Stochastic Optimisation - Problem Sheet 1

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Answer 3)
Let $X_1, X_2, \dots$ by iid random variables with distribution Bern(p) with $p \in [0,1]$. Let $q \in [0,1]$ with $q > p$.
The moment generating function of X is $\mathcal{M}_X(t) := \mathbb{E}[e^{tX}] = (p + e^t + (1-p))$.
By applying *Chernoff Bounds* we have that

$$\begin{array}{rcl} \mathbb{P}\left(\sum_{i=1}^n X_i > nq\right) & \leq & \inf_{\theta>0} e^{-nq\theta}(\mathbb{E}[e^{\theta X}])^n \\ & = & \inf_{\theta>0} e^{-nq\theta}(pe^\theta + (1-p))^n \end{array}$$

Consider the natural log of the right hand side and define

$$f := -nq\theta + \ln(pe^\theta + 1 - p)$$

Since the natural log is a monotonically increasing function, $\inf_{\theta>0} e^f$ is equal to the RHS of above.
We have

$$\begin{array}{rrcl} & \frac{\partial f}{\partial \theta} & = & -nq + n\frac{pe^\theta}{pe^\theta + 1 - p} \\ \text{Setting} & \frac{\partial f}{\partial \theta} & = & 0 \\ \implies & 0 & = & -nq + n\frac{pe^\theta}{pe^\theta + 1 - p} \\ \implies & q & = & \frac{pe^\theta}{pe^\theta + 1 - p} \\ \implies & pe^\theta + 1 - p & = & \frac{p}{q}e^\theta \\ \implies & e^\theta & = & \frac{1-p}{\frac{p}{q} - p} \\ & & = & \frac{q - qp}{p - qp} \\ \text{Since} & q & > & p \\ \implies & q - qp & > & p - qp \\ \implies & \frac{q-qp}{p-qp} & > & 1 \\ \implies & \ln\left(\frac{q-qp}{p-qp}\right) & > & 0 \end{array}$$

Thus $\underset{\theta;\theta>0}{\text{argmin}}(f) = \ln\left(\frac{q-qp}{p-qp}\right)$. This means

$$\begin{array}{rcl}
\inf_{\theta>0} f &=& -nq\ln\left(\frac{q-qp}{p-qp}\right)+n\ln\left(p\cdot\frac{q-qp}{p-qp}+1-p\right)\\
&=& -n\left[q\ln\left(\frac{q(1-p)}{p(1-q)}\right)-\ln\left(p\cdot\frac{q(1-p)}{p(1-q)}+1-p\right)\right]\\
&=& -n\left[q\ln\left(\frac{q}{p}\right)+q\ln\left(\frac{1-p}{1-q}\right)-\ln\left(\frac{q(1-p)}{1-q}+1-p\right)\right]\\
&=& -n\left[q\ln\left(\frac{q}{p}\right)-q\ln\left(\frac{1-q}{1-p}\right)-\ln\left(\frac{1-p}{1-q}\right)\right]\\
&=& -n\left[q\ln\left(\frac{q}{p}\right)+(1-q)\ln\left(\frac{1-q}{1-p}\right)\right]\\
&=& -nK(q;p)\\
\implies \inf_{\theta>0} e^{-nq\theta}(pe^\theta+(1-p))^n &=& \exp(-nK(q;p))\\
\implies \mathbb{P}\left(\sum_{i=1}^n X_i>nq\right) &\leq& \exp(-nK(q;p))
\end{array}$$

Answer 6a)
Let $Z\sim N(0,1)$.

$$\begin{array}{rcl}
\mathbb{E}[e^{\theta Z}] &=& \int_{-\infty}^\infty e^{\theta x}f_Z(x)dx\\
&=& \int_{-\infty}^\infty e^{\theta x}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}x^2}dx\\
&=& \int_{-\infty}^\infty \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}(x^2-2x\theta)}dx\\
&=& \int_{-\infty}^\infty \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}(x-\theta)^2+\frac{1}{2}\theta^2}dx\\
&=& e^{\frac{1}{2}\theta^2}\int_{-\infty}^\infty \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}(x-\theta)^2}dx\\
&=& e^{\frac{1}{2}\theta^2}\int_{-\infty}^\infty f_Y(x)dx \quad\text{where}\quad Y\sim N(\theta,1)\\
&=& e^{\frac{1}{2}\theta^2}\cdot 1\\
&=& e^{\frac{1}{2}\theta^2}
\end{array}$$

Answer 6b)
Let $X_1,X_2,\dots$ be iid random variables with distribution $N(\mu,\sigma^2)$ for $\mu,\sigma\in\mathbb{R}$. Let $\gamma\in\mathbb{R}$ st $\gamma>\mu$.
The moment generating function of X is $\mathcal{M}_X(t):=\mathbb{E}[e^{tX}]=e^{\mu\theta+\frac{1}{2}\sigma^2\theta^2}$.
By applying *Chernoff Bounds* we have that

$$\begin{array}{rcl}
\mathbb{P}\left(\sum_{i=1}^n X_i>n\gamma\right) &\leq& \inf_{\theta>0} e^{-n\gamma\theta}(\mathbb{E}[e^{X\theta}])^n\\
&=& \inf_{\theta>0} e^{-n\theta(\gamma-\mu-\frac{1}{2}\sigma^2\theta)}
\end{array}$$

Consider the natural log of the right hand side and define $f := -n\theta(\gamma - \mu - \frac{1}{2}\sigma^2\theta)$.

$$\begin{array}{rrcl} & \frac{\partial f}{\partial \theta} & = & -n(\gamma - \mu - \frac{1}{2}\sigma^2\theta) - n\theta(-\frac{1}{2}\sigma^2) \\ & & = & -n(\gamma - \mu - \sigma^2\theta) \\ \text{Setting} & \frac{\partial f}{\partial \theta} & = & 0 \\ \implies & \gamma - \mu - \sigma^2\theta & = & 0 \\ \implies & \theta & = & \frac{\gamma - \mu}{\sigma^2} \\ \text{Since} & \gamma > \mu & \& & \sigma^2 > 0 \\ \implies & 0 & < & \frac{\gamma - \mu}{\sigma^2} = \theta \end{array}$$

Thus $\underset{\theta;\theta>0}{\text{argmin}}(f) = \frac{\gamma - \mu}{\sigma^2}$. This means

$$\begin{array}{rrcl} & \inf_{\theta>0} f & = & -n\left(\frac{\gamma - \mu}{\sigma^2}\right)\left(\gamma - \mu - \frac{1}{2}(\gamma - \mu)\right) \\ & & = & -n\frac{(\gamma - \mu)^2}{2\sigma^2} \\ \implies & \inf_{\theta>0} e^{-n\theta\left(\gamma - \mu - \frac{1}{2}\sigma^2\theta\right)} & = & \exp\left(-n\frac{(\gamma - \mu)^2}{2\sigma^2}\right) \\ \implies & \mathbb{P}\left(\sum_{i=1}^n X_i > n\gamma\right) & \leq & \exp\left(-n\frac{(\gamma - \mu)^2}{2\sigma^2}\right) \end{array}$$