Multivariable Calculus - Problem Sheet 2

Dom Hutchinson

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Question 4

Question 4.1
Compute the gradient of $f(\mathbf{r}) = \mathbf{a} \circ \mathbf{r}$, where $\mathbf{a} \in \mathbb{R}^3$ is a fixed vector.

My Solution 4.1

$$\begin{array}{rcl} f(\mathbf{r}) & = & \mathbf{a} \circ \mathbf{r} \\ & = & a_1 x + a_2 y + a_3 z \\ \implies \nabla f(\mathbf{r}) & = & (a_1, a_2, a_3) \end{array}$$

Question 4.2
Compute the divergence of $\mathbf{v}(\mathbf{r}) = \nabla r^n$, where $r = |\mathbf{r}|$.
For which value of n does the divergence vanish?

My Solution 4.2

$$\begin{array}{rrcl} & \mathbf{v}(\mathbf{r}) & = & \nabla r^n \\ \text{Since} & r & = & \sqrt{x^2+y^2+z^2} \\ \implies & \nabla \circ \mathbf{v}(\mathbf{r}) & = & \nabla \circ \nabla r^n \\ & & = & \nabla \circ \nabla (x^2+y^2+z^2)^{n/2} \\ & & = & \nabla \circ \left(xn(x^2+y^2+z^2)^{\frac{n}{2}-1}, yn(x^2+y^2+z^2)^{\frac{n}{2}-1}, zn(x^2+y^2+z^2)^{\frac{n}{2}-1} \right) \\ & & = & \left(n(x^2+y^2+z^2)^{\frac{n}{2}-1} + xn(xn-2x)(x^2+y^2+z^2)^{\frac{n}{2}-2} \right) \\ & & + & \left(n(x^2+y^2+z^2)^{\frac{n}{2}-1} + yn(yn-2y)(x^2+y^2+z^2)^{\frac{n}{2}-2} \right) \\ & & + & \left(n(x^2+y^2+z^2)^{\frac{n}{2}-1} + zn(zn-2z)(x^2+y^2+z^2)^{\frac{n}{2}-2} \right) \\ & & = & 3n(x^2+y^2+z^2)^{\frac{n}{2}-1} + n(x^n - 2x^2 + y^2 n - 2y^2 + z^2 n - 2z^2)(x^2+y^2+z^2)^{\frac{n}{2}-2} \\ & & = & n(x^2+y^2+z^2)^{\frac{n}{2}-2} \left(3(x^2+y^2+z^2) + (x^n - 2x^2 + y^2 n - 2y^2 + z^2 n - 2z^2) \right) \\ & & = & n(x^2+y^2+z^2)^{\frac{n}{2}-2} \left((n+1)(x^2+y^2+z^2) \right) \end{array}$$

This equals zero if $n = 0$ or $n = -1$.
Question 4.3
Compute the curl of $\mathbf{v}(\mathbf{r}) = \boldsymbol{\omega} \times \mathbf{r}$, where $\boldsymbol{\omega} \in \mathbb{R}^3$ is a fixed vector.

My Solution 4.3

$$\begin{aligned}
\mathbf{v}(\mathbf{r}) &= \boldsymbol{\omega} \times \mathbf{r} \\
&= \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e})3 \\ \omega_1 & \omega_2 & \omega_3 \\ x & y & z \end{vmatrix} \\
&= \left(\begin{vmatrix} \omega_2 & \omega_3 \\ y & z \end{vmatrix}, \begin{vmatrix} \omega_1 & \omega_3 \\ x & z \end{vmatrix}, \begin{vmatrix} \omega_1 & \omega_2 \\ x & y \end{vmatrix} \right) \\
&= (\omega_2 z - \omega_3 y,\ \omega_1 z - \omega_3 x,\ \omega_1 y - \omega_2 x) \\
\text{Then} \quad \nabla \times \mathbf{v} &= \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ \partial x & \partial y & \partial z \\ \omega_2 z - \omega_3 y & \omega_1 z - \omega_3 x & \omega_1 y - \omega_2 x \end{vmatrix} \\
&= \Bigg(\frac{\partial}{\partial y}(\omega_1 y - \omega_2 x) - \frac{\partial}{\partial z}(\omega_1 z - \omega_3 x) \\
&\quad , \frac{\partial}{\partial x}(\omega_1 y - \omega_2 x) - \frac{\partial}{\partial z}(\omega_2 z - \omega_3 y) \\
&\quad , \frac{\partial}{\partial x}(\omega_1 z - \omega_3 x) - \frac{\partial}{\partial y}(\omega_2 z - \omega_3 y) \Bigg) \\
&= (\omega_1 - \omega_1, -\omega_2 - \omega_2, -\omega_3 - (-\omega_3)) \\
&= (0, -2\omega_2, 0)
\end{aligned}$$

Question 7

Without doing any calculations, how do you know the following result is false?

$$\nabla \circ (\mathbf{u} \times \mathbf{v}) = \mathbf{u} \circ (\nabla \times \mathbf{v}) + \mathbf{v} \circ (\nabla \times \mathbf{u})$$

Find a corrected version of this result

My Solution 7

The cross product is anticommutative, but this identity shows it to be commutative.
The correct identity is

$$\nabla \circ (\mathbf{u} \times \mathbf{v}) = \mathbf{v} \circ (\nabla \times \mathbf{u}) - \mathbf{u} \circ (\nabla \times \mathbf{v})$$

Question 8

Show that, for vector fields $\mathbf{u}(\mathbf{r}), \mathbf{v}(\mathbf{r})$ both in $\mathbb{R}^3$

$$\nabla \times (\mathbf{u} \times \mathbf{v}) = (\nabla \circ \mathbf{v})\mathbf{u} - (\nabla \circ \mathbf{u})\mathbf{v} + (\mathbf{v} \circ \nabla)\mathbf{u} - (\mathbf{u} \circ \nabla)\mathbf{v}$$

My Solution 8

$$\begin{aligned}
[\nabla \times (\mathbf{u} \times \mathbf{v})]_i &= \varepsilon_{ijk} \frac{\partial}{\partial x_j} [\mathbf{u} \times \mathbf{v}]_k \\
&= \varepsilon_{ijk} \frac{\partial}{\partial x_j} \varepsilon_{klm}(u_l v_m) \\
&= \varepsilon_{ijk} \varepsilon_{klm} \frac{\partial}{\partial x_j}(u_l v_m) \\
&= \varepsilon_{kij} \varepsilon_{klm} \left(\frac{\partial u_l}{\partial x_j} v_m + u_l \frac{\partial v_m}{\partial x_j} \right) \\
&= (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}) \left(\frac{\partial u_l}{\partial x_j} v_m + u_l \frac{\partial v_m}{\partial x_j} \right) \\
&= \left(\frac{\partial u_i}{\partial x_j} v_j + u_i \frac{\partial v_j}{\partial x_j} \right) - \left(\frac{\partial u_j}{\partial x_j} v_i + u_j \frac{\partial v_i}{\partial x_j} \right) \\
&= \big([(\mathbf{v} \circ \nabla)\mathbf{u}]_i + [(\nabla \circ \mathbf{v})\mathbf{u}]_i \big) - \big([(\nabla \circ \mathbf{u})\mathbf{v}]_i + [(\mathbf{u} \circ \nabla)\mathbf{v}]_i \big)
\end{aligned}$$

This is true for $i = 1, 2, 3$ so identity holds.