Multivariable Calculus - Problem Class 3

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Question 1.2.3
Caclulate $(\mathbf{u}\circ\nabla)\mathbf{u}$ in cyclindrical coordinates, where $\mathbf{u} = u_r\hat{\mathbf{r}} + u_\theta\hat{\boldsymbol{\theta}} + u_z\hat{\mathbf{z}}$.

My Solution 1.2.3

$$
\begin{array}{lrcl}
\text{N.B.} & \nabla & = & \sum_{\alpha=1}^{3} \frac{\mathbf{q}_\alpha}{h_\alpha}\frac{\partial}{\partial q_\alpha} \\
 & & = & \hat{\boldsymbol{r}}\frac{\partial}{\partial r} + \frac{\hat{\boldsymbol{\theta}}}{r}\frac{\partial}{\partial \theta} + \hat{\boldsymbol{z}}\frac{\partial}{\partial z} \\
\& & \hat{\boldsymbol{r}} & = & (\cos\theta, \sin\theta, 0) \\
 & \hat{\boldsymbol{\theta}} & = & (-\sin\theta, \cos\theta, 0) \\
 & \hat{\boldsymbol{z}} & = & (0, 0, 1)
\end{array}
$$

$$
\begin{array}{rcl}
(\mathbf{u}\circ\nabla)\mathbf{u} & = & \left(u_r\frac{\partial}{\partial r} + \frac{u_\theta}{r}\frac{\partial}{\partial\theta} + u_z\frac{\partial}{\partial z}\right)\left(u_r\hat{\boldsymbol{r}} + u_\theta\hat{\boldsymbol{\theta}} + u_z\hat{\boldsymbol{z}}\right) \\
 & & \text{Since } \hat{\boldsymbol{r}} \text{ \& } \hat{\boldsymbol{\theta}}\text{depend on } \theta \\
 & = & \left(u_r\frac{\partial}{\partial r} + \frac{u_\theta}{r}\frac{\partial}{\partial\theta} + u_z\frac{\partial}{\partial z}\right)\hat{\boldsymbol{r}} + \left(u_r^2\frac{\partial\hat{\boldsymbol{r}}}{\partial r} + \frac{u_\theta u_r}{r}\frac{\partial\hat{\boldsymbol{r}}}{\partial\theta} + u_z u_r\frac{\partial\hat{\boldsymbol{r}}}{\partial z}\right) \\
 & + & \left(u_r\frac{\partial}{\partial r} + \frac{u_\theta}{r}\frac{\partial}{\partial\theta} + u_z\frac{\partial}{\partial z}\right)\hat{\boldsymbol{\theta}} + \left(u_r u_\theta\frac{\partial\hat{\boldsymbol{\theta}}}{\partial r} + \frac{u_\theta^2}{r}\frac{\partial\hat{\boldsymbol{\theta}}}{\partial\theta} + u_z u_\theta\frac{\partial\hat{\boldsymbol{\theta}}}{\partial z}\right) \\
 & + & \left(u_r\frac{\partial}{\partial r} + \frac{u_\theta}{r}\frac{\partial}{\partial\theta} + u_z\frac{\partial}{\partial z}\right)\hat{\boldsymbol{z}}
\end{array}
$$

Question 1.3.1
Let $f(r)$ be a smooth scalar-valued function of $r = |\mathbf{r}|$, and let $\mathbf{a}\in\mathbb{R}^3$ be a constance vector. Calculate

$$\nabla\times(\mathbf{r}\times\mathbf{a}f(r))$$

My Solution 1.3.1

$$
\begin{array}{rcl}
[\nabla\times(\mathbf{r}\times\mathbf{a}f(r))]_i & = & \varepsilon_{ijk}\frac{\partial}{\partial x_j}[\mathbf{r}\times\mathbf{a}f(r)]_k \\
 & = & \varepsilon_{ijk}\frac{\partial}{\partial x_j}\varepsilon_{klm}x_l f(r)a_m \\
 & = & \varepsilon_{kij}\varepsilon_{klm}\frac{\partial}{\partial x_j}(x_l f(r)a_m) \\
 & = & [\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}]a_m\left(f(r)\frac{\partial x_l}{\partial x_j} + x_l\frac{\partial f(r)}{\partial x)j}\right) \\
 & = & [\delta_{il}a_j - a_i\delta_{jl}]\left(f(r)\delta_{jl} + x_l\frac{\partial r}{\partial x_j}\frac{\partial f(r)}{\partial r}\right) \\
 & = & [\delta_{il}a_j - a_i\delta_{jl}]\left(f(r)\delta_{jl} + \frac{x_l x_j}{r}f'(r)\right) \\
 & = & \delta_{ij}a_j f(r) - 3a_i f(r) + \frac{a_j x_i x_j}{r}f'(r) - \frac{a_i f'(r)}{r}x_j x_j \\
 & = & a_i f(r) - 3a_i f(r) + \frac{f'(r)}{r}(\mathbf{a}\circ\mathbf{r})x_i - rf'(r)a_i \\
\implies \nabla\times(\mathbf{a}f(r)) & = & -2f(r)\mathbf{a} - rf'(r)\mathbf{a} + \frac{f'(r)}{r}(\mathbf{a}\circ\mathbf{r})x_i
\end{array}
$$

N.B. - $\delta_{ij}\times\delta_{ij} = 3$ & $\sum_{j=1}^{3}x_j^2 = r^2$

Question 1.3.2
Let $\mathbf{u}$ be a vector field in $C(\mathbb{R}^3, \mathbb{R}^3)$. Show that

$$\mathbf{u} \times (\nabla \times \mathbf{u}) = \frac{1}{2}\nabla(\mathbf{u} \circ \mathbf{u}) - (\mathbf{u} \circ \nabla)\mathbf{u}$$

My Solution 1.3.2

$$\begin{array}{rcl}
[\mathbf{u} \times (\nabla \times \mathbf{u})]_i & = & \varepsilon_{ijk} u_j [\nabla \times \mathbf{u}_k \\
& = & \varepsilon_{ijk} u_j \varepsilon_{klm} \frac{\partial}{\partial x_i} u_m \\
& = & \varepsilon_{kij} \varepsilon_{klm} u_j \frac{\partial}{\partial x_i} u_m \\
& = & [\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}] u_j \frac{\partial}{\partial x_i} u_m \\
& = & u_j \frac{\partial}{\partial x_i} u_j - u_j \frac{\partial}{\partial x_j} u_i \\
& = & \left[\frac{\partial}{\partial x_i}\left(\frac{1}{2}u_j^2\right)\right] - (\mathbf{u} \circ \nabla) u_i \\
& = & \left[\frac{1}{2}\nabla(\mathbf{u} \circ \mathbf{u}) - (\mathbf{u} \circ \nabla)\mathbf{u}\right]_i
\end{array}$$