Multivariable Calculus - Problem Sheet 1

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Question 3
Let $\mathbf{F} : \mathbb{R}^2 \to \mathbb{R}^3$ be given by

$$\mathbf{F}(\mathbf{x}) = (x_1^2 x_2, sin(x_1 + x)2, e^{x_1 x_2})$$

Question 3.1
Compute the matrix $\mathbf{F}'(\mathbf{x})$ of partial derivatives, $\frac{\partial F_i}{\partial x_j}$.

My Solution 3.1

$$\mathbf{F}'(\mathbf{x}) \quad = \quad \underline{\begin{pmatrix} 2x_1x_2 & x_1^2 \\ \cos(x_1 + x_2) & \cos(x_1 + x_2) \\ x_2 e^{x_1 x_2} & x_1 e^{x_1 x_2} \end{pmatrix}}$$

Question 3.2
Computer the directional derivative, $D_{\mathbf{v}}\mathbf{F}(\mathbf{x})$, where $\mathbf{v} = (1, 2)$.

My Solution 3.2

$$\begin{aligned}
\|\mathbf{v}\| &= \sqrt{5} \\
D_{\mathbf{v}}\mathbf{F}(\mathbf{x}) &= \frac{1}{\|\mathbf{v}\|} \begin{pmatrix} \frac{d}{dt}(x_1 + t)^2(x_2 + 2t)|_{t=0} \\ \frac{d}{dt}\sin(x_1 + x_2 + 3t)|_{t=0} \\ \frac{d}{dt} e^{(x_1+t)(x_2+2t)}|_{t=0} \end{pmatrix} \\
&= \frac{1}{\sqrt{5}} \begin{pmatrix} 2(x_1 + t)(x_2 + 2t) + 2(x_1 + t)^2|_{t=0} \\ 3\cos(x_1 + x_2 + 3t)|_{t=0} \\ (2x_1 + x_2 + 4t)e^{(x_1+t)(x_2+2t)}|_{t=0} \end{pmatrix} \\
&= \underline{\frac{1}{\sqrt{5}} \begin{pmatrix} 2x_1x_2 + 2x_1^2 \\ 3\cos(x_1 + x_2) \\ (2x_1 + x_2)e^{x_1 x_2} \end{pmatrix}}
\end{aligned}$$

Question 3.3
Compare the result from 3.2, evaluated at $\mathbf{x} = \mathbf{x}_0 = (1, 1)$, to the vector $\mathbf{F}'(\mathbf{x}_0)\mathbf{v}$.

My Solution 3.3

$$\begin{aligned}
D_{\mathbf{v}}\mathbf{F}(\mathbf{x}_0) &= \tfrac{1}{\sqrt{5}}\begin{pmatrix} 4 \\ 3\cos(2) \\ 3e \end{pmatrix} \\
\mathbf{F}'(\mathbf{x}_0)\mathbf{v} &= \begin{pmatrix} 2 & 1 \\ \cos(2) & \cos(2) \\ e & e \end{pmatrix}\begin{pmatrix} 1 \\ 2 \end{pmatrix} \\
&= \underline{\begin{pmatrix} 4 \\ 3\cos(2) \\ 3e \end{pmatrix}}
\end{aligned}$$

Question 4
Let $\mathbf{F}, \mathbf{H} : \mathbb{R}^2 \to \mathbb{R}^3, \mathbf{G} : \mathbb{R}^3 \to \mathbb{R}^3$. Define

$$\mathbf{F}(\mathbf{x}) = A\mathbf{x} \text{ where } A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\mathbf{G}(\mathbf{x}) = (x_1x_2, x_2x_3, \sin(x_1x_2x_3))$$

$$\mathbf{H} = \mathbf{G} \circ \mathbf{F}$$

Question 4.1
Use the chain rule to calculate $\mathbf{H}'(1,1)$.

My Solution 4.1

$$\begin{aligned}
& & \mathbf{F}(1,1) &= (3,3,1) \\
\& & \mathbf{G}' &= \begin{pmatrix} x_2 & x_1 & 0 \\ 0 & x_3 & x_2 \\ x_2x_3\cos(x_1x_2x_3) & x_1x_3\cos(x_1x_2x_3) & x_1x_2\cos(x_1x_2x_3) \end{pmatrix} \\
\implies & & (\mathbf{G}' \circ \mathbf{F})(1,1) &= \mathbf{G}'(3,3,1) \\
& & &= \begin{pmatrix} 3 & 3 & 0 \\ 0 & 1 & 3 \\ 3\cos(9) & 3\cos(9) & 9\cos(9) \end{pmatrix} \\
\text{Chain rule} & & \mathbf{H}' &= (\mathbf{G}' \circ \mathbf{F})\mathbf{F}' \\
& & &= \begin{pmatrix} 3 & 3 & 0 \\ 0 & 1 & 3 \\ 3\cos(9) & 3\cos(9) & 9\cos(9) \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 2 & 1 \\ 1 & 0 \end{pmatrix} \\
& & &= \underline{\begin{pmatrix} 9 & 9 \\ 5 & 1 \\ 18\cos(9) & 9\cos(9) \end{pmatrix}}
\end{aligned}$$

Question 4.2
Calculate $\mathbf{H}'(1,1)$ directly.

My Solution 4.2

$$\begin{array}{rrcl}
 & \mathbf{H} & = & ((x_1+2x_2)(2x_1+x_2), x_1(2x_1+x_2), \sin(x_1(x_1+2x_2)(2x_1+x_2))) \\
\implies & \mathbf{H}' & = & \begin{pmatrix} (2x_1+x_2)+2(x_1+2x_2) & 2(2x_1+x_2)+(x_1+2x_2) \\ (2x_1+x_2)+2x_1 & x_1 \\ i & j \end{pmatrix} \\
\textit{where} & i & = & [(x_1+2x_2)(2x_1+x_2)+x(2x_1+x_2)+2x_1(x_1+2x_2)]\cos(x_1(x_1+2x_2)(2x_1+x_2))]] \\
\& & j & = & [2x_1(2x_1+x_2)+x_1(x_1+2x_2)]\cos(x_1(x_1+2x_2)(2x_1+x_2)) \\
\implies & \mathbf{H}'(1,1) & = & \begin{pmatrix} 3+2\times 3 & 2\times 3+3 \\ 3+2 & 1 \\ [3\times 3+1\times 3+2\times 3]\cos(1\times 3\times 3) & [2\times 3+1\times 3]\cos(1\times 3\times 3) \end{pmatrix} \\
 & & = & \underline{\begin{pmatrix} 9 & 9 \\ 5 & 1 \\ 18\cos(9) & 9\cos(9) \end{pmatrix}}
\end{array}$$

Question 5

Consider the coupled nonlinear system of equations given by

$$x^3 + e^y = s, \quad \cos x + xy = t$$

When $(s,t) = (1,1)$ the solution is given by $(x,y) = (0,0)$. Can we guarantee unique solutions (x,y) for values of (s,t) near $(1,1)$?

My Solution 5

$$\begin{array}{lrcl}
\text{Let} & \mathbf{F}:\mathbb{R}^2\to\mathbb{R}^2 \quad st & & \mathbf{F}=(s,t) \\
\implies & \mathbf{F}' & = & \begin{pmatrix} 3x^2 & e^y \\ -\sin(x)+y & x \end{pmatrix} \\
\implies & J_{\mathbf{F}} & = & 3x^3 + e^y(\sin(x)+y) \\
\implies & J_{\mathbf{F}}(0,0) & = & 0 + 1(sin(0)+0) \\
 & & = & \underline{0}
\end{array}$$

Since $J_{\mathbf{F}}(0,0)$ is not invertible, by the *inverse function theorem*, we <u>cannot</u> guarantee unique solutions near $(1,1)$.