Stochastic Optimisation - Assessed Problem Sheet 1

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Answer 1.
Let $X_1, \dots, X_n$ be IID random variables each modelling how many drinks each attendee drank after their first (We assume all attendees have at least one drink). For this scenario we have that $\mathbb{E}[X_i] = \frac{7}{2} - 1 = \frac{5}{2}$ and $X_i \geq 0$ for all i.

Thus, by *Markov's Inequality*, we have that

$$\forall\ i \in [1, 100] \quad \mathbb{P}(X_i \geq 6 - 1) \leq \frac{\mathbb{E}[X_i]}{6-1} = \frac{5/2}{5} = \frac{1}{2}$$

Meaning, out of the 100 attendees at most $100 \times \frac{1}{2} = 50$ attendees had at least 6 drinks.

Consider a scenario where fifty guests drank 3 drinks and the other fifty drank 4 drinks. This scenarios fulfils the requirements that all attendees have at least one drink and on average each attendee had $\frac{7}{2}$ drinks. In this scenario no attendees have at least 6 drinks, thus the lower bound for the number of attendees who had at least 6 drink is 0.

The number of attendees who had at least 6 drinks is between 0 and 50.

Answer 2.
Let $X \sim \text{Poisson}(\lambda)$ where λ is unknown, π_0 be the prior distribution for λ and $\pi_1(\cdot|n)$ be the posterior distribution for λ, given the value n was sampled from X. This means

$$\pi_1(\lambda|n) \propto \pi_0(\lambda) p_\lambda(n)$$

where $p_\lambda(n) := \mathbb{P}(X = n)$ given $X \sim \text{Poisson}(\lambda)$.

Suppose $\pi_0 \sim \text{Gamma}(\alpha, \beta)$ and note that

$$p_\lambda(n) = \frac{\lambda^n e^{-\lambda}}{n!} \quad \text{and} \quad \pi_0(\lambda) = \frac{\lambda^{\alpha-1} e^{-\lambda\beta}}{\Gamma(\alpha)\beta^\alpha}$$

As we are considering proportionality wrt λ, we can ignore terms which do not involve λ. Giving

$$p_\lambda(n) \propto \lambda^n e^{-\lambda} \quad \text{and} \quad \pi_0(\lambda) \propto \lambda^{\alpha-1} e^{-\lambda\beta}$$

Using these results we can build an expression for the posterior $\pi_1(\cdot|n)$

$$\begin{array}{rcl} \pi_1(\lambda|n) & \propto & \pi_0(\lambda) p_\lambda(n) \\ & \propto & \left(\lambda^{\alpha-1} e^{-\lambda\beta}\right) \cdot \left(\lambda^n e^{-\lambda}\right) \\ & = & \lambda^{n+\alpha-1} e^{-\lambda(\beta+1)} \end{array}$$

By comparing this expression to that of a Gamma distribution we have that

$$\pi_1(\lambda|n) \sim \text{Gamma}(\alpha + n, \beta + 1)$$

Answer 3.
Consider a two-armed bandit where the rewards from each arm are modelled by IID random variables X_1, X_2 each with distribution Poisson(λ_i) with means λ_1, λ_2 unknown.

Here I give a version of the Thompson Sampling algorithm for solving the multi-armed bandit problem for this bandit, with a round limit T.

I. Define a Gamma(α, β) distribution prior for the mean of each arm, with the values of α, β chosen arbitrarily (Perhaps $\alpha = \beta = 1$).

II. To start the t^{th} round, sample $\hat{\mu}_1(t)$ from the prior for arm one and $\hat{\mu}_2(t)$ from the prior for arm two.

III. If $\hat{\mu}_1(t) \geq \hat{\mu}_2(t)$ then play arm one; otherwise, play arm two. Let n denote the observed reward from the played arm.

IV. Suppose the prior for the mean of the played arm at the start of this round was a Gamma(α_t, β_t) distribution. Define the posterior for the mean of the played arm to be a Gamma($\alpha_t + n, \beta_t + 1$) distribution.

V. For the non-played arm, define the posterior for its mean to be the same as its prior at the start of this round.

VI. Repeat steps II.-V. until T rounds have been played. Use the posteriors from round t as the priors for round $t + 1$.

N.B. After round t the posterior for the mean of arm i will be a Gamma($\alpha + Y_t, \beta + N_t$) distribution where Y_t is the sum-total reward received from playing arm i in the first t rounds and N_t is the number of times arm i was played in the first t rounds.