Stochastic Optimisation - Notes

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NOTE - *Markov Chain* typically refers to the discrete setting; whilst *Markov Process* typically refers to the continuous setting.

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1 Multi-Armed Bandit

1.1 The Problem

Example 1.1 - *Motivating Example*
Consider having a group of patients and several treatments they could be assigned to. How best do you go about determining which treatment is best? The obvious approach is to assign some of the patients randomly and then assign the rest to the best treatment, but how much evidence is sufficient? And how likely are you to choose a sub-optimal treatment?

Definition 1.1 - *Multi-Armed Bandit Problem*
An agent is faced with a choice of K actions. Each (discrete) time step the agent plays action i they receive a reward from the random real-valued distribution ν_i. Each reward is independent of the past. The distributions $\nu_1, \dots, \nu_K$ are unknown to the agent.
In the *Multi-Armed Bandit Problem* the agent seeks to maximise a measure of long-run reward.

Remark 1.1 - *Informal Definition of Multi-Armed Bandit Problem*
Given a finite set of actions and a random reward for each action, how best do we learn the reward distribution and maximise reward in the long-run.

Definition 1.2 - *Formal Definition of Multi-Armed Bandit Problem*
Consider a sequence of (unknown) mutually independent random variables $\{X_i(t)\}_{i\in[1,K]}$, with $t \in \mathbb{N}$. Consider $X_i(t)$ to be the distribution of rewards an agent would receive if they performed action i at time t. Since the rewards are independent of the past $X_i(t), X_i(t+1), \dots$ are IID random variables. The *Multi-Armed Bandit Problem* tasks us to find the greatest expected reward from all the actions.

$$\mu^* := \max_{i=1}^{K} \mu_i \quad \text{where } \mu_i = \mathbb{E}(X_i(t))$$

There are a number of ways to formalise this objective.

Remark 1.2 - *Assumptions*
For the *Multi-Armed Bandit Problem* we make the following assumptions about the set up

- When action i is played only the realisation of $X_i(t)$ is observed and none of $X_j(t),\ j \neq i$, are observed. Thus when the agent's t^{th} action is played only the rewards of actions $\{1, \dots, t-1\}$ are known to the agent.
- The agent has access to an external source of randomness which is used to choose it's next action.

Definition 1.3 - *Strategy, $I(\cdot)$*
Our agent's strategy $I : \mathbb{N} \to [1, K]$ is a function which determines which action the agent shall make at a given point in time. The strategy can use the knowledge gained from previous actions & their rewards only.

$$I(t) = I\big(t, \underbrace{\{I(s)\}_{\in[1,t)}}_{\text{Prev. Actions}}, \underbrace{\{X_{I(s)}(s)\}_{\in[1,t)}}_{\text{Prev. Rewards}}\big) \in [1, K]$$

Definition 1.4 - *Long-Run Average Reward Criterion, X_**
For a strategy $I(\cdot)$ we define the following measure for *Long-Run Average Reward*

$$X_* = \lim_{T\to\infty} \inf \frac{1}{T} \sum_{t=1}^{T} \mathbb{E}(X_{I(t)})$$

The *Infinum* is taken as there is no guarantee the limit exists (depending on the strategy), typically we will only deal with strategies where this limit exists.
Most strategies as based only on realisations of $\{X_i(s)\}_{s\in[1,t)}$, thus $\mathbb{E}(X_{I(t)}) \leq \mu^*$ and thus $X_* \leq \mu^*$. A strategy $I(\cdot)$ is *Optimal* if $X_* = \mu^*$.

Remark 1.3 - *It is not hard to find an Optimal Strategy in the (very) long run, so we are going to look at Regret Minimisation First.*

1.2 Regret Minimisation

Definition 1.5 - *Regret,* R_n
Regret is a measure of how much reward was lost during the first n time steps. The *Regret* R_n of a strategy $\{I(t)\}_{t\in\mathbb{N}}$ in the first n time steps is given by

$$\begin{aligned} R_n &= \max_{k=1}^{K} \sum_{t=1}^{n} \mathbb{E}[\underbrace{X_k(t)}_{\text{Best Pos}} - \underbrace{X_{I(t)}(t)}_{\text{Actual}}] \\ &= n\mu^* - \sum_{t=1}^{n} \mathbb{E}\big[X_{I(t)}(t)\big] \end{aligned}$$

Regret only involves expectation and thus can be learnt from observations. We want to produce a strategy where *Total Regret* grows sub-linearly.(i.e. $R_T/T \overset{T\to\infty}{\longrightarrow} 0$)

Remark 1.4 - *Minimising the growth rate of R_T with T is quite hard.*
The best achievable regret scales as $R_T \sim c\log T$ (i.e. $R_T/c\log T \overset{T\to\infty}{\longrightarrow} 1$) where c depends on the reward distributions $X_1(t), \dots, X_K(t)$.

Definition 1.6 - *Pseudo-Regret,* $\tilde{R}_n$
Pseudo-Regret $\tilde{R}_n$ is a less popular alternative to *Regret* R_n. The *Pseudo-Regret* R_n of a strategy $\{I(t)\}_{t\in\mathbb{N}}$ in the first n time steps is given by

$$\tilde{R}_n = \max_{k=1}^{K} \sum_{t=1}^{n} \big(X_k(t) - X_{I(t)}(t)\big)$$

Pseudo-Regret includes intrinsic randomness (which is independent of the past) and thus cannot be learnt from observations.

1.3 Best Arm Identification for Bernoulli Distribution

Example 1.2 - *Best Arm Identification for Bernoulli Bandits*
Consider a bandit with two *Bernoulli* arms: $\{X_1(t)\}_{t\in\mathbb{N}}$ IID RVs with distribution $\text{Bern}(\mu_1)$; and, $\{X_2(t)\}_{t\in\mathbb{N}}$ IID RVs with distribution $\text{Bern}(\mu_2)$.
Suppose $\mu_1 > \mu_2$ (i.e. arm 1 is better). Let the player play each arm n times and declare the arm with the greatest empirical mean to be the better arm. *What is the probability of choosing the wrong arm (Arm 2)?.*

An error occurs if $\sum_{t=1}^{n} X_2(t) \geq \sum_{t=1}^{n} X_1(t)$ and thus we want to calculate the probability of this event.
Define $\{Y(t)\}_{t\in\mathbb{N}}$ st $Y(t) := \{X_2(t) - X_1(t)$. This means $Y(t) \in \{-1, 0, 1\} \subset [-1, 1]$.
To use *Hoeffding's inequality* we need to scale Y to be in $[0, 1]$, so we define

$Z(t) := \frac{1}{2}(Y(t)+1)$. We have $\mathbb{E}(Z(t)) = \frac{1}{2}(1+\mu_2-\mu_1)$ and an error occurs if $\sum_{t=1}^n Y(t) > 0 \iff \sum_{t=1}^n Z(t) \geq \frac{n}{2}$. By *Hoeffding's Inequality*

$$\begin{array}{rcll}
\mathbb{P}(\text{error}) & = & \mathbb{P}\left(\displaystyle\sum_{i=1}^n Z(t) \geq \frac{n}{2}\right) & \\
& = & \mathbb{P}\left(\left(\displaystyle\sum_{i=1}^n Z(t)\right) - \frac{n}{2}(1+\mu_2-\mu_1) \geq \frac{n}{2}(\mu_1-\mu_2)\right) & \text{subtracting } \mu \text{ from both sides} \\
& = & \mathbb{P}\left(\displaystyle\sum_{i=1}^n \left(X_i - \underbrace{\frac{1}{2}(1+\mu_2-\mu_1)}_{\mu}\right) \geq n\underbrace{\frac{1}{2}(\mu_1-\mu_2)}_{t}\right) & \text{arranging for Hoeffding's} \\
& \leq & \exp\left(-2n\cdot\frac{1}{4}(\mu_1-\mu_2)^2\right) \quad \text{by Hoeffding's Inequality} & \\
& = & \exp\left(-\frac{n}{2}(\mu_1-\mu_2)^2\right) &
\end{array}$$

1.4 Heuristic

2 Probability

Definition 2.1 - *Random Process*
A *Random Process* is a collection of random variables indexed by time $\{X_t\}_{t\in T}$ (e.g. flipping a coin several times). Each of these random variables can take a value from a state space S. A random process a *Discrete Time Process* if the index set T is discrete. A random process a *Continuous Time Process* if the index set T is continuous.

2.1 Probability Inequalities

Remark 2.1 - *We can use the moments of a random variable to determine bounds on the probability of it taking values in a certain set.*
Theorem 2.1 - *Markov's Inequality*
Let X be a non-negative random variable. Then

$$\forall\ c > 0 \quad \mathbb{P}(X \geq c) \leq \frac{\mathbb{E}(X)}{c}$$

Proof

Consider an event A and define its indicator $\mathbb{1}(A)(\omega) := \begin{cases} 1 & \omega \in A \\ 0 & \omega \notin A \end{cases}$. Fix $c > 0$, then

$$\begin{array}{rrcl}
& \mathbb{E}(X) & \geq & \mathbb{E}[X\mathbb{1}(X \geq c)] \\
& & \geq & \mathbb{E}[c\mathbb{1}(X \geq c)] \\
& & = & c\mathbb{P}(X \geq c) \\
\implies & \mathbb{P}(X \geq c) & \leq & \frac{1}{c}\mathbb{E}(X)
\end{array}$$

Theorem 2.2 - *Chebyshev's Inequality*
Let X be a random-variable with finite mean and variance. Then

$$\forall\ c > 0 \quad \mathbb{P}(|X - \mathbb{E}(X)| \geq c) \leq \frac{\text{Var}(X)}{c^2}$$

Proof
Note that the events $|X - \mathbb{E}(X)| \geq c$ and $(X - \mathbb{E}(X))^2 \geq c^2)$ are equivalent. Note that $\text{Var}([X - \mathbb{E}(X)]^2) = \text{Var}(X)$. Then the result follows by *Markov's Inequality.*

Theorem 2.3 - *Chebyshev's Inequality for Sum of IIDs*
Let $X_1, \dots, X_n$ be IID random variables with finite mean μ and finite variance σ^2.

$$\forall\ c > 0 \quad \mathbb{P}\left(\left|\left(\sum_{i=1}^n X_i\right) - n\mu\right| \geq nc\right) \leq \frac{\sigma^2}{nc^2}$$

Proof
This is proved by extending the proof of `Theorem 2.2` and noting that the variance of a sum of IIDs is the sum of the individual variances.

Theorem 2.4 - *Chernoff Bounds*
Let X be a random variable whose moment-generating function $\mathbb{E}[e^{\theta X}]$ is finite $\forall\theta$. Then

$$\forall\ c \in \mathbb{R} \quad \mathbb{P}(X \geq c) \leq \inf_{\theta>0} e^{-\theta c}\mathbb{E}(e^{\theta X}) \quad \text{and} \quad \mathbb{P}(X \leq c) \leq \inf_{\theta<0} e^{-\theta c}\mathbb{E}(e^{\theta X})$$

Proof
Note that the events $X \geq c$ and $e^{\theta X} \geq e^{\theta c}$ are equivalent for all $\theta > 0$. The result follows by applying *Markov's Inequality* to $r^{\theta X}$ and taking the best bound over all possible θ.

$$\begin{array}{rcl} \mathbb{P}(X \geq c) & = & \mathbb{P}(e^{\theta X} \geq e^{\theta c}) \\ & \leq & e^{-\theta c}\mathbb{E}(e^{\theta X}) \\ & \leq & \inf_{\theta<0} e^{-\theta c}\mathbb{E}(e^{\theta X}) \end{array}$$

Theorem 2.5 - *Chernoff Bounds for Sum of IIDs*
Let $X_1, \dots, X_n$ be IID random variables. Then $\forall\ c \in \mathbb{R}$

$$\begin{array}{rl} \mathbb{P}\left(\displaystyle\sum_{i=1}^n X_i \geq nc\right) & \leq \displaystyle\inf_{\theta>0} e^{-n\theta c}\left(\mathbb{E}\left[e^{\theta X}\right]\right)^n \\ \mathbb{P}\left(\displaystyle\sum_{i=1}^n X_i \leq nc\right) & \leq \displaystyle\inf_{\theta<0} e^{-n\theta c}\left(\mathbb{E}\left[e^{\theta X}\right]\right)^n \end{array}$$

Theorem 2.6 - *Jensen's Inequality*
Let f be a *Convex Function* and X be a random variable. Then

$$\mathbb{E}[f(X)] \geq f(\mathbb{E}[X])$$

Theorem 2.7 - *Bound on Moment Generating Function*
Let X be a random variable taking values in $[0, 1]$ with finite expected value μ. Then we can bound the MGF of the centred random variable with

$$\forall\ \theta \in \mathbb{R} \quad \mathbb{E}\left[e^{\theta(X-\mu)}\right] \leq e^{\theta^2/8}$$

Proof (of weaker version)
Let X_1 be an independent copy of X, so both have mean μ. We can easily verify that $f(x) = e^{\theta x}$ is a convex function for all $\theta \in \mathbb{R}$. By *Jensen's Inequality* to $f(\cdot)$ and X_1

$$\mathbb{E}[e^{-\theta X_1}] \geq e^{-\theta\mathbb{E}[X_1]} = e^{-\theta\mu} \quad (1)$$

Consequently

$$\begin{array}{rrll} & \mathbb{E}[e^{\theta(X-X_1)}] & = & \mathbb{E}[e^{\theta X}]\cdot\mathbb{E}[e^{-\theta X_1}] & \text{by independence} \\ & & \geq & \mathbb{E}[e^{\theta X}]\cdot e^{-\theta\mu} & \text{by (1)} \\ & & = & \mathbb{E}[e^{\theta(X-\mu)}] & \\ \implies & \mathbb{E}[e^{\theta(X-X_1)}] & \geq & \mathbb{E}[e^{\theta(X-\mu)}] & \end{array}$$

Since $X, X_1 \in [0,1]$ then $(X-X_1)\in[-1,1]$. As X, X_1 have the same distribution $\mathbb{E}(X-X_1)=0$ and the distribution is symmetric around the mean.
Define random variable S which is independent of X, X_1 and takes values $\{-1,1\}$, each with probability $p=\frac{1}{2}$. $S(X-X_1)$ has the same distribution as $(X-X_1)$ due to independence of S and symmetry of $(X-X_1)$. Hence

$$\begin{array}{rrll} & \mathbb{E}[e^{\theta(X-X_1)}] & = & \mathbb{E}[e^{\theta S(X-X_1)}] & \text{by identical distribution} \\ & & \leq & \mathbb{E}[e^{\theta S}] & \text{(2) since } (X-X_1)\in[-1,1] \\ & & = & \frac{1}{2}(e^{\theta}+e^{-\theta}) & \text{by def. of expectation} \\ \implies & \mathbb{E}[e^{\theta(X-X_1)}] & \leq & \frac{1}{2}(e^{\theta}+e^{-\theta}) & \end{array}$$

Note that $f(x)=e^x+e^{-x}$ is increasing for $x\in(0,\infty)$; decreasing for $x\in(-\infty,0)$; and symmetric around 0.
Using a *Taylor Series* we an observed that

$$\begin{array}{rrll} & \frac{1}{2}(e^{\theta}-e^{-\theta}) & = & \sum\limits_{n=0}^{\infty}\frac{\theta^{2n}}{(2n)!} & \text{by Taylor expansion of} e^x \\ & & \leq & \sum\limits_{n=0}^{\infty}\dfrac{(\theta^2/2)^n}{n!} & \\ & & \overset{\text{def.}}{=} & e^{\theta^2/2} & \\ \implies & \frac{1}{2}(e^{\theta}+e^{-\theta}) & \leq & e^{\theta^2/2} & \end{array}$$

Combining all these results we get

$$\begin{array}{rl} & \mathbb{E}[e^{\theta(X-\mu)}]\leq\mathbb{E}[e^{\theta(X-X_1)}]\leq\frac{1}{2}(e^{\theta}+e^{-\theta})\leq e^{\theta^2/2} \\ \implies & \mathbb{E}[e^{\theta(X-\mu)}]\leq e^{\theta^2/2} \end{array}$$

$\square$

Theorem 2.8 - *Hoeffding's Theorem*
Let $X_1,\dots,X_n$ be IID random variables taking values in $[0,1]$ and with finite expected value μ. Then

$$\forall\ t>0\quad \mathbb{P}\left(\sum_{i=1}^{n}(X_i-\mu)>nt\right)\leq e^{-2nt^2}$$

Proof
From *Chernoff's Bound* we have that

$$\forall\ \theta>0\quad \mathbb{P}\left(\sum_{i=1}^{n}(X_i-\mu)>nt\right)\leq e^{-\theta nt}\left(\mathbb{E}[e^{\theta(X-\mu)}]\right)^n$$

Using `Theorem 2.7` to bound the moment generating function, we get

$$\forall\ \theta>0\quad \mathbb{P}\left(\sum_{i=1}^{n}(X_i-\mu)>nt\right)\leq e^{-\theta nt}\cdot e^{n\frac{\theta^2}{8}}=e^{n\left(-\theta t+\frac{1}{8}\theta^2\right)}$$

Thus, by taking logs and rearranging, we get

$$\forall\ \theta > 0 \quad \frac{1}{n}\log\mathbb{P}\left(\sum_{i=1}^{n}(X_i-\mu) > nt\right) \leq -\theta t + \frac{\theta^2}{8}$$

We have that $-\theta t + \frac{\theta^2}{8}$ is minimised at $\theta = 4t$ which is positive if t is positive. Thus, by applying this bound and substituting $\theta = 4t$ we get

$$\forall\ \theta > 0 \quad \mathbb{P}\left(\sum_{i=1}^{n}(X_i-\mu) > nt\right) \leq e^{n\left(-4t^2+\frac{1}{8}(16t^2)\right)} = e^{n(-4t^2+2t^2)} = e^{-2nt^2}$$

□

2.2 Markov Processes

Definition 2.2 - *Markov Property*
A random process has the *Markov Property* if the conditional probability of a future state only depends on the current state.

$$\mathbb{P}(X_{t+1} = y | X_t = x_t, X_{t-1} = x_{t-1}) = \mathbb{P}(X_{t+1} = y | X_t = x_t)$$

A random process with the *Markov Property* is called a *Markov Process/Chain.*

Remark 2.2 - *On this course we only deal with discrete time markov chains*
Definition 2.3 - *Transience*
A state $x \in S$ is *Transient* if $\mathbb{P}(\exists\ t > 0 : X_t = x | X_0 = x) < 1$. The number of times the markov chain returns to a transient state is finite, with probability 1.

Definition 2.4 - *Recurrent*
A state $x \in S$ is *Recuurent* if $\mathbb{P}(\exists\ t > 0 : X_t = x | X_0 = x) = 1$. The number of times the markov chain returns to a recurrent state is infinite, with probability 1.
Every markov chain, with a finite state space S, has a recurrent communicating class.

Definition 2.5 - *Communication Class*
We say $y \in S$ is *Accessible* from $x \in S$ if $\exists\ t \geq 0$ st $[P^t]_{xy} > 0$.
We say x and y *communicate* (denoted xCy) if: x is *accessible* from y and y is *accessible* from x.
Communication is an *equivalence relation* on the state space S. Hence, *communication* partitions S into equivalence classes called *Communication Classes.* All elements of a *Communication Class* communicate with all other elements in the class, it is possible for elements to be accessible from another class but not for those elements to *communicate.*
If one state in a *Communicating Class* is *Transient/Recurrent* then all states are in that class.
If a *Markov Chain* has only one communicating class it is called *Irreducible.*

2.2.1 Discrete Time Markov Chains

Proposition 2.1 - *Characterising a Discrete Time Markov Process*
A *Discrete Time Markov Process* can be characterised by the set of all 1-step conditional probabilities

$$\mathbb{P}(X_{t+1} = y | X_t = x)\ \forall\ x, y \in S$$

A markov chain is *time-homogeneous* if the 1-step conditionals only depend on x, y and not on t ($\mathbb{P}(X_{t+1} = y|X_t = x) = \mathbb{P}(X_1 = t|X_0 = x)$). The 1-step conditional probabilities of a *time-homogeneous markov process* can be specified in an $|S| \times |S|$ matrix P where

$$p_{x,y} = \mathbb{P}(X_{t+1} = y|X_t = x)$$

P is a *Stochastic Matrix.*

Proposition 2.2 - *n-Step Transition Probabilities from 1-Step Transition Matrix*
Let P be the 1-step transition matrix for a *time-homogeneous.*
The 2-step transition probabilities (ie $\mathbb{P}(X_{t+2} = z|X_t = x)$) can be found as

$$\begin{aligned}
\mathbb{P}(X_{t+2} = z|X_t = x) &= \mathbb{P}(X_2 = z|X_0 = x) && \text{by time-homogeneity} \\
&= \textstyle\sum_{y \in S} \mathbb{P}(X_2 = z, X_1 = y|X_t = x) \\
&= \textstyle\sum_{y \in S} \mathbb{P}(X_1 = y|X_0 = x)\mathbb{P}(X_2 = z|X_1 = t, X_0 = x) \\
&= \textstyle\sum_{y \in S} \mathbb{P}(X_1 = y|X_0 = x)\mathbb{P}(X_2 = z|X_1 = y) \\
&= \textstyle\sum_{y \in S} p_{xy}p_{yz} \\
&\equiv [P^2]_{xz}
\end{aligned}$$

This can be generalise for the n-step transition probabilities with

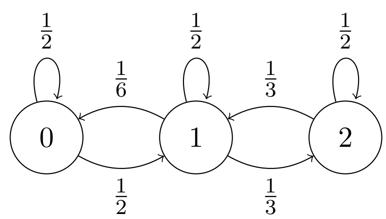
$$\mathbb{P}(X_{t+n} = z|X_t = x) = [P^n]_{xz}$$

Proposition 2.3 - *Any Joint Probability from 1-Step Transition Matrix*
For a time homogeneous markov process the joint distribution of any transition can be computed by considering the individual steps of the transition.

$$\mathbb{P}(X_{n_0} = x_0, X_{n_1} = x_1, X_{n_2} = x_2, \dots) = \mathbb{P}(X_{n_0} = x_0) \cdot [P^{n_0 - n_1}]_{x_0 x_1} \cdot [P^{n_1 - n_2}]_{x_1 x_2} \cdots$$

Proposition 2.4 - *State Diagram Representation*
A graph/automata can be drawn to represent the transition probability matrix P. A node is assigned for each member of the state space S and an arrow is drawn between each pair of nodes (x, y) where $P_{xy} \neq 0$. Generally the value of P_{xy} is denoted on the arrow.

$$P = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{6} & \frac{1}{2} & \frac{1}{3} \\ 0 & \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$



Definition 2.6 - *Invariant Distribution*
Let $\boldsymbol{\mu}(t)$ denote the probability distribution of random variable X (i,e, $\mu_x(t) = \mathbb{P}(X_t = x)$). Then

$$\begin{aligned}
& \boldsymbol{\mu}(t+1) &= \mathbb{P}(X_{t+1} = y) = \textstyle\sum_{x \in S} \mathbb{P}(X_t = x, X_{t+1} = y) = \sum_{x \in S} \mu_x(t) p_{xy} = \boldsymbol{\mu}(t)P \\
\implies & \boldsymbol{\mu}(t+1) &= \boldsymbol{\mu}(t)P
\end{aligned}$$

A distribution $\boldsymbol{\pi}$ on the state space is called an *Invariant Distribution* if $\boldsymbol{\pi} = \boldsymbol{\pi}P$. If X_t has distribution $\boldsymbol{\pi}$ so will $X_{t+1}, \dots$. Every markov chain with a *finite* state space S has an *invariant distribution.* (Not necessarily true if S is infinite).

Proposition 2.5 - *Finding an Invariant Distribution*
If an *Invariant Distribution* it is easy to find by solving $\boldsymbol{\pi}P = \boldsymbol{\pi}$ and using normalising constant $\sum_{x\in S}\pi_x = 1$.

Remark 2.3 - *If a Markov chain is irreducible, its invariant distribution (if one exists) is unique*
If a *Markov Chain* is irreducible and has a finite state space, then it has a unique invariant distribution.

Example 2.1 - *Markov Chains*

- The *Asymmetric Simple Random Walk* on $\mathbb{Z}$ is $\underbrace{\text{irreducible}}_{\text{obvious}}$, $\underbrace{\text{transient}}_{\text{not obvious}}$ and has no invariant distribution.
- The *Symmetric Simple Random Walk* on $\mathbb{Z}$ is $\underbrace{\text{irreducible}}_{\text{obvious}}$, $\underbrace{\text{recurrent}}_{\text{not obvious}}$ and has no invariant distribution.

Theorem 2.9 - *Ergodic Theorem for Markov Chains*
Let $\{X_t\}_{t\in\mathbb{N}}$ be an irreducible markov chain on state space S (not necessarily finite) with unique invariant distribution $\boldsymbol{\pi}$. Then

$$\forall\ x \in S \quad \lim_{t\to\infty}\frac{1}{t}\sum_{s=1}^{t}\mathbb{1}(X_s = x) = \pi_x$$

i.e. The fraction of time spend in state $x \in S$ tends to π_x in the long run.

Definition 2.7 - *Period*
The *Period* of a state $x \in S$ is the greatest common divisor of all possible return times to x

$$\text{Period}(x) := \gcd\big(\{t > 0 : \mathbb{P}(X_t = x|X_0 = x) > 0\}\big)$$

A state $x \in S$ is *Aperiodic* if $\text{Period}(x) = 1$. An irreducible markov chain is *aperiodic* if all its states are aperiodic.
All states in a *communicating class* have the same period.

Proposition 2.6 - *Marginal Distribution of Irreducible, Aperiodic Markov Chain*
If an irreducible, aperiodic Markov Chain has an invariant distribution $\boldsymbol{\pi}$, then

$$\forall\ x \in S \quad \mu_x(t) \overset{t\to\infty}{\longrightarrow} \pi_x$$

Definition 2.8 - *Reversibility*
A markov chain $\{X_t\}_{t\in\mathbb{Z}}$ is *Reversible* if all joint distributions are the same forwards and backwards in time. (i.e. the distribution of the chain is the same if it was reversed).
An irreducible markov chain $\{X_t\}_{t\in\mathbb{Z}}$ with transition matrix P is reversible iff

$$\exists\ \boldsymbol{\pi} \quad \text{st} \quad \pi_x p_{xy} = \pi_y p_{yx}\ \forall\ x, y \in S$$

This is the *Local/Detailed Balance Equation.* Note that this is a system of $\binom{|S|}{2}$ equations which need to be consistent for reversibility to exist.

2.2.2 Continuous Time Markov Process

Definition 2.9 - *Continuous Time Markov Process*
A stochastic process $\{X_t\}_{t\in\mathbb{R}}$ is a *Continuous Time Markov Process* on state space s if

$$\forall\ s<t\ \&\ x,y\in S\quad \mathbb{P}(X_t=y|X_s=x,X_u,u\leq s)=\mathbb{P}(X_t=y|X_s=x)$$

ie future values only depend on the present value and not past.
If $\forall\ t,s,x,y\ \mathbb{P}(X_t=y|X_s=x)$ depends only on $x,y,t-s$ (observed values & change in time) then the process is *Time-Homogeneous.*
For *Time-Homogenous Markov Processes* we let $P(t)$ denote the stochastic matrix with the probability of each possible transition after t time $[P(t)]_{xy}=\mathbb{P}(X_t=y|X_0=x)$.

Remark 2.4 - *A Time-Homogenous Markov Process is completely described by its initial condition and the family of transition probability matrices* $\{P(t):t\geq 0$
This set of matrices $\{P(t):t\geq 0$ is uncountably large.

Definition 2.10 - *Chapman-Kolmogorov Equations*
For a *Time-Homogeneous Markov Process* the family of stochastic matrices $\{P(t):t\geq 0$ satisfy the following:

i). $P(0)=I$;

ii). $P(t+s)=P(t)P(s)=P(s)P(t)$

Hence

$$\frac{d}{dt}P(t):=\lim_{\delta\to0}\frac{P(t+\delta)-P(t)}{\delta}=\lim_{\delta\to0}\frac{\overbrace{P(t)P(\delta)}^{\text{ii)}}-P(t)}{\delta}=P(t)\lim_{\delta\to0}\frac{(P(\delta)-I)}{\delta}$$

Suppose that $Q:=\lim_{\delta\to0}\dfrac{P(\delta)-P(0)}{\delta}=\lim_{\delta\to0}\dfrac{P(\delta)-\overbrace{I}^{\text{i)}}}{\delta}$ exists.
Then $P(t)$ solve the following differential equations, known as the *Chapman-Kolmogorov Equations*

$$\frac{d}{dt}P(t)=\underbrace{P(t)Q}_{\text{forward eqn.}}=\underbrace{QP(t)}_{\text{backward eqn.}}$$

The solution to these equations is

$$P(t)=P(0)e^{Qt}=e^{Qt}P(0)=e^{Qt}I=e^{Qt}$$

N.B. Q is called the *Rate Matrix* or *Infinesimal Generator* of the markov process.

Proposition 2.7 - *Properties of the Rate Matrix, Q*
Let Q be the rate matrix of a *continuous-time markov process.* Q has the following properties

- If $n\neq y$ then $q_{xy}:=\lim_{\delta\to0}\dfrac{[P(\delta)]_{xy}-0}{\delta}\geq 0$. (The off-diagonal elements are non-negative).
- $\forall\ x\in S,\ \sum_{y\in S}q_{xy}=\lim_{\delta\to0}\dfrac{1-1}{\delta}=0$. The rows of Q sum to 0.
- Thus, the diagonal entries q_{xx} are negative. (We denote $-q_{xx}$ by q_x)

Proposition 2.8 - *Interpreting the Rate Matrix Q*
Let Q be the rate matrix of a *continuous-time markov process.*
If the markov process enters state x at time t, it will remain in x for a random time which is distributed $\text{Exp}(q_x)$. (Note that $q_x := -q_x x$).
It the jumps to state y with probability $\frac{q_x y}{q_x}$, independent of the past.

Definition 2.11 - *Invariant Distributions*
Suppose a *Continuous-Time Markov Process* starts with distribution $\boldsymbol{\mu}(0)$ on state space S (i.e. $\mathbb{P}(X_0 = x) = [\boldsymbol{\mu}(0)]_x$). Then, the distribution of X_t is $\boldsymbol{\mu}(t) := \boldsymbol{\mu}(0)P(t) = \boldsymbol{\mu}(0) \underbrace{e^{Qt}}_{\text{CK Eqns}}$.
If there exists a distribution $\boldsymbol{\pi}$ on the state space S st $\forall\ t \geq 0\ \boldsymbol{\pi} = \boldsymbol{\pi}P(t) = \boldsymbol{\pi} \underbrace{e^{Qt}}_{\text{CK Eqns}}$, then $\boldsymbol{\pi}$ is an *Invariant Distribution.* This distribution is invariant wrt time.
If a markov process has a finite state space then it definitely has an invariant distribution.

Proposition 2.9 - *Finding an Invariant Distribution*
Starting with $\boldsymbol{\pi} = \boldsymbol{\pi}e^{Qt}$ we find that differentiating wrt t and then evaluating at time $t = 0$ we get $\mathbf{0} = \boldsymbol{\pi}Q$ (The Global Balance Equations). This system of equations can be solved to find an *Invariant Distribution.*
A markov process is *reversible* iff there exists a distribution $\boldsymbol{\pi}$ on S which satisfies $\boldsymbol{\pi}_x q_{xy} = \boldsymbol{\pi}_y q_{yx}\ \forall\ x, y \in S$ (Local balance equations). Solving this system of equations will also find an *Invariant Distribution* but it is not guaranteed to have a solution.

Theorem 2.10 - *Ergodic Theorem*
Let $[X_t]_{t\in\mathbb{R}^+}$ is an *Irreducible Markov Process* on a state space S and has an invariant distribution $\boldsymbol{\pi}$. Then

$$\forall\ x \in S \quad \pi_x = \lim_{t\to\infty} \frac{1}{t} \int_0^t \mathbb{1}(X_s = x)ds$$

Moreover, for an arbitrary initial distribution $\boldsymbol{\mu}(0)$, $\boldsymbol{\mu}(t)$ converges to $\boldsymbol{\pi}$ pointwise (i.e. $\mu_x(t) \overset{t\to\infty}{\longrightarrow} \pi_x$)

2.2.3 Poisson Process

Definition 2.12 - *Counting Process*
A *Counting Process* is a stochastic process $\{N(t)\}_{t\in\mathbb{R}}$ st

i). $N(0) = 0$ and $N(t) \in \mathbb{Z}$ for all $t \geq 0$; and,

ii). $N(t)$ is a non-decreasing function of t.

Definition 2.13 - *Independent Increments*
A *Process* $\{N(t)\}_{t\in\mathbb{R}^+}$ is said to have *Independent Increments* if $\forall\ s \in (0, t),\quad (N(t) - N(s))$ is independent of $\{N(u) : u \in [0, s]\}$.

Definition 2.14 - *Poisson Process*
A *Poisson Process* is a *counting process* $\{N(t)\}_{t\in\mathbb{R}^+}$ which has independent increments and <u>at least one</u> of the following equivalence statements are true

- $\forall\ t \in [0, t] \quad (N(t) - N(s)) \sim \text{Po}(\lambda(t - s))$.
- $\mathbb{P}(N(t + \delta) - N(t) = 1) = \lambda\delta + o(\delta)$ <u>and</u>
  $\mathbb{P}(N(t + \delta) - N(t) = 0) = 1 - \lambda\delta + o(\delta)$ <u>and</u>
  $\mathbb{P}(N(t + \delta) - N(t) \geq 2) = o(\delta)$.

- The <u>times between successive increments</u> of the process $N(\cdot)$ are iid $\text{Exp}(\lambda)$ random variables.

The parameter $\lambda \in \mathbb{R}^{>0}$ is called the *rate* of the poisson process. *Poisson Processes* are continuous time markov chains.

Example 2.2 - *Poisson Process*
Counting the number of cars which have passed a given point over time.

Proposition 2.10 - *Properties of Poisson Processes*
Define $\{N(t)\}_{t\in\mathbb{R}^+}$ to be a *Poisson Process* with rate λ. Then the following properties hold

i). The counting process $\{N(\beta t)\}_{t\in\mathbb{R}^+}$, with $\beta > 0$, is a Poisson Process with rate $\beta\lambda$.

ii). If $\{N_1(t)\}_{t\in\mathbb{R}^+}$ and $\{N_2(t)\}_{t\in\mathbb{R}^+}$ are independent poisson processes with rates λ_1 and λ_2, respectively, then $\{N(t) := N_1(t) + N_2(t)\}_{t\in\mathbb{R}^+}$ is a poisson process with rate $\lambda := \lambda_1 + \lambda_2$.

iii). Let $X_1, X_2, \dots$ be a sequence of iid $\text{Bern}(p)$ random vairables, independent of $N(\cdot)$. Define $N_1(t) := \sum_{i=1}^{N(t)} X_i$ and $N_2(t) := \sum_{i=1}^{N(t)}(1 - X_i)$ (These are called *Bernoulli Thinnings*). These assign increments in $N(\cdot)$ randomly to either N_1 or N_2 (with probability p).
Then, $N_1(\cdot)$ and $N_2(\cdot)$ are independent poisson processes with rates λp and $\lambda(1-p)$, respectively.

0 Reference

Definition 0.1 - *Stochastic Matrix*
A matrix is called a *Stochastic matrix* if:

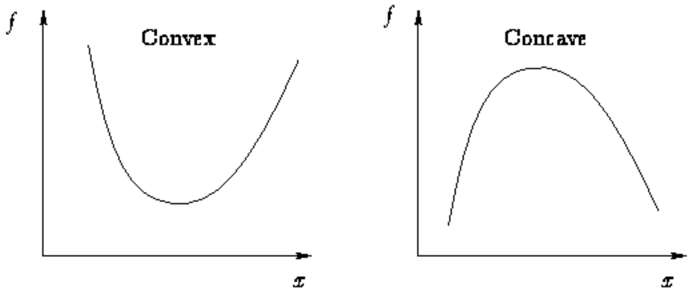
i). All elements are non-negative.

ii). All rows sum to 1.

Definition 0.2 - *Convex Function*
A function $f : \mathbb{R} \to (\mathbb{R} \cup \{+\infty\})$ is *Convex* if, $\forall\ x, y \in \mathbb{R},\ \alpha \in [0, 1]$, we have

$$f(\alpha x + (1-\alpha)y) \leq \alpha f(x) + (1-\alpha)f(y)$$

A smooth function f is convex iff f is twice differentiable and $f''(x) \geq 0\ \forall\ x \in \mathbb{R}$.
Visually, a function is convex if you can draw a line between any two points on the function and the function lies below the line.



Definition 0.3 - *Equivalence Relation*
A relation is an *Equivalence Relation* if it is

i). Reflexive: $i \leftrightarrow i$.

ii). Symmetric: If $i \to j$ then $j \to i$.

iii). Transitive: If $i \to j$ and $j \to k$ then $i \to k$.

Definition 0.4 - *Simple Random Walk*
A *Simple Random Walk* is a random walk which moves only one step at a time. (i.e. $X_{t+1} = X_t \pm 1$). A probability p is defined for $\mathbb{P}(X_{t+1} = X_t + 1)$, this means $1 - p$ is the probability of stepping in the other direction. A *Simple Random Walk* is *Assymetric* if $p \neq p - 1$.

Definition 0.5 - *Matrix Exponential, e^X*
Let X be a matrix then we define the *Matrix Exponential* as $e^X := I + X + \frac{X^2}{2!} + \dots$

0.1 Notation

$p_{x,y}$	$\mathbb{P}(X_{t+1} = x \vert X_t = y)$ for a *time homogenous markov process*

0.1.1 Asymptotic Notation

Definition 0.6 - *Oh Notation*
Let $f, g : \mathbb{R}^+ \to \mathbb{R}$.
We say $f = o(g)$ (little oh of g) at 0 if $\frac{f(x)}{g(x)} \xrightarrow{x \to 0} 0$.
We say $f = O(g)$ (big oh of g) at 0 if $\exists\ c > 0$ st $|f(x)| \leq c|g(x)|$ in a neighbourhood of 0.
$f = o(g)$ at infinity and $f = O(g)$ at infinity are defined analogously.

Definition 0.7 - *Omega Notation*
Let $f, g : \mathbb{R}^+ \to \mathbb{R}$.
We say $g = \omega(g)$ if $o(f)$ and we say $f = \Omega(g)$ if $g = O(f)$.

Example 0.1 - *Oh & Omega Notation*
Define $f(x) = x,\ g(x) = \sin(x)$ and $h(x) = x^2$.
Then, $g = O(f)$ at 0 and $g = o(f)$ at infinity. $h = o(f)$ at 0 and $h = \omega(f)$ at infinity.