Stochastic Optimisation - Notes

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NOTE - *Markov Chain* typically refers to the discrete setting; whilst *Markov Process* typically refers to the continuous setting.

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1 Multi-Armed Bandit

1.1 The Problem

Example 1.1 - *Motivating Example*
Consider having a group of patients and several treatments they could be assigned to. How best do you go about determining which treatment is best? The obvious approach is to assign some of the patients randomly and then assign the rest to the best treatment, but how much evidence is sufficient? And how likely are you to choose a sub-optimal treatment?

Definition 1.1 - *Multi-Armed Bandit Problem*
An agent is faced with a choice of K actions. Each (discrete) time step the agent plays action i they receive a reward from the random real-valued distribution ν_i. Each reward is independent of the past. The distributions $\nu_1, \dots, \nu_K$ are unknown to the agent.
In the *Multi-Armed Bandit Problem* the agent seeks to maximise a measure of long-run reward.

Remark 1.1 - *Informal Definition of Multi-Armed Bandit Problem*
Given a finite set of actions and a random reward for each action, how best do we learn the reward distribution and maximise reward in the long-run.

Definition 1.2 - *Formal Definition of Multi-Armed Bandit Problem*
Consider a sequence of (unknown) mutually independent random variables $\{X_i(t)\}_{i\in[1,K]}$, with $t \in \mathbb{N}$. Consider $X_i(t)$ to be the distribution of rewards an agent would receive if they performed action i at time t. Since the rewards are independent of the past $X_i(t), X_i(t+1), \dots$ are IID random variables. The *Multi-Armed Bandit Problem* tasks us to find the greatest expected reward from all the actions.

$$\mu^* := \max_{i=1}^{K} \mu_i \quad \text{where } \mu_i = \mathbb{E}(X_i(t))$$

There are a number of ways to formalise this objective.

Definition 1.3 - *Strategy, $I(\cdot)$*
Our agent's strategy $I : \mathbb{N} \to [1, K]$ is a function which determines which action the agent shall make at a given point in time. The strategy can use the knowledge gained from previous actions & their rewards only.

$$I(t) = I\big(t, \underbrace{\{I(s)\}_{\in[1,t)}}_{\text{Prev. Actions}}, \underbrace{\{X_{I(s)}(s)\}_{\in[1,t)}}_{\text{Prev. Rewards}}\big) \in [1, K]$$

Definition 1.4 - *Long-Run Average Reward Criterion, X_**
For a strategy $I(\cdot)$ we define the following measure for *Long-Run Average Reward*

$$X_* = \lim_{T\to\infty} \inf \frac{1}{T} \sum_{t=1}^{T} \mathbb{E}(X_{I(t)})$$

The *Infinum* is taken as there is no guarantee the limit exists (depending on the strategy), typically we will only deal with strategies where this limit exists.
Most strategies as based only on realisations of $\{X_i(s)\}_{s\in[1,t)}$, thus $\mathbb{E}(X_{I(t)}) \leq \mu^*$ and thus $X_* \leq \mu^*$. A strategy $I(\cdot)$ is *Optimal* if $X_* = \mu^*$.

Remark 1.2 - *It is not hard to find an Optimal Strategy in the (very) long run, so we are going to look at Regret Minimisation First.*
Proposition 1.1 - *Mathematical Model & Assumptions for Multi-Armed Bandit Problem*
Model:

- Bandit has K bernoulli arms.
- $X_i(t) \in \mathbb{R}$ is the reward obtained by played arm $i \in [1, K]$ at time step $t \in \mathbb{N}$.

Assumptions:

- $X_1(\cdot), X_2(\cdot), \dots$ are mutually independent sequences.
- For each i $\{X_i(t)\}_{t\in\mathbb{N}}$ is a sequence of iid Bern(μ_i) random variables

We define the following quantities to make analysis easier

- $I(t) \in [1, K]$. The index of the arm played in time t;
- $N_j(t) := \sum_{s=1}^{t} \mathbb{1}(I(s) = j)$. The number of times arm j has been played in the first t rounds;
- $S_j(t) := \sum_{s=1}^{t} X_j(s)\mathbb{1}(I(s) = j)$. The total reward from arm j in the first t rounds. This is a Binomial random variable independent Bin$(N_j(t), \mu_j)$;
- $\hat{\mu}_{j,n} := \frac{S_j(t)}{N_j(t)}$. The sample mean reward from arm j in the first n plays of arm j.

Definition 1.5 - *Policy*
A *policy* is a family of functions f_t which specify what arm is to be played in round t. f_t should depend on the information available at time t $\left\{I(s), X_{I(s)}(s) : s \in [1, t-1]\right\}$.

Randomised policies are allowed. So, in addition to the history up to time t, f_t can depend upon a $U(t) \sim U[0, 1]$ random variable which is independent of $X_i(\cdot)$. Thus

$$I(t) = f_t\big(\underbrace{I(1), \dots, I(t-1)}_{\text{arms chosen}}, \underbrace{X_{I(1)}(1), \dots, X_{I(t-1)}(t-1)}_{\text{observed rewards}}, \underbrace{U(t)}_{\text{randomness}}\big)$$

We want to find the best policy (ie one which minimises the regret)

1.2 Regret Minimisation

Definition 1.6 - *Regret, R_n*
Regret is a measure of how much reward was lost during the first n time steps. The *Regret* R_n of a strategy $\{I(t)\}_{t\in\mathbb{N}}$ in the first n time steps is given by

$$\begin{aligned} R_n &= \max_{k=1}^{K} \sum_{t=1}^{n} \mathbb{E}[\underbrace{X_k(t)}_{\text{Best Pos}} - \underbrace{X_{I(t)}(t)}_{\text{Actual}}] \\ &= n\mu^* - \sum_{t=1}^{n} \mathbb{E}\big[X_{I(t)}(t)\big] \end{aligned}$$

Regret only involves expectation and thus can be learnt from observations. We want to produce a strategy where *Total Regret* grows sub-linearly.(i.e. $R_T/T \overset{T\to\infty}{\longrightarrow} 0$)

Remark 1.3 - *Minimising the growth rate of R_T with T is quite hard.*
The best achievable regret scales as $R_T \sim c \log T$ (i.e. $R_T / c \log T \overset{T \to \infty}{\longrightarrow} 1$) where c depends on the reward distributions $X_1(t), \dots, X_K(t)$.

Definition 1.7 - *Pseudo-Regret, $\tilde{R}_n$*
Pseudo-Regret $\tilde{R}_n$ is a less popular alternative to *Regret* R_n. The *Pseudo-Regret* R_n of a strategy $\{I(t)\}_{t \in \mathbb{N}}$ in the first n time steps is given by

$$\tilde{R}_n = \max_{k=1}^{K} \sum_{t=1}^{n} \left(X_k(t) - X_{I(t)}(t) \right)$$

Pseudo-Regret includes intrinsic randomness (which is independent of the past) and thus cannot be learnt from observations.

1.3 Best Arm Identification for Bernoulli Distribution

Example 1.2 - *Best Arm Identification for Bernoulli Bandits*
Consider a bandit with two *Bernoulli* arms: $\{X_1(t)\}_{t \in \mathbb{N}}$ IID RVs with distribution $\text{Bern}(\mu_1)$; and, $\{X_2(t)\}_{t \in \mathbb{N}}$ IID RVs with distribution $\text{Bern}(\mu_2)$.
Suppose $\mu_1 > \mu_2$ (i.e. arm 1 is better). Let the player play each arm n times and declare the arm with the greatest empirical mean to be the better arm. *What is the probability of choosing the wrong arm (Arm 2)?*.

An error occurs if $\sum_{t=1}^{n} X_2(t) \geq \sum_{t=1}^{n} X_1(t)$ and thus we want to calculate the probability of this event.
Define $\{Y(t)\}_{t \in \mathbb{N}}$ st $Y(t) := \{X_2(t) - X_1(t)$. This means $Y(t) \in \{-1, 0, 1\} \subset [-1, 1]$.
To use *Hoeffding's inequality* we need to scale Y to be in $[0, 1]$, so we define $Z(t) := \frac{1}{2}(Y(t) + 1)$. We have $\mathbb{E}(Z(t)) = \frac{1}{2}(1 + \mu_2 - \mu_1)$ and an error occurs if $\sum_{t=1}^{n} Y(t) > 0 \iff \sum_{t=1}^{n} Z(t) \geq \frac{n}{2}$. By *Hoeffding's Inequality*

$$\begin{array}{rcll}
\mathbb{P}(\text{error}) & = & \mathbb{P}\left(\sum_{i=1}^{n} Z(t) \geq \frac{n}{2} \right) & \\
& = & \mathbb{P}\left(\left(\sum_{i=1}^{n} Z(t) \right) - \frac{n}{2}(1 + \mu_2 - \mu_1) \geq \frac{n}{2}(\mu_1 - \mu_2) \right) & \text{subtracting } \mu \text{ from both sides} \\
& = & \mathbb{P}\left(\sum_{i=1}^{n} \left(X_i - \underbrace{\frac{1}{2}(1 + \mu_2 - \mu_1)}_{\mu} \right) \geq n \underbrace{\frac{1}{2}(\mu_1 - \mu_2)}_{t} \right) & \text{arranging for Hoeffding's} \\
& \leq & \exp\left(-2n \cdot \frac{1}{4}(\mu_1 - \mu_2)^2 \right) & \text{by Hoeffding's Inequality} \\
& = & \exp\left(-\frac{n}{2}(\mu_1 - \mu_2)^2 \right) &
\end{array}$$

1.4 Heuristic

Remark 1.4 - *How many tests?*
Suppose an agent is comparing two arms and is given a finite time horizon T after in which they must choose the best arm. The obvious strategy is to perform each task N times and then choose the arm with the greatest empirical mean. But, how do we choose N to minimise

regret over time T?

Proposition 1.2 - *Naïve Heuristic (Single Test)*
Consider a 2-armed bandit & the following Heuristic

Play each arm once. Pick the arm with the greatest sample mean reward (breaking ties arbitrarily) and playing that arm on all subsequent rounds.

This heuristic picks the wrong arm with probability $\mu_2(1-\mu_1)$. In this case the wrong arm is played $T-1$ times, giving a bounded regret

$$\mathcal{R}(T) \geq \underbrace{\mu_2(1-\mu_1)}_{\text{prob of wrong choice}} \cdot \underbrace{(\mu_1-\mu_2)}_{\text{Loss}} \cdot \underbrace{(T-1)}_{\#\text{ steps}}$$

This regret grows linearly in T.

Theorem 1.1 - *Chernoff Bound of a Binomial Random Variable*
Let $X \sim \text{Bin}(n, \alpha$ with $n \in \mathbb{N},\ \alpha \in (0,1)$. Then

$$\forall\ \beta > \alpha \quad \mathbb{P}(X \geq Xn) \leq e^{-nK(\beta;\alpha)}$$

where

$$K(\beta;\alpha) := \begin{cases} \beta \ln\left(\frac{\beta}{\alpha}\right) + (1-\beta)\ln\left(\frac{1-\beta}{1-\alpha}\right) & \text{if } \beta \in [0,1] \\ +\infty & \text{otherwise} \end{cases}$$

with $x\ln(x) := 0$ if $x = 0$.
Similarly

$$\forall\ \beta < \alpha \quad \mathbb{P}(X \leq \beta n) \leq e^{-nK(\beta;\alpha)}$$

Note that $K(\cdot;\cdot)$ is known as both *relative entropy* and *Kullback-Leibler Divergence*

Proposition 1.3 - *Better Heuristic (N Tests)*
Consider a 2-armed bandit problem & the following heuristic

Play each arm $N < \frac{T}{2}$. Pick the arm with the greatest sample mean reward (breaking ties arbitrarily) and playing that arm on all subsequent rounds.

Note that $S_1(n)$ & $S_2(n)$ are *binomial* random variables with distributions $\text{Bin}(N,\mu_1)$, $\text{Bin}(N,\mu_2)$ respectively. And, $S_1(n)$ and $S_2(n)$ are independent of each other. Thus for $\beta \in (\mu_2, \mu_1)$

$$\mathbb{P}(S_1(N) < \beta N,\ S_2(N) > \beta N) \quad \leq \quad e^{-N(K(\beta;\mu_1)+K(\beta;\mu_2))} = e^{-NJ(\mu_1,\mu_2)}$$

where

$$J(\mu_1,\mu_2) = \inf_{\beta\in[\mu_2,\mu_1]} \big(K(\beta;\mu_1) + K(\beta;\mu_2)\big)$$

The values of β which solve $J(\cdot;\cdot)$ describe the most likely ways for the event $(S_1(N) < S_2(N)$ to occur (ie the wrong decision is made).

Proposition 1.4 - *Optimal N*
For the situation described in `Proposition 1.2` we want to find N which minimises regret, given a total time horizon of T.

If the right decision is made in the end, regret only occurs during exploration and is equal to $N \cdot (\mu_1 - \mu_2)$ (since the wrong arm is played N times).

However, if the wrong decision is made in the end, regret is equal to $(T - N) \cdot (\mu_1 - \mu_2)$.

Thus, the overall regret up to time T is

$$\begin{aligned}\mathcal{R}(T) &= \underbrace{(T-2N)(\mu_1-\mu_2)\mathbb{P}\big(S_1(N) < S_2(N)\big)}_{\text{if wrong decision made}} + \underbrace{N(\mu_1-\mu_2)}_{\text{guaranteed regret}} \\ &\simeq (\mu_1-\mu_2)(N + Te^{-NJ(\mu_1,\mu_2)})\end{aligned}$$

This expression is minimised for N close to the solution of $1 = TJ(\mu_1,\mu_2)e^{-NJ(\mu_1,\mu_2)}$ (ie when $N = \frac{\ln T}{J(\mu_1,\mu_2)} + O(1)$).
The corresponding regret is

$$\mathcal{R}(T) = \frac{\mu - 1 - \mu_2}{J(\mu_1,\mu_2)}\ln(T) + O(1)$$

If $\mu_1 \simeq \mu_2$ then $J(\mu_1,\mu_2) \simeq (\mu_1 - \mu - 2)^2$ and the above regret becomes $\mathcal{R}(T) = \frac{\ln(T)}{\mu_1 - \mu_2} + O(1)$.

1.5 UCB Algorithm

Remark 1.5 - *UCB Algorithm*
The *Upper Confidence Bound Algorithm* is a *frequentist* algorithm for solving the multi-armed bandit problem.

Remark 1.6 - *Motivation*
The problem with the heuristics in `Proposition 1.2,1.3` is that they treat the sample mean as the true mean (*Certainty Equivalence*), which is not great.

Suppose we observed sample mean reward for arm i of $\hat{\mu}_{i,n}$ after n plays. How far from the true value can μ_i be?

$$\mathbb{P}(\mu_i > \hat{\mu}_{i,n} + x) \leq e^{-2nx^2} \text{ by Hoeffding's Inequality}$$

Suppose the inequality holds with equality (ie greatest possible probability). Then for some chosen $\delta \in [0,1]$

$$x = \sqrt{\frac{1}{2n}\ln\left(\frac{1}{\delta}\right)} \implies \mathbb{P}(\mu_i > \hat{\mu}_{i,n} + x) = \delta \quad \text{since } \delta = e^{-2nx^2}$$

This suggests a heuristic:

$$\text{Play arm which maximises} \quad \hat{\mu}_{i,N_i(t)} + \sqrt{\frac{1}{2N_i(t)}\ln\left(\frac{1}{\delta}\right)}$$

where you choose $\delta \in [0,1]$ based on how lucky you feel. This quantity is the upper bound of a $1-\delta$ confidence interval for the value of μ_i.

This heuristic allows for our choice to be changed any number of times.

Definition 1.8 - *UCB(α) Algorithm*
Consider the set up of the multi-armed bandit problem in `Proposition 1.1` and wlog that $\mu_1 > \mu_2 \geq \cdots \geq \mu_K$.
Consider a k-armed bandit and let $\alpha > 0$.

i). In the first K rounds, play each arm once.

ii). At the end of each round $t \geq K$ compute the *UCB(α)* index of each arm i defined as

$$\hat{\mu}_{i,N_i(t)} + \sqrt{\frac{\alpha \ln(t)}{2N_i(t)}}$$

iii). In round $t+1$ play the arm with the greatest index (breaking ties arbitrarily)

$$I(t+1) = \text{argmax}_{i \in [1,K]} \left\{ \hat{\mu}_{i,N_i(t)} + \sqrt{\frac{\alpha \ln(t)}{2N_i(t)}} \right\}$$

1.5.1 Analysis

Theorem 1.2 - *Upper Bound on Regret*
Consider a K-armed bandit and define $\Delta_i := \mu_1 - \mu_i$.

If the *UCB(α)* algorithm is used, with $\alpha > 1$, then the regret in the first T rounds is bounded above by

$$\mathcal{R} \leq \sum_{i=2}^{K} \left(\frac{\alpha+1}{\alpha-1} \Delta_i + \frac{2\alpha}{\Delta_i} \ln(T) \right)$$

This bounds grows logarithmically in T, which is very good.

If α is taken to be large, then the regret grows faster (bad). If α is small, the constant term dominates for smaller values of T (constant term blows up close to 1).

You should choose a value a bit larger than 1 (often $\alpha = 2$).

NOTE this is proved at the end of this subsubsection `Proof 1.4`.

Theorem 1.3 - *When a sub-optimal arm is played*
Consider apply *UCB(α)* to a k-armed bandit and define $\Delta_i := \mu_1 - \mu_i$. Let $s \geq K$ (so we have completed the first stage of UCB) and suppose $I(s+1) = j \neq 1$ (ie arm at time $s+1$ is suboptimal). Then one of the following is true:

i). $\hat{\mu}_{1,N_1(s)} \leq \mu_1 - \sqrt{\frac{\alpha \ln(s)}{2N_1(s)}}$. The sample mean reward on the optimal arm is much smaller than the true mean.

ii). $\hat{\mu}_{j,N_j(s)} \geq \mu_j + \sqrt{\frac{\alpha \ln(s)}{2N_j(s)}}$. The sample mean reward on arm j is much larger than its true mean.

iii). $N_j(s) < \frac{2\alpha \ln(s)}{\Delta_j^2}$. Arm j has been played very few times.

Proof 1.1 - *Theorem 1.3*
This is a proof by contradiction.

Suppose $I(s+1)=j\neq 1$ but that none of the three inequalities holds. Then

$$\begin{array}{rcll}\underbrace{\hat{\mu}_{1,N_1(s)}+\sqrt{\dfrac{\alpha\ln(s)}{2N_1(s)}}}_{\text{UCB}(\alpha)\text{ index }1} & > & \mu_1 & \text{by not i)}\\ & = & \mu_j+\Delta_j & \text{by def. of }\Delta_j\\ & \geq & \mu_j+\sqrt{\dfrac{2\alpha\ln(s)}{N_j(s)}} & \text{by not iii)}\\ & \geq & \hat{\mu}_{1,N_1(s)}-\sqrt{\dfrac{\alpha\ln(s)}{2N_1(s)}}+\sqrt{\dfrac{2\alpha\ln(s)}{N_j(s)}} & \text{by not ii)}\\ & \geq & \hat{\mu}_{1,N_1(s)}+\left(\sqrt{2}-\frac{1}{\sqrt{2}}\right)\sqrt{\dfrac{\alpha\ln(s)}{N_1(s)}} & \\ & = & \underbrace{\hat{\mu}_{j,N_j(s)}+\sqrt{\dfrac{\alpha\ln(s)}{2N_j(s)}}}_{\text{UCB}(\alpha)\text{ index j}} & \end{array}$$

But, this implies that the *UCB(α)* index of arm 1 at the end of round s is greater than that of arm j. Hence arm j would not be played in time slot $s+1$. $\square$

Theorem 1.4 - *Counting Lemma*
Let $\{I(t)\}_{t\in\mathbb{N}}$ be a $\{0,1\}$-valued sequence and $N(t):=\sum_{s=1}^t I(s)$. Then

$$\forall\, t,u\in\mathbb{N}\quad N(t)\leq u+\sum_{s=u+1}^t I(s)\mathbb{1}\{N(s-1)\geq u\}$$

with an empty sum defined to be zero.

Proof 1.2 - *Theorem 1.4*
Fix $t,u\in\mathbb{N}$. There are two possibilities

Case 1 $N(t)\leq u$. (Have not reached u yet)

Case 2 $\exists\ s\in[1,t]$ st $N(s)>u$. (Already reached u). Let s^* denote the smallest such s. Then it must be true that $N(s^*-1)=u$ and $s^*\geq u+1$. Hence

$$\begin{array}{rcl}N(t) & = & \displaystyle\sum_{s=1}^{s^*-1}I(s)+\sum_{s=s^*}^t I(s)\\ & = & \displaystyle N(s^*-1)+\sum_{s=s^*}^t I(s)\,\underbrace{\mathbb{1}\{N(s-1)\geq u\}}_{\text{true for all in sum}}\\ & \leq & \displaystyle u+\sum_{s=u+1}^t I(s)\mathbb{1}\{N(s-1)\geq u\}\quad\text{since }s^*\geq u+1\end{array}$$

$\square$

Proof 1.3 - *Theorem 1.2*
Fix $t\in\mathbb{N}$ adn take $u_{t,j}:=\left\lceil\frac{2\alpha\ln(t)}{\Delta_j^2}\right\rceil$.
By `Theorem 1.4` we have that

$$N_j(t)\leq u+\sum_{s=u+1}^t\mathbb{1}\{(N_j(s-1)\geq u_{t,j})\ \&\ (I(s)=j)\}$$

Both sides involve random variables. Taking expectations we get

$$\mathbb{E}[N_j(t)] \leq u + \sum_{s=u}^{t-1} \mathbb{P}\big\{(N_j(s) \geq u_{y,j}) \ \& \ (I(s+1)=j)\big\}$$

By `Theorem 1.3` and the definition of u, *IF* $I(s+1)=j$ *and* $N_j(s) \geq u$ *then*

$$\hat{\mu}_{1,N_1(s)} \leq u_1 - \sqrt{\frac{\alpha \ln(s)}{2N_1(s)}} \quad \text{or} \quad \hat{\mu}_{j,N_j(s)} > \mu_j + \sqrt{\frac{\alpha \ln(s)}{2N_j(s)}}$$

Thus

$$\mathbb{E}[N_j(t)] \leq u_{t,j} + \sum_{s=u_{t,j}}^{t-1} \left[\underbrace{\mathbb{P}\left(\hat{\mu}_{1,N_1(s)} \leq \mu_1 - \sqrt{\frac{\alpha \ln(s)}{2N_1(s)}}\right)}_{\hat{\mu}_1 \text{ is unusually small}} + \underbrace{\mathbb{P}\left(\hat{\mu}_{j,N_j(s)} > \mu_j - \sqrt{\frac{\alpha \ln(s)}{2N_j(s)}}\right)}_{\hat{\mu}_j \text{ is unusually large}} \right]$$

By *Hoeffding's Inequality*

$$\begin{array}{rcl} \mathbb{E}[N_j(t)] & \leq & u + \displaystyle\sum_{s=u}^{t-1} 2s^{-\alpha} \\ & \leq & u + \displaystyle\int_{u-1}^{\infty} 2s^{-\alpha} ds \quad \text{assumption } \alpha > 1 \text{ required here} \\ & = & u + \dfrac{2(u-1)^{-(\alpha-1)}}{\alpha-1} \\ & \leq & u + \dfrac{2}{\alpha-1} \quad \text{since } u \geq 2 \implies (u-1)^{-(\alpha-1)} \leq 1 \end{array}$$

Thus

$$\forall\ j \in [2,K] \quad \mathbb{E}[N_j(t)] \leq u + \frac{2}{\alpha-1} \leq \frac{2\alpha \ln(t)}{\Delta_j^2} + 1 + \frac{2}{\alpha-1}$$

A regret of $\Delta_j := \mu_1 - \mu_j$ is incurred every time arm j is played. Hence the total regret up to time t is bounded by

$$\begin{array}{rcl} \mathcal{R}(t) & := & \displaystyle\sum_{i=2}^{K} \Delta_i \mathbb{E}[N_i(t)] \\ & \leq & \displaystyle\sum_{i=2}^{K} \left(\frac{2\alpha \ln(t)}{\Delta_i} + \frac{\alpha+1}{\alpha-1}\Delta_i\right) \end{array}$$

$\square$

1.5.2 Improvements?

Remark 1.7 - *The regret of UCB grows logarithmically with T. No other algorithm can do better.*
Further, the constant factor of $\ln(T)$ used is almost optimal. This shall now be shown.

Definition 1.9 - *Strongly Consistent*
A strategy for the multi-armed bandit problem is said to be *strongly consistent* if its regret satisfies $\mathcal{R}(T) = o(T^\alpha)$ for all $\alpha > 0$. (i.e. its regret grows slower than any fractional power of T).

The *UCB(α)* algorithm is strongly consistent for all $\alpha > 1$ as its regret grows logarithmically with T.

Theorem 1.5 - *Lai and Robbins*
Consider a K-armed bandit, where the rewards from arm i are iid Bern(μ_i) and rewards from distinct arms are mutually independent. Then, for any *strongly consistent* strategy, the number of times that a sub-optimal arm i is played up to time T, $N_i(T)$ satisifies

$$\liminf_{T\to\infty}\frac{\mathbb{E}[N_i(T)]}{\ln(T)} \geq \frac{1}{K(\mu_i;\mu^*)}$$

where $\mu* := \max_{i=1}^K \mu_i$ and $K(q;p)$ is the *KL-Divergence* of a Bern(q) distribution wrt a Bern(p) distribution.

Proposition 1.5 - *Lower bound on Regret*
Here we derive a lower bound for the regret of any strongly consistent strategy from the multi-armed bandit problem.

$$\begin{aligned}\liminf_{T\to\infty}\frac{\mathcal{R}(T)}{\ln(T)} &= \liminf_{T\to\infty}\frac{\sum_{i;\mu_i<\mu^*} *\mu^* - \mu_i)\mathbb{E}[N_i(T)]}{\ln(T)} \\ &\geq \sum_{i;\mu_i<\mu^*}\frac{\mu^*-\mu_i}{K(\mu_i;\mu^*)} & \text{by } \texttt{Theorem 1.6}\end{aligned}$$

Proposition 1.6 - *Comparing to Lower bound of UCB(α)*
We showed that the regret of the *UCB(α)* algorithm satisfies

$$\limsup_{T\to\infty}\frac{\mathcal{R}(T)}{\ln(T)} \leq \sum_{i;\mu_i<\mu^*}\frac{2}{\mu^*-\mu_i}$$

To compare this to the result in `Proposition 1.5` we use *Pinsker's Inequality.* (Proof in homework).

We see thjat the upper bound on the regret achieved by *UCB(α)* is approximately four times greater than the lower bound on the best regret achievable by any algorithm. This is very good.

Theorem 1.6 - *Concentration Inequalities for Sample Means*

$$\begin{aligned}\mathbb{P}\left(\hat{\mu}_{j,N_j(s)} \geq u_j + \sqrt{\frac{\alpha\ln s}{2N_j(s)}}\right) &\leq e^{-\alpha\ln s} = s^{-\alpha} \\ \mathbb{P}\left(\hat{\mu}_{1,N_1(s)} \leq u_1 - \sqrt{\frac{\alpha\ln s}{2N_1(s)}}\right) &\leq e^{-\alpha\ln s} = s^{-\alpha}\end{aligned}$$

Proof 1.4 - *Theorem 1.5*
The proof is immediate from *Hoeffding's Inequality*, which is applicable since the X_j are iid and take values in $\{0,1\} \subseteq [0,1]$.

Remark 1.8 - *Is there an algorithm which achieves lower regret?*
No. There is no algorithm which has regret growing slower than $\ln(T)$.

1.6 Thompson Sampling

Remark 1.9 - *Thompson Sampling*
Thompson Sampling is a *Bayesian* algorithm for the multi-armed bandit problem. It was one of the first algorithms for solving the problem, but remains on of the best as it is asymptotically optimal.

Theorem 1.7 - *Relationship between Beta & Gamma Distribution*
Let $X \sim \text{Gamma}(\alpha, \lambda)$ and $Y \sim \text{Gamma}(\beta, \lambda)$ (ie shared scale parameter but different shape parameters). Then

$$V := \frac{X}{X+Y} \sim \text{Beta}(\alpha, \beta)$$

Proof 1.5 - *Theorem 1.7*
Consider the map $(X, Y) \mapsto \dfrac{X}{X+Y}$. This maps $\mathbb{R}^2 \to \mathbb{R}$ (reduces dimensions) and thus we cannot directly use the formula to compute the density of V.
We introduce an auxiliary random variable $W = X$. Now $(X, Y) \mapsto (V, W) := g(X, Y) = \left(\dfrac{X}{X+Y}, X\right)$. Note that X and Y are non-negative random variables. Hence, $V \in [0, 1]$ and $W \in \mathbb{R}^+$.
For $(v, w) \in [0, 1] \times \mathbb{R}^+$ we want to find all solutions of $g(x, y) = (v, w)$. Clearly $(x, y) = \left(w, w\dfrac{1-v}{v}\right)$ in a unique solution. The joint density of (V, W) is given by the formula

$$f_{V,W}(v, w) = \sum_{(x,y):g(x,y)=(v,w)} \frac{f_{X,Y}(x, y)}{|\det(J_g(x, y))|} \quad (1)$$

Using the density function for *Gamma* random variables and the independence of X and Y

$$f_{X,Y}(x, y) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1} y^{\beta-1} e^{-x-y}$$

Next, we compute the *Jacobian* of g and its determinant

$$J_g(x, y) = \begin{pmatrix} \dfrac{y}{(x+y)^2} & \dfrac{-x}{(x+y)^2} \\ 1 & 0 \end{pmatrix} \quad \implies \quad |\det(J_g(x, y))| = \frac{x}{(x+y)^2} = \frac{v^2}{w}$$

Substituting the definition of $f_{X,Y}$, the results of the *Jacobian* and the solution $(x, y) = \left(w, w\dfrac{1-v}{v}\right)$ into (1) we get

$$\begin{aligned} f_{V,W}(v, w) &= \frac{1}{v^2/w} \frac{1}{\Gamma(\alpha)\Gamma(\beta)} w^{\alpha-1} e^{-w} \left(\frac{w(1-v)}{v}\right)^{\beta-1} e^{-w(1-v)/v} \\ &= \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \frac{w}{v^2} w^{\alpha-1} \left(w\frac{1-v}{v}\right)^{\beta-1} e^{-w/v} \end{aligned}$$

We are only interested in the marginal distribution of V

$$\begin{array}{rcl} f_V(v) & = & \int_{w=0}^{\infty} f_{V,W}(v,w)dw \\ & = & \frac{(1-v)^{\beta-1}v^{\alpha}-1}{\Gamma(\alpha)\Gamma(\beta)}\int_{w=0}^{\infty}\left(\frac{w}{v}\right)^{\alpha+\beta-1}e^{-w/v}\frac{dw}{v} \\ & = & \frac{(1-v)^{\beta-1}v^{\alpha-1}}{\Gamma(\alpha)\Gamma(\beta)}\int_{u=0}^{\infty}u^{\alpha+\beta-1}e^{-u}du \qquad\qquad \text{where } u:=\frac{w}{v} \\ & = & \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}v^{\alpha-1}(1-v)^{\beta-1} \\ & \sim & \text{Beta}(\alpha,\beta) \end{array}$$

$\square$

Remark 1.10 - *Beta*$(1,1)$ *is equivalent to Uniform*$[0,1]$
Proposition 1.7 - *Thompson Sampling Algorithm*
Consider a K-armed bandit with independent Bernoulli arms with parameters $\mu_1,\dots,\mu_K$. *Thompson Sampling* follows the following process.

i). Define a prior distribution Beta$(1,1)$ for the parameter of each arm.

ii). At the start of round t, sample $\hat{\mu}_i(t)$ for $i\in[1,K]$, from the corresponding prior distributions.

iii). Play the arm with the greatest sample value $I(t)\in\text{argmax}_{i\in[1,K]}\hat{\mu}_i(t)$ (breaking ties arbitrarily).

iv). Compute a posterior distribution for that parameter, based on the observed reward. Beta$(\alpha+1,\beta)$ if a reward is given and Beta$(\alpha,\beta+1)$ it a reward is not given. (The priors for the other arms are the same as their priors as no result was observed).

v). Keep repeating ii)-iv) using the posterior distributions calculated in round t as the priors for round $t+1$.

The endpoint of the algorithm is when time runs out.

Proposition 1.8 - *Choosing the Prior Distribution for Thompson Sampling*
We use *Beta* distributions as the prior for the unknown parameters μ_i of the *Bernoulli* reward distributions. This choice is convenient because, if the prior has a *Beta* distribution then so does the posterior distribution, after observing the reward (i.e. it is a *Conjugate Prior*).

Remark 1.11 - *Beta Distributions are conjugate priors for Bernoulli Random Variables*
Remark 1.12 - *Thompson Sampling on other Reward Distributions*
Thompson Sampling can be extended to non-bernoulli reward distributions. However, the ease of implementation depends on how easy it is to sample from the posterior distribution. *Conjugate priors* should be used to make this easier. Bounds on the regret are not known in all cases.

Theorem 1.8 - *Posterior from a Beta Distribution Prior of a Bernoulli Random Variable*
Let $X\sim$ Bernoulli(μ) and assume a prior distribution $\mu\sim$ Beta(α,β). Then, the posterior distribution of μ given $X=1$ is Beta$(\alpha+1,\beta)$ and given $X=0$ is Beta$(\alpha,\beta+1)$.

Proof 1.6 - *Theorem 1.8*

Let f_μ denote the density of μ. By out assumption of a Beta(α,β) prior we have

$$f_\mu(p)=\frac{\Gamma(\alpha+\beta)}{\gamma(\alpha)\Gamma(\beta)}p^{\alpha-1}(1-p)^{\beta-1}\quad\text{for } p\in[0,1]$$

Also $\mathbb{P}(X=1|\mu=p)=p=1-\mathbb{P}(X=0|\mu=p)$.

Hence, by *Bayes's Theorem*, the posterior density conditional on $X=1$ satisfies

$$\begin{array}{rcl} f_\mu(p|X=1) & \propto & f_\mu(p)\mathbb{P}(X=1|p) \\ & \propto & p^\alpha(1-p)^{\beta-1} \end{array}$$

Where the constant of proportionality is determined by the fact the density integrates to 1.

We can recognise this distribution as the pdf of a Beta$(\alpha+1,\beta)$. Similarly, it can be shown that $f_\mu(\cdot|X=0)$ is the density of a Beta$(\alpha,\beta+1)$ random variable. □

Theorem 1.9 -
Let $X\sim$ Beta(α,β) for $\alpha,\beta\in\mathbb{N}$ and $Y\sim$ Bin$(\alpha+\beta-1,p)$ for $p\in(0,1)$. Then

$$\mathbb{P}(X>p)=\mathbb{P}(Y\leq\alpha-1)$$

Proof 1.7 - *Theorem 1.9*
Consider this result. Let $\{N_t\}_{t\geq0}$ be a poisson process with intensity $\lambda>0$ and $n,t\in\mathbb{N}$.Then, conditional on the event $N_t=n$, the unordered increment times on $[0,t]$ are mutually independent and uniformly distributed on $[0.t]$.

Let $X\sim$ Beta(α,β) then we can write $X=\frac{V}{V+W}$ where $V\sim$ Gamma$(\alpha,)$, $W\sim$ Gamma(β,γ). If α,β are integered values then we can interpret V as the time of the α^{th} increment of a unit rate Poisson process and $(V+W)$ as the time of the $(\alpha+\beta)^{th}$ increment.

Consequently, conditional on $(V+W)=\tau$, the poisson process N_t has exactly $\alpha+\beta-1$ increments in the interval $[0,\tau)$. By the fact result above, the unordered times of these increments are iid uniformly distributed on $[0,\tau)$.

The event $\{X>p\}$ is the same as the event $\{V>p\tau\}$, conditional on $(V+W)=\tau$. This means that at most $\alpha=1$ increments occur in $[0,p\tau]$. As the increments are IID uniform in $[0,\tau)$ and there are $\alpha+\beta-1$ in total, the number of increments in $[0,p\tau]$ has a Bin$(\alpha+\beta-1,p)$ distribution.

THus the events $\{X>p\}$, $\{V>p\tau|V+W=\tau\}$ and $\{Y\leq\alpha-1\}$ all have the same probability. □

1.6.1 Analysis

Remark 1.13 - *Analysis of Thompson Sampling is Hard*
The main challenge is to deal with the situation where there is an initial run of bad luck on the optimal arm. This causes the posterior for the optimal arm to be biased towards small values. Hence, the optimal arm is not played very often meaning it takes a long time to recover from the initial bad luck.

For contrast, we only need to worry about plays of sub-optimal arms after they have been played sufficiently often, by which time the posterior is concentrated around the true parameter value.

Theorem 1.10 - *Bound on Regret*
The regret of *Thompson Sampling* applied to a multi-armed bandit with two *Bernoulli* arms is bounded as

$$\mathcal{R}(T) \leq \frac{40\ln(T)}{\Delta} + c$$

where Δ is the arm gap and c is some arbitrary constant which depends on Δ (but, importantly, not T).

The proof to this is not given in full, but some useful lemmas are shown.

Remark 1.14 - *Posterior Distribution over Time*
As the number of times each arm is played increases, its posterior distribution concentrates increasingly sharply around the true parameter value.

Proposition 1.9 - *Number of times wrong arm is played*
In the analysis of a multi-armed bandit with two *Bernoulli arms* we assume that the second arm (with parameter μ_2) is the worse of the two arms. Thus to bound regret it suffices to bound the number of times the second arm is played.

Fix a time horizon T and define $L = \left\lceil \frac{24\ln(T)}{\Delta^2} \right\rceil$, $\tau = \inf\{t \in [0,T] : N_2(t) \geq L\}$. Then, for $\tau \leq t \leq T$

$$\mathbb{P}\left(\theta_2(t) \geq \mu_2 + \frac{\Delta}{2}\right) \leq \frac{2}{T^3}$$

Where $\theta_i(t)$ is the value sampled from the prior of μ_i at time t.

Proof 1.8 - *Proposition 1.9*
By the definition of τ if $t \geq \tau$ then $N_2(t) \geq L$. Thus

$$\begin{aligned}
&\mathbb{P}\left(\theta_2(t) \geq \mu_2 + \frac{\Delta}{2}\right) \\
= \ &\mathbb{P}\left(\theta_2(t) \geq \mu_2 + \frac{\Delta}{2}, \frac{S_2(t)}{N_2(t)} \leq \mu_2 + \frac{\Delta}{4}\right) + \mathbb{P}\left(\theta_2(t) \geq \mu_2 + \frac{\Delta}{2}, \frac{S_2(t)}{N_2(T)} > \mu_2 + \frac{\Delta}{4}\right) \\
\leq \ &\mathbb{P}\left(\theta_2(t) \geq \mu_2 + \frac{\Delta}{2} \Big| \frac{S_2(t)}{N_2(t)} \leq \mu_2 + \frac{\Delta}{4}\right) + \mathbb{P}\left(\theta_2(t) \geq \mu_2 + \frac{\Delta}{2}, \frac{S_2(t)}{N_2(T)} > \mu_2 + \frac{\Delta}{4}\right) \quad (1)
\end{aligned}$$

We now bound these two terms.

Firstly, conditional on the number of times the second arm is player $N_2(T)$, the total reward from these plays $S_2(t)$ is the sum of $N_2(t)$ independent $\text{Bern}(\mu_2)$ random variables. Hence, using *Hoeffding's Inequality* we have

$$\mathbb{P}\left(\frac{S_2(t)}{N_2(t)} > \mu_2 + \frac{\Delta}{4} \Big| N_2(t)\right) \leq \exp\left(-2N_2(t)\frac{\Delta^2}{16}\right)$$

As we have assumed that $N_2(t) \geq L \geq \frac{1}{\Delta^2}(24\ln(T))$ we can conclude that

$$\mathbb{P}\left(\frac{S_2(t)}{N_2(t)} > \mu_2 + \frac{\Delta}{4}\right) \leq \exp\left(-3\ln(T)\right) = \frac{1}{T^3} \quad (2)$$

Next, we note that conditional on $S_2(t)$ and $N_2(t)$, the distribution of $\theta_2(t)$ is $\text{Beta}(S_2(t) + 1, N_2(t) - S_2(t) + 1)$. Consequently, by `Theorem 1.9`, we have that

$$\mathbb{P}\left(\theta_2(t) \geq \mu_2 + \frac{\Delta}{2}\right) = \mathbb{P}\left(\text{Bin}\left(N_2(t) + 1, \mu_2 + \frac{\Delta}{2}\right) \leq S_2(t)\right)$$

Applying *Hoeffding's inequality* to the RH term, we see that for $N_2(t) \geq L$ we have

$$\begin{array}{rcl}\mathbb{P}\left(\theta_2(t) \geq \mu_2 + \frac{\Delta}{2} \middle| \frac{S_2(t)}{N_2(t)} \leq \mu_2 + \frac{\Delta}{4}\right) & \leq & \exp\left(-2\big(N_2(t)+1\big)\frac{\Delta^2}{16}\right) \\ & \leq & \exp\left(-\frac{L\Delta^2}{8}\right) \\ & \leq & \frac{1}{T^3} \end{array} \quad (3)$$

Substituting (2) and (3) into (1), we can conclude that if $t \geq \tau$ (ie $N_2(t) \geq L$) then

$$\mathbb{P}\left(\theta_2(t) \geq \mu_2 + \frac{\Delta}{2}\right) \leq \frac{2}{T^3}$$

□

1.7 Genie Analysis

Remark 1.15 - *Genie*
Here we analyse a simpler version of Thompson Sampling for a 2-armed bandit. We assume that the value of μ_1 is known, but it is not known whether it is greater than μ_2.

This means we now only have a prior/posterior for μ_2 and we compare $\theta_2(t)$ (the value sampled from the prior of μ_2) to the true value of μ_1.

It is likely that this scenario should be more successful than the standard scenario, thus we can only find an upper bound on the regret of the normal scenario.

Theorem 1.11 - *Times Sub-optimal arm is played*
Fix $T \in \mathbb{N}$ and define $L := \left\lceil \frac{2\ln(T)}{\Delta^2} \right\rceil$, $\tau := \inf\{t \in [1,T] : N_2(t) \geq L\}$. τ is the first time that arm two has been played at least L times. The number of plays of arm two after τ is bounded as

$$\forall\ t \geq \tau \quad \mathbb{P}(\theta_2(t) \geq \mu_1) \leq \frac{2}{T}$$

This means $\mathbb{E}[\text{plays of arm two after time } \tau] = (T-\tau)\frac{2}{T} \leq 2$

Proof 1.9 - *Theorem 1.11*
Define the events

$$A_t := \{\theta_2(t) \geq \mu_1\} \quad B_t := \left\{\frac{S_2(t)}{N_2(t)} \leq \mu_2 + \frac{\Delta}{2}\right\}$$

A_t is the event that the sample from the prior of μ_2 is greater than μ_1. B_t is the event the observed rewards from arm 2 at closer to μ_2 than μ_1.

If $t \geq \tau$, then $N_2(t) \geq L$ and Hoeffding's inequality yields

$$\begin{array}{rcll}\mathbb{P}(B_t^c) & \leq & \exp\left(-2N_t\frac{\Delta^2}{4}\right) & \\ & \leq & \exp\left(-\frac{\Delta^2 L}{2}\right) & \text{by def. } L \\ & \leq & e^{-\ln(T)} & \\ & = & \frac{1}{T} & \end{array}$$

We can bound $\mathbb{P}(A_t)$ as follows

$$\begin{array}{rcl}\mathbb{P}(A_t) & = & \mathbb{P}(A_t \cap B_t) + \mathbb{P}(A_t \cap B_t^c) \\ & = & \mathbb{P}(A_t|B_t)\mathbb{P}(B_t) + \mathbb{P}(A_t|B_t^c)\mathbb{P}(B_t^c) \\ & \leq & \mathbb{P}(A_t|B_t) + \mathbb{P}(B_t^c) \end{array}$$

We now want to bound $\mathbb{P}(A_t|B_t)$.

Since $\theta_2(t+1)$ is sampled from the posterior distribution of μ_2 after t rounds we have

$$\theta_2(t+1) \sim \text{Beta}\left(1 + \underbrace{S_2(t)}_{\#\text{ successes}}, 1 + \underbrace{N_2(t) - S_2(t)}_{\#\text{ failures}}\right)$$

Hence, by `Theorem 1.9`

$$\mathbb{P}\big(\theta_2(t+1) \geq \mu_1 \big| S_2(t), N_2(t)\big) = \mathbb{P}\big(\text{Bin}(N_2(t)+1, \mu_1) \leq S_2(t)\big)$$

By Hoeffding's inequality, if $S_2(t) < \mu_1 \cdot N_2(t)$, then

$$\mathbb{P}\big(\text{Bin}(N_2(t)+1, \mu_1) \leq S_2(t)\big) \leq \exp\left(-2N_2(t)\left(\mu_1 - \frac{S_2(t)}{N_2(t)}\right)^2\right)$$

Conditioning on the event B_t, we have

$$\begin{array}{rcl} S_2(t) & \leq & \left(\mu_2 + \frac{\Delta}{2}\right) N_2(t) \\ & = & \left(\mu_1 - \frac{\Delta}{2}\right) N_2(t) \end{array}$$

Hence

$$\begin{array}{rcl} \mathbb{P}(A_t|B_t, N_2(t)) & = & \mathbb{P}(\theta_2(t+1) \geq \mu_1 | B_t, N_2(t)) \\ & \leq & \exp\left(-\frac{2N_2(t)\Delta^2}{4}\right) \end{array}$$

Recall, by the definition of τ, $\forall\ t \geq \tau,\ N_2(t) \geq L$.
Hence, $\forall\ t \geq \tau$

$$\begin{array}{rcl} \mathbb{P}(A_t|B_t) & \leq & \exp\left(-\frac{L\Delta^2}{2}\right) \\ & \leq & e^{-\ln T} \\ & = & \frac{1}{T} \end{array}$$

We have already showed $\forall\ t \geq \tau,\ \mathbb{P}(B_t^c) \leq \frac{1}{T}$.

Combining these results, we get

$$\forall\ t \geq \tau, \quad \mathbb{P}(A_t) \leq \mathbb{P}(A_t|B_t) + \mathbb{P}(B_t^c) \leq \frac{2}{T}$$

This is claim on `Theorem 1.11` $\square$

Proposition 1.10 - *Bound of Regret*
Using `Theorem 1.11` above we can bound the regret as

$$\mathcal{R}(T) \leq \Delta \cdot (L+2)$$

where $L+2$ is the most time arm two is played in the first T time steps.

2 Probability

Definition 2.1 - *Random Process*
A *Random Process* is a collection of random variables indexed by time $\{X_t\}_{t \in T}$ (e.g. flipping a coin several times). Each of these random variables can take a value from a state space S. A random process a *Discrete Time Process* if the index set T is discrete. A random process a *Continuous Time Process* if the index set T is continuous.

2.1 Probability Inequalities

Remark 2.1 - *We can use the moments of a random variable to determine bounds on the probability of it taking values in a certain set.*

Theorem 2.1 - *Markov's Inequality*

Let X be a non-negative random variable. Then

$$\forall\ c > 0 \quad \mathbb{P}(X \geq c) \leq \frac{\mathbb{E}(X)}{c}$$

Proof

Consider an event A and define its indicator $\mathbb{1}(A)(\omega) := \begin{cases} 1 & \omega \in A \\ 0 & \omega \notin A \end{cases}$. Fix $c > 0$, then

$$\begin{array}{rcl} \mathbb{E}(X) & \geq & \mathbb{E}[X\mathbb{1}(X \geq c)] \\ & \geq & \mathbb{E}[c\mathbb{1}(X \geq c)] \\ & = & c\mathbb{P}(X \geq c) \\ \implies \mathbb{P}(X \geq c) & \leq & \frac{1}{c}\mathbb{E}(X) \end{array}$$

Theorem 2.2 - *Chebyshev's Inequality*

Let X be a random-variable with finite mean and variance. Then

$$\forall\ c > 0 \quad \mathbb{P}(|X - \mathbb{E}(X)| \geq c) \leq \frac{\mathrm{Var}(X)}{c^2}$$

Proof

Note that the events $|X - \mathbb{E}(X)| \geq c$ and $(X - \mathbb{E}(X))^2 \geq c^2)$ are equivalent. Note that $\mathrm{Var}([X - \mathbb{E}(X)]^2) = \mathrm{Var}(X)$. Then the result follows by *Markov's Inequality.*

Theorem 2.3 - *Chebyshev's Inequality for Sum of IIDs*

Let $X_1, \dots, X_n$ be IID random variables with finite mean μ and finite variance σ^2.

$$\forall\ c > 0 \quad \mathbb{P}\left(\left|\left(\sum_{i=1}^n X_i\right) - n\mu\right| \geq nc\right) \leq \frac{\sigma^2}{nc^2}$$

Proof

This is proved by extending the proof of `Theorem 2.2` and noting that the variance of a sum of IIDs is the sum of the individual variances.

Theorem 2.4 - *Chernoff Bounds*

Let X be a random variable whose moment-generating function $\mathbb{E}[e^{\theta X}]$ is finite $\forall \theta$. Then

$$\forall\ c \in \mathbb{R} \quad \mathbb{P}(X \geq c) \leq \inf_{\theta > 0} e^{-\theta c}\mathbb{E}(e^{\theta X}) \quad \text{and} \quad \mathbb{P}(X \leq c) \leq \inf_{\theta < 0} e^{-\theta c}\mathbb{E}(e^{\theta X})$$

Proof

Note that the events $X \geq c$ and $e^{\theta X} \geq e^{\theta c}$ are equivalent for all $\theta > 0$. The result follows by applying *Markov's Inequality* to $r^{\theta X}$ and taking the best bound over all possible θ.

$$\begin{array}{rcl} \mathbb{P}(X \geq c) & = & \mathbb{P}(e^{\theta X} \geq e^{\theta c}) \\ & \leq & e^{-\theta c}\mathbb{E}(e^{\theta X}) \\ & \leq & \inf_{\theta < 0} e^{-\theta c}\mathbb{E}(e^{\theta X}) \end{array}$$

Theorem 2.5 - *Chernoff Bounds for Sum of IIDs*
Let $X_1, \dots, X_n$ be IID random variables. Then $\forall\ c \in \mathbb{R}$

$$\begin{aligned}\mathbb{P}\left(\sum_{i=1}^n X_i \geq nc\right) &\leq \inf_{\theta>0} e^{-n\theta c}\left(\mathbb{E}\left[e^{\theta X}\right]\right)^n \\ \mathbb{P}\left(\sum_{i=1}^n X_i \leq nc\right) &\leq \inf_{\theta<0} e^{-n\theta c}\left(\mathbb{E}\left[e^{\theta X}\right]\right)^n\end{aligned}$$

Theorem 2.6 - *Jensen's Inequality*
Let f be a *Convex Function* and X be a random variable. Then

$$\mathbb{E}[f(X)] \geq f(\mathbb{E}[X])$$

Theorem 2.7 - *Bound on Moment Generating Function*
Let X be a random variable taking values in $[0,1]$ with finite expected value μ. Then we can bound the MGF of the centred random variable with

$$\forall\ \theta \in \mathbb{R} \quad \mathbb{E}\left[e^{\theta(X-\mu)}\right] \leq e^{\theta^2/8}$$

Proof (of weaker version)
Let X_1 be an independent copy of X, so both have mean μ. We can easily verify that $f(x) = e^{\theta x}$ is a convex function for all $\theta \in \mathbb{R}$. By *Jensen's Inequality* to $f(\cdot)$ and X_1

$$\mathbb{E}[e^{-\theta X_1}] \geq e^{-\theta\mathbb{E}[X_1]} = e^{-\theta\mu} \quad (1)$$

Consequently

$$\begin{array}{rrcll} & \mathbb{E}[e^{\theta(X-X_1)}] & = & \mathbb{E}[e^{\theta X}]\cdot\mathbb{E}[e^{-\theta X_1}] & \text{by independence} \\ & & \geq & \mathbb{E}[e^{\theta X}]\cdot e^{-\theta\mu} & \text{by (1)} \\ & & = & \mathbb{E}[e^{\theta(X-\mu)}] & \\ \implies & \mathbb{E}[e^{\theta(X-X_1)}] & \geq & \mathbb{E}[e^{\theta(X-\mu)}] & \end{array}$$

Since $X, X_1 \in [0,1]$ then $(X-X_1) \in [-1,1]$. As X, X_1 have the same distribution $\mathbb{E}(X-X_1) = 0$ and the distribution is symmetric around the mean.
Define random variable S which is independent of X, X_1 and takes values $\{-1,1\}$, each with probability $p = \frac{1}{2}$. $S(X-X_1)$ has the same distribution as $(X-X_1)$ due to independence of S and symmetry of $(X-X_1)$. Hence

$$\begin{array}{rcll} \mathbb{E}[e^{\theta(X-X_1)}] & = & \mathbb{E}[e^{\theta S(X-X_1)}] & \text{by identical distribution} \\ & \leq & \mathbb{E}[e^{\theta S}] & \text{(2) since } (X-X_1)\in[-1,1] \\ & = & \frac{1}{2}(e^{\theta}+e^{-\theta}) & \text{by def. of expectation} \\ \implies \mathbb{E}[e^{\theta(X-X_1)}] & \leq & \frac{1}{2}(e^{\theta}+e^{-\theta}) & \end{array}$$

Note that $f(x) = e^x + e^{-x}$ is increasing for $x \in (0,\infty)$; decreasing for $x \in (-\infty, 0)$; and symmetric around 0.
Using a *Taylor Series* we an observed that

$$\begin{array}{rcll} \frac{1}{2}(e^{\theta}-e^{-\theta}) & = & \sum\limits_{n=0}^{\infty}\frac{\theta^{2n}}{(2n)!} & \text{by Taylor expansion of} e^x \\ & \leq & \sum\limits_{n=0}^{\infty}\dfrac{(\theta^2/2)^n}{n!} & \\ & \overset{\text{def.}}{=} & e^{\theta^2/2} & \\ \implies \frac{1}{2}(e^{\theta}+e^{-\theta}) & \leq & e^{\theta^2/2} & \end{array}$$

Combining all these results we get

$$\begin{aligned} & \mathbb{E}[e^{\theta(X-\mu)}] \leq \mathbb{E}[e^{\theta(X-X_1)}] \leq \tfrac{1}{2}(e^{\theta}+e^{-\theta}) \leq e^{\theta^2/2} \\ \implies & \mathbb{E}[e^{\theta(X-\mu)}] \leq e^{\theta^2/2} \end{aligned}$$

$\square$

Theorem 2.8 - *Hoeffding's Theorem*
Let $X_1,\dots,X_n$ be IID random variables taking values in $[0,1]$ and with finite expected value μ. Then

$$\forall\ t>0 \quad \mathbb{P}\left(\sum_{i=1}^n (X_i-\mu) > nt\right) \leq e^{-2nt^2}$$

Proof
From *Chernoff's Bound* we have that

$$\forall\ \theta>0 \quad \mathbb{P}\left(\sum_{i=1}^n (X_i-\mu) > nt\right) \leq e^{-\theta nt}\left(\mathbb{E}[e^{\theta(X-\mu)}]\right)^n$$

Using `Theorem 2.7` to bound the moment generating function, we get

$$\forall\ \theta>0 \quad \mathbb{P}\left(\sum_{i=1}^n (X_i-\mu) > nt\right) \leq e^{-\theta nt}\cdot e^{n\frac{\theta^2}{8}} = e^{n\left(-\theta t+\frac{1}{8}\theta^2\right)}$$

Thus, by taking logs and rearranging, we get

$$\forall\ \theta>0 \quad \frac{1}{n}\log\mathbb{P}\left(\sum_{i=1}^n (X_i-\mu) > nt\right) \leq -\theta t+\frac{\theta^2}{8}$$

We have that $-\theta t+\frac{\theta^2}{8}$ is minimised at $\theta=4t$ which is positive if t is positive. Thus, by applying this bound and substituting $\theta=4t$ we get

$$\forall\ \theta>0 \quad \mathbb{P}\left(\sum_{i=1}^n (X_i-\mu) > nt\right) \leq e^{n\left(-4t^2+\frac{1}{8}(16t^2)\right)} = e^{n(-4t^2+2t^2)} = e^{-2nt^2}$$

$\square$

2.2 Markov Processes

Definition 2.2 - *Markov Property*
A random process has the *Markov Property* if the conditional probability of a future state only depends on the current state.

$$\mathbb{P}(X_{t+1}=y|X_t=x_t,X_{t-1}=x_{t-1}) = \mathbb{P}(X_{t+1}=y|X_t=x_t)$$

A random process with the *Markov Property* is called a *Markov Process/Chain.*

Remark 2.2 - *On this course we only deal with discrete time markov chains*
Definition 2.3 - *Transience*
A state $x\in S$ is *Transient* if $\mathbb{P}(\exists\ t>0 : X_t=x|X_0=x)<1$. The number of times the markov chain returns to a transient state is finite, with probability 1.

Definition 2.4 - *Recurrent*
A state $x \in S$ is *Recuurent* if $\mathbb{P}(\exists\ t > 0 : X_t = x | X_0 = x) = 1$. The number of times the markov chain returns to a recurrent state is infinite, with probability 1.

Every markov chain, with a finite state space S, has a recurrent communicating class.

Definition 2.5 - *Communication Class*
We say $y \in S$ is *Accessible* from $x \in S$ if $\exists\ t \geq 0$ st $[P^t]_{xy} > 0$.

We say x and y *communicate* (denoted xCy) if: x is *accessible* from y and y is *accessible* from x.

Communication is an *equivalence relation* on the state space S. Hence, *communication* partitions S into equivalence classes called *Communication Classes*. All elements of a *Communication Class* communicate with all other elements in the class, it is possible for elements to be accessible from another class but not for those elements to *communicate*.

If one state in a *Communicating Class* is *Transient/Recurrent* then all states are in that class.

If a *Markov Chain* has only one communicating class it is called *Irreducible*.

2.2.1 Discrete Time Markov Chains

Proposition 2.1 - *Characterising a Discrete Time Markov Process*
A *Discrete Time Markov Process* can be characterised by the set of all 1-step conditional probabilities

$$\mathbb{P}(X_{t+1} = y | X_t = x)\ \forall\ x, y \in S$$

A markov chain is *time-homogeneous* if the 1-step conditionals only depend on x, y and not on t ($\mathbb{P}(X_{t+1} = y | X_t = x) = \mathbb{P}(X_1 = t | X_0 = x)$). The 1-step conditional probabilities of a *time-homogeneous markov process* can be specified in an $|S| \times |S|$ matrix P where

$$p_{x,y} = \mathbb{P}(X_{t+1} = y | X_t = x)$$

P is a *Stochastic Matrix*.

Proposition 2.2 - *n-Step Transition Probabilities from 1-Step Transition Matrix*
Let P be the 1-step transition matrix for a *time-homogeneous*.

The 2-step transition probabilities (ie $\mathbb{P}(X_{t+2} = z | X_t = x)$) can be found as

$$\begin{array}{rcll}
\mathbb{P}(X_{t+2} = z | X_t = x) & = & \mathbb{P}(X_2 = z | X_0 = x) & \text{by time-homogeneity} \\
& = & \sum_{y \in S} \mathbb{P}(X_2 = z, X_1 = y | X_t = x) & \\
& = & \sum_{y \in S} \mathbb{P}(X_1 = y | X_0 = x)\mathbb{P}(X_2 = z | X_1 = t, X_0 = x) & \\
& = & \sum_{y \in S} \mathbb{P}(X_1 = y | X_0 = x)\mathbb{P}(X_2 = z | X_1 = y) & \\
& = & \sum_{y \in S} p_{xy} p_{yz} & \\
& \equiv & [P^2]_{xz} &
\end{array}$$

This can be generalise for the n-step transition probabilities with

$$\mathbb{P}(X_{t+n} = z | X_t = x) = [P^n]_{xz}$$

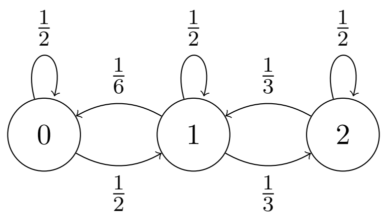
Proposition 2.3 - *Any Joint Probability from 1-Step Transition Matrix*

For a time homogeneous markov process the joint distribution of any transition can be computed by considering the individual steps of the transition.

$$\mathbb{P}(X_{n_0} = x_0, X_{n_1} = x_1, X_{n_2} = x_2, \dots) = \mathbb{P}(X_{n_0} = x_0) \cdot [P^{n_0 - n_1}]_{x_0 x_1} \cdot [P^{n_1 - n_2}]_{x_1 x_2} \cdots$$

Proposition 2.4 - *State Diagram Representation*
A graph/automata can be drawn to represent the transition probability matrix P. A node is assigned for each member of the state space S and an arrow is drawn between each pair of nodes (x, y) where $P_{xy} \neq 0$. Generally the value of P_{xy} is denoted on the arrow.

$$P = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{6} & \frac{1}{2} & \frac{1}{3} \\ 0 & \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$



Definition 2.6 - *Invariant Distribution*
Let $\boldsymbol{\mu}(t)$ denote the probability distribution of random variable X (i,e, $\mu_x(t) = \mathbb{P}(X_t = x)$). Then

$$\begin{array}{rrcl} & \boldsymbol{\mu}(t+1) & = & \mathbb{P}(X_{t+1} = y) = \sum_{x \in S} \mathbb{P}(X_t = x, X_{t+1} = y) = \sum_{x \in S} \mu_x(t) p_{xy} = \boldsymbol{\mu}(t) P \\ \implies & \boldsymbol{\mu}(t+1) & = & \boldsymbol{\mu}(t) P \end{array}$$

A distribution $\boldsymbol{\pi}$ on the state space is called an *Invariant Distribution* if $\boldsymbol{\pi} = \boldsymbol{\pi} P$. If X_t has distribution $\boldsymbol{\pi}$ so will $X_{t+1}, \dots$. Every markov chain with a *finite* state space S has an *invariant distribution.* (Not necessarily true if S is infinite).

Proposition 2.5 - *Finding an Invariant Distribution*
If an *Invariant Distribution* it is easy to find by solving $\boldsymbol{\pi} P = \boldsymbol{\pi}$ and using normalising constant $\sum_{x \in S} \pi_x = 1$.

Remark 2.3 - *If a Markov chain is irreducible, its invariant distribution (if one exists) is unique*
If a *Markov Chain* is irreducible and has a finite state space, then it has a unique invariant distribution.

Example 2.1 - *Markov Chains*

- The *Asymmetric Simple Random Walk* on $\mathbb{Z}$ is $\underbrace{\text{irreducible}}_{\text{obvious}}$, $\underbrace{\text{transient}}_{\text{not obvious}}$ and has no invariant distribution.
- The *Symmetric Simple Random Walk* on $\mathbb{Z}$ is $\underbrace{\text{irreducible}}_{\text{obvious}}$, $\underbrace{\text{recurrent}}_{\text{not obvious}}$ and has no invariant distribution.

Theorem 2.9 - *Ergodic Theorem for Markov Chains*
Let $\{X_t\}_{t\in\mathbb{N}}$ be an irreducible markov chain on state space S (not necessarily finite) with unique invariant distribution $\boldsymbol{\pi}$. Then

$$\forall\ x \in S \quad \lim_{t\to\infty} \frac{1}{t}\sum_{s=1}^{t} \mathbb{1}(X_s = x) = \pi_x$$

i.e. The fraction of time spend in state $x \in S$ tends to π_x in the long run.

Definition 2.7 - *Period*
The *Period* of a state $x \in S$ is the greatest common divisor of all possible return times to x

$$\text{Period}(x) := \text{gcd}\big(\{t > 0 : \mathbb{P}(X_t = x | X_0 = x) > 0\}\big)$$

A state $x \in S$ is *Aperiodic* if Period$(x) = 1$. An irreducible markov chain is *aperiodic* if all its states are aperiodic.

All states in a *communicating class* have the same period.

Proposition 2.6 - *Marginal Distribution of Irreducible, Aperiodic Markov Chain*
If an irreducible, aperiodic Markov Chain has an invariant distribution $\boldsymbol{\pi}$, then

$$\forall\ x \in S \quad \mu_x(t) \overset{t\to\infty}{\longrightarrow} \pi_x$$

Definition 2.8 - *Reversibility*
A markov chain $\{X_t\}_{t\in\mathbb{Z}}$ is *Reversible* if all joint distributions are the same forwards and backwards in time. (i.e. the distribution of the chain is the same if it was reversed).

An irreducible markov chain $\{X_t\}_{t\in\mathbb{Z}}$ with transition matrix P is reversible iff

$$\exists\ \boldsymbol{\pi} \quad \text{st} \quad \pi_x p_{xy} = \pi_y p_{yx}\ \forall\ x, y \in S$$

This is the *Local/Detailed Balance Equation.* Note that this is a system of $\binom{|S|}{2}$ equations which need to be consistent for reversibility to exist.

2.2.2 Continuous Time Markov Process

Definition 2.9 - *Continuous Time Markov Process*
A stochastic process $\{X_t\}_{t\in\mathbb{R}}$ is a *Continuous Time Markov Process* on state space s if

$$\forall\ s < t\ \&\ x, y \in S \quad \mathbb{P}(X_t = y | X_s = x, X_u, u \leq s) = \mathbb{P}(X_t = y | X_s = x)$$

ie future values only depend on the present value and not past.

If $\forall\ t, s, x, y$ $\mathbb{P}(X_t = y|X_s = x)$ depends only on $x, y, t - s$ (observed values & change in time) then the process is *Time-Homogeneous.*

For *Time-Homogenous Markov Processes* we let $P(t)$ denote the stochastic matrix with the probability of each possible transition after t time $[P(t)]_{xy} = \mathbb{P}(X_t = y|X_0 = x)$.

Remark 2.4 - *A Time-Homogenous Markov Process is completely described by its initial condition and the family of transition probability matrices* $\{P(t) : t \geq 0$

This set of matrices $\{P(t) : t \geq 0$ is uncountably large.

Definition 2.10 - *Chapman-Kolmogorov Equations*
For a *Time-Homogeneous Markov Process* the family of stochastic matrices $\{P(t) : t \geq 0$ satisfy the following:

i). $P(0) = I$;

ii). $P(t+s) = P(t)P(s) = P(s)P(t)$

Hence

$$\frac{d}{dt}P(t) := \lim_{\delta\to 0}\frac{P(t+\delta)-P(t)}{\delta} = \lim_{\delta\to 0}\frac{\overbrace{P(t)P(\delta)}^{\text{ii)}}-P(t)}{\delta} = P(t)\lim_{\delta\to 0}\frac{(P(\delta)-I)}{\delta}$$

Suppose that $Q := \lim_{\delta\to 0}\frac{P(\delta)-P(0)}{\delta} = \lim_{\delta\to 0}\frac{P(\delta)-\overbrace{I}^{\text{i)}}}{\delta}$ exists.
Then $P(t)$ solve the following differential equations, known as the *Chapman-Kolmogorov Equations*

$$\frac{d}{dt}P(t) = \underbrace{P(t)Q}_{\text{forward eqn.}} = \underbrace{QP(t)}_{\text{backward eqn.}}$$

The solution to these equations is

$$P(t) = P(0)e^{Qt} = e^{Qt}P(0) = e^{Qt}I = e^{Qt}$$

N.B. Q is called the *Rate Matrix* or *Infinesimal Generator* of the markov process.

Proposition 2.7 - *Properties of the Rate Matrix, Q*
Let Q be the rate matrix of a *continuous-time markov process.* Q has the following properties

- If $n \neq y$ then $q_{xy} := \lim_{\delta\to 0}\frac{[P(\delta)]_{xy}-0}{\delta} \geq 0$. (The off-diagonal elements are non-negative).
- $\forall\ x \in S$, $\sum_{y\in S} q_{xy} = \lim_{\delta\to 0}\frac{1-1}{\delta} = 0$. The rows of Q sum to 0.
- Thus, the diagonal entries q_{xx} are negative. (We denote $-q_{xx}$ by q_x)

Proposition 2.8 - *Interpreting the Rate Matrix Q*
Let Q be the rate matrix of a *continuous-time markov process.*

If the markov process enters state x at time t, it will remain in x for a random time which is distributed $\text{Exp}(q_x)$. (Note that $q_x := -q_x x$).

It the jumps to state y with probability $\frac{q_x y}{q_x}$, independent of the past.

Definition 2.11 - *Invariant Distributions*
Suppose a *Continuous-Time Markov Process* starts with distribution $\boldsymbol{\mu}(0)$ on state space S (i.e. $\mathbb{P}(X_0 = x) = [\boldsymbol{\mu}(0)]_x$). Then, the distribution of X_t is $\boldsymbol{\mu}(t) := \boldsymbol{\mu}(0)P(t) = \boldsymbol{\mu}(0)\underbrace{e^{Qt}}_{\text{CK Eqns}}$.

If there exists a distribution $\boldsymbol{\pi}$ on the state space S st $\forall\ t \geq 0$ $\boldsymbol{\pi} = \boldsymbol{\pi}P(t) = \boldsymbol{\pi}\underbrace{e^{Qt}}_{\text{CK Eqns}}$, then $\boldsymbol{\pi}$ is an *Invariant Distribution.* This distribution is invariant wrt time.

If a markov process has a finite state space then it definitely has an invariant distribution.

Invariant Distributions are not guaranteed to be unique.

Proposition 2.9 - *Finding an Invariant Distribution*
Starting with $\boldsymbol{\pi} = \boldsymbol{\pi} e^{Qt}$ we find that differentiating wrt t and then evaluating at time $t = 0$ we get $\mathbf{0} = \boldsymbol{\pi} Q$ (The Global Balance Equations). This system of equations can be solved to find an *Invariant Distribution.*

A markov process is *reversible* iff there exists a distribution $\boldsymbol{\pi}$ on S which satisfies $\boldsymbol{\pi}_x q_{xy} = \boldsymbol{\pi}_y q_{yx}\ \forall\ x, y \in S$ (Local balance equations). Solving this system of equations will also find an *Invariant Distribution* but it is not guaranteed to have a solution.

Theorem 2.10 - *Ergodic Theorem*
Let $[X_t]_{t\in\mathbb{R}^+}$ is an *Irreducible Markov Process* on a state space S and has an invariant distribution $\boldsymbol{\pi}$. Then

$$\forall\ x \in S \quad \pi_x = \lim_{t\to\infty} \frac{1}{t} \int_0^t \mathbb{1}(X_s = x) ds$$

Moreover, for an arbitrary initial distribution $\boldsymbol{\mu}(0)$, $\boldsymbol{\mu}(t)$ converges to $\boldsymbol{\pi}$ pointwise (i.e. $\mu_x(t) \overset{t\to\infty}{\longrightarrow} \pi_x$)

2.2.3 Poisson Process

Definition 2.12 - *Counting Process*
A *Counting Process* is a stochastic process $\{N(t)\}_{t\in\mathbb{R}}$ st

i). $N(0) = 0$ and $N(t) \in \mathbb{Z}$ for all $t \geq 0$; and,

ii). $N(t)$ is a non-decreasing function of t.

Definition 2.13 - *Independent Increments*
A *Process* $\{N(t)\}_{t\in\mathbb{R}^+}$ is said to have *Independent Increments* if $\forall\ s \in (0, t), \quad (N(t) - N(s))$ is independent of $\{N(u) : u \in [0, s]\}$.

Definition 2.14 - *Poisson Process*
A *Poisson Process* is a *counting process* $\{N(t)\}_{t\in\mathbb{R}^+}$ which has independent increments and <u>at least one</u> of the following equivalence statements are true

- $\forall\ t \in [0, t] \quad (N(t) - N(s)) \sim \text{Po}(\lambda(t - s))$.
- $\mathbb{P}(N(t + \delta) - N(t) = 1) = \lambda\delta + o(\delta)$ <u>and</u>
  $\mathbb{P}(N(t + \delta) - N(t) = 0) = 1 - \lambda\delta + o(\delta)$ <u>and</u>
  $\mathbb{P}(N(t + \delta) - N(t) \geq 2) = o(\delta)$.
- The <u>times between successive increments</u> of the process $N(\cdot)$ are iid $\text{Exp}(\lambda)$ random variables.

The parameter $\lambda \in \mathbb{R}^{>0}$ is called the *rate* of the poisson process. *Poisson Processes* are continuous time markov chains.

Example 2.2 - *Poisson Process*

Counting the number of cars which have passed a given point over time.

Proposition 2.10 - *Properties of Poisson Processes*
Define $\{N(t)\}_{t\in\mathbb{R}^+}$ to be a *Poisson Process* with rate λ. Then the following properties hold

i). The counting process $\{N(\beta t)\}_{t\in\mathbb{R}^+}$, with $\beta > 0$, is a Poisson Process with rate $\beta\lambda$.

ii). If $\{N_1(t)\}_{t\in\mathbb{R}^+}$ and $\{N_2(t)\}_{t\in\mathbb{R}^+}$ are independent poisson processes with rates λ_1 and λ_2, respectively, then $\{N(t) := N_1(t) + N_2(t)\}_{t\in\mathbb{R}^+}$ is a poisson process with rate $\lambda := \lambda_1 + \lambda_2$.

iii). Let $X_1, X_2, \dots$ be a sequence of iid Bern(p) random vairables, independent of $N(\cdot)$. Define $N_1(t) := \sum_{i=1}^{N(t)} X_i$ and $N_2(t) := \sum_{i=1}^{N(t)}(1 - X_i)$ (These are called *Bernoulli Thinnings*). These assign increments in $N(\cdot)$ randomly to either N_1 or N_2 (with probability p).
Then, $N_1(\cdot)$ and $N_2(\cdot)$ are independent poisson processes with rates λp and $\lambda(1-p)$, respectively.

2.3 Transformation of Random Varaibles

Example 2.3 - *Discrete Case*
Consider rolling a fair die where $\Omega := \{1,2,3,4,5,6\}$ and $\forall\ \omega \in \Omega \quad \mathbb{P}(\omega) = \frac{1}{6}$.
Let $X(\omega) = \omega\ \forall\ \omega \in \Omega$ so the pmf of X is given by

$$p_X(i) = \frac{1}{6}\ \forall\ i \in \{1,\dots,6\}$$

Consider $Y := X^2$. The pmf of Y is straightforward to work out as each value of X maps to a unique value of Y

$$P_Y(i) = \frac{1}{6} \text{ for } \sqrt{i} \in \{1,\dots,6\}$$

Consider $Z := (X-2)^2$. This is not quite so simple as multiple values of X map to the same value of Z.

$$p_Z(1) = \frac{2}{6}; \quad p_Z(i) = \frac{1}{6} \text{ for } i \in \{0,4,9,16\}$$

Example 2.4 - *Continuous Case*
Let $X \sim \text{Uniform}[0,1]$ and define $Y := 2X$. We have

$$f_X(x) = \begin{cases} 1 & x \in [0,1] \\ 0 & \text{otherwise} \end{cases} \quad F_X(x) = \begin{cases} 0 & x < 0 \\ x & x \in [0,1] \\ 1 & x > 1 \end{cases}$$

Now consider Y the cdf is

$$F_Y(y) := \mathbb{P}(Y \le y) = \mathbb{P}(2X \le y) = \mathbb{P}\left(X \le \frac{y}{2}\right) = F_X\left(\frac{y}{2}\right)$$

We then obtain the pdf for Y by differentiation and the chain rule.

$$f_Y(y) = F_Y'(y) = \underbrace{\frac{1}{2}F_X'\left(\frac{y}{2}\right)}_{\text{chain rule}} = \frac{1}{2}f_X\left(\frac{y}{2}\right) = \begin{cases} \frac{1}{2} & \frac{y}{2} \in [0,1] \\ 0 & \text{otherwise} \end{cases}$$

Proposition 2.11 - *Increasing Functions*
Let X be a random variable and $Y := g(X)$ where $g : \mathbb{R} \to \mathbb{R}$ is a strictly increasing function. Then, g is invertible on its range (denoted g^{-1}). Thus the cdfs for X and Y are related as

$$F_Y(y) = \underbrace{\mathbb{P}(g(X) \leq y) = \mathbb{P}(X \leq g^{-1}(y))}_{\text{as } g \text{ is increasing}} = F_X(g^{-1}(y))$$

Differentiating this gives us the pdfs

$$f_Y(y) = f_X\left(g^{-1}(y)\right)\frac{dg^{-1}(y)}{dy} = \left.\frac{f_X(x)}{g'(x)}\right|_{x=g^{-1}(y)}$$

The probability mass of X in the interval $(x, x + dx)$ gets mapped to the interval $\big(g(x), g(x + dx)\big)$. By *Taylor Expansion* of g we get $g(x + dx) \simeq g(x) + g'(x)dx$

Proposition 2.12 - *General Mappings - Single Random Variable*
Let X be a (discrete) random variable and define $Y := g(x)$ for any $g : \mathbb{R} \to \mathbb{R}$ which is differentiable.

There is a contribution $\frac{f_X(x)}{|g'(x)|}$ from each x such that $g(x) = y$. Where $g'(x)$ is positive or negative does not matter, as it only determines where the pre-image of $(y, y+dy)$ is of the form $(x, x+dx)$ or $(x - dx, x)$. Only the relative widths of the intervals matters for the contribution.

By summing the contributions from all solutions of $g(x) = y$ we get

$$p_Y(y) = \sum_{x:g(x)=y} \frac{f_X(x)}{|g'(x)|}$$

This formula is valid so long as the set $\{x : g(x) = y\}$ is countable. If it is not countable, then Y is a continuous RV (or is a mixed random variable)

Proposition 2.13 - *General Mappings - Random Vectors*
Consider the random vector $\mathbf{X} := (X_1, \dots, X_d)$ with joint density $f_{\mathbf{X}}$ and define $\mathbf{Y} := g(\mathbf{X})$ for any differentiable $g : \mathbb{R}^d \to \mathbb{R}^d$.

Let $J_g(\mathbf{x})$ denote the *Jacobian* of g at $\mathbf{x}$. Then

$$f_{\mathbf{Y}}(\mathbf{y}) = \sum_{\mathbf{x}\in\mathbb{R}^d:g(\mathbf{x})=\mathbf{y}} \frac{f_{\mathbf{X}}(\mathbf{x})}{|\det(J_g(\mathbf{x}))}$$

Let $\mathbf{x}$ solve $g(\mathbf{x}) = \mathbf{y}$. In a neighbourhood of $\mathbf{x}$, g is approximately a linear function. By *Taylor Expansion*

$$g(\mathbf{x}') \simeq f(\mathbf{x}) + J_g(\mathbf{x})(\mathbf{x}' - \mathbf{x}) = \mathbf{y} + J_g(\mathbf{x})(\mathbf{x}' - \mathbf{x})$$

for $\mathbf{x}'$ in a small enough neighbourhood of $\mathbf{x}$.

0 Reference

Definition 0.1 - *Stochastic Matrix*
A matrix is called a *Stochastic matrix* if:

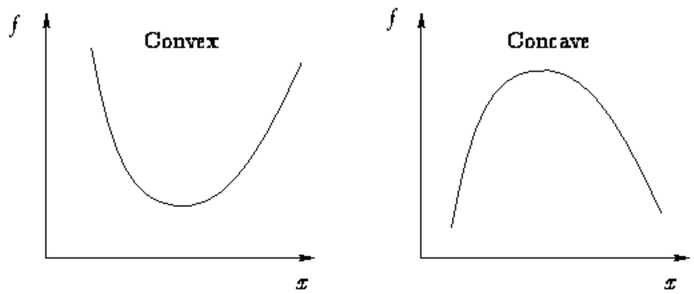
i). All elements are non-negative.

ii). All rows sum to 1.

Definition 0.2 - *Convex Function*
A function $f : \mathbb{R} \to (\mathbb{R} \cup \{+\infty\})$ is *Convex* if, $\forall\ x, y \in \mathbb{R},\ \alpha \in [0, 1]$, we have

$$f(\alpha x + (1 - \alpha)y) \leq \alpha f(x) + (1 - \alpha)f(y)$$

A smooth function f is convex iff f is twice differentiable and $f''(x) \geq 0\ \forall\ x \in \mathbb{R}$.
Visually, a function is convex if you can draw a line between any two points on the function and the function lies below the line.



Definition 0.3 - *Equivalence Relation*
A relation is an *Equivalence Relation* if it is

i). Reflexive: $i \leftrightarrow i$.

ii). Symmetric: If $i \to j$ then $j \to i$.

iii). Transitive: If $i \to j$ and $j \to k$ then $i \to k$.

Definition 0.4 - *Simple Random Walk*
A *Simple Random Walk* is a random walk which moves only one step at a time. (i.e. $X_{t+1} = X_t \pm 1$). A probability p is defined for $\mathbb{P}(X_{t+1} = X_t + 1)$, this means $1 - p$ is the probability of stepping in the other direction. A *Simple Random Walk* is *Assymetric* if $p \neq p - 1$.

Definition 0.5 - *Matrix Exponential, e^X*
Let X be a matrix then we define the *Matrix Exponential* as $e^X := I + X + \frac{X^2}{2!} + \dots$

Theorem 0.1 - *Pinsker's Inequality*
For two distributions Bern(p) and Bern(q)

$$K(q; p) \geq 2(q - p)^2$$

Definition 0.6 - *Jacobian Matrix*
The *Jacobian Matrix* is the first-order partial-derivatives of a multidimensional function $\mathbf{f}:\mathbb{R}^n\to\mathbb{R}^m$ wrt each parameter.

$$J_f := \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \dots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \dots & \frac{\partial f_2}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1} & \frac{\partial f_m}{\partial x_2} & \dots & \frac{\partial f_m}{\partial x_n} \end{pmatrix}$$

0.1 Notation

$p_{x,y}$	$\mathbb{P}(X_{t+1}=x\vert X_t=y)$ for a *time homogenous markov process.*
$I(t)$	Arm played in round t.
$N_i(t)$	$\sum_{s=1}^t\{I(s)=i\}$ Number of times arm i was played in first t rounds.
$S_i(t)$	$\sum_{s=1}^t X_i(s)\{I(s)=i\}$ Total reward from arm i in first t rounds.
$\hat{\mu}_{i,N_i(t)}$	$\frac{S_i(t)}{N_i(t)}$ sample mean reward from arm i in first t rounds.
Δ_i	$\mu^*-\mu_i$ the arm gaps from a K-armed bandit.
$X_i(t)$	RV modelling the result if arm i was played on round t.

0.1.1 Asymptotic Notation

Definition 0.7 - *Oh Notation*
Let $f,g:\mathbb{R}^+\to\mathbb{R}$.

We say $f=o(g)$ (little oh of g) at 0 if $\frac{f(x)}{g(x)}\overset{x\to 0}{\longrightarrow}0$.

We say $f=O(g)$ (big oh of g) at 0 if $\exists\ c>0$ st $|f(x)|\leq c|g(x)|$ in a neighbourhood of 0.

$f=o(g)$ at infinity and $f=O(g)$ at infinity are defined analogously.

Definition 0.8 - *Omega Notation*
Let $f,g:\mathbb{R}^+\to\mathbb{R}$.

We say $g=\omega(g)$ if $o(f)$ and we say $f=\Omega(g)$ if $g=O(f)$.

Example 0.1 - *Oh & Omega Notation*
Define $f(x)=x,\ g(x)=\sin(x)$ and $h(x)=x^2$.
Then, $g=O(f)$ at 0 and $g=o(f)$ at infinity. $h=o(f)$ at 0 and $h=\omega(f)$ at infinity.