Multivariable Calculus - Problem Sheet 3

Dom Hutchinson

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Question 3.1
Let $\phi(\mathbf{r}) = f(r)$, where f is a function of a single variable, $r = |\mathbf{r}|$. Computer $\Delta\phi$.

My Solution 3.1

$$\begin{array}{rcl} =\Delta\phi &=& \sum_{\alpha=1}^{3} \frac{\partial^2}{\partial x_\alpha^2} f(r) \\ &=& \sum_{\alpha=1}^{3} \frac{\partial}{\partial x_\alpha}\left(\frac{\partial}{\partial x_\alpha} f(r)\right) \\ &=& \sum_{\alpha=1}^{3} \frac{\partial}{\partial x_\alpha}\left(\frac{\partial r}{\partial x_\alpha}\frac{\partial f}{\partial r}\right) \\ &=& f' \sum_{\alpha=1}^{3} \frac{\partial^2}{\partial x_\alpha^2} r \\ &=& f' \sum_{\alpha=1}^{3} \frac{\partial}{\partial x_\alpha} \frac{x_\alpha}{r} \\ &=& f' \sum_{\alpha=1}^{3} \left[\frac{1}{r} - x_\alpha\left(\frac{x_\alpha}{r^3}\right)\right] \\ &=& \frac{f'}{r} \sum_{\alpha=1}^{3} \left[1 - \frac{x_\alpha^2}{r^2}\right] \\ &=& \frac{f'}{r}\left[3 - \frac{r^2}{r^2}\right] \\ &=& 2\frac{f'}{r} \end{array}$$

Question 3.2
Let $\boldsymbol{\mu} \in \mathbb{R}^3$ be a nonzero vector. Compute $\Delta\frac{\boldsymbol{\mu} \circ \mathbf{r}}{r^3}$.

My Solution 3.2

$$\begin{array}{rcl} \Delta\frac{\boldsymbol{\mu}\circ\mathbf{r}}{r^3} &=& \nabla\circ\left(\nabla\frac{\boldsymbol{\mu}\circ\mathbf{r}}{r^3}\right) \\ \nabla\frac{\boldsymbol{\mu}\circ\mathbf{r}}{r^3} &=& \left(\frac{\mu_1 x}{r^3}, \frac{\mu_2 y}{r^3}, \frac{\mu_3 z}{r^3}\right) \\ &=& \left(\frac{\mu_1 r^3 - (\mu_1 x)(3xr)}{r^6}, \frac{\mu_2 r^3 - (\mu_2 y)(3yr)}{r^6}, \frac{\mu_3 r^3 - (\mu_3 z)(3zr)}{r^6}\right) \\ \nabla\circ\left(\nabla\frac{\boldsymbol{\mu}\circ\mathbf{r}}{r^3}\right) &=& \frac{\mu_1(-4x)r^5 - \mu_1(r^2-3x^2)5xr^3}{r^{10}} + \frac{\mu_2(-4y)r^5 - \mu_2(r^2-3y^2)5yr^3}{r^{10}} \\ &+& \frac{\mu_3(-4z)r^5 - \mu_3(r^2-3z^2)5zr^3}{r^{10}} \\ &=& \frac{3}{r^7}\left(\mu_1 x(5x^2-3r^2) + \mu_2 y(5y^2-3r^2) + \mu_3 z(5z^2-3r^2)\right) \\ &=& \frac{3}{r^7}\mu_i x_i(5x_i^2 - 3r^2) \end{array}$$

Question 4
This question concerns the transformation to elliptical coordinates (μ, ν) given by the relation

$$x = a\cosh\mu\cos\nu \quad y = a\sinh\mu\sin\nu$$

where $mu \in [0, \infty)$, $\nu \in [0, 2\pi)$.

Question 4.1
Show that curves of constant μ correspond to ellipses in the (x, y) plane and that curves of

constant ν are hyperbolae.

My Solution 4.1
Set μ to be constant.
Then $\exists$ constants $b, c \in \mathbb{R}$ such that $\cosh\mu = b$ & $\sinh\mu = c$.
Then $x = ab\cos\nu$ & $y = ac\sin\nu$.
The general formula for an ellipse is

$$\frac{x^2}{\alpha^2} + \frac{y^2}{\beta^2} = 1$$

Set $\alpha = ab$ & $\beta = ac$.

$$\begin{aligned} \implies \tfrac{x^2}{\alpha^2} + \tfrac{y^2}{\beta^2} &= \tfrac{(ab\cos\nu)^2}{(ab)^2} + \tfrac{(ac\sin\nu)^2}{(ac)^2} \\ &= \tfrac{a^2b^2\cos^2\nu}{a^2b^2} + \tfrac{a^2c^2\sin^2\nu}{a^2c^2} \\ &= \cos^2\nu + \sin^2\nu \\ &= 1 \end{aligned}$$

Thus, for a constant μ, this curves corresponds to the general formula of an ellipse.

Set ν to be constant.
Then $\exists$ constants $d, e \in \mathbb{R}$ such that $\cos\nu = d$ & $\sin\nu = e$.
Then $x = ad\cosh\mu$ & $y = ae\sinh\mu$.
The general formula for a hyperbola is

$$\frac{y^2}{\gamma^2} - \frac{x^2}{\delta^2} = 1$$

Set $\gamma = ae$ & $\delta = ad$.

$$\begin{aligned} \tfrac{y^2}{\gamma^2} - \tfrac{x^2}{\delta^2} &= \tfrac{(ae\sinh\mu)^2}{(ae)^2} - \tfrac{(ad\cosh\mu)^2}{(ad)^2} \\ &= \sinh^2\mu - \cosh^2\mu \\ &= 1 \end{aligned}$$

Thus, for a constant ν, this curve corresponds to the general formula for a hyperbola.

Question 4.2
Derive a basis $\hat{\boldsymbol{\mu}}$, $\hat{\boldsymbol{\nu}}$ for elliptical coordinates and, in doing so, show that the scal factors are

$$h_\mu = h_\nu = a\sqrt{\sinh^2\mu + \sin^2\nu}$$

Also confirm that $\hat{\boldsymbol{\mu}}$ and $\hat{\boldsymbol{\nu}}$ are orthogonal.

My Solution 4.2

$$\begin{aligned}
\mathbf{r} &= (a\cosh\mu\cos\nu, a\sinh\mu\sin\nu) \\
h_\mu &= |\tfrac{\partial\mathbf{r}}{\partial\mu}| \\
&= \sqrt{(a\sinh\mu\cos\nu)^2 + (a\cosh\mu\sin\nu)^2} \\
&= a\sqrt{\sinh^2\mu\cos^2\nu + \cosh^2\mu\sin^2\nu} \\
&= a\sqrt{\sinh^2\mu\cos^2\nu + (1+\sinh^2\mu)\sin^2\nu} \\
&= a\sqrt{\sinh^2\mu(\cos^2\nu + \sin^2\nu) + \sin^2\nu} \\
&= a\sqrt{\sinh^2\mu + \sin^2\nu}
\end{aligned}$$

$$\begin{aligned}
h_\nu &= |\tfrac{\partial\mathbf{r}}{\partial\nu}| \\
&= \sqrt{(-a\cosh\mu\sin\nu)^2 + (a\sinh\mu\cos\nu)^2} \\
&= a\sqrt{\cosh^2\mu\sin^2\nu + \sinh^2\mu\cos^2\nu} \\
&= a\sqrt{\sinh^2\mu + \sin^2\nu}
\end{aligned}$$

$$\begin{aligned}
\hat{\boldsymbol{\mu}} &= \tfrac{1}{h_\mu}\tfrac{\partial\mathbf{r}}{\partial\mu} \\
&= \tfrac{1}{a\sqrt{\sinh^2\mu+\sin^2\nu}}(a\sinh\mu\cos\nu,\ a\cosh\mu\sin\nu) \\
&= \tfrac{1}{\sqrt{\sinh^2\mu+\sin^2\nu}}(\sinh\mu\cos\nu,\ \cosh\mu\sin\nu)
\end{aligned}$$

$$\begin{aligned}
\hat{\boldsymbol{\nu}} &= \tfrac{1}{h_\nu}\tfrac{\partial\mathbf{r}}{\partial\nu} \\
&= \tfrac{1}{a\sqrt{\sinh^2\mu+\sin^2\nu}}(-a\cosh\mu\sin\nu,\ a\sinh\mu\cos\nu) \\
&= \tfrac{1}{\sqrt{\sinh^2\mu+\sin^2\nu}}(-\cosh\mu\sin\nu,\ \sinh\mu\cos\nu)
\end{aligned}$$

$$\begin{aligned}
\hat{\boldsymbol{\mu}}\circ\hat{\boldsymbol{\nu}} &= \tfrac{1}{\sinh^2\mu+\sin^2\nu}\left[(\sinh\mu\cos\nu)(-\cosh\mu\sin\nu) + (\cosh\mu\sin\nu)(\sinh\mu\cos\nu)\right] \\
&= \tfrac{1}{\sinh^2\mu+\sin^2\nu}[0] \\
&= 0
\end{aligned}$$

Since $\hat{\boldsymbol{\mu}}\circ\hat{\boldsymbol{\nu}} = 0$ then $\hat{\boldsymbol{\mu}}$ & $\hat{\boldsymbol{\nu}}$ are orthogonal.

Question 4.3
Calculate the Jacobian determinant. Is the mapping of coordinates always invertible? If not, when is it non-invertible?

My Solution 4.3

$$\begin{aligned}
J_{\mathbf{r}} &= \begin{vmatrix} h_\mu[\hat{\boldsymbol{\mu}}]_1 & h_\nu[\hat{\boldsymbol{\nu}}]_1 \\ h_\mu[\hat{\boldsymbol{\mu}}]_2 & h_\nu[\hat{\boldsymbol{\nu}}]_2 \end{vmatrix} \\
&= \begin{vmatrix} a\sinh\mu\cos\nu & -a\cosh\mu\sin\nu \\ a\cosh\mu\sin\nu & a\sinh\mu\cos\nu \end{vmatrix} \\
&= (a\sin\mu\cos\nu)^2 + (a\cosh\mu\sin\nu)^2 \\
&= a^2\left[\sinh^2\mu\cos^2\nu + \cosh^2\mu\sin^2\nu]\right] \\
&= a^2\left[\sinh^2\mu + \sin^2\nu\right]
\end{aligned}$$

This coordinate mapping is not always invertible.
It is not invertible if $(\mu,\nu) = (0,\pi n)\ \forall\ n\in\mathbb{N}$.

Question 4.4
Express ∇f in elliptical coordinates.

My Solution 4.4

$$
\begin{aligned}
\nabla f(x,y) &= \nabla f(a\cosh\mu\cos\nu,\ a\sinh\mu\sin\nu)\\
&= \frac{\hat{\boldsymbol{\mu}}}{h_\mu}\frac{\partial f}{\partial\mu} + \frac{\hat{\boldsymbol{\nu}}}{h_\nu}\frac{\partial f}{\partial\nu}\\
&= \frac{1}{a(\sinh^2\mu+\sin^2\nu)}\left[(a\sinh\mu\cos\nu,\ a\cosh\mu\sin\nu)\frac{\partial f}{\partial\mu} + (-a\cosh\mu\sin\nu,\ a\sinh\mu\cos\nu)\frac{\partial f}{\partial\nu}\right]\\
&= \frac{1}{\sinh^2\mu+\sin^2\nu}\left[(\sinh\mu\cos\nu,\ \cosh\mu\sin\nu)\frac{\partial f}{\partial\mu} + (-\cosh\mu\sin\nu,\ \sinh\mu\cos\nu)\frac{\partial f}{\partial\nu}\right]
\end{aligned}
$$

Question 4.5

Find Δf in elliptical coordiantes.

My Solution 4.5

$$
\begin{aligned}
\Delta f &= \nabla\circ(\nabla f)\\
&= \frac{\partial}{\partial\mu}(\nabla f_1) + \frac{\partial}{\partial\nu}(\nabla f_2)\\
&= \frac{\partial}{\partial\mu}\left[\frac{1}{(\sinh^2\mu+\sin^2\nu)}(\sinh\mu\cos\nu\frac{\partial f}{\partial\mu} - \cosh\mu\sin\nu\frac{\partial f}{\partial\mu})\right]\\
&+ \frac{\partial}{\partial\nu}\left[\frac{1}{(\sinh^2\mu+\sin^2\nu)}(\cosh\mu\sin\nu\frac{\partial f}{\partial\nu} + \sinh\mu\cos\nu\frac{\partial f}{\partial\nu})\right]\\
&= \left[\frac{2\sinh\mu\cosh\mu}{(\sinh^2\mu+\sin^2\nu)^2}\left(\sinh\mu\cos\nu\frac{\partial f}{\partial\mu} - \cosh\mu\sin\nu\frac{\partial f}{\partial\mu}\right)\right.\\
&+ \left.\frac{1}{(\sinh^2\mu+\sin^2\nu)}\left(\cosh\mu\cos\nu\frac{\partial f}{\partial\mu} + \sinh\mu\cos\nu\frac{\partial^2 f}{\partial\mu^2} - \sinh\mu\sin\nu\frac{\partial f}{\partial\mu} - \cosh\mu\sin\nu\frac{\partial^2 f}{\partial\mu^2}\right)\right]\\
&+ \left[\frac{-2\sin\nu\cos\nu}{(\sinh^2\mu+\sin^2\nu)^2}(\cosh\mu\sin\nu\frac{\partial f}{\partial\nu} + \sinh\mu\cos\nu\frac{\partial f}{\partial\nu})\right.\\
&+ \left.\frac{1}{(\sinh^2\mu+\sin^2\nu)}(\cosh\mu\cos\nu\frac{\partial f}{\partial\nu} + \cosh\mu\sin\nu\frac{\partial^2 f}{\partial\nu^2} - \sinh\mu\sin\nu\frac{\partial f}{\partial\nu} + \sinh\mu\cos\nu\frac{\partial^2 f}{\partial\mu^2})\right]\\
&= \left[\frac{(2\sinh^2\mu\cosh\mu\cos\nu - 2\sinh\mu\cosh^2\mu\sin\nu) + (\sinh^2\mu+\sin^2\nu)(\cosh\mu\cos\nu - \sinh\mu\sin\nu)}{(\sinh^2\mu+\sin^2\nu)^2}\right]\frac{\partial f}{\partial\mu}\\
&+ \left[\frac{(-2\cosh\mu\sin^2\nu\cos\nu - 2\sinh\mu\sin\nu\cos^2\nu) + (\sinh^2\nu+\sin^2\nu)(\cosh\mu\cos\nu - \sinh\mu\sin\nu)}{(\sinh^2\mu+\sin^2\nu)^2}\right]\frac{\partial f}{\partial\nu}\\
&+ \left[\frac{\sinh\mu\cos\nu - \cosh\mu\sin\nu}{(\sinh^2\mu+\sin^2\nu)}\right]\frac{\partial^2 f}{\partial\mu^2} + \left[\frac{\cosh\mu\sin\nu + \sinh\mu\cos\nu}{(\sinh^2\mu+\sin^2\nu)}\right]\frac{\partial^2 f}{\partial\nu^2}\\
&= \left[\frac{(2\sinh^2\mu\cosh\mu\cos\nu - 2\sinh\mu\cosh^2\mu\sin\nu) + (\sinh^2\mu+\sin^2\nu)(\cosh\mu\cos\nu) - (\cosh^2\mu - \cos^2\nu)(\sinh\mu\sin\nu)}{(\sinh^2\mu+\sin^2\nu)^2}\right]\frac{\partial f}{\partial\mu}\\
&+ \left[\frac{(-2\cosh\mu\sin^2\nu\cos\nu - 2\sinh\mu\sin\nu\cos^2\nu) + (\sinh^2\mu+\sin^2\nu)(\cosh\mu\cos\nu) - (\cosh^2\mu - \cos^2\nu)(\sinh\mu\sin\nu)}{(\sinh^2\mu+\sin^2\nu)^2}\right]\frac{\partial f}{\partial\nu}\\
&+ \left[\frac{a(\hat{\boldsymbol{\mu}}_1+\hat{\boldsymbol{\nu}}_1)}{h_\mu}\right]\frac{\partial^2 f}{\partial\mu^2} + \left[\frac{a(\hat{\boldsymbol{\mu}}_2+\hat{\boldsymbol{\nu}}_2)}{h_\mu}\right]\frac{\partial^2 f}{\partial\nu^2}\\
&= \left[\frac{-\sinh^2\mu\cosh\mu\cos\nu + \sinh\mu\cosh^2\mu\sin\nu + \cosh\mu\sin^2\nu\cos\nu + \sinh\mu\sin\nu\cos^2\nu}{(\sinh^2\mu+\sin^2\nu)^2}\right]\frac{\partial f}{\partial\mu}\\
&+ \left[\frac{-\cosh\mu\sin^2\nu\cos\nu - \sinh\mu\sin\nu\cos^2\nu + \sinh^2\mu\cosh\mu\cos\nu - \sinh\mu\cosh^2\mu\sin\nu}{(\sinh^2\mu+\sin^2\nu)^2}\right]\frac{\partial f}{\partial\nu}\\
&+ \left[\frac{a(\hat{\boldsymbol{\mu}}_1+\hat{\boldsymbol{\nu}}_1)}{h_\mu}\right]\frac{\partial^2 f}{\partial\mu^2} + \left[\frac{a(\hat{\boldsymbol{\mu}}_2+\hat{\boldsymbol{\nu}}_2)}{h_\mu}\right]\frac{\partial^2 f}{\partial\nu^2}\\
&= \left[\frac{3a^3\sinh\mu\cosh\mu(\hat{\boldsymbol{\mu}}_1-\hat{\boldsymbol{\mu}}_2) + a^3\sin\nu\cos\nu(\hat{\boldsymbol{\mu}}_1+\hat{\boldsymbol{\mu}}_2)}{h_\mu^3}\right]\frac{\partial f}{\partial\mu} + \left[\frac{a^3\sinh\mu\cosh\mu(\hat{\boldsymbol{\nu}}_1+\hat{\boldsymbol{\nu}}_2) + a^3\sin\nu\cos\nu(\hat{\boldsymbol{\nu}}_1-\hat{\boldsymbol{\nu}}_2)}{h_\nu^3}\right]\frac{\partial f}{\partial\nu}\\
&+ \left[\frac{a(\hat{\boldsymbol{\mu}}_1+\hat{\boldsymbol{\nu}}_1)}{h_\mu}\right]\frac{\partial^2 f}{\partial\mu^2} + \left[\frac{a(\hat{\boldsymbol{\mu}}_2+\hat{\boldsymbol{\nu}}_2)}{h_\mu}\right]\frac{\partial^2 f}{\partial\nu^2}
\end{aligned}
$$