Computer Practical 1

Statistics 2

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2 Binomial Maximum Likelihood Estimators

Let $Y \sim \text{Binomial}(n, p)$. The maximum likelihood estimate for p is $\hat{p}(Y) = \frac{Y}{n}$. $\hat{p}$ is an unbiased estimator.

Question 1

Let $X_1, \dots, X_n \overset{\text{iid}}{\sim} \text{Bernoulli}(p)$. Then

$$\begin{array}{rcl} \text{Var}(\hat{p}) & = & \text{Var}\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right) \\ & = & \frac{1}{n^2}\text{Var}\left(\sum_{i=1}^{n} X_i\right) \\ & = & \frac{1}{n^2}.n\text{Var}\left(X_1\right) \\ & = & \frac{1}{n}p(1-p) \end{array}$$

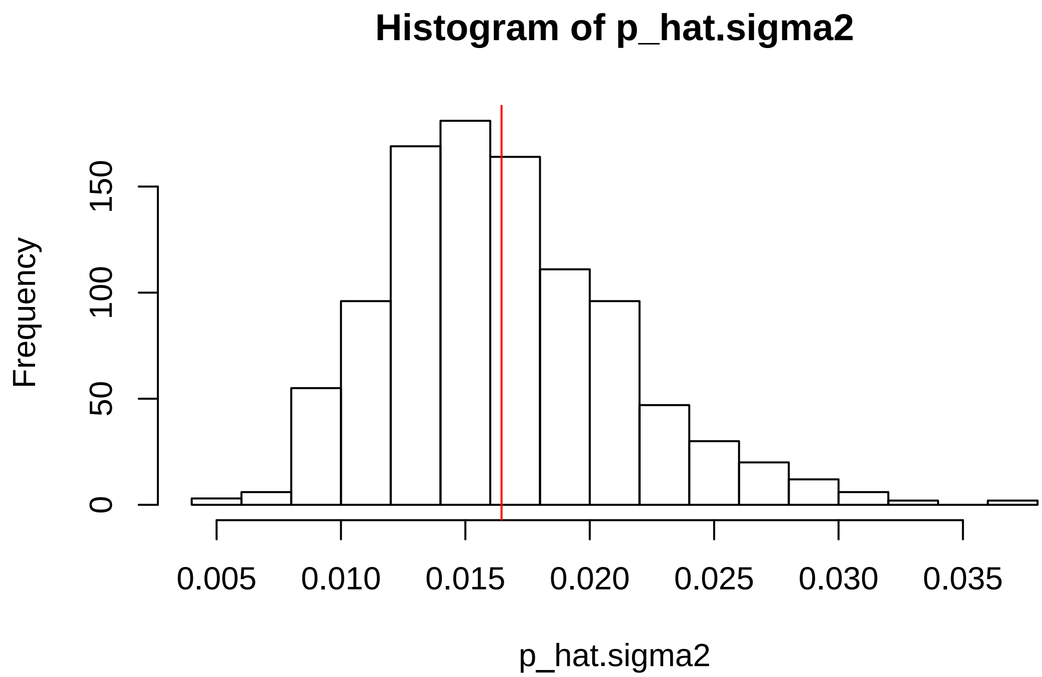
Question 2

```r
n<-13; p<-.31; sample_size<-25; trials<-1000
var_hat=(1/n)*p*(1-p)
var_hat
```

```
## [1] 0.01645385
```

```r
phat<-function(Y) {
  Y/n
}

x_values<-rbinom(n=sample_size*trials,size=n,prob=p)
x_samples<-matrix(x_values,nrow=sample_size)
p_hat.samples<-apply(x_samples,1,phat)
p_hat.sigma2=apply(p_hat.samples,1,var)
hist(p_hat.sigma2,breaks=20)
abline(v=var_hat,col="red")
```



3 Clinic Data

```
year.data<-read.csv("year_data.csv")
knitr::kable(year.data)
```

year	births	deaths	clinic
1841	3036	237	1
1842	3287	518	1
1843	3060	274	1
1844	3157	260	1
1845	3492	241	1
1846	4010	459	1
1841	2442	86	2
1842	2659	202	2
1843	2739	164	2
1844	2956	68	2
1845	3241	66	2
1846	3754	105	2

Let $Y_i \sim \text{Binomial}(n_i, p_i)$ model the number of deaths in clinic i where n_i is the total number of births in clinic i & p_i is the mortality rate for clinic i. Assume Y_1 & Y_2 are independent.

```
n1 <- sum(year.data[year.data$clinic==1,]$births) # number of births in clinic 1
y1 <- sum(year.data[year.data$clinic==1,]$deaths) # number of deaths in clinic 1
cat("Number of births in clinic 1:",prettyNum(n1,big.mark=","),"\nNumber of deaths in clinic 1:",pretty
```

```
## Number of births in clinic 1: 20,042
```

```
## Number of deaths in clinic 1: 1,989
```

```
n2 <- sum(year.data[year.data$clinic==2,]$births) # number of births in clinic 2
y2 <- sum(year.data[year.data$clinic==2,]$deaths) # number of deaths in clinic 2
cat("Number of births in clinic 2:",prettyNum(n2,big.mark=","),"\nNumber of deaths in clinic 2:",prettyN
```

```
## Number of births in clinic 2: 17,791
## Number of deaths in clinic 2: 691
```

Question 3

```
p1_hat=y1/n1
p2_hat=y2/n2

cat("p1_hat:",p1_hat,"\np2_hat:",p2_hat)
```

```
## p1_hat: 0.09924159
## p2_hat: 0.03883986
```

Quetion 4

Assume that $p = p_1 = p_2$ and define $W := \hat{p}_1(Y_1) - \hat{p}_2(Y_2)$. Then

$$\begin{array}{rcl}
\mathbb{E}(W) & = & \mathbb{E}(\hat{p}_1(Y_1) - \hat{p}_2(Y_2)) \\
& = & \mathbb{E}(\hat{p}_1(Y_1)) - \mathbb{E}(\hat{p}_2(Y_2)) \\
& = & p_1 - p_2 \\
& = & p - p \\
& = & 0 \\
\mathrm{Var}(W) & = & \mathrm{Var}(\hat{p}_1(Y_1) - \hat{p}_2(Y_2)) \\
& = & \mathrm{Var}(\hat{p}_1(Y_1)) + \mathrm{Var}(\hat{p}_2(Y_2)) \\
& = & \frac{1}{n_1}p_1(1-p_1) + \frac{1}{n_2}p_2(1-p_2) \\
& = & \frac{1}{n_1}p(1-p) + \frac{1}{n_2}p(1-p) \\
& = & \frac{n_1+n_2}{n_1 n_2}p(1-p)
\end{array}$$

Question 5

Suppose $p = p_1 = p_2$. We have $\hat{p} = \frac{1989+691}{20042+17791} = \frac{2680}{37833} = 0.0708376$.

$$\begin{array}{rcl}
\mathbb{P}(|W - \mu_W| \geq \hat{p}_1(y_1) - \hat{p}_1(y_2)) & = & \mathbb{P}(|W - 0| \geq 0.0992416 - 0.0388399) \\
& = & \mathbb{P}(|W| \geq 0.0604017) \\
& \leq & \dfrac{\sigma_W^2}{0.0604017^2} \text{ by Chebyshev's Inequality} \\
& = & \dfrac{1}{0.0604017^2} \times \dfrac{n_1+n_2}{n_1 n_2}\hat{p}(1-\hat{p}) \\
& = & \dfrac{1}{0.0604017^2} \times \dfrac{20042+17791}{20042 \times 17791} \times 0.0708376 \times 0.9291624 \\
& = & 0.0019142
\end{array}$$

Thus it is very unlikely to observe these two mortality rates, assuming the underlying rate is the same.

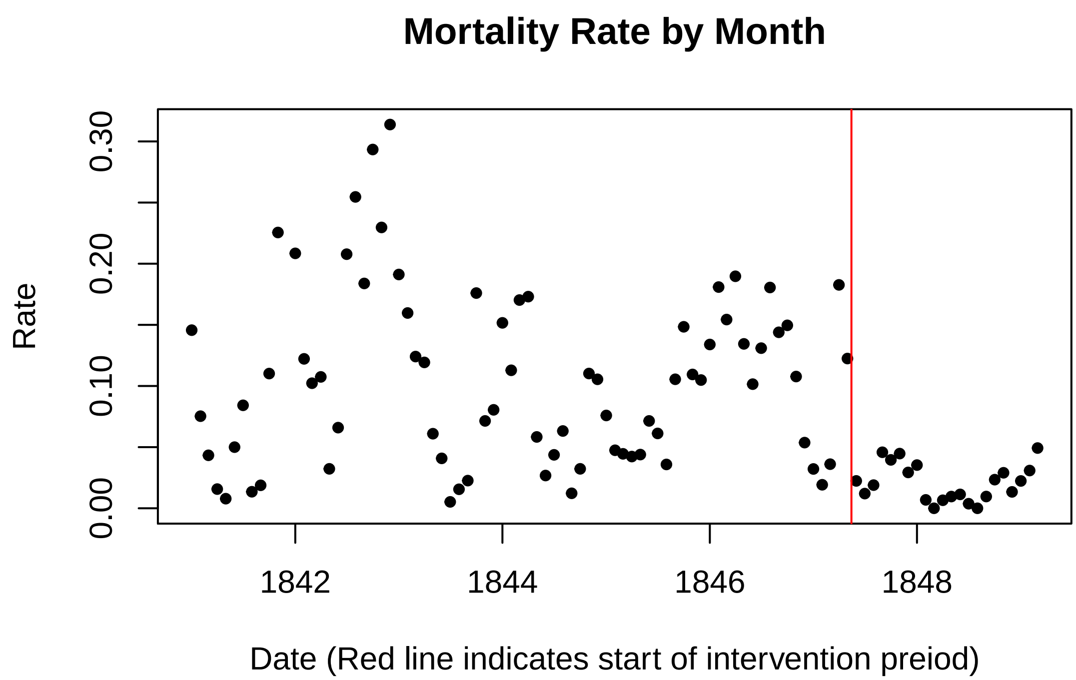
4 Intervention: Chlorine Hand Washing

```
month.data<-read.csv("month_data.csv")
month.data<-month.data[!is.na(month.data$births),]
```

```r
month.data$rate<-month.data$deaths/month.data$births
month.data$date<-as.Date(month.data$date)

intervention.date<-as.Date("1847-05-15")

plot(month.data$date, month.data$rate, pch=20, main="Mortality Rate by Month", xlab="Date (Red line ind
abline(v=intervention.date, col="red")
```




```r
before.intervention<-month.data[month.data$date< intervention.date,]
n1<-sum(before.intervention$births)
y1<-sum(before.intervention$deaths)
cat("Number of births before intervention:",prettyNum(n1,big.mark=","),"\nNumber of deaths before inter
```

```
## Number of births before intervention: 19,571
## Number of deaths before intervention: 2,060
```

```r
after.intervention <-month.data[month.data$date>=intervention.date,]
n2<-sum(after.intervention$births)
y2<-sum(after.intervention$deaths)
cat("Number of births after intervention:",prettyNum(n2,big.mark=","),"\nNumber of deaths after interven
```

```
## Number of births after intervention: 6,595
## Number of deaths after intervention: 142
```

Question 6

```r
p1_hat<-y1/n1
p2_hat<-y2/n2

cat("p1_hat:",p1_hat,"\np2_hat:",p2_hat)
```

```
## p1_hat: 0.1052578
## p2_hat: 0.02153146
```

Define random variable $W = \hat{p}_1(Y_1) - \hat{p}_2(Y_2)$.

Suppose $p = p_1 = p_2$. We have $\hat{p} = \frac{2060+142}{19571+6595} = \frac{2202}{26166} = 0.084155$.

$$\begin{aligned}
\mathbb{P}(|W - \mu_W| \geq \hat{p}_1(y_1) - \hat{p}_1(y_2)) &= \mathbb{P}(|W - 0| \geq 0.1052578 - 0.0215315) \\
&= \mathbb{P}(|W| \geq 0.0837263) \\
&\leq \frac{\sigma_W^2}{0.0837263^2} \text{ by Chebyshev's Inequality} \\
&= \frac{1}{0.0837263^2} \times \frac{n_1 + n_2}{n_1 n_2} \hat{p}(1 - \hat{p}) \\
&= \frac{1}{0.0837263^2} \times \frac{19571 + 6595}{19571 \times 6595} \times 0.084155 \times 0.915845 \\
&= 0.0022289
\end{aligned}$$

Thus it is very unlikely to observe these two mortality rates, assuming the underlying rate is the same.

5 A First Logistic Regression

```
x1<-c(1,0)
x2<-c(1,1)

sigma<-function(z) {
  1/(1+exp(-z))
}
```

Question 7