Multivariable Calculus - Problem Class 2

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Question 1

Question 1.1
Explain what is meant by a linear map.

My Solution 1.1
A map, $\mathbf{F}:\mathbb{R}^m\to\mathbb{R}^n$, is linear if $\forall\ \mathbf{x},\mathbf{y}\&\lambda,\mu\in\mathbb{R}\ \mathbf{F}(\lambda\mathbf{x}+\mu\mathbf{y}))=\lambda\mathbf{F}(\mathbf{x})+\mu\mathbf{F}(\mathbf{y})$.

Question 1.2
Is $\mathbf{f}(\mathbf{x})=(x_2+x_3,x_1+x_2)$ a linear map. Explain why.

My Solution 1.2
Yes.
Let $a\in\mathbb{R}$ and $\mathbf{x},\mathbf{y}\in\mathbb{R}^3$.

$$\begin{aligned}
\mathbf{f}(a\mathbf{x}+\mathbf{y}) &= \mathbf{f}(a(x_1,x_2,x_3)+(y_1,y_2,y_3))\\
&= ((ax_1+y_1,ax_2+y_2,ax_3+y_3))\\
&= ((ax_2+y_2)+(ax_3+y_3),(ax_1+y_1)+(ax_2+y_2))\\
&= (a(x_2+x_3)+(y_2+y_3),a(x_1+x_2)+(y_1+y_2))\\
&= a(x_2+x_3,x_1+x_2)+(y_2+y_3,y_1+y_2)\\
&= a\mathbf{f}(\mathbf{x})+\mathbf{f}(\mathbf{y})
\end{aligned}$$

Question 1.3
Is $\mathbf{f}(\mathbf{x})=(x_2x_3,x_1x_2)$ a linear map. Explain why.

My Solution 1.3
No.
Let $\mathbf{x},\mathbf{y}\in\mathbb{R}^3$.

$$\begin{aligned}
\mathbf{f}(\mathbf{x}+\mathbf{y}) &= \mathbf{f}((x_1,x_2,x_3)+(y_1,y_2,y_3))\\
&= \mathbf{f}((x_1+y_1,x_2+y_2,x_3+y_3))\\
&= (x_2y_2x_3y_3,x_1y_1x_2y_2)\\
&\neq (x_2x_3,x_1x_2)+(y_2y_3,y_1y_2)\\
&= \mathbf{f}(\mathbf{x})+\mathbf{f}(\mathbf{y})
\end{aligned}$$

Question 2
Let $\mathbf{F}:\mathbb{R}^3\to\mathbb{R}^2$ be defined by $\mathbf{F}(x)=(x_2x_3,x_1x_2)$.
Find the derivative of $\mathbf{F}$ in the direction $(1,-1,1)$ using two independent methods.

My Solution 2

$$\begin{aligned}
\mathbf{F}' &= \begin{pmatrix} \frac{\partial F_1}{\partial x_1} & \frac{\partial F_1}{\partial x_2} & \frac{\partial F_1}{\partial x_3} \\ \frac{\partial F_2}{\partial x_1} & \frac{\partial F_2}{\partial x_2} & \frac{\partial F_2}{\partial x_3} \end{pmatrix} \\
&= \begin{pmatrix} 0 & x_3 & x_2 \\ x_2 & x_1 & 0 \end{pmatrix} \\
\implies \mathbf{F}'(1,-1,1) &= \begin{pmatrix} 0 & 1 & -1 \\ -1 & 1 & 0 \end{pmatrix}
\end{aligned}$$

Question 3

Let

$$\mathbf{F}(u,v,w) = (v^2 + uw, u^2 + w^2, u^2v - w^3)$$

and

$$\mathbf{G}(x,y) = (xy^3, x^2 - y^2, 3x + 5y)$$

and define $\mathbf{H}(x,y) = (\mathbf{F} \circ \mathbf{G})(x,y)$. Compute $\mathbf{H}'(-1,1)$.

My Solution 3

$$\begin{aligned}
\text{By Chain Rule} \qquad \mathbf{H} &= (\mathbf{F}' \circ \mathbf{G})\mathbf{G}' \\
\mathbf{G}' &= \begin{pmatrix} y^3 & 3xy^2 \\ 2x & -2y \\ 3 & 5 \end{pmatrix} \\
\implies \qquad \mathbf{G}'(-1,1) &= \begin{pmatrix} 1 & -3 \\ -2 & -2 \\ 3 & 5 \end{pmatrix} \\
\mathbf{F}' &= \begin{pmatrix} w & 2v & u \\ 2u & 0 & 2w \\ 2uv & u^2 & -3w^2 \end{pmatrix} \\
\mathbf{G}(-1.1) &= (-1,0,2) \\
\implies \qquad (\mathbf{F}' \circ \mathbf{G})(-1,1) &= \mathbf{F}'(-1,0,2) \\
&= \begin{pmatrix} 2 & 0 & -1 \\ -2 & 0 & 4 \\ 0 & 1 & -12 \end{pmatrix} \\
\implies \qquad \mathbf{H}' &= \begin{pmatrix} 2 & 0 & -1 \\ -2 & 0 & 4 \\ 0 & 1 & -12 \end{pmatrix} \begin{pmatrix} 1 & -3 \\ -2 & -2 \\ 3 & 5 \end{pmatrix} \\
&= \begin{pmatrix} -1 & -11 \\ 10 & 26 \\ -38 & -62 \end{pmatrix}
\end{aligned}$$

Question 4

Given

$$z = f\left(\frac{x+y}{x-y}\right)$$

show that

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 0$$

My Solution 4

$$\begin{aligned}
x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y} &\equiv x\frac{\partial}{\partial x}\left(\frac{x+y}{x-y}\right)f'\left(\frac{x+y}{x-y}\right)+y\frac{\partial}{\partial y}\left(\frac{x+y}{x-y}\right)f'\left(\frac{x+y}{x-y}\right)\\
&= x\left[\frac{1}{x-y}-\frac{x+y}{(x-y)^2}\right]f'\left(\frac{x+y}{x-y}\right)+y\left[\frac{1}{x-y}-\frac{x+y}{(x-y)^2}\right]f'\left(\frac{x+y}{x-y}\right)\\
&= x\left[\frac{-2y}{(x-y)^2}\right]f'\left(\frac{x+y}{x-y}\right)+y\left[\frac{2x}{(x-y)^2}\right]f'\left(\frac{x+y}{x-y}\right)\\
&= \left[\frac{-2xy+2xy}{(x-y)^2}\right]f'\left(\frac{x+y}{x-y}\right)\\
&= 0
\end{aligned}$$

Question 5

Show that the pair of equations

$$x^2-y^2-u^3+v^2+4=0 \quad 2xy+y^2-2u^2+3v^4+8=0$$

determine local functions $u(x,y)$ and $v(x,y)$ defined for $(u,v)=(2,1)$ such that $(x,y)=(2,-1)$. Computer $\frac{\partial u}{\partial x}$ at $(x,y)=(2,-1),\ (u,v)=(2,1)$.

My Solution 5.1

$$\begin{aligned}
\tfrac{\partial}{\partial x}(x^2-y^2-u^3+v^2+4) &= 2x-3\tfrac{\partial u}{\partial x}u^2+2\tfrac{\partial v}{\partial x}v=0\\
\tfrac{\partial}{\partial y}(x^2-y^2-u^3+v^2+4) &= -2y-3\tfrac{\partial u}{\partial y}u^2+2\tfrac{\partial v}{\partial y}v=0\\
\tfrac{\partial}{\partial x}(2xy+y^2-2u^2+3v^4+8) &= 2y-4\tfrac{\partial u}{\partial x}u+12\tfrac{\partial v}{\partial x}v^3=0\\
\tfrac{\partial}{\partial y}(2xy+y^2-2u^2+3v^4+8) &= 2x+2y-4\tfrac{\partial u}{\partial y}u+12\tfrac{\partial v}{\partial y}v^3=0
\end{aligned}$$

These can be simplied to

$$\begin{pmatrix}2x & -2y\\ 2y & 2x+2y\end{pmatrix}+\begin{pmatrix}-3u^2 & 2v\\ -4 & 12v^3\end{pmatrix}\begin{pmatrix}\frac{\partial u}{\partial x} & \frac{\partial u}{\partial y}\\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y}\end{pmatrix}=\begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix}$$

WTS $\begin{pmatrix}-3u^2 & 2v\\ -4 & 12v^3\end{pmatrix}$ is non-singular so $\begin{pmatrix}\frac{\partial u}{\partial x} & \frac{\partial u}{\partial y}\\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y}\end{pmatrix}$ exists.

$$\begin{aligned}
\text{Set } (u,v)=(2,1) &\implies \begin{pmatrix}-3u^2 & 2v\\ -4 & 12v^3\end{pmatrix}=\begin{pmatrix}-12 & 2\\ -8 & 12\end{pmatrix}\\
\begin{vmatrix}-12 & 2\\ -8 & 12\end{vmatrix} &= -144+16=-128\neq 0
\end{aligned}$$

So solutions $u(x,y),\ v(x,y)$ exists close to $(u,v)=(2,1),\ (x,y)=(2,-1)$ by implicit function theorem.

My Solution 5.2

$$\begin{aligned}
\begin{pmatrix}u_x & u_y\\ v_x & v_y\end{pmatrix} &= \begin{pmatrix}-3u^2 & 2v\\ 4u & 12v^3\end{pmatrix}^{-1}\begin{pmatrix}2x & -2y\\ 2y & 2x+2y\end{pmatrix}\\
&= \frac{-1}{-128}\begin{pmatrix}21v^3 & 2v\\ 4u & -3u^2\end{pmatrix}\begin{pmatrix}2x & -2y\\ 2y & 2x+2y\end{pmatrix}\\
\implies \qquad u_x &= \tfrac{-1}{-128}[(12v^3)(2x)+(2v)(2y)]\\
&= \tfrac{-1}{-128}[(12)(4)+(2)(-1)]\\
&= \frac{23}{64}
\end{aligned}$$