Stochastic Optimisation - Problem Sheet 1

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October 13, 2020

Answer 3)
Let $X_1, X_2, \dots$ by iid random variables with distribution $\text{Bern}(p)$ with $p \in [0,1]$. Let $q \in [0,1]$ with $q > p$.
The moment generating function of X is $\mathcal{M}_X(t) := \mathbb{E}[e^{tX}] = (p + e^t + (1-p))$.
By applying *Chernoff Bounds* we have that

$$\begin{aligned}\mathbb{P}\left(\sum_{i=1}^n X_i > nq\right) &\leq \inf_{\theta>0} e^{-nq\theta}(\mathbb{E}[e^{\theta X}])^n \\ &= \inf_{\theta>0} e^{-nq\theta}(pe^\theta + (1-p))^n\end{aligned}$$

Consider the natural log of the right hand side and define

$$f := -nq\theta + \ln(pe^\theta + 1 - p)$$

Since the natural log is a monotonically increasing function, $\inf\limits_{\theta>0} e^f$ is equal to the RHS of above.
First I shall derive $\underset{\theta;\theta>0}{\text{argmin}}(f)$

$$\frac{\partial f}{\partial \theta} = -nq + n\frac{pe^\theta}{pe^\theta + 1 - p}$$

$$\begin{aligned}\text{Setting} && \frac{\partial f}{\partial \theta} &= 0 \\ \implies && 0 &= -nq + n\frac{pe^\theta}{pe^\theta + 1 - p} \\ \implies && q &= \frac{pe^\theta}{pe^\theta + 1 - p} \\ \implies && pe^\theta + 1 - p &= \frac{p}{q}e^\theta \\ \implies && e^\theta &= \frac{1-p}{\frac{p}{q} - p} \\ && &= \frac{q - qp}{p - qp} \\ \implies && \theta &= \ln\left(\frac{q-qp}{p-qp}\right)\end{aligned}$$

$$\begin{aligned}\text{Since} && q &> p \\ \implies && q - qp &> p - qp \\ \implies && \frac{q-qp}{p-qp} &> 1 \\ \implies && \ln\left(\frac{q-qp}{p-qp}\right) &> 0\end{aligned}$$

Thus $\underset{\theta;\theta>0}{\text{argmin}}(f) = \ln\left(\frac{q-qp}{p-qp}\right)$. This means

$$\begin{array}{rcl}
\inf_{\theta>0} f &=& -nq\ln\left(\frac{q-qp}{p-qp}\right) + n\ln\left(p\cdot\frac{q-qp}{p-qp}+1-p\right) \\
&=& -n\left[q\ln\left(\frac{q(1-p)}{p(1-q)}\right) - \ln\left(p\cdot\frac{q(1-p)}{p(1-q)}+1-p\right)\right] \\
&=& -n\left[q\ln\left(\frac{q}{p}\right) + q\ln\left(\frac{1-p}{1-q}\right) - \ln\left(\frac{q(1-p)}{1-q}+1-p\right)\right] \\
&=& -n\left[q\ln\left(\frac{q}{p}\right) - q\ln\left(\frac{1-q}{1-p}\right) - \ln\left(\frac{1-p}{1-q}\right)\right] \\
&=& -n\left[q\ln\left(\frac{q}{p}\right) + (1-q)\ln\left(\frac{1-q}{1-p}\right)\right] \\
&=& -nK(q;p) \\
\implies \quad \inf_{\theta>0} e^{-nq\theta}(pe^\theta+(1-p))^n &=& \exp(-nK(q;p)) \\
\implies \quad \mathbb{P}\left(\sum_{i=1}^n X_i > nq\right) &\leq& \exp(-nK(q;p))
\end{array}$$

Answer 6a)
Let $Z\sim N(0,1)$.

$$\begin{array}{rcl}
\mathbb{E}[e^{\theta Z}] &=& \int_{-\infty}^\infty e^{\theta x}f_Z(x)dx \\
&=& \int_{-\infty}^\infty e^{\theta x}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}x^2}dx \\
&=& \int_{-\infty}^\infty \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}(x^2-2x\theta)}dx \\
&=& \int_{-\infty}^\infty \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}(x-\theta)^2+\frac{1}{2}\theta^2}dx \\
&=& e^{\frac{1}{2}\theta^2}\int_{-\infty}^\infty \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}(x-\theta)^2}dx \\
&=& e^{\frac{1}{2}\theta^2}\int_{-\infty}^\infty f_Y(x)dx \quad \text{where} \quad Y\sim N(\theta,1) \\
&=& e^{\frac{1}{2}\theta^2}\cdot 1 \\
&=& e^{\frac{1}{2}\theta^2}
\end{array}$$

Answer 6b)
Let $X_1, X_2, \dots$ be iid random variables with distribution $N(\mu,\sigma^2)$ for $\mu,\sigma\in\mathbb{R}$. Let $\gamma\in\mathbb{R}$ st $\gamma>\mu$.
The moment generating function of X is $\mathcal{M}_X(t) := \mathbb{E}[e^{tX}] = e^{\mu\theta+\frac{1}{2}\sigma^2\theta^2}$.
By applying *Chernoff Bounds* we have that

$$\begin{array}{rcl}
\mathbb{P}\left(\sum_{i=1}^n X_i > n\gamma\right) &\leq& \inf_{\theta>0} e^{-n\gamma\theta}(\mathbb{E}[e^{X\theta}])^n \\
&=& \inf_{\theta>0} e^{-n\theta(\gamma-\mu-\frac{1}{2}\sigma^2\theta)}
\end{array}$$

Consider the natural log of the right hand side and define $f := -n\theta(\gamma-\mu-\frac{1}{2}\sigma^2\theta)$.
Since the natural log is a monotonically increasing function, $\inf_{\theta>0} e^f$ is equal to the RHS of above.

First I shall derive $\underset{\theta;\theta>0}{\text{argmin}}(f)$

$$\begin{array}{rcl}\frac{\partial f}{\partial\theta} & = & -n(\gamma-\mu-\frac{1}{2}\sigma^2\theta)-n\theta(-\frac{1}{2}\sigma^2)\\ & = & -n(\gamma-\mu-\sigma^2\theta)\end{array}$$

$$\begin{array}{rrcl}\text{Setting} & \frac{\partial f}{\partial\theta} & = & 0\\ \implies & \gamma-\mu-\sigma^2\theta & = & 0\\ \implies & \theta & = & \frac{\gamma-\mu}{\sigma^2}\\ & & & \\ \text{Since} & \gamma>\mu & \& & \sigma^2>0\\ \implies & 0 & < & \frac{\gamma-\mu}{\sigma^2}=\theta\end{array}$$

Thus $\underset{\theta;\theta>0}{\text{argmin}}(f)=\frac{\gamma-\mu}{\sigma^2}$. This means

$$\begin{array}{rrcl} & \inf_{\theta>0} f & = & -n\left(\frac{\gamma-\mu}{\sigma^2}\right)\left(\gamma-\mu-\frac{1}{2}(\gamma-\mu)\right)\\ & & = & -n\frac{(\gamma-\mu)^2}{2\sigma^2}\\ \implies & \inf_{\theta>0} e^{-n\theta\left(\gamma-\mu-\frac{1}{2}\sigma^2\theta\right)} & = & \exp\left(-n\frac{(\gamma-\mu)^2}{2\sigma^2}\right)\\ \implies & \mathbb{P}\left(\sum_{i=1}^n X_i>n\gamma\right) & \leq & \exp\left(-n\frac{(\gamma-\mu)^2}{2\sigma^2}\right)\end{array}$$