Compupter Practical 3

Statistics 2

Dom Hutchinson

```
options(warn=-1)
diabetes<-read.csv("data/diabetes_data.csv",header=T) # load data set

missing<-function(v){ # Replace missing values with median of non-zero values
   med<-median(v[v>0])
   v[v==0]<-med
   return(v)
}

diabetes$Glucose<-missing(diabetes$Glucose)
diabetes$BloodPressure<-missing(diabetes$BloodPressure)
diabetes$Insulin<-missing(diabetes$Insulin)
diabetes$SkinThickness<-missing(diabetes$SkinThickness)
diabetes$BMI<-missing(diabetes$BMI)
```

Question 1

Here I shall test the hypotheses

$$H_0 : \text{BloodPressure data can be modelled by a } \mathcal{N}(\mu, \sigma^2) \text{ distribution}$$
$$H_1 : \text{BloodPressure data can } \textbf{not} \text{ be modelled by a } \mathcal{N}(\mu, \sigma^2) \text{ distribution}$$

I shall use the maximum likelihood estimates for μ and σ^2.

$$\hat{\mu}_{\text{MLE}} = \frac{1}{n}\sum_{i=1}^{n} x_i \text{ and } \hat{\sigma}^2 = \frac{1}{n}\sum_{i=1}^{n}(x_i - \bar{x})^2$$

Define the test statistic

$$T_{\text{pearson}}(\mathbf{X}) = \sum_{j=1}^{m} \frac{(o_j - e_j)^2}{e_j} \to_{\mathcal{D}} \chi^2_r$$

where o_j is the number of observations of class j and e_j is the expected number of observations of class j, assuming H_0 is true.
In this case $r = m - 3 = 7 - 3 = 4$ since we have seven classes of observations but three constraints on these classes due to the assumption of a normal distribution (mean, variance & sum of class sizes).

```
breaks<-c(-Inf,seq(45,95,by=10),Inf) # Quantise data
obs<-table(cut(diabetes$BloodPressure,breaks))

# Perform Pearson's Goodness of Fit Test
n<-length(diabetes$BloodPressure) # number of data points
r<-length(obs)-3 # degrees of freedom
mu<-sum(diabetes$BloodPressure)/n # MLE mean
sigma<-sqrt(sum((diabetes$BloodPressure-mu)^2)/n) # MLE variance

exp<-n*(pnorm(breaks[-1],mean=mu,sd=sigma)-pnorm(breaks[-length(breaks)],mean=mu,sd=sigma)) # expected
round(cbind(obs,exp),1) # return data table
```

```
##           obs   exp
## (-Inf,45]   9   9.0
## (45,55]    44  48.7
## (55,65]   155 150.1
## (65,75]   271 241.9
## (75,85]   183 204.3
## (85,95]    83  90.4
## (95, Inf]  23  23.6
```

```
t_obs<-sum((obs-exp)^2/exp) # observered test statistic
p_val<-1-pchisq(t_obs,df=r) # p-value
cat("mu=",mu,"\nsigma=",sigma,"\ndf=",r,"\nt_obs=",t_obs,"\np_val=",p_val,sep="") # return test values
```

```
## mu=72.38672
## sigma=12.08876
## df=4
## t_obs=6.959031
## p_val=0.1380692
```

Here the p-value (0.1380692) is not statistically significant enough to reject H_0, for a reasonable significance level.
Thus we accept that *BloodPressure* can be modelled by a Normal distribution.

Question 2

Let $X_1, \ldots, X_n \overset{\text{iid}}{\sim} \text{Normal}(\mu, 12^2)$ model the *blood pressure* of members of the study.
Here I shall test the hypotheses

$$H_0 : \mu = 70 \text{ against } H_1 : \mu > 70$$

Define test statistic

$$T(\mathbf{X}) = \frac{\bar{\mathbf{X}} - \mu_0}{\sigma/\sqrt{n}} = \frac{\bar{\mathbf{X}} - 70}{12/\sqrt{768}} \to_{\mathcal{D}} \text{Normal}(0, 1)$$

This convergence in distribution is a result of the central limit theorem.

```
mu<-70; sigma<-12
n<-length(diabetes$BloodPressure) # number of data points
x_bar<-mean(diabetes$BloodPressure)

t_obs<-(x_bar-mu)/(sigma/sqrt(n)) # observered test statistic
p_val<-1-pnorm(t_obs) # p-value
cat("mu=",mu,"\nsigma=",sigma,"\nx_bar=",x_bar,"\nt_obs=",t_obs,"\np_val=",p_val,sep="") # display test
```

```
## mu=70
## sigma=12
## x_bar=72.38672
## t_obs=5.511891
## p_val=1.774995e-08
```

Here the p-value (1.7749953×10^{-8}) is statistically significant enough to reject H_0, for a reasonable significance level.
Thus we accept the alternative hypothesis, that $\mu > 70$.

Question 3

Let $Y_i \overset{\text{ind}}{\sim} \text{Bernoulli}(\sigma(\theta^T x_i))$ for $i \in [1, n]$ $\pi_i = \mathbb{P}(Y_i = 1)$ where $\sigma(z) := \frac{1}{1+e^{-z}}$.
Then

$$
\begin{aligned}
\pi_i &:= \mathbb{P}(Y_i = 1) \\
&= \sigma(\theta^T x_i) \\
&:= \frac{1}{1 + e^{-\theta^T x_i}} \\
&= \frac{1}{1 + e^{-\sum_{j=1}^{d} \theta_j x_{ij}}} \\
\implies \ln \pi_i &= \ln\left(\frac{1}{1 + e^{-\sum_{j=1}^{d} \theta_j x_{ij}}}\right) \\
&= \ln 1 - \ln(1 + e^{-\sum_{j=1}^{d} \theta_j x_{ij}}) \\
&= -\ln(1 + e^{-\sum_{j=1}^{d} \theta_j x_{ij}}) \\
\text{and} \quad \ln(1 - \pi_i) &= \ln\left(1 - \frac{1}{1 + e^{-\sum_{j=1}^{d} \theta_j x_{ij}}}\right) \\
&= \ln\left(\frac{e^{-\sum_{j=1}^{d} \theta_j x_{ij}}}{1 + e^{-\sum_{j=1}^{d} \theta_j x_{ij}}}\right) \\
&= \ln(e^{-\sum_{j=1}^{d} \theta_j x_{ij}}) - \ln(1 + e^{-\sum_{j=1}^{d} \theta_j x_{ij}}) \\
&= -\left(\sum_{j=1}^{d} \theta_j x_{ij}\right) - \ln(1 + e^{-\sum_{j=1}^{d} \theta_j x_{ij}}) \\
\implies \ln \frac{\pi_i}{1 - \pi_i} &= \ln(\pi_i) - \ln(1 - \pi) \\
&= -\ln(1 + e^{-\sum_{j=1}^{d} \theta_j x_{ij}}) + \left(\sum_{j=1}^{d} \theta_j x_{ij}\right) + \ln(1 + e^{-\sum_{j=1}^{d} \theta_j x_{ij}}) \\
&= \sum_{j=1}^{d} \theta_j x_{ij}
\end{aligned}
$$

```
# sigmoid function
sigma<-function(z) {
   1/(1+exp(-z))
}

# Log likelihood
ell<-function(theta,X,y) {
   p<-as.vector(sigma(X%*%theta))
   sum(y*log(p)+(1-y)*log(1-p))
}

# score function
score<-function(theta,X,y) {
   p<-as.vector(sigma(X%*%theta))
   as.vector(t(X)%*%(y-p))
}

# MLE
maximise.ell<-function(ell,score,X,y,theta0) {
   optim.out<-optim(theta0, fn=ell, gr=score, X=X, y=y, method="BFGS", control=list(fnscale=-1, maxit=1
   return(list(theta=optim.out$par, value=optim.out$value))
}
```

Question 4

Here I shall test whether the variables *BloodPressure, SkinThichness, Insulin* and *Age* are statistically significant to the development of diabetes.
To do so I shall test the hypotheses

$$H_0 : \boldsymbol{\theta} := (\theta_3, \theta_4, \theta_5, \theta_8) = \mathbf{0} \text{ against } H_1 : \boldsymbol{\theta} \neq \mathbf{0}$$

Consider the likelihood ratio statistic

$$\Lambda_n := \frac{L(\hat{\boldsymbol{\theta}}_0; \boldsymbol{x})}{L(\hat{\boldsymbol{\theta}}_{\text{MLE}}; \boldsymbol{x})}$$

where $\hat{\boldsymbol{\theta}}_0$ is the maximum likelihood estimator under the null hypothesis and $\hat{\boldsymbol{\theta}}_{\text{MLE}}$ is the maximum likelihood estimator for the full model.
Define test statistic

$$T_n(\mathbf{X}) := -2\Lambda_n = -2[\ell(\hat{\boldsymbol{\theta}}_0; \boldsymbol{X}) - \ell(\hat{\boldsymbol{\theta}}_{\text{MLE}}; \boldsymbol{X})] \sim \chi^2_r$$

where $r = 4$ since the null hypothesis specifies restrictions on four variables.

```
X_rest<-cbind(1,as.matrix(diabetes[,c(1,2,6,7)])) # Variables we are not testing (ie assuming others=0)
X_full<-cbind(1,as.matrix(diabetes[,1:8])) # all variables
Y<-diabetes[,9] # outcomes

theta_hat_0.value<-maximise.ell(ell,score,X_rest,Y,rep(0,5))$value # MLE under H0
theta_hat_mle.value<-maximise.ell(ell,score,X_full,Y,rep(0,9))$value # MLE for full model
cat("ell(theta_hat_0): ",theta_hat_0.value,"\nell(theta_hat_mle): ",theta_hat_mle.value,sep="") # outpu
```

```
## ell(theta_hat_0): -358.1828
## ell(theta_hat_mle): -356.4209
```

Using these results we can calculate an observed test statistic

$$T_n(\mathbf{x}) = -2[\ell(\hat{\boldsymbol{\theta}}_0; \boldsymbol{x}) - \ell(\hat{\boldsymbol{\theta}}_{\text{MLE}}; \boldsymbol{x})] = -2[(-358.18) - (-356.42)] = 3.52$$

Since $T_n(\mathbf{X}) \sim \chi^2_4$ we have an observered p-value of

$$p(\mathbf{x}) := \mathbb{P}(T_n(\mathbf{X}) \geq T_n(\mathbf{x}); H_0) = \mathbb{P}(\chi^2_4 \geq 3.52) = 0.4743$$

Using the code described in the epilogue we can confirm this calculation.

```
model1<-glm(Y~X_full,family=binomial) #full model
model2<-glm(Y~X_rest,family=binomial) #restricted model
suppressMessages(library(lmtest)) # load library
lrtest(model1, model2) # perform linear regression test
```

```
## Likelihood ratio test
##
## Model 1: Y ~ X_full
## Model 2: Y ~ X_rest
##   #Df  LogLik Df  Chisq Pr(>Chisq)
## 1   9 -356.42
## 2   5 -358.18 -4 3.5237     0.4743
```

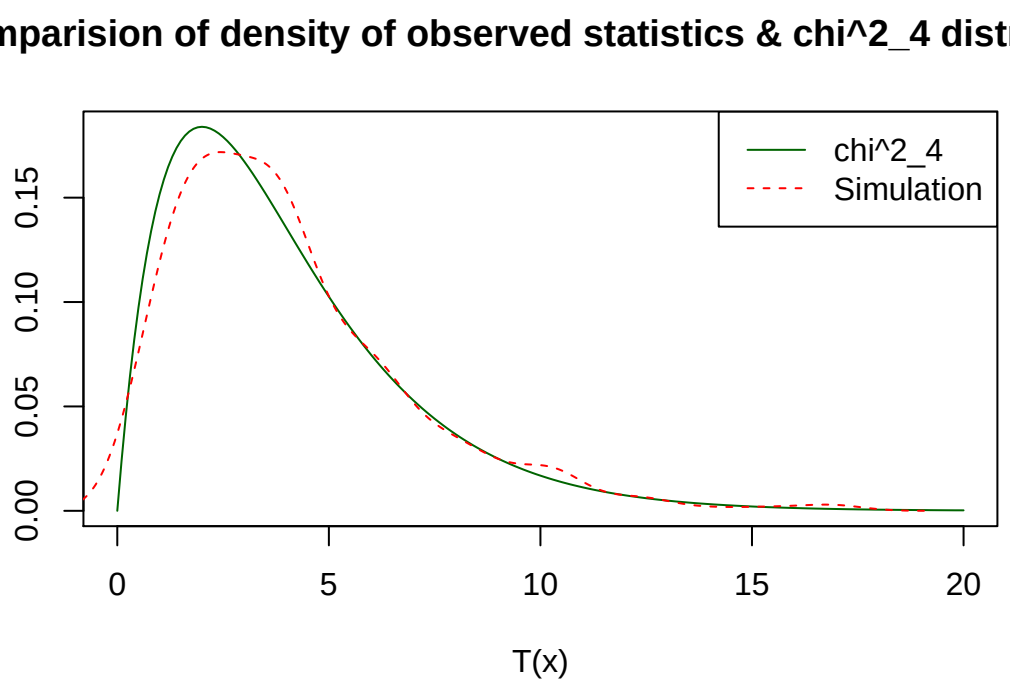
Here the p-value (0.4743) is not statistically significant enough to reject H_0, for a reasonable significance level. Thus we accept that the variables *BloodPressure, SkinThichness, Insulin* and *Age* are not statistically significant for the development of diabetes and thus $(\theta_3, \theta_4, \theta_5, \theta_8) = \mathbf{0}$.

Question 5

```
set.seed(779543035) # Set RNG seed

generate.ys<-function(X,theta) { # Generate new outcomes from data
   n<-dim(X)[1]
   rbinom(n,size=1,prob=sigma(X%*%theta))
}

simulate<-function(theta_hat_mle) {
   new_Y<-generate.ys(X_rest,theta_hat_mle) # Generate new outcomes

   theta_hat_0.value<-maximise.ell(ell,score,X_rest,new_Y,rep(0,5))$value # MLE under H0
   theta_hat_mle.value<-maximise.ell(ell,score,X_full,new_Y,rep(0,9))$value # MLE under full model

   t_obs<--2*(theta_hat_0.value-theta_hat_mle.value) # observed test statistic
}

n_trials<-500
theta_hat_mle<-maximise.ell(ell,score,X_rest,Y,rep(0,5))$theta
simulation.raw<-sapply(1:n_trials, function(i) simulate(theta_hat_mle)) # Run simulation
```

a)

```
x<-seq(0,20,0.1)
plot(x,dchisq(x,4),type="l",col="darkgreen",xlab="T(x)",ylab=""
     ,main="Comparision of density of observed statistics & chi^2_4 distribution") # Plot chi^2_m distr
lines(density(simulation.raw),col="red",lty=2) # Plot distribution of obsered test statistics
legend("topright",legend=c("chi^2_4","Simulation"),lty=1:2,col=c("darkgreen","red"))
```



b)

Here I shall test the hypotheses

$$H_0 : -2\ln\Lambda_n \sim \chi^2_4 \text{ against } H_1 : -2\ln\Lambda_n \not\sim \chi^2_4$$

This shall be done using Pearson's Goodness-of-Fit test. Usint test statistic

$$T_{\text{pearson}}(\mathbf{X}) = \sum_{j=1}^{m} \frac{(o_j - e_j)^2}{e_j} \to_{\mathcal{D}} \chi^2_r$$

In this case $r = 12 - 1 = 11$ since I split the observed data into twelve classes and there is a single constraint on these classes (sum of class sizes).

```
breaks<-c(-Inf,seq(1,11,by=1),Inf) # quantise data
obs<-table(cut(simulation.raw,breaks))

exp<-n_trials*(pchisq(breaks[-1],4)-pchisq(breaks[-length(breaks)],4)) # Expected number of observation
round(cbind(obs,exp),1) # Return data table
```

```
##           obs  exp
## (-Inf,1]   35 45.1
## (1,2]      81 87.0
## (2,3]      91 89.0
## (3,4]      88 75.9
## (4,5]      62 59.4
## (5,6]      40 44.1
## (6,7]      35 31.6
## (7,8]      20 22.2
## (8,9]      15 15.2
```

```
## (9,10]      8 10.3
## (10,11]    13  6.9
## (11, Inf]  12 13.3
```

```
r<-length(obs)-1
t_obs<-sum((obs-exp)^2/exp) # observered test statistic
p_val<-1-pchisq(t_obs,df=r) # p-value
cat("df=",r,"\nt_obs=",t_obs,"\np_val=",p_val,sep="") # return test values
```

```
## df=11
## t_obs=11.68184
## p_val=0.3880274
```

Here the p-value (0.3880274) is not statistically significant enough to reject H_0, for a reasonable significance level.
Thus we accept that the test statistic $-2\ln\Lambda_n$ is distributed according to χ^2_4.