}$$

Proof

$$\begin{aligned}
\mathbf{u} \times \mathbf{v} &= \begin{pmatrix} u_2v_3 - u_3v_2 \\ u_3v_1 - u_1v_3 \\ u_1v_2 - u_2v_1 \end{pmatrix} \\
\mathbf{v} \times \mathbf{u} &= \begin{pmatrix} u_3v_2 - u_2v_3 \\ u_1v_3 - u_3v_1 \\ u_2v_1 - u_1v_2 \end{pmatrix} \\
&= \begin{pmatrix} -(u_2v_3 - u_3v_2) \\ -(u_3v_1 - u_1v_3) \\ -(u_1v_2 - u_2v_1) \end{pmatrix} \\
&= -\begin{pmatrix} u_2v_3 - u_3v_2 \\ u_3v_1 - u_1v_3 \\ u_1v_2 - u_2v_1 \end{pmatrix} \\
&= -\mathbf{u} \times \mathbf{v}
\end{aligned}$$

Remark 2.2 - *Values of Levi-Civita Tensor in* $\mathbb{R}^3$

$$\begin{aligned}
\mathcal{E}_{123} = \mathcal{E}_{231} = \mathcal{E}_{312} &= 1 \\
\mathcal{E}_{231} = \mathcal{E}_{132} = \mathcal{E}_{321} &= -1 \\
\text{All other cases} &= 0
\end{aligned}$$

Proposition 2.1 - *Cross Product with Levi-Civita Tensor Notation*
By expanding the first component of the cross product we find

$$\begin{aligned}
[\mathbf{u} \times \mathbf{v}]_1 &= u_2v_3 - u_3v_2 \\
&= 0.u_1v_1 + 0.u_1v_2 + 0.u_1v_3 \\
&+ 0.u_2v_1 + 0.u_2v_2 + 1.u_2v_3 \\
&+ 0.u_3v_1 - 1.u_3v_2 + 0.u_3v_3 \\
&= \textstyle\sum_{j=1}^3 \sum_{k=1}^3 \mathcal{E}_{1jk} u_j v_k \\
&= \mathcal{E}_{1jk} u_j v_k
\end{aligned}$$

By considering the second & third components we find

$$[\mathbf{u} \times \mathbf{v}]_i = \mathcal{E}_{ijk} u_j v_k$$

Proposition 2.2 - *Double Levi-Civita Tensor Product*
The formula for the product of two *Levi-Civita tensor*s with one shared variable is given by

$$\mathcal{E}_{ijk}\mathcal{E}_{ilm} = \delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl}$$

This is found by considering all combinations.
There are 81 combinations but it is easy to discount most the zero-valued ones quickly.

Remark 2.3 - *Proving a Vector identity*
When proving a vector identity it is best to do so by comparing components.
i.e. Show the i^{th} component of $LHS = i^{th}$ component of rHS.

Theorem 2.2 - *Vector Triple Product*

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}$$

Proof

$$\begin{array}{rcl}
[\mathbf{a} \times (\mathbf{b} \times \mathbf{c})]_i & = & \mathcal{E}_{ijk} a_j [\mathbf{b} \times \mathbf{c}]_k \\
& = & \mathcal{E}_{ijk} a_j \mathcal{E}_{klm} b_i c_m \\
\text{By cycling } \mathcal{E}_{ijk} & = & \mathcal{E}_{ijk} \mathcal{E}_{klm} a_j b_i c_m \\
& = & (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}) a_j b_l c_m \\
& = & (\delta_{il} b_i)(\delta_{jm} c_m a_j) - (\delta_{im} c_m)(\delta_{jl} b_l a_j) \\
& = & b_i c_j a_j - c_i b_j a_j \\
& = & (\mathbf{c} \cdot \mathbf{a}) b_i - (\mathbf{b} \cdot \mathbf{a}) c_i \\
& = & [(\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}]_i
\end{array}$$

This is true for $i = 1, 2, 3$.
Since it holds for all components, the result holds.

2.2 Scalar & Vector Fields

Definition 2.1 - *Scalar Field*
A *scalar field* on $\mathbb{R}^3$ is a function $f : \mathbb{R}^3 \to \mathbb{R}$.

Definition 2.2 - *Vector Field*
A *vector field* on $\mathbb{R}^3$ is a map $\mathbf{v} : \mathbb{R}^3 \to \mathbb{R}^3$.

$$\mathbf{v}(\mathbf{r}) = (v_1(\mathbf{r}), v_2(\mathbf{r}), v_3(\mathbf{r}))$$

where $v_i(\mathbf{r})$ are scalar fields for $i = 1, 2, 3$.

Remark 2.4 - *Some Physical Applications of Scalar & Vector Fields*
Scalar Applications

i) Temperature, $T(\mathbf{q})$;

ii) Density, $\rho(\mathbf{r})$;

iii) Electric Charge Density, $q(\mathbf{r})$.

Vector Applications

i) Velocity, $\mathbf{v}(\mathbf{r})$;

ii) Displacement, $\mathbf{s}(\mathbf{r})$.

All of these quantities are governed by differential equations in space (and time) & require integration to find.

2.3 Gradient

Definition 2.3 - *Gradient of Scalar Fields*
A *gradient* shows the direction of greatest increase of a scalar field.

$$\nabla f := \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right) \quad \text{OR} \quad [\nabla f]_i = \frac{\partial f}{\partial x_i}$$

N.B. - The gradient is only a property of a scalar field, never a vector field.

Proposition 2.3 - *Gradient shows Direction of Greatest Increase*
When $\nabla f \neq \mathbf{0}$ it shows the direction of greatest increase of f.
Proof
Let $\mathbf{v} \in \mathbb{R}$ *st* $|\mathbf{v}\| = 1$.
The rate of change of f in the direction of $\mathbf{v}$ is $D_{\mathbf{v}} f = \nabla f \circ \mathbf{v} = |\nabla f|\ |\mathbf{v}| \cos\theta$
where θ is the angle between $\mathbf{v}$ & ∇f.
This value is maximised when $\theta = 0$.
In this case $\mathbf{v}$ is in the direction of ∇f.

Remark 2.5 - *Gradient of Scalar Fields that only depends on Magnitude*
Let $f : \mathbb{R}^3 \to \mathbb{R}$ be a scalar field & $g : \mathbb{R} \to \mathbb{R}$ be a linear map st $f(\mathbf{r}) \equiv g(r) = g(|\mathbf{r}|)$.
Then

$$\nabla g = \hat{\mathbf{r}} g'(r)$$

Proof

$$\begin{array}{rrcl} & [\nabla g]_i & = & \frac{\partial}{\partial x_i} g(r) \\ & \text{By chain rule} & = & \frac{\partial r}{\partial x_i} \cdot \frac{\partial g}{\partial r} \\ & & = & [\nabla r]_i g'(r) \\ & & = & \frac{x_i}{r} g'(r) \\ \implies & \nabla g & = & \frac{\mathbf{r}}{r} g'(r) \\ & & = & \hat{\mathbf{r}} g'(r) \end{array}$$

Remark 2.6 - *Gradient is Perpendicular to Surface*
The gradient of a function is perpendicular to the level surface of the function.
Proof
Let $\mathbf{c}(t)$ lie on surface S.

$$\begin{array}{lrcl} \implies & f(\mathbf{c}(t)) & = & C\ \forall\ t \\ \implies & \frac{d}{dt} f(\mathbf{c}(t)) & = & 0 \\ \implies & \frac{d}{dt}\mathbf{c}(t) \circ \nabla f|_{\mathbf{c}(t)} & = & 0 \end{array}$$

Let $\mathbf{c}(t)$ be a point & $\mathbf{c}(t + \delta t)$ be a point in the future.

$$\begin{array}{lrcl} \implies & d\mathbf{c} & = & \mathbf{c}(t+\delta t) - \mathbf{c}(t) \\ \implies & \frac{d\mathbf{c}}{dt} & = & \lim_{\delta t \to 0} \frac{\mathbf{c}(t+\delta t) - \mathbf{c}(t)}{dt} \end{array}$$

So $\frac{ds}{dt}$ lies in S.
Since $\nabla f \circ \frac{ds}{dt} = 0$ and the dot product of two vectors is 0 iff they are perpendicular
Then ∇f must be perpendicular to S.

2.4 Divergence

Definition 2.4 - *Divergence*
The *Divergence* of a vector field, $\mathbf{v}(\mathbf{r}) : \mathbb{R}^3 \to \mathbb{R}^3$, is the scalar field, $\mathbb{R}^3 \to \mathbb{R}$, given by

$$\begin{aligned} \nabla \circ \mathbf{v} &:= \tfrac{\partial v_1}{\partial x_1} + \tfrac{\partial v_2}{\partial x_2} + \tfrac{\partial v_3}{\partial x_3} \\ &\equiv \tfrac{\partial v_i}{\partial x_i} \\ &\equiv \delta_i v_i \end{aligned}$$

Remark 2.7 - *Divergence is not the dot product*
Divergence is not a dot product since ∇ is not a numerical vector, rather a differential operator.
N.B. - $\mathbf{v} \circ \nabla \equiv v_1 \frac{\partial}{\partial x_1} + v_2 \frac{\partial}{\partial x_2} + v_3 \frac{\partial}{\partial x_3} \not\equiv \nabla \circ \mathbf{v}$.

Remark 2.8 - *Interpretation of Divergence*
Divergence measures the expansion, when $\nabla \circ \mathbf{v} > 0$, or contraction, when $\nabla \circ \mathbf{v} < 0$, of a field.

Definition 2.5 - *Curl*
The *Curl* of a vector field, $\mathbf{v} : \mathbb{R}^3 \to \mathbb{R}^3$, is the vector field on $\mathbb{R}^3$ given by

$$\begin{aligned} \nabla \times \mathbf{v} &:= \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \partial x & \partial y & \partial z \\ v_1 & v_2 & v_3 \end{vmatrix} \\ &= \hat{\mathbf{x}} \left(\frac{\partial v_3}{\partial y} - \frac{\partial v_2}{\partial z} \right) + \hat{\mathbf{y}} \left(\frac{\partial v_1}{\partial z} - \frac{\partial v_3}{\partial x} \right) \hat{\mathbf{z}} \left(\frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y} \right) \\ \iff [\nabla \times \mathbf{v}]_i &= \varepsilon_{ijk} \tfrac{\partial v_k}{\partial x_j} \end{aligned}$$

Remark 2.9 - *Interpretation of Curl*
Curl measures local rotation of a vector field at a given point.
N.B. - When $\nabla \times \mathbf{v} = \mathbf{0}$ there is no local rotation.

Remark 2.10 - *Curl of Position Vector*

$$\begin{aligned} [\nabla \times \mathbf{r}]_i &= \varepsilon_{ijk} \tfrac{\partial}{\partial x_j} x_k \\ &= \varepsilon_{ijk} \delta_{jk} \\ &= \varepsilon_{ijj} \\ &= 0 \\ \implies \nabla \times \mathbf{r} &= \mathbf{0} \end{aligned}$$

2.5 Second Order Differential Operators

Proposition 2.4 - *Valid Second Order Differential Operators*
From the 3 first order differential operators, there are 9 possible second order differential operators.
However, due to their mappings, only 5 are valid.

i) $\nabla \times (\nabla)$ - $(\mathbb{R}^3 \to \mathbb{R}^3) \to (\mathbb{R}^3 \to \mathbb{R})$.

ii) $\nabla \circ (\nabla)$ - $(\mathbb{R}^3 \to \mathbb{R}^3) \to (\mathbb{R}^3 \to \mathbb{R}^3)$

iii) $\nabla(\nabla \circ)$ - $(\mathbb{R}^3 \to \mathbb{R}^3) \to (\mathbb{R}^3 \to \mathbb{R}^3)$.

iv) $\nabla \circ (\nabla \times)$ - $(\mathbb{R}^3 \to \mathbb{R}^3) \to (\mathbb{R}^3 \to \mathbb{R}^)$.

v) $\nabla \times (\nabla \times)$ - $(\mathbb{R}^3 \to \mathbb{R}^3) \to (\mathbb{R}^3 \to \mathbb{R}^3)$

Proposition 2.5 - *Null Second Order Differential Operators*
Of these 5 valid operators, 2 always produce $\mathbf{0}$ as their result.

i) For any scalar field, $f : \mathbb{R}^3 \to \mathbb{R}$, $\nabla \times (\nabla f) = \mathbf{0}$.
Proof

$$\begin{array}{rrcl} & [\nabla \times \nabla f]_i &=& \varepsilon_{ijk}\frac{\partial}{\partial x_j}[\nabla f]_k \\ &&=& \varepsilon_{ijk}\frac{\partial}{\partial x_j}\frac{\partial f}{\partial x_k} \\ &&\equiv& \varepsilon_{ijk}\frac{\partial}{\partial x_k}\frac{\partial f}{\partial x_j} \\ &&=& -\varepsilon_{ikj}\frac{\partial}{\partial x_k}\frac{\partial f}{\partial x_j} \\ &&=& -[\nabla \times \nabla f]_i \\ \implies & [\nabla \times \nabla f]_i &=& 0 \end{array}$$

This is true for $i = 1, 2, 3$ so $\nabla \times \nabla f = \mathbf{0}$.

iv) For any vector field, $\mathbf{v} : \mathbb{R}^3 \to \mathbb{R}^3$, $\nabla \circ (\nabla \times \mathbf{v}) = 0$.
Proof

$$\begin{array}{rrcl} \text{Since} & \frac{\partial}{\partial x_i}[\nabla \times \mathbf{v}]_i &=& \frac{\partial}{\partial x_i}\varepsilon_{ijk}\frac{\partial}{\partial x_j}v_k \\ &&=& -\frac{\partial}{\partial x_j}\varepsilon_{jik}\frac{\partial}{\partial x_i}v_k \\ &&=& -[\nabla \circ (\nabla \times \mathbf{v})]_i \\ \implies & \frac{\partial}{\partial x_i}[\nabla \times \mathbf{v}]_i &=& 0 \end{array}$$

This is true for $i = 1, 2, 3$ so $\nabla \circ (\nabla \times \mathbf{v}) = \mathbf{0}$.

Theorem 2.3 - *Product Rules For Second Order Differential Operator*

i) $\nabla \circ (f\mathbf{v}) = f\nabla \circ \mathbf{v} + \nabla f \circ \mathbf{v}$.
Proof

$$\begin{array}{rcl} \nabla \circ (f\mathbf{v}) &=& \frac{\partial}{\partial x_i}(fv_i) \\ &=& f\frac{\partial v_i}{\partial x_i} + v_i\frac{\partial f}{\partial x_i} \\ &=& f\nabla \circ \mathbf{v} + \mathbf{v} \circ \nabla f \end{array}$$

ii) $\nabla \times (f\mathbf{v}) = f\nabla \times \mathbf{v} + \nabla f \times \mathbf{v}$.
Proof

$$\begin{array}{rcl} [\nabla \times (f\mathbf{v})]_i &=& \varepsilon_{ijk}\frac{\partial}{\partial x_j}(fv_k) \\ &=& \varepsilon_{ijk}f\frac{\partial v_k}{\partial x_j} + \varepsilon_{ijk}\frac{\partial f}{\partial x_j}v_k \\ &=& f[\nabla \times \mathbf{v}]_i + [\nabla f \times \mathbf{v}]_i \end{array}$$

iii) $\nabla \circ (f\nabla g) = f\Delta g + \nabla f \circ \nabla g$.
Proof

$$\begin{array}{rcl} \nabla \circ (f\nabla g) &=& \frac{\partial}{\partial x_i}(f\nabla g_i) \\ &=& f\frac{\partial \nabla g_i}{\partial x_i} + \nabla g\frac{\partial f}{\partial x_i} \\ &=& f\nabla \circ \nabla g + \nabla g \circ \nabla f \\ &=& f\Delta g + \nabla f \circ \nabla g \end{array}$$

2.6 The Laplacian

Definition 2.6 - *Laplacian*
The *Laplacian* of a scalar field, $f(\mathbf{r}) : \mathbb{R}^3 \to \mathbb{R}$, is the scalar field defined by

$$\begin{aligned} \Delta f &:= \nabla \circ (\nabla f) \\ &= \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) f \\ &= (\nabla \circ \nabla) f \\ &\equiv \frac{\partial^2}{\partial x_i^2} f \end{aligned}$$

Remark 2.11 - *The Laplacian of a Vector Field*
Since *the Laplacian* can be applied to scalar components $v_1(\mathbf{r}),\ v_2(\mathbf{r}),\ v_3(\mathbf{r})$.
Then we can apply it to a vector field $\mathbf{v}(\mathbf{r}) = (v_1(\mathbf{r}), v_2(\mathbf{r}), v_3(\mathbf{r}))$.

$$\Delta \mathbf{v} := (\Delta v_1,\ \Delta v_2,\ \Delta v_3)$$

Proposition 2.6 - *The Laplacian is Anti-commutative*

$$\Delta \mathbf{v} = \nabla(\nabla \circ \mathbf{v}) - \nabla \times (\nabla \times \mathbf{v})$$

Proof

$$\begin{aligned} [\nabla \times (\nabla \times \mathbf{v})]_i &= \varepsilon_{ijk} \tfrac{\partial}{\partial x_j} [\nabla \times \mathbf{v}]_k \\ &= \varepsilon_{ijk} \tfrac{\partial}{\partial x_j} \varepsilon_{klm} \tfrac{\partial}{\partial x_l} v_m \\ &= \varepsilon_{kij} \varepsilon_{klm} \tfrac{\partial}{\partial x_j} \tfrac{\partial}{\partial x_l} v_m \\ &= (\delta_{il}\delta_{jm} - \delta_{im}\delta_{lj}) \tfrac{\partial}{\partial x_j} \tfrac{\partial}{\partial x_l} v_m \\ &= \tfrac{\partial}{\partial x_i} \tfrac{\partial}{\partial x_j} v_j - \tfrac{\partial}{\partial x_j} \tfrac{\partial}{\partial x_j} v_i \\ &= [\nabla(\nabla \circ \mathbf{v}) - \Delta \mathbf{v}]_i \end{aligned}$$

Proposition 2.7 - *Product Rule for The Laplacian*
Let $f(\mathbf{r}) : \mathbb{R}^3 \to \mathbb{R}$ be a scalar field & $\mathbf{v}(\mathbf{r}) : \mathbb{R}^3 \to \mathbb{R}^3$ be a vector field.
Then

$$\nabla \circ (f(\mathbf{v})) = f(\nabla \circ \mathbf{v}) + (\nabla f) \circ \mathbf{v}$$

Proof

$$\begin{aligned} \nabla \circ (f(\mathbf{v})) &= \tfrac{\partial}{\partial x_i} (f(v)_i) \\ &= f(\tfrac{\partial v_i}{\partial x_i}) + v_i \tfrac{\partial f}{\partial x_i} \\ &= f(\nabla \circ \mathbf{v}) + (\nabla f) \circ \mathbf{v} \end{aligned}$$

2.7 Curvilinear Co-Ordinate Systems

Definition 2.7 - *Curvilinear Co-Ordinates*
Curvilinear Co-Ordinates are defined by a map

$$\mathbf{r} : \mathbb{R}^3 \to \mathbb{R}^3$$

which takes $\mathbf{q} \in \mathbb{R}^3$, representing variables of a co-ordinate system, as an input.

$$\mathbf{r} = \mathbf{r}(\mathbf{q}) = (x(\mathbf{q}), y(\mathbf{q}), z(\mathbf{q}))$$

N.B. - The same ideas hold for two dimensional co-ordinate systems.

Definition 2.8 - *Metric Coefficients / Scale Factors*
Let $\mathbf{r}(\mathbf{q})$ be a curvilinear co-ordinate system.
The *Metric Coefficients* of $\mathbf{r}$ are

$$h_i := \left|\frac{\partial \mathbf{r}}{\partial q_i}\right|$$

Proposition 2.8 - *Polar Curvilinear Co-Ordinates*
By considering polar co-ordinates, which are two dimensional, we have

$$\begin{array}{rrcl} & \mathbf{q} & = & (r,\theta) \\ \implies & \mathbf{r}(\mathbf{q}) & = & (x(\mathbf{q}), y(\mathbf{q})) \\ & & = & (x(r,\theta), y(r,\theta)) \\ & & = & (r\cos\theta, r\sin\theta) \end{array}$$

Definition 2.9 - *Co-Ordinate Surfaces*
Co-Ordinate Surfaces are surfaces where $q_i = c$ for $i = 1, 2, 3$ and a constant c.

Definition 2.10 - *Co-Ordinate Curves*
Co-Ordinates Curves are the curves formed by the intersection of two *co-ordinate surfaces*.

Definition 2.11 - *Co-Ordinate Axes*
Co-ordinate Axes are determined by the tangents of *co-ordinate curves* at a point where three *co-ordinate surfaces* intersect.
N.B. - These are not, generally, fixed in space.

Proposition 2.9 - *Expressing Points in Space Using Curvilinear Co-Ordinates*
Consider the point $\mathbf{r} = \mathbf{r}(q_1, q_2, q_3)$.
We can determine the rate of change of $\mathbf{r}$ in the direction $\hat{\mathbf{q}}_1$ as

$$\hat{\mathbf{q}}_1 := \frac{\partial \mathbf{r}}{\partial q_1}(q_1, q_2, q_3)$$

Thus we can express any point in space as

$$\mathbf{r} = q_1\hat{\mathbf{q}}_1 + q_2\hat{\mathbf{q}}_2 + q_3\hat{\mathbf{q}}_3$$

where $\hat{\mathbf{q}}_\alpha = \frac{1}{h_\alpha}.\frac{\partial \mathbf{r}}{\partial q_\alpha}$ and are unit vectors.

Remark 2.12 - *Uniqueness of Curvilinear Co-Ordinates*
The inverse function theorem tells us that there is a unique map from one system to another, **iff** the Jacobian determinant of this map is non-zero.

$$J_{\mathbf{r}} = \begin{vmatrix} h_1[\hat{\mathbf{q}}_1]_1 & h_2[\hat{\mathbf{q}}_2]_1 & h_3[\hat{\mathbf{q}}_3]_1 \\ h_1[\hat{\mathbf{q}}_1]_2 & h_2[\hat{\mathbf{q}}_2]_2 & h_3[\hat{\mathbf{q}}_3]_2 \\ h_1[\hat{\mathbf{q}}_1]_3 & h_2[\hat{\mathbf{q}}_2]_3 & h_3[\hat{\mathbf{q}}_3]_3 \end{vmatrix} \neq 0$$

Definition 2.12 - *Orthogonal Curvilinear Co-Ordinate System*
An *Orthogonal Curvilinear Co-Ordinate System* is a co-ordinate system formed by three mutually perpendicular unit vectors along the co-ordinate axes $\hat{\mathbf{q}}_1$, $\hat{\mathbf{q}}_2$, $\hat{\mathbf{q}}_3$.
N.B. - The convention is for these systems to be *right-handed*, so $\hat{\mathbf{q}}_1 = \hat{\mathbf{q}}_2 \times \hat{\mathbf{q}}_3$.

Proposition 2.10 - *Curvilinear Co-Ordinates of Orthogonal Linear Maps*
Let $R \in M_3(\mathbb{R})$ be a *orthogonal matrix* and $\mathbf{r} = \mathbf{r}(\mathbf{q}) = R\mathbf{q}$ such that $x_i = R_{ij}q_j$.
Then $\mathbf{r}'(\mathbf{q}) = R$.

$$\implies h_i = \left|\frac{\partial \mathbf{r}}{\partial q_i}\right| = \sqrt{R_{1i}^2 + R_{2i}^2 + R_{3i}^2}\ \forall\ i \in [1, 2, 3]$$

Since $R^TR = I \implies h_i = 1\ \forall\ i \in [1,2,3]$.
So $\hat{\mathbf{q}}_j = (R_{1j}, R_{2j}, R_{3j})$.

Proposition 2.11 - *Curvilinear Co-Ordinate System of Cylindrical Polars*
In three dimensions cylindrical polars are defined by

$$(x,y,z) = \mathbf{r}(r,\theta,z) = (r\cos\theta, r\sin\theta, z)$$

Then

$$\begin{aligned}\frac{\partial \mathbf{r}}{\partial r} &= (\cos\theta, \sin\theta, 0)\\ \frac{\partial \mathbf{r}}{\partial \theta} &= (-r\sin\theta, r\cos\theta, 0)\\ \frac{\partial \mathbf{r}}{\partial z} &= (0,0,1)\end{aligned}$$

and

$$\begin{aligned}h_r &= \sqrt{\cos^2\theta + \sin^2\theta + 0^2}\\ &= 1\\ h_\theta &= \sqrt{r^2\cos^2\theta + r^2\sin^2\theta + 0^2}\\ h_z &= 1\end{aligned}$$

Thus the curvilinear co-ordinate system is

$$\begin{aligned}\hat{\mathbf{r}} &= (\cos\theta, \sin\theta, 0)\\ \hat{\boldsymbol{\theta}} &= (-\sin\theta, \cos\theta, 0)\\ \hat{\mathbf{z}} &= (0,0,1)\end{aligned}$$

Remark 2.13 - *Curvilinear Co-Ordinate System of Spherical Polars*
In three dimension, spherical polars are defined by

$$(x,y,z) = \mathbf{r}(r,\phi,\theta) = (r\sin\phi\cos\theta, r\sin\phi\sin\theta, r\cos\phi)$$

Then

$$\begin{aligned}\frac{\partial \mathbf{r}}{\partial r} &= (\sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi)\\ \frac{\partial \mathbf{r}}{\partial \phi} &= (r\cos\phi\cos\theta, r\cos\phi\sin\theta, -r\sin\phi)\\ \frac{\partial \mathbf{r}}{\partial \theta} &= (-r\sin\phi\sin\theta, r\sin\phi\cos\theta, 0)\end{aligned}$$

And

$$\begin{aligned}h_r &= \sqrt{\sin^2\phi(\sin^2\theta + \cos^2\theta) + \cos^2\phi}\\ &= 1\\ h_\phi &= r\\ h_\theta &= r\sin\phi\end{aligned}$$

Thus the curvilinear co-ordinate system is

$$\begin{aligned}\hat{\mathbf{r}} &= (\sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi)\\ \hat{\boldsymbol{\phi}} &= (\cos\phi\cos\theta, \cos\phi\sin\theta, -\sin\phi)\\ \hat{\boldsymbol{\theta}} &= (-\sin\theta, \cos\theta, 0)\end{aligned}$$

2.8 Transformation of the Gradient

Proposition 2.12 - *Gradient as Cartesian Vector*
The differential operator ∇ can be given as a Cartesian vector

$$\begin{aligned}\nabla &= \hat{\mathbf{x}}\frac{\partial}{\partial x} + \hat{\mathbf{y}}\frac{\partial}{\partial y} + \hat{\mathbf{z}}\frac{\partial}{\partial z}\\ &\equiv \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)\end{aligned}$$

Proposition 2.13 - *Transformation of Gradient*
Consider $f(\mathbf{r}) \equiv f(\mathbf{r}(\mathbf{q}))$.
Then for $\alpha = 1, 2, 3$

$$\begin{aligned} \frac{1}{h_\alpha}\frac{\partial f}{\partial q_\alpha} &= \frac{1}{h_\alpha}\frac{\partial x_i}{\partial q_\alpha}\frac{\partial f}{\partial x_i} \\ &= [\hat{\mathbf{q}}_\alpha]_i[\nabla f]_i \\ &= \hat{\mathbf{q}}_\alpha.\nabla f \end{aligned}$$

If $\mathbf{u} = u_1\hat{\mathbf{q}}_1 + u_2\hat{\mathbf{q}}_2 + u_3\hat{\mathbf{q}}_3$ then by orthogonality of $\hat{\mathbf{q}}_i$ $u_i = \mathbf{u} \circ \hat{\mathbf{q}}_i$.
Let $\mathbf{u} = \nabla f$

$$\implies \nabla = \frac{1}{h_i}\frac{\partial f}{\partial q_i}\hat{\mathbf{q}}_i \quad i = 1, 2, 3$$

So

$$\nabla = \sum_{\alpha=1}^{3}\frac{\hat{\mathbf{q}}_\alpha}{h_\alpha}\frac{\partial}{\partial q_\alpha}$$

2.9 Transformation of Divergence

Theorem 2.4 - *Gradient of Unit Vectors of Basis of Curvilinear Co-Ordinate System*
Let $\hat{\mathbf{q}}_\alpha$ for $\alpha = 1, 2, 3$ be the unit vector basis for a curvilinear co-ordinate system.
From $\nabla = \sum_{\alpha=1}^{3}\frac{\hat{\mathbf{q}}_\alpha}{h_\alpha}\frac{\partial}{\partial q_\alpha}$ we have

$$\begin{aligned} \nabla q_\beta &= \textstyle\sum_{\alpha=1}^{3}\frac{\hat{\boldsymbol{q}}_\alpha}{h_\alpha}\frac{\partial q_\beta}{\partial q_\alpha} \\ &= \textstyle\sum_{\alpha=1}^{3}\frac{\hat{\boldsymbol{q}}_\alpha}{h_\alpha}\delta_{\alpha\beta} \\ \nabla q_\beta &= \frac{\hat{\boldsymbol{q}}_\beta}{h_\beta} \end{aligned}$$

Theorem 2.5

$$\begin{array}{lrcl} \text{By Product Rule} & \nabla \times (q_2\nabla q_3) &=& q_2\nabla \times \nabla q_3 + \nabla q_2 \times \nabla q_3 \\ \text{Since } \nabla \times \nabla = 0 & &=& \nabla q_2 \times \nabla q_3 \\ \text{By } \textbf{Theorem 2.5} & &=& \frac{\hat{\mathbf{q}}_2}{h_2} \times \frac{\hat{\mathbf{q}}_3}{h_3} \\ \text{Right} - \text{Handed} & &=& \frac{\hat{\mathbf{q}}_1}{h_2h_3} \end{array}$$

Theorem 2.6

$$\begin{array}{rrcll} & \nabla \times \nabla q_\beta &=& 0 & \\ \implies & \nabla \times \left(\frac{\hat{\mathbf{q}}_\beta}{h_\beta}\right) &=& \mathbf{0} & \text{Theorem 2.4} \\ \text{Since} & \nabla \circ (\nabla \times q_2\nabla q_3) &=& 0 & \\ \implies & \nabla \circ \left(\frac{\hat{\mathbf{q}}_1}{h_2h_3}\right) &=& 0 & \text{Theorem 2.5} \\ \& & \nabla \circ \left(\frac{\hat{\mathbf{q}}_2}{h_3h_1}\right) &=& 0 & \text{Cyclic Permutations} \\ \& & \nabla \circ \left(\frac{\hat{\mathbf{q}}_3}{h_1h_2}\right) &=& 0 & \end{array}$$

Proposition 2.14 - *Transformation of Divergence*

Not examinable.

$$\begin{array}{lrcl}
 & \nabla \circ \mathbf{u} & = & \nabla \circ (u_1\hat{\mathbf{q}}_1 + u_2\hat{\mathbf{q}}_2 + u_3\hat{\mathbf{q}}_3) \\
 & & = & \nabla \circ \left(u_1 h_2 h_3 \circ \frac{\hat{\mathbf{q}}_1}{h_2 h_3}\right) + \left(u_2 h_3 h_1 \circ \frac{\hat{\mathbf{q}}_2}{h_3 h_1}\right) + \left(u_3 h_1 h_2 \circ \frac{\hat{\mathbf{q}}_3}{h_1 h_2}\right) \\
 & & = & \left[u_1 h_2 h_3 \nabla \circ \left(\frac{\hat{\mathbf{q}}_1}{h_2 h_3}\right) + \nabla(u_1 h_2 h_3) \circ \left(\frac{\hat{\mathbf{q}}_1}{h_2 h_3}\right)\right] \\
 & & + & \left[u_2 h_3 h_1 \nabla \circ \left(\frac{\hat{\mathbf{q}}_2}{h_3 h_1}\right) + \nabla(u_2 h_3 h_1) \circ \left(\frac{\hat{\mathbf{q}}_2}{h_3 h_1}\right)\right] \\
 & & + & \left[u_3 h_1 h_2 \nabla \circ \left(\frac{\hat{\mathbf{q}}_3}{h_1 h_2}\right) + \nabla(u_3 h_1 h_2) \circ \left(\frac{\hat{\mathbf{q}}_3}{h_1 h_2}\right)\right] \\
\text{Null Identity \& } \textbf{Theorem 2.6} & & = & \frac{\hat{\mathbf{q}}_1}{h_1 h_2} \circ \sum_{\alpha=1}^{3} \left[\frac{\hat{\mathbf{q}}_\alpha}{h_\alpha} \frac{\partial(u_1 h_2 h_3)}{\partial q_\alpha}\right] \\
 & & + & \frac{\hat{\mathbf{q}}_2}{h_2 h_3} \circ \sum_{\alpha=1}^{3} \left[\frac{\hat{\mathbf{q}}_\alpha}{h_\alpha} \frac{\partial(u_2 h_3 h_1)}{\partial q_\alpha}\right] \\
 & & + & \frac{\hat{\mathbf{q}}_3}{h_3 h_1} \circ \sum_{\alpha=1}^{3} \left[\frac{\hat{\mathbf{q}}_\alpha}{h_\alpha} \frac{\partial(u_3 h_1 h_2)}{\partial q_\alpha}\right] \\
\text{By Orthoganality} & \nabla \circ \mathbf{u} & = & \frac{1}{h_1 h_2 h_3}\left[\frac{\partial(u_1 h_2 h_3)}{\partial q_1} + \frac{\partial(u_2 h_3 h_1)}{\partial q_2} + \frac{\partial(u_3 h_1 h_2)}{\partial q_3}\right]
\end{array}$$

2.10 Transformation of curl

Proposition 2.15 - *Transformation of Curl*
Not examinable.

$$\begin{array}{rcl}
\nabla \times \mathbf{u} & = & \nabla \times (u_1\hat{\mathbf{q}}_1 + u_2\hat{\mathbf{q}}_2 + u_3\hat{\mathbf{q}}_3) \\
 & = & \nabla \times \left(u_1 h_1 \frac{\hat{\mathbf{q}}_1}{h_1}\right) + \nabla \times \left(u_2 h_2 \frac{\hat{\mathbf{q}}_2}{h_2}\right) + \nabla \times \left(u_3 h_3 \frac{\hat{\mathbf{q}}_3}{h_3}\right) \\
 & = & \nabla(u_1 h_1) \times \left(\frac{\hat{\mathbf{q}}_1}{h_1}\right) + (u_1 h_1)\left(\nabla \times \frac{\hat{\mathbf{q}}_1}{h_1}\right) \\
 & + & \nabla(u_2 h_2) \times \left(\frac{\hat{\mathbf{q}}_2}{h_2}\right) + (u_2 h_2)\left(\nabla \times \frac{\hat{\mathbf{q}}_2}{h_2}\right) \\
 & + & \nabla(u_3 h_3) \times \left(\frac{\hat{\mathbf{q}}_3}{h_3}\right) + (u_3 h_3)\left(\nabla \times \frac{\hat{\mathbf{q}}_3}{h_3}\right) \\
 & = & \sum_{\alpha=1}^{3} \left[\frac{\hat{\mathbf{q}}_\alpha}{h_\alpha} \frac{\partial(u_1 h_1)}{\partial q_\alpha} \times \left(\frac{\hat{\mathbf{q}}_1}{h_1}\right)\right] \\
 & + & \sum_{\alpha=1}^{3} \left[\frac{\hat{\mathbf{q}}_\alpha}{h_\alpha} \frac{\partial(u_2 h_2)}{\partial q_\alpha} \times \left(\frac{\hat{\mathbf{q}}_2}{h_2}\right)\right] \\
 & + & \sum_{\alpha=1}^{3} \left[\frac{\hat{\mathbf{q}}_\alpha}{h_\alpha} \frac{\partial(u_3 h_3)}{\partial q_\alpha} \times \left(\frac{\hat{\mathbf{q}}_3}{h_3}\right)\right] \\
\text{Since} & & \hat{\mathbf{q}}_1 \times \hat{\mathbf{q}}_1 = 0,\ \hat{\mathbf{q}}_2 \times \hat{\mathbf{q}}_1 = -\hat{\mathbf{q}}_3,\ \hat{\mathbf{q}}_3 \times \hat{\mathbf{q}}_1 = \hat{\mathbf{q}}_2 \\
 & = & \left[\frac{\hat{\mathbf{q}}_2}{h_1 h_3}\frac{\partial(h_1 u_1)}{\partial q_3} - \frac{\hat{\mathbf{q}}_3}{h_2 h_1}\frac{\partial(h_1 u_1)}{\partial q_2}\right] + \left[\frac{\hat{\mathbf{q}}_3}{h_2 h_1}\frac{\partial(h_2 u_2)}{\partial q_1} - \frac{\hat{\mathbf{q}}_1}{h_3 h_2}\frac{\partial(h_2 u_2)}{\partial q_3}\right] + \left[\frac{\hat{\mathbf{q}}_1}{h_3 h_2}\frac{\partial(h_3 u_3)}{\partial q_2} - \frac{\hat{\mathbf{q}}_2}{h_1 h_3}\frac{\partial(h_3 u_3)}{\partial q_1}\right] \\
 & = & \frac{\hat{\mathbf{q}}_1}{h_2 h_3}\left(\frac{\partial(h_3 u_3)}{\partial q_2} - \frac{\partial(h_2 u_2)}{\partial q_3}\right) + \frac{\hat{\mathbf{q}}_2}{h_3 h_1}\left(\frac{\partial(h_1 u_1)}{\partial q_3} - \frac{\partial(h_3 u_3)}{\partial q_1}\right) + \frac{\hat{\mathbf{q}}_3}{h_1 h_2}\left(\frac{\partial(h_2 u_2)}{\partial q_1} - \frac{\partial(h_1 u_1)}{\partial q_2}\right) \\
\nabla \times \mathbf{u} & = & \frac{1}{h_1 h_2 h_3}\begin{vmatrix} h_1\hat{\mathbf{q}}_1 & h_2\hat{\mathbf{q}}_2 & h_3\hat{\mathbf{q}}_3 \\ \frac{\partial}{\partial q_1} & \frac{\partial}{\partial q_2} & \frac{\partial}{\partial q_3} \\ h_1 u_1 & h_2 u_2 & h_3 u_3 \end{vmatrix}
\end{array}$$

3 Integral Theorems of Vector Calculus

Remark 3.1 - *Fundamental Theorem of Scalar Calculus*
From scalar calculus the *Fundamental Theorem of Calculus* is

$$\int_a^b f'(x)dx = [f(x)]_a^b = f(b) - f(a)$$

3.1 The Integral of a Scalar Field

Definition 3.1 - *Path*
A *path* is a map $\mathbf{p} : [t_1, t_2] \to \mathbb{R}^3$ such that $t \mapsto \mathbf{p}(t)$.
It connects the point $\mathbf{p}(t_1)$ to $\mathbf{p}(t_2)$ along the curve C.

N.B. - We say the curve C is *parametrised* by the path.

Definition 3.2 - *Line Integral*
Let $f:\mathbb{R}^3\to\mathbb{R}$ be a scalar field & $\mathbf{p}(t)$ be a path along a curve C for $t\in[t_1,t_2]$.
The *Line Integral* of f along C is found by

$$\int_C f(\mathbf{r})ds=\int_{t_1}^{t_2} f(\mathbf{p}(t)).|\mathbf{p}'(t)|dt$$

Since $\mathbf{r}=\mathbf{p}(t)$ on C and $d\mathbf{r}=\mathbf{p}'(t)dt$.

Remark 3.2 - *Parametrisation is not Unique*
The result of a line integral does not depend on how it is parametrised.
Proof
Consider $t=g(u)$ for $t_1<t<t_2$ such that $t_1=g(u_1)$ & $t_2=g(u_2)$.
Then

$$\begin{aligned}\int_{t_1}^{t_2} f(\mathbf{p}(t))|\mathbf{p}'(t)|dt &= \int_{u_1}^{u_2} f(\mathbf{p}(g(u)))|\mathbf{p}'(g(u))|g'(u)du\\ &= \int_{u_1}^{u_2} f(\mathbf{q}(u))|\mathbf{q}'(u)|du\end{aligned}$$

Set $\mathbf{q}(u)=\mathbf{p}(g(u))$

$$\begin{aligned}\mathbf{q}'(u) &= g'(u)\mathbf{p}'(g(u))\\ &= \int_{u_1}^{u_2} f(\mathbf{q}(u))|\mathbf{q}'(u)|du'\end{aligned}$$

So results are the same.

Remark 3.3 - *Direction and Line Integral Value*
The value of the line integral depends upon the direction of the path C.

$$\int_{-C} f(\mathbf{r})ds=-\int_C f(\mathbf{r})ds$$

N.B. - This is the same with standard integrals where $\displaystyle\int_{x_1}^{x_2} f(x)ds=-\int_{x_2}^{x_1} f(x)dx$.

3.2 The Line Integral of a Vector Field

Definition 3.3 - *Line Integral of Vector Field*
Let $\mathbf{F}(\mathbf{r}):\mathbb{R}^3\to\mathbb{R}^3$ be a vector field & $\mathbf{p}(t)$ be a path along a curve C for $t\in[t_1,t_2]$.
Then the *Line Integral* of $\mathbf{F}$ along C is found by

$$\int_C \mathbf{F}(\mathbf{r})\circ d\mathbf{r}=\int_{t_1}^{t_2}\mathbf{F}(\mathbf{p}(t))\circ\mathbf{p}'(t)dt$$

N.B. - The value depends on direction like scalar fields.

Remark 3.4 - *Parametrising Line Integral of Vector Field*
The line integral of a vector field **is** independent of how it is parametrised.

Proposition 3.1 - *Fundamental Theorem of Calculus for Line Integrals*
Let $f(\mathbf{r})$ be a scalar field & C be a curve in $\mathbb{R}^3$ parametrised by $\mathbf{p}(t)$ for $t\in[t_1,t_2]$.
Then

$$\int_C \nabla f(\mathbf{r})\circ d\mathbf{r}=f(\mathbf{p}(t_2))-f(\mathbf{p}(t_1))$$

Proof
We known $\int_C \nabla f(\mathbf{r}) \circ d\mathbf{r} = \int_C \nabla f(\mathbf{p}(t)) \circ \mathbf{p}'(t)dt$
However

$$\begin{aligned} \tfrac{d}{dt} f(\mathbf{p}(t)) &= \mathbf{p}'(t) \circ \nabla f \\ &= \int_{t_1}^{t_2} \tfrac{d}{dt} f(\mathbf{p}(t))dt \\ &= [f(\mathbf{p}(t))]_{t_1}^{t_2} \\ &= f(\mathbf{p}(t_2)) - f(\mathbf{p}(t_1)) \end{aligned}$$

3.3 Surface Integrals of Scalar & Vector Fields

Definition 3.4 - *Closed Path*
A *Closed Path* is a path $\mathbf{p}(t)$ for $t \in [t_1, t_2]$ where $\mathbf{p}(t_1) = \mathbf{p}(t_2)$.

Definition 3.5 - *Simple Path*
A *Simple Path* is a path $\mathbf{p}(t)$ for $t \in [t_1, t_2]$ that does not intersect except at $\mathbf{p}(t_1) = \mathbf{p}(t_2)$.

Definition 3.6 - *Boundary,* $\mathbb{R}^2$
The *boundary,*$\partial D \subset \mathbb{R}^2$, of a plane $D \subset \mathbb{R}^2$ is a collection of points which form a perimeter of D.
N.B. - This should be a simple closed path.

Definition 3.7 - *Boundary,* $\mathbb{R}^3$
The *boundary,* $\partial S \subset \mathbb{R}^3$, of a surface $S \subset \mathbb{R}^3$ is a mapping from the boundary ∂D. *N.B.* - This should be a simple closed path.

Remark 3.5 - *Path along Plane Boundary & Surface Boundary*
Let $S \subset \mathbb{R}^3$ be a surface and $D \subset \mathbb{R}^2$ be a plane.
Let $\mathbf{c}(t) \in \mathbb{R}^2$ be a simple closed path along ∂D.
Then $\mathbf{p}(t) = \mathbf{s}(\mathbf{c}(t))$ is the simple closed path along ∂S.

Remark 3.6 - *Parameter sing Surface Integrals*
Let $D \subset \mathbb{R}^2$ and $\bar{D} = D \cup \partial D$.
Define a map $\mathbf{s} : \bar{D} \to \mathbb{R}^3$ st $(u, v) \mapsto \mathbf{s}(u, v)$ where $\frac{\partial \mathbf{s}}{\partial u}$ & $\frac{\partial \mathbf{s}}{\partial v}$ are linearly independent of D.
A *surface* $S \subset \mathbb{R}^3$ is defined parametrically as

$$S = \{\mathbf{s}(u, v) | (u, v) \in D\}$$

Definition 3.8 - *Integral of a Scalar Field over a Surface*
The *Integral of a Scalar Field,* $f(\mathbf{r})$, over a surface, S, is given by

$$\int_S f(\mathbf{s})dS = \int_S f(\mathbf{r})|d\mathbf{S}|$$

where $d\mathbf{S} = \hat{\mathbf{n}}dS$ with $\hat{\mathbf{n}}$ begin a normal unit vector to S.
N.B. - This shows that a surface element is defined by its area, dS, & direction, $\hat{\mathbf{n}}$.

Remark 3.7 - *Set up to Computing Surface Integrals*
Let $\mathbf{s}(u + du, v)$, $\mathbf{s}(u, v + dv)$ & $\mathbf{s}(u, v)$ lie on the surface $S \subset \mathbb{R}^3$.
Assuming du & dv become vanishingly small we can use a Taylor expansion to show

$$\mathbf{s}(u + du, v) - \mathbf{s}(u, v) = \mathbf{s}(u, v) + du\frac{\partial \mathbf{s}}{\partial u}(u, v) - \mathbf{s}(u, v) + hot = du\frac{\partial \mathbf{s}}{\partial u}$$

$$\mathbf{s}(u, v + dv) - \mathbf{s}(u, v) = \mathbf{s}(u, v) + dv\frac{\partial \mathbf{s}}{\partial v}(u, v) - \mathbf{s}(u, v) + hot = dv\frac{\partial \mathbf{s}}{\partial v}$$

$du\frac{\partial \mathbf{s}}{\partial u}$ & $dv\frac{\partial \mathbf{s}}{\partial v}$ lie on the surface S and on the area

$$\hat{\mathbf{n}}ds = \frac{\partial \mathbf{s}}{\partial u}du \times \frac{\partial \mathbf{s}}{\partial v}dv \equiv \mathbf{N}(u,v)dudv$$

Thus

$$\mathbf{N}(u,v) = \frac{\partial \mathbf{s}}{\partial u} \times \frac{\partial \mathbf{s}}{\partial v}$$

N.B. - $\hat{\mathbf{n}}$ can point in two opposing directions, however this does not affect $|d\mathbf{S}|$.

Proposition 3.2 - *Computing Surface Integrals*

$$\int_S f(\mathbf{r})dS = \int_D f(\mathbf{s}(u,v))|\mathbf{N}(u,v)|dudv$$

Remark 3.8 - *Surface Integral to find Physical area*
Set $f(\mathbf{s}) = 1$ then $\displaystyle\int_S dS$ produces the physical area of the surface, S.

Definition 3.9 - *Integral of a Vector Field over a Surface*
Let $\mathbf{v}(\mathbf{r}) : \mathbb{R}^3 \to \mathbb{R}^3$ be a vector field and $S \subset \mathbb{R}^3$ be a surface.
The integral of $\mathbf{v}(\mathbf{r})$ over S is defined by

$$\begin{aligned}\int_S \mathbf{v}(\mathbf{r}) \circ d\mathbf{S} &\equiv \int_S \mathbf{v}(\mathbf{r}) \circ \hat{\mathbf{n}}dS \\ &= \int_S \mathbf{v}(\mathbf{s}(u,v)) \circ \mathbf{N}(u,v)dudv\end{aligned}$$

Remark 3.9 - *Direction of Surface Integrals*
Surface integrals have direction.
This direction, $\hat{\mathbf{n}}$, is in one of two opposing directions for a given surface, S
We must be told which direction to use during computation, otherwise we will receive different results.

3.4 Stokes' Theorem

Definition 3.10 - *Right Hand Thumb Rule*
Used to ensure the direction of a surface and its boundary are oriented correctly.
Point right hand thumb in direction of normal to surface, S, then the direction of its boundary,∂S, should follow the way the rest of the fingers curl.

Theorem 3.1 - *Stokes' Theorem*
Let $\mathbf{F}$ be a vector field in $\mathbb{R}^3$ and $S \subset \mathbb{R}^3$ be a surface with boundary ∂S.
Stokes' Theorem states

$$\int_S (\nabla \times \mathbf{F}) \circ d\mathbf{S} = \int_{\partial S} \mathbf{F} \circ d\mathbf{r}$$

N.B. - S and ∂S must be oriented consistently according to the right hand thumb rule.

3.5 Green's Theorem in the Plane

Remark 3.10 - *Green's Theorem vs Stokes' Theorem*
Green's Theorem is an application of Stokes' Theorem over two dimensions.

Theorem 3.2 - *Green's Theorem in the Plane*
Let $S \subset \mathbb{R}^3$ be a surface on $z = 0$ & $\mathbf{A} = (A_1(x,y), A_2(x,y), 0)$.
Then
$$\int_S \left(\frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y}\right) dxdy = \int_{\partial S} A_1 dx + A_2 dy$$

Proof

$$\begin{aligned}\partial \mathbf{S} &= \hat{\mathbf{z}}\partial S \equiv dx\ dy \\ \nabla \times \mathbf{A} &= \begin{vmatrix}\hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \partial x & \partial y & \partial z \\ A_1 & A_2 & 0\end{vmatrix} \\ &= \left(\frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y}\right)\hat{\mathbf{z}}\end{aligned}$$

Since $\frac{A_i}{dz} = 0$ for $i = 1, 2$.
By Stokes' Theorem
$$\int_S \left(\frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y}\right) dxdy = \int_{\partial S} \mathbf{A} \circ d\mathbf{r}$$
Since $d\mathbf{r} = dx\hat{\mathbf{x}} + dy\hat{\mathbf{y}}$ & ∂S is anti-clockwise by the right hand thumb rule.
$$\int_{\partial S} \mathbf{A} \circ d\mathbf{r} = \int_{\partial S} A_1 dx + A_2 dy$$

3.6 Volume Integrals

Definition 3.11 - *Volume Integral*
Let $f : \mathbb{R}^3 \to \mathbb{R}$ be a scalar field and $V \subset \mathbb{R}^3$.
The *Volume Integral* of f over V is given by
$$\int_V f(\mathbf{r})dV \equiv \iiint f(x,y,z)dxdydz$$

Definition 3.12 - *Volume Elements*
A *Volume Element* allow us to integrate a function wrt to a volume in various co-ordinate systems, by providing another volume & a scaling factor.
N.B. - They don't have a direction.

Proposition 3.3 - *Volume Integrals of a Scalar Field*
Under the transformation of co-ordinates from $\mathbf{r}(x,y,z)$ to a curvilinear system of $\mathbf{q}(q_1,q_2,q_3)$ defined by $\mathbf{r}(\mathbf{q})$
$$\int_V f(\mathbf{r})dxdydz = \int_{V_q} f(\mathbf{r}(\mathbf{q}))\left|\frac{\partial(x,y,z)}{\partial(q_1,q_2,q_3)}\right| dq_1dq_2dq_3$$
where V_q is mapped by $\mathbf{r}$ into V.
N.B. - The scale factor here is the Jacobian Determinant of this mapping.
Proof (*Sketch*)
Consider the elemental volume
$$\begin{aligned}dV &= dxdydz \\ &= (\hat{\mathbf{z}}dz) \circ ((\hat{\mathbf{x}}dx) \times (\hat{\mathbf{y}}dy))\end{aligned}$$

Under the mapping, the mapped elemental volume dV_q is defined by a parallelepiped with sides given by
$$\frac{\partial \mathbf{r}}{\partial q_1}dq_1,\ \frac{\partial \mathbf{r}}{\partial q_2}dq_2,\ \frac{\partial \mathbf{r}}{\partial q_3}dq_3$$

The volume of dV_q is therefore

$$|(\hat{\mathbf{q}}_3 h_4 dq_3) \circ ((\hat{\mathbf{q}}_1 h_1 dq_1) \times (\hat{\mathbf{q}}_2 h_2 dq_2))| = |J_{\mathbf{r}}| dq_1 dq_2 dq_3$$

Remark 3.11 - *Scaling Factor with Orthonormal Directions*
Since $\mathbf{q}_\alpha = \dfrac{1}{h_\alpha}\dfrac{\partial \mathbf{r}}{\partial q_\alpha}$ and using a result from *curvilinear co-ordinates*
We get

$$\text{If } \hat{\mathbf{q}}_j \text{ are orthonormal then } |J_{\mathbf{r}}| = h_1 h_2 h_3$$

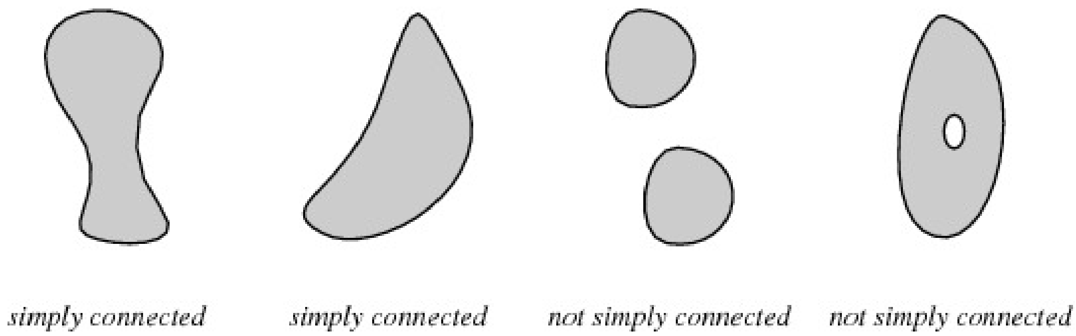
Proposition 3.4 - *Physical Volume from Volume Integral*
Let $V \subset \mathbb{R}^3$.
By setting $f = 1$ then $\displaystyle\int_V dV$ derives the physical volume of the volume V.

Remark 3.12 - *Volume Integrals of Common Curvilinear Systems*
Volume integrals for Cartesian, spherical & cylindrical co-ordinates.

$$\iiint f(x,y,z)dxdydz = \iiint f(r,\phi,\theta).r^2 sin\phi.drd\phi d\theta = \iiint f(r,\theta,z).r.dr.d\theta dz$$

3.7 Divergence Theorem

Definition 3.13 - *Simply Connected Region*
A region V is *simply connected* if all paths in V can be transformed onto all other paths in V continuously without ever leaving V.



Proposition 3.5 - *Divergence Theorem*
Let $\mathbf{F} : \mathbb{R}^3 \to \mathbb{R}^3$ be a vector field and $V \subset \mathbb{R}^3$ be a simply connected domain with boundary ∂V.
The *Divergence Theorem* states

$$\int_V \nabla \circ \mathbf{F} dV = \int_{\partial V} \mathbf{F} \circ d\mathbf{S} = \int_{\partial V} \mathbf{F} \circ \hat{\mathbf{n}} dS$$

where $\hat{\mathbf{n}}$ points out of V.
N.B. - Proof is non-examinable.

3.8 Green's Identities

Remark 3.13 - *Not a Green's Identity*
Let $\mathbf{F} = \nabla f$.

Then $\nabla \circ \mathbf{F} = \nabla \circ \nabla f = \Delta f$.
By the divergence theorem

$$\implies \int_V \Delta f dV = \int_{\partial V} \hat{\mathbf{n}} \nabla f dS$$

Proposition 3.6 - *Green's First Identity*
Let $f, g : \mathbb{R}^3 \to \mathbb{R}$ be scalar fields $\mathbf{F} : \mathbb{R}^3 \to \mathbb{R}^3$.
Define $\mathbf{F} = g\nabla f$.
Then *Green's First Identity* states

$$\int_V (\nabla g \circ \nabla f) + (g\Delta f) dV = \int_{\partial V} g\hat{\mathbf{n}} \circ \nabla f dS$$

Proposition 3.7 - *Green's Second Identity*
By subtracting the result of *Green's First Identity* we have

$$\int_V (g\Delta f - f\Delta g) dV = \int_{\partial V} (g\hat{\mathbf{n}} \circ \nabla f - f\hat{\mathbf{n}} \circ \nabla g) dS$$

0 Reference

0.1 Notation

Notation 0.1 - *Basis vectors*
Basis vectors are donated by having a hat,$\hat{}$, over them.
Example The basis vectors of $\mathbb{R}^3$ in Cartesian co-ordinates are

$$\begin{array}{rcccl}\hat{\mathbf{x}} & = & (1,0,0) & \equiv & \hat{\mathbf{e}}_1 \\ \hat{\mathbf{y}} & = & (0,1,0) & \equiv & \hat{\mathbf{e}}_2 \\ \hat{\mathbf{z}} & = & (0,0,1) & \equiv & \hat{\mathbf{e}}_3\end{array}$$

Notation 0.2 - *Boundary*
For a region $A \subset \mathbb{R}^n$ the *boundary* of A is denoted as

$$\partial A$$

Notation 0.3 - *Closed Loop Integral*
Let C be a *closed curve.*
An integral along C can be denoted as

$$\int_C \nabla f \circ d\mathbf{r} \equiv \oint_C \nabla f \circ d\mathbf{r}$$

Notation 0.4 - *Curl*
The *Curl* of a vector field $\mathbf{v} : \mathbb{R}^3 \to \mathbb{R}^3$ is denoted as

$$\nabla \times \mathbf{v}$$

Notation 0.5 - *Direction*
Let $\mathbf{v} \in \mathbb{R}^n$. The direction in $\mathbf{v}$ is defined as

$$\hat{\mathbf{v}} = \frac{\mathbf{v}}{|\mathbf{v}|}$$

Notation 0.6 - *Directional Derivative*
The *Directional Derivative* of the map $\mathbf{F} : \mathbb{R}^m \to \mathbb{R}^n,\ m, n \in \mathbb{N}$ at the point $\mathbf{x}_0 \in \mathbb{R}^m$ in the direction $\hat{\mathbf{v}} \in \mathbb{R}^m$ is denoted by

$$D_{\hat{\mathbf{v}}}\mathbf{F}(\mathbf{x}_0)$$

Notation 0.7 - *Divergence*
The *Divergence* of a vector field $\mathbf{v} : \mathbb{R}^3 \to \mathbb{R}^3$ is denoted as

$$\nabla \circ \mathbf{v}$$

Notation 0.8 - *Einstein Summation Convention*
The *Einstein Summation Convention* is used to simplify summation formulae.
When indices are repeated it can be assumed that a summation is being used, thus the $\sum$ notation can be dropped.

$$\sum_{i=1}^{n} x_i y_i \implies x_i y_i$$

But, *Greek* characters are used as indices when we don't want to imply summation.

Notation 0.9 - *Gradient*
The *Gradient* of a scalar field $f : \mathbb{R}^3 \to \mathbb{R}$ is denoted as

$$\nabla f$$

Notation 0.10 - *Isolating Vector Components*
Let $\mathbf{v} \in \mathbb{R}^n$ for $n \in \mathbb{N}$ and $i \in [1, n]$.
To isolate the i^{th} component of the vector $\mathbf{v}$ we use

$$[\mathbf{v}]_i$$

Notation 0.11 - *Jacobian Determinant*
The *Jacobian determinant* of the vector map $\mathbf{F}$ is denoted

$$J_{\mathbf{F}}$$

Notation 0.12 - *Kronecker Delta*
The *Kronecker Delta* function of two variables which returns 1 if they are the same, and 0 otherwise.

$$\delta_{ij} = \begin{cases} 1 \text{ if } i = j \\ 0 \text{ otherwise} \end{cases}$$

Notation 0.13 - *Laplacian*
The *Laplacian* of a scalar field $f : \mathbb{R}^3 \to \mathbb{R}$ is denoted as

$$\Delta f$$

N.B. - Occasionally it is denoted as $\nabla^2 f$.

Notation 0.14 - *Levi-Civita Tensor*
The *Levi-Civita Tensor* describes the antisymmetric nature of vectors.
It is denoted in the form

$$\mathcal{E}_{ijk} \quad i, j, k \in \mathbb{N}$$

It is defined by the following rules

i) $\mathcal{E}_{123} = 1$.

ii) $\mathcal{E}_{ijk} = 0$ if any suffices are repeated.

iii) Reversing any suffices negates the value of $\mathcal{E}_{ijk}$.

N.B. - This is occasionally know as *antisymmetric symbol.*

Notation 0.15 - *Line Integral of Scalar Field*
The *Line Integral* of a scalar field $f : \mathbb{R}^3 \to \mathbb{R}$ along a curve C is denoted

$$\int_C f(\mathbf{r}) ds$$

where $ds = |d\mathbf{r}|$.

Notation 0.16 - *Line Integral of Vector Field*
the *Line Integral* of a vector field $\mathbf{F} : \mathbb{R}^3 \to \mathbb{R}^3$ along a curve C is denoted as.

$$\int_C \mathbf{F}(\mathbf{r}) \circ d\mathbf{r}$$

N.B. - $\circ$ is dot product here.

Notation 0.17 - *Position Vector,* $\mathbb{R}^3$
The general *position vector* in $\mathbb{R}^3$ is often denoted by

$$\mathbf{r} := (x, y, z)$$

Notation 0.18 - *Vector Notation*
$\mathbf{x} \in \mathbb{R}^m$ for $m \in \mathbb{N}$ is used to denoted the vector $\begin{pmatrix} x_1 \\ \vdots \\ x_m \end{pmatrix}$, $x_i \in \mathbb{R}\ \forall\ i = \{1, \dots, m\}$.

Notation 0.19 - *Volume Integral of a Scalar Field*
Let $f(\mathbf{r})$ be a scalar field.
The *volume integral* of $f(\mathbf{r})$ over $V \subset \mathbb{R}^3$ is denoted

$$\int_V f(\mathbf{r})dV \equiv \iiint_V f(x, y, z)dxdydz$$

0.2 Definitions

Definition 0.1 - *Anti-Symmetric*
An operation is *anti-symmetric* if reversing the order of its variables does not change the magnitude of the result, *but* does swap the sign.
Example - Cross-product.

Definition 0.2 - *Associative*
An operation is *associative* if changing the order of its variables does not alter the outcome.
Example - Addition.

Definition 0.3 - *Cross Product*
Let $\mathbf{u}, \mathbf{v} \in \mathbb{R}^3$ st $\mathbf{u} = (u_1, u_2, u_3)$ & $\mathbf{v} = (v_1, v_2, v_3)$.
The *Cross product* of $\mathbf{u}$ & $\mathbf{v}$ is defined

$$\begin{aligned} \mathbf{u} \times \mathbf{v} &:= \begin{vmatrix} \hat{\mathbf{e}}_1 & \hat{\mathbf{e}}_2 & \hat{\mathbf{e}}_3 \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} \\ &= \hat{\mathbf{e}}_1 \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} - \hat{\mathbf{e}}_2 \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix} + \hat{\mathbf{e}}_3 \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} \end{aligned}$$

Definition 0.4 - *Direction*
A *Direction* is a vector of unit length

$$|\mathbf{v}| = 1$$

Definition 0.5 - *Dot Product*
Let $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ st $\mathbf{u} = (u_1, \dots, u_n)$ & $\mathbf{v} = (v_1, \dots, v_n)$.
The *Dot Product* of $\mathbf{u}$ & $\mathbf{v}$ is defined

$$\mathbf{u} \cdot \mathbf{v} := u_i v_i = \sum_{i=1}^{n} u_i v_i$$

Definition 0.6 - *Hessian Matrix*
Let $f : \mathbb{R}^n \to \mathbb{R}$ be a scalar function.
The *Hessian Matrix* is the square matrix representation of the second-order partial derivatives

of f.

$$\mathbf{H} = \begin{pmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_n} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial^2 x_2} & \cdots & \frac{\partial^2 f}{\partial x_2 \partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 f}{\partial x_n \partial x_1} & \frac{\partial^2 f}{\partial x_n \partial x_2} & \cdots & \frac{\partial^2 f}{\partial x_n^2} \end{pmatrix}$$

Definition 0.7 - *Hyperbolic Trigonometric Functions*

$$\sinh x = \frac{e^x - e^{-x}}{2} \quad \cosh x = \frac{e^x + e^{-x}}{2} \quad \tanh x = \frac{\sinh x}{\cosh x}$$
$$\text{cosech } x = \frac{1}{\sinh x} \quad \text{sech } x = \frac{1}{\cosh x} \quad \coth x = \frac{1}{\tanh x}$$

Definition 0.8 - *Level Curves/Surface*
Level curves/surfaces are lines where $f(x, y) = c$ for some constant $c \in \mathbb{R}$.
N.B. - Consider contour lines on an OS map.

Definition 0.9 - *Multi-Linear*
A multi-variable function is *multi-linear* if it can be separated into multiple functions, each taking one variable, without altering the outcome.
i.e. $f(\lambda x_1 + \mu x_2, y) = \lambda f(x_1, y) + \mu f(x_2, y)$.

Definition 0.10 - *Norm*
Let $\mathbf{v} \in \mathbb{R}^n$ for $n \in \mathbb{N}$.
The *Norm* of $\mathbf{v}$ is the strictly positive length of a $\mathbf{v}$.

$$|\mathbf{v}| = \sqrt{\mathbf{v} \cdot \mathbf{v}} = \sqrt{\sum_{i=1}^{n} v_i^2}$$

Definition 0.11 - *Orthogonal Matrix*
A matrix, $A \in M(\mathbb{R})$ is *orthogonal* if

$$A^T A = I \implies A^T = A^{-1}$$

0.3 Theorems

Theorem 0.1 - *Chain Rule*

$$\begin{aligned} [f(g(x))]' &= f'(g(x)).g'(x) \\ (\mathbf{F} \circ \mathbf{G})' &= (\mathbf{F}' \circ \mathbf{G})\mathbf{G}' \\ \frac{dy}{dx} &= \frac{dy}{dt}\frac{dt}{dx} \end{aligned}$$

Theorem 0.2 - Common Curvilinear Co-Ordinate Systems

$$\begin{array}{rcccl} \textit{Cylindrical} \quad (x, y, z) &=& \mathbf{r}(r, \theta, z) &=& (r\cos\theta, r\sin\theta, z) \\ \textit{Polar} \quad (x, y) &=& \mathbf{r}(r, \theta) &=& (r\cos\theta, r\sin\theta) \\ \textit{Spherical} \quad (x, y, z) &=& \mathbf{r}(r, \phi, \theta) &=& (r\sin\phi\cos\theta, r\sin\phi\theta, r\cos\phi) \end{array}$$

Theorem 0.3 - *Derivative of Exponent*

$$[e^{f(x)}]' = f(x)' e^{f(x)}$$

Theorem 0.4 - *Derivative of Logarithms*

$$(\log_b f)' = \frac{f'}{\ln(b).f'} \quad (\ln f)' = \frac{f'}{f}$$

Theorem 0.5 - *Derivative of Hyperbolic Trigonometric Functions - Hyperbolic*

$$\begin{array}{rcl|rcl} \frac{d}{dx}\sinh x &=& \cosh x & \frac{d}{dx}\text{cosech } x &=& -\text{cosech } x\coth x \\ \frac{d}{dx}\cosh x &=& \sinh x & \frac{d}{dx}\text{sech } x &=& -\text{sech } x\tanh x \\ \frac{d}{dx}\tanh x &=& \text{sech }^2 x & \frac{d}{dx}\coth x &=& -\text{cosech }^2 x \end{array}$$

Theorem 0.6 - *Derivative of Functions - Inverse*

$$\begin{array}{rcl|rcl} \frac{d}{dx}\sin^{-1}x &=& \frac{1}{\sqrt{1-x^2}} & \frac{d}{dx}\text{cosec }^{-1}x &=& \frac{-1}{|x|\sqrt{x^2-1}} \\ \frac{d}{dx}\cos^{-1}x &=& \frac{-1}{\sqrt{1-x^2}} & \frac{d}{dx}\sec^{-1}x &=& \frac{1}{|x|\sqrt{x^2-1}} \\ \frac{d}{dx}\tan^{-1}x &=& \frac{1}{1+x^2} & \frac{d}{dx}\cot^{-1}x &=& \frac{-1}{1+x^2} \end{array}$$

Theorem 0.7 - *Derivative of Trigonometric Functions - Standard*

$$\begin{array}{rcl|rcl} \frac{d}{dx}\sin x &=& \cos x & \frac{d}{dx}\text{cosec } x &=& -\text{cosec } x\cot x \\ \frac{d}{dx}\cos x &=& -\sin x & \frac{d}{dx}\sec x &=& \tan x\sec x \\ \frac{d}{dx}\tan x &=& \sec^2 x & \frac{d}{dx}\cot x &=& -\text{cosec }^2 x \end{array}$$

Theorem 0.8 - *Euler's Formula*

$$e^{ix} = \cos x + i\sin x$$

Theorem 0.9 - *Integration by Parts*

$$\int_a^b udv = [uv]_a^b - \int_a^b vdu \iff \int_a^b udv + vdu = \int_a^b u\frac{dv}{dx}dx + v\frac{du}{dx}dx = \int_a^b \frac{d}{dx}(uv)dx = [uv]_b^a$$

Theorem 0.10 - *Product Rule*

$$(f.g)' = fg' + f'g$$

Theorem 0.11 - *Quotient Rule*

$$\left(\frac{f}{g}\right)' = \frac{f'g - g'f}{g^2}$$

Theorem 0.12 - *Trigonometric Identities - Hyperbolic*

$$\begin{array}{rcl|rcl} \cosh^2\theta - \sinh^2\theta &=& 1 & \tanh^2\theta + \text{sech }^2\theta &=& 1 \\ \coth^2\theta - \text{cosech }^2\theta &=& 1 & && \\ \sinh(\theta\pm\phi) &=& \sinh\theta\cosh\phi \pm \cosh\theta\sinh\phi & \sinh(2\theta) &=& 2\sinh\theta\cosh\theta \\ \cosh(\theta\pm\phi) &=& \cosh\theta\cosh\phi \pm \sinh\theta\sinh\phi & \cosh(2\theta) &=& \cosh^2\theta + \sinh^2\theta \\ \tanh(\theta\pm\phi) &=& \frac{\tanh\theta \pm \tanh\phi}{1 \pm \tanh\theta\tanh\phi} & && \\ \sinh^2\theta &=& \frac{\cosh 2\theta + 1}{2} & \cosh^2\theta &=& \frac{\cosh 2\theta - 1}{2} \end{array}$$

Theorem 0.13 - *Trigonometric Identities - Standard*

$$
\begin{array}{rcl|rcl}
\sin^2\theta+\cos^2\theta & = & 1 & & & \\
\sin(\theta\pm\phi) & = & \sin\theta\cos\phi\pm\cos\theta\sin\phi & \sin(2\theta) & = & 2\sin\theta\cos\theta \\
\cos(\theta\pm\phi) & = & \cos\theta\cos\phi\mp\sin\theta\sin\phi & \cos(2\theta) & = & \cos^2\theta-\sin^2\theta \\
\tan(\theta\pm\phi) & = & \dfrac{\tan\theta\pm\tan\phi}{1\mp\tan\theta\tan\phi} & \tan(2\theta) & = & \dfrac{2\tan\theta}{1-\tan^2\theta} \\
\sin^2\theta & = & \dfrac{1-\cos 2\theta}{2} & \cos^2\theta & = & \dfrac{1+\cos 2\theta}{2} \\
2\cos\theta\cos\phi & = & \cos(\theta-\phi)+\cos(\theta+\phi) & 2\sin\theta\sin\phi & = & \cos(\theta-\phi)-\cos(\theta+\phi) \\
2\sin\theta\cos\phi & = & \sin(\theta+\phi)+\sin(\theta-\phi) & 2\cos\theta\sin\phi & = & \sin(\theta+\phi)-\sin(\theta-\phi) \\
\sin(-x) & = & -\sin(x) & & &
\end{array}
$$