# Problems Sheet 8

Statistics 2

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```
xtab<-c(109,65,22,3,1,0)
xobs<-rep(0:5,xtab)
n<-length(xobs)
xobs.bar<-mean(xobs)
```

## Question 1

Let $\mathbf{X} := (X_1, \dots, X_{200}) \overset{\text{iid}}{\sim} \text{Poisson}(\lambda)$ be a random vector representing the number of deaths each year & $\mathbf{x} := (x_1, \dots, x_{200})$ be a realisation of $\mathbf{X}$.
Here we shall test $H_0 : \lambda \leq 0.5$ against $H_1 : \lambda > 0.5$.
Define test statistic $T(\mathbf{X}) = \frac{1}{n}\sum_{i=1}^n X_i$, from the data we have

$$T(\mathbf{x}) = -\frac{1}{200}\sum_{i=1}^{200} x_i = \frac{122}{200} = -0.61$$

Since $\sum_{i=1}^n X_i$ is an equivalent test statistic to the Neyamn-Pearson Test statistic then $T(\mathbf{X})$ is an equivalent test statistic too.
Thus $T(\mathbf{X})$ is the uniformly most powerful test statistic for $H_0$ & $H_1$ and has associated $p$-value

$$\begin{aligned}
p(\mathbf{X}) &= \mathbb{P}(T(\mathbf{X}) \geq T(\mathbf{x}); 0.5) \\
&= \mathbb{P}\left(\tfrac{1}{200}\textstyle\sum_{i=1}^{200} X_i \geq 0.61; 0.5\right) \\
&= \mathbb{P}\left(\textstyle\sum_{i=1}^{200} X_i \geq 122; 0.5\right) \\
&= \mathbb{P}\left(\textstyle\sum_{i=1}^{200} X_i \geq 122; 0.5\right) \\
&= \mathbb{P}(Y \geq 122) \text{ by independence where } Y \sim \text{Poisson}(200 \times 0.5) = \text{Poisson}(100) \\
&= 1 - \mathbb{P}(Y < 122) \\
&= 0.0180734
\end{aligned}$$

## Question 2

### a)

We can approximate $Y \sim \text{Poisson}(100)$ as $\tilde{Y} \approx \text{Normal}(100, 100)$. Then

$$p(\mathbf{X}) = 1 - \mathbb{P}(Y < 122) \approx 1 - \mathbb{P}(\tilde{Y} < 122) = 0.4129356$$

### b)

```
lambda<-.5
trials<-1000

samples.raw<-sapply(1:trials, function(i) sum(rpois(200,lambda)))
count<-sum(samples.raw<=122)
p<-1-count/trials
p
```

```
## [1] 0.012
```

# Question 3

Let $\mathbf{X} := (X_1, \dots, X_{200}) \overset{\text{iid}}{\sim} \text{Poisson}(\lambda)$ be a random vector representing the number of deaths each year & $\mathbf{x} := (x_1, \dots, x_{200})$ be a realisation of $\mathbf{X}$.
I shall test $H_0 : \lambda = 0.55$ against $H_1 : \lambda \neq 0.55$ at a significance level of 5% where $\Theta := \mathbb{R}^{>0}$.
Thus $\Theta_0 := \{0.55\}$ and is nested in $\Theta$. Note that $\dim(\Theta) = 1$ and $\dim(\Theta_0) = 0$.
Since $\Theta_0$ is nested within $\Theta$ we have

$$T_n(\mathbf{X}) := -2 \ln \Lambda_n(\mathbf{X}) \to_{\mathcal{D}(\cdot;\theta)} \chi^2_{\dim(\Theta)-\dim(\Theta_0)} = \chi^2_{1-0} = \chi^2_1 \text{ where } \Lambda_n(\mathbf{x}) := \frac{\sup_{\lambda \in \Theta_0} f_n(\mathbf{x}; \lambda)}{\sup_{\lambda \in \Theta} f_n(\mathbf{x}; \lambda)}$$

We shall us $T_n(\mathbf{X})$ as our test statistic.
We have

$$\begin{aligned} T(\mathbf{X}) &:= -2 \ln \Lambda_n(\mathbf{X}) \\ &= -2 \ln \tfrac{\sup_{\lambda=0.55} f_n(\mathbf{x};\lambda)}{\sup_{\lambda \in \mathbb{R}^{>0}} f_n(\mathbf{x};\lambda)} \\ &= -2 \left[ \ell_n(\mathbf{x}; \lambda) - \ell_n(\mathbf{x}; \hat{\lambda}_{MLE}) \right] \end{aligned}$$

Since I am constructing $\alpha = 0.95$ confidence interval we wish to retain $\lambda$ if

$$T_n(\mathbf{x}) := -2 \left[ \ell_n(\lambda; \mathbf{x}) - \ell_n(\hat{\lambda}_{MLE}; \mathbf{x}) \right] < \chi^2_{1,\alpha}$$

This is the confidence set

$$\begin{aligned} C(\mathbf{x}) &= \left\{ \lambda \in \mathbb{R}^{>0} : T_n(\mathbf{x}) < \chi^2_{1,\alpha} \right\} \\ &= \left\{ \lambda \in \mathbb{R}^{>0} : -2 \left[ \ell_n(\lambda; \mathbf{x}) - \ell_n(\hat{\lambda}_{MLE}; \mathbf{x}) \right] < \chi^2_{1,\alpha} \right\} \\ &= \left\{ \lambda \in \mathbb{R}^{>0} : \ell_n(\lambda; \mathbf{x}) > \ell_n(\hat{\lambda}_{MLE}; \mathbf{x}) - \tfrac{1}{2}\chi^2_{1,\alpha} \right\} \end{aligned}$$

There is a rule of thumb that at the 5 signifcance level the confidence set can be approximated as

$$C(\mathbf{x}) = \left\{ \lambda \in \mathbb{R}^{>0} : \ell_n(\lambda; \mathbf{x}) \geq \ell(\hat{\theta}_n; \mathbf{x}) - 2 \right\}$$

We can estimate the bounds of this set using r to calculate $\ell_n(\lambda; \mathbf{x})$ at regular intervals.

```r
ell <- function(lambda) {
  stopifnot(all(lambda > 0))
  n <- length(xobs)
  n * (-lambda + mean(xobs) * log(lambda))
}

ell.mle<-optimise(ell,interval=c(0,1),maximum=TRUE)$objective
cat("ell.mle=",ell.mle,sep="")
```

```
## ell.mle=-182.3042
```

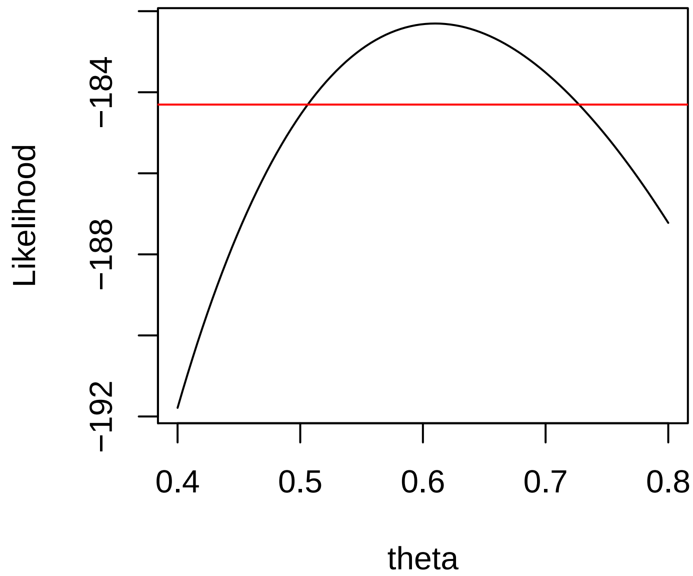
Giving

$$\begin{aligned} C(\mathbf{x}) &= \left\{ \ell_n(\lambda; \mathbf{x}) > \ell_n(\hat{\lambda}_{MLE}; \mathbf{x}) - 2 \right\} \\ &= \{ \ell_n(\lambda; \mathbf{x}) > -182.3041513 - 2 \} \\ &= \{ \ell_n(\lambda; \mathbf{x}) > -184.3041513 \} \end{aligned}$$

```r
x<-seq(0.4,.8,length.out=100)
y<-sapply(x,function(p) ell(p))
z<-x[which(y>ell.mle-2)] # The values where the inequality holds
cat("lower:",min(z),"\nupper:",max(z),sep="")
```

```
## lower:0.5090909
## upper:0.7232323
```

```r
plot(x,y,type="l",xlab="theta",ylab="Likelihood")
abline(h=ell.mle-2,col="red")
```




Thus

$$C(\mathbf{x}) = \{\theta \in [0.5090909, 0.7232323]\}$$

We accept $H_0$ in this case.

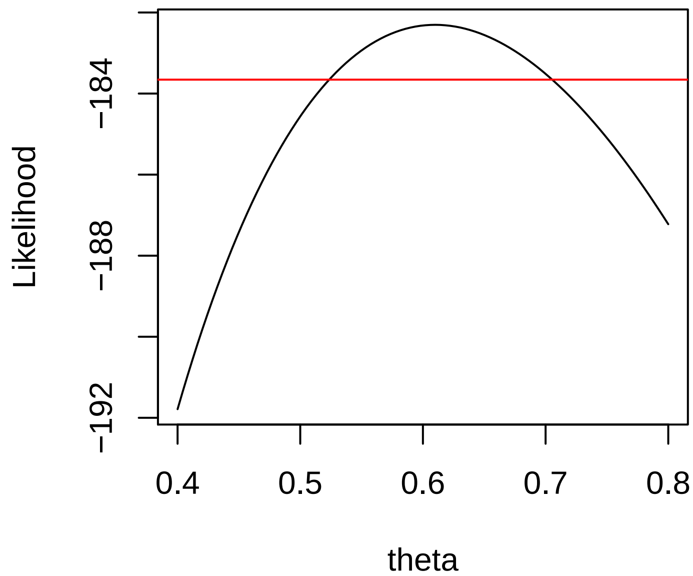
### b)

```r
alpha=0.9
quantile<-qchisq(alpha,1)

z<-x[which(y>ell.mle-.5*quantile)] # The values where the inequality holds
cat("chisq_alpha=",quantile,"\nlower:",min(z),"\nupper:",max(z))
```

```
## chisq_alpha= 2.705543
## lower: 0.5252525
## upper: 0.7030303
```

```r
plot(x,y,type="l",xlab="theta",ylab="Likelihood")
abline(h=ell.mle-.5*quantile,col="red")
```




Thus

$$\begin{aligned} C(\mathbf{x}) &= \left\{\lambda \in \mathbb{R}^{>0} : \ell_n(\lambda;\mathbf{x}) > \ell_n(\hat{\lambda}_{MLE};\mathbf{x}) - \tfrac{1}{2}2.7055435\right\} \\ &= \left\{\lambda \in \mathbb{R}^{>0} : \ell_n(\lambda;\mathbf{x}) > -182.3041513 - 1.3527717\right\} \\ &= \left\{\lambda \in \mathbb{R}^{>0} : \ell_n(\lambda;\mathbf{x}) > -183.656923\right\} \\ &= \{\lambda \in [0.5252525, 0.7030303] \text{ from R} \end{aligned}$$

We accept $H_0$ in this case.