Statistics 2 - Problem Sheet 4

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Question - 1.

For the following statistical models derive the score function, $\ell'(\theta;x)$, and the Fisher Information, $I(\theta)$. Assume each model to be regular.

Question 1.1 - $X \sim \text{Geometric}(p)$

Answer 1.1
Let $X \sim \text{Geometric}(p)$ with p unknown. Then

$$\begin{array}{rrcl}
\text{We have} & L(p;x) & \propto & f(x;p) \\
& & = & (1-p)^{x-1}p \\
\implies & \ell(p;x) & = & \ln f(x;p) \\
& & = & (x-1)\ln(1-p) + \ln p + c \\
\text{By definition} & \ell'(p;x) & = & \frac{d}{dp}\ell(p;x) \\
& & = & -\frac{x-1}{1-p} + \frac{1}{p} \\
& & = & \frac{1-x}{1-p} + \frac{1}{p} \\
\text{By definition} & I(p) & = & \mathbb{E}(\ell'(p;X)^2;p) \\
& & = & \text{Var}(\ell'(p;X);p) \text{ by regularity conditions} \\
& & = & \text{Var}\left(\frac{1-X}{1-p} + \frac{1}{p};p\right) \\
& & = & \text{Var}\left(\frac{1-X}{1-p};p\right) \\
& & = & \frac{1}{(1-p)^2}\text{Var}\left(1-X;p\right) \\
& & = & \frac{1}{(1-p)^2}\text{Var}\left(X;p\right) \\
& & = & \frac{1}{(1-p)^2} \times \frac{1-p}{p^2} \\
& & = & \frac{1}{p^2(1-p)}
\end{array}$$

Question 1.2 - $X \sim \text{Binomial}(n,p)$ where n is known.

Answer 1.2
Let $X \sim \text{Binomial}(n,p)$ with n known & p unknown. Then

$$\begin{array}{rrcl}
\text{We have} & L(p;x,n) & \propto & f(x;p,n) \\
& & = & p^x(1-p)^{n-x} \\
\implies & \ell(p;x,n) & = & x\ln p + (n-x)\ln(1-p) \\
\implies & \ell'(p;x,n) & = & \frac{x}{p} - \frac{n-x}{1-p} + c \\
& & = & \frac{x}{p} + \frac{x-n}{1-p}
\end{array}$$

$$\begin{array}{rcl} I(p) & = & \text{Var}(\ell'(p;X,n);p) \\ & = & \text{Var}\left(\frac{X}{p}+\frac{X-n}{1-p};p\right) \\ & = & \frac{1}{p^2(1-p)^2}\text{Var}(X(1-p)+(X-n)p;p) \\ & = & \frac{1}{p^2(1-p)^2}\text{Var}(X-np;p) \\ & = & \frac{1}{p^2(1-p)^2}\text{Var}(X;p) \\ & = & \frac{1}{p^2(1-p)^2}np(1-p) \\ & = & \frac{n}{p(1-p)} \end{array}$$

Question 1.3 - $X \sim \text{Normal}(\mu,\sigma^2)$ where σ^2 is known.

Answer 1.3
Let $X \sim \text{Normal}(\mu,\sigma^2)$ with σ^2 known & μ unknown. Then

$$\begin{array}{rrcl} \text{We have} & \ell(\mu;x,\sigma^2) & = & n\ln\sigma^2+\frac{1}{\sigma^2}(x-\mu)^2+c \\ \implies & \ell'(\mu;x,\sigma^2) & = & \frac{\partial}{\partial\mu}\ell(\mu;x,\sigma^2) \\ & & = & -\frac{1}{\sigma^2}2(x-\mu) \\ & I(\mu) & = & \text{Var}(\ell'(\mu;X,\sigma^2);\mu) \\ & & = & \text{Var}\left(-\frac{2}{\sigma^2}(X-\mu);\mu\right) \\ & & = & \left(\frac{2}{\sigma^2}\right)^2\text{Var}(X-\mu;\mu) \\ & & = & \frac{4}{\sigma^4}\text{Var}(X;\mu) \\ & & = & \frac{4}{\sigma^4}\sigma^2 \\ & & = & \frac{4}{\sigma^2} \end{array}$$

Question 1.4 - $X \sim \text{Pareto}(x_0,\theta)$ where x_0 is known.

Answer 1.4
Let $X \sim \text{Pareto}(x_0,\theta)$ with x_0 known & θ unknown. Then

$$\begin{array}{rrcl} \text{We have} & L(\theta;x,x_0) & \propto & \frac{\theta x_0^\theta}{x^{\theta+1}}\mathbb{1}\{x\geq x_0\} \\ \implies & \ell(\theta;x,x_0) & = & \ln\theta+\theta\ln x_0-(\theta+1)\ln x+c \\ \implies & \ell'(\theta;x,x_0) & = & \frac{1}{\theta}+\ln x_0-\ln x \\ \implies & \ell''(\theta;x,x_0) & = & -\frac{1}{\theta^2} \\ & I(\theta) & = & -\mathbb{E}(\ell''(\theta;X,x_0);\theta) \\ & & = & -\left(-\frac{1}{\theta^2}\right) \\ & & = & \frac{1}{\theta^2} \end{array}$$

Question - 2.

For each of the following verify that $\mathbb{E}(\ell'(\theta;X);\theta)=0$.

Question 2.1 - $X\sim$ Geometric(p)

Answer 2.1
Let $X\sim$ Geometric(p)

$$\begin{array}{rrcl} & \ell'(p;x) & = & \frac{1-x}{1-p}+\frac{1}{p} \\ \implies & \mathbb{E}(\ell'(p;X);p) & = & \mathbb{E}\left(\frac{1-X}{1-p}+\frac{1}{p};p\right) \\ & & = & \frac{1}{1-p}\mathbb{E}(1-X)+\frac{1}{p} \\ & & = & \frac{1}{1-p}\left(1-\frac{1}{p}\right)+\frac{1}{p} \\ & & = & \frac{p-1}{p(1-p)}+\frac{1}{p} \\ & & = & \frac{p-1+(1-p)}{p(1-p)} \\ & & = & 0 \end{array}$$

Question 2.2 - $X\sim$ Binomial(n,p) where n is known.

Answer 2.2
Let $X\sim$ Binomial(n,p) with n known.

$$\begin{array}{rrcl} & \ell'(p;x,n) & = & \frac{x}{p}+\frac{x-n}{1-p} \\ \implies & \mathbb{E}(\ell'(p;X,n);p) & = & \mathbb{E}\left(\frac{X}{p}+\frac{X-n}{1-p};x,n\right) \\ & & = & \frac{1}{p}\mathbb{E}(X)+\frac{1}{1-p}\mathbb{E}(X-n) \\ & & = & \frac{1}{p}(np)+\frac{1}{1-p}(np-n) \\ & & = & n+n\frac{p-1}{1-p} \\ & & = & n-n \\ & & = & 0 \end{array}$$

Question 2.3 - $X\sim$ Normal(μ,σ^2) where σ^2 is known.

Answer 2.3
Let $X\sim$ Normal(μ,σ^2) with σ^2 known and μ unknown.

$$\begin{array}{rrcl} & \ell'(\mu;x,\sigma^2) & = & -\frac{2}{\sigma^2}(x-\mu) \\ \implies & \mathbb{E}(\ell'(\mu;X,\sigma^2);\mu) & = & \mathbb{E}\left(-\frac{2}{\sigma^2}(X-\mu);\mu\right) \\ & & = & -\frac{2}{\sigma^2}\mathbb{E}(X-\mu;\mu) \\ & & = & -\frac{2}{\sigma^2}(\mu-\mu) \\ & & = & 0 \end{array}$$

Question 2.4 - $X\sim$ Pareto(x_0,θ) where x_0 is known.

Answer 2.4
Let $X\sim$ Pareto(x_0,θ) with x_0 is known & θ unknown.

$$\begin{array}{rrcl} & \ell'(\theta;x,x_0) & = & \frac{1}{\theta}+\ln x_0-\ln x \\ \implies & \mathbb{E}(\ell'(\theta;X,x_0);\theta) & = & \mathbb{E}\left(\frac{1}{\theta}+\ln x_0-\ln X;\theta\right) \\ & & = & \frac{1}{\theta}+\ln x_0-\mathbb{E}(\ln X;\theta) \\ & & = & \frac{1}{\theta}+\ln x_0-(\ln x_0+\frac{1}{\theta}) \\ & & = & 0 \end{array}$$

Question - 3.

In the case of $X \sim \text{Pareto}(x_0, \theta)$ where both x_0 & θ are unknown, explain why the Fisher Information Regularity Conditions are not met.

Answer 3
Let $X \sim \text{Pareto}(x_0, \theta)$ with both x_0 & θ unknown.
The Fisher Information Regularity Conditions require $L'(\theta, x_0; x)$ to exist $\forall\ x \in \mathcal{X}$.
We notice that

$$\begin{array}{rcl} L'(\theta, x_0; x) & = & \left(\frac{\partial}{\partial\theta} L(\theta, x_0; x) \quad \frac{\partial}{\partial x_0} L(\theta, x_0; x)\right) \\ & \propto & \left(\frac{\partial}{\partial\theta} \frac{\theta x_0^\theta}{x^{\theta+1}} \mathbb{1}\{x \geq x_0\} \quad \frac{\partial}{\partial x_0} \frac{\theta x_0^\theta}{x^{\theta+1}} \mathbb{1}\{x \geq x_0\}\right) \end{array}$$

Consider the derivative wrt x_0.
This contains an indicator function, $\mathbb{1}\{x \geq x_0\}$, which depends upon the variable we are deriving wrt.
Since the indicator function is discontinuous at x_0 it is not differentiable.
Thus $L'(\theta, x_0; x)$ does not exist in this case, further the Fisher Information Regularity Conditions are not met in the case of $X \sim \text{Pareto}(x_0, \theta)$ with x_0 & θ unknown. $\square$