Statistics 2 - Problem Sheet 3

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October 22, 2019

Question - 1.

This question is about maximum likelihood estimates of transformations of the parameter.

Question 1.1

Suppose $\mathbf{X} \overset{\text{iid}}{\sim} \text{Exponential}(\lambda)$ where $\lambda \in \Theta = \mathbb{R}^{>0}$.
Consider the transformed parameter $\tau = \frac{1}{1+\lambda}$.
Write the likelihood function in terms of τ, then maximise it to find $\hat{\tau}_{ml}$.
Confirm that $\hat{\tau}_{mle} = \frac{1}{1+\hat{\lambda}_{ml}}$.

Answer 1.1

We have $f_{X_i}(x_i;\lambda) = \lambda e^{-\lambda x_i}\ \forall\ i \in [1,n]$ since $\lambda > 0$.

$$
\begin{array}{rrcl}
 & \tau & = & \frac{1}{1+\lambda} \\
\implies & 1+\lambda & = & \frac{1}{\tau} \\
\implies & \lambda & = & \frac{1}{\tau} - 1 = \frac{1-\tau}{\tau} \\
 & L(\lambda;\mathbf{x}) & = & \prod\limits_{i=1}^n f_{X_i}(x_i;\lambda) \\
 & & = & \prod\limits_{i=1}^n \lambda e^{-\lambda x_i} \\
 & & = & \lambda^n e^{-\lambda \sum_{i=1}^n x_i} \\
\implies & L(\tau;\mathbf{x}) & = & \left(\frac{1-\tau}{\tau}\right)^n e^{\frac{\tau-1}{\tau}\sum_{i=1}^n x_i} \\
\implies & \ell(\tau;\mathbf{x}) & = & n\ln(1-\tau) - n\ln\tau + \frac{\tau-1}{\tau}\sum\limits_{i=1}^n x_i \\
\implies & \frac{\partial}{\partial\tau}\ell(\tau;\mathbf{x}) & = & -\frac{n}{1-\tau} - \frac{n}{\tau} + \frac{\tau-(\tau-1)}{\tau^2}\sum\limits_{i=1}^n x_i \\
 & & = & -\frac{n}{1-\tau} - \frac{n}{\tau} + \frac{1}{\tau^2}\sum\limits_{i=1}^n x_i \\
\text{Setting} & 0 & = & \frac{\partial}{\partial\tau}\ell(\hat{\tau};\mathbf{x}) \\
\implies & 0 & = & -\frac{n}{1-\hat{\tau}} - \frac{n}{\hat{\tau}} + \frac{1}{\hat{\tau}^2}\sum\limits_{i=1}^n x_i \\
\implies & 0 & = & -n\hat{\tau}^2 - n\hat{\tau}(1-\hat{\tau}) + (1-\hat{\tau})\sum\limits_{i=1}^n x_i \\
 & & = & -n\hat{\tau}^2 - n\hat{\tau} + n\hat{\tau}^2 + \sum\limits_{i=1}^n x_i - \hat{\tau}\sum\limits_{i=1}^n x_i \\
\implies & \sum\limits_{i=1}^n x_i & = & \hat{\tau}\left(n + \sum\limits_{i=1}^n x_i\right) \\
\implies & \hat{\tau} & = & \dfrac{\sum_{i=1}^n x_i}{n+\sum_{i=1}^n x_i} \\
 & & = & \dfrac{1}{1+\frac{n}{\sum_{i=1}^n x_i}}
\end{array}
$$

$$\begin{array}{rrcl}
\text{We have} & L(\lambda;\mathbf{x}) & = & \lambda^n e^{-\lambda\sum_{i=1}^n x_i} \\
\implies & \ell(\lambda;\mathbf{x}) & = & n\ln\lambda - \lambda\sum_{i=1}^n x_i \\
\implies & \frac{\partial}{\partial\lambda}\ell(\lambda;\mathbf{x}) & = & \frac{n}{\lambda} - \sum_{i=1}^n x_i \\
\text{Setting} & 0 & = & \frac{\partial}{\partial\hat{\lambda}}\ell(\lambda;\mathbf{x}) \\
 & & = & \frac{n}{\hat{\lambda}} - \sum_{i=1}^n x_i \\
\implies & \hat{\lambda} & = & \frac{n}{\sum_{i=1}^n x_i} \\
\implies & \hat{\tau} & = & \frac{1}{1+\hat{\lambda}}
\end{array}$$

Question 1.2
If $X \sim \text{Poisson}(\lambda)$ where $\lambda \in \Theta = \mathbb{R}^{>0}$ then $\mathbb{P}(X=0;\lambda) = e^{-\lambda}$.
Explain why the maximum likelihood estimate of this probability is $e^{-\hat{\lambda}_{ml}}$.

Answer 1.2
Define $\tau(\lambda) = e^{-\lambda}$. Then $\hat{\tau}(\lambda) = e^{-\hat{\lambda}}$.

Question - 2.

Let $X_1, X_2, \dots \overset{\text{iid}}{\sim} \text{Uniform}[0,1]$ each with probability density function $f(x) = \mathbb{1}\{0 \le x \le 1\}$.

Question 2.1 - Calculate the mean and variance of $\ln(X_1)$.

Answer 2.1

$$\begin{array}{rcl}
\mathbb{E}(\ln(X_1)) & = & \int f(x)\ln(x)dx \\
 & = & \int \mathbb{1}\{0 \le x \le 1\} ln(x)dx \\
 & = & \int_0^1 \ln(x)dx \\
 & = & [x\ln(x) - x]_0^1 \\
 & = & (1\ln(1) - 1) - (0\ln(0) - 0) \\
 & = & (-1) - (0) \\
 & = & -1 \\
\text{var}(\ln(X_1)) & = & \mathbb{E}(\ln(X_1)^2) - \mathbb{E}(\ln(X_1))^2 \\
 & = & \int f(x)\ln(x)^2 dx - 1 \\
 & = & \int \mathbb{1}\{0 \le x \le 1\}\ln(x)^2 dx - 1 \\
 & = & \int_0^1 \ln(x)^2 dx - 1 \\
 & = & [x\ln(x)^2]_0^1 - \int_0^1 2\ln(x)dx - 1 \\
 & = & [x\ln(x)^2 - 2(x\ln(x) - x)]_0^1 - 1 \\
 & = & (\ln(1)^2 - 2\ln(1) + 2) - 0 - 1 \\
 & = & 2 - 1 \\
 & = & 1
\end{array}$$

Question 2.2 - By taking logs, find a random variable X such that as $n \to \infty$

$$(X_1, \dots, X_n)^{\frac{1}{\sqrt{n}}} e^{\sqrt{n}} \to_{\mathcal{D}} X$$

Answer 2.2
Let $X_1, X_2, \dots$ be iid random variables and define $Y_n = (X_1 \times \dots \times X_n)^{1/\sqrt{n}} e^{\sqrt{n}}$.Then

$$\begin{array}{rcl} lnY_n &=& \frac{1}{\sqrt{n}}\left(\sum_{i=1}^{n} \ln(X_i)\right) + \sqrt{n} \\ \implies \quad \mathbb{P}(\ln Y_n \leq y) &=& \mathbb{P}\left(\frac{1}{\sqrt{n}}\left(\sum_{i=1}^{n} \ln(X_i)\right) + \sqrt{n} \leq y\right) \\ &=& \mathbb{P}\left(\sqrt{n}\ln(X_1) + \sqrt{n} \leq y\right) \end{array}$$

Want to find X st $\lim_{n\to\infty} \mathbb{P}(\sqrt{n}(\ln(X_1) + 1) \leq y) = \mathbb{P}(X \leq y)$ (I think?)
$\dots$