Statistics 2 - Notes

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1 Estimation

1.1 Introduction

Definition 1.1 - *Probabiltiy Space,* $(\Omega, \mathcal{F}, \mathbb{P})$
A mathematical construct for modelling the real world. A *Probabilty Space* has three elements

i) Ω - Sample space.

ii) $\mathcal{F}$ - Set of events.

iii) $\mathbb{P}$ - Probability measure.

and most fulfil the following conditions

i) $\Omega \in \mathcal{F}$;

ii) $\forall\ A \in \mathcal{F} \implies A^c \in \mathcal{F}$;

iii) $\forall\ A_0, \dots, A_n \in \mathcal{F} \implies \left(\bigcup_i A_i\right) \in \mathcal{F}$;

iv) $\mathbb{P}(\Omega) = 1$; and,

v) $\mathbb{P}\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mathbb{P}(A_i)$ for disjoint $A_1, A_2, ...$ (Countable Additivity).

Definition 1.2 - *Random Variable*
A function which maps an event in the sample space to a value *e.g.* $X : \Omega \to \mathbb{R}$.

Remark 1.1 - *Probability Density Function for iid Random Variable Vector*
For $\mathbf{X} \sim f_n(\cdot; \theta)$ where each component of $\mathbf{X}$ is independent and identically distribution the probability density function of $\mathbf{X}$ is

$$f_n(\mathbf{x}; \theta) = \prod_{i=1}^{n} f(x_i; \theta)$$

Definition 1.3 - *Expectation*
The mean value for a random variable. For rv X

$$\mathbb{E}(X) := \sum_{x \in \chi} x f_X(x) \quad \& \quad \mathbb{E}(X) := \int_{\mathbb{R}} x f_X(x) dx$$

Theorem 1.1 - *Expection of a Function*
For a function $g : \mathbb{R} \to \mathbb{R}$ and rv X with pmf f_X

$$\mathbb{E}(g(X)) := \sum_{g(x) \in \chi} x f_X(x) \quad \& \quad \mathbb{E}(g(X)) := \int_{\mathbb{R}} g(x) f_X(x) dx$$

Theorem 1.2 - *Expectation of a Linear Operator*
For rv X with pmf f_X & $a, b \in \mathbb{R}$

$$\mathbb{E}(aX + b) = a\mathbb{E}(X) + b$$

Definition 1.4 - *Variance*
For rv X

$$\mathrm{Var}(X) := \mathbb{E}\big[(X - \mathbb{E}(X))^2\big] = \mathbb{E}(X^2) - \mathbb{E}(X)^2$$

Theorem 1.3 - *Variance of a Linear Operator*
For rv X and $a, b \in \mathbb{R}$

$$\text{Var}(aX + b) = a^2\text{Var}(X)$$

Definition 1.5 - *Moment of a Random Variable*
For rv X the n^{th} moment of X is defined as $\mathbb{E}(X^n)$.
N.B. - $\mathbb{E}(X^n) \neq \mathbb{E}(X)^n$.

Definition 1.6 - *Covariance*
For rv X & Y

$$\text{Cov}(X, Y) := \mathbb{E}\big[(X - \mathbb{E}(X))(Y - \mathbb{E}(Y))\big] = \mathbb{E}(XY) - \mathbb{E}(X)\mathbb{E}(Y)$$

Theorem 1.4 - *Properties of Covaraince*
Let X & Y be independent random varaibles

i) $\text{Cov}(X, X) = \text{Var}(X)$;

ii) $\text{Cov}(X, Y) = 0$

Theorem 1.5 - *Variance of two Random Variables with linear operators*

$$\text{Var}(aX + bY) = a^2\text{Var}(X) + b^2\text{Var}(Y) + 2ab\text{Cov}(X, Y)$$

Theorem 1.6 - *Independent Random Variables*
Random variables $X_1, \ldots, X_n$ are independent iff

$$\mathbb{P}(X_1 \leq a_1, \ldots, X_n \leq a_n) = \prod_{i=1}^{n} \mathbb{P}(X_i \leq a_i) \ \forall \ a_1, \ldots, a_n \in \mathbb{R}$$

1.2 The Likelihood Function

Definition 2.1 - *Likelihood Function*
Define $\mathbf{X} \sim f_n(\cdot; \theta^*)$ for some unknown $\theta^* \in \Theta$ and let $\mathbf{x}$ be an observation of $\mathbf{X}$.
A *Likelihood Function* is any function, $L(\cdot; \mathbf{x}) : \Theta \to [0, \infty)$, which is proportional to the PMF/PDF of the obeserved realisation $\mathbf{x}$.

$$L(\theta; \mathbf{x}) := C f_b(\mathbf{x}; \theta) \ \forall \ C > 0$$

N.B. Sometimes this is called the *Observed* Likelihood Function since it is dependent on observed data.

Definition 2.2 - *Log-Likelihood Function*
Let $\mathbf{X} \sim f_n(\cdot; \theta^*)$ for some unknown $\theta^* \in \Theta$ and $\mathbf{x}$ be an observation of $\mathbf{X}$.
The *Log-Likelihood Function* is the natural log of a *Likelihood Function*

$$\ell(\theta; \mathbf{x}) := \ln f_n(\mathbf{x}; \theta) + C, \ C \in \mathbb{R}$$

Theorem 2.1 - *Multidiensional Transforms*
Let $\mathbf{X}$ be a continuous random vector in $\mathbb{R}^n$ with PDF $f_{\mathbf{X}}$; $g : \mathbb{R}^n \to \mathbb{R}^n$ be a continuous diferentiable bijection; and, $h := g^{-1}$.
Then $\mathbf{Y} = g(\mathbf{X})$ is a continuous random vector and its PDF is

$$f_{\mathbf{Y}}(\mathbf{y}) = f_{\mathbf{X}}(h(\mathbf{y})H_h(\mathbf{Y})$$

where

$$J_h := \left|\det\left(\frac{\partial h}{\partial \mathbf{y}}\right)\right|$$

Proposition 2.1 - *Invaraince of Likelihood Function by bijective transformation of the observations independent of θ*
Let $g : \mathbb{R}^n \to \mathbb{R}^n$ be a bijetive transformation which is independent of θ; and $\mathbf{Y} := g(\mathbf{X})$.
Then $\mathbf{Y}$ is a random variable with PDF/PMG

$$f_{\mathbf{Y}}(\mathbf{y};\theta) \propto f_{\mathbf{X}}(g^{-1}(\mathbf{y});\theta)$$

Hence, if $\mathbf{y} = g(\mathbf{x})$ then $L_{\mathbf{Y}}(\theta;\mathbf{y}) \propto L_{\mathbf{X}}(\theta;\mathbf{x})$

Proof 2.1 - *Proposition 2.1*
Let $g : \mathbb{R}^n \to \mathbb{R}^n$ be a bijective transformation which is independent of θ; $h := g^{-1}$; $\mathbf{X}, \mathbf{Y}$ be a rvs st $\mathbf{Y} := g(\mathbf{X})$.

i) *Discrete Case* - Consider the case when $\mathbf{X}$ is a discrete rv. Then

$$\begin{aligned} f_{\mathbf{Y}}(y;\theta) &= \mathbb{P}(\mathbf{Y} = \mathbf{y};\theta) \\ &= \mathbb{P}(g^{-1}(\mathbf{Y}) = g^{-1}(\mathbf{y});\theta) \\ &= \mathbb{P}(h(\mathbf{Y}) = h(\mathbf{y});\theta) \\ &= \mathbb{P}(\mathbf{X} = h(\mathbf{y});\theta) \\ &= f_{\mathbf{X}}(g^{-1}(\mathbf{y});\theta) \end{aligned}$$

ii) *Continuous Case* - Consider the case when $\mathbf{X}$ is a continuous rv.
Then, by **Theorem 2.1**

$$f_{\mathbf{Y}}(\mathbf{y};\theta) = f_{\mathbf{X}}(g^{-1}(\mathbf{y});\theta) J_{g^{-1}}(\mathbf{y})$$

Since $J_{g^{-1}}$ does not depend on θ this case is solved.

Thus in botoh cases $L_{\mathbf{Y}}(\theta; y) = f_{\mathbf{Y}}(y;\theta) \propto f_{\mathbf{X}}(g^{-1}(\mathbf{y});\theta) = L_{\mathbf{X}}(\theta;\mathbf{x})$. □

1.3 Maximum Likelihood Estimates

Definition 3.1 - *Maximum Likelihood Estimate*
Let $\mathbf{X} \sim f_n(\cdot;\theta)$; and $\mathbf{x}$ be a realisation of $\mathbf{X}$.
The *Maximum Likelihood Estimate* is the value $\hat{\theta} \in \Theta$ st

$$\forall\ \theta \in \Theta\ f_n(\mathbf{x};\hat{\theta}) \geq f_n(\mathbf{x},\theta)$$

Equivalently

$$\forall\ \theta \in \Theta\ L(\hat{\theta};\mathbf{x}) \geq L(\theta;\mathbf{x}) \quad \text{or} \quad \ell(\hat{\theta};\mathbf{x}) \geq \ell(\theta;\mathbf{x})$$

i.e. $\hat{\theta}(\mathbf{x}) := \text{argmax}_\theta(L(\theta;\mathbf{x}))$.

Remark 3.1 - *The Maximum Likelihood Estimate may <u>not</u> be unique*

Example 3.1 - *MLE for Uniform Distribution*
Consider $\mathbf{X} \overset{\text{iid}}{\sim} U[0,\theta]$ for $\theta > 0$.
Then

$$\begin{aligned} L(\theta;\mathbf{x}) &\propto f_n(\mathbf{x};\theta) \\ &= \prod_{i=1}^{n} \frac{1}{\theta} \mathbb{1}\{x_i \in [0,\theta] \\ &= \frac{1}{\theta^n} \prod_{i=1}^{n} \mathbb{1}\{x_i \in [0,\theta] \\ \implies \qquad \hat{\theta} &= \max\{x_i : x_i \in \mathbf{x}\} \end{aligned}$$

Remark 3.2 - *MLE of Reparameterisation*
Define $\tau(\theta) : \mathbb{R} \to \mathbb{R}$. Then

$$\hat{\tau} = \tau(\hat{\theta})$$

N.B. We often write $\tilde{f}$ to represent the pmf when τ is taken as a parameter rateher than θ. *i.e.* $f(x;\theta) = \tilde{f}(x;\tau(\theta))$.

Theorem 3.1 - *Invariance of MLE under bijective Reparameterisation*
Let $g : \Theta \to G$ be a bijective transformation of the statisitcal parameter θ.
Let $\mathbf{X} \sim f(\cdot;\theta) = \tilde{f}(\cdot; g(\theta))$ for some θ, and let $\mathbf{x}$ be a realisation of $\mathbf{X}$.

$$\text{If } \hat{\theta} \text{ s an MLE of } \theta \text{ then } \hat{\tau} = g(\hat{\theta}) \text{ is an MLE of } \tau.$$

Proof 3.1 - *Theorem 3.1*
This is a proof by contradiction.
Suppose $\exists\ \tau^* \in G st \tilde{f}(x;\tau^*) > \tilde{f}(x;\tau^*)$ We know that $\forall\ \theta \in \Theta,\ f(x;\theta) = \tilde{f}(x; g(\theta))$ and $\forall\ \tau \in G,\ f(x; g^{-1}(\tau)) = \tilde{f}(x;\tau)$.
We deduce that

$$\begin{aligned} f(x; g^{-1}(\tau^*)) &= \tilde{f}(x;\tau^*) \\ &> \tilde{f}(x;\hat{\tau}) \text{ by assumption} \\ &= f(x; g^{-1}(\hat{\tau})) \\ &= f(x;\hat{\theta}) \end{aligned}$$

This contradicts the assumption that $\hat{\theta}$ is an maximum likelihood estimate of θ.
□

Remark 3.3 - *Not all Reparameterisations are Bijective*
When reparameterisations $g : \mathbb{R} \to \mathbb{R}$ is not bijective it is helpful to consider the *induced likelihood*

$$L^*(\tau;\mathbf{x}) := \max_{\theta \in G_\tau} L(\theta;\mathbf{x}) \text{ where } G_\tau := \{\theta : g(\theta) = \tau\}$$

Since this reduces the domain to only where g is bijective.

1.4 Determinig MLEs - The Tractable Case

Proposition 4.1 - *Differentiable Likelihood in the continuous case - Multivariate*
When $L(\theta;\mathbf{x}$ is differentiable one can find MLEs by considering its extrema. This is done equating & solving the cases when the gradient is zero, *i.e.* $\nabla L(\theta;\mathbf{x}) = 0$, and then checking whether this is a maximum or minimum point.
A point is a local minimum if the Hessian at the point is *Negative Definite i.e.* $x^T A x < 0\ \forall\ x \neq \mathbf{0}$.

Example 4.1 - *MLE of Normal Distribution*

Let $\mathbf{X} \overset{\text{iid}}{\sim} \mathcal{N}(\mu, \sigma^2)$

$$\begin{aligned}
& & L(\mu,\sigma^2;\mathbf{x}) &= \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x_i-\mu)^2}{2\sigma^2}} \\
\implies & & \ell(\mu,\sigma^2;\mathbf{x}) &= C - \tfrac{n}{2}\ln(\sigma^2) - \tfrac{1}{2\sigma^2}\textstyle\sum_{i=1}^{n}(x_i-\mu)^2 \\
\implies & & \nabla\ell(\mu,\sigma^2;\mathbf{x}) &= \left(\frac{-1}{\sigma^2}\sum_{i=1}^{n}(x_i-\mu), \quad -\frac{n}{2\sigma^2} + \frac{1}{2\sigma^4}\sum_{i=1}^{n}(x_i-\mu)^2\right) \\
\text{Setting} & & \frac{-1}{\sigma^2}\sum_{i=1}^{n}(x_i-\mu) &= 0 \\
\implies & & \hat{\mu} &= \frac{1}{n}\textstyle\sum_{i=1}^{n} x_i = \bar{x} \\
\text{Setting} & -\frac{n}{2\sigma^2} + \frac{1}{2\sigma^4}\sum_{i=1}^{n}(x_i-\mu)^2 &= 0 \\
\implies & & \hat{\sigma}^2 &= \frac{1}{n}\sum_{i=1}^{n}(x_i\hat{\mu})^2
\end{aligned}$$

We now want to check whether $(\hat{\mu}, \hat{\sigma^2})$ is a minimum.

$$\begin{aligned}
\nabla^2\ell(\mu,\sigma^2;\mathbf{x}) &= \begin{pmatrix} \dfrac{\partial^2\ell(\mu,\sigma^2;\mathbf{x})}{\partial\mu^2} & \dfrac{\partial^2\ell(\mu,\sigma^2;\mathbf{x})}{\partial\mu\partial\sigma^2} \\ \dfrac{\partial^2\ell(\mu,\sigma^2;\mathbf{x})}{\partial\mu\sigma^2} & \dfrac{\partial^2\ell(\mu,\sigma^2;\mathbf{x})}{\partial(\sigma^2)^2} \end{pmatrix} \\
&= \begin{pmatrix} -\dfrac{n}{\hat{\sigma}^2} & 0 \\ 0 & -\dfrac{n}{2\hat{\sigma}^4} \end{pmatrix}
\end{aligned}$$

Since $\begin{pmatrix} z_1 & z_2 \end{pmatrix}\begin{pmatrix} -a & 0 \\ 0 & -b \end{pmatrix}\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = -az_1^2 - bz_2^2 < 0 \ \forall\ a, b > 0$ and we have $\frac{n}{\hat{\sigma}^2},\ \frac{n}{2\hat{\sigma}^4} > 0$ then we can conclude that $\nabla^2\ell$ is negative definite.
Thus $\hat{\mu} = \bar{x}$ & $\hat{\sigma}^2 = \dfrac{1}{n}\displaystyle\sum_{i=1}^{n}(x_i\hat{\mu})^2$ is an MLE for the normal distribution.

Example 4.2 - *MLE for Capture-Recapture Model*
Suppose you are wanting to calculate the unknown size of a population, n. The Capture-Recapture Model is one technique that can be used. You tag $t \leq n$ members of the population; wait for a while; then recapture $c \leq n$ members of which $x \leq \min\{t, c\} \leq n$ are tagged.
With t, c, x known produce a MLE for n.
We first work out the associated probability distribution for X, the population size. We have

i) $\binom{t}{x}$ ways of choosing x members among the tagged ones;

ii) $\binom{n-t}{c-x}$ ways of choosing the remaining members among the non-tagged ones;

iii) $\binom{n}{c}$ ways of choosing c members in a population of n individuals.

Thus

$$f_X(x;n) = \frac{\binom{t}{x}\binom{n-t}{c-x}}{\binom{n}{c}}$$

This means that $X \sim \text{Hypergeometric}(t, n, c)$ with t & c known.

Now we calculate the MLE for X

$$\begin{aligned} L(n;x) &= f_X(x;n) \\ &= \frac{\binom{t}{x}\binom{n-t}{c-x}}{\binom{n}{c}} \\ &= \frac{\frac{t!}{x!(t-x)!}\frac{(n-t)!}{(c-x)!(n-t-c+x)!}}{\frac{n!}{c!(n-c)!}} \end{aligned}$$

Now we consider $L(n;x) = 0$ when $x > \min\{t, c\}$. We want to indetify values of n for which $L(n;x) \geq L(n-1;x)$.
Consider $n - 1 \geq \min\{t, c\} \implies L(n-1;x) > 0$

$$\begin{aligned} \text{Let } r(n) &:= \frac{L(n;x)}{L(n-1;x)} \\ &= \frac{n-t}{n-t-c+x}\frac{n-c}{n} \\ \Rightarrow \quad 1 &\leq r(n) \\ \Leftrightarrow \quad 1 &\leq \frac{n-t}{n-t-c+x}\frac{n-c}{n} \\ \Leftrightarrow \quad n(n-t-c+x) &\leq (n-t)(n-c) \\ \Leftrightarrow \quad n^2 - nt - cn + xn &\leq n^2 - nt - cn + ct \\ \Leftrightarrow \quad xn &\leq ct \\ \Leftrightarrow \quad x &\leq \frac{ct}{n} \end{aligned}$$

So $L(n;x)$ is increasing for $n \leq \left\lfloor \frac{ct}{x} \right\rfloor$ & decreasing for $n > \left\lfloor \frac{ct}{x} \right\rfloor$.
Consequently $\hat{n}_{\text{MLE}}(x) = \left\lfloor \frac{tc}{x} \right\rfloor$

1.5 Statistics and Estimators

Definition 5.1 - *Statistic*
Given some data $\mathbf{x}$ a statistic is a fucntion of the data $T(\mathbf{x})$.
N.B. A statistic cannot depend on an unknown statistical parameter.

Definition 5.2 - *Estimate*
Let $\mathbf{X} \sim f_n(\cdot;\theta^*)$ with $\theta^* \in \Theta$ and $\mathbf{x}$ be a realisation of $\mathbf{X}$.
An *Estimate* θ^* is a statistic $\hat{\theta}(\mathbf{x}) = T(\mathbf{x})$ which is intended to approximate the real value of θ^*.
N.B. An *Estimate* is a real value & thus is hard to evaluate.

Definition 5.3 - *Estimator*
Let $\mathbf{X} \sim f_n(\cdot;\theta^*)$ with $\theta^* \in \Theta$ and $\mathbf{x}$ be a realisation of $\mathbf{X}$.
An *Estimator* of θ^* is $\hat{\theta}$ where $\hat{\theta}(\mathbf{x})$ is an *estimate*.
N.B. We call $T(\mathbf{X})$ an estimator. This is a random variable.

Definition 5.4 - *Distribution of an Estimator*
Let $\mathbf{X}|sim f_n(\cdot;\theta^*)$ with $\theta^* \in \Theta \subseteq \mathbb{R}$.
If $\hat{\theta}(\mathbf{X})$ is a real-valued random variable, we can write its CDF as

$$\begin{aligned} F_{\hat{\theta}(\mathbf{X})}(t;\theta^*) &= \mathbb{P}(\hat{\theta}(\mathbf{X}) \leq t;\theta^*) \\ &= \int_{\mathcal{X}^n} \mathbb{1}\{\hat{\theta}(\mathbf{x}) \leq t\} f_n(\mathbf{x};\theta^*) d\mathbf{x} \end{aligned}$$

Remark 5.1 - *Estimator dependends upon true value*
The distribution of $\hat{theta}(\mathbf{X})$ depends on the distribution of $\mathbf{X}$ which in turn depends upon the

distribution of θ^*.
Thus the distribution of an estimator depends on the true parameter of the variable it is estimating.

Remark 5.2 - *Estimator Distribution & Sample Size*
As sample size increasesthe distribution of an estimator may converge to a more standard distribution (*e.g.* Normal, Poisson).
Definition 5.5 - *Bias*
Bias is a measure of how much an estimator deviates from the true value, on average.

$$\begin{aligned}\text{Bias}(\hat{\theta};\theta^*) &:= \mathbb{E}(\hat{\theta}(\mathbf{X}) - \theta^*;\theta^*) \\ &= \mathbb{E}(\hat{\theta};\theta^*) - \mathbb{E}(\theta^*;\theta^*) \\ &= \mathbb{E}(\hat{\theta};\theta^*) - \theta^*\end{aligned}$$

Definition 5.6 - *Unbiased Estimator*
An *Estimator*, $\hat{\theta}$, is said to be *Unbiased* if $\forall\ \theta \in \Theta,\ \text{Bias}(\hat{\theta};\theta) = 0$.
Equivalently $\mathbb{E}(\hat{\theta};\theta) = \theta$.

Definition 5.7 - *Mean Square Error*
The *Mean Square Error* of an estimator is the mean of the squared error associated with rv $\hat{\theta}$.

$$MSE(\hat{\theta};\theta^*) := \mathbb{E}\left[(\hat{\theta}(\mathbf{X}) - \theta^*)^2;\theta^2\right]$$

Proposition 5.1 - *Simplifcation of MSE Formula*
The MSE is a combination of variance & bias.

$$\begin{aligned}MSE(\hat{\theta};\theta^*) &= \mathbb{E}\left[(\hat{\theta}(\mathbf{X}) - \theta^*)^2;\theta^2\right] \\ &= \mathbb{E}\left[\left\{\hat{\theta} - \mathbb{E}(\hat{\theta};\theta^*)\right\}^2;\theta^*\right] + \left(\mathbb{E}(\hat{\theta} - \theta^*;\theta^*\right)^2 \\ &= \text{Var}(\hat{\theta};\theta^*) + \text{Bias}(\hat{\theta};\theta^*)^2\end{aligned}$$

Example 5.1 - *Sample mean as an Estimator*
Let $\mathbf{X} \overset{\text{iid}}{\sim} \text{Poisson}(\lambda^*)$.
Suppose we are using the sample mean, $\hat{\lambda}(\mathbf{x}) := \frac{1}{n}\sum_{i=1}^n x_i$, as an estimate of λ^*. We first want to show this estimator is *Unbiased*

$$\begin{aligned}\mathbb{E}(\hat{\lambda};\lambda) &= \mathbb{E}\left(\frac{1}{n}\sum_{i=1}^n X_i;\lambda\right) \\ &= d\frac{1}{n}\sum_{i=1}^n \mathbb{E}(X_i;\lambda) \\ &= \tfrac{1}{n}n\lambda \\ &= \lambda\end{aligned}$$

Thus $\hat{\lambda}$ is unbiased.
Now we consider the MSE of $\hat{\lambda}$

$$\begin{aligned}MSE(\hat{\lambda};\lambda) &= \text{Var}(\hat{\lambda};\lambda) \\ &= \text{Var}\left(\frac{1}{n}\sum_{i=1}^n X_i;\lambda\right) \\ &= \frac{1}{n^2}\sum_{i=1}^n \text{Var}(X_i;\lambda) \\ &= \tfrac{1}{n^2}n\lambda \\ &= \tfrac{\lambda}{n}\end{aligned}$$

This shows that as the sample size increases the MSE of $\hat{\lambda}$ converges to 0.

1.6 Probabilistic Convergence

Remark 6.1 - *Motivation*
Here we consider the properties of a maximum likelihood estimators as the sample size increases.

Theorem 6.1 - *Markov's Inequality*
For a *non-negative* random variable X and a constant $a > 0$

$$\mathbb{P}(X \geq a) \leq \frac{\mathbb{E}(X)}{a}$$

Proof 6.1 - *Markov's Inequality*
Consider continuous X. We have

$$\begin{aligned} a\mathbb{P}(X \geq a) &= a\int_a^\infty f_X(x)dx \\ &\leq \int_a^\infty x f_X(x)dx \\ &\leq \int_0^\infty x f_X(x)dx \\ &= \mathbb{E}(X) \\ \implies a\mathbb{P}(X \geq a) &= \mathbb{E}(X) \\ \implies \mathbb{P}(X \geq a) &\leq \frac{\mathbb{E}(X)}{a} \end{aligned}$$

□

Theorem 6.2 - *Chebyshev's Inequality*
Let $\mu = \mathbb{E}(X)$ and $\sigma^2 = \text{Var}(X)$. Then

$$\forall\ a > 0,\ \mathbb{P}(|X - \mu| \geq a) \leq \frac{\sigma^2}{a^2}$$

Proof 6.2 - *Chebyshev's Inequality*
We have

$$\begin{aligned} \mathbb{P}(|X - \mu| \geq a) &= \mathbb{P}(|X - \mu|^2 \geq a^2) \\ &\leq \frac{\mathbb{E}\left((X-\mu)^2\right)}{a^2} \text{ By Markov's Inequality} \\ &= \frac{\sigma^2}{a^2} \end{aligned}$$

□

Definition 6.1 - *Convergence in Probability*
We say the sequence of random variables $\{Z_n\}_{n\in\mathbb{N}}$ converges in probability to the random variable Z if

$$\forall\ \varepsilon > 0,\ \lim_{n\to\infty} \mathbb{P}(|Z_n - Z| > \varepsilon) = 0$$

N.B. This is denoted $Z_n \to_{\mathbb{P}} Z$.
N.B. The random variables $\{Z_n\}_{n\in\mathbb{N}}$ & Z must be in the same probability space.

Theorem 6.3 - *Weak Law of Large Numbers*
If $\{X_n\}_{n\in\mathbb{N}}$ are idependent & identically distributed and $\mathbb{E}(X_1) = \mu < \infty$ then

$$Z_n = \frac{1}{n}\sum_{i=1}^n X_i \to_{\mathbb{P}} \mu$$

N.B. This is an example of Convergence in Probability.

Definition 6.2 - *Convergence in Distribution*
We say the sequence of random variables $\{Z_n\}_{n\in\mathbb{N}}$ converges in distribution to random variable Z if

$$\forall\ z \in \mathrm{Z} \text{ where } \mathbb{P}(Z \leq z) \text{ is continuous, } \lim_{n\to\infty} \mathbb{P}(Z_n \leq z) = \mathbb{P}(Z \leq z)$$

N.B. This is denoted $Z_n \to_{\mathcal{D}} Z$.
N.B. The random variables $\{Z_n\}_{n\in\mathbb{N}}$ & Z need not be in the same probability space.

Remark 6.2 - *Equivalent Statements to Convergence in Distribution*
Saying that $Z_n \to_{\mathcal{D}} Z$ is equivalent to saying that

$$\forall\ z \in \mathrm{Z} \text{ where } F_Z(z) \text{ is continuous, } \lim_{n\to\infty} F_{Z_n}(z) = F_Z(z)$$

Theorem 6.4 - *Central Limit Theorem*
If $\{X_n\}_{n\in\mathbb{N}}$ are idependent & identically distributed, $\mathbb{E}(X_1) = \mu < \infty$ and $\mathrm{Var}(X_1) = \sigma^2 < \infty$ then

$$\frac{\sqrt{n}}{\sigma}(Z_n - \mu) \to_{\mathcal{D}} Z \sim \mathrm{Normal}(0, 1)$$

Theorem 6.5 - *Convergence in Probability & Distribution*
Convergence in probabilitiy $\implies$ Convergence in distribution, **but** the opposite is not necessarily true.

Theorem 6.6 - *Convergence in Probability & Distribution to a Constant*
Convergence in distribution to a constant **and** convergence in probability to a constant are equivalent.

Example 6.1 -
Let $X \sim \mathrm{Bernoulli}\left(\frac{1}{2}\right)$ and $\{X_n\}_{n\in\mathbb{N}}$ be a sequence of random variables where $X_i := (1-X)+\frac{1}{n}$.
We have

$$F_X(x) = \begin{cases} 0 & ,\ x < 0 \\ \frac{1}{2} & ,\ x \in [0, 1) \\ 1 & , x \geq 1 \end{cases} \qquad F_{X_n}(x) = \begin{cases} 0 & ,\ x < \frac{1}{n} \\ \frac{1}{2} & ,\ x \in \left[\frac{1}{n}, 1 + \frac{1}{n}\right) \\ 1 & , x \geq 1 + \frac{1}{n} \end{cases}$$

Clearly $F_{X_n}(x) \to F_X(x)$ at all points at which F_X is continuous (*i.e.* $x \in \mathbb{R}\backslash\{0, 1\}$).
Thus $X_n \to_{\mathcal{D}} X$.

Theorem 6.7 - *Continuous Mapping Theorem*
Let $g : Z \to G$ be a *continuous* function. Then

i) If $Z_n \to_{\mathbb{P}} Z$, then $g(Z_n) \to_{\mathbb{P}} g(Z)$;

ii) If $Z_n \to_{\mathcal{D}} Z$, then $g(Z_n) \to_{\mathcal{D}} g(Z)$

Theorem 6.8 - *Slutsky's Theorem*
Let $\{Y_n\}_{n\in\mathbb{N}}$ & $\{Z_n\}_{n\in\mathbb{N}}$ be sequences of random variables, Y be a random variable & $c \in \mathbb{R}\backslash 0$ be a constant.
If $Y_n \to_{\mathcal{D}} Y$ and $Z_n \to_{\mathcal{D}} c$, then

i) $Y_n + Z_n \to_{\mathcal{D}} Y + c$;

ii) $Y_n Z_n \to_{\mathcal{D}} Yc$; and,

iii) $\frac{Y_n}{Z_n} \to_{\mathcal{D}} \frac{Y}{c}$.

Definition 6.3 - *Convergence in Quadratic Mean*
Let $\{Z_n\}_{n\in\mathbb{N}}$ be a sequence of random variables & Z be a random variable.
We say that $\{Z_n\}_{n\in\mathbb{N}}$ *Converges in Quadratic Mean* to the random variable Z if

$$\lim_{n\to\infty} \mathbb{E}\left[(Z_n - Z)^2\right] = 0$$

N.B. This is denoted $Z_n \to_{qm} Z$.

Theorem 6.9 - *If $Z_n \to_{qm} Z$ then $Z_n \to_{\mathbb{P}} Z$*

Proof 6.3 - *Theorem 5.9*
Fix any $\varepsilon > 0$. We have

$$\begin{aligned} \mathbb{P}(|Z_n - Z| > \varepsilon) &= \mathbb{P}(|Z_n - Z|^2 > \varepsilon^2) \\ &\leq \tfrac{1}{\varepsilon^2}\mathbb{E}\left[(Z_n - Z)^2\right] \text{ by Markov's Inequality} \\ &\to 0 \text{ since } Z_n \to_{qm} Z. \end{aligned}$$

Hence $Z_n \to_{\mathbb{P}} Z$. □

1.7 Probabilistic Convergence & Estimators

Definition 7.1 - *Consistency of a Sequence of Estimators*
A sequence of estimators, $\{\hat{\theta}_n(\cdots) : \chi^n \to \Theta\}$, are said to be *Consistent* if

$$\forall\ \theta \in \Theta \text{ with } \mathbf{X}_n \sim f_n(\cdot;\theta),\ \hat{\theta}_n(\mathbf{X}_n) \to_{\mathbb{P}(\cdot;\theta)} \theta$$

Remark 7.1 - *Consistency of a Sequence of Estimators*

i) In numerous situations one will talk about the consistency of *the* estimator, *e.g.* for the MLE, but also for the mean, etc. This implicitly refers to the corresponding sequence of MLEs, sequence of means, etc.

ii) Note the $\mathbb{P}(\cdot;\theta)$ in the limit above, and in particular the dependence on θ. This is often omitted in practice, you should however not forget what the symbols actually mean.

iii) Quadratic mean / Mean Square convergence $\implies$ consistency.
That is, if the MSE of the estimator converges to 0, the estimator is consistent.

Example 7.1 - *Consistency of Flipping Coins*
Let $\mathbf{X} \overset{\text{iid}}{\sim} \text{Bernoulli}(\theta^*)$ for some $\theta^* \in [0,1]$.
The maximum likelihood estimate and method of moments for $\hat{\theta}_n$ are the sample mean.

$$\hat{\theta}_n(X_1,\ldots,X_n) = \frac{1}{n}\sum_{i=1}^{n} X_i$$

By the *Weak Law of Large Numbers* we have that *consistency* of $\{\hat{\theta}n\}$, since $\mathbb{E}(X_1) = \theta^*$.

Example 7.2 - *Crude Confidence Interval when Flipping Coins*
Let $\mathbf{X} \overset{\text{iid}}{\sim} \text{Bernoulli}(\theta^*)$ for some $\theta^* \in [0,1]$ and define $\hat{\theta}_n := \hat{\theta}_n(X_1,\ldots,X_n)$.
We shall produce a *confidence interval* for θ^*.

$$\mathbb{E}(\hat{\theta}_n;\theta^*) = \theta^* \quad \text{and} \quad \text{Var}(\hat{\theta}_n;\theta^*) = \frac{\theta^*(1-\theta^*)}{n}$$

$$\begin{aligned}
\mathbb{P}\left(|\hat{\theta}_n - \theta^*| \geq \varepsilon; \theta^*\right) &\leq \tfrac{\theta^*(1-\theta^*)}{n\varepsilon^2} \quad \text{by Chebyshev's Inequality} \\
\text{We don't know } \theta^*, \text{ but can deduce that } \theta^*(1-\theta^*) &\leq \tfrac{1}{4} \\
\implies \mathbb{P}\left(|\hat{\theta}_n - \theta^*| \geq \varepsilon; \theta^*\right) &\leq \tfrac{1}{4n\varepsilon^2} \\
\text{Define } \alpha &:= \tfrac{1}{4n\varepsilon^2} \\
\implies \mathbb{P}\left(|\hat{\theta}_n - \theta^*| \geq \tfrac{1}{2\sqrt{n\alpha}}; \theta^*\right) &\leq \alpha \\
\implies \mathbb{P}\left(\hat{\theta}_n - \tfrac{1}{2\sqrt{n\alpha}} < \theta^* < \hat{\theta}_n + \tfrac{1}{2\sqrt{n\alpha}}; \theta^*\right) &\geq 1-\alpha
\end{aligned}$$

This means the random interval $(\hat{\theta}_n - \frac{1}{2\sqrt{n\alpha}},\ \hat{\theta}_n + \frac{1}{2\sqrt{n\alpha}}; \theta^*)$ contains θ^* with probability $1-\alpha$. We can note that the interval decreases as n increases, and increases as α decreases. *N.B.* $\hat{\theta}_n$ is a random variable, while θ^* is not.

Example 7.3 - *Assymptotically Exact Confidence Interval when Flipping Coins*
This is an improvement on the bound produced in **Example 5.3**.
Let $\mathbf{X} \overset{\text{iid}}{\sim} \text{Bernoulli}(\theta^*)$ for some $\theta^* \in [0,1]$, $W \sim \text{Normal}(0,1)$ and define $\hat{\theta}_n := \hat{\theta}_n(X_1, \ldots, X_n)$. We shall show that

$$\frac{\sqrt{n}(\hat{\theta}_n - \theta^*)}{\sqrt{\hat{\theta}_n(1-\hat{\theta}_n)}} \to_{\mathcal{D}} W$$

We know that $\text{Var}(X_1) = \theta^*(1-\theta^*)$.
By the *Weak Law of Large Numbers* $\hat{\theta}_n \to_{\mathbb{P}} \theta^*$.
By the *Central Limit Theorem*

$$\frac{\sqrt{n}(\hat{\theta}_n - \theta^*)}{\sqrt{\hat{\theta}_n(1-\hat{\theta}_n)}} \to_{\mathcal{D}} W$$

Define $Y_n = \dfrac{\sqrt{n}(\hat{\theta}_n - \theta^*)}{\sqrt{\theta^*(1-\theta^*)}}$ and $Z_n = \dfrac{\sqrt{\theta^*(1-\theta^*)}}{\sqrt{\hat{\theta}_n(1-\hat{\theta}_n)}}$.
By the *Continuous Mapping Theorem* tells us that $Z_n \to_{\mathcal{D}} 1$ and $Z_n \to_{\mathbb{P}} 1$.
Hence, by *Slutsky's Theorem*

$$\frac{\sqrt{n}(\hat{\theta}_n - \theta^*)}{\sqrt{\hat{\theta}_n(1-\hat{\theta}_n)}} = Y_n Z_n \to_{\mathcal{D}} W$$

This gives us random interval

$$\left(\hat{\theta}_n - z_{\alpha/2}\sqrt{\frac{\hat{\theta}_n(1-\hat{\theta}_n)}{n}}, \hat{\theta}_n + z_{\alpha/2}\sqrt{\frac{\hat{\theta}_n(1-\hat{\theta}_n)}{n}}\right)$$

This interval captures θ^* asymptotically (in n) with probability $1-\alpha$.
N.B. $z_\alpha = \Phi^{-1}(1-\alpha)$ where Φ is the cumulative denisty function of a $\text{Normal}(0,1)$.

1.8 The Fisher Information

Remark 8.1 - *Motivation*
In the next part of the content we shall show that given $\mathbf{X}_n \overset{\text{iid}}{\sim} f(\cdot; \theta^*)$ then for sufficiently regular models

i) There exists a lower bound on the achievable performance of any estimate of θ^*.

ii) A scaled & centered sequence of maximum likelihood estimators $\{\hat{\theta}_n(\mathbf{X}_n)\}$ become asymptotically normal as $n \to \infty$.

Remark 8.2 - *Measuring Performance of Estimator*
We measure the performance of an estimator $\hat{\theta}$ in terms of variance, since its mean should be θ^*. Lower variance indicates better performance.

Definition 8.1 - *The Score Function*
Let $\ell(\theta; x) := \ln f(x; \theta)$.
The *Score Function* is a measure of the sensitivity of the likelihood function wrt θ

$$\ell'(\theta; x) := \frac{d}{d\theta}\ell(\theta, ; x) = \frac{\frac{d}{d\theta}\ln f(x;\theta)}{\ln f(x;\theta)} = \frac{\ln L'(\theta; x)}{\ln L(\theta; x)}$$

Remark 8.3 - *θ^* is a turning point of $\ell(\theta; x)$*
Note that under the *Fisher Information Regularity Conitions* we have that $\forall\ \theta \in \Theta$

$$\begin{aligned}
\mathbb{E}(\ell'(\theta; X); \theta) &= \int_S \frac{\frac{d}{d\theta}f(x;\theta)}{f(x;\theta)} f(x;\theta)dx \\
&= \int_S \frac{d}{d\theta} f(x;\theta)dx \\
&= \frac{d}{d\theta}\int_S f(x;\theta)dx \\
&= \tfrac{d}{d\theta}(1) \\
&= 0
\end{aligned}$$

This shows that we expect the derivative to equal 0 at θ^*. Further, this means θ^* is a turning point of the log-likelihood function (hopefully a maximum).

Example 8.1 - *Application of **Remark 6.3***
Let $X \sim \text{Poisson}(\theta)$. Then $f_X(x;\theta) = \frac{\theta^x}{x!}e^{-\theta}\mathbb{1}\{x \in \mathbb{N}\}$.

$$\begin{aligned}
\implies \qquad \ell(\theta; x) &= -\theta + x\ln\theta - \ln x! \\
\implies \qquad \ell'(\theta; x) &= -1 + \tfrac{x}{\theta} \\
\implies \quad \mathbb{E}(\ell'(\theta; X); \theta) &= -1 + \tfrac{\theta}{\theta} \\
&= 0
\end{aligned}$$

Definition 8.2 - *Fisher Information Regularity Conditions*
Let Θ be an open interval in $\mathbb{R}$ and $f(x;\theta)$ be a pmf/pdf.
Below are conditions which a model is required to meet in order to be considered sufficiently regular such that *Fisher Information* can be drawn from it.

i) Both $L'(\theta; x) = \frac{d}{d\theta}f(x;\theta)$ and $L''(\theta; x) = \frac{d^2}{d\theta^2}f(x;\theta)$ exist for any $x \in \mathcal{X}$.

ii) $\forall\ \theta \in \Theta$ the set $S := \{x \in \mathcal{X} :\ f(x;\theta) > 0\}$ does not depend on $\theta \in \Theta$.

iii) The idenity below exists

$$\int_S \frac{d}{d\theta}f(x;\theta)dx = \frac{d}{d\theta}\int_S f(x;\theta)dx = 0$$

Definition 8.3 - *Fisher Information*
Fisher Information is a technique for measuring the amount of information that an observable random variable X carries about an unknown parameter θ upon which the probability of X depends.
Let $X \sim f(\cdots; \theta)$. Then the *Fisher Information* for any $\theta \in \Theta$ is

$$I(\theta) := \mathbb{E}(\ell'(\theta; X)^2; \theta) \geq 0$$

N.B. This is the *Expectation of the score, squared ≡ Second moment of the score.*

Remark 8.4 - *Fisher Information*

i) *Fisher Information* is a function of the parameter, θ, not the data, X.

ii) $I(\theta)$ can be thought of as being the average *information* brought by a single observation X about θ, assuming $X \sim f(\cdot;\theta)$.

iii) Since $\forall\ \theta \in \Theta,\ \mathbb{E}(\ell'(\theta;X);\theta) = 0$ then

$$I(\theta) = \text{Var}(\ell'(\theta;X);\theta)$$

The variance of the score.

Example 8.2 - *Fisher Information of Poisson*
Let $X \sim \text{Poisson}(\theta)$.
From **Example 6.1** we kown that $\ell'(\theta;x) = -1 + \frac{x}{\theta}$. Then

$$\begin{aligned} I(\theta) &= \text{Var}(\ell'(\theta;X);\theta) \\ &= \text{Var}\left(-1 + \tfrac{X}{\theta};\theta\right) \\ &= \text{Var}\left(\tfrac{X}{\theta};\theta\right) \\ &= \tfrac{1}{\theta^2}\text{Var}(X;\theta) \\ &= \tfrac{1}{\theta^2}.\theta \text{ since } X \sim \text{Poisson}(\theta) \\ &= \tfrac{1}{\theta} \end{aligned}$$

Theorem 8.1 - *Alternative Expression of Fisher Information*
Let $f(x;\theta)$ be a pmf/pdf which statisfies the conditions of **Definition 6.2**. If

$$\forall\ \theta \in \Theta \quad \int_{\mathcal{X}} \frac{d^2}{d\theta^2} f(x;\theta)dx = \frac{d}{d\theta}\int_{\mathcal{X}} \frac{d}{d\theta} f(x;\theta)dx$$

Then

$$I(\theta) = -\mathbb{E}\left(\frac{d^2}{d\theta^2}\ell(\theta;X);\theta\right)$$

N.B. $\dfrac{d}{d\theta}\displaystyle\int_{\mathcal{X}} \frac{d}{d\theta} f(x;\theta)dx = 0$ by the regularity conditions.

Proof 8.1 - *Theorem 6.1*
By the *Quotient Rule*

$$\begin{aligned} \tfrac{d^2}{d\theta^2}\ell(\theta;x) &= \frac{d}{d\theta}\frac{\frac{d}{d\theta}f(x;\theta}{f(x;\theta)} \\ &= \frac{\frac{d^2}{d\theta^2}f(x;\theta)}{f(x;\theta)} - \left(\frac{\frac{d}{d\theta}f(x;\theta)}{f(x;\theta)}\right)^2 \end{aligned}$$

Consequently

$$\begin{aligned} \mathbb{E}\left(\tfrac{d^2}{d\theta^2}\ell(\theta;X);\theta\right) &= \int_S \frac{\frac{d^2}{d\theta^2}f(x;\theta)}{f(x;\theta}f(x;\theta)dx - \int_S \left(\frac{\frac{d}{d\theta}f(x;\theta)}{f(x;\theta)}\right)^2 f(x;\theta)dx \\ &= \int_S \frac{d^2}{d\theta^2}f(x;\theta)dx - \int_S \ell'(\theta;x)^2 f(x;\theta)dx \\ &= 0 - \mathbb{E}(\ell'(\theta;X)^2;\theta) \\ &= -I(\theta) \\ \implies \qquad I(\theta) &= -\mathbb{E}\left(\tfrac{d^2}{d\theta^2}\ell(\theta;X);\theta\right) \end{aligned}$$

□

1.9 Efficiency and The Cramer-Rao Bound

Definition 9.1 - *IID Score Function*
Let $\mathbf{X} \overset{\text{iid}}{\sim} f(\cdot;\theta)$ for some $\theta \in \Theta$. Then the *Score Function* is

$$\ell_n'(\theta;\mathbf{x}) := \frac{d}{d\theta}\ell_n(\theta;\mathbf{x}) \text{ where } l_n(\theta;\mathbf{x}) := \ln f_n(\mathbf{x};\theta) = \sum_{i=1}^{n} \ell(\theta;x_i)$$

N.B. $\frac{d}{d\theta} l_n(\theta;\mathbf{x}) = \frac{d}{d\theta}\sum \ell(\theta;x_i) = \sum \ell'(\theta;x_i)$.

Definition 9.2 - *IID Fisher Information*
Let $\mathbf{X} \overset{\text{iid}}{\sim} f(\cdot;\theta)$ for some $\theta \in \Theta$. Then the *Fisher Information* is

$$I_n(\theta) := \mathbb{E}(l_n'(\theta;\mathbf{X})^2;\theta) = \text{Var}(l_n'(\theta;\mathbf{X});\theta)$$

Theorem 9.1 - *Relationship between IID Fisher Information & Fisher Information*
Consider the sitatution where $\forall\ \theta \in \Theta,\ f_n(\mathbf{x};\theta) = \prod_{i=1}^{n} f(x_i;\theta)$. Then

$$\forall\ \theta \in \Theta,\ I_n(\theta) = nI(\theta)$$

Proof 9.1 - *Theorem 7.1*
Let $\mathbf{X} \overset{iid}{\sim} f(\cdot;\theta)$. Then

$$\begin{aligned} I_n(\theta) &= \text{Var}(\ell_n'(\theta;\mathbf{X});\theta) \\ &= \text{Var}\left(\sum_{i=1}^{n} \ell'(\theta;X_i);\theta\right) \\ &= n\text{Var}\left(\sum_{i=1}^{n} \ell'(\theta;X_1);\theta\right) \\ \implies I_n(\theta) &= nI(\theta) \end{aligned}$$

□

Theorem 9.2 - *Cauchy-Schwarz Inequality for Expectation*
Let X & Y be real-valued random variables in the same probability space. Then

$$\mathbb{E}(XY)^2 \leq \mathbb{E}(X^2)\mathbb{E}(Y^2)$$

Proof 9.2 - *Theorem 7.2*
If $\mathbb{E}(Y^2) = 0$ then $\mathbb{P}(Y = 0) = 1$ so $\mathbb{E}(XY) = 0$ and the statement holds.
Thus, assume $\mathbb{E}(Y^2) > 0$ and define $\lambda := \dfrac{\mathbb{E}(XY)}{\mathbb{E}(Y^2)}$. Then

$$\begin{aligned} 0 &\leq \mathbb{E}(X - \lambda Y)^2) \\ &= \mathbb{E}(X^2) - 2\lambda\mathbb{E}(XY) + \lambda^2\mathbb{E}(Y^2) \\ &= \mathbb{E}(X^2) - 2\frac{\mathbb{E}(XY)^2}{\mathbb{E}(Y^2)} + \frac{\mathbb{E}(XY)^2}{\mathbb{E}(Y^2)} \\ &= \mathbb{E}(X^2) - \frac{\mathbb{E}(XY)^2}{\mathbb{E}(Y^2)} \\ \implies \mathbb{E}(XY)^2 &\leq \mathbb{E}(X^2)\mathbb{E}(Y^2) \end{aligned}$$

□

Theorem 9.3 - *Covaraince Inequality*
Let X and Y be real-valued random variables in the same probability space. Then

$$\text{Cov}(X,Y)^2 \leq \text{Var}(X)\text{Var}(Y)$$

Proof 9.3 - *Theorem 7.3*
Let $W = X - \mathbb{E}(X)$ and $Z = Y - \mathbb{E}(Y)$ giving $\mathbb{E}(WZ) = \text{Cov}(X,Y)$, $\mathbb{E}(W^2) = \text{Var}(X)$ and $\mathbb{E}(Z^2) = \text{Var}(Y)$.
By applying the *Cauchy-Schwarz inequality* we get

$$\text{Cov}(X,Y)^2 = \mathbb{E}(WZ)^2 \leq \mathbb{E}(W^2)\mathbb{E}(Z^2) = \text{Var}(X)\text{Var}(Y) \iff \text{Cov}(X,Y)^2 \leq \text{Var}(X)\text{Var}(Y)$$

Remark 9.1 - *Correlation value*
The result in **Theorem 7.3** is the reason why correlation is valued in $[-1,1]$.

$$\text{Corr}(X,Y) = \frac{\text{Cov}(X,Y)}{\sqrt{\text{Var}(X)\text{Var}(Y)}}$$

Theorem 9.4 - *Cramer-Rao Inequality - Scalar Parameter*
Let $\mathbf{X}_n \overset{\text{iid}}{\sim} f(\cdot;\theta)$ and assume the *Fisher Information Regularity Conditions* hold.
Let $\hat{\theta}_n(\cdot)$ be an estimator of θ with expectation $m(\theta) := \mathbb{E}(\hat{\theta}_n(\mathbf{X}_n);\theta)$ which statisfies

$$\forall\ \theta \in \Theta,\ \underbrace{\frac{d}{d\theta}\int \hat{\theta}_n(\mathbf{x}) f_n(\mathbf{x};\theta) d\mathbf{x}}_{\mathbb{E}(\hat{\theta}_n)} = \int \hat{\theta}_n(\mathbf{x}) \frac{d}{d\theta} f_n(\mathbf{x};\theta) d\mathbf{x}$$

Then

$$\forall\ \theta \in \Theta, \quad \text{Var}(\hat{\theta}_n(\mathbf{X});\theta) \geq \frac{m'(\theta)^2}{nI(\theta)}$$

Proof 9.4 - *Theorem 7.4*
We notice that

$$\begin{aligned} m'(\theta) &= \tfrac{d}{d\theta}\mathbb{E}(\hat{\theta}_n(\mathbf{X}_n);\theta) \\ &= \tfrac{d}{d\theta}\textstyle\int_{S^n} \hat{\theta}_n(\mathbf{x}_n) f_n(\mathbf{x}_n;\theta) d\mathbf{x}_n \end{aligned}$$

The clever part of this proof is to observe that

$$\begin{aligned} \text{Var}(\hat{\theta}_n(\mathbf{X}_n);\theta) nI(\theta) &= \text{Var}(\hat{\theta}_n(\mathbf{X}_n);\theta)\text{Var}(\ell_n(\theta;\mathbf{X}_n);\theta) \\ &\geq \text{Cov}(\hat{\theta}_n(X_n), \ell_n'(\theta;\mathbf{X}_n);\theta)^2 \text{ by Covariance Inequality} \end{aligned}$$

Thus

$$\begin{aligned} \text{Cov}(\hat{\theta}_n(X_n), \ell_n'(\theta;\mathbf{X}_n);\theta)^2 &= \mathbb{E}(\hat{\theta}_n(X_n)\ell_n'(\theta;\mathbf{X}_n);\theta) - \mathbb{E}(\hat{\theta}_n(\mathbf{X}_n);\theta)\mathbb{E}(\ell_n'(\theta;\mathbf{X}_n);\theta) \\ &= \mathbb{E}(\hat{\theta}_n(X_n)\ell_n'(\theta;\mathbf{X}_n);\theta) - \mathbb{E}(\hat{\theta}_n(\mathbf{X}_n);\theta) \times 0 \\ &= \mathbb{E}(\hat{\theta}_n(X_n)\ell_n'(\theta;\mathbf{X}_n);\theta) \\ &= \int_{S^n} \hat{\theta}_n(\mathbf{x}_n)\ell_n'(\theta;\mathbf{x}_n) f_n(\mathbf{x}_n;\theta) d\mathbf{x}_n \\ &= \int_{S^n} \hat{\theta}_n(\mathbf{x}_n) \frac{\frac{d}{d\theta} f_n(\mathbf{x}_n;\theta)}{f_n(\mathbf{x}_n;\theta)} f_n(\mathbf{x}_n;\theta) d\mathbf{x}_n \\ &= \int_{S^n} \hat{\theta}_n(\mathbf{x}_n) \tfrac{d}{d\theta} f_n(\mathbf{x}_n;\theta) \\ &= \tfrac{d}{d\theta}\int_{S^n} \hat{\theta}_n(\mathbf{x}_n) f_n(\mathbf{x}_n;\theta) d\mathbf{x}_n \text{ by regularity assumption} \\ &= m'(\theta) \\ \implies \qquad \text{Var}(\hat{\theta}_n(X_n);\theta) nI(\theta) &\geq m'(\theta)^2 \end{aligned}$$

Proposition 9.1 - *Useful result from Cramer-Rao Inequality*
If $\hat{\theta}_n(\mathbf{X}_n)$ is an unbiased estimator (*i.e.* $m(\theta) = \theta$) then

$$\text{Var}(\hat{\theta}_n(\mathbf{X}_n);\theta) = MSE(\hat{\theta}_n(\mathbf{X}_n);\theta) \geq \frac{1}{nI(\theta)}$$

This shows there is a lower bound on the possible performance of an estimator.

Definition 9.3 - *Efficient Estimator*
An *Estimator* is said to be *Efficient* when its variance is equal to the *Cramer-Rao lower bound* $\forall\ \theta^*$.

Example 9.1 - *Efficient Coin Flipping*
Let $\mathbf{X} \overset{\text{iid}}{\sim} \text{Bernoulli}(\theta)$ with $\theta \in [0,1]$, this corresponds to flipping a coin n times and considering each flip the random variable $X : \{H,T\} \to \{0,1\}$ such that $X(H) = 1$ and $X(T) = 0$ with probability distribution such that $\mathbb{P}(X = 1;\theta) = \theta$ and $\mathbb{P}(X = 0;\theta) = 1 - \theta$. We consider the intuitive estimator of θ

$$\hat{\theta}_n := \hat{\theta}_n(\mathbf{X}_n) := \frac{1}{n}\sum_{i=1}^{n} X_i$$

The estimator is unbiased $\forall\ n \in \mathbb{N}$ and its variance is

$$\text{Var}(\hat{\theta}_n;\theta) = \frac{\text{Var}(X_1;\theta)}{n} = \frac{\mathbb{E}(X_1^2;\theta) - \mathbb{E}(X_1;\theta)^2}{n} = \frac{\theta - \theta}{n} = \frac{\theta(1-\theta)}{n}$$

Now we consider the *Cramer-Rao bound*

$$\begin{array}{rrcl} \text{We find} & L(\theta;x) & = & \theta^x(1-\theta)^{1-x} \\ \implies & \ell(\theta;x) & = & x\ln\theta + (1-x)\ln(1-\theta) \\ \implies & \ell'(\theta;x) & = & \frac{x}{\theta} - \frac{1-x}{1-\theta} \\ \implies & \ell''(\theta;x) & = & -\frac{x}{\theta^2} - \frac{1-x}{(1-\theta)^2} \end{array}$$

Thus we can use $I(\theta) = -\mathbb{E}(\ell''(\theta;X);\theta)$

$$\begin{array}{rrcl} \implies & I(\theta) & = & -\mathbb{E}\left(-\frac{X}{\theta^2} - \frac{1-X}{(1-\theta)^2};\theta\right) \\ & & = & \mathbb{E}\left(\frac{X}{\theta^2} + \frac{1-X}{(1-\theta)^2};\theta\right) \\ & & = & \frac{\theta}{\theta^2} + \frac{1-\theta}{(1-\theta)^2} \\ & & = & \frac{1}{\theta} + \frac{1}{1-\theta} \\ & & = & \frac{1}{\theta(1-\theta)} \\ & I_n(\theta) & = & nI(\theta) \text{ Since } X_1, X_2, \dots \text{ are iid} \end{array}$$

The *Cramer-Rao bound* for the variance is

$$\frac{1}{nI(\theta} = \frac{\theta(1-\theta)}{n}$$

Thus our estimator is efficient.

1.10 Asymptotic Distribution of the Maximum Likelihood Estimator

Theorem 10.1 -
Suppose that $\mathbf{X}_n \overset{\text{iid}}{\sim} f(\cdot;\theta^*)$ for some $\theta^* \in \Theta$ and assume that

i) The sequence of maximum likelihood estiamtors $\{\hat{\theta}_n(\mathbf{X}_n)\}$ is consistent;

ii) The *Fisher Information Regularity Conditions* (**Definition 6.2**) hold and $I(\theta^*) = -\mathbb{E}[\ell''(\theta;X);\theta] > 0$.

iii) $\exists\ C(\cdot) : \mathcal{X} \to [0,\infty)$ such that $\mathbb{E}(C(X_1);\theta^*) < \infty$, $\Xi \subset \Theta$ an open set containing θ^* and $\Delta(\cdot) : \Xi \to [0,\infty)$ continuous at 0 st $\Delta(0) == 0$, st $\forall\ \theta, \theta', x \in \Xi^2 \times \mathcal{X}$.

$$|\ell''(\theta;x) - \ell(\theta';x)| \leq C(x)\Delta(\theta - \theta')$$

Then $\forall\ \theta^* \in \Theta$

$$\sqrt{nI(\theta^*)}(\hat{\theta}n(\mathbf{X}_n) - \theta^*) \to_{\mathcal{D}(;\theta^*)} Z \sim \text{Normal}(0,1)$$

Theorem 10.2 -
Under the conditions of **Theorem 8.1**, with $\hat{\theta}_n := \hat{\theta}_n(\mathbf{X})$ the maximum likelihood etimator

$$\ell'_n(\hat{\theta}_n; \mathbf{X}) = \ell'_n(\theta^*; \mathbf{X}) + (\hat{\theta}_n - \theta^8)\{\ell''_n(\theta^*; \mathbf{X}) + R_n\}$$

where $\frac{1}{n}R_n \to_{\mathbb{P}(\cdot;\theta^*)} 0$.

Proof 10.1 - *Theorem 8.1*
By **Theorem 8.2** $\ell'_n(\hat{\theta}_n; \mathbf{X}) = \ell'_n(\theta^*; \mathbf{X}) + (\hat{\theta}_n - \theta^8)\{\ell''_n(\theta^*; \mathbf{X}) + R_n\}$ where $\frac{1}{n}R_n \to_{\mathbb{P}(\cdot;\theta^*)} 0$.
Since $\hat{\theta}_n$ is the maximum likelihood estimator & the *Fisher Information Regularity Conditions* hold, the score at $\ell'(\hat{\theta}_n; X) = 0$.
Hence, $0 = \ell''(\hat{\theta}_n; X) = \ell'_n(\theta; X) + (\hat{\theta}_n - \theta^*)\{\ell''(\theta; X) + R_n\}$.
Rearranging & rescalling by $\sqrt{n}$ gives

$$\sqrt{n}(\hat{\theta}_n - \theta^*) = \frac{\frac{1}{\sqrt{n}}\ell'(\theta^*; X)}{-\frac{1}{\sqrt{n}}\{\ell''(\theta^*; X) + R_n} =: \frac{U_n}{V_n - \frac{R_n}{n}}$$

Recall that $\ell'_n(\theta^*; X) = \sum_{i=1}^{n} \ell'(\theta; X_i)$ and $\ell''_n(\theta^*; X) = \sum_{i=1}^{n} \ell''(\theta^*; X_i)$.
Since $\mathbb{E}(\ell'(\theta^*; X_i); \theta^*) = 0$ and $\text{Var}(\ell'(\theta^*; X_i); \theta^*) = I(\theta^*)$
$\implies U_n \to_{\mathcal{D}(\cdot;\theta^*)} U \sim \text{Normal}(0, I(\theta^*))$ by the *Central Limit Theorem.*
We observed that $V_n \to_{\mathbb{P}(\cdot;\theta^*)} I(\theta^*)$ by the *Weak Law of Large Numbers* since $\mathbb{E}(-\ell''(\theta^*; X_i); \theta^*) = I(\theta^*)$.
It follows that $V_n - \frac{1}{n}R_n \to_{\mathbb{P}(\cdot;\theta^*)} I(\theta^*)$ by *Slutsky's Theorem.*
Using *Slutsky's Theorem* again

$$\sqrt{n}(\hat{\theta}_n - \theta^*) = \frac{U_n}{V_n - \frac{1}{n}R_n} \to_{\mathcal{D}(\cdot;\theta^*)} \frac{\sqrt{I(\theta^*)}}{I(\theta^*)} Z \text{ where } Z \sim \text{Normal}(0,1)$$

We can rewrite this as

$$\sqrt{nI(\theta^*)}(\hat{\theta}_n - \theta^*) \to_{\mathcal{D}(\cdot;\theta^*)} Z \sim \text{Normal}(0,1)$$

Proof 10.2 - *Theorem 8.2*
This is an non-examinable, sketch proof of ***Theorem 8.2****.*
By the regularity conditions and the mean alue theorem

$$\frac{\ell'_n(\theta; \mathbf{x}) - \ell'_n(\theta^*; \mathbf{x})}{\theta - \theta^*} = \ell''_n(\tilde{\theta}; \mathbf{x})$$

for some $\tilde{\theta} \in (\theta, \theta^*)$. Hence, we deduce that

$$\begin{aligned} \ell'_n(\theta; \mathbf{x}) - \ell'_n(\theta^*; \mathbf{x}) &= (\theta - \theta^*)\ell''_n(\tilde{\theta}; \mathbf{x}) \\ &= (\theta - \theta^*)\{\ell''_n(\theta^*; \mathbf{x}) + [\ell''_n(\tilde{\theta}; \mathbf{x}) - \ell_n(\theta^*; \mathbf{x})]\} \\ &= (\theta - \theta^*)\{\ell''_n(\theta; \mathbf{x}) + R_n(\theta, \theta^*, \mathbf{x})\} \end{aligned}$$

Now we replace θ with the maximum likelihood estimator $\hat{\theta}_n := \hat{\theta}_n(\mathbf{X})$. We find

$$\ell'(\hat{\theta}_n; \mathbf{X}) = \ell'_n(\theta^*; \mathbf{X}) + (\hat{\theta}_n - \theta^*)\{\ell''_n(\theta^*; \mathbf{X}) + R_n(\hat{\theta}_n, \theta^*, \mathbf{x}\}$$

and we need to analyse R_n.
Since $\hat{\theta}_n \to_{\mathbb{P}(\cdot;\theta^*)} \theta^*$ we can take n large enough that $\mathbb{P}(\hat{\theta}_n \in \Xi; \theta^*)$ with arbitrarily high probability.

On the event $\{\hat{\theta} \in \Xi\}$ and we have $\{\tilde{\theta}_n \in \Xi\}$ since $\tilde{\theta}_n \in (\hat{\theta}_n, \theta^*)$ and

$$\begin{aligned} |\tfrac{1}{n}R_n| &= \tfrac{1}{n}|\ell_n''(\tilde{\theta}_n; \mathbf{X}) - \ell_n''(\theta^*; \mathbf{X})| \\ &= \frac{1}{n}\left|\sum_{i=1}^{n} \ell''(\tilde{\theta}_n; X_i) - \ell''(\theta^*; X_i)\right| \\ &\leq \frac{1}{n}\sum_{i=1}^{n}\left|\ell''(\tilde{\theta}_n; X_i) - \ell''(\theta^*; X_i)\right| \\ &\leq \Delta(\tilde{\theta}_n - \theta^*)\left\{\frac{1}{n}\sum_{i=1}^{n} C(X_i)\right\} \end{aligned}$$

from the smoothness condition on ℓ''.
From the *Weak Law of Large Numbers*

$$\frac{1}{n}\sum_{i=1}^{n} C(X_i) \to_{\mathbb{P}(\cdot;\theta^*)} \mathbb{E}(C(X_1); \theta^*) < \infty$$

and from the consistency of $\{\hat{\theta}_n\}$ and $\{\tilde{\theta}_n\}$ and continuity of $\Delta(\cdot)$ we have by the *Continuous Mapping Theorem*

$$\Delta(\tilde{\theta}_n - \theta^*) \to_{\mathbb{P}(\cdot;\theta^*)} 0$$

Hence, $\frac{1}{n}R_n \to_{\mathbb{P}(\cdot;\theta^*)} 0$ □

Definition 10.1 - *Asyptically Efficient*
A sequence of estimators $\{\hat{\theta}_n(\mathbf{X})\}$ is *Asymptotically Efficient* if either its mean-squared error converges to the *Cramer-Rao Lower Bound*

$$\forall \theta \in \Theta,\ n\text{MSE}(\hat{\theta}_n(\mathbf{X}_n); \theta) \underset{n\to\infty}{\longrightarrow} \frac{1}{I)\theta)}$$

or $\hat{\theta}_n$ is *Asumptotically Normally Distributed* in the sense of **Theorem 8.1**

$$\forall\ \theta \in \Theta,\ \sqrt{nI(\theta)}(\hat{\theta} - \theta) \to_{\mathcal{D}(\cdot;\theta)} Z$$

N.B. The variance of $\frac{Z}{\sqrt{(nI(theta^*)}}$ is exactly $\frac{1}{nI(\theta)}$.

Theorem 10.3 -
Under the conditions of **Theorem 8.1** the maximum likelihood estimator is *asymptotically efficient.*

Definition 10.2 - *Regular Statistical Model*
Any *Statistical Model* which satisfies the condition of **Theorem 8.1** is a *Regular Statistical Model.*

Remark 10.1 - *Why use MLE over others*
Due to the *Asymptotic Efficieny* of maximum likelihood estimators it is beter to use them in *Regular Statistical Models.*

1.11 Confidence Sets Around the Maximum Likelihood Estimator

Definition 11.1 - *Coverage of an Interval*
Let $\mathbf{X} \sim f_n(\cdot; \theta),\ \theta \in \Theta = \mathbb{R},\ L(\cdot) : \mathcal{X}^n \to \Theta$ and $U(\cdot) : \mathcal{X}^n \to \Theta$ where $\forall\ \mathbf{x} \in \mathcal{X}^n,\ L(\mathbf{x}) < U(\mathbf{x})$.
Then, $\forall \theta \in \Theta$ the coverage $C_{\mathcal{I}}(\theta)$ of the random interval $\mathcal{I}(\mathbf{X}) := [L(\mathbf{X}), U(\mathbf{X})]$ at θ is

$$C_{\mathcal{I}}(\theta) := \mathbb{P}(\theta \in [L(\mathbf{X}), U(\mathbf{X})]; \theta) = \mathbb{P}(L(\mathbf{X}) \leq \theta \leq U(\mathbf{X}); \theta)$$

Remark 11.1 - *Coverage of an Interval in Words*
$C_{\mathcal{I}}(\theta)$ is the probabiltiy that the deterministic quantity θ falls into the random interval $\mathcal{I}(\mathbf{X})$ under the probability distribution $\mathbb{P}(\cdot;\theta)$ wher $\mathbf{X} \sim f_n(\cdot;\theta)$.

Remark 11.2 - *Multi-Dimensional Coverage*
We can extend *Coverage of an Interval* to the multi-dimensional case by considering confidence sets and then considering the probability $\mathbb{P}(\theta \in \mathcal{I}(\mathbf{X});\theta)$.

Definition 11.2 - *Confidence Interval*
$\forall\ \alpha \in [0,1]$ we say that an inerval $\mathcal{I}(\mathbf{X}) := [L(\mathbf{X}), U(\mathbf{X})]$ is a $1-\alpha$ confidence interval if $\forall\ \theta \in \Theta$ its coverage is at least $1-\alpha$ or more formally $\inf_{\theta\in\Theta} C_{\mathcal{I}}(\theta) \geq 1-\alpha$.

Remark 11.3 - *Exact Confidence Interval*
If $C_{\mathcal{I}}(\theta) = 1-\alpha\ \forall\ \theta \in \Theta$ then $\mathcal{I}$ is an exact $1-\alpha$ confidence interval.

Definition 11.3 - *Observed Confidence Interval*
For an interval $\mathcal{I}(\cdot) = [L(\cdot), U(\cdot)]$ with $L : \mathcal{X}^n \to \Theta$ and $U : \mathcal{X}^n \to \Theta$, and a realisation $\mathbf{x}$, the corresponding *Observed Confidence Interval* is $\mathcal{I}(\mathbf{x})$.
N.B. Nothing interesting can be said about the probability that $\theta \in \mathcal{I}(\mathbf{x})$ since θ and $\mathcal{I}(\mathbf{x})$ are deterministic.

Notation 11.1 - *Quantile of Normal*$(0,1)$
For any $\beta \in (0,1)$ let $z_\beta \in \mathbb{R}$ be such that for $Z \sim \text{Normal}(0,1)$, $1-\Phi(z_\beta) = \mathbb{P}(Z > z_\beta) = \beta$.

Example 11.1 - *Confidence interval for the mean of a Normal Distribution*
Let $\mathbf{X} \overset{\text{iid}}{\sim} \text{Normal}(\mu, \sigma^2)$ for $\theta = (\mu, \sigma^2) \in \mathbb{R} \times \mathbb{R}^{\geq 0}$ and wher σ^2 is known.
Consider the estimator $\hat{\mu}_n = \hat{\mu}_n(\mathbf{X}) = \frac{1}{n}\sum_{i=1}^n X_i$ of μ. Then we know that the following non-asymptotic result holds.
We have $\frac{1}{n}\sum_{i=1}^n X_i \sim \text{Normal}(\mu, \frac{\sigma^2}{n}$. Thus

$$\frac{\frac{1}{n}\sum_{i=1}^n X_i - \mu}{\sqrt{\sigma^2/n}} \sim \text{Normal}(0,1)$$

Then

$$\begin{aligned}
\forall\ \alpha \in (0,1) \quad , \quad & \mathbb{P}\left(z_{1-\alpha/2} \leq \frac{\hat{\mu}_n(\mathbf{X}) - \mu}{\sqrt{\sigma^2/n}} \leq z_{\alpha/2}; \mu \right) \\
= \ & \mathbb{P}\left(\frac{\hat{\mu}_n(\mathbf{X}) - \mu}{\sqrt{\sigma^2/n}} \leq z_{\alpha/2} \right) - \mathbb{P}\left(\frac{\hat{\mu}_n(\mathbf{X}) - \mu}{\sqrt{\sigma^2/n}} \leq z_{1-\alpha/2} \right) \\
= \ & \left(1 - \tfrac{\alpha}{2}\right) - \left(1 - \left(1 - \tfrac{\alpha}{2}\right)\right) \\
= \ & 1 - \alpha
\end{aligned}$$

By symmetry we notice that $z_{1-\frac{\alpha}{2}} = -z_\alpha 2$.
By rearranging we have the equivalence of events

$$\left\{ -z_{\alpha/2} \leq \frac{\hat{\mu}_n(\mathbf{X}) - \mu}{\sqrt{\sigma^2/n}} \leq z_{\alpha/2} \right\} = \left\{ \hat{\mu}_n(\mathbf{X}) - z_{\alpha/2}\frac{\sigma}{\sqrt{n}} \leq \mu \leq \hat{\mu}_n(\mathbf{X}) + z_{\alpha/2}\frac{\sigma}{\sqrt{n}} \right\}$$

To rearrange we separate into two events & treat then seperately

$$\begin{aligned}
\left\{ \frac{\hat{\mu}_n(\mathbf{X}) - \mu}{\sigma/\sqrt{n}} \leq z_{\alpha/2} \right\} &= \left\{ \frac{\hat{\mu}_n(\mathbf{X})}{\sigma/\sqrt{n}} - z_{\alpha/2} \leq \frac{\mu}{\sigma/sqrtn} \right\} \\
&= \left\{ \mu \geq \hat{\mu}_n(\mathbf{X}) - \tfrac{\sigma}{\sqrt{n}} z_{\alpha/2} \right\}
\end{aligned}$$

Similarly

$$\begin{aligned}\left\{-z_{\alpha/2} \leq \frac{\hat{\mu}_n(X)-\mu}{\sqrt{\sigma^2/n}}\right\} &= \left\{\frac{\mu}{\sigma/\sqrt{n}} \leq \frac{\hat{\mu}_n(X)}{\sigma/\sqrt{n}} + z_{\alpha/2}\right\} \\ &= \left\{\mu \leq \hat{\mu}_n(X) + z_{\alpha/2}\frac{\sigma}{\sqrt{n}}\right\}\end{aligned}$$

So the interval $\mathcal{I}(X) = [L(X), U(X)]$ where $L(\mathbf{x}) = \bar{x} - z_{\alpha/2}\frac{\sigma}{\sqrt{n}}$ and $U(\mathbf{x}) = \bar{x} + z_{\alpha/2}\frac{\sigma}{\sqrt{n}}$ is an $1-\alpha$ exact confidence interval.

Remark 11.4 - *Confidence Intervals with unknown σ^2*
When σ^2 is unknown we can defined $\{\hat{\sigma}_n^2\}_{n\in\mathbb{N}}$ to be a consistent sequence of estimators of σ^2 (*e.g.* the sample variance)

$$\hat{\sigma}_n^2 := \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \hat{\mu}_n(\mathbf{X}))^2$$

1.12 Asymptotic Approximation of Confidence Intervals

Theorem 12.1 -
Assume $\mathbf{X} \sim f(\cdot;\theta^*)$. Let $\{\hat{\theta}_n\}_{n\in\mathbb{N}}$ be a consistent sequence of estimators of θ^* and assume that $\{\hat{\theta}_n\}$ is asymptotically normal in the sense that

$$\exists\ \sigma^2 > 0 \text{ st } \frac{\hat{\theta}_n(\mathbf{X}) - \theta^*}{\sqrt{\sigma^2/n}} \to_{\mathcal{D}(\cdot;\theta^*)} Z \sim \text{Normal}(0,1)$$

Then $\forall\ \alpha \in (0,1),\ \mathcal{I}_n(\mathbf{X}) - [L_n(\mathbf{X}), U_n(\mathbf{X})]$ is an asymptotically exact $1-\alpha$ condifence interval, where $L_n(\mathbf{x}) := \hat{\theta}_n(\mathbf{x}) - z_{\alpha/2}\frac{\sigma}{\sqrt{n}}$ and $U_n(\mathbf{x}) := \hat{\theta}(\mathbf{x}) + z_{\alpha/2}\frac{\sigma}{\sqrt{n}}$.

Proof 12.1 - *Theorem 10.1*
Let $\{W_n\}_{n\in\mathbb{N}}$ be defined by $W_n := \frac{\hat{\theta}_n(X)-\theta^*}{\sqrt{\sigma^2/n}}$.
Since $W_n \to_{\mathcal{D}(\cdot;\theta^*)} Z \sim \text{Normal}(0,1)$ we have

$$\begin{aligned}\mathbb{P}(-z_{\alpha/2} \leq W_n \leq z_{\alpha/2}) &= F_{W_n}(z_{\alpha/2}) - F_{W_n}(-z_{\alpha/2}) \\ &\underset{n\to\infty}{\longrightarrow} \Phi(z_{\alpha/2}) - \Phi(-z_{\alpha/2}) \\ &= 1-\alpha\end{aligned}$$

Similary to before we have the equivalence of events

$$\{-z_{\alpha/2} \leq W_n \leq z_{\alpha/2}\} = \left\{\hat{\theta}_n - z_{\alpha/2}\frac{\sigma}{\sqrt{n}} \leq \theta^* \leq \hat{\theta}_n + z_{\alpha/2}\frac{\sigma}{\sqrt{n}}\right\}$$

So $\lim_{n\to\infty}\mathbb{P}\left(\hat{\theta}_n(X) - z_{\alpha/2}\frac{\sigma}{\sqrt{n}} \leq \theta^* \leq \hat{\theta}_n(X) + z_{\alpha/2}\frac{\sigma}{\sqrt{n}};\theta^*\right) = 1-\alpha$
Remark 12.1 - *Theorem 10.1*
The confidence interval is only asymptotically exact. For finite n, the overage of the confidence interval will be different from $1-\alpha$ but the difference will converge to 0 as n increases. In practice σ^2 may be unknown, in these cases substitute for a consistent sequence of estimators of σ^2.
Theorem 12.2 -
Assum $\mathbf{X} \sim f(\cdot;\theta^*)$ let $\{\hat{\theta}_n\}_{n\in\mathbb{N}}$ be a consistent sequence of estimators of θ^* and assume that $\{\hat{\theta}_n\}$ is asymptotically normal in the sense that

$$\exists\ \sigma^2 > 0 \text{ st } \frac{\hat{\theta}_n(\mathbf{X}) - \theta^*}{\sqrt{\sigma^2/n}} \to_{mathcalD(\cdot;\theta^*)} Z \sim \text{Normal}(0,1)$$

Assume also that $\{\hat{\sigma}_n^2\}_{n\in\mathbb{N}}$ is a consistent sequence of estimators of σ^2. Then $\forall\ \alpha \in (0,1)$, $\mathcal{I}_n(\mathbf{X}) = [L_n(\mathbf{X}), U_n(\mathbf{X})]$ is an asymptotically exact $1-\alpha$ confidence interval, where $L_n(\mathbf{x}) := \hat{\theta}_n(\mathbf{x}) - z_{\alpha/2}\sqrt{\hat{\sigma}_n^2(\mathbf{x})/n}$ and $U_n(\mathbf{x}) := \hat{\theta}_n(\mathbf{x}) + z_{\alpha/2}\sqrt{\hat{\sigma}_n^2(\mathbf{x})/n}$.

Proof 12.2 - *Theorem 10.2*
Define $W_n := \frac{\hat{\theta}_n - \theta^*}{\sqrt{\hat{\sigma}_n^2(X)/n}} = \frac{\hat{\theta}_n(X)-\theta^*}{\sqrt{\sigma^2/n}} - \sqrt{\frac{\sigma^2}{\hat{\sigma}_n^2(X)}}$.
By consistency of $\{\hat{\sigma}_n^2\}_{n\in\mathbb{N}}$ and the *Continuous Mapping Theorem*

$$\sqrt{\frac{\sigma^2}{\hat{\sigma}_n^2(X)}} \to_{\mathbb{P}(\cdot;\theta^*)} 1$$

By *Slutsky's Theorem*

$$W_n \to_{\mathcal{D}(\cdot;\theta^*)} Z \sim \text{Normal}(0,1)$$

The rest of the proof is the same as for **Theorem 10.1**.

Remark 12.2 - *Theorem 10.2*
For a given n the quality of the normal approximatioin will be affected by this additional approximation. One may find that for less accurate estimators of σ^2, the n required for the confidence interval to have almost the right coverage will be higher.

1.13 Estimating the Information for Maximum Likelihood Estimates

Remark 13.1 - *Applying Theorem 10.2 to sequences of MLEs for regular statistical models*
When dealing with *Maximum Likeihood Estimators* for regular statistical models we have that $\sigma^2 = 1/I(\theta^*)$ thus

$$\sqrt{nI(\theta^*)}(\hat{\theta}_n - \theta^*) \to_{\mathcal{D}(\cdot;\theta^*)} Z \sim \text{Normal}(0,1)$$

However the *Fisher Information* is unknown so we consider two cases

i) When the expectation, $I(\theta^*) = -\mathbb{E}(\ell''(\theta^*; X_1); \theta^*)$, can be calculated. In this case we replace θ^* with $\hat{\theta}_n$ in the equation.

ii) When the expectation **cannot** be calcualted we invoke the *Weak Law of Large Numbers* and onsider the sequence of estimators, $J_n(\hat{\theta}_n) := -\frac{1}{n}\sum_{i=1}^n \ell''(\hat{\theta}_n; X_i)$.

Theorem 13.1 - *Case i)*
Assume $\{\hat{\theta}_n\}$ is a sequence of *Maximum Likelihood Estimators* st $\hat{\theta}_n \to_{\mathbb{P}(\cdot;\theta^*)} \theta^*$ and I is a continuous function of θ. Then $I(\hat{\theta}_n) \to_{\mathbb{P}(\cdot;\theta^*)} I(\theta^*)$.
N.B. The proof of this follows directly from the *Continous Mapping Function.*

Remark 13.2 - *Theorem 11.1*
It is only necessary for I to be continuous in the neighbourhood of θ^*. This is due to an extension of the *Continuous Mapping Theorem* that states

$$\text{If } X_n \to_{\mathbb{P}} X \text{ and } g \text{ is a function with discontinuity set } D \text{ then}$$
$$\mathbb{P}(X \in D) = 0 \implies (X_n) \to_{\mathbb{P}} g(X).$$

Theorem 13.2 - *Case ii)*
Assume that $\{\hat{\theta}_n\}$ is a sequence of *Maximum Likelihood Estimators* st

i) $\hat{\theta}_n \to_{\mathbb{P}(\cdot;\theta^*)} \theta^*$;

ii) $I(\theta) = -\mathbb{E}(\ell''(\theta; X); \theta)\ \forall\ \theta \in \Theta$

iii) $\exists\, C : \mathcal{X} \to [0,\infty)$ st $\mathbb{E}(C(X_1);\theta^*) < \infty$, $\Xi \subset \Theta$ is an open set containing θ^* and $\Delta(\cdot) : \Xi \to [0,\infty)$ is continuous at 0 st $\Delta(0) = 0$, and st $\forall\ \theta, \theta^*, x \in \Xi^2 \times \mathcal{X}$ $|\ell''(\theta; x) - \ell''(\theta'; x)| \leq C(x)\Delta(\theta - \theta')$

Then

$$J_n(\hat{\theta}_n) \to_{\mathbb{P}(\cdot;\theta^*)} I(\theta^*)$$

Proof 13.1 - *Theorem 11.2*
Consider the following decomposition

$$\begin{aligned} J_n(\hat{\theta}) - I(\theta^*) &= -\tfrac{1}{n}\sum_{i=1}^{n} \ell''(\hat{\theta}_n; X_i) - I(\theta^*) \\ &= T_1 + T_2 \\ \text{Where } T_1 &= -\tfrac{1}{n}\sum_{i=1}^{n} \ell''(\hat{\theta}_n; X_i) + \tfrac{1}{n}\sum_{i=1}^{n} \ell''(\theta^*; X_i) \\ \text{and } T_2 &= -\left\{\tfrac{1}{n}\sum_{i=1}^{n} \ell''(\theta^*; X_i\right\} - I(\theta^*) \end{aligned}$$

Now the first term can be upper bounded as follows (for sufficiently large n, with arbitrary large probabiltiy the second inequality holds)

$$\begin{aligned} |T_1| &= \left|-\tfrac{1}{n}\sum_{i=1}^{n} \ell''(\hat{\theta}); X_i) + \tfrac{1}{n}\sum_{i=1}^{n} \ell''(\theta^*; X_i)\right| \\ &\leq \tfrac{1}{n}\sum_{i=1}^{n} \left|\ell''(\hat{\theta}_n; X_i) - \ell''(\theta^*; X_i)\right| \\ &\leq \Delta(\theta\theta_n - \theta^*)\tfrac{1}{n}\sum_{i=1}^{n} C(X_i) \end{aligned}$$

By the *Weak Law of Large Numbers*

$$\frac{1}{n}\sum_{i=1}^{n} C(X_i) \to_{\mathbb{P}(\cdot;\theta^*)} \mathbb{E}(C(X_1);\theta^*)$$

by the assumed consistency of $\{\hat{\theta}_n\}_{n\in\mathbb{N}}$ and continuity of Δ we have that

$$\Delta(\hat{\theta}_n - \theta^*) \to_{\mathbb{P}(\cdot;\theta^*)} 0$$

Consequently $T_1 \overset{n\to\infty}{\longrightarrow}_{\mathbb{P}(\cdot;\theta^*)} 0$.
By the *Weak Law of Large Numbers* we have

$$\begin{aligned} & -\tfrac{1}{n}\sum_{i=1}^{n} \ell''(\theta^*; X_i) \quad \to_{\mathbb{P}(\cdot;\theta^*)} \quad I(\theta^*) \\ \implies \quad T_2 = -\tfrac{1}{n}\sum_{i=1}^{n} \ell''(\theta^*; X_i) - I(\theta^*) \quad &\to_{\mathbb{P}(\cdot;\theta^*)} \quad 0 \end{aligned}$$

Since $T_1 \overset{n\to\infty}{\longrightarrow}_{\mathbb{P}(\cdot;\theta^*)} 0$ and $T_2 \overset{n\to\infty}{\longrightarrow}_{\mathbb{P}(\cdot;\theta^*)} 0$ we deduce from the earlier decomposition that

$$J_n(\hat{\theta}_n) \to_{\mathbb{P}(\cdot;\theta^*)} I(\theta^*)$$

□

Remark 13.3 - *Summary*
Whenever **Theorem 8.1** holds for a sequence of *Maximum Likelihood Estimators*

$$\textit{i.e. } \sqrt{nI(\theta^*)}(\hat{\theta}_n - \theta^*) \to_{\mathcal{D}(\cdot;\theta^*)} Z \sim \text{Normal}(0,1)$$

we can replace $I(\theta^*)$ with one of two options

i) $I(\hat{\theta}_n)$ whenever

(a) $I(\theta)$ is continuous in a neighbourhood of θ^*; and,

(b) The interval $[L(\mathbf{X}), U(\mathbf{X})]$ with $L(\mathbf{x}) := \hat{\theta}_n - z_{\alpha/2}\sqrt{nI(\hat{\theta})n)}$ and $U(\mathbf{x}) := \hat{\theta}_n + z_{\alpha/2}\sqrt{nI(\hat{\theta})n)}$ is an asymptotically exact $1-\alpha$ confidence interval for $\theta*$.

ii) $J_n(\hat{\theta}_n) := -\frac{1}{n}\sum_{i=1}^{n} \ell''(\hat{\theta}_n; X_i)$ whenever

(a) The assumptions of **Theorem 11.2** hold; and,

(b) The interval $[L(\mathbf{X}), U(\mathbf{X})]$ with $L(\mathbf{x}) := \hat{\theta}_n - z_{\alpha/2}\sqrt{nJ_n(\hat{\theta}_n}$ and $U(\mathbf{x}) := \hat{\theta}_n + z_{\alpha/2}\sqrt{nJ_n(\hat{\theta}_n}$ is an asymptotically exact $1-\alpha$ confidence interval for θ^*

Example 13.1 - *Coin Flipping*
Here the new results for this chapter are applied in order to simplfy methods used in previous examples when finding confidence intervals & upper bounds on θ^*.
The sequence of estimators $\hat{\theta}_n := \frac{1}{n}\sum_{i=1}^{n} X_i$ is consistent by the *Weak Law of Large Numbers* and the conditions for asymptotic normality hold $\forall\ \theta \in \Theta$. Hence

$$\sqrt{nI(\theta^*)}(\hat{\theta}_n - \theta^*) \to_{\mathcal{D}(\cdot;\theta^*)} Z \sim \text{Normal}(0,1)$$

We can compute the *Fisher Information* $\forall\ \theta \in \Theta$. We have

$$\begin{array}{rrcl} & \ell'(\theta(x) & = & \frac{x}{\theta} - \frac{1-x}{1-\theta} \\ \text{and} & \ell''(\theta; x) & = & -\frac{x}{\theta^2} - \frac{1-x}{(1-\theta)^2} \\ \implies & I(\theta) & = & \frac{1}{\theta} + \frac{1}{1-\theta} \\ & & = & \frac{1}{\theta(1-\theta)} \end{array}$$

In practice θ^* is unknown so we replace $I(\theta^*)$ with $I(\hat{\theta}_n)$ to give the asymptotically exact confidence interval, $[L(\mathbf{X}), U(\mathbf{X})]$ where

$$L(\mathbf{X}) = \hat{\theta}_n - z_{\alpha/2}\sqrt{\frac{\hat{\theta}_n(1-\hat{\theta}_n)}{n}} \text{ and } U(\mathbf{X}) = \hat{\theta}_n + z_{\alpha/2}\sqrt{\frac{\hat{\theta}_n(1-\hat{\theta}_n)}{n}}$$

If we did not know how to computer $I(\theta)$ we could instead compute

$$\begin{array}{rcl} J_n(\hat{\theta}_n) & = & -\frac{1}{n}\sum_{i=1}^{n} \ell''(\hat{\theta}_n; X_i) \\ & = & -\frac{1}{n}\sum_{i=1}^{n} \left\{ -\frac{X_i}{\hat{\theta}_n^2} - \frac{1-X_i}{(1-\hat{\theta}_n)^2} \right\} \\ & = & \frac{1}{\hat{\theta}_n^2}\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right) + \frac{1}{(1-\hat{\theta}_n)^2}\left(1 - \frac{1}{n}\sum_{i=1}^{n} X_i\right) \\ & = & \frac{\hat{\theta}_n}{\hat{\theta}_n^2} + \frac{1-\hat{\theta}_n}{(1-\hat{\theta}_n)^2} \\ & = & \frac{1}{\hat{\theta}_n(1-\hat{\theta}_n)} \end{array}$$

In this case $J_n(\hat{\theta}_n) = I(\hat{\theta}_n)$, this is not always true.

Definition 13.1 - *Observed Fisher Information*
Let $\mathbf{X} \overset{\text{iid}}{\sim} f(\cdot;\theta^*)$ be a vector of n random variables.
The *Observed FIsher Information* at θ is

$$nJ_n(\theta) = -\ell''(\theta; \mathbf{X}) = -\sum_{i=1}^{n} \ell''(\theta; X_i)$$

N.B. $\mathbb{E}(J_n(\theta^*);\theta^*) = I(\theta^*)$ and that it differs from the *Fisher Information* (under the *Fisher Information Regularity Conditions* by not being an expectation.

1.14 Transformations and Confidence Intervals

Definition 14.1 - *Wald Approach*
The confidence intevals seen so far fit the *Wald Approach.*
If $\mathbf{X} \overset{\text{iid}}{\sim} f(\cdot;\theta^*)$ where $\theta^* \in \Theta \subset \mathbb{R}$ then one can define a confidence interval for θ^* using the asymptotic distribution of the *Maximum Likelihood Estimator*

$$L(\mathbf{x}) = \hat{\theta}_n - z_{\alpha/2}\sqrt{nI(\theta^*)} \text{ and } U(\mathbf{x}) = \hat{\theta}_n + z_{\alpha/2}\sqrt{nI(\theta^*)}$$

which ensures that as $n \to \infty$, $\mathbb{P}(\theta^* \in [L(\mathbf{X}), U(\mathbf{X})] \to 1-\alpha$.

Proposition 14.1 - *Transfomed Confidence Interval - Increasing*
Let $\tau := g(\theta)$ be a bijective, continously differentiable & increasing function.
This gives a direct transformation of $[L(\mathbf{x}), U(\mathbf{x})]$ to $[g(L(\mathbf{x})), g(U(\mathbf{x}))]$.

$$\textit{i.e. } \{\mathbf{x} \in \mathcal{X}^n : L(\mathbf{x}) \leq \theta^* \leq U(\mathbf{x})\} = \{\mathbf{x} \in \mathcal{X}^n : g(L(\mathbf{x})) \leq \tau^* \leq g(U(\mathbf{x}))\}$$

Consequently

$$\begin{aligned} \mathbb{P}(\theta^* \in [L(\mathbf{X}), U(\mathbf{X})]; \theta^*) &= \mathbb{P}(\tau^* \in [g(L(\mathbf{X})), g(U(\mathbf{X}))] \\ &\to 1-\alpha \text{ as } n \to \infty \end{aligned}$$

i.e. $[g(L(\mathbf{X})), g(U(\mathbf{X}))]$ is an asymptotically exact $1-\alpha$ for τ^*.

Proposition 14.2 - *Trasformed Confidence Interval - Decreasing*
Let $\tau := g(\theta)$ be a bijective, continously differentiable & decreasing function.
This gives a direct transformation of $[L(\mathbf{X}), U(\mathbf{X})]$ to $[g(U(\mathbf{X})), g(L(\mathbf{X}))]$ which is an asymptotically exact $1-\alpha$ confidence interval for τ^*.

Remark 14.1 - *Deriving Reparametised Confidence Intervals*
We can obtain a reparametised *Confidence Interval* by working with the reparameterised likelihood, $\tilde{f}(\mathbf{x};\tau) := f(\mathbf{x}; g^{-1}(\tau))$. Now we can find $\tilde{L}(\mathbf{x})$ and $\tilde{U}(\mathbf{x})$ directly.

Theorem 14.1 -
Assume $X \in f(\cdot;\theta)$ for $\theta \in \Theta \subseteq \mathbb{R}$ and let $\tau := g(\theta)$ where g is bijective & continuosly diferentiable.
The *Fisher Infromatoin* for the parameterisation $\tilde{f}(x;\tau) := f(x; g^{-1}(\tau))$ is

$$\tilde{I}(\tau) = \frac{I(\theta)}{g'(\theta)^2}$$

Proof 14.1 - *Theorem 12.1*
Since $\tilde{f}(x;\tau) = f(x; g^{-1}(\tau))$ the log-likelihood for *tau* is

$$\tilde{\ell}(\tau; x) = \ln \tilde{f}(x;\tau) = \ln f(x; g^{-1}(\tau))$$

The score is therefore

$$\begin{aligned} \tilde{\ell}'(\tau; x) &= \tfrac{d}{d\tau} \ln f(x; g^{-1}(\tau)) \\ &= \tfrac{d}{d\theta} \ln f(x; g^{-1}(\tau)) \times \tfrac{d}{d\tau} g^{-1}(\tau) \\ &= \ell'(g^{-1}(\tau); x) \times \frac{1}{g'(g^{-1}(\tau))} \\ &= \frac{\ell'(\theta; x)}{g'(\theta)} \end{aligned}$$

No we use the definition of *Fisher Information*

$$\begin{aligned}\tilde{I}(\tau) &= \mathbb{E}(\tilde{\ell}'(\tau;X)^2;\tau)\\ &= \mathbb{E}\left(\frac{\ell'(\theta;X)^2}{g'(\theta)^2};\theta\right)\\ &= \frac{1}{g'(\theta)^2}\mathbb{E}\left(\ell'(\theta;X)^2;\theta\right)\\ &= \frac{I(\theta)}{g'(\theta)^2}\end{aligned}$$

Remark 14.2 -
As a consequence, for regular statistical models

$$\sqrt{n\tilde{I}(\tau^*)}(\hat{\tau}_n-\tau^*)\to_{\mathcal{D}(\cdot;\tau^*)} Z\sim \text{Normal}(0,1)$$

is equivalent to

$$\sqrt{\frac{nI(\theta^*)}{g'(\theta^*)^2}}(\hat{\tau}_n-\tau^*)\to_{\mathcal{D}(\cdot;\theta^*)} Z\sim \text{Normal}(0,1)$$

which leads to

$$\begin{aligned}\tilde{L}(\mathbf{x}) &= \hat{\tau}_n - z_{\alpha/2}\sqrt{\frac{g'(\theta^*)^2}{nI(\theta^*)}}\\ \tilde{U}(\mathbf{x}) &= \hat{\tau}_n + z_{\alpha/2}\sqrt{\frac{g'(\theta^*)^2}{nI(\theta^*)}}\end{aligned}$$

N.B. This is not necessarily the same *Confidence Interval* as obtained by transforming $[L(\mathbf{x}),U(\mathbf{x})]$ directly.

Example 14.1 -
Consider $\mathbf{X}\overset{\text{iid}}{\sim}\text{Normal}(\mu,1)$.
We know that the *Maximum Likelihood Estiamtor* of μ is $\bar{X}\sim\text{Normal}\left(\mu,\frac{1}{n}\right)$.
A $1-\alpha$ *Confidence Interval* for μ is

$$\left[\bar{X}-\frac{z_{\alpha/2}}{\sqrt{n}},\bar{X}+\frac{z_{\alpha/2}}{\sqrt{n}}\right]$$

Consider the parameterisation $\tau=\frac{1}{\mu}$. This corresponds to $g(x)=\frac{1}{x}$ which is bijective & continuously differentiable except at 0, and is decreasing.
Hence, a $1-\alpha$ exact *Confidence Interval* for τ is

$$\left[\frac{1}{\bar{X}+z_{\alpha/2}/\sqrt{n}},\frac{1}{\bar{X}-z_{\alpha/2}/\sqrt{n}}\right]$$

Consider the two ways to find an asymptotically $1-\alpha$ *Exact Confidence Interval* for τ. After direct calculations we find that

$$\tilde{\ell}''(\tau;x)=-\frac{3}{\tau^4}+\frac{2x}{\tau^3}$$

So

$$\tilde{I}(\tau):=i\mathbb{E}(\tilde{\ell}''(\tau;X);\tau)=\frac{3}{\tau^3}-\frac{2}{\tau^4}=\frac{1}{\tau^4}$$

Noting that the *Maximum Likelihood Estimator* for τ is $1/\bar{X}$ we find that

$$\sqrt{\frac{n}{\tau^4}}\left(\frac{1}{\bar{X}}-\tau\right)\to_{\mathcal{D}(\cdot;\tau)} Z\sim\text{Normal}(0,1)$$

so an asumptotically exact $1-\alpha$ *Confidence Interval* is

$$\left[\frac{1}{\bar{X}} - z_{\alpha/2}\frac{\tau^2}{\sqrt{n}}, \frac{1}{\bar{X}} + z_{\alpha/2}\frac{\tau^2}{\sqrt{n}}\right]$$

Alternatively, instead of working out $\tilde{I}(\tau)$ as above, we could use **Theorem 12.1** to find that

$$\tilde{I}(\tau) = \frac{I(\theta)}{g'(\theta)^2}, \quad \theta = g^{-1}(\tau) = \frac{1}{\tau}$$

Since $I(\theta) = 1$ and $g(\theta) = 1/\theta \implies g'(\theta) = -1/\theta^2 = -\tau^2$, we have

$$\tilde{I}(\tau) = \frac{1}{(-1/\theta^2)^2} = \frac{1}{(-\tau^2)^2} = \frac{1}{\tau^4}$$

Remark 14.3 - *Example 12.1*

i) The transformed *Confidence Interval* is exact, which the second *Confidence Interval* is not since $\sqrt{n/\tau^4}\left(\frac{1}{\bar{X}} - \tau\right)$ is not exactly normally distributed, but only asymptotically so.

ii) The transformed *Confidence Interval* is not generally centred at $\hat{\tau}$.

iii) This serves as an example that convergence in distribution says nothing about convergence of moments. In particular, you can derive that $\frac{1}{\bar{X}}$ does not have a mean for any $\mu \in \mathbb{R}$.

1.15 Likelihood Ratio Confidence Sets - Wilk's Approach

Remark 15.1 - *Motivation*
Consider a *Wald Confidence Interval* $\mathcal{I}(\theta^*)$.
It is possible for some $\theta \notin \mathcal{I}(\theta^*)$ to have a greater likelihood interval than some $\theta' \in \mathcal{I}(\theta^*)$. It is possible $\exists\ \theta \in \mathcal{I}(\theta^*)$ st $L(\theta; \mathbf{x}) = 0$.
Wald Confidence Intervals are not invariant under reparameterisation.
These features of *Wald Confidence Intervals* motivate why we may wish to consider a different type of *Confidence Interval*.

Definition 15.1 - *Likelihood Ratio*
Define $\mathbf{X} \overset{\text{iid}}{\sim} f(\cdot; \theta^*)$ for some $\theta^* \in \Theta$, let $\{\hat{\theta}_i\}$ be a sequence of consitent *Maximum Likelihood Estimators* of $\theta^* \in \Theta$.
Define $\forall\ \mathbf{x} \in \mathcal{X}^n$ the *Likelihood Ratio*

$$\Lambda_n(\mathbf{x}) := \frac{L(\theta^*; \mathbf{x})}{L(\hat{\theta}_n; \mathbf{x})} \in [0, 1]$$

Theorem 15.1 -
Define $\mathbf{X} \overset{\text{iid}}{\sim} f(\cdot; \theta^*)$ for some $\theta^* \in \Theta$, let $\{\hat{\theta}_i\}$ be a sequence of consitent *Maximum Likelihood Estimators* of $\theta^* \in \Theta$ and assume that the conditions of **Theorem 8.1** hold (implying asymptotic normality). Then

$$-2\ln\Lambda_n(\mathbf{X}_n) \to_{\mathcal{D}(\cdot;\theta^*)} Z^2 \sim \chi^2_1$$

Remark 15.2 -
We observe that

$$-2\ln\Lambda_n(\mathbf{x}) = -2\left(\ell(\theta^*; \mathbf{x}) - \ell(\hat{\theta}_n; \mathbf{x})\right) = 2\left(\ell(\hat{\theta}_n; \mathbf{x}) - \ell(\theta^*; \mathbf{x})\right)$$

i.e. This is twie the difference of the log-likelihoods for $\hat{\theta}_n$ and θ^*.

Definition 15.2 - *Confidence Sets*
Define $\mathcal{X}^2_{1,\alpha}$ to be the number st $\mathbb{P}(W \leq \chi^2_{1,\alpha}) = 1-\alpha$ for $W \sim \chi^2_1$. The *Confidence Sets*

$$C(\mathbf{X}_n) := \left\{\theta \in \Theta : 2\left[\ell(\hat{\theta}_n; \mathbf{X}_n) - \ell(\theta; \mathbf{X}_n)\right] \leq \chi^2_{1,\alpha}\right\} \subseteq \Theta$$

are asymptotically exact $1-\alpha$ *Confidence Sets* for θ^* since

$$\mathbb{P}(\theta^* \in C(\mathbf{X}_n; \theta^*) = \mathbb{P}(-2\ln\Lambda_n(\mathbf{X}_n) \leq \chi^2_{1,\alpha}; \theta^*) \underset{n\to\infty}{\longrightarrow} 1-\alpha$$

Remark 15.3 - *Interpretting Confidence Sets*
$C(\mathbf{x}_n)$ cntains the values θ st $\ell(\theta; \mathbf{x}_n)$ is not too muuch less than $\ell(\hat{\theta}_n; \mathbf{x}_n)$. Hence, these confidence intervals contain those values of θ with the greatest likelihood values.

Remark 15.4 -
The cobserved confidence set $C(\mathbf{x})$ is defined implicitly, and finding an explicit representaiton of such sets might not be easy This difficulty explains why *Wald's Approach* has been historically populat, despite its shortcomings. However, with the help of a computer, it is often easy to determine $C(\mathbf{x})$ numerically.
Proof 15.1 - *Theorem 13.1*
Consider the second order *Taylor Expansion* of $\ell_n(\theta; x) = \ln f_n(x; \theta)$

$$\ell_n(\theta; x) = \ell_n(\theta_0; x) + (\theta - \theta_0)\ell_n'(\theta_0; x) + \frac{(\theta-\theta_0)^2}{2}\ell_n''(\bar{\theta}; x) \text{ for some } \bar{\theta} \in [\theta, \theta_0]$$

Rearranging we find

$$\ell_n(\theta; x) - \ell_n(\theta_0; x) = (\theta - \theta_0)\ell_n'(\theta_0; x) + \frac{(\theta-\theta_0)^2}{2}\ell_n''(\bar{\theta}; x)$$

Take $\theta = \theta^*$ and $\theta_0 = \hat{\theta}_n$.
Since $\ell_n'(\hat{\theta}_n; x) - \ell_n(\hat{\theta}_n; x)$ then

$$\begin{aligned}
\ln\Lambda_n(x) &= \ell_n(\theta^*; x) - \ell_n(\hat{\theta}_n; x) \\
&= \frac{(\theta^* - \hat{\theta}_n)^2}{2}\ell_n''(\bar{\theta}_n; x) \text{ for some } \bar{\theta}_n \in [\theta^*, \hat{\theta}_n] \\
\implies -2\ln\Lambda(x) &= -(\theta^*\hat{\theta}_n)^2\ell_n''(\bar{\theta}_n; x) \\
&= -\left[\sqrt{nI(\theta^*)}\right]^2(\theta^* - \hat{\theta}_n)^2\tfrac{1}{nI(\theta^*)}\ell_n''(\bar{\theta}_n; x)
\end{aligned}$$

Consider the random variable $-2\ln\Lambda(X)$. Then we have

$$\sqrt{nI(\theta^*)}(\hat{\theta}_n(X) - \theta^*) \to_{\mathcal{D}(\cdot;\theta^*)} Z \sim \text{Normal}(0,1)$$

By the *Continuous Mapping Theorem*

$$\left[\sqrt{nI(\theta^*)}\right]^2(\hat{\theta}_n - \theta^*)^2 \to_{\mathcal{D}(\cdot;\theta^*)} Z^2$$

Since $\bar{\theta}_n \in [\theta^*, \hat{\theta}_n]$

$$-\frac{1}{n}\ell_n''(\bar{\theta}_n; X) \to_{\mathbb{P}(\cdot;\theta^*)} I(\theta^*)$$

By *Slutsky's Theorem*

$$-2\ln\Lambda_n(X) \to_{\mathcal{D}(\cdot;\theta^*)} Z^2 \sim \chi^2_1$$

□

Remark 15.5 - *A Rule of Thumb*
Under the assumptions of **Theorem 13.1**, the set

$$\left\{\theta \in \Theta : \ell(\theta; \mathbf{x}) \geq \ell(\hat{\theta}_n; \mathbf{x}) - 2\right\}$$

is an asymptotically approximate 95% confidence set for θ^*.

Proof 15.2 - *Remark 13.1*
We have $\chi^2_{0.05} = 3.84$.
The result follows from the approximation $1.92 \approx 2$ □

1.16 Transformation Invariant Confidence Sets

Remark 16.1 - *Motivation*
Here we investigate whether the likelihood ratio approach to determining confidence sets is invariant to transformations, in contrast to *Wald's Approach.*
Consider the reparameterisation of the likelihood in terms of $\tau := g(\theta)$ where $g : \Theta \to G$ is bijective. We have

$$\tilde{f}(\mathbf{x}; \tau) := f(\mathbf{x}; g^{-1}(\tau)) = f(\mathbf{x}; \theta$$

We can now define

$$C(\mathbf{x}) := \left\{\theta \in \Theta : -2\left[\ell(\theta; \mathbf{x}) - \ell(\hat{\theta}_n; \mathbf{x}\right] \leq \chi^2_{1,\alpha}\right\} \text{ and } \tilde{C}(\mathbf{x}) := \left\{\theta \in \Theta : -2\left[\tilde{\ell}(\theta; \mathbf{x}) - \tilde{\ell}(\hat{\theta}_n; \mathbf{x}\right] \leq \chi^2_{1,\alpha}\right\}$$

We want to know whether $\theta \in C(\mathbf{x}) \iff g(\theta \in \tilde{C}(\mathbf{x}) \ \forall \ \mathbf{x} \in \chi^n$.
i.e. $C(\mathbf{x})$ & $\tilde{C}(\mathbf{x})$ define the same sets up to reparameterisation.

Theorem 16.1 -
Let $\mathbf{X} \sim f(\cdot; \theta^*)$, C and $\tilde{C}$ defined as in **Remark 14.1**.
Assume that $g : \Theta \to G$ is bijective Then

$$\forall \ \mathbf{x} \in \chi^n \text{ and } \theta^* \in \Theta, \ \theta \in C(\mathbf{x}) \iff g(\theta \in \tilde{C}(\mathbf{x})$$

Thus

$$\mathbb{P}(\theta^* \in C(\mathbf{X}); \theta^*) = \mathbb{P}(g(\theta^*) \in \tilde{C}(\mathbf{X}); \tau = g(\theta^*))$$

Proof 16.1 - *Theorem 14.1*
Let $\mathbf{x} \in \chi^n$ be arbitrary.
Everything rests on the observation that

$$\forall \ \theta \in \Theta, \ \ell(\theta; \mathbf{x}) = \ln f(\mathbf{x}; \theta) = \ln f(\mathbf{x}; g(\theta) = \tilde{\ell}(g(\theta; \mathbf{x})$$

and similary

$$\forall \ \tau \in G, \ \tilde{\ell}(\tau; \mathbf{x}) = \ln \tilde{f}(\mathbf{x}; \tau) = \ln f(\mathbf{x}; g^{-1}(\tau)) = \ell(g^{-1}(\tau); \mathbf{x})$$

Note that $g(\hat{\theta}_n)$ is the *Maximum Likelihood Estimate* of τ.
Assume $\theta \in C(\mathbf{x})$. Then

$$-2\left[\ell(\theta; \mathbf{x}) - \ell(\hat{\theta}_n; \mathbf{x})\right] \leq \chi^2_{1,\alpha}$$

Thus

$$-2\left[\tilde{\ell}(g(\theta); \mathbf{x}) - \tilde{\ell}(g(\hat{\theta}_n); \mathbf{x})\right] \leq \chi^2_{1,\alpha}$$

Thus $g(\theta \in \tilde{C}(\mathbf{x})$.
So $\theta \in C(\mathbf{x}) \implies g(\theta) \in \tilde{C}(\mathbf{x})$.

Similarly, assume that $g(\theta) \in \tilde{C}(\mathbf{x})$. Thus

$$-2\left[\ell(\theta;\mathbf{x}) - \ell(\hat{\theta}_n;\mathbf{x})\right] \leq \chi^2_{1,\alpha}$$

Thus $\theta \in C(\mathbf{x})$.
So $\theta \in C(\mathbf{x}) \iff g(\theta) \in \tilde{X}(\mathbf{x})$.

For the last part, this correspondence implies that

$$\{\mathbf{x} \in \chi^n; \theta^* \in C(\mathbf{x})\} = \{\mathbf{x} \in \chi^2 : g(\theta^*) \in \tilde{C}(\mathbf{x}\}$$

Thus, we can conclude from the equivalnce of the events

$$\{\theta^* \in C(\mathbf{X}) = \{g(\theta^*) \in \tilde{C}(\mathbf{X})\}$$

2 Testing

2.1 Introduction to Hypothesis Tests

Remark 1.1 - *Motivation*
Hypothesis testing allows us to make decisions about a parameter, rather than just estimating a rangle of values.

Definition 1.1 - *Hypothesis Testing*
Hypothesis Testing is a process for deciding which of two competing hypotheses, H_0 or H_1, is more consistent with an observation $\mathbf{x} = (x_1, \ldots, x_n)$ of $\mathbf{X} = (X_1, \ldots, X_n) \sim f(\cdot;\theta)$.
Remark 1.2 - *Difference to Statistics 1*
In Statistics 1 we always had the null hypothesis be $H_0 = \mu$. Now we consider a more general case where

i) $\mathbf{X} \sim f(\cdot;\theta)$ where $\theta \in \Theta$ is unknown.

ii) We have an observation $\mathbf{x}$ of $\mathbf{X}$;

iii) We have forumulate a null hypothesis concerning possible values of θ (*e.g.* $H_0 : \theta \in \Theta_0$)

iv) We have an alternative hypothsis, $H_1 : \theta \in \Theta_1 = \Theta \backslash \Theta_0$.

Definition 1.2 - *Simple Hypothesis*
A *Simple Hypothesis* is a hypothesis H_i of the form $H_i : \theta = \theta_i$ where θ_i is a specified value, equivalenetly $H_i : \theta \in \Theta_i = \{\theta_i\}$.

Definition 1.3 - *Composite Hypothesis*
A *Composite Hypothesis* is a hypothsis H_i of the form $H_i : \theta \in \Theta_i$ where Θ_i is not a singleton. (*i.e.* $|\Theta_i| > 1$).

Definition 1.4 - *One-Sided Test*
Let θ be a scalar & $\theta_0 \in \Theta$ be a specified value.
A *One-Sided Test* is a hypothesis test of the form

$$H_0 : \theta \leq \theta_0 \text{ and } H_1 : \theta > \theta_0$$

or

$$H_0 : \theta \geq \theta_0 \text{ and } H_1 : \theta < \theta_0$$

Definition 1.5 - *Two-Sided Test*
Let θ be a scalar & $\theta_0 \in \Theta$ be a specified value.
A *Two-Sided Test* is a hypothesis test of the form

$$H_0 : \theta = \theta_0 \text{ and } H_1 : \theta \neq \theta_0$$

Definition 1.6 - *Test Statistic*
A *Test Statistic* is an operation on an observation which we use to determine the outcome of a hypothesis test. Using the distribution of specified *Test Statistic* we can determine the likelihood of see a certain observation under the null-hypothesis & thus the likelihood of the null-hypothesis being true.
N.B. A test statistic has the signature $T : \chi^n \to \mathbb{R}$.

Definition 1.7 - *Critical Value*
The *Critical Value*, $c \in \mathbb{R}$, is an explicit value which if the value of a test statistic T exceeds it (*i.e.* $T(\mathbf{x}) \geq c$) we reject the null-hypothesis.

Definition 1.8 - *Critical Region*
The *Critical Region* is the sets of observations which cause us to reject the null hypothesis.

$$R := \{\mathbf{x} \in \chi^n : T(\mathbf{x}) \geq c\}$$

where T is a *Test Statistic* & c is a *Critical Value*.
N.B. $\chi^n = R \cup R^c$.

2.2 Hypothesis Testing - Significance and Power

Definition 2.1 - *Type I & Type II Error*
Type I Error occurs when H_0 is rejected, when in fact it is true.
Type II Error occurs where H_0 is accepted, when in fact it is false.
Consider the table below

	Retain H_0	**Reject** H_0
H_0 **is True**	Correct	*Type I Error*
H_1 **is True**	*Type II Error*	Correct

Definition 2.2 - *Significance Level*
Significance Level is the rate at which we allow *Type I Error*s to occur

$$\alpha = \mathbb{P}(\text{Type I Error}) \in [0, 1]$$

Typically this is the level of inprobability at which we reject the null hypothesis.
N.B. Common *Significance Levels* are $\alpha = 0.05, 0.01$.

Example 2.1 - *Testing the mean of a normal sample*
Suppose that $\mathbf{X} \overset{\text{iid}}{\sim} \text{Normal}(\mu, \sigma^2)$ and we want ot test

$$H_0 : \mu \leq 0 \text{ and } H_1 : \mu > 0$$

We consider the test statistic $T(\mathbf{x}) = \frac{1}{n}\sum_{i=1}^n x_i = \bar{x}$ with critical region

$$R := \{\mathbf{x} \in \chi^n : \bar{x} \geq c\} \text{ for } c \in \mathbb{R}$$

We want to find $c \in \mathbb{R}$ st $\mathbb{P}(X \in R; \mu \in \Theta_0) \leq \alpha \implies \mathbb{P}(\bar{x} \geq c; \mu \in \Theta_0) \leq \alpha)$.
We know that $\bar{X} \sim \text{Normal}(\mu, sigma^2/n)$.

Hence $\dfrac{\bar{X}-\mu}{\sigma/\sqrt{n}} \sim \text{Normal}(0,1)$.
We have

$$\mathbb{P}(\bar{X} \geq c; \mu) = \mathbb{P}\left(\frac{\sqrt{n}(\bar{X}-\mu)}{\sigma} \geq \frac{(c-\mu)\sqrt{n}}{\sigma}; \mu\right) = 1 - \Phi\left(\frac{\sqrt{n}(c-\mu)}{\sigma}\right)$$

We want to ensure that

$$\begin{array}{rrcl} & \mathbb{P}(\bar{X} \geq c; \mu \in \Theta_0) & \leq & \alpha \\ \iff & 1 - \Phi\left(\frac{\sqrt{n}(c-\mu)}{\sigma}\right) & \leq & \alpha \\ \iff & \frac{\sqrt{n}(c-\mu)}{\sigma} & \geq & \Phi^{-1}(1-\alpha) \\ \iff & c & \geq & \mu + \frac{\sigma}{\sqrt{n}}\Phi^{-1}(1-\alpha) \end{array}$$

Now observe that, for a fixed c¡ and considering $\mu \leq 0$ and $\mu \in \Theta_0$

$$\mathbb{P}(\bar{X} \geq c; \mu \in \Theta_0) \leq \mathbb{P}(\bar{X} \geq c; \mu = 0)$$

Thus we can esure that

$$\sup_{\mu \in \theta_0} \mathbb{P}(\bar{X} \geq c; \mu) = \alpha$$

by taking $c = \frac{\sigma}{\sqrt{n}}\Phi^{-1}(1-\alpha)$.

Remark 2.1 - *Change in Critical Value*
Critical Value, c, decreases as number of sample, n, increases.
Critical Value, c, increases as variance, σ, increases.

Remark 2.2 -
Significance Level, α, is directly related to the phrase *"statistical significance"*. *Statistical Significance* relates only to the *Type I Error* rate.

2.2.1 Power

Definition 2.3 - *Power Function*
Let $\mathbf{X} \sim f(\cdot; \theta^*)$, $T(\cdot)$ be a test statistic & c be the critical value of T.
The power function, $\pi(\cdot; T, c) : \Theta \to [0,1]$, is the probabiltiy of rejecting H_0 when th etrue value of the parameter is $\theta \in \Theta$.

$$\pi(\theta; T, c) := \mathbb{P}(\mathbf{X} \in R; \theta) = \mathbb{P}(T(\mathbf{X}) \geq c; \theta)$$

Remark 2.3 -
For a given $\theta \in \Theta_1$, the probability of a *Type II Error* occuring is $1 - \pi(\theta; T, c)$.

Remark 2.4 -

i) THe power is non-increasing in c, regardless of whether $\theta \in \Theta_0$ or $\theta \in \Theta_1$.

ii) To make the probabiltiy of *Type I Error* tend to 0 we should make c very large so we rarely reject H_0.

iii) If c is really large, we will rarely reject H_0 even if $\theta \in \Theta_1$. Thus the *Power* is low and the probabilitiy of *Type II Error* is high.

Notation 2.1 -
When it is clear from context what test, $T(\cdot)$, and critical value, c, we are referring to then we may write $\pi(\theta)$ in place of $\pi(\theta; T, c)$.

Example 2.2 - *Testing the Mean of a Normal Sample - Continued*
Suppose that $\mathbf{X} \overset{\text{iid}}{\sim} \text{Normal}(\mu, \sigma^2)$ and we want to test

$$H_0 : \mu \leq 0 \text{ and } H_1 : \mu > 0$$

We consider the test statistic $T(\mathbf{x}) = \bar{x}$ with critical region $R = \{\mathbf{x} \in \mathcal{X}^n : \bar{x} \geq c\}$ for some $c \in \mathbb{R}$.
The *Power Function* of this test is

$$\pi(\mu; T, c) = \mathbb{P}(\bar{X} \geq c; \mu)$$

We have already derived that $\dfrac{\bar{X} - \mu}{\sigma/\sqrt{n}} \sim \text{Normal}(0, 1)$. Hence

$$\begin{aligned}
\mathbb{P}(\bar{X} \geq c; \mu) &= \mathbb{P}\left(\frac{\bar{X} - \mu}{\sigma/\sqrt{n}} \geq \frac{c - \mu}{\sigma/\sqrt{n}}\right) \\
&= \mathbb{P}\left(Z \geq \frac{c - \mu}{\sigma/\sqrt{n}}; \mu\right) \\
&= 1 - \Phi\left(\frac{c - \mu}{\sigma/\sqrt{n}}\right) \\
&= \Phi\left(\frac{\mu - c}{\sigma/\sqrt{n}}\right)
\end{aligned}$$

Definition 2.4 - *Size of a Test*
The size of a test is the greatest possible probability of making a *Type I Error*

$$\alpha = \sup_{\theta \in \Theta_0} \pi(\theta; T, c)$$

N.B. It is the maximium power under the null-hypothesis.

Remark 2.5 -
Generally we choose a critical value c so that th etest has size α.

Definition 2.5 - *Significance Level of a Test*
A test has level α if its size is less than or equal to α. The corresponding test is called a *Level α Test.*

Definition 2.6 -
When $\Theta_0 = \{\theta_0\}$ (*i.e.* simple) then $\alpha = \pi(\theta_0; T, c)$ is the significan level.

Definition 2.7 -
When $\Theta_1 = \{\theta_1\}$ (*i.e.* simple) then $10\pi(\theta_1; T, c)$ is the probability of *Type II Error.*

Example 2.3 - *Testing the mean of a normal sample - Continued*
Suppose that $\mathbf{X} \overset{\text{iid}}{\sim} \text{Normal}(\mu, \sigma^2)$ and that we want to test

$$H_0; \mu \leq 0 \text{ and } H_1 : \mu > 0$$

We consider the test statistic $T(\mathbf{x}) = \bar{x}$ with critical region R.
A test of size α is obtained by choosing

$$c = \frac{\sigma}{\sqrt{n}} \Phi^{-1}(1 - \alpha) = \frac{\sigma}{\sqrt{n}} z_\alpha$$

So we consider the fact that $c = \frac{\sigma}{\sqrt{n}} z_\alpha$ and we obtain

$$\mathbb{P}\left(\bar{X} \geq \frac{\sigma}{\sqrt{n}} z_\alpha; \mu\right) = 1 - \Phi\left(z_\alpha - \frac{\mu\sqrt{n}}{\sigma}\right)$$

This gives the power $\forall\ \mu \in \mathbb{R}$ and we are interested in particular in it for $\mu > 0$.

2.3 Designing Tests - Neyman-Peason Approach

Remark 3.1 - *Plan for Testing at Significance Level, α*

i) Define a model $f(\cdot;\theta)$ for $\theta \in \Theta$

ii) Define a null hypothesis $H_0 : \theta \in \Theta_0$ and an alternative hypothesis $H_1 : \theta \in \Theta_1 = \Theta \backslash \Theta_1$

iii) Define a test statistic $T(\mathbf{x})$.

iv) Choose a critical value, c, st $\sup_{\theta \in \Theta_0} \mathbb{P}(T(\mathbf{X}) \geq c; \theta) \leq \alpha$.

N.B. The value of c is determined the value of α (which we set).
Theorem 3.1 - *Neyman-Pearson Lemma*
Suppose we test $H_0 : \theta = \theta_0$ against $H_1 : \theta = \theta_1$ and use the *Likelihodd Ratio Test Statistic*

$$T_{NP}(\mathbf{x}) := \frac{f_n(\mathbf{x};\theta_1}{f_n(\mathbf{x};\theta_0)} = \frac{L(\theta_1;\mathbf{x})}{L(\theta_0;\mathbf{x})}$$

Let hte *Critical Value*, $c_{NP} \geq 0$, be st the test has size α

$$\mathbb{P}(T_{NP} \geq c_{NP}; \theta_0) = \alpha$$

Then, this test is the most poweful level α test.
i.e. Among all tests with level α, this test maximises the power function.

Proof 3.1 - *Theorem 2.1*
Cibsuder for an arbitrary level α test (T, c), the linear combination of *Type I Errors* and *Type II Errors.*

$$\phi(T, c) := c_{NP}\alpha(T, c) + \beta(T, c)$$

where $\alpha(T, c) = \mathbb{P}(T(\mathbf{X}) \geq c; \theta_0) = \mathbb{P}(\text{Type I Error}$ and $\beta(T, c) = \mathbb{P}(T(\mathbf{X}) < c; \theta_1) = 1 - \mathbb{P}(T(\mathbf{X}) \geq c; \theta_1) = \mathbb{P}(\text{Type II Error.}$
Then

$$\begin{aligned}
\phi(T, c) &= c_{NP}\alpha(T, c) + \beta(T, c) \\
&= c_{NP}\mathbb{P}(T(\mathbf{X}) \geq c; \theta_0) + [1 - \mathbb{P}(\mathbf{X}) \geq c; \theta_1)] \\
&= \left[c_{NP} \int \mathbb{1}\{T(\mathbf{x}) \geq c\} f_n(\mathbf{x}; \theta_0) d\mathbf{x}\right] + \left[1 - \int \mathbb{1}\{T(\mathbf{x}) \geq c\} f_n(\mathbf{x}; \theta) d\mathbf{x}\right] \\
&= 1 + \int \mathbb{1}\{T(\mathbf{x}) \geq c\} \left[c_{NP} f_n(\mathbf{x}; \theta_0) - f_n(\mathbf{x}; \theta_1)\right] d\mathbf{x} \\
&= 1 + \int \mathbb{1}\{T(\mathbf{x}) \geq c\} \left[c_{NP} - \frac{f_n(\mathbf{x}; \theta_1)}{f_n(\mathbf{x}; \theta_0)}\right] f_n(\mathbf{x}; \theta_0) d\mathbf{x} \\
&= 1 + \int \mathbb{1}\{T(\mathbf{x}) \geq c\}(c_{NP} - T_{NP}(\mathbf{x})) f_n(\mathbf{x}; \theta_0) d\mathbf{x}
\end{aligned}$$

Now consider the difference

$$\phi(T, c) - \phi(T_{NP}, c_{NP}) = \int \big(\mathbb{1}\{T(\mathbf{x}) \geq c\} - \mathbb{1}\{T_{NP}(\mathbf{x})\} \geq c_{NP}\}\big)(c_{NP} - T_{NP}(\mathbf{x})) f_n(\mathbf{x}; \theta_0) d\mathbf{x}$$

We observe that

$$\mathbb{1}\{T_{NP}(\mathbf{x}) \geq c_{NP}\} = 1 \iff c_{NP} - T_{NP}(\mathbf{x}) \leq 0$$

and

$$\mathbb{1}\{T_{NP}(\mathbf{x}) \geq c_{NP}\} = 0 \iff c_{NP} - T_{NP}(\mathbf{x}) > 0$$

Thus

$$\forall\ \mathbf{x} \in \mathcal{X}^n, \quad [\mathbb{1}\{T(\mathbf{x}) \geq c\} - \mathbb{1}\{T_{NP}(\mathbf{x}) \geq c_{NP}\}](c_{NP} - T_{NP}(\mathbf{x})) \geq 0$$

and hence as the integral of a non-negative function

$$\phi(T, c) - \phi(T_{NP}, c_{NP}) \geq 0$$

We have established

$$\begin{aligned} 0 &\leq \phi(T, c) - \phi(T_{NP}, c_{NP}) \\ &= c_{NP}\alpha(T, c) + \beta(T, c) - c_{NP}\alpha(T_{NP}, c_{NP}) - \beta(T_{NP}, c_{NP}) \\ &= \underbrace{c_{NP}}_{\geq 0}[\alpha(T, c) - \alpha(T_{NP}, c_{NP})] + \underbrace{\beta(T, c) - \beta(T_{NP}, c_{NP})}_{\geq 0} \end{aligned}$$

Since (T, c) specifides an α level test, we know $\alpha(T, c) \geq c$ while (T_{NP}, c_{NP}) specifies a size α test so $\alpha(T_{NP}, c_{NP}) = \alpha$.
It follows that

$$\alpha(T, c) - \alpha(T_{NP}, c_{NP})$$

so we have

$$\beta(T, c) - \beta(T_{NP}, c_{NP}) \geq 0$$

which means (T_{NP}, c_{NP})'s *Type II Error* rate is no higher than (T, c).
Since (T, c) is an arbitrary α level test, we conclude that (T_{NP}, c_{NP}) is the most powerful test with level α. □

Remark 3.2 - *Neyman-Pearson with Non-Continuous Random Variable*
If $T(\mathbf{X})$ is not a continuous random variable, then it is possible that no such c_{NP} exists. In this situtation we perform an appropriate randomised test, and this will also be the most powerful size α test.
N.B. The detials of this are not covered in this course.

Definition 3.1 - *Neyman-Peason Procedure*
For **Theorem 2.1** we can deduce the *Neyman-Peason Procedure* for testing two simple hypotheses

i) Choose the *Likelihood Ratio* as the *Test Statistic*

$$T(\mathbf{x}) = \frac{f_n(\mathbf{x}; \theta_1)}{f_n(\mathbf{x}; \theta_0)} = \frac{L(\theta_1; \mathbf{x})}{L(\theta_0; \mathbf{x})}$$

ii) Choose a critical value c in order to target a particular significance level, α, st

$$\alpha = \pi(\theta_0) = \mathbb{P}(T(\mathbf{X}) \geq c; \theta_0)$$

iii) Compute the *Power*

$$\pi(\theta_1, T, c) = \mathbb{P}(T(\mathbf{X}) \geq c; \theta_1)$$

iv) Compute $T(\mathbf{x})$ and report whether $T(\mathbf{x}) \geq c$ as well as the power $\pi(\theta_1, T, c)$ or the *Type II Error* rate $1 - \pi(\theta_1, T, c)$

Remark 3.3 - *Limitations of Neyman-Pearson Approach*

i) Often just rejecting H_0 or retaining H_0 is not satisfactory, we may want more information.

ii) It is not obvious how to calibrate a likelihood ration test (*i.e.* TO find the critical value or compute the power function).

2.4 Testing - p-Values, Equivalent Test Statistics and Computing the Power Function

Remark 4.1 - *Motivation for p-Value*
Many studies prefer not to select in advance just one significance level α, or they may wish to reposrt something more informative than a binary decision. IN suc cases, they can report the p-value associated with the observed test statistic.

Definition 4.1 - *p-Value*
Let $\mathbf{X} \sim f_n(\cdot;\theta^*)$ for some $\theta^* \in \Theta$.
The *p-Value* for a test with test statistic $T(\mathbf{x})$ is the probability of seeing a test statistic $T(\mathbf{X})$ at least as extreme as $T(\mathbf{x})$.

$$p(\mathbf{x}) := \sup_{\theta_0 \in \Theta_0} \mathbb{P}(\underbrace{T(\mathbf{X})}_{\text{RV}} \geq \underbrace{T(\mathbf{x})}_{\text{Observed}} ; \theta_0)$$

Equivalently, $p(\mathbf{x})$ is the smallest significance level at which we would reject H_0.

Remark 4.2 - *p-Value Intuition*
Intuitively, *p-value* is a measure of the eveidence against H_0. The smaller it is, the less likely it is that $\mathbf{x}$ is a realisation of $\mathbf{X} \sim f(\cdot;\theta_0)$, resuliting in strong evidence against H_0.
N.B. A large p-value is not evidence in favour of H_0, nor is it necessarily evidence in favour of H_1 as H_1 is not involved at all when comuting the p-value.

Remark 4.3 - *Standard Caution*
$p(\mathbf{x})$ is *not* the probability that H_0 is true. It is the probability to observe the data we observed if θ_0 is true.

Remark 4.4 - *Distribution of p-Value*
When using a simple null hypothesis $\Theta_0 = \{\theta_0\}$ and assuming $T(\mathbf{X})$ is a continuous random variable when $\mathbf{X} \sim f(\cdot;\theta_0)$, the distribution of $p(\mathbf{X})$ is in fact uniform under the null hypothesis.

Example 4.1 - *Normal*
THe model is $\mathbf{X} \overset{\text{iid}}{\sim} \text{Normal}(\mu, 1)$ and we want to test $H_0 : \mu = \mu_0 < 0$ against $H_1 : \mu = \mu_1 > 0$.
The p-value for $T(\mathbf{x}) = \bar{x} = \frac{1}{n}\sum x_i$ is

$$p(\mathbf{x}) := \sup_{\mu \in \Theta_0} \mathbb{P}\left(\frac{1}{n}\sum_{i=1}^n X_i \geq T(\mathbf{x}) = \bar{x}; \mu\right) = \mathbb{P}\left(\frac{1}{n}\sum_{i=1}^n X_i \geq T(\mathbf{x}) = \bar{x}; \mu\right)$$

A very large postive value of the empirical mean leads to a small p-value and is an indication of how unlikely it is to have observed $\mathbf{x}$ if it was a realisation of $\mathbf{X} \overset{\text{iid}}{\sim} \text{Normal}(\mu_0, 1)$.
A large p-value is not an argument in favour of H_0, in fact it could suggest that $T(\mathbf{x})$ is an unlikely realisation under H_0.
We have already calculate this kind of expression under the null hypothesis $\bar{X} \sim \text{Normal}(\mu_0, \frac{1}{n})$ so

$$\begin{aligned} \mathbb{P}(\bar{X} \geq c; \mu) &= \mathbb{P}(\sqrt{n}(\bar{X} - \mu_0) \geq \sqrt{n}(c - \mu_0); \mu_0) \\ &= \mathbb{P}(Z \geq \sqrt{n}(c - \mu_0)) \\ &= 1 - \Phi(\sqrt{n}(c - \mu_0)) \end{aligned}$$

It follows that

$$p(\mathbf{x}) = 1 - \Phi(\sqrt{n}(\bar{x} - \mu_0))$$

Definition 4.2 - *Equivalent Statistics*
A statistic $T'(\mathbf{x})$ is equivalent to $T(\mathbf{x})$ if $\forall$ critical values $c \in \mathbb{R}$ of $T(\cdot)$ we can find $c' \in \mathbb{R}$ we can find $c' \in \mathbb{R}$ st $\forall\ \mathbf{x} \in \mathcal{X}^n$

$$T(\mathbf{x}) \geq c \iff T'(\mathbf{x}) \geq c'$$

Equivalently, $\forall\ c \in \mathbb{R}$ there exist $c' \in \mathbb{R}$ such that the corresponding critical regions of $T(\cdot)$ and $T'(\cdot)$ respectively conincide

$$\{\mathbf{x} \in \mathcal{X}^n : T(\mathbf{x}) \geq c\} = \{\mathbf{x} \in \mathcal{X}^n : T'(\mathbf{x}) \geq c'\}$$

Proposition 4.1 - *Proving Equivalence*
To verify that $T'(\mathbf{x})$ is an *Equivalent Statistic* to $T(\mathbf{x})$ it is sufficient to factorise $T(\mathbf{x})$ as

$$T(\mathbf{x}) = Mf(T'(\mathbf{x}))$$

where M is independent of $\mathbf{x}$ and f is increasing & bijective.

Proof 4.1 - *Proposition 4.1*

$$\begin{aligned} T(\mathbf{x}) \leq c &\Leftrightarrow Mf(T'(\mathbf{x})) \geq c \\ &\Leftrightarrow f(T'(\mathbf{x})) \geq \tfrac{c}{M} \\ &\Leftrightarrow T'(\mathbf{x}) \leq \underbrace{f^{-1}(c/M)}_{c'} \end{aligned}$$

Example 4.2 - *Geometric Example*
Let that $\mathbf{X} \overset{\text{iid}}{\sim}$ Geometric(p) so that $f(x;p) = (1-p)^{x-1}p\mathbb{1}\{x \in \mathbb{N}\backslash\{0\}\}$.
Suppose that we want to test $H_0 : p = p_0$ against $H_1 : p = p_1$ with $p_0 > p_1$.

$$T_{NP}(\mathbf{x}) = \frac{f_n(\mathbf{x};p_1)}{f_n(\mathbf{x};p_0)} = \frac{\prod_{i=1}^n f(x_i;p_1)}{\prod_{i=1}^n f(x_i;p_0)} = \prod_{i=1}^n \frac{f(\mathbf{x}_i;p_1)}{f(\mathbf{x}_i;p_0)}$$

So for $x \in X$

$$\frac{f(x;p_1)}{f(x;p_0)} = \frac{(1-p_1)^{x-1}p_1}{(1-p_0)^{x-1}p_0} = \left(\frac{1-p_1}{1-p_0}\right)^x \left(\frac{1-p_1}{1-p_0}\right)^{-1} \left(\frac{p_1}{p_0}\right)$$

So

$$T_{NP}(\mathbf{x}) = \left(\frac{1-p_1}{1-p_0}\right)^{\sum x_i} \left(\frac{1-p_1}{1-p_0}\right)^{-n} \left(\frac{p_1}{p_0}\right)^n = \left(\frac{1-p_1}{1-p_0}\right)^{n\bar{x}} \underbrace{\left(\frac{1-p_1}{1-p_0}\right)^{-n} \left(\frac{p_1}{p_0}\right)^n}_{M}$$

Note that

$$p_0 > p_1 \implies 1 - p_0 < 1 - p_1 \implies \frac{1-p_1}{1-p_0} > 1$$

So $\left(\frac{1-p_1}{1-p_0}\right)^{n\bar{x}}$ is increasing with $\bar{x}$.
It follows that $T'(\mathbf{x}) = \bar{x}$ is an equivalent test statistic to T_{NP}.
If $\mathbf{X} \overset{\text{iid}}{\sim}$ Geometric(p) then $n\bar{x} \sim$ Negative-Binomical(n,p).
Hence we can compute c_{NP} or compute the power function.

Example 4.3 - *Normal Example*
The model is $\mathbf{X} \overset{\text{iid}}{\sim} \text{Normal}(\mu, 1)$ and we want to test $H_0 : \mu = 0$ against $H_1 : \mu = 1$.
The *Neyman-Pearson Test Statistic* is

$$\begin{aligned} T_{NP}(\mathbf{x}) &= \frac{f_n(\mathbf{x}; \mu = 1)}{f_n(\mathbf{x}; \mu = 0)} \\ &= \frac{\left(\frac{1}{\sqrt{2\pi}}\right)^n e^{-\frac{1}{2}\sum(x_i - 1)^2}}{\left(\frac{1}{\sqrt{2\pi}}\right)^n e^{-\frac{1}{2}\sum(x_i - 0)^2}} \\ &= e^{-\frac{1}{2}\left(\sum x_i^2 - 2\sum x_i + n - \sum x_i^2\right)} \\ &= e^{-\frac{1}{2}(-2n\bar{x} + n)} \\ &= \underbrace{e^{-\frac{n}{2}}}_{M} e^{n\bar{x}} \end{aligned}$$

Since T_{NP} is increasing in terms of $\bar{x}$, $\bar{x}$ is an equivalet test statistic to T_{NP}.
To relate $T_{NP} \geq c_{NP}$ with $T(\mathbf{x}) = \bar{x} \geq c$.
We have

$$\begin{aligned} T_{NP}(\mathbf{x}) \geq c_{NP} &\iff e^{n\bar{x} - \frac{n}{2}} \geq c_{NP} \\ &\iff n\bar{x} - \tfrac{n}{2} \geq \ln c_{NP} \\ &\iff \bar{x} - \tfrac{1}{2} \geq \tfrac{1}{n} \ln c_{NP} \\ &\iff \bar{x} \geq \underbrace{\frac{1}{2} + \frac{1}{n} \ln c_{NP}}_{c} \end{aligned}$$

So $c_{NP} = e^{n(c - \frac{1}{n})}$.
Now we can compute the power function.
We know that $\bar{X} \sim \text{Normal}(\mu, \frac{1}{n})$ for $\alpha \in (0, 1)$.
We find c by solving

$$\begin{aligned} & \pi(\mu_0; T, c) = \alpha \\ \implies & \mathbb{P}(\bar{X} \geq c; \mu_0) = \alpha \\ \implies & \mathbb{P}\left(Z \geq \tfrac{c - 0}{1/\sqrt{n}}\right) = \alpha \\ \implies & 1 - \Phi(c\sqrt{n}) = \alpha \\ \implies & \Phi(c\sqrt{n}) = 1 - \alpha \\ \implies & c\sqrt{n} = \Phi^{-1}(1 - \alpha) \\ \implies & c = \frac{\Phi^{-1}(1 - \alpha)}{\sqrt{n}} \\ & \quad = \frac{z_\alpha}{\sqrt{n}} \end{aligned}$$

Hence $c_{NP} = e^{n\left(\frac{z_\alpha}{\sqrt{n}} - \frac{1}{2}\right)}$ We can also compute *Type II Error* probability

$$\begin{aligned} 1 - \pi(1) &= \mathbb{P}(\bar{X} < c; \mu = 1) \\ &= \mathbb{P}\left(Z < \tfrac{c - 1}{1/\sqrt{n}}\right) \\ &= \Phi(\sqrt{n}c - \sqrt{n}) \\ &= \Phi(z_\alpha - \sqrt{n}) \overset{n \to \infty}{\longrightarrow} 0 \end{aligned}$$

2.5 Uniformly Most Powerful Tests

Definition 5.1 - *Uniformly Most Powerful Test*
Consider a test involving composite hypothese $H_0 : \theta \leq \theta_0$ against $H_1 : \theta > \theta_0$.
A *Uniformly Most Powerful Test* is a test (T, c) which has the largest power $\pi(\theta; T, c)$ among all possible tests, uniformly in $\theta \in \Theta_1$. That is a (T, c) st $\forall \theta \in \Theta_1$ and any test statistic (T', c')

$$\pi(\theta; T, c) \geq \pi(\theta; T', c')$$

Remark 5.1 -
The *Type II Error Rate* depends on a specific value of $\theta \in \Theta_1$. Typcally, the *Type II Error Rate* is close to $1-\alpha$ for values of $\theta \in \Theta_1$ *"very close to being in "* Θ_0. *i.e.* $\pi(\theta; T, c) \approx \alpha$ for $\theta = \theta_0 + \varepsilon$ for ε very small.

Theorem 5.1 -
Let $\Theta_1 = \{\theta : \theta > \theta_0\}$ for some $\theta_0 \in \Theta$.
Assume that for the simple hypothese

$$H_0' : \theta = \theta_1 \quad \text{against} \quad H_1' : \theta = \theta_2$$

The *Neyman-Pearson Test Statistic*

$$T_{NP}(\mathbf{x}) = \frac{f_n(\mathbf{x}; \theta_2)}{f_n(\mathbf{x}; \theta_1)}$$

is equivalent to the same test statistic $T(\mathbf{x})$ for any $\theta_1 < \theta_2$ and $T(\mathbf{x})$ does not depend on θ_1 or θ_2.
Then $T(\mathbf{x})$ is the uniformly most powerful test statistic for

$$H_0 : \theta \leq \theta_0 \quad \text{against} \quad H_1 : \theta > \theta_0$$

and the associated p-value is

$$p(\mathbf{x}) = \mathbb{P}(T(\mathbf{X}) \geq T(\mathbf{x}); \theta_0)$$

Example 5.1 - *Poisson*
Let $\mathbf{X} \overset{\text{iid}}{\sim} \text{Poisson}(\lambda)$ for some $\lambda > 0$ and we want to test $H_0 : \lambda \leq \lambda_0$ against $H_1 : \lambda > \lambda_0$.
Compute the p-value associated with this test.
Consider $H_0 : \lambda = \lambda_0$ & $H_1 : \lambda = \lambda_1$ where $\lambda_1 > \lambda_0$.

$$T_{NP}(\mathbf{x}) = \prod_{i=1}^{n} \left(\frac{e^{-\lambda_1} \lambda_1^{x_i} (x_i!)^{-1}}{e^{-\lambda_0} \lambda_0^{x_i} (x_i!)^{-1}} \right) = e^{-n(\lambda_1 - \lambda_0)} \left(\frac{\lambda_1}{\lambda_0} \right)^{\sum x_i}$$

Since $\lambda_1 > \lambda_0$ we have $T_{NP}(\mathbf{x})$ is an increasing function in terms of $\sum x_i$.
So $S_n := \sum x_i$ is an equivalent test statistic and does not depend on λ_0 or λ_1.
Hence, S_n is a *Uniformly Most Powerful Test.* To find the p-value

$$p(\mathbf{x}) = \mathbb{P}(S_n(\mathbf{X}) \geq S_n(\mathbf{x}); \lambda_0)$$

We use λ_0 in this scenario since it is the vlaue which is most likely to produce extreme values, in general, for $T(\mathbf{X})$.
We know that $S_n(\mathbf{X}) = \sum X_i \sim \text{Poisson}(n\lambda)$. So

$$p(\mathbf{x}) = \sum_{k \geq S_n(\mathbf{x})} e^{-\lambda_0 n} (n\lambda_0)^k \frac{1}{k!}$$

Alternatively
If n is alrge $S_n(\mathbf{X}) \simeq \text{Normal}(n\lambda, n\lambda)$.
So

$$p(\mathbf{x}) \simeq \mathbb{P}\left(Z \geq \frac{S_n(\mathbf{x}) - n\lambda_0}{\sqrt{n\lambda_0}} \right) = 1 - \Phi\left(\frac{S_n(\mathbf{x}) - n\lambda_0}{\sqrt{n\lambda_0}} \right)$$

where $Z \sim \text{Normal}(0, 1)$.

Example 5.2 - *Geometric*
Let $\mathbf{X} \overset{\text{iid}}{\sim} \text{Geometric}(p)$ we have already shown that for two simple hypotheses $\bar{X}$ is equivalent

to the likelihood ratio test statistic when $p_0 > p_1$.
It follows that $T(\mathbf{x}) = S_n(\mathbf{x})$ is equivalent.
We have noticed that $S_n(\mathbf{X}) \sim \text{NegBinomial}(n, p)$.
We shall compute the p-value assocaited to hypotheses $H_0 : p \leq p_0$ against $H_1 : p > p_0$.
From **Example 2.4.2** we have that for $H_0' : p = p_0$ against $H_1' : p = p_1$

$$T_{NP}(\mathbf{x}) = \left(\frac{p_1}{p_0}\right)^n \left(\frac{1-p_1}{1-p_0}\right)^{-n} \left(\frac{1-p_1}{1-p_0}\right)^{n\bar{x}}$$

For $P_1 > p_0$ we see that T_{NP} is a decreasing function (since the last two terms are < 0) in terms of $\sum X_i =: S_n$.
Hence it is increasing in terms of $T(\mathbf{x}) := -S_n(\mathbf{x})$.
Since S_n is independent of p_0 & p_1 we have that $-S_n$ is a *Unifromly Most Powerful Test.*

$$p\text{-value} := p(\mathbf{x}) - \mathbb{P}(-S_n(\mathbf{X}) \geq -S_n(\mathbf{x}); p_0) = \mathbb{P}(S_n(\mathbf{X}) \leq S_n(\mathbf{x}); p_0)$$

where $S_n(\mathbf{X}) \sim \text{NegativeBinomial}(n, p_0)$.

Remark 5.2 - *Uniformly Most Powerful Tests need not exist*
In general, *Uniformly Most Powerful Tests* need not exist.
It might be the case that (T_1, c_1) is best for, say, $\theta_{1,1} \in \Theta_1$.
i.e. $\forall\ (T', c')$

$$\pi(\theta_{1,1}; T_1, c_1) \geq \pi(\theta_{1,1}; T', c')$$

but $\exists\ (T_2, c_2)$ st

$$\pi(\theta_{1,2}; T_2, c_2) > \pi(\theta_{1,2}; T_1, c_1) \quad \theta_{1,2} \in \Theta_1$$

i.e. (T_2, c_2) is better than (T_1, c_1).

2.6 Generalised Likelihood Ratio Test

Remark 6.1 - *Generalised Tests*
In the most general case we would like to test $H_0 : \theta \in \Theta_0$ against $H_1 : \theta \in \Theta_1$.
There is no guarantee of the existence of an optimal test statistic.

Proposition 6.1 - *Generalised Likelihood Ratio Test*
We can generalie the likelihood ratio test for simple hypotheses from

$$T_{NP}(\mathbf{x}) = \frac{f_n(\mathbf{x}; \theta_1)}{f_n(\mathbf{x}; \theta_0)}$$

to

$$T_{\text{suggested}}(\mathbf{x}) := \frac{\sup_{\theta \in \Theta_1}(f_n(\mathbf{x}; \theta)}{\sup_{\theta \in \Theta_0}(f_n(\mathbf{x}; \theta)}$$

N.B. The generalised simple hypothese are $\Theta_i = \{\theta_i\}$ for $\theta_i \in \Theta$.

Definition 6.1 - *Likelihood Ratio*
We define a *Likelihood Ratio*

$$\begin{aligned}
\Lambda_n(\mathbf{x}) &:= \frac{\sup_{\theta \in \Theta_0} f_n(\mathbf{x}; \theta)}{\sup_{\theta \in \Theta} f_n(\mathbf{x}; \theta)} \\
&= \frac{\sup_{\theta \in \Theta_0} f_n(\mathbf{x}; \theta)}{f_n(\mathbf{x}; \underbrace{\hat{\theta}_n}_{\text{MLE}})} \\
&= \min\left\{ \underbrace{1}_{\hat{\theta}_n \in \Theta_0}, \underbrace{\frac{\sup_{\theta \in \Theta_0} f_n(\mathbf{x}; \theta)}{\sup_{\theta \in \Theta_1} f_n(\mathbf{x}; \theta)}}_{\hat{\theta}_n \notin \Theta_0} \right\}
\end{aligned}$$

Remark 6.2 - *Likelihood Ratio*

i) The denominiator corresponds to plugging in the *Maximum Likelihood Estimate* in the likelihood (assuming it exists and is unique).

ii) The last equality follows from the fact that $\sup_{\theta\in\Theta} f_n(\mathbf{x};\theta) \geq \sup_{\theta\in\Theta_0} f_n(\mathbf{x};\theta)$.
if the inequality is strict then

$$\sup_{\theta\in\Theta} f_n(\mathbf{x};\theta) = \sup_{\theta\in\Theta_1} f_n(\mathbf{x};\theta) > \sup_{\theta\in\Theta_0} f_n(\mathbf{x};\theta)$$

and if it is an equality then $\Lambda_n(\mathbf{x}) = 1$.

Definition 6.2 - *Nested Parameter Space*
Let $\Theta \subseteq \mathbb{R}^d$ and define $\phi = (\phi_1, \phi_2) : \Theta \to \Phi_1 \times \Phi_2$ to a *continuously differentiable bijection* with $\Phi_1 \subseteq \mathbb{R}^r$ and $\Phi_2 \subseteq \mathbb{R}^{d-r}$.
Θ_0 is said to be *Nested* within Θ if for some $c \in \Phi_1 \subseteq \mathbb{R}^r$

$$\Theta_0 = \{\theta \in \Theta : \phi_1(\theta) = c\}$$

N.B. $\dim(\Theta_0) = d - r$.

Example 6.1 - *Nested Parameter Space*
Suppose $\mathbf{X} \overset{\text{iid}}{\sim} \text{Normal}(\mu, \sigma^2)$ then $\boldsymbol{\Theta} = \{\mu, \sigma^2\}$.
Suppose $\phi_1(\mu, \sigma^2) = \mu$ & $\phi_2(\mu, \sigma^2) = \sigma^2$ then $\Phi_1 = \mathbb{R}$ & $\Phi_2 = \mathbb{R}^+ \subseteq \mathbb{R}$.
Then $\Theta_0 := \{(\mu, \sigma^2) : \phi_1(\mu, \sigma^2) = 0\}$ is *Nested* in Θ.

Theorem 6.1 - *Distribution of Test Statistics in Nested Parameter Spaces*
Let $\Theta \subset \mathbb{R}^d$ and $\mathbf{X} \overset{\text{iid}}{\sim} f(\cdot;\theta)$ for $\theta \in \Theta_0$ (*i.e.* H_0 is true) where Θ_0 is nested in Θ. Then

$$T_n(\mathbf{X}) = -2\ln\Lambda_n(\mathbf{X}) \to_{\mathcal{D}(\cdot;\theta)} W \sim \chi^2_r$$

with $r = \dim(\Theta) - \dim(\Theta_0)$.
N.B. The proof is not covered in this course but relies on the taulor expansion of the likelihood.

Remark 6.3 - *If value of the null hypothesis is fixed,* $H_0 : \theta = 0$*, then* $dim(\Theta_0) = 0$

Example 6.2 -
Let $\mathbf{X} \overset{\text{iid}}{\sim} \text{Poisson}(\lambda^*)$ for some $\lambda^* > 0$.
Consider the test

$$H_0 : \lambda = \lambda_0 \quad \text{against} \quad H_1 : \lambda \neq \lambda_0$$

Then

$$\begin{aligned} T_n(\mathbf{x}) &= -2\ln\left(\frac{f_n(\mathbf{x};\lambda_0)}{f_n(\mathbf{x};\hat{\lambda}_n)}\right) \text{ where } \hat{\lambda}_n \text{ is MLE of } \lambda \\ &= -2\ln[\ell_n(\lambda;\mathbf{x}) - \ell_n(\hat{\lambda}_n;\mathbf{x})] \end{aligned}$$

We know

$$\begin{aligned} f_n(\mathbf{x};\lambda) &= \prod_{i=1}^n \frac{e^{-\lambda}\lambda^{x_i}}{x_i!} \text{ by independence} \\ &\propto e^{-n\lambda}\lambda^{n\bar{x}} \\ \implies \ell_n(\lambda;\mathbf{x}) &= c - n\lambda + n\bar{x}\ln\lambda \\ \text{Taking } \ell_n'(\lambda;\mathbf{x}) &= 0 \\ \implies -n + \tfrac{\bar{x}n}{\lambda} &= 0 \\ \implies \hat{\lambda} &= \bar{x} \end{aligned}$$

We confirm $\hat{\lambda}$ is a *Maximum Likelihood Estimate* by showing $\ell_n''(\bar{x}; x) < 0$.
So

$$\begin{aligned} T_n(\mathbf{x}) &= -2\left[-n(\lambda_0 - \hat{\lambda}_n) + n\bar{x}\ln\tfrac{\lambda_0}{\hat{\lambda}_n}\right] \\ &= -2\left[-n(\lambda_0 - \bar{x}) + n\bar{x}\ln\tfrac{\lambda_0}{\bar{x}}\right] \end{aligned}$$

We know that

$$T_n(\mathbf{X}) \sim \chi^2_{1-0} = \chi^2_1 \text{ under } H_0$$

since $\dim(\Theta) = 1$ and $\dim(\Theta_0) = 0$.

Definition 6.3 - *Two Sided Test*
A *Two Sided Hypothesis Test* is a hypothesis test where the alternative hypothese covers two separate regions of the parameter space. The general form is

$$H_0 : \theta = \theta_0 \quad \text{against} \quad H_1 : \theta \neq \theta_0$$

Remark 6.4 - *Connection to Confidence Intervals*
Recall that a test of size approximately α is obtained by retain H_0 if

$$T_n(\mathbf{x}) = -2\left[\ell_n(\theta_0; \mathbf{x}) - \ell_n(\hat{\theta}_n; \mathbf{x})\right] < \chi^2_{r,\alpha}$$

The values of θ_0 which lead to the retention of H_0 are

$$\begin{aligned} C(\mathbf{x}) &= \left\{\theta : -2[\ell_n(\theta; \mathbf{x}) - \ell_n(\hat{\theta}_n; \mathbf{x})] < \chi^2_{r,\alpha}\right\} \\ &= \left\{\theta : \ell_n(\theta; \mathbf{x}) > \ell_n(\hat{\theta}_n; \mathbf{x}) - \tfrac{1}{2}\chi^2_{r,\alpha}\right\} \end{aligned}$$

This is an asymptotically exact $1 - \alpha$ confidence set for θ.

2.7 Categorical Distributions and Pearson's χ^2 Test

Definition 7.1 - *Categorical Distribution*
Let Y be a random variable which takes one value from a finite set $\{1, \dots, m\}$ where each value represents a unique category.
Let $\mathrm{p} := (p_1, \dots, p_m)$ be a vector where $p_i = \mathbb{P}(Y = i)$ then

$$Y \sim \text{Categorical}(\mathbf{p})$$

N.B. $\sum_{i=1}^m p_i = 1,\ p_i \in [0, 1]\ \forall\ i \in [1, m]$ and p is a vector of $m - 1$ free parameters.

Remark 7.1 - *Bernoulli*$(p) \sim$ *Categorical*$(1 - p, p)$

Definition 7.2 - *Counts*
Let $Y_1, \dots, Y_n \overset{\text{iid}}{\sim} \text{Categorical}(\mathbf{p})$.
We define the random variable N_k to model the number of times k occurs in a realisation of $Y_1, \dots, Y_n$.

$$N_k := \sum_{i=1}^n \mathbb{1}\{Y_i = k : i \in [1, n]\}$$

This definition gives rise to the random vector

$$\mathbf{X} := (N_1, \dots, N_m) \sim \text{Multinomial}(n, \mathbf{p})$$

with

$$\mathbb{P}(N_1 = n_1, \dots, N_m = n_m; \mathbf{p}) = \mathbb{1}\left\{\sum_{i=1}^m n_i = n\right\}\left\{\frac{n!}{\prod_{i=1}^m n_i}\right\}\prod_{i=1}^n p_i^{n_i}$$

N.B. $\mathbb{E}(N_i) = np_i$ and $\text{Var}(N_i) = np_i(1 - p_i)$.

2.7.1 Generalised Likelihood Ratio Test Statistic

Definition 7.3 - *Simplex*
TODO Expand this
$\mathcal{S}_m$ is a simplex of probability vectors of length m if

$$\mathcal{S}_m := \left\{ (_1, \dots, p_m) \in [0,1]^m : \sum_{i=1}^m p_i = 1 \right\}$$

Proposition 7.1 - *Generalised Likelihood Ratio*

$$\Lambda_n(\mathbf{x}) = \frac{f_n(\mathbf{x}; \mathbf{p}_0)}{\sup_{\mathbf{p} \in \mathcal{S}_m} f_n(\mathbf{x}; \mathbf{p})}$$

Theorem 7.1 - *Maximum Likelihood Estimate for Multinomial* $\boldsymbol{p}$
Let $\mathbf{X} = (N_1, \dots, N_m) \sim \text{Multinomial}(n, \mathbf{p}^*)$ for some $\mathbf{p}^* \in \mathcal{S}_m$ and $\mathbf{x} = (n_1, \dots, n_m)$ be a realisation of $\mathbf{X}$.
The maximum likelihood estimate of $\mathbf{p}^*$ is

$$\hat{\mathbf{p}}(\mathbf{x}) = (\hat{p}_1(\mathbf{x}), \dots, \hat{p}_m(\mathbf{x})) = \left(\frac{n_1}{,} \dots, \frac{n_m}{n} \right)$$

Proof 7.1 - *Theorem 8.1*
Note that

$$\sum_{i=1}^m p_i = 1 \implies p_m = 1 - \sum_{i=1}^{m-1} p_i$$

Hence there are only $m-1$ independent variables and

$$\begin{aligned} L(\mathbf{p}, \mathbf{x}) &= L(p_1, \dots, p_{m-1}; \mathbf{x}) \\ &\propto \prod_{j=1}^m p_j^{n_j} \\ &= \left(\prod_{j=1}^{m-1} p_j^{n_j} \right) \left(1 - \sum_{i=1}^{m-1} p_i \right)^{n_m} \end{aligned}$$

So

$$\ell(p_1, \dots, p_{m-1}; \mathbf{x}) = C + \left(\sum_{i=1}^{m-1} n_j \ln p_j \right) + n_m \ln \left(1 - \sum_{i=1}^{m-1} p_i \right)$$

Now for $k = 1, \dots, m-1$.

$$\begin{aligned} \text{Setting } \frac{\partial}{\partial p_k} \ell(p_1, \dots, p_{m-1}; \mathbf{x}) &= \frac{n_k}{p_k} - \frac{n_m}{1 - \sum_{i=1}^{m-1} p_i} \\ &= 0 \\ \implies \frac{n_k}{p_k} &= \frac{n_m}{p_m} \ \forall \ k \in [1, m] \end{aligned}$$

So $\frac{n_1}{p_1} = \dots = \frac{n_m}{p_m} = c$ and $\sum_{i=1}^m p_i = 1$.

$$\implies \sum_{i=1}^m \frac{n_i}{c} = 1 \implies \sum_{i=1}^m n_i = c \implies n = c$$

Hence $\frac{n_k}{p_k} = n \implies \hat{p}_j = \frac{n_k}{n} \ \forall \ k \in [1, m]$.
In order to confirm that this is a maximum we will show that $\ell(\mathbf{p}; \mathbf{x})$ is concave.
i.e. for $\lambda \in [0,1]$ $\ell(\lambda \mathbf{p} + (1-\alpha)\mathbf{p}'; \mathbf{x}) \geq \lambda \ell(\mathbf{p}; \mathbf{x}) + (1-\lambda)\ell(\mathbf{p}'; \mathbf{x})$.

$$\begin{aligned} \ell(\lambda \mathbf{p} + (1-\lambda)\mathbf{p}'; \mathbf{x}) &= \textstyle\sum_{i=1}^m n_i \ln(\lambda p_i + (1-\lambda) p_i') \\ &\geq \textstyle\sum_{i=1}^m n_i \left[\lambda_i \ln p_i + (1-\lambda) \ln p_i' \right] \text{ since } \ln x \text{ is concave} \\ &= \lambda \textstyle\sum_{i=1}^m n_i \ln p_i + n_i (1-\lambda) \ln p_i' \\ &= \lambda \ell(\mathbf{p}; \mathbf{x}) + (1-\lambda) \ell(\mathbf{p}'; \mathbf{x}) \end{aligned}$$

Thus concave.
It follows that

$$\Lambda_n(\mathbf{x}) = \frac{f_n(\mathbf{x};\mathbf{p}_0)}{\sup_{\mathbf{p}\in\mathcal{S}_m} f_n(\mathbf{x};\mathbf{p})} = \prod_{i=1}^{m} \frac{p_{0,i}^{n_i}}{\hat{p}_i^{n_i}} = \prod_{i=1}^{m} \frac{p_{0,i}^{n_i}}{(n_i/n)^{n_i}}$$

so that

$$T_n(\mathbf{x}) = -2\ln\Lambda_n(\mathbf{x}) = -2\sum_{i=1}^{m} n_i\{\ln p_{0,i} - \ln(n_i/n)\}$$

is the *Generalised Likelihood Ratio* test statistic. From the general theorem

$$T_n(\mathbf{x}) \to_{\mathcal{D}(\cdot;\mathbf{p}_0} \chi^2_{m-1}$$

since $\dim(\mathcal{S}_m) = m-1$.
Many people rewrite this statistic as

$$\begin{aligned} T_n(\mathbf{x}) &= 2\sum_{j=1}^{m} o_j \ln\left(\frac{0_j}{e_j}\right) \\ &= 2\sum_{j=1}^{m} n_j \ln\left(\frac{n_j/n}{p_{0,j}}\right) \\ &= -2\sum_{j=1}^{m} n_j \ln\left(\frac{n_j}{np_{0,j}}\right) \end{aligned}$$

where $o_j = n_j$ is the observered number in category j and $e_j = np_{0,j}$ is the expected number in category j. □

2.7.2 Pearson's χ^2 Test Statistic

Definition 7.4 - *Pearson's χ^2 Test Statistic*
Let $\mathbf{x} \sim$ Categoritcal($\mathbf{p}$) where $\mathbf{p} := (p_0, \dots, p_m)$ and $\mathbf{x}$ is a relisation of $\mathbf{X}$.
We define *Pearson's χ^2 Test Statistic* as

$$T_{\text{Pearson}}(\mathbf{x}) := \sum_{j=1}^{m} \frac{(n_j - np_j)^2}{np_j} = \sum_{j=1}^{m} \frac{(o_j - e_j)^2}{e_j} \to_{\mathcal{D}(\cdot;\mathbf{p})} \chi^2_{m-1}$$

where o_j is the number of observations of category j and e_j is the expected number of observations of category j. *N.B.* TODO - something about degrees of freedom.

2.8 Goodness-of-Fit Example - Mendel's Peas

Gregor Mendel, the father of modern genetics, performed experiments to test his theory of inheritance in the 1850s & 60s.
There are $k = 4$ types of peas

1. Round Yellow.
2. Wrinked Yellow.
3. Round Green, and
4. Wrinkled Green,

According to Medel's theory of inheritance, the number of peas of each type in the second generation of a crossing experiment should be an observatio of a Multinomial$(n, \mathbf{p}_0)$ random vector where

$$\mathbf{p}_0 = \left(\frac{9}{16}, \frac{3}{16}, \frac{3}{16}, \frac{1}{16}\right)$$

This consitutes the null hypothesis.
In Mendel's experiment, with $n = 556$, he recored

$$\mathbf{x} = (n_1, \dots, n_4) = (315, 101, 108, 32)$$

The test statistic is

$$\begin{aligned} w &= \sum_{i=1}^{k} \frac{(n_i - np_{0,i})^2}{np_{0,i}} \\ &= 0.47 \end{aligned}$$

The p-value is therefore the probability that a $\chi^2_{k-1} = \chi^2_3$ random variable exceeds 0.47

$$p(\mathbf{x}) = \mathbb{P}(\chi^2_3 > 0.47) = 0.925$$

We would not reject the null hypothesis for any reasonable α, so the data does not contadict his theory. In fact the P-value is quite large and there is some controversy regarding whether Mendel's results are *too good*.

Ronald Fisher looked at Mendel's data for a sequence of m similar experiments and discovered that they seemed all to obtain this kind of unusually good agreement. Each test statistic w_i is modelled approximately as an observation of an independent $\chi^2_{k_i-1}$ random variable where k_i is eh number of possible outcomes in experiment i.
Since the sum of independent $\chi^2_{k_i-1}$ random variables is $\chi^2_{\sum_i(k_i-1)}$ random variable, one can use the sum of the test statistics as a test statistic and compute the p-value

$$p(\mathbf{x}) = \mathbb{P}\left(Y \geq \sum_{i=1}^{m} w_i\right) \text{ where } Y \sim \chi^2_{\sum_i(k_1-1)}$$

The fact he obtaind a *pooled* χ^2 test statistic of $\sum_{i=1}^m w_i = 42$ and a a value of $\sum_{i=1}^m (k_i - 1) = 84$.

$$p(\mathbf{x}) = \mathbb{P}(Y \geq 42) \approx 0.999965 \text{ where } Y \sim \chi^2_{84}$$

But Fisher's H_0 was that Mendel's data was *gathered honestly* and assumed that Mendel's theory is true. And his alternative hypthesis H_1 was that the data was <u>not</u> collected honestly. Thus his p-value is in fact

$$\mathbb{P}(-Y \geq -42) = \mathbb{P}(Y \leq 42) = 0.000035$$

So Fisher's H_0 is rejected with a very low α.
Therea are many who belive taht neither Mendel, nor his assistants, did anything untoward in their experiments but it is an interesting applicaiton nonetheless.

3 Bayesian Inference

3.1 Bayesian Inference

Remark 1.1 - *Overview of Bayesian Approach*

1. Probabilities can descibe degrees of belief, *i.e.* subjective statements. Therefore we can use probability to model things beyond just data. For example, if I flip a coin and hold it concealed in my hand I could say *the probability that the coin has turned up heads is* $\frac{1}{2}$.

2. We can make subjective statements about a model parameter θ. IN particular, we can model it as a random variable with a partituclar *prior* distribution, before receiving the data. Large data sets can elimate the *prior*.

3. Inferences about θ can be conducted on the basis of the conditional distribution of the random variable θ given the data $\mathbf{x}$. THis condtional, or *posterior*, distribution will be the object through which we define point estimates and interval estimates.

Remark 1.2 - *Exentision to Remark 1.1 iii)*
We do not need to consider *hypothetical repeated experiments* when conducting Bayesian inference. The impact of the data and the model will be solely through the posterior distribution, which in turn will be defined in terms of he *prior* and the observe dlikelihood function.

Example 1.1 - *Priors are Useful*
Consider the these three statistical experiments which produce the same result in the classical, *Frequentist*, approach but are different in the *Bayesian Approach*

1. A lady who drinks milk in her tea claims to be able to tell which was poured first, the teat or the milk. In ten trials, she determines correctly whether it was tea or milk which was poured in the cup first.

2. A music expert claims to be able to tell whether a page of music was written by Haydn or by Mozart. In ten trials conducted, he correctly determines the composer every time.

3. A drunken friend says he can predict the outcome of a fair coin-flip. In ten trials, he is right every time.

Using a *frequentist* aproach $T(\mathbf{X}) := \sum_i X_i$ appears to be a good test statistic and each of these experiments produces the same p-value under this test statistic.
However by considering each experiment we note that we have *prior* beliefs about how likely each one is to be true.
These beliefs are not encoded in our *frequentist* test statistic.

Definition 1.1 - *β-Distribution*
Let $X \sim \text{Beta}(\alpha, \beta)$.
A *continuous* random variable with shape parameters $\alpha, \beta > 0$. Then

$$\begin{aligned} f_X(x) &\propto x^{\alpha-1}(1-x)^{\beta-1}\mathbb{1}\{x \in [0,1]\} \\ \mathbb{E}(X) &= \frac{\alpha}{\alpha+\beta} \\ \text{Var}(X) &= \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)} \\ \mathcal{M}_X(t) &= 1 + \sum_{k=1}^{\infty}\left(\prod_{r=0}^{k-1}\frac{\alpha+r}{\alpha+\beta+r}\right)\frac{t^k}{k!} \end{aligned}$$

Proposition 1.1 - *Usefulness of β-Distribution*
β-Distributions are used to model our *prior* beliefs about unknown parameters.
Suppose θ is an unknown parameter then we can encode our *prior* belief for the distribution of θ as

$$\vartheta \sim \text{Beta}(\alpha, \beta)$$

for some shape parmeters $\alpha, \beta > 0$.
N.B. To determine the values of α & β we define the mean & variane we want and then solve the simultaneous equations this produces.

Theorem 1.1 - *Bayes' Theorem*
Let $\{B_i\}_{i\in\mathbb{N}}$ be a collection of events wich partitions Ω. *i.e.* $\bigcup_i B_i = \Omega$ and $B_i \bigcap B_j = \emptyset \; \forall \; i \neq j$. Then

$$\begin{aligned} \mathbb{P}(A) &= \textstyle\sum_i \mathbb{P}(A|B_i)\mathbb{P}(B_i) \\ \text{and} \quad \mathbb{P}(B_j|A) &= \frac{\mathbb{P}(A|B_j)\mathbb{P}(B_j)}{\mathbb{P}(A)} \\ &= \frac{\mathbb{P}(A|B_j)\mathbb{P}(B_j)}{\sum_i \mathbb{P}(A|B_i)\mathbb{P}(B_i)} \end{aligned}$$

Example 1.2 - *Using a Prior*
Consider a bag containing a ball of unknown colour which may be either black or white (We shall refer to the is unknown colour as $\theta \in \{w, b\}$).
A white ball is added to the bag, then a ball is drawn at random , the ball we pick is white.
What is the probability that the remaining ball is white, or in other words the probabiltiy that $\vartheta = w$?

We model our prior belief abut the colour of the ball as

$$\mathbb{P}(\vartheta = w) = \mathbb{P}(\vartheta = b) = \frac{1}{2}$$

We observe $X = W$ so we can *update* the distribution of ϑ via conditional probabilties

$$\begin{aligned} \mathbb{P}(\vartheta = w|X = w) &= \frac{\mathbb{P}(X = w|\vartheta = w)\mathbb{P}(\vartheta = w)}{\mathbb{P}(X = w|\vartheta = w)\mathbb{P}(\vartheta = 2) + \mathbb{P}(X = w|\vartheta = b)\mathbb{P}(\vartheta = b)} \\ &= \frac{1 \times \frac{1}{2}}{1 \times \frac{1}{2} + \frac{1}{2} \times \frac{1}{2}} \\ &= \frac{2}{3} \end{aligned}$$

Hence the posterior distribution for ϑ is

$$\mathbb{P}(\vartheta = \theta|X = 2) = \begin{cases} \frac{2}{3}, & \vartheta = w \\ \frac{1}{3}, & \vartheta = b \end{cases}$$

Theorem 1.2 - *Generalisation of Prior Beliefs*
Suppose ϑ is a continuous random varible.
The density of the *prior*, $\mathbb{P}(\theta)$ can be generalised as

$$\begin{aligned} p(\theta|\mathbf{x}) &= \frac{f_n(\mathbf{x};\theta)p(\theta)}{\int_\Theta f_n(\mathbf{x};\psi)p(\psi)d\psi} &\propto \; f_n(\mathbf{x};\theta)p(\theta) \\ &= \frac{L(\theta;\mathbf{x})p(\theta)}{\int_\Theta L(\psi;\mathbf{x})p(\psi)d\psi} &\propto \; L(\theta;\mathbf{x})p(\theta) \end{aligned}$$

This shows the *posterior* distribution combines the information of the *prior* & that carried by the *data* (given the model).

Example 1.3 - *Bernoulli Likelihood with Beta Prior*
Let $\mathbf{X} \overset{\text{iid}}{\sim} \text{Bernoulli}(\theta)$ and suppose a $\text{Beta}(\alpha, \beta)$ distribution is used for a prior distriution of θ. Here we derive the posterior distribution $\mathbb{P}(\theta|\mathbf{x})$.
Consider the likelihood

$$\begin{aligned} L(\theta;\mathbf{x}) &\propto \prod_{i=1}^{n} \theta^{x_i}(1-\theta)^{1-x_i} \text{ a Bernoulli Distribution} \\ &= \theta^{\sum x_i}(1-\theta)^{n-\sum x_i} \\ &= \theta^{n\bar{x}}(1-\theta)^{n(1-\bar{x})} \end{aligned}$$

Since the posterior statisifies $\mathbb{P}(\theta|\mathbf{x}) \propto L(\theta;\mathbf{x})\mathbb{P}(\theta)$. Hence

$$\begin{aligned}\mathbb{P}(\theta;\mathbf{x}) &\propto \left[\theta^{n\bar{x}}(1-\theta)^{n(1-\bar{x})}\right]\cdot\left[\theta^{\alpha-1}(1-\theta)^{\beta-1}\right]\\ &= \theta^{n\bar{x}+\alpha-1}(1-\theta)^{n(1-\bar{x})+\beta-1}\\ &\sim \text{Beta}(n\bar{x}+\alpha, n(1-\bar{x})+\beta)\end{aligned}$$

Consider the affect this had on the mean

$$\begin{aligned}\text{Prior} \quad \mathbb{E}(\theta) &= \frac{\alpha}{\alpha+\beta}\\ \text{Posterior} \quad \mathbb{E}(\theta|\mathbf{x}) &= \frac{n\bar{x}+\alpha}{\alpha+\beta+n}\\ &= \bar{x}\left(\frac{n}{\alpha+\beta+n}\right)+\frac{\alpha}{\alpha+\beta+n}\\ &\underset{n\to\infty}{\longrightarrow} \bar{x}\end{aligned}$$

Remark 1.3 - *Calculating Posterior*
To calculate the *posterior*, $\mathbb{P}(\theta|\mathbf{x})$, precisely we can use that $\int p(\theta|\mathbf{x})d\theta = 1$ if θ is continuous, and $\sum p(\theta|\mathbf{x}) = 1$ is θ is discrete.
However, in many cases this is unnecessary as we can identitfy the distribution through the expressioni of the *posterior* in terms of θ.

3.2 Posterior Distributions

Definition 2.1 - *Conjugacy*
A *prior* is said to be *Conjugate* if its distribution is of the same family of distribtions as the *posterior*'s.

Proposition 2.1 - *Choosing a Posterior Distribution*
It can be hard to known what to choose as a *Posterior* distribution as there is often an overwhelming amount of information which can be incorparted into this decision.
Some common choices are

i) *Posterioir mean* of the distribtuion, $\int_\Theta \theta p(\theta|\mathbf{x})d\theta$.
N.B. useful if the posterior distribution is *unimodal*.

ii) The *Maximum a Posteriori Estimator*, $\text{argmax}_{\theta\in\Theta} p(\theta|\mathbf{x})$.
N.B. Can be misleading in some contrived situations.

iii) The *median*, or more general quantiles.

iv) *Posterior Variance*.

Example 2.1 - *Uniform Prior*
The number of male births in paris where $x = 251,527$, while the number of female births was $y = 241,945$.
Here we want to assess whether the probability of a male birth, θ, is above $\frac{1}{2}$ or not.
Consider a binomial distribution Binomial$(x+y,\theta)$.
Since we have a lack of prior information about the true value of θ we shall model it using a uniform distribution. This is equivalent to a Beta$(1,1)$ distribution.
This yields a posterior distribution

$$(\vartheta|X=x) \sim \text{Beta}(251528, 241946)$$

and from this we can compute a p-value

$$p(\mathbf{x}) = \mathbb{P}(\vartheta > \frac{1}{2}|X = x) \approx 1 - 1.15 \times 10^{-42}$$

This is very large thus we conclude that θ is most likely greater than $\frac{1}{2}$.

Theorem 2.1 - *Uniform*$[0,1] \sim$ *Beta*$(1,1)$

Definition 2.2 - *Posterior Expected Loss*
Let $p(\theta|x)$ denote the *Posterior* density.
Then the *Posterior Expected Loss* of an estimate $\hat{\theta}$ of θ is

$$R(\hat{\theta}|\mathbf{x}) := \int L(\theta, \hat{\theta})\mathbb{P}(\theta|\mathbf{x})d\theta$$

where $L(\theta, \hat{\theta})$ is a non-negative *Loss Function*.
N.B. AKA *Posterior Risk*.
N.B. $\hat{\theta}$ is a fixed value.

Definition 2.3 - *Bayes Estimate*
A *Bayes Estimate* is obtained by minimising the *Posterior Risk*

$$\hat{\theta}_{\text{Bayes}} = \text{argmin}_{\theta\in\Theta} R(\hat{\theta}|\mathbf{x})$$

Theorem 2.2 - *Squared Error Loss*
Define a loss function $L(\theta, \hat{\theta}) = (\theta - \hat{\theta})^2$ and suppose $\int \theta^2 p(\theta|\mathbf{x})d\theta < \infty$ (*i.e.* is finite). Then

$$\hat{\theta}_{\text{Bayes}} = \int_\Theta \theta p(\theta|\mathbf{x})d\theta = \text{Posterior Mean}$$

Proof 2.1 - *Theorem 2.2*
For simplicity define $\pi(\theta) := \mathbb{P}(\theta|\mathbf{x})$, $s := \int \theta^2\pi(\theta)d\theta$ and $m = \int \theta\pi(\theta)d\theta$. Then

$$\begin{aligned} R(\hat{\theta}) &= \int L(\theta, \hat{\theta})\mathbb{P}(\theta|\mathbf{x})d\theta \\ &= \int(\theta - \hat{\theta})^2\pi(\theta)d\theta \\ &= \int(\theta^2 - 2\hat{\theta}\theta + \hat{\theta}^2)\pi(\theta)d\theta \\ &= \int \theta^2\pi(\theta)d\theta - 2\hat{\theta}\int \theta\pi(\theta)d\theta + \hat{\theta}\underbrace{\int \pi(\theta)d\theta}_{=1} \\ &= s - 2\hat{\theta}m + \hat{\theta}^2 \end{aligned}$$

We want $\text{argmin}_{\hat{\theta}} R(\hat{\theta})$.
Since $R(\hat{\theta})$ is quadratic we find that

$$\hat{\theta}_{\min} = -\frac{-2m}{2 \times 1} = m$$

Remark 2.1 -
If we consider *Squared Error Loss* then th eposteriori mean minisise the risk and the minimised risk is

$$T(\theta, \hat{\theta}_{\text{Bayes}})1 = \int(\theta - \hat{\theta}_{\text{Bayes}})^2 p(\theta|\mathbf{x})d\theta = \text{Var}_{p(\theta|\mathbf{x})}(\theta)$$

which is the *Posterior Variance*.

Proposition 2.2 - *Alternative Loss Functions*

1. If $L(\theta, \hat{\theta}) := |\theta - \hat{\theta}|$ then $\hat{\theta}_{\text{Bayes}}$ is the posterior median.
2. If $L(\theta, \hat{\theta}) := \mathbb{1}\{theta \neq \hat{\theta}\}$ and θ is discrete, then $\hat{\theta}_{\text{Bayes}}$ is the posterior mode.

Proof 2.2 - *Proposition 2.2 i)*
First we compute $R(\hat{\theta}|\mathbf{x})$

$$\begin{aligned} R(\hat{\theta}) &= \int_{-\infty}^{\infty} |\theta - \hat{\theta}| p(\theta|\mathbf{x}) d\theta \\ &= \int_{-\infty}^{\hat{\theta}} (\hat{\theta} - \theta) p(\theta|\mathbf{x}) d\theta + \int_{\hat{\theta}}^{\infty} (\theta - \hat{\theta}) p(\theta|\mathbf{x}) d\theta \\ &= \hat{\theta} \int_{-\infty}^{\hat{\theta}} p(\theta|\mathbf{x}) d\theta - \hat{\theta} \int_{-\infty}^{\hat{\theta}} \theta p(\theta|\mathbf{x}) d\theta + \int_{\hat{\theta}}^{\infty} \theta p(\theta|\mathbf{x}) d\theta - \hat{\theta} \int_{\hat{\theta}}^{\infty} p(\theta|\mathbf{x}) d\theta \end{aligned}$$

We want to find the minimum of $R(\hat{\theta}|\mathbf{x})$

$$\begin{aligned} R'(\hat{\theta}) &= \tfrac{d}{d\hat{\theta}} R(\hat{\theta}) \\ &= \int_{-\infty}^{\hat{\theta}} p(\theta|\mathbf{x}) d\theta + \hat{\theta} p(\hat{\theta}|\mathbf{x}) - \hat{\theta} p(\hat{\theta}|\mathbf{x}) - \hat{\theta} p(\hat{\theta}|\mathbf{x}) - \int_{\hat{\theta}}^{\infty} p(\theta|\mathbf{x}) d\theta + \hat{\theta} p(\hat{\theta}|\mathbf{x}) \\ &= \int_{-\infty}^{\hat{\theta}} p(\theta|\mathbf{x}) d\theta - \int_{\hat{\theta}}^{\infty} p(\theta|\mathbf{x}) d\theta \end{aligned}$$

Setting $R'(\hat{\theta}) = 0$

$$\int_{-\infty}^{\hat{\theta}} p(\theta|\mathbf{x}) d\theta = \int_{\hat{\theta}}^{\infty} p(\theta|\mathbf{x}) d\theta \implies \hat{\theta} = \text{Median}$$

Taking $R''(\hat{\theta}) = p(\hat{\theta}|\mathbf{x}) + p(\hat{\theta}|\mathbf{x}) > 0 \ \forall \ \hat{\theta}$ thus this is a minimum.

3.3 Credible Intervals

Definition 3.1 - *Symmetric $1-\alpha$ Credible Interval*
In the scalar case we find $\theta_1, \theta_2 \in \Theta$ st

$$\int_{-\infty}^{\theta_1} p(\theta|\mathbf{x}) d\theta = \alpha/2 \text{ and } \int_{\theta_2}^{\infty} p(\theta|\mathbf{x}) d\theta = \alpha/2$$

The interval (θ_1, θ_2) is called a *Symmetric $1-\alpha$ Credible Interval.*

Definition 3.2 - *High Posterior Density Set*
The *High Posterior Density Set*, $\mathcal{HPD}(1-\alpha)$, is the level set

$$\mathcal{HPD}_v := \{\theta \in \Theta : p(\theta|\mathbf{x}) \geq v\}$$

where v is chosen st

$$\mathbb{P}(\theta \in \mathcal{HPD}_v|\mathbf{x}) = \int_{\mathcal{HPD}_v} p(\theta|\mathbf{x}) d\theta = 1 - \alpha$$

N.B. This is a set, not an inverval.

Theorem 3.1 - *The $\mathcal{HPD}(1-\alpha)$ redivle set is the smallest subet of Θ containing exactly $1-\alpha$ of the total probability.*

Remark 3.1 - *High Posterior Density Sets are harder to determine without a computer than Symmetric Credible Intervals*

Example 3.1 -
We have established that for the Beta/Bernoulli Model, with statistical parameter θ, the posterior for n observations is Beta$(n\bar{x}+\alpha, n(1+\bar{x})+\beta)$ distribution whose mean and variance are

$$\mathbb{E}(\theta|\mathbf{x}) = \frac{n\bar{x}+\alpha}{n+\alpha+\beta} = \frac{\bar{x}+\alpha/n}{1+(\alpha+\beta)/n}$$

and

$$\text{Var}(\theta|\mathbf{x}) = \frac{(n\bar{x}+\alpha)[n(1-\bar{x})+\beta]}{(n+\alpha+\beta)^2(n+\alpha+\beta+1)} = \frac{(\bar{x}+\alpha/n)[(1-\bar{x})+\beta/n]}{[1+(\alpha+\beta)/n]^2[n+\alpha+\beta+1]}$$

In this example it is easier to see that the Bayes estimator with squared error loss is

$$\hat{\theta}_{\text{Bayes}}(\mathbf{X}) + \frac{n\bar{X}+\alpha}{n+\alpha+\beta}$$

and hence that

$$\hat{\theta}_{\text{Bayes}}(\mathbf{X}) \to_{\mathbb{P}(\cdot;\theta)} \theta \text{ by the weak law of large numbers}$$

and the fact that $\frac{\alpha}{n}$ & $\frac{\alpha+\beta}{n}$ both converge to 0 as $n \to \infty$.
Similaryy, the posterior variance is

$$\frac{\bar{X}+\alpha/n)[(1-\bar{X})+\beta/n]}{[1+(\alpha+\beta)/n]^2[n+\alpha+\beta+1]} \to_{\mathbb{P}(\cdot;\theta)} 0$$

since $\bar{X} \to_{\mathbb{P}(\cdot;\theta)} \theta$.
This is a rudimentary frequentist analysis of a Bayes Estimator.

3.4 Bayesian Hypothesis Testing

0 Appendix

Definition 0.1 - *Gradient*

$$\nabla f(\boldsymbol{\theta}; \mathbf{x}) := \left(\frac{\partial f(\boldsymbol{\theta}; \mathbf{x})}{\partial \theta_1}, \dots, \frac{\partial f(\boldsymbol{\theta}; \mathbf{x})}{\partial \theta_n} \right)$$

Definition 0.2 - *Hessian*

$$\nabla f(\boldsymbol{\theta}; \mathbf{x}) := \begin{pmatrix} \frac{\partial^2 f(\boldsymbol{\theta}; \mathbf{x})}{\partial \theta_1^2} & \frac{\partial^2 f(\boldsymbol{\theta}; \mathbf{x})}{\partial \theta_1 \partial \theta_2} & \cdots & \frac{\partial^2 f(\boldsymbol{\theta}; \mathbf{x})}{\partial \theta_n \theta_1} \\ \frac{\partial^2 f(\boldsymbol{\theta}; \mathbf{x})}{\partial \theta_1^2} & \frac{\partial^2 f(\boldsymbol{\theta}; \mathbf{x})}{\partial \theta_1 \theta_2} & \cdots & \frac{\partial^2 f(\boldsymbol{\theta}; \mathbf{x})}{\partial \theta_n \theta_2} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 f(\boldsymbol{\theta}; \mathbf{x})}{\partial \theta_1 \partial \theta_n} & \frac{\partial^2 f(\boldsymbol{\theta}; \mathbf{x})}{\partial \theta_2 \theta_n} & \cdots & \frac{\partial^2 f(\boldsymbol{\theta}; \mathbf{x})}{\partial \theta_n^2} \end{pmatrix}$$

Theorem 0.1 - *Minimum of a Quadratic*
Consider the quadratic $ax^2 + bx + c$. We have that

$$x_{\min} = -\frac{b}{2a}$$

0.1 Notation

Notation	Denotes
$Z_n \to_{\mathbb{P}} Z$	$\{Z_n\}_{n\in\mathbb{N}}$ converges in *Probabilitiy* to random variable Z.
$Z_n \to_{\mathcal{D}} Z$	$\{Z_n\}_{n\in\mathbb{N}}$ converges in *Distribution* to random variable Z.
$Z_n \to_{qm} Z$	$\{Z_n\}_{n\in\mathbb{N}}$ converges in *Quadratic Mean* to random variable Z.
$\theta \in \Theta \subseteq \mathbb{R}^{d_\theta}$	Scalar or vector parameter characterising a probability distribution
$\hat{\theta}$	Estimation for the value of the parameter θ
θ^*	True value of the paramter θ
$\mathbb{P}$	Probability measure $\mathbb{P} : \mathcal{F} \to [0, 1]$
Ω	Sample space
X	Scalar random variable
$\mathcal{F}$	Sigma field (Set of events)
χ	Support of rv XX. A set set X is definitely in it *i.e.* $\mathbb{P}(X \in \chi; \theta) = 1$
$\mathbf{X}$	Vector consiting of scalar random variables

0.2 R

Command	Result
hist(a)	Plots a histogram of the values in array a
mean(a)	Returns the mean value of array a
rbinom(s,n,p)	Samples n of $Bi(n, p)$ random variables
rep(v,n)	Produces an array of size n where each entry has value v
$x \leftarrow v$	Maps value v to variable x

0.3 Probability Distributions

Definition 3.1 - *Binomial Distribution*
Let X be a discrete random variable modelled by a *Binomial Distribution* with n events and rate of success p.

$$\begin{aligned} p_X(k) &= \tbinom{n}{k} p^k (1-p)^{n-k} \\ \mathbb{E}(X) np = &\ \& \quad Var(X) = np(1-p) \end{aligned}$$

Definition 3.2 - *Gamma Distribution*
Let T be a continuous randmo variable modelled by a *Gamma Distribution* with shape parameter α & scale parameter λ. Then

$$\begin{aligned} f_T(x) &= \frac{\lambda^\alpha x^{\alpha-1} e^{-\lambda x}}{\Gamma(\alpha)} \quad \text{for } x > 0 \\ \mathbb{E}(T) = \frac{\alpha}{\lambda} \quad \& \quad & Var(T) = \frac{\alpha}{\lambda^2} \end{aligned}$$

N.B. $\alpha, \lambda > 0$.

Definition 3.3 - *Exponential Distribution*
Let T be a continuous random variable modelled by a *Exponential Distribution* with parameter λ. Then

$$\begin{aligned} f_T(t) &= \mathbb{1}\{t \geq 0\}.\lambda e^{-\lambda t} \\ F_T(t) &= \mathbb{1}\{t \geq 0\}.\left(1 - e^{-\lambda t}\right) \\ \mathbb{E}(X) = \tfrac{1}{\lambda} \quad \& \quad & Var(X) = \tfrac{1}{\lambda^2} \end{aligned}$$

N.B. Exponential Distribution is used to model the wait time between decays of a radioactive source.

Definition 3.4 - *Normal Distribution*
Let X be a continuous random variable modelled by a *Normal Distribution* with mean μ & variance σ^2.
Then

$$\begin{aligned} f_X(x) &= \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \\ F_X(x) &= \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{x} e^{-\frac{(y-\mu)^2}{2\sigma^2}} dy \\ M_X(\theta) &= e^{\mu\theta + \sigma^2\theta^2(1/2)} \\ \mathbb{E}(X) = \mu \quad \& \quad & Var(X) = \sigma^2 \end{aligned}$$

Definition 3.5 - *Poisson Distribution*
Let X be a discrete random variable modelled by a *Poisson Distribution* with parameter λ. Then

$$\begin{aligned} p_X(k) &= \frac{e^{-\lambda}\lambda^k}{k!} \qquad \text{For } k \in \mathbb{N}_0 \\ \mathbb{E}(X) = \lambda \quad \& \quad & Var(X) = \lambda \end{aligned}$$

N.B. Poisson Distribution is used to model the number of radioactive decays in a time period.