

# Stochastic Optimisation - Assessed Problem Sheet 1

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## Answer 1.

Let  $X_1, \dots, X_n$  be IID random variables each modelling how many drinks each attendee drank after their first (We assume all attendees have at least one drink). For this scenario we have that  $\mathbb{E}[X_i] = \frac{7}{2} - 1 = \frac{5}{2}$  and  $X_i \geq 0$  for all  $i$ .

Thus, by *Markov's Inequality*, we have that

$$\forall i \in [1, 100] \quad \mathbb{P}(X_i \geq 6 - 1) \leq \frac{\mathbb{E}[X_i]}{6 - 1} = \frac{5/2}{5} = \frac{1}{2}$$

Meaning, out of the 100 attendees at most  $100 \times \frac{1}{2} = 50$  attendees had at least 6 drinks.

Consider a scenario where fifty guests drank 3 drinks and the other fifty drank 4 drinks. This scenario fulfils the requirements that all attendees have at least one drink and on average each attendee had  $\frac{7}{2}$  drinks. In this scenario no attendees have at least 6 drinks, thus the lower bound for the number of attendees who had at least 6 drinks is 0.

The number of attendees who had at least 6 drinks is between 0 and 50.

## Answer 2.

Let  $X \sim \text{Poisson}(\lambda)$  where  $\lambda$  is unknown,  $\pi_0$  be the prior distribution for  $\lambda$  and  $\pi_1(\cdot|n)$  be the posterior distribution for  $\lambda$ , given the value  $n$  was sampled from  $X$ . This means

$$\pi_1(\lambda|n) \propto \pi_0(\lambda)p_\lambda(n)$$

where  $p_\lambda(n) := \mathbb{P}(X = n)$  given  $X \sim \text{Poisson}(\lambda)$ .

Suppose  $\pi_0 \sim \text{Gamma}(\alpha, \beta)$  and note that

$$p_\lambda(n) = \frac{\lambda^n e^{-\lambda}}{n!} \quad \text{and} \quad \pi_0(\lambda) = \frac{\lambda^{\alpha-1} e^{-\lambda\beta}}{\Gamma(\alpha)\beta^\alpha}$$

As we are considering proportionality wrt  $\lambda$ , we can ignore terms which do not involve  $\lambda$ . Giving

$$p_\lambda(n) \propto \lambda^n e^{-\lambda} \quad \text{and} \quad \pi_0(\lambda) \propto \lambda^{\alpha-1} e^{-\lambda\beta}$$

Using these results we can build an expression for the posterior  $\pi_1(\cdot|n)$

$$\begin{aligned} \pi_1(\lambda|n) &\propto \pi_0(\lambda)p_\lambda(n) \\ &\propto \left(\lambda^{\alpha-1} e^{-\lambda\beta}\right) \cdot \left(\lambda^n e^{-\lambda}\right) \\ &= \lambda^{n+\alpha-1} e^{-\lambda(\beta+1)} \end{aligned}$$

By comparing this expression to that of a Gamma distribution we have that

$$\pi_1(\lambda|n) \sim \text{Gamma}(\alpha + n, \beta + 1)$$

**Answer 3.**

Consider a two-armed bandit where the rewards from each arm are modelled by IID random variables  $X_1, X_2$  each with distribution  $\text{Poisson}(\lambda_i)$  with means  $\lambda_1, \lambda_2$  unknown.

Here I give a version of the Thompson Sampling algorithm for solving the multi-armed bandit problem for this bandit, with a round limit  $T$ .

- I. Define a  $\text{Gamma}(\alpha, \beta)$  distribution prior for the mean of each arm, with the values of  $\alpha, \beta$  chosen arbitrarily (Perhaps  $\alpha = \beta = 1$ ).
- II. To start the  $t^{\text{th}}$  round, sample  $\hat{\mu}_1(t)$  from the prior for arm one and  $\hat{\mu}_2(t)$  from the prior for arm two.
- III. If  $\hat{\mu}_1(t) \geq \hat{\mu}_2(t)$  then play arm one; otherwise, play arm two. Let  $n$  denote the observed reward from the played arm.
- IV. Suppose the prior for the mean of the played arm at the start of this round was a  $\text{Gamma}(\alpha_t, \beta_t)$  distribution. Define the posterior for the mean of the played arm to be a  $\text{Gamma}(\alpha_t + n, \beta_t + 1)$  distribution.
- V. For the non-played arm, define the posterior for its mean to be the same as its prior at the start of this round.
- VI. Repeat steps II.-V. until  $T$  rounds have been played. Use the posteriors produced in round  $t$  as the priors for round  $t + 1$ .

*N.B.* After round  $t$  the posterior for the mean of arm  $i$  will be a  $\text{Gamma}(\alpha + Y_t, \beta + N_t)$  distribution where  $Y_t$  is the sum-total reward received from playing arm  $i$  in the first  $t$  rounds and  $N_t$  is the number of times arm  $i$  was played in the first  $t$  rounds.