Multivariable Calculus - Problem Class 6

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Question 2
Calculate

$$\int_C \mathbf{r} \circ d\mathbf{r}$$

where C is any curve connecting the point $\mathbf{r}_1$ to $\mathbf{r}_2$.

My Solution 2
Method 1

$$\begin{array}{rcl} \int_C \mathbf{r} \circ d\mathbf{r} & = & \int_C d\left(\frac{1}{2}\mathbf{r} \circ \mathbf{r}\right) \\ & = & \left[\frac{1}{2}\mathbf{r} \circ \mathbf{r}\right]_{\mathbf{r}_2}^{\mathbf{r}_1} \\ & = & \frac{1}{2}|\mathbf{r}_2|^2 - \frac{1}{2}|\mathbf{r}_1|^2 \end{array}$$

Method 2

$$\begin{array}{lrcl} \text{Let} & f & = & \frac{1}{2}\mathbf{r} \circ \mathbf{r} \\ \implies & \nabla f & = & \mathbf{r} \\ \text{Since} & \nabla\left(\frac{1}{2}\mathbf{r} \circ \mathbf{r}\right) & = & \frac{1}{2}\nabla(r^2) \\ & & = & \frac{2r}{2}\nabla r \\ & & = & r\left(\frac{\mathbf{r}}{r}\right) \\ & & = & \mathbf{r} \end{array}$$

Question 4
Let ϕ be a scalar field. use the divergence theorem to show that

$$\int_V \nabla\phi dV = \int_{\partial V} \phi\hat{\mathbf{n}} dS$$

My Solution 4
The Divergence Theorem states

$$\int_V \nabla \circ \mathbf{f} dV = \int_{\partial V} \mathbf{f} \circ d\mathbf{S}$$

Let $\mathbf{f} = \phi(\mathbf{r})\mathbf{a}$ where $\mathbf{a} \in \mathbb{R}^3$ is constant.
Consider

$$\begin{array}{rcl} \nabla \circ \mathbf{f} & = & \nabla \circ (\phi(\mathbf{r})\mathbf{a}) \\ & = & (\nabla\phi) \circ \mathbf{a} + \phi(\nabla \circ \mathbf{a}) \\ & = & (\nabla\phi) \circ \mathbf{a} \end{array}$$

By applying the divergence theorem

$$\mathbf{a} \circ \int_V \nabla\phi dV = \mathbf{a} \circ \int_{\partial V} \phi d\mathbf{S}$$

This holds $\forall$ **a** so

$$\int_V \nabla\phi dV = \int_{\partial V} \phi d\mathbf{S} = \int_{\partial V} \phi\hat{\mathbf{n}} dS$$

Question 5.1
A vector field **F** is given by $\mathbf{F}(x,y,z) = (xy, yz, xz)$. Calculate $\nabla \circ \mathbf{F}$.

My Solution 5.1

$$\nabla \circ \mathbf{F}(x) = x + y + z$$

Question 5.2
The volume, V of a tetrahedron is bounded by four triangular surfaces formed by the intersection of the planes $x = 0$, $y = 0$, $z = 0$ and $x + y + z = 1$.
Calculate the volume integral

$$\int_V \nabla \circ \mathbf{F} dV$$

My Solution 5.2

$$\begin{aligned} \textstyle\int_V \nabla \circ \mathbf{F} dV &= \int_{x=0}^{x=1}\int_{y=0}^{y=1-x}\int_{z=0}^{z=1-x-y} (x+y+z) dz dy dx \\ &= \int_{x=0}^{x=1}\int_{y=0}^{y=1-x} \left((x+y)(1-x-y) + \frac{1}{2}(1-x-y)^2\right) dy dz \\ &= \dots \end{aligned}$$

Question 5.3
State the divergence theorem and use it ot calculate the value of the integral in 5.2 by an independent method.

My Solution 5.3
$\int_V \nabla \circ \mathbf{F} dV = \int_{\partial V} \mathbf{F} \circ d\mathbf{S}$
where $\partial V = S_1 \cup S_2 \cup S_3 \cup S_4$.
Let S_1 be the volume surface in $x = 0$ plane
$\implies \mathbf{F}(x,y,z) = (0, yz, 0)$ and $\partial\mathbf{S} = \hat{\mathbf{n}} dS = -\hat{\mathbf{x}} dS$
$\implies \mathbf{F} \circ d\mathbf{S} = 0$ on S_1.
Similarly $\mathbf{F} \circ d\mathbf{S} = 0$ on S_2 & S_3.
$\implies \int_{\partial V} \mathbf{F} \circ d\mathbf{S} = \int_{S_4} \mathbf{F} \circ d\mathbf{S}$.
Let $D = \{(x,y) | 0 < x < 1, 0 < y < 1 - x\}$ and $\mathbf{s}(x,y) = (x, y, 1-x-y)$ for $(x,y) \in D$.
Now mechanical

Question 6
If $\mathbf{f}(\mathbf{r}) = (0, x, 0)$ and $\mathbf{g}(\mathbf{r}) = (-y, 0, 0)$ show that, for any closed curve $C \in \mathbb{R}^3$

$$\int_C \mathbf{f} \circ d\mathbf{r} = \int_C \mathbf{g} \circ d\mathbf{r}$$

My Solution 6

Method 1

Consider $\int_C (\mathbf{f} - \mathbf{g}) \circ d\mathbf{r}$

We want to show $\mathbf{f} - \mathbf{g} = \nabla f$ so that we can use the Fundamental Theorem of Calculus.

This requires $\frac{\partial f}{\partial x} = g \implies f = yx$ and $\frac{\partial f}{\partial y} = x \implies f = xy$.

These both produce the same result, so hold.

Set $\nabla(x, y) = \mathbf{f} - \mathbf{g}$.

$$\implies \int_C (\mathbf{f} - \mathbf{g}) d\mathbf{r} = \int_C \nabla \mathbf{f} \circ d\mathbf{r} = 0$$

Since it is a closed loop.

Method 2

Set $\mathbf{A} = \mathbf{f} - \mathbf{g} = (y, x, 0)$

$$\begin{array}{rrcl}
\implies & \int_C \mathbf{A} \circ d\mathbf{r} & = & \int_C \nabla \times \mathbf{A} \circ d\mathbf{S} \quad \text{Stokes}' \text{ Theorem} \\
& \nabla \times \mathbf{A} & = & \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \partial x & \partial y & \partial z \\ y & x & 0 \end{vmatrix} \\
& & = & (0, 0, 1 - 1) \\
& & = & \mathbf{0} \\
\implies & \int_C \mathbf{A} \circ d\mathbf{r} & = & \int_S \nabla \times \mathbf{A} \circ d\mathbf{S} \\
& & = & \int_S 0 d\mathbf{S} \\
& & = & 0
\end{array}$$