

Statistics 2 - Notes

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Question - 1.

Derive the likelihood function and log-likelihood function for the following distributions.

Question 1.1 - $\mathbf{X} \stackrel{\text{iid}}{\sim} \text{Poisson}(\lambda)$ with λ unknown.

Answer 1.1

Let $\mathbf{X} \stackrel{\text{iid}}{\sim} \text{Poisson}(\lambda)$ with λ unknown. Then

$$\begin{aligned} L(\lambda; \mathbf{x}) &\propto f_n(\mathbf{x}; \lambda) \\ &= \prod_{i=1}^n f(x_i; \lambda) \\ &= \prod_{i=1}^n e^{-\lambda} \lambda^{x_i} (x_i!)^{-1} \\ &= e^{-\lambda n} \lambda^{\sum_{i=1}^n x_i} \prod_{i=1}^n (x_i!)^{-1} \\ &\propto e^{-\lambda n} \lambda^{\sum_{i=1}^n x_i} \\ \ell(\lambda; \mathbf{x}) &\propto \ln L(\lambda; \mathbf{x}) \\ &= -\lambda n + \left(\sum_{i=1}^n x_i \right) (\ln \lambda) \end{aligned}$$

Question 1.2 - $X \sim \text{Binomial}(n, p)$ with n & p unknown.

Answer 1.2

Let $X \sim \text{Binomial}(n, p)$ with n & p unknown. Then

$$\begin{aligned} L(n, p; x) &\propto f(x; n, p) \\ &= \binom{n}{x} p^x (1-p)^{n-x} \\ \ell(n, p; x) &\propto \ln L(n, p; x) \\ &= \ln \binom{n}{x} + x \ln p + (n-x) \ln(1-p) \end{aligned}$$

Question 1.3 - $\mathbf{X} \stackrel{\text{iid}}{\sim} \text{Normal}(\mu, \sigma^2)$ with μ & σ unknown.

Answer 1.3

Let $\mathbf{X} \stackrel{\text{iid}}{\sim} \text{Normal}(\mu, \sigma^2)$ with μ & σ unknown. Then

$$\begin{aligned}
 L(\mu, \sigma^2; \mathbf{x}) &\propto f_n(\mathbf{x}; \mu, \sigma^2) \\
 &= \prod_{i=1}^n (2\pi\sigma)^{-\frac{1}{2}} e^{-\frac{(x_i - \mu)^2}{2\sigma^2}} \\
 &= (2\pi\sigma)^{-\frac{n}{2}} \prod_{i=1}^n e^{-\frac{(x_i - \mu)^2}{2\sigma^2}} \\
 &\propto \sigma^{-\frac{n}{2}} \prod_{i=1}^n e^{-\frac{(x_i - \mu)^2}{2\sigma^2}} \\
 \ell(\mu, \sigma^2; \mathbf{x}) &\propto \ln L(\mu, \sigma^2; \mathbf{x}) \\
 &= -\frac{n}{2} \ln \sigma - \frac{1}{2\sigma^2} \sum_{i=1}^n (x_i - \mu)^2 \\
 &\propto n \ln \sigma + \frac{1}{\sigma^2} \sum_{i=1}^n (x_i - \mu)^2
 \end{aligned}$$

Question 1.4 - $\mathbf{X} \stackrel{\text{iid}}{\sim} \text{Uniform}(a, b)$ with a & b unknown.

Answer 1.4

Let $\mathbf{X} \stackrel{\text{iid}}{\sim} \text{Uniform}(a, b)$ with a & b unknown. Then

$$\begin{aligned}
 L(a, b; \mathbf{x}) &\propto f_n(\mathbf{x}; a, b) \\
 &= \prod_{i=1}^n f(x_i; a, b) \\
 &= \prod_{i=1}^n \frac{1}{b-a} \\
 &= \frac{1}{(b-a)^n} \\
 \ell(a, b; \mathbf{x}) &\propto -n \ln(b-a) \\
 &\propto -\ln(b-a)
 \end{aligned}$$

Question - 2.

Suppose that $X_1, X_2, X_3 \stackrel{\text{iid}}{\sim} \text{Exponential}(\lambda)$ represent the lifetimes of three lightbulbs in my living room. I installed them $2\frac{1}{2}$ years ago, and one has just blown (the others are still working). Show that the likelihood function corresponding to this is

$$L(\lambda; 2.5) \propto \lambda e^{-7.5\lambda}$$

Answer 2

Let $X_1, X_2, X_3 \stackrel{\text{iid}}{\sim} \text{Exponential}(\lambda)$.

Define $Y = \min(X_1, X_2, X_3)$. Then

$$\begin{aligned}
 F_Y(y; \lambda) &= \mathbb{P}(Y \leq y) \\
 &= 1 - \mathbb{P}(Y > y) \\
 &= 1 - \mathbb{P}(X_1 > y, X_2 > y, X_3 > y) \\
 &= 1 - \mathbb{P}(X_1 > y)^3 \\
 &= 1 - (e^{-\lambda y})^3 \\
 &= 1 - e^{-3\lambda y} \\
 \implies f_Y(y; \lambda) &= F'_Y(y; \lambda) \\
 &= 3\lambda y e^{-3\lambda y} \\
 \implies L(\lambda; 2.5) &\propto f_Y(2.5; \lambda) \\
 &= 3\lambda e^{-7.5\lambda} \\
 &\propto \lambda e^{-7.5\lambda}
 \end{aligned}$$

Question - 3.

Suppose I construct a random variable X in the following way. First, I toss a biased coin with probability of heads p . If it comes up heads, then $X = 0$, otherwise $X \sim \text{Poisson}(\lambda)$. We call this the *Zero-Inflated Poisson* distribution, and write $X \sim \text{ZIP}(p, \lambda)$. Derive the probability mass function for X and the likelihood function for a realisation \mathbf{x} of $\mathbf{X} \stackrel{\text{iid}}{\sim} \text{ZIP}(p, \lambda)$.

Answer 3

Let $X \sim \text{ZIP}(p, \lambda)$, $Y \sim \text{Bernoulli}(p)$ & $Z \sim \text{Poisson}(\lambda)$. Then

$$\begin{aligned}
 f_X(x) &= \begin{cases} \mathbb{P}(Y = 1) + \mathbb{P}(Z = 0|Y = 0) & , x = 0 \\ \mathbb{P}(Z = x|Y = 0) & , x \in \mathbb{N} \\ 0 & \text{otherwise} \end{cases} \\
 &= \begin{cases} p + \mathbb{P}(Z = 0)\mathbb{P}(Y = 0) & , x = 0 \\ \mathbb{P}(Z = x)\mathbb{P}(Y = 0) & , x \in \mathbb{N} \\ 0 & \text{otherwise} \end{cases} \\
 &= \begin{cases} p + e^{-\lambda}\lambda^0(0!)^{-1}(1-p) & , x = 0 \\ (e^{-\lambda}\lambda^x(x!)^{-1})(1-p) & , x \in \mathbb{N} \\ 0 & \text{otherwise} \end{cases} \\
 &= \begin{cases} p + e^{-\lambda}(1-p) & , x = 0 \\ \frac{1}{x!}(e^{-\lambda}\lambda^x)(1-p) & , x \in \mathbb{N} \\ 0 & \text{otherwise} \end{cases} \\
 &= p\mathbb{1}\{x = 0\} + \frac{1}{x!}(e^{-\lambda}\lambda^x)(1-p)
 \end{aligned}$$

Let $\mathbf{X} \stackrel{\text{iid}}{\sim} \text{ZIP}(p, \lambda)$ with p & λ unknown. Then

$$\begin{aligned}
 L(p, \lambda; \mathbf{x}) &\propto f_n(\mathbf{x}; p, \lambda) \\
 &= \prod_{i=1}^n \left[p\mathbb{1}\{x = 0\} + \frac{1}{x!}(e^{-\lambda}\lambda^x)(1-p) \right]
 \end{aligned}$$