Statistics 2 - Problem Sheet 1

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Question - 1.

Derive the likelihood function and log-likelihood function for the following distributions.

Question 1.1 - $\mathbf{X} \overset{\text{iid}}{\sim} \text{Poisson}(\lambda)$ with λ unknown.

Answer 1.1
Let $\mathbf{X} \overset{\text{iid}}{\sim} \text{Poisson}(\lambda)$ with λ unknown. Then

$$\begin{array}{rcl} L(\lambda;\mathbf{x}) & \propto & f_n(\mathbf{x};\lambda) \\ & = & \prod_{i=1}^n f(x_i;\lambda) \\ & = & \prod_{i=1}^n e^{-\lambda}\lambda^{x_i}(x_i!)^{-1} \\ & = & e^{-\lambda n}\lambda^{\sum_{i=1}^n x_i}\prod_{i=1}^n (x_i!)^{-1} \\ & \propto & e^{-\lambda n}\lambda^{\sum_{i=1}^n x_i} \\ \ell(\lambda;\mathbf{x}) & = & \ln L(\lambda;\mathbf{x}) \\ & = & -\lambda n + \left(\sum_{i=1}^n x_i\right)(\ln\lambda) + c \end{array}$$

Question 1.2 - $X \sim \text{Binomial}(n,p)$ with n & p unknown.

Answer 1.2
Let $X \sim \text{Binomial}(n,p)$ with n & p unknown. Then

$$\begin{array}{rcl} L(n,p;x) & \propto & f(x;n,p) \\ & = & \binom{n}{x}p^x(1-p)^{n-x} \\ \ell(n,p;x) & = & \ln L(n,p;x) \\ & = & ln\binom{n}{x} + x\ln p + (n-x)\ln(1-p) + c \end{array}$$

Question 1.3 - $\mathbf{X} \overset{\text{iid}}{\sim}$ Normal(μ, σ^2) with μ & σ unknown.

Answer 1.3
Let $\mathbf{X} \overset{\text{iid}}{\sim}$ Normal(μ, σ^2) with μ & σ unknown. Then

$$\begin{array}{rcl} L(\mu,\sigma^2);\mathbf{x}) & \propto & f_n(\mathbf{x};\mu,\sigma^2) \\ & = & \prod_{i=1}^n (2\pi\sigma^2)^{-\frac{1}{2}} e^{-\frac{(x_i-\mu)^2}{2\sigma^2}} \\ & = & (2\pi\sigma^2)^{-\frac{n}{2}} \prod_{i=1}^n e^{-\frac{(x_i-\mu)^2}{2\sigma^2}} \\ & \propto & (\sigma^2)^{-\frac{n}{2}} e^{-\frac{1}{2\sigma^2}\sum_{i=1}^n (x_i-\mu)^2} \\ \ell(\mu,\sigma^2;\mathbf{x}) & = & \ln L(\mu,\sigma^2;\mathbf{x}) \\ & = & -\frac{n}{2}\ln\sigma^2 - \frac{1}{2\sigma^2}\sum_{i=1}^n (x_i-\mu)^2 + c \end{array}$$

Question 1.4 - $\mathbf{X} \overset{\text{iid}}{\sim}$ Uniform(a, b) with a & b unknown.

Answer 1.4
Let $\mathbf{X} \overset{\text{iid}}{\sim}$ Uniform(a, b) with a & b unknown. Then

$$\begin{array}{rcl} L(a,b;\mathbf{x}) & \propto & f_n(\mathbf{x};a,b) \\ & = & \prod_{i=1}^n \begin{cases} \frac{1}{b-a} & , x_i \in [a,b] \\ 0 & \text{otherwise} \end{cases} \\ & = & \left(\frac{1}{b-a}\right)^n \mathbb{1}\{\forall\ x_i \in \mathbf{x},\ x_i \in [a,b]\} \\ \ell(a,b;\mathbf{x}) & = & \begin{cases} -n\ln(b-a) + c & , \text{ if } \forall\ x_i \in \mathbf{x},\ x_i \in [a,b] \\ \text{undefined} & \text{otherwise} \end{cases} \end{array}$$

Question - 2.

Suppose that $X_1, X_2, X_3 \overset{\text{iid}}{\sim}$ Exponential(λ) represent the lifetimes of three lightbulbs in my living room. I installed then $2\frac{1}{2}$ years ago, and one has just blown (the others are still working). Show that the likelihood function corresponding to this is

$$L(\lambda; 2.5) \propto \lambda e^{-7.5\lambda}$$

Answer 2
Let $X_1, X_2, X_3 \overset{\text{iid}}{\sim}$ Exponential(λ).
Define $Y = \min(X_1, X_2, X_3)$. Then

$$\begin{array}{rrcll} & F_Y(y;\lambda) & = & \mathbb{P}(Y \leq y) & \\ & & = & 1 - \mathbb{P}(Y > y) & \\ & & = & 1 - \mathbb{P}(X_1 > y, X_2 > y, X_3 > y) & \text{By independece} \\ & & = & 1 - \mathbb{P}(X_1 > y)^3 & \text{By identical distribution} \\ & & = & 1 - (e^{-\lambda y})^3 & \\ & & = & 1 - e^{-3\lambda y} & \\ \implies & f_Y(y;\lambda) & = & F_Y'(y;\lambda) & \\ & & = & 3\lambda y e^{-3\lambda y} & \\ \implies & L(\lambda; 2.5) & \propto & f_Y(2.5;\lambda) & \\ & & = & 3\lambda e^{-7.5\lambda} & \\ & & \propto & \lambda e^{-7.5\lambda} & \end{array}$$

Question - 3.

Suppose I construct a random variable X in the following way. First, I toss a biased coin with probability of heads p. If it comes up heads, then $X = 0$, otherwise $X \sim \text{Poisson}(\lambda)$. We call this the *Zero-Inflated Poisson* distribution, and write $X \sim ZIP(p, \lambda)$. Derive the probabilitiy mass function for X and the likelihood function for a realisation $\mathbf{x}$ of $\mathbf{X} \overset{\text{iid}}{\sim} \text{ZIP}(p, \lambda)$.

Answer 3
Let $X \sim \text{ZIP}(p, \lambda)$, $Y \sim \text{Bernoulli}(p)$ & $Z \sim \text{Poisson}(\lambda)$. Then

$$\begin{array}{rcl}
f_X(x) & = & \begin{cases} \mathbb{P}(Y=1) + \mathbb{P}(Z=0|Y=0) & ,x=0 \\ \mathbb{P}(Z=x|Y=0) & ,x \in \mathbb{N} \\ 0 & \text{otherwise} \end{cases} \\
& = & \begin{cases} p + \mathbb{P}(Z=0)\mathbb{P}(Y=0) & ,x=0 \\ \mathbb{P}(Z=x)\mathbb{P}(Y=0) & ,x \in \mathbb{N} \\ 0 & \text{otherwise} \end{cases} \\
& = & \begin{cases} p + e^{-\lambda}\lambda^0(0!)^{-1}(1-p) & ,x=0 \\ (e^{-\lambda}\lambda^x(x!)^{-1})(1-p) & ,x \in \mathbb{N} \\ 0 & \text{otherwise} \end{cases} \\
& = & \begin{cases} p + e^{-\lambda}(1-p) & ,x=0 \\ \frac{1}{x!}(e^{-\lambda}\lambda^x)(1-p) & ,x \in \mathbb{N} \\ 0 & \text{otherwise} \end{cases} \\
& = & p\mathbb{1}\{x=0\} + \frac{1}{x!}(e^{-\lambda}\lambda^x)(1-p)
\end{array}$$

Let $\mathbf{X} \overset{\text{iid}}{\sim} \text{ZIP}(p, \lambda)$ with p & λ unknown. Then

$$\begin{array}{rcl}
L(p, \lambda; \mathbf{x}) & \propto & f_n(\mathbf{x}; p, \lambda) \\
& = & \prod_{i=1}^n \left[p\mathbb{1}\{x=0\} + \frac{1}{x!}(e^{-\lambda}\lambda^x)(1-p)\right]
\end{array}$$