Statistics 2 - Problem Sheet 5

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Question - 1.

Suppose that $\mathbf{X} \overset{\text{iid}}{\sim} \text{Poisson}(\theta^*)$. Consider the observed Fisher Infromation evaluated at $\hat{\theta}_n$ and show that $J_n(\hat{\theta}_n) = 1/\bar{x}$. Find an asymptotically exact $1-\alpha$ confidence interval for θ^*.

Answer 1
Let $\mathbf{X} \overset{\text{iid}}{\sim} \text{Poisson}(\theta^*)$ and consider $\hat{\theta}_n = \hat{\theta}_n(\mathbf{X}) := \frac{1}{n}\sum_{i=1}^n X_i$. Then

$$\begin{array}{rrcl}
 & \ell'(\theta;\mathbf{x}) & = & -n + \frac{1}{\theta}\sum\limits_{i=1}^n x_i \\
\implies & ell''(\theta;\mathbf{x}) & = & -\frac{1}{\theta^2}\sum\limits_{i=1}^n x_i \\
\text{We have} & I(\theta) & = & -\mathbb{E}(\ell''(\hat{\theta};X);\hat{\theta}) \\
 & & = & -\mathbb{E}\left(-\frac{1}{\hat{\theta}^2}\sum\limits_{i=1}^n X_i;\hat{\theta}\right) \\
 & & = & \dfrac{1}{\hat{\theta}^2}\sum\limits_{i=1}^n \mathbb{E}(X_i;\hat{\theta}) \\
 & & = & \dfrac{1}{\hat{\theta}^2}n\hat{\theta} \\
 & & = & \dfrac{n}{\hat{\theta}} \\
 & & = & \dfrac{n}{\bar{x}} \\
\text{and} & J_n(\hat{\theta}_n) & := & -\dfrac{1}{n}\sum\limits_{i=1}^n \ell''(\hat{\theta}_n;X_i) \\
 & & = & -\dfrac{1}{n}\sum\limits_{i=1}^n \left(-\frac{1}{\hat{\theta}_n^2};X_i\right) \\
 & & = & \dfrac{1}{n\hat{\theta}^2}\sum\limits_{i=1}^n X_i \\
 & & = & \dfrac{\bar{x}}{\hat{\theta}^2} \\
 & & = & \dfrac{1}{\bar{x}}
\end{array}$$

From Theorem 12.2 in notes we have that $[L(\mathbf{X}), U(\mathbf{X})]$ is an asymptotically exact $1-\alpha$ confidence interval of θ^* where

$$L(\mathbf{x}) := \hat{\theta}_n - \frac{z_{\alpha/2}}{\sqrt{nJ_n(\hat{\theta}_n)}} = \bar{x} - z_{\alpha/2}\sqrt{\frac{\bar{x}}{n}} \text{ and } U(\mathbf{x}) := \hat{\theta}_n + \frac{z_{\alpha/2}}{\sqrt{nJ_n(\hat{\theta}_n)}} = \bar{x} + z_{\alpha/2}\sqrt{\frac{\bar{x}}{n}}$$

Question - 2.

Suppose you are given the observations

$$\mathbf{x} = (11, 23, 20, 11, 15, 29, 20, 16, 15, 14)$$

presumed to come from some independent, identically distributed random sample $\mathbf{X}$.

Question 2.1
Compute an approximate 95% observed confidence interval for the expectation of μ of X, assumed to be such that $|\mathbb{E}(X)| < \infty$ and $\sigma^2 = \text{Var}(X) < \infty$, explaining all of your reasoning carfully. Note that we are not assuming any model for the observations, except for the independence, and that neither μ nor σ^2 are known.
Hint - Start with the cetnral limit theorem for $\bar{X}_n = \frac{1}{n}\sum_{i=1}^n X_i$.

Answer 2.1
Let $\mathbf{X}$ be an independent, indentically distributed sample of an unknown model with its mean & variance unknown but finite.
Consider the maximum likelihood estimate $\hat{\mu} := \bar{X}_n = \frac{1}{n}\sum_{i=1}^n X_i$.
By the Central Limit Theorem we have

$$\frac{\hat{\mu} - \mu^*}{\sqrt{n/\sigma^2}} \to_{\mathcal{D}} Z \sim \text{Normal}(0,1)$$

Thus

$$\begin{aligned} \mathbb{P}\left(z_{-\alpha/2} \leq \frac{\hat{\mu}_n - \mu^*}{\sqrt{n/\sigma^2}} \leq z_{\alpha/2}; \mu\right) &= 1 - \alpha \\ \implies \mathbb{P}\left(\hat{\mu}_n - z_{\alpha/2}\frac{\sigma}{\sqrt{n}} \leq \mu^* \leq \hat{\mu}_n + z_{\alpha/2}\frac{\sigma}{\sqrt{n}}\right) &= 1 - \alpha \text{ by rearrangement} \end{aligned}$$

Since σ^2 is unknown we consider a consistent sequence of estimators $\{\hat{\sigma}_n^2\}_{n\in\mathbb{N}}$ where

$$\sigma_n^2 = \frac{1}{n-1}\sum_{i=1}^n (X_i - \hat{\mu}_n)^2$$

From the data we have $n = 10$, $\hat{\mu}_{10} = \frac{174}{10}$ & $\hat{\sigma}_{10}^2 = \frac{1}{9} \times \frac{1432}{5} = \frac{1432}{45}$.
Since we want a 95% confidence interval we set $1 - \alpha = 0.95 \implies \alpha = 0.05$.
Thus $z_{0.975} = 1.960$ we have

$$\begin{aligned} \mathbb{P}\left(\frac{174}{10} - 1.96\sqrt{\frac{1432/45}{10}} \leq \mu^* \leq \frac{174}{10} + 1.960\sqrt{\frac{1432/45}{10}}\right) &= 0.95 \\ \equiv \qquad \mathbb{P}\left(\mu^* \in [13.9, 20.9]\right) &= 0.95 \end{aligned}$$

Question 2.2
Now suppose you are also told that $\mathbf{X} \overset{\text{iid}}{\sim} \text{Poisson}(\theta)$. Compute another approximate 95% confidence interval for the expectaiton of X, using the Wald approach.

Answer 2.2
Suppose $\mathbf{X} \overset{\text{iid}}{\sim} \text{Poisson}(\theta)$.
By Wald's Approach we have that $\mathcal{I}(\mu^*) := [L(\mathbf{X}), U(\mathbf{X})]$ is a $1 - \alpha$ confidence interval of μ where

$$L(\mathbf{x}) := \hat{\mu}_n - \frac{z_{\alpha/2}}{\sqrt{nI(\mu^*)}} \text{ and } U(\mathbf{x}) := \hat{\mu}_n + \frac{z_{\alpha/2}}{\sqrt{nI(\mu^*)}}$$

From question 1 we have $I(\mu^*) = n/\bar{x}$ for a poisson iid random sample. Thus, using data from previous part, we have

$$\begin{aligned} \mathcal{I}(\mu^*) &= \left[\hat{\mu}_n - z_{\alpha/2}\sqrt{\frac{\bar{x}}{n^2}}, \hat{\mu}_n + z_{\alpha/2}\sqrt{\frac{\bar{x}}{n^2}}\right] \\ &= \left[\frac{174}{10} - 1.96\sqrt{\frac{174/10}{100}}, \frac{174}{10} + 1.96\sqrt{\frac{174/10}{100}}\right] \\ &= [16.6, 18.2] \end{aligned}$$

Question 2.3 - Which of the two confidence intervals do you prefer?

Answer 2.3 - The second since it is a tighter bound on μ^* and encodes information about the possible model of the data, rather than just the data.