Multivariable Calculus - Problem Sheet 4

Dom Hutchinson

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Question 4
Calculte the integral $\int_C \mathbf{F}\circ d\mathbf{r}$ when $\mathbf{F}=(-x^2y, xy^2, 0)$ and C is a square in teh (x,y) plane with vertices at $(0,0),(l,0),(l,l),(0,l)$ which is oriented anticlockwise.

My Solution 4
C can be describe by

$$\begin{array}{rcl}\mathbf{p}_1(t) &=& (t,0,0),\\ \mathbf{p}_2(t) &=& (l,t,0),\\ \mathbf{p}_3(t) &=& (l-t,l,0)\ \&\\ \mathbf{p}_4(t) &=& (0,l-t,0) \text{ for } t\in[0,1]\end{array}$$

Then

$$\begin{array}{rcl}\mathbf{p}'_1(t) &=& (1,0,0),\\ \mathbf{p}'_2(t) &=& (0,1,0),\\ \mathbf{p}'_3(t) &=& (-1,0,0)\ \&\\ \mathbf{p}'_4(t) &=& (0,-1,0).\end{array}$$

And

$$\begin{array}{rcccl}\mathbf{F}(\mathbf{p}_1(t)) &=& \mathbf{F}(t,0,0) &=& \mathbf{0},\\ \mathbf{F}(\mathbf{p}_2(t)) &=& \mathbf{F}(l,t,0) &=& (-l^2t, lt^2, 0),\\ \mathbf{F}(\mathbf{p}_3(t)) &=& \mathbf{F}(l-t,l,0) &=& (-l(l-t)^2, (l-t)l^2, 0)\ \&\\ \mathbf{F}(\mathbf{p}_4(t)) &=& \mathbf{F}(0,l-t,0) &=& \mathbf{0}.\end{array}$$

$$\begin{array}{rcl}\displaystyle\int_C \mathbf{F}(\mathbf{r})\circ d\mathbf{r} &=& \displaystyle\sum_{i=1}^{4}\int_0^1 \mathbf{F}(\mathbf{p}_i(t))\circ\mathbf{p}'_i(t)dt\\ &=& \displaystyle 0+\int_0^1(-l^2t, lt^2, 0)\circ(0,1,0)dt+\int_0^1(-l(l-t)^2,(l-t)l^2,0)\circ(-1,0,0)dt+0\\ &=& \displaystyle\int_0^1 lt^2+l(l-t)^2dt\\ &=& \displaystyle\int_0^1 2lt^2+l^3-2l^2t\ dt\\ &=& \displaystyle\frac{2lt^3}{3}+l^3t-l^2t^2\Big|_0^1\\ &=& \displaystyle\frac{2l}{3}\end{array}$$

Question 6
Let $\mathbf{F}(\mathbf{r})=(-y^2,x,z^2)$ and define a curve C in $\mathbb{R}^3$ to be the intersection of the cylinder $x^2+y^2=1$ and the plane $y+z=2$.

Question 6.1
Compute

$$\int_C \mathbf{F}\circ d\mathbf{r}$$

My Solution 6.1
The equation $x^2 + y^2 = 1$ can be parametrised as $x(\theta) = \cos(\theta),\ y(\theta) = \sin(\theta)$ for $\theta \in [0, 2\pi)$.
Since $y + z = 2$ then $z = 2 - y \equiv z(\theta) = 2 - \sin(\theta)$.
Thus, C can be descibe by $\mathbf{p}(\theta) = (\cos\theta, \sin\theta, 2 - \sin\theta)$ for $\theta \in [0, 2\pi)$. Then

$$\begin{array}{rrcl}
 & \mathbf{p}'(\theta) & = & (-\sin\theta, \cos\theta, 2 - \cos\theta) \\
\& & \mathbf{F}(\mathbf{p}(\theta)) & = & (-\sin^2\theta, \cos\theta, (2 - \sin\theta)^2) \\
\implies & \int_C \mathbf{F}(\mathbf{r}) \circ d\mathbf{r} & = & \int_0^{2\pi} \mathbf{F}(\mathbf{p}(\theta)) \circ \mathbf{p}'(\theta) d\theta \\
 & & = & \int_0^{2\pi} -\sin^3\theta + \cos^2\theta + (2 - \cos\theta)(2 - \sin\theta)^2 d\theta \\
 & & = & \int_0^{2\pi} -\sin\theta(1 - \cos^2\theta) + \frac{1}{2}(1 + \cos 2\theta) + 2(2 - \sin\theta)^2 - \cos\theta(2 - \sin\theta)^2 d\theta \\
 & & = & \int_0^{2\pi} -\sin\theta + \sin\theta\cos^2\theta + \frac{1}{2} + \frac{1}{2}\cos 2\theta + 8 - 8\sin\theta + 2\sin^2\theta - \cos\theta(2 - \sin\theta)^2 d\theta \\
 & & = & \cos\theta - \frac{1}{3}(\cos^3\theta) + \frac{\theta}{2} + \frac{1}{4}(\sin 2\theta) + 8\theta - 8\cos\theta + \theta - \frac{1}{2}(\sin 2\theta) + \frac{1}{3}(2 - \sin\theta)^3 \Big|_0^{2\pi} \\
 & & = & [1 - \frac{1}{3} + \pi + 0 + 16\pi - 8 + 2\pi - 0 + \frac{8}{3}] - [1 - \frac{1}{3} + 0 + 0 + 0 - 8 + 0 + 0 + \frac{8}{3}] \\
 & & = & 19\pi
\end{array}$$

Question 6.2
Compute

$$\int_S \nabla \times \mathbf{F} \circ d\mathbf{S}$$

where S is a surface of your choice which boundary ∂S coinciding with the curve C. Hence show that Stokes' theorem is satisfied.

My Solution 6.2