

# Problems Sheet 6

## Statistics 2

Dom Hutchinson

### Question 1

Let  $\mathbf{X} \sim \text{Exponential}(\lambda)$  with  $\lambda > 0$  unknown. Let  $m$  denote the median of  $X \sim \text{Exponential}(\lambda)$  from a realisation  $\mathbf{x}$  of  $\mathbf{X}$ .

a)

By the definition of  $m$  as the median, we have

$$\begin{aligned} \mathbb{P}(X \leq m) &= \frac{1}{2} \\ \Rightarrow 1 - e^{-\lambda m} &= \frac{1}{2} \\ \Rightarrow e^{-\lambda m} &= \frac{1}{2} \\ \Rightarrow -\lambda m &= \ln \frac{1}{2} \\ \Rightarrow m &= -\frac{\ln \frac{1}{2}}{\lambda} \\ &= \frac{1}{\lambda} \ln 2 \end{aligned}$$

Define  $\hat{m}(\mathbf{x})$  to be the maximum likelihood estimate of  $m$ .

Then  $-\frac{\ln \frac{1}{2}}{\lambda}$  is a maximum likelihood estimate of  $m$  since mapping from  $\lambda$  to  $m$  is bijective.

Since,  $\hat{\lambda} = \frac{1}{\bar{x}} \Rightarrow -\frac{\ln \frac{1}{2}}{1/\bar{x}} = -\bar{x} \ln \frac{1}{2}$  is the maximum likelihood estimate of  $m$ , the median.

b)

```
n<-7; lambda<-3.3; N<-10000
# Takes 7 samples from Exp(3.3) and records median (10_000 times)
t_estimates<-sapply(1:N,function(i) median(rexp(n,rate=lambda)))
```

```
t_star=-log(.5)/lambda
cat("t_star=",t_star,"\n",sep="")
```

```
## t_star=0.2100446
```

```
# Bias
```

```
t_bias=mean(t_estimates)-t_star
```

```
# Mean Square Error
```

```
t_mse=var(t_estimates)+t_bias^2
```

```
# Print values
```

```
cat("Bias=",t_bias,"\n",sep="")
```

```
## Bias=0.0189104
```

```
cat("MSE=",t_mse,"\n",sep="")
```

```
## MSE=0.0139587
```

c)

```
# Calculates maximum likelihood estimate for median
m_hat<-function(x) {
  -log(.5)*mean(x)
}

mle_estimates<-sapply(1:N,function(i) m_hat(rexp(n,rate=lambda)))

# Bias
mle_bias=mean(mle_estimates)-t_star

# Mean Square Error
mle_mse=var(mle_estimates)+mle_bias^2

# Print values
cat("Bias=",mle_bias,"\n",sep="")

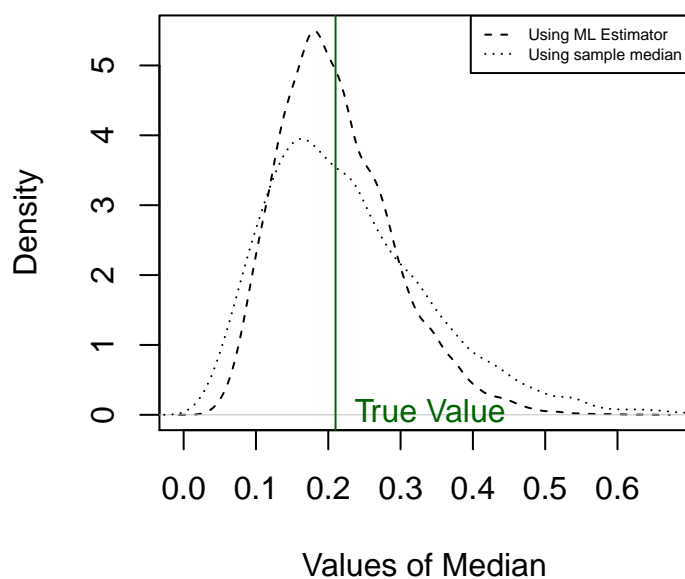
## Bias=0.001486258
cat("MSE=",mle_mse,"\n",sep="")

## MSE=0.006393609
```

d)

```
# Plot sample densities
plot(density(mle_estimates),lty=2,main="Estimators of the median, n=7",xlab="Values of Median")
abline(v=t_star,col="darkgreen")
text(t_star, 0, labels = "True Value", pos = 4, col = "darkgreen")
lines(density(t_estimates),type="l",lty=3)
legend("topright",legend=c("Using ML Estimator","Using sample median"),lty=c(2,3),cex=.5)
```

## Estimators of the median, $n=7$



Both estimators are biased. Since the distribution of the Maximum Likelihood estimator is much more concentrated around the true value of the median, it is the better estimator in the case when  $n = 7$ .

e)

```
n<-100; lambda<-3.3; N<-10000

t_estimates<-sapply(1:N,function(i) median(rexp(n,rate=lambda)))
mle_estimates<-sapply(1:N,function(i) m_hat(rexp(n,rate=lambda)))

# Bias
t_bias=mean(t_estimates)-t_star
mle_bias=mean(mle_estimates)-t_star

# Mean Square Error
t_mse=var(t_estimates)+t_bias^2
mle_mse=var(mle_estimates)+mle_bias^2

# Print values
cat("Bias of Sample Median=",t_bias,"\n",sep="")

## Bias of Sample Median=0.001296839
cat("Bias of MLE=",mle_bias,"\n",sep="")

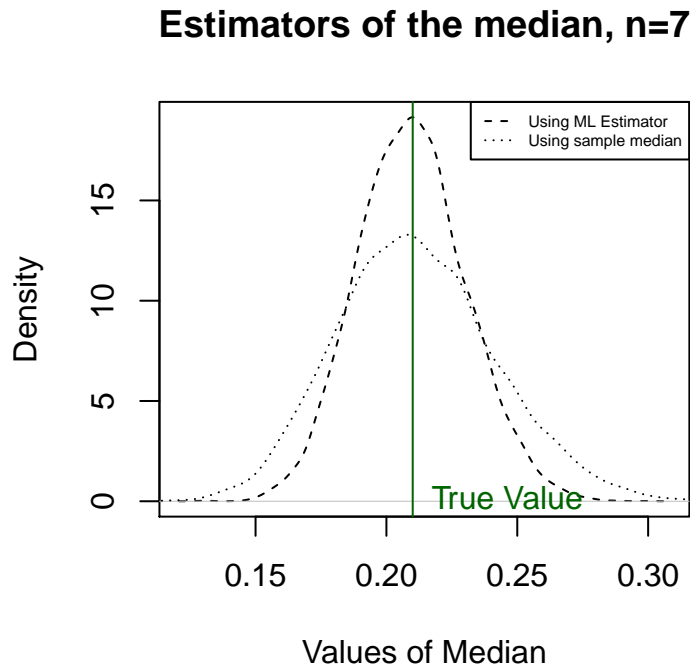
## Bias of MLE=-0.0003179905
cat("MSE of Sample Median=",t_mse,"\n",sep="")

## MSE of Sample Median=0.0009197369
```

```
cat("MSE of MLE=",mle_mse,"\n",sep="")

## MSE of MLE=0.0004323206

# Plot sample densities
plot(density(mle_estimates),lty=2,main="Estimators of the median, n=7",xlab="Values of Median")
abline(v=t_star,col="darkgreen")
text(t_star, 0, labels = "True Value", pos = 4, col = "darkgreen")
lines(density(t_estimates),type="l",lty=3)
legend("topright",legend=c("Using ML Estimator","Using sample median"),lty=c(2,3),cex=.5)
```



Both distributions appear less biased. Again the Maximum Likelihood estimate is the better estimator since more of its mass is concentrated around the true value of the median.

## Question 2

Let  $X_1, \dots, X_n \stackrel{\text{iid}}{\sim} \text{Cauchy}(\theta)$  with  $\theta \in \mathbb{R}$  an unknown location parameter.

a)

Define  $Y_i := \mathbb{1}$  for  $i \in [1, n]$ . Since  $X_i \sim \text{Cauchy}(\theta)$  and the Cauchy distribution is symmetric around  $\theta$  then

$$\mathbb{P}(X_i \geq \theta) = \frac{1}{2}$$

Thus

$$\mathbb{P}(Y_i = y) = \begin{cases} \frac{1}{2}, & y = 1 \\ \frac{1}{2}, & y = 0 \end{cases}$$

This is equivalent to the definition of a Bernoulli  $\left(\frac{1}{2}\right)$  random variable. Since  $X_i$  are iid and  $Y_i$  depends on only one, unique variable then  $Y_i$  are iid. We recognise  $B := \sum_{i=1}^n Y_i$  as the sum of  $n$  Bernoulli  $\left(\frac{1}{2}\right)$  random variables, which is the definition of a Binomial  $\left(n, \frac{1}{2}\right)$  random variable.

**b)**

**i)**

Suppose  $Y_{(i)} = 0$ .

Then  $Y_{(j)} = 0 \forall j \in [1, i]$  by definition of an order statistic.

Since  $||[1, i]|| = i$  then  $B \leq n - i$ .

Thus  $Y_{(i)} = 0 \implies B \leq n - i$ .

Suppose  $B \leq n - i$ .

Then  $\sum_{j=1}^n Y_j \leq n - i$ .

Therefore, there exists at least  $i$  elements,  $j \in [1, n]$ , where  $Y_j = 0$ .

By the definition of an order statistic  $Y_{(i)} = 0$ .

Thus  $B \leq n - i \implies Y_{(i)} = 0$ .

Thus  $Y_{(i)} = 0 \iff B \leq n - i$ . □

**ii)**

Suppose  $Y_{(i)} = 1$ .

Then  $Y_{(j)} = 1 \forall j \in [i, n]$  by definition of an order statistic.

Since  $||[i, n]|| = n - i + 1$  then  $B \geq n - i + 1$ .

Thus  $Y_{(i)} = 1 \implies B \geq n - i + 1$ .

Suppose  $B \geq n - i + 1$ .

Then  $\sum_{j=1}^n Y_j \geq n - i + 1$ . Therefore, there exists at least  $n - i + 1$  elements,  $j \in [1, n]$ , where  $Y_j = 1$ .

This means that at most  $n - (n - i + 1) = i - 1$  elements,  $j \in [1, n]$ , where  $Y_j = 0$ .

By the definition of an order statistic  $Y_{(i)} = 1$ .

Thus  $B \geq n - i + 1 \implies Y_{(i)} = 1$ .

Thus  $B \geq n - i + 1 \iff Y_{(i)} = 1$ . □

**c)**

Consider  $\mathbb{P}(X_{(k+1)} < \theta \leq X_{(n-k)}; \theta)$ . We have

$$\begin{aligned}
\mathbb{P}(X_{(k+1)} < \theta \leq X_{(n-k)}; \theta) &\equiv \mathbb{P}(X_{(k+1)} < \theta, \theta \leq X_{(n-k)}; \theta) \\
&\equiv \mathbb{P}(Y_{(k+1)} = 0, Y_{(n-k)} = 1) \\
&\equiv \mathbb{P}(B \leq n - (k+1), B \geq n - (n-k) + 1) \\
&\equiv \mathbb{P}(B \leq n - k + 1, B \geq k + 1) \\
&\equiv \mathbb{P}(k + 1 \leq B \leq n - k + 1) \\
&\equiv \mathbb{P}(k < B < n - k) \text{ since } B \text{ is discrete}
\end{aligned}$$

**d)**

Let  $\alpha = 0.05$  and  $n = 25$ .

Want to find  $k$  st  $\mathbb{P}(X_{(k+1)} < \theta \leq X_{(25-k)}; \theta) \geq 0.95$ .

This is equivalent to finding  $k$  st  $\mathbb{P}(k < B < 25 - k) \geq 0.95$  We know that  $B \sim \text{Binomial}(25, \frac{1}{2})$ . Thus

$$\begin{aligned}
0.95 &\leq \mathbb{P}(k < B < 25 - k) \\
&= \mathbb{P}(k + 1 \leq B \leq 24 - k) \\
&= \sum_{i=k+1}^{24-k} \mathbb{P}(B = i) \\
&= \sum_{i=k+1}^{24-k} \binom{25}{i} \left(\frac{1}{2}\right)^i \left(\frac{1}{2}\right)^{25-i} \\
&= \frac{1}{2^{25}} \sum_{i=k+1}^{24-k} \binom{25}{i} \\
&= \frac{2}{2^{25}} \sum_{i=k+1}^{12} \binom{25}{i} \text{ by symmetry}
\end{aligned}$$

By performing calculations we find that

$$\frac{1}{2^{24}} \sum_{i=8}^{12} \binom{25}{i} \geq 0.95$$

Thus  $k + 1 = 8 \implies k = 7$  produces the tightest  $1 - \alpha$  confidence interval for the true value of  $\theta$  when  $n = 25$  and  $\alpha = 0.05$ .