

Statistics 2 - Problem Sheet 3

Dom Hutchinson

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Question - 1.

This question is about maximum likelihood estimates of transformations of the parameter.

Question 1.1

Suppose $\mathbf{X} \stackrel{\text{iid}}{\sim} \text{Exponential}(\lambda)$ where $\lambda \in \Theta = \mathbb{R}^{>0}$.

Consider the transformed parameter $\tau = \frac{1}{1+\lambda}$.

Write the likelihood function in terms of τ , then maximise it to find $\hat{\tau}_{ml}$.

Confirm that $\hat{\tau}_{mle} = \frac{1}{1+\hat{\lambda}_{ml}}$.

Answer 1.1

We have $f_{X_i}(x_i; \lambda) = \lambda e^{-\lambda x_i} \forall i \in [1, n]$ since $\lambda > 0$.

$$\begin{aligned}
 \tau &= \frac{1}{1+\lambda} \\
 \implies 1 + \lambda &= \frac{1}{\tau} \\
 \implies \lambda &= \frac{1}{\tau} - 1 = \frac{1-\tau}{\tau} \\
 L(\lambda; \mathbf{x}) &= \prod_{i=1}^n f_{X_i}(x_i; \lambda) \\
 &= \prod_{i=1}^n \lambda e^{-\lambda x_i} \\
 &= \lambda^n e^{-\lambda \sum_{i=1}^n x_i} \\
 \implies L(\tau; \mathbf{x}) &= \left(\frac{1-\tau}{\tau}\right)^n e^{\frac{\tau-1}{\tau} \sum_{i=1}^n x_i} \\
 \implies \ell(\tau; \mathbf{x}) &= n \ln(1-\tau) - n \ln \tau + \frac{\tau-1}{\tau} \sum_{i=1}^n x_i \\
 \implies \frac{\partial}{\partial \tau} \ell(\tau; \mathbf{x}) &= -\frac{n}{1-\tau} - \frac{n}{\tau} + \frac{\tau-1}{\tau^2} \sum_{i=1}^n x_i \\
 &= -\frac{n}{1-\tau} - \frac{n}{\tau} + \frac{1}{\tau^2} \sum_{i=1}^n x_i \\
 \text{Setting } 0 &= \frac{\partial}{\partial \tau} \ell(\hat{\tau}; \mathbf{x}) \\
 \implies 0 &= -\frac{n}{1-\hat{\tau}} - \frac{n}{\hat{\tau}} + \frac{1}{\hat{\tau}^2} \sum_{i=1}^n x_i \\
 \implies 0 &= -n\hat{\tau}^2 - n\hat{\tau}(1-\hat{\tau}) + (1-\hat{\tau}) \sum_{i=1}^n x_i \\
 &= -n\hat{\tau}^2 - n\hat{\tau} + n\hat{\tau}^2 + \sum_{i=1}^n x_i - \hat{\tau} \sum_{i=1}^n x_i \\
 \implies \sum_{i=1}^n x_i &= \hat{\tau} \left(n + \sum_{i=1}^n x_i \right) \\
 \implies \hat{\tau} &= \frac{\sum_{i=1}^n x_i}{n + \sum_{i=1}^n x_i} \\
 &= \frac{1}{1 + \frac{n}{\sum_{i=1}^n x_i}} \\
 \text{We have } L(\lambda; \mathbf{x}) &= \lambda^n e^{-\lambda \sum_{i=1}^n x_i} \\
 \implies \ell(\lambda; \mathbf{x}) &= n \ln \lambda - \lambda \sum_{i=1}^n x_i \\
 \implies \frac{\partial}{\partial \lambda} \ell(\lambda; \mathbf{x}) &= \frac{n}{\lambda} - \sum_{i=1}^n x_i \\
 \text{Setting } 0 &= \frac{\partial}{\partial \lambda} \ell(\hat{\lambda}; \mathbf{x}) \\
 &= \frac{n}{\hat{\lambda}} - \sum_{i=1}^n x_i \\
 \implies \hat{\lambda} &= \frac{n}{\sum_{i=1}^n x_i} \\
 \implies \hat{\tau} &= \frac{1}{1 + \hat{\lambda}}
 \end{aligned}$$

Question 1.2

If $X \sim \text{Poisson}(\lambda)$ where $\lambda \in \Theta = \mathbb{R}^{>0}$ then $\mathbb{P}(X = 0; \lambda) = e^{-\lambda}$.

Explain why the maximum likelihood estimate of this probability is $e^{-\hat{\lambda}_{ml}}$.

Answer 1.2

Define $\tau(\lambda) = e^{-\lambda}$. Then $\hat{\tau}(\lambda) = e^{-\hat{\lambda}}$.

Question - 2.

Let $X_1, X_2, \dots \stackrel{\text{iid}}{\sim} \text{Uniform}[0, 1]$ each with probability density function $f(x) = \mathbb{1}\{0 \leq x \leq 1\}$.

Question 2.1 - Calculate the mean and variance of $\ln(X_1)$.

Answer 2.1

$$\begin{aligned}
 \mathbb{E}(\ln(X_1)) &= \int f(x) \ln(x) dx \\
 &= \int \mathbb{1}\{0 \leq x \leq 1\} \ln(x) dx \\
 &= \int_0^1 \ln(x) dx \\
 &= [x \ln(x) - x]_0^1 \\
 &= (1 \ln(1) - 1) - (0 \ln(0) - 0) \\
 &= (-1) - (0) \\
 &= -1 \\
 \text{var}(\ln(X_1)) &= \mathbb{E}(\ln(X_1)^2) - \mathbb{E}(\ln(X_1))^2 \\
 &= \int f(x) \ln(x)^2 dx - 1 \\
 &= \int \mathbb{1}\{0 \leq x \leq 1\} \ln(x)^2 dx - 1 \\
 &= \int_0^1 \ln(x)^2 dx - 1 \\
 &= [x \ln(x)^2]_0^1 - \int_0^1 2 \ln(x) dx - 1 \\
 &= [x \ln(x)^2 - 2(x \ln(x) - x)]_0^1 - 1 \\
 &= (\ln(1)^2 - 2 \ln(1) + 2) - 0 - 1 \\
 &= 2 - 1 \\
 &= 1
 \end{aligned}$$

Question 2.2 - By taking logs, find a random variable X such that as $n \rightarrow \infty$

$$(X_1, \dots, X_n)^{\frac{1}{\sqrt{n}}} e^{\sqrt{n}} \rightarrow_{\mathcal{D}} X$$

Answer 2.2