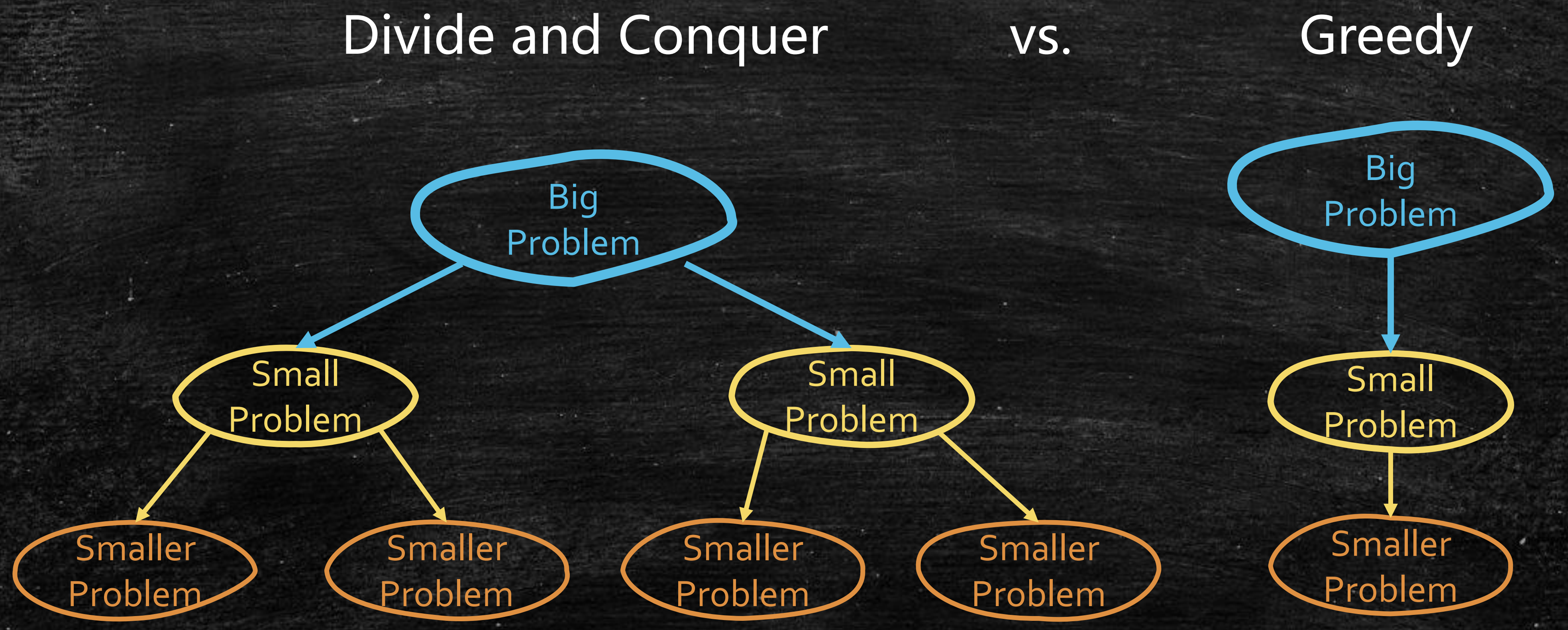
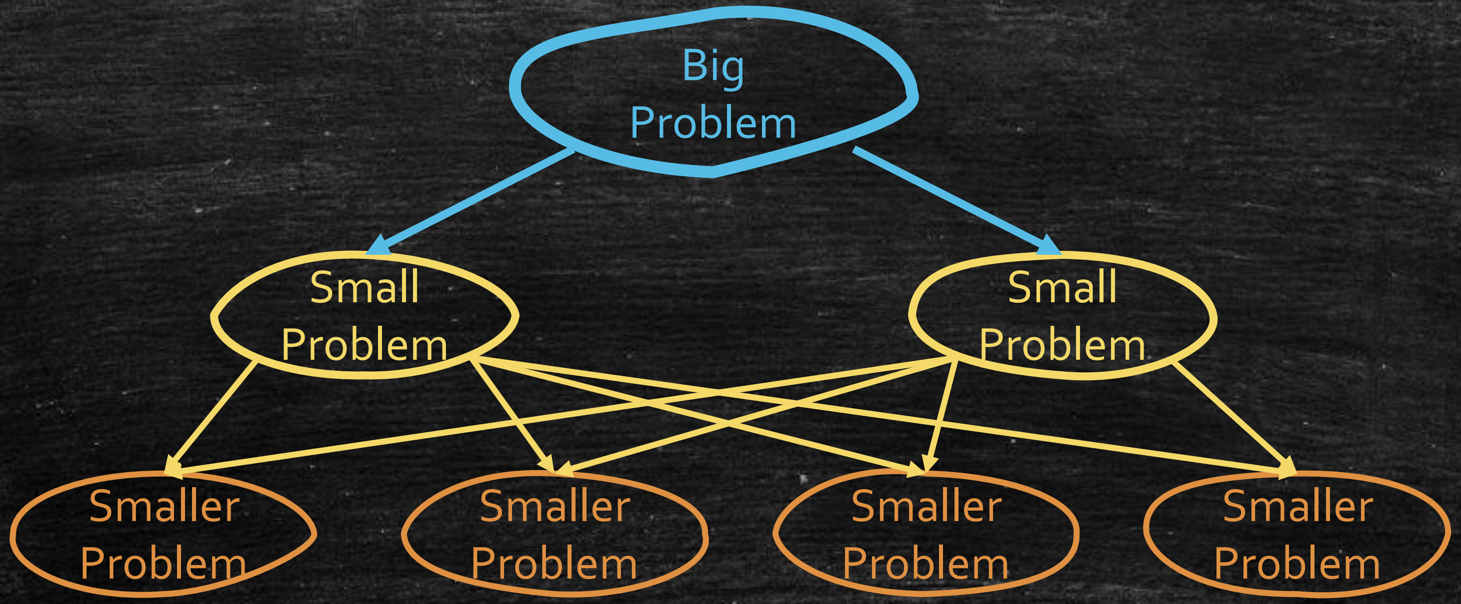
Dynamic Programming

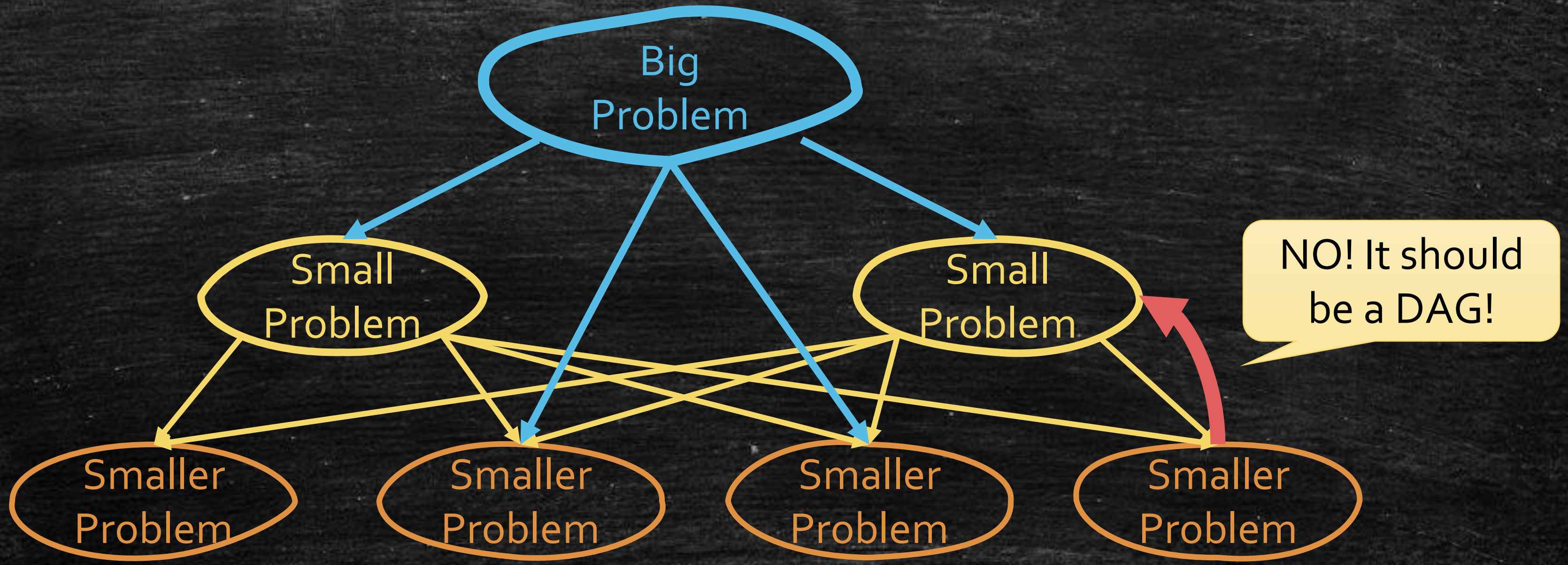
Recall Divide and Conquer vs. Greedy



Dynamic Programming vs. Divide and Conquer



Dynamic Programming



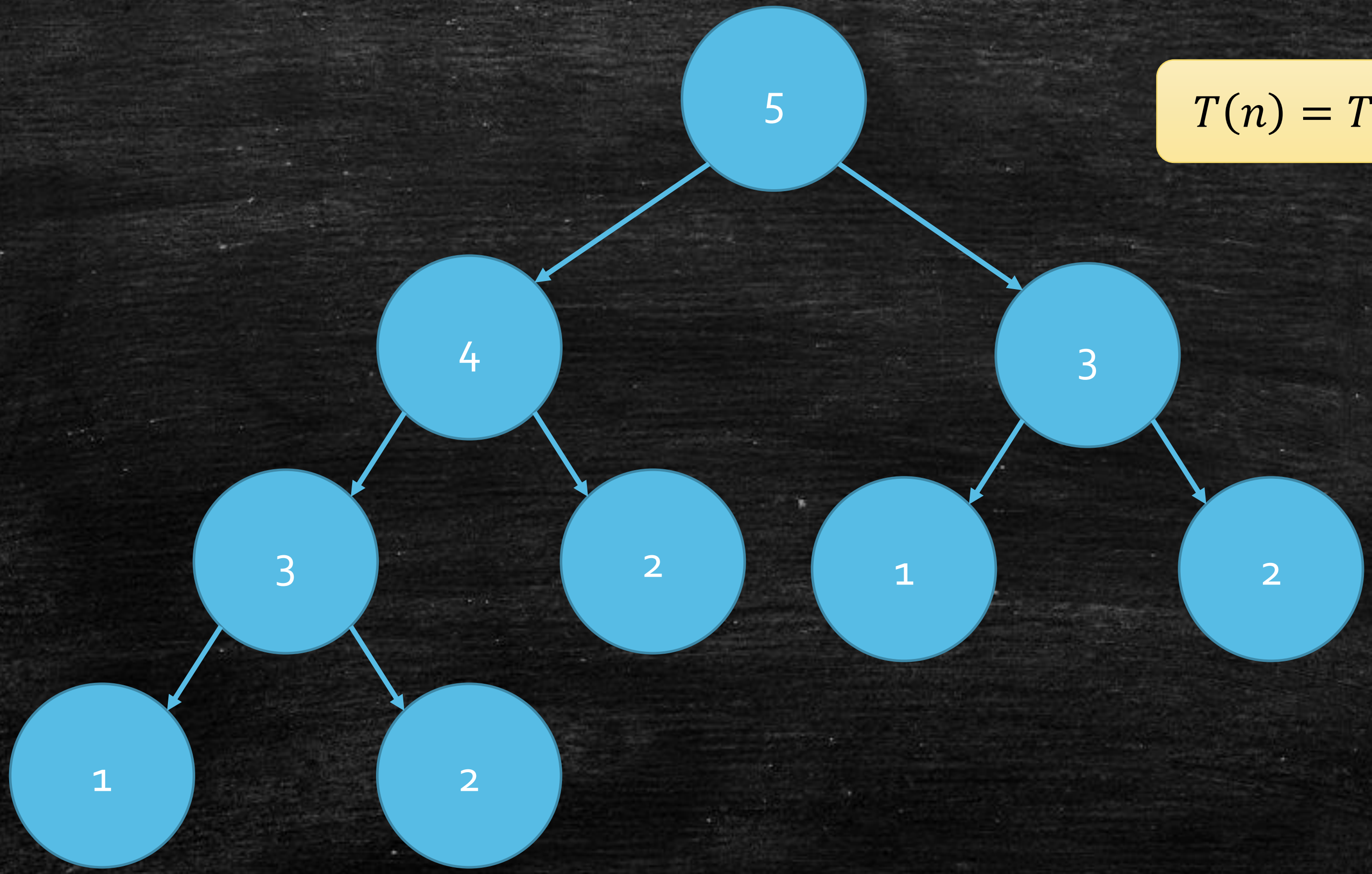
An Easy Example

- Fibonacci
- $Fib(n) = Fib(n-1) + Fib(n-2)$
- Solve Recursively

Fibonacci

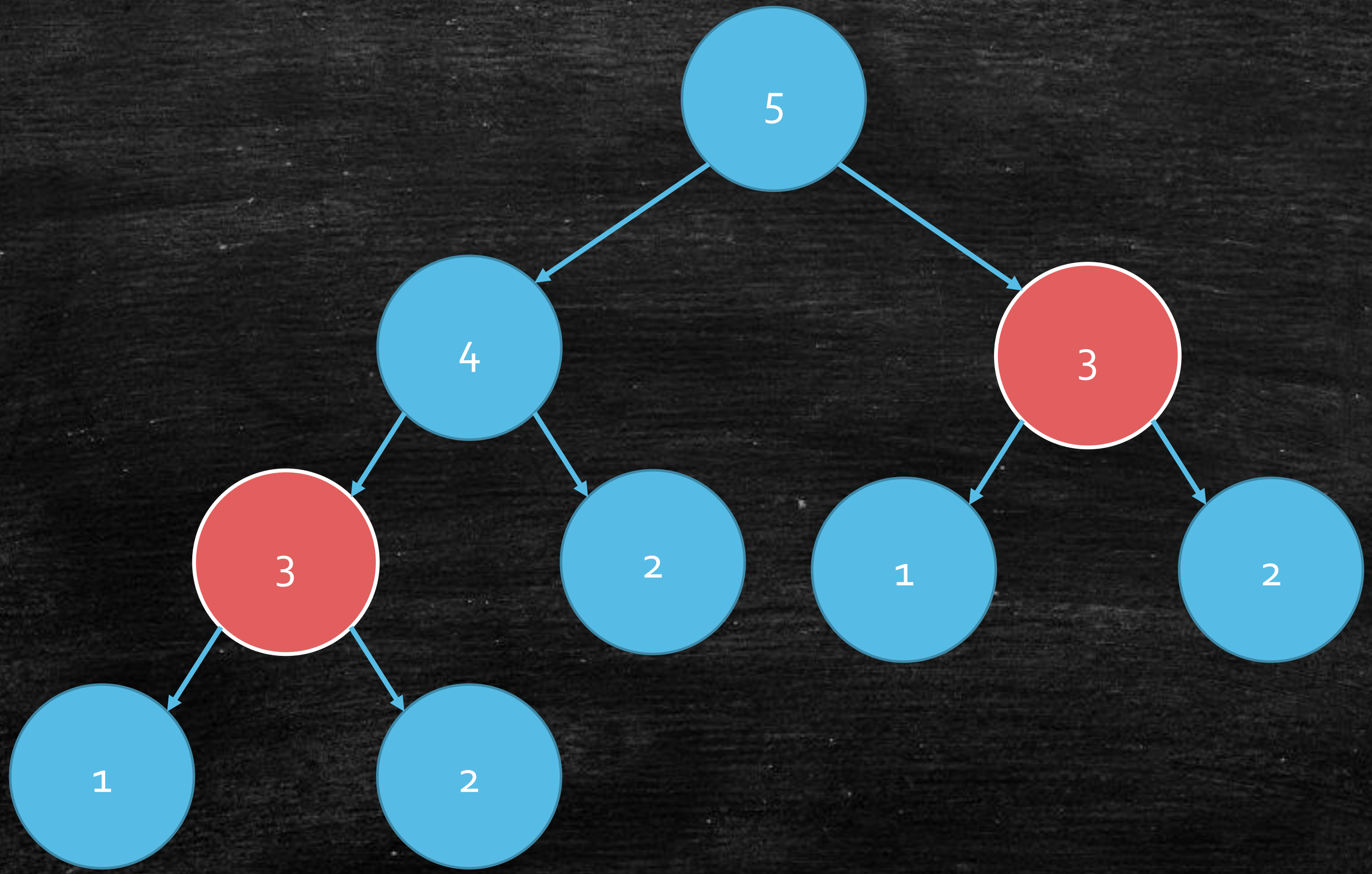
```
function fib(n)
    if n>1
        return fib(n − 1) + fib(n − 2)
    else
        return 1
```

Recursive Tree

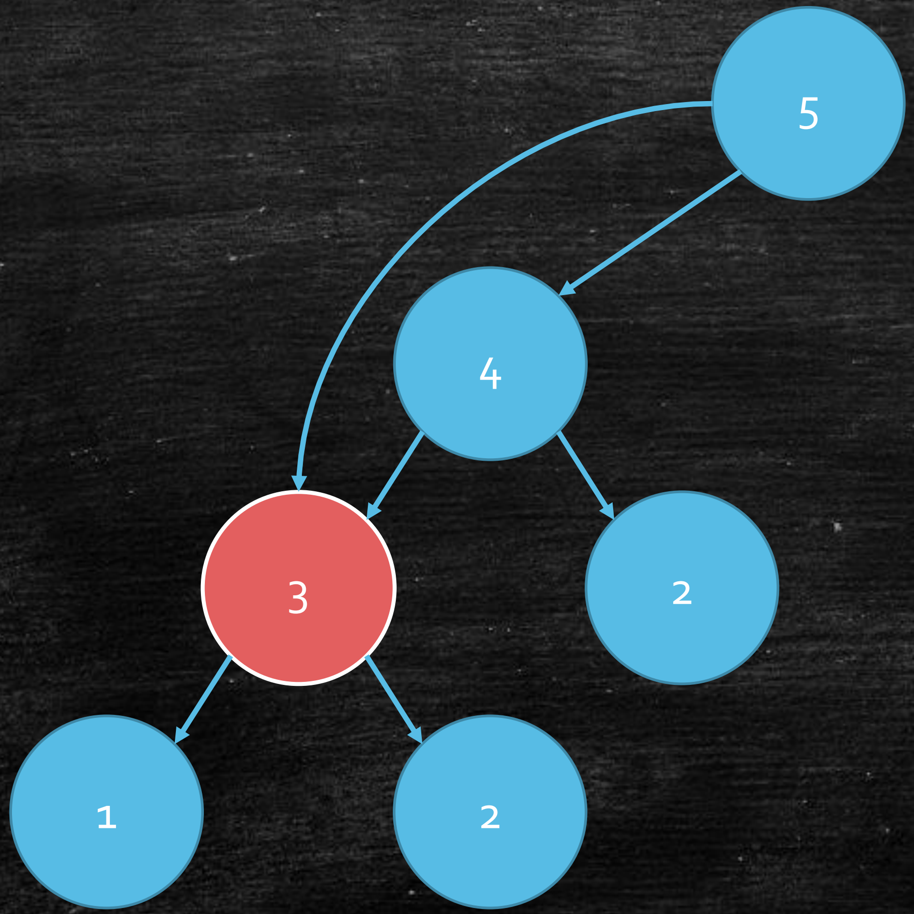


$$T(n) = T(n-1) + T(n-2)$$

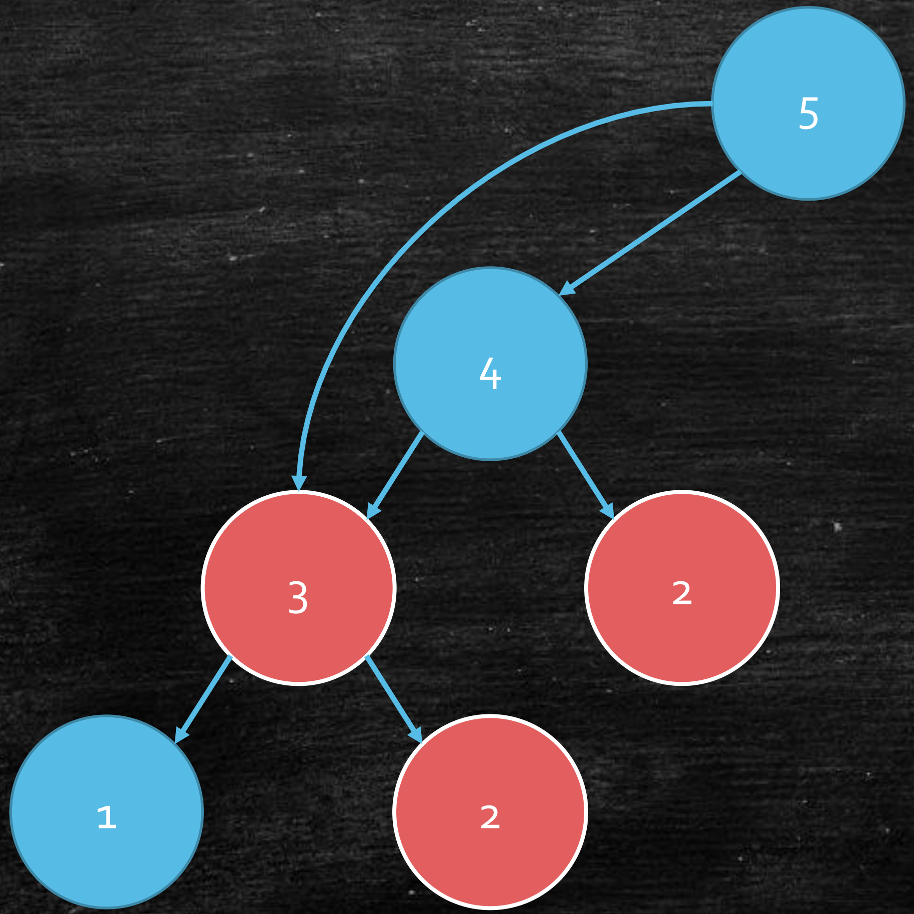
Improvement



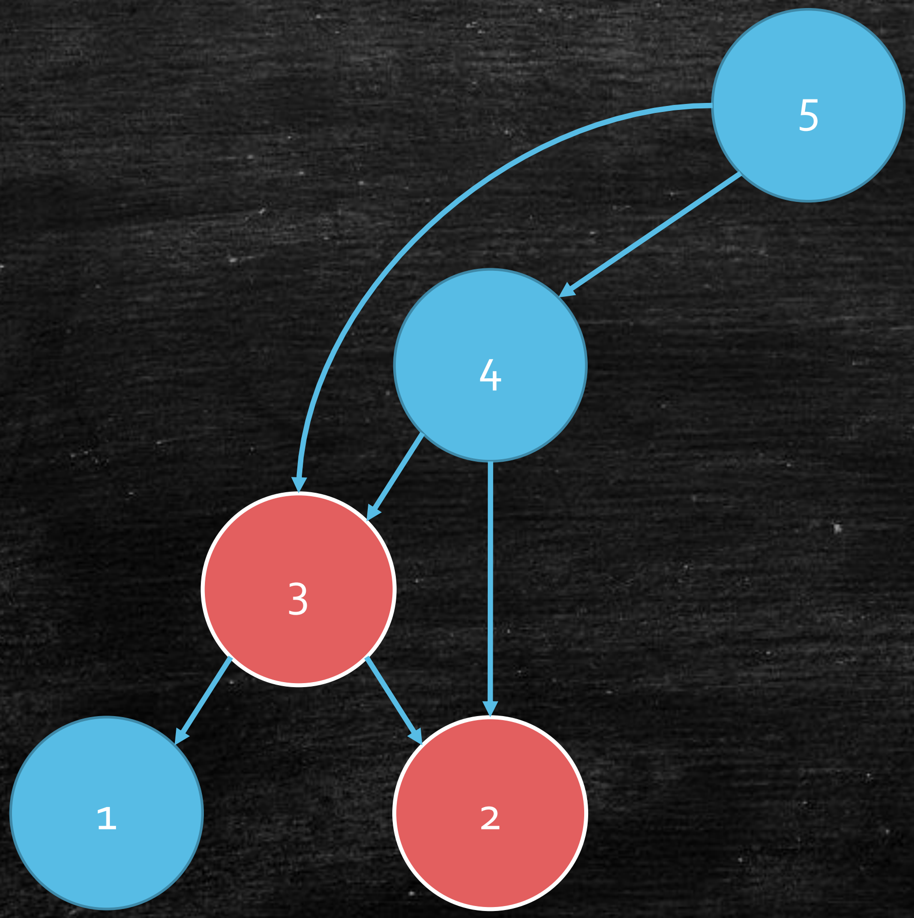
Improvement



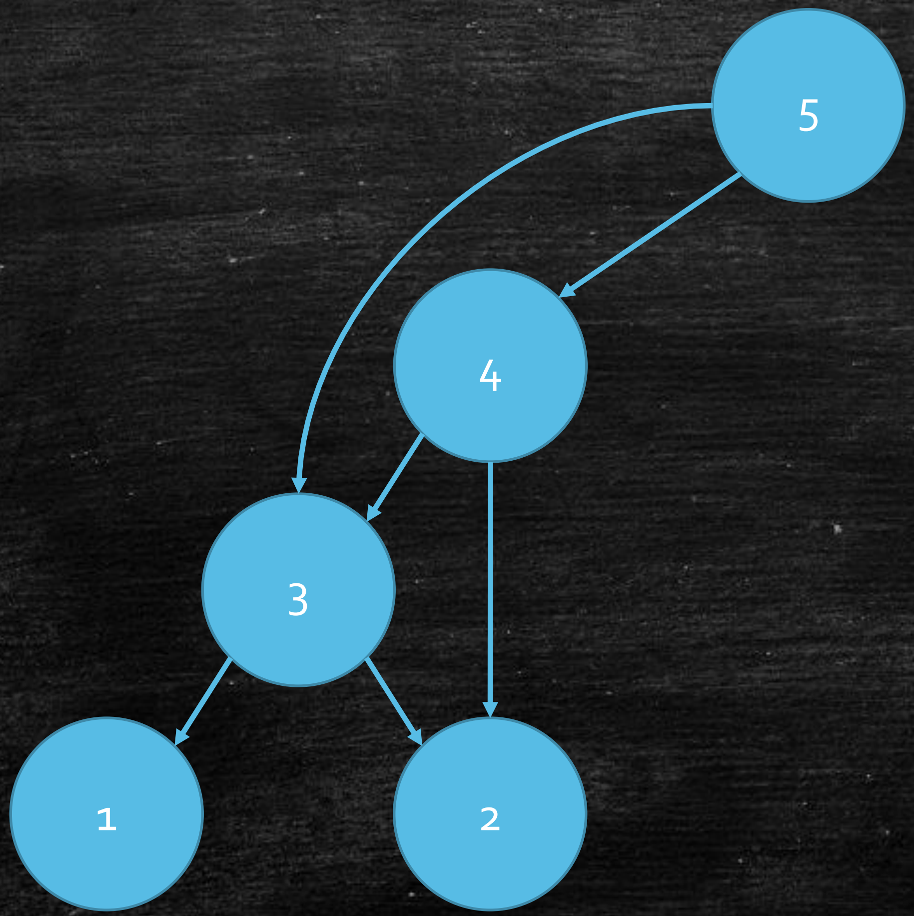
Improvement



Improvement



Improvement



It becomes a DAG!

Implement: memoization

Fibonacci

function $fib(n)$

  Check whether n is stored, if yes then directly return.

  if n>1

    return & store $fib(n-1)+fib(n-2)$

  else

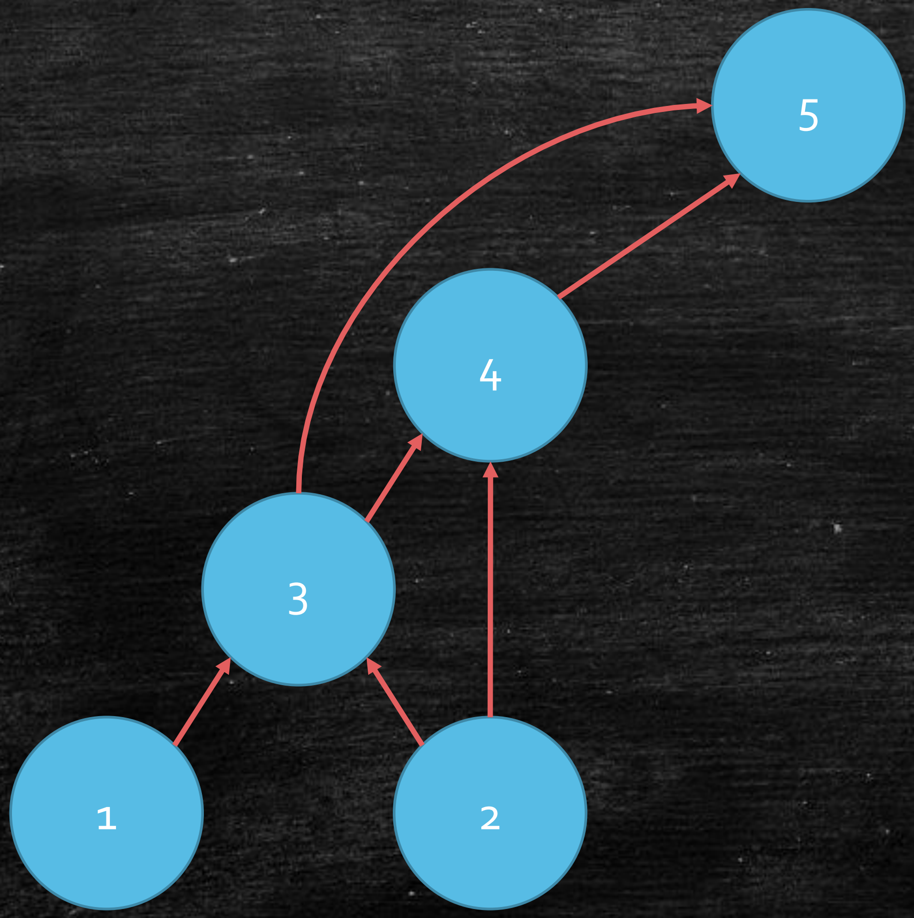
    return & store 1

Each i

- Calculate once
- Checked twice

Totally: $O(n)$

Improvement



Reverse the graph.
Solve them by the topological order.

Implement: DP

- Observation
  - If we know $fib(1) \dots fib(i-1)$.
  - $fib(i)$ can be calculated in constant time.
- DP: calculate all status by a topological order.

Fibonacci $O(n)$

function $fib(n)$
  $fib[0] = fib[1] = 1$
  for i = 2 to n
    $fib[i] = fib[i-1] + fib[i-2]$
  return $fib[n]$

Guideline for DP design

- Design a **recursive** Algorithm.
- Merge the **common** subproblems.
- Check whether we are in a **DAG** and find the **topological order** of this DAG. (usually, by hand.)
- Solve & store the subproblems by the topological order.

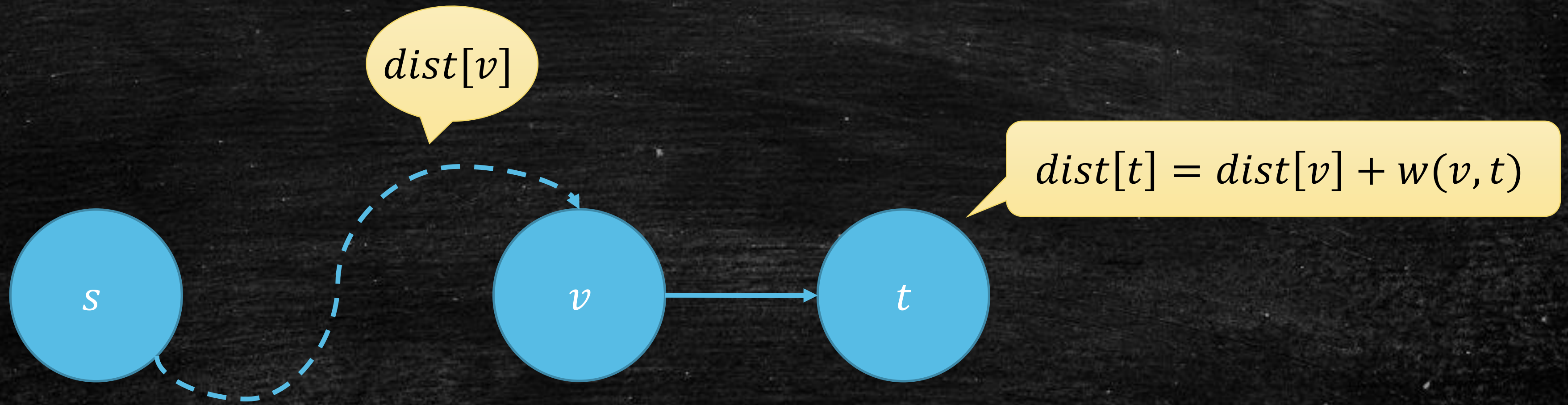
Let us use the guideline

Shortest Path in DAGs

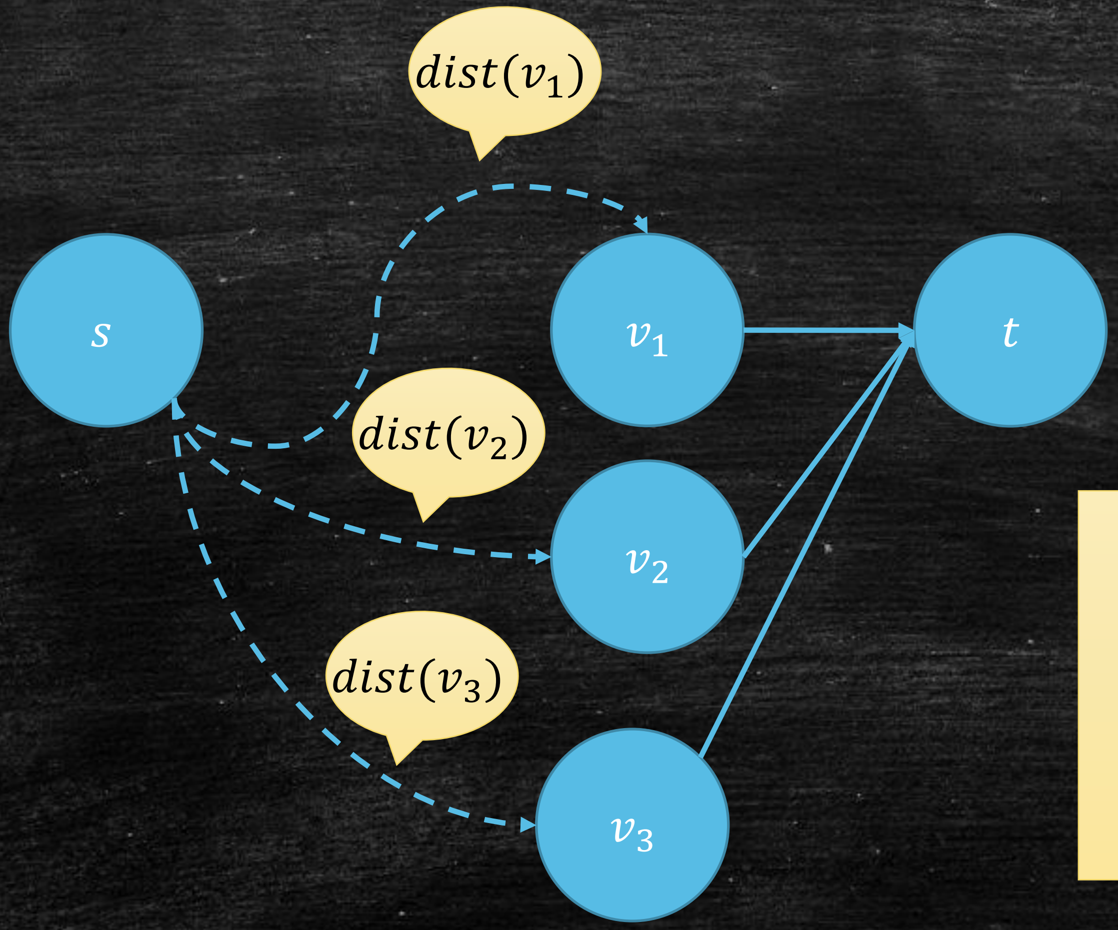
- **Input:** A Directed Acyclic Graph (DAG) $G = (V, E)$, a start vertex $s \in V$, and a weight function $w(e)$ for all $e \in E$. (possible non-negative)
- **Output:** the distance from s to every $v \in V$.

Important Fact

- Not restricted in DAG!
- Used in Dijkstra, BFS, Bellman-Ford.

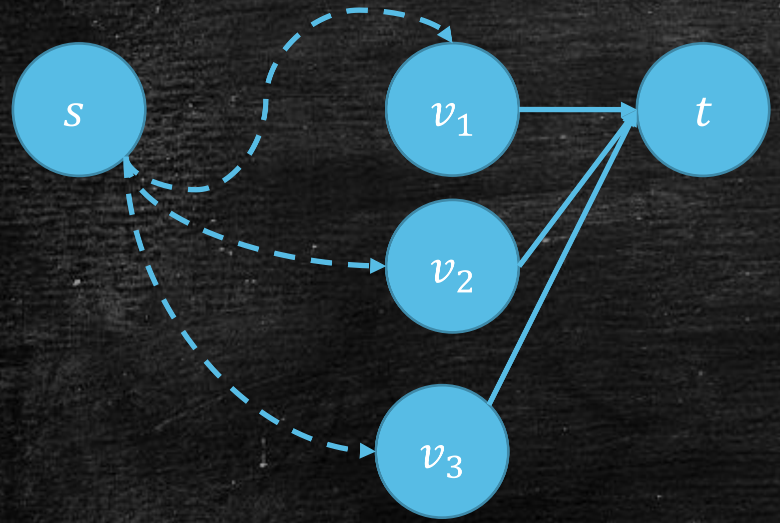


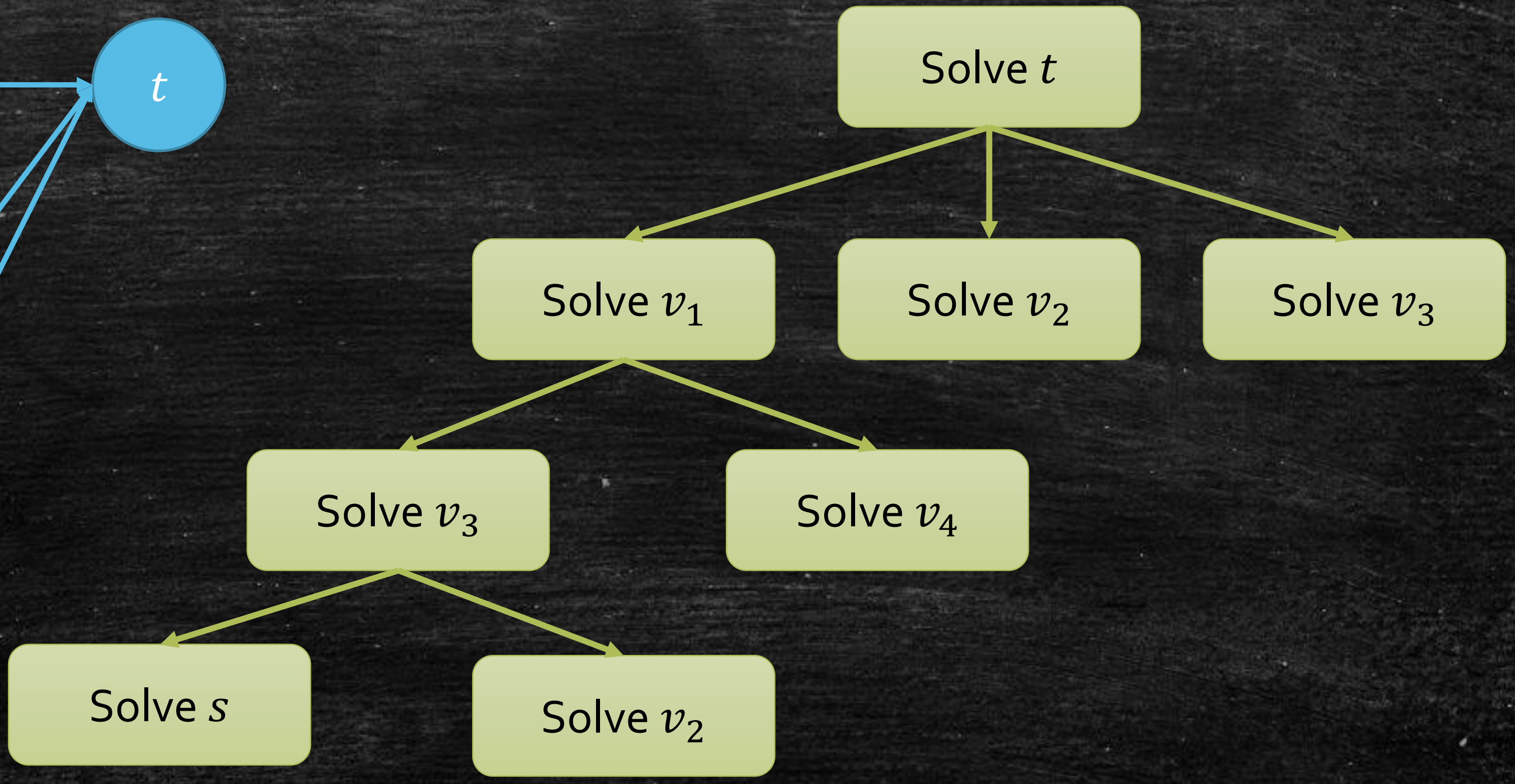
Solve $s \to t$ distance recursively!



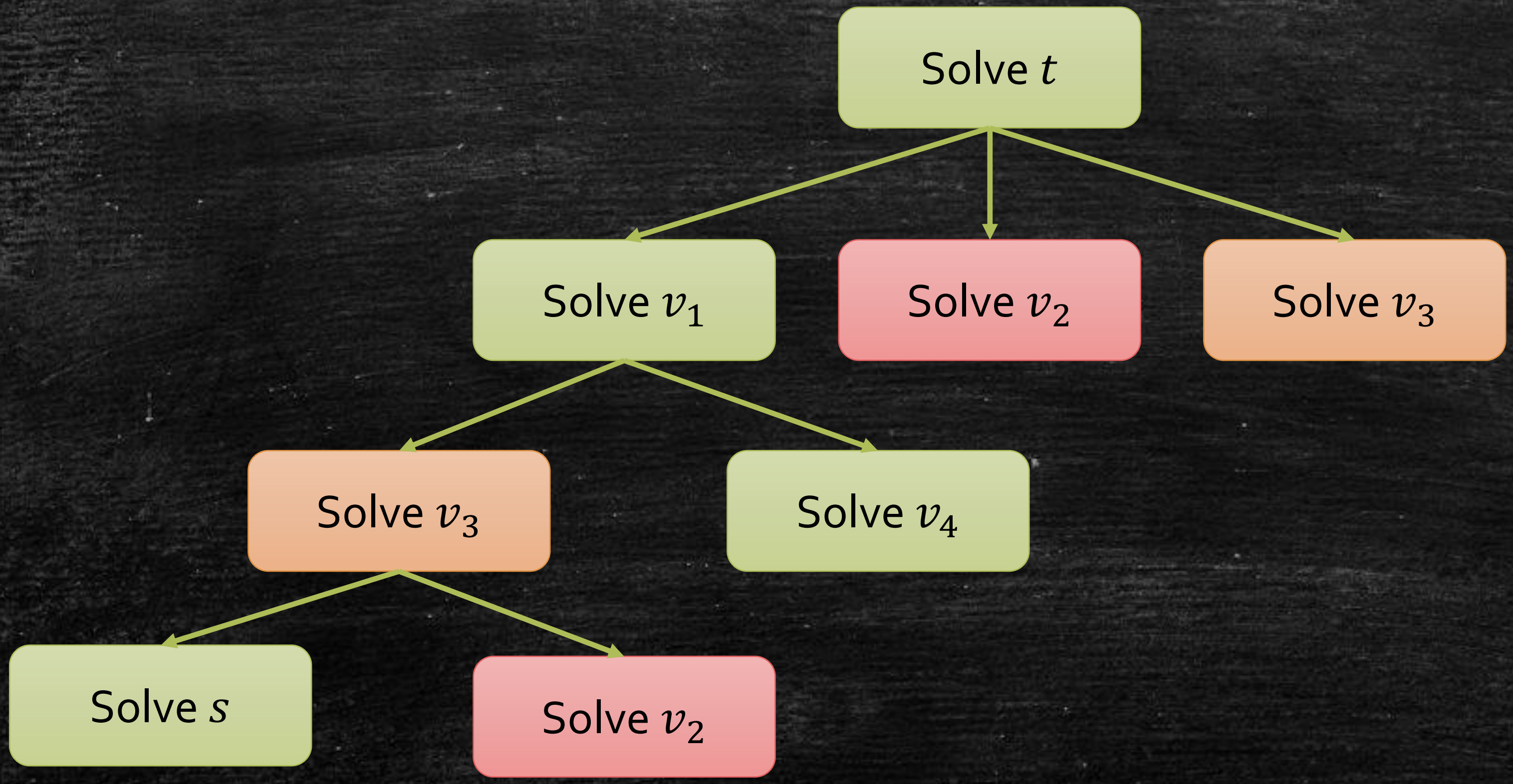
$$dist(t) = \min \begin{cases} dist(v_1) + w(v_1, t) \\ dist(v_2) + w(v_2, t) \\ dist(v_3) + w(v_3, t) \end{cases}$$

A recursive method to solve $dist[t]$.

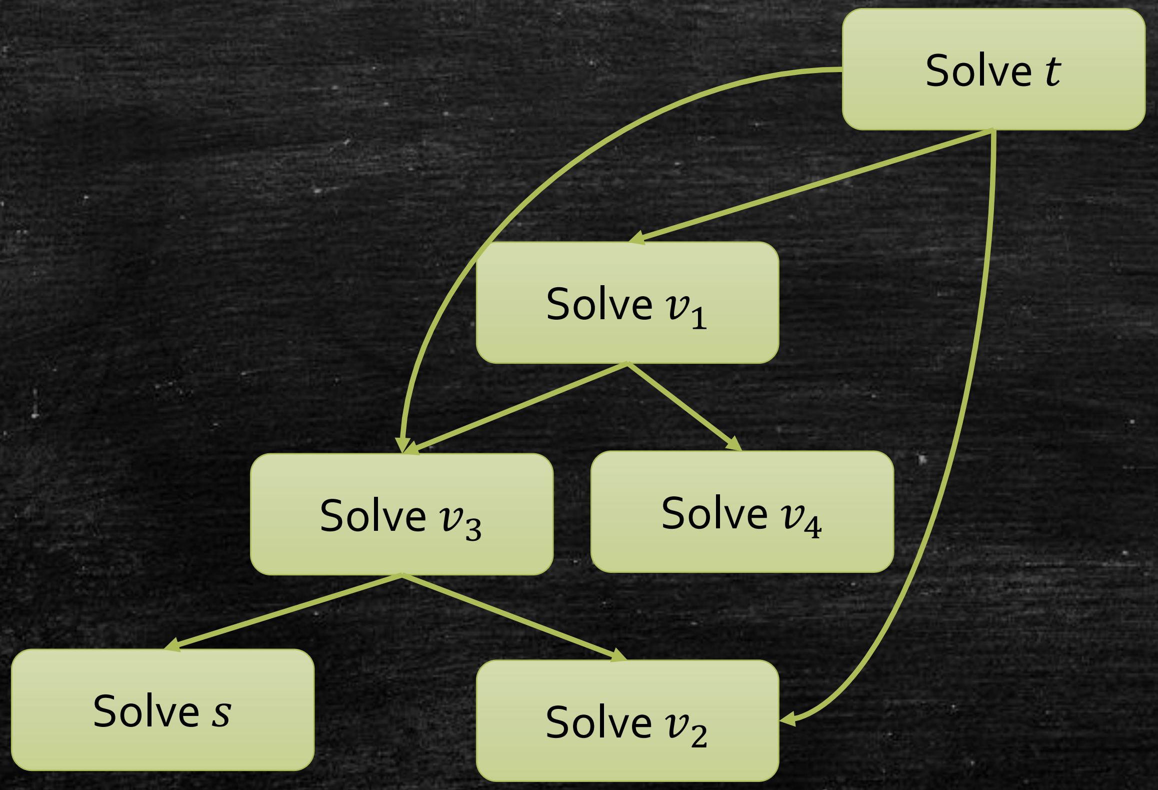




Merge common subproblems

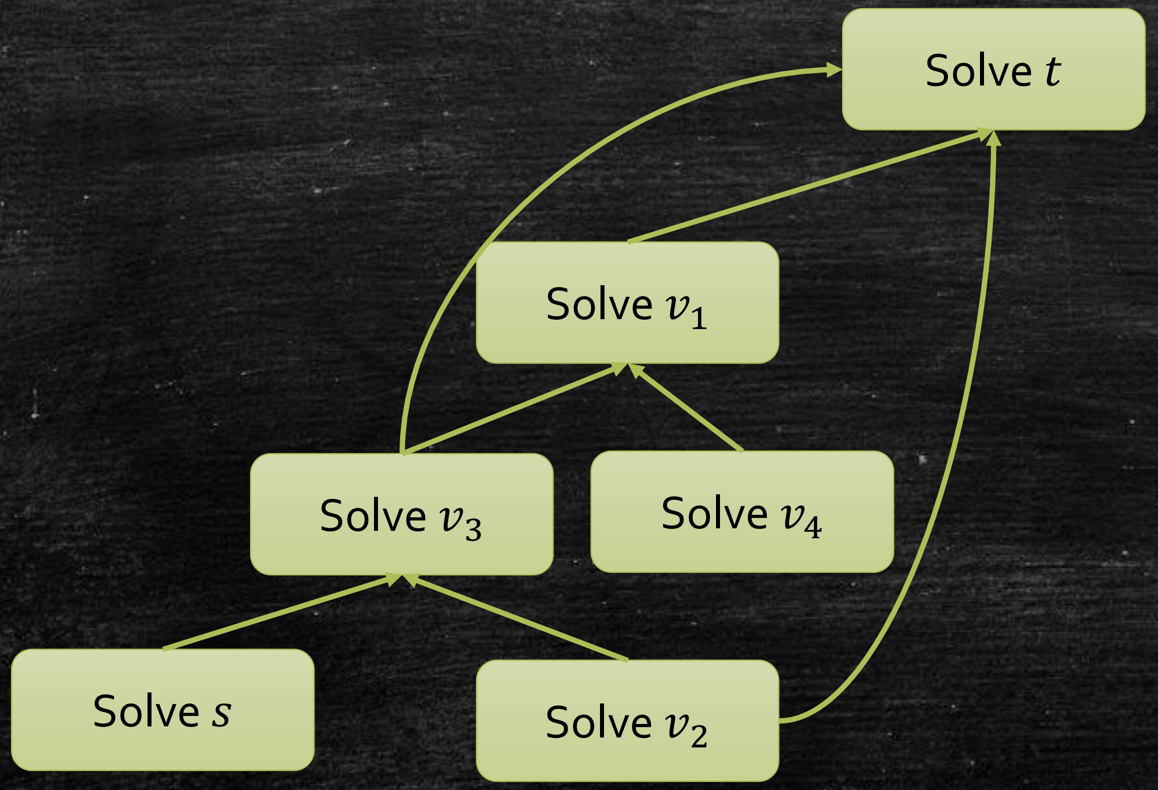


Merge common subproblems



We have at most n subproblems!

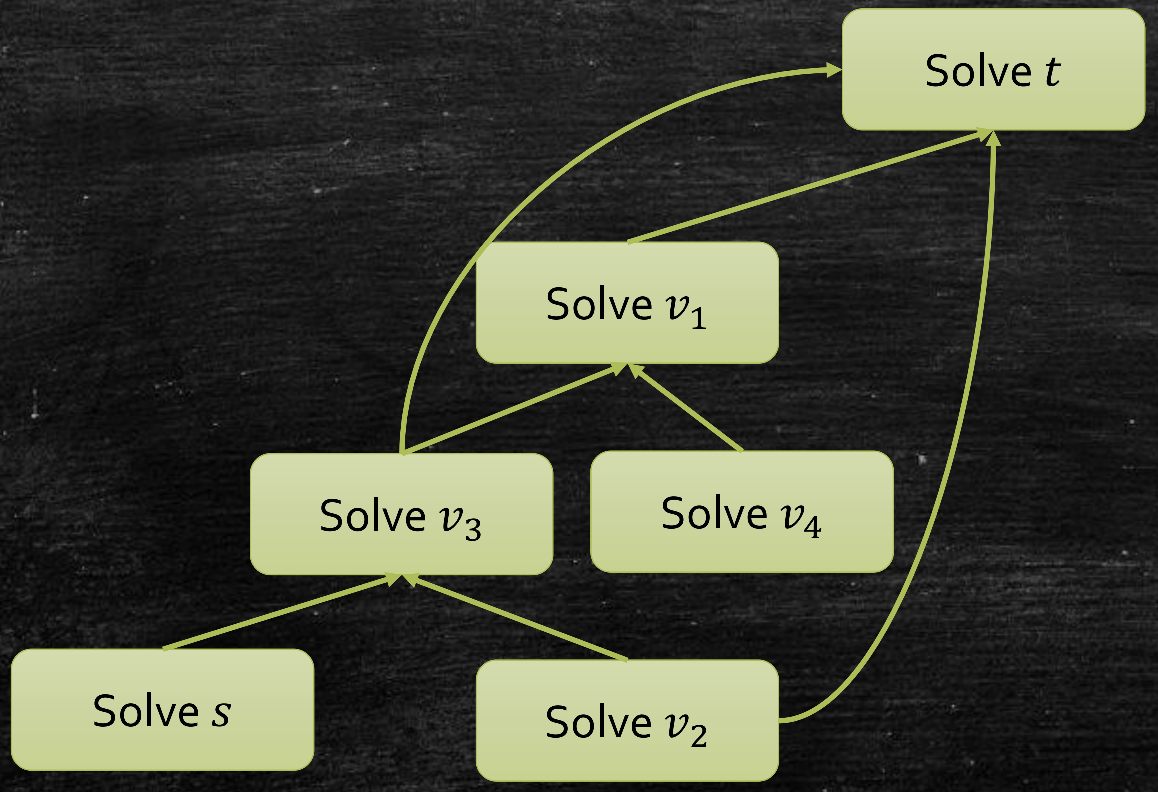
Are we in a DAG?



We have at most n subproblems!

It is exactly a DAG, because it is G!

Solve it by topological order!

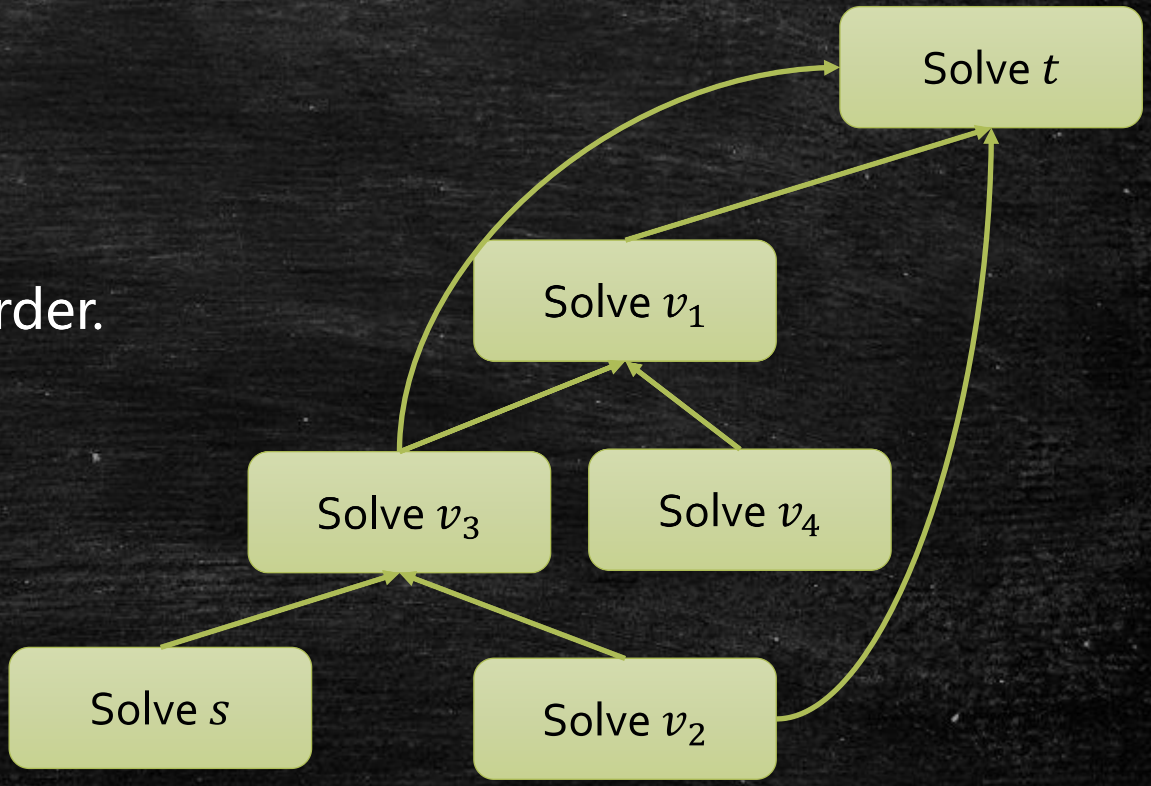


We have at most n subproblems!

It is exactly a DAG, because it is G!

Solve it by topological order!

- Plan
  - Find a Topological Order of V.
    - $O(|V| + |E|)$
  - $dist[s] = 0$.
  - Solve & record $dist[u]$ by the order.
  - Solve $dist[u] = \min\{$
    - $dist[v_1] + w(v_1, u)$
    - $dist[v_2] + w(v_2, u)$
    - $dist[v_3] + w(v_3, u)$
    - $\ldots\}$
  - $O(|V| + |E|)$



What about the correctness?

- We can easily check the correctness of DP Algorithms by induction.
- Base case:
  - Check our initialization: $dist[s] = 0$.
- Induction:
  - Assume $dist[u_i]$ is correct for all $i < k$.
  - $dist[u_k]$ can be solved correctly by the min of
    - $dist[v_1] + w(u_k, v_1)$
    - $dist[v_2] + w(u_k, v_2)$
    - $dist[v_3] + w(u_k, v_3)$
    - …

The topological order make a feasible induction order!

A simpler guideline

- Find subproblems.
- Check whether we are in a **DAG** and find the **topological order** of this DAG. (usually, by hand.)
- Solve & store the subproblems by the topological order.

More DP algorithms!

Longest Increasing Subsequence

- **Input:** A sequence $a_1, a_2, \dots, a_n$.
- **Output:** the Longest Increasing Subsequence (LIS)
  - $a_{i_1} < a_{i_2} < a_{i_3} \dots < a_{i_k}$
  - $i_1 < i_2 < i_3 \dots < i_k$

1	5	13	2	6	24	15	23	2	16

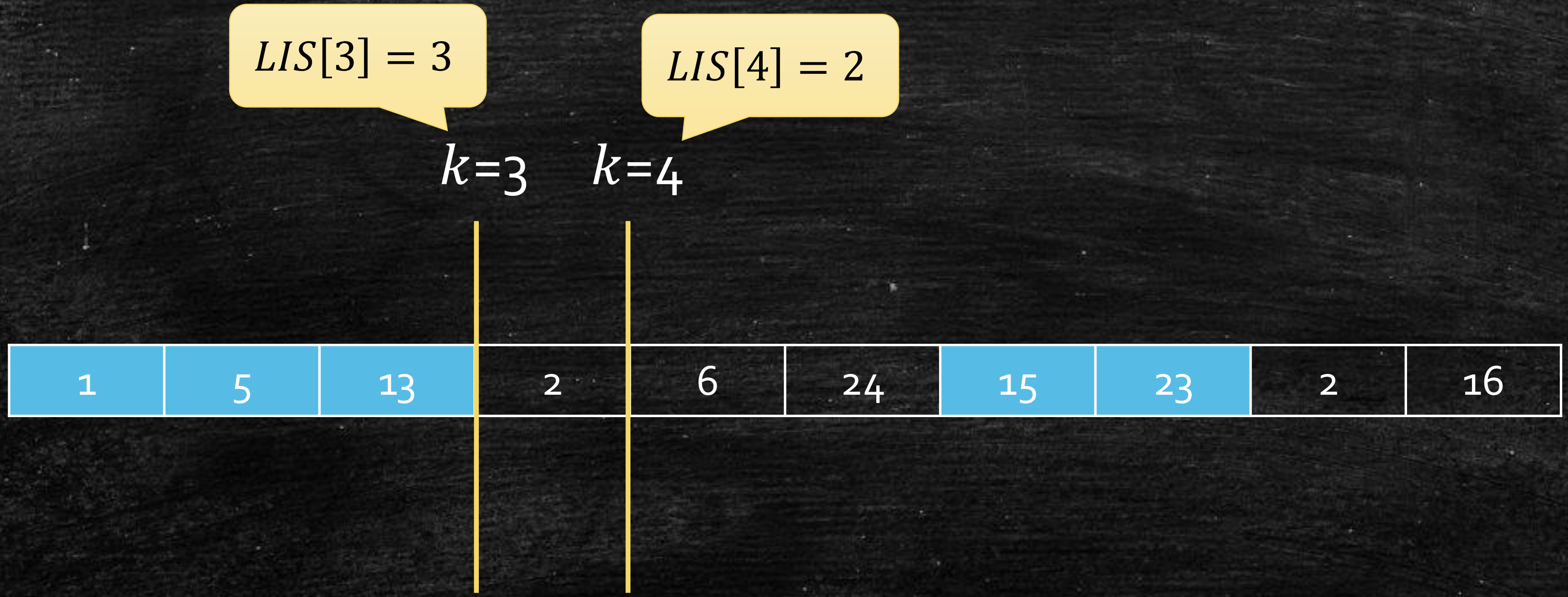
Solve LIS recursively!

- Enumerate the last number a_i.
  - $LIS = \max_{\mathrm{i} \leq n} LIS(a_i)$

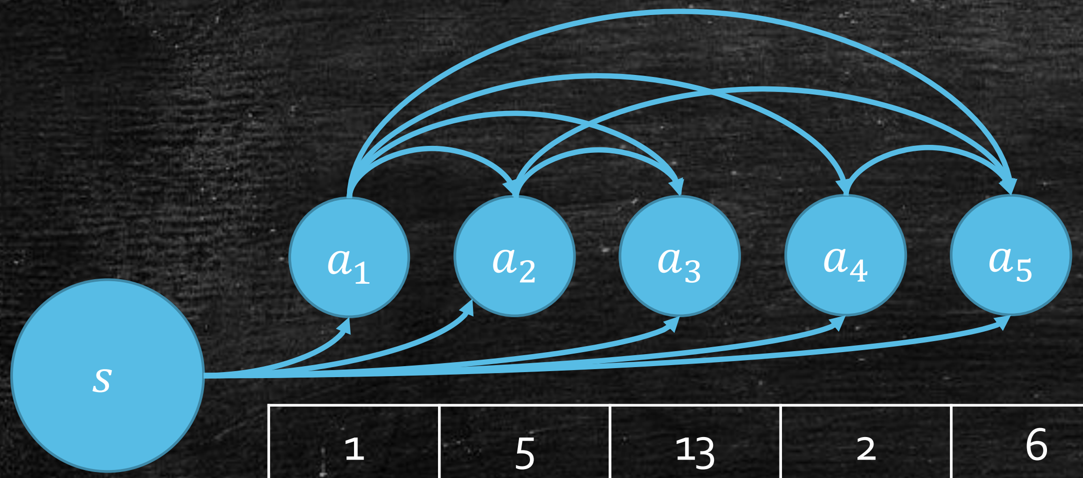
1	5	13	2	6	24	15	23	2	16

Define subproblems

- $LIS[k]$: the Longest Increasing Subsequence ended by a_k.



Another view of the problem.

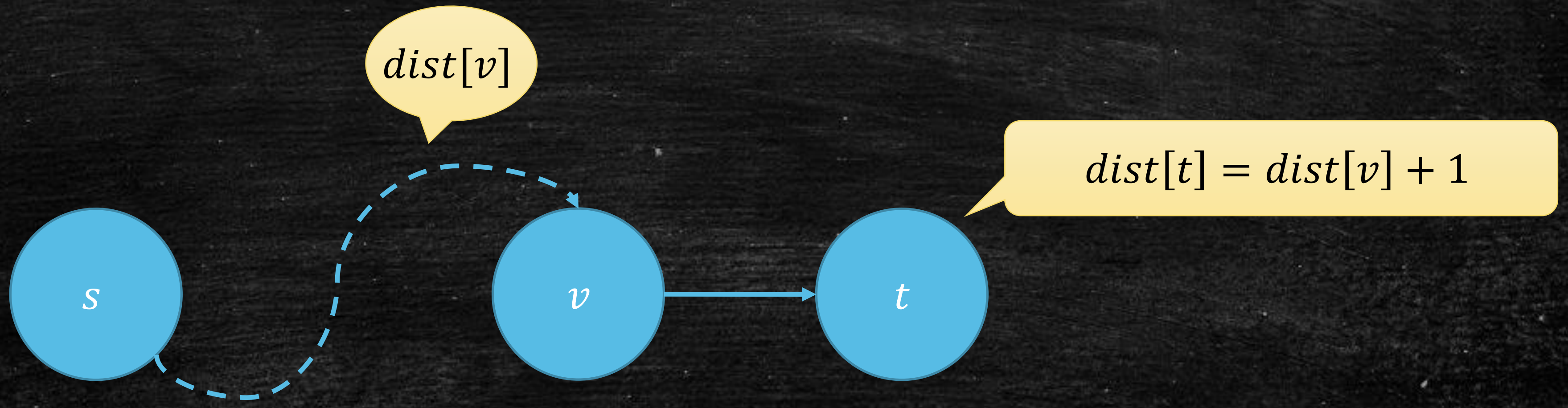


1	5	13	2	6	24	15	23	2	16

$LIS[k]$ can be viewed as the longest path from s to a_k.

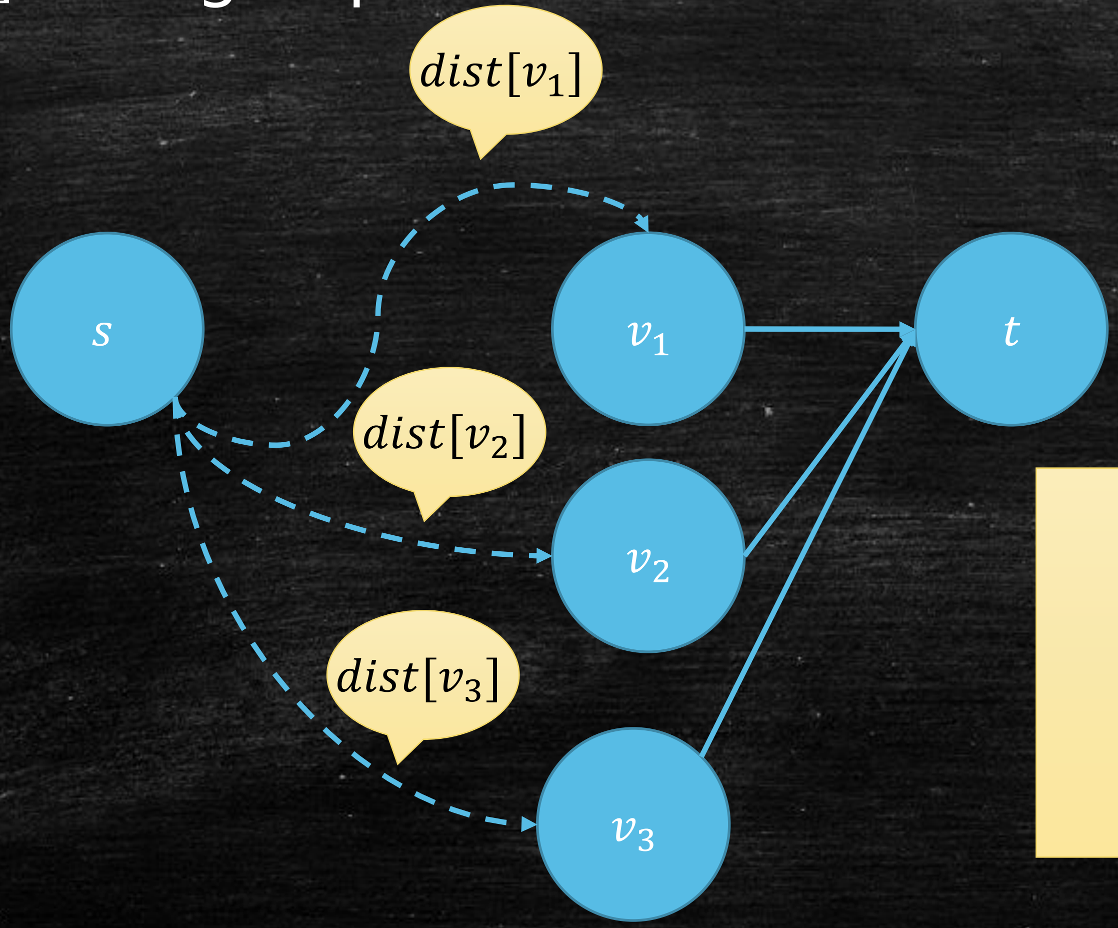
Important Fact

- $dist[v] \rightarrow$ longest path from s to v.



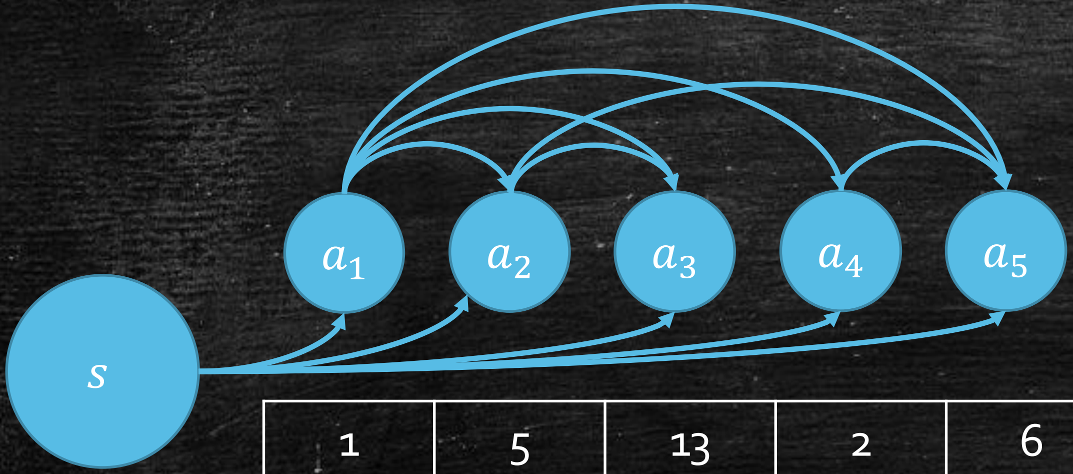
Important Fact

- $dist[v] \rightarrow$ longest path from s to v.



$$dist[t] = \max \begin{cases} dist[v_1] + 1 \\ dist[v_2] + 1 \\ dist[v_3] + 1 \end{cases}$$

Another view of the problem.

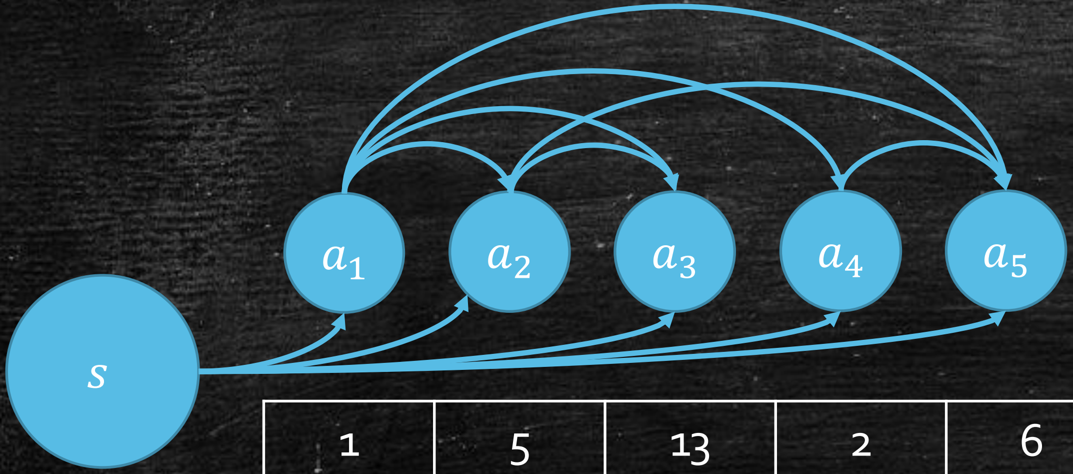


1	5	13	2	6	24	15	23	2	16

$LIS[k]$ can be viewed as the longest path from s to a_k.

$1 \ldots n$ is a topological order!

Another view of the problem.



1	5	13	2	6	24	15	23	2	16

Longest Increasing Subsequence

function $LIS(n)$

$\quad \text{lis}[0] = 0$

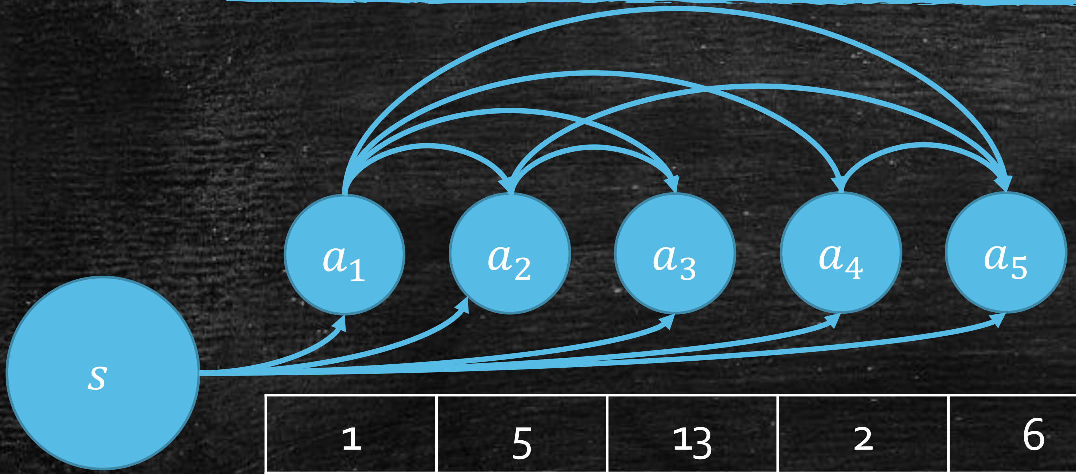
$\quad$ **for** i = 1 to n

$$\text{lis}[i] = \max_{a_j < a_i, j < i} \{lis[j] + 1\}$$

$\quad$ **return** $\max_{1 \le i \le n} lis[i]$

$s = a_0 = -\infty$

Another view of the problem.



	1	5	13	2	6	24	15	23	2	16
LIS	1	-	-	-	-	-	-	-	-	-

Longest Increasing Subsequence

function $LIS(n)$

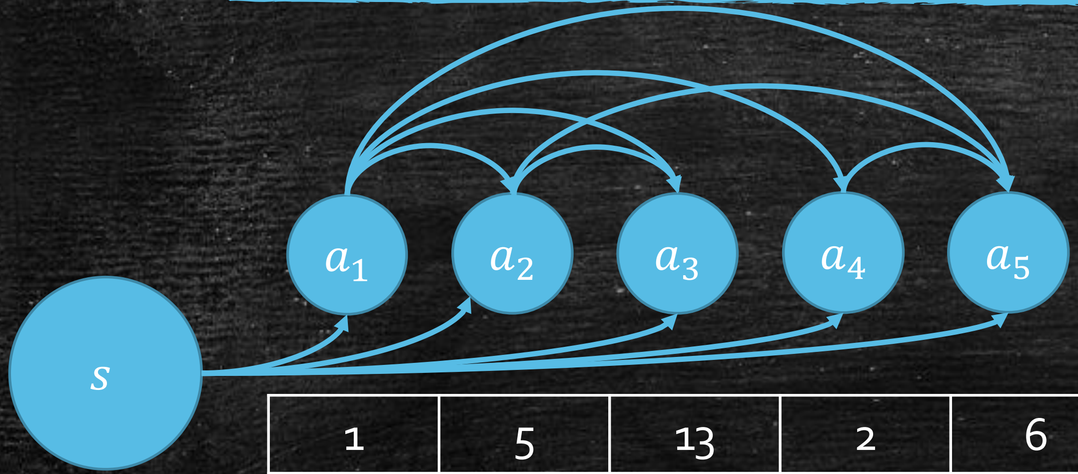
$\text{lis}[0] = 0$

for i = 1 to n

$\text{lis}[i] = \max_{a_j < a_i, j < i} \{lis[j] + 1\}$

return $\max_{1 \le i \le n} lis[i]$

Another view of the problem.



	1	5	13	2	6	24	15	23	2	16
LIS	1	2	-	-	-	-	-	-	-	-

Longest Increasing Subsequence

function $LIS(n)$

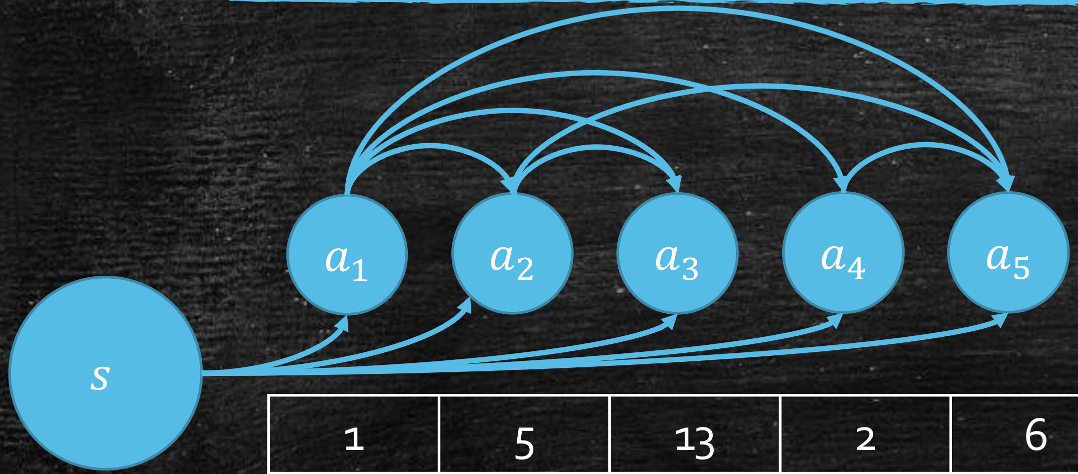
$\text{lis}[0] = 0$

for i = 1 to n

$$\text{lis}[i] = \max_{a_j < a_i, j < i} \{lis[j] + 1\}$$

return $\max_{1 \le i \le n} lis[i]$

Another view of the problem.



	1	5	13	2	6	24	15	23	2	16
LIS	1	2	3	-	-	-	-	-	-	-

Longest Increasing Subsequence

function $LIS(n)$

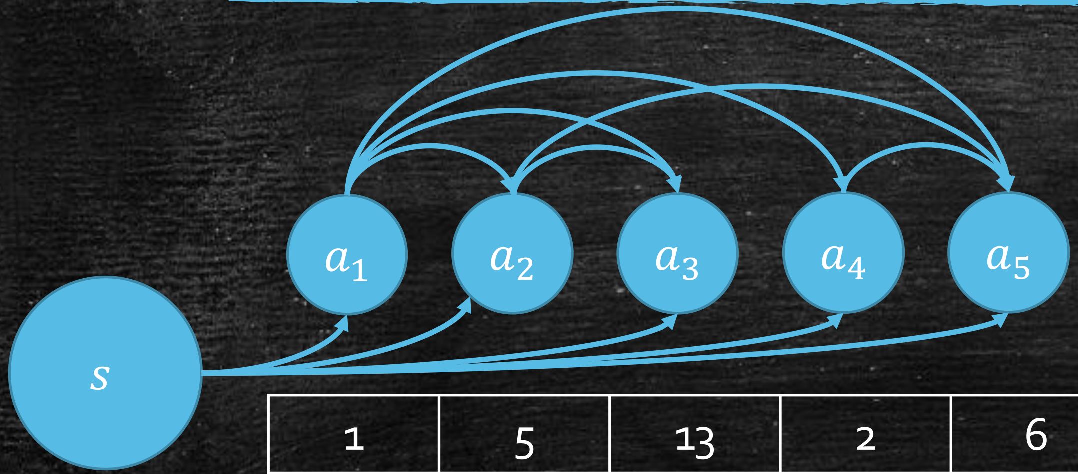
$\quad \text{lis}[0] = 0$

$\quad$ **for** i = 1 to n

$\quad\quad \text{lis}[i] = \max\limits_{a_j<a_i, j<i} \{lis[j] + 1\}$

$\quad$ **return** $\max\limits_{1\leq \text{i} \leq \text{n}} lis[i]$

Another view of the problem.



	1	5	13	2	6	24	15	23	2	16
LIS	1	2	3	2	-	-	-	-	-	-

Longest Increasing Subsequence

function $LIS(n)$

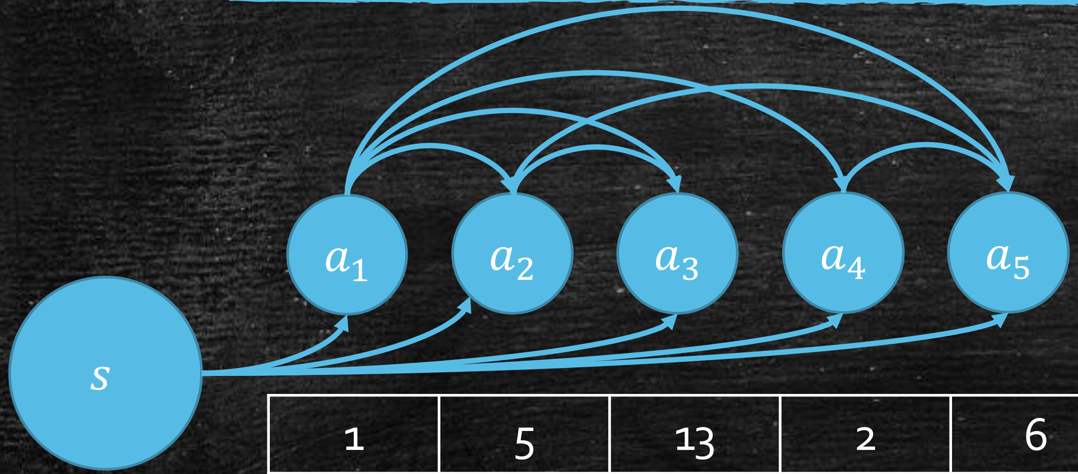
  $\text{lis}[0] = 0$

  for i = 1 to n

    $\text{lis}[i] = \max_{a_j < a_i, j < i} \{lis[j] + 1\}$

  return $\max_{1 \le i \le n} lis[i]$

Another view of the problem.



	1	5	13	2	6	24	15	23	2	16
LIS	1	2	3	2	3	-	-	-	-	-

Longest Increasing Subsequence

$O(n^2)$

function $LIS(n)$

  $\text{lis}[0] = 0$

  for i = 1 to n

    $\text{lis}[i] = \max\limits_{a_j < a_i, j < i} \{lis[j] + 1\}$

  return $\max\limits_{1 \le i \le n} lis[i]$

$s = a_0 = -\infty$

Edit Distance

- Motivation: How to change from one string to another?
- Allowed operations
  - Insertion: insert a character to a specific location.
  - Deletion: delete a character from a specific location.
  - Replacement: rewrite a character at a specific location.
- Change SNOWY to SUNNY?
  - SNNWY
  - SNNY
  - SUNNY

Another View

- Allowed operations
  - Alignment: Insert space with 0 cost.
  - Insertion: **rewrite** a character from a space at a specific location.
  - Deletion: **rewrite** a character to a space at a specific location.
  - Replacement: **rewrite** a character at a specific location.

S	N	O	W	Y

S	U	N	N	Y

Another View

- Allowed operations
  - Alignment: Insert space with 0 cost.
  - Insertion: **rewrite** a character from a space at a specific location.
  - Deletion: **rewrite** a character to a space at a specific location.
  - Replacement: **rewrite** a character at a specific location.

S	N	O	W	Y

S	U	N	N	Y

Another View

- Allowed operations
  - Alignment: Insert space with 0 cost.
  - Insertion: **rewrite** a character from a space at a specific location.
  - Deletion: **rewrite** a character to a space at a specific location.
  - Replacement: **rewrite** a character at a specific location.

Change alignment

S	N	O	W	_	Y

S	U	N	N	Y	_

Another View

- Allowed operations
  - Alignment: Insert space with 0 cost.
  - Insertion: **rewrite** a character from a space at a specific location.
  - Deletion: **rewrite** a character to a space at a specific location.
  - Replacement: **rewrite** a character at a specific location.

The same as before.

S	_	N	O	W	Y

S	U	N	N	_	Y

Optimization

- What is the minimized cost to change from a string to another? (it is symmetric)
- We call it the **Edit Distance** of the two string.
- Usage
  - Quantifying how dissimilar two strings are.

Edit Distance Calculation

- **Input:** two strings
  - $X: x_1, x_2, \dots, x_m$
  - $Y: y_1, y_2, \dots, y_n$
- **Output:** the edit distance between x and y.
- Another view
  - Find the best alignment!

Find out the subproblems

Imagine the best alignment from the tail.

X	?	?	?	?	?

Y	?	?	?	?	?

Find out the subproblems

Case 1

X	?	?	?	?	x_m

Y	?	?	?	?	y_n

Find out the subproblems

Case 2

X	?	?	?	?	_
Y	?	?	?	?	y_n

Find out the subproblems

Case 3

X	?	?	?	?	x_m
Y	?	?	?	?	_

Find out the subproblems

Case 1

The best alignment for $X[1{\sim}m-1]$ and $Y[1{\sim}n-1]$.

Plus one cost if $x_m \neq y_n$

X	?	?	?	?	x_m
Y	?	?	?	?	y_n

Find out the subproblems

Case 2

The best alignment for $X[1\sim m]$ and $Y[1\sim n-1]$.

Plus one cost

X	?	?	?	?	_
Y	?	?	?	?	y_n

Find out the subproblems

Case 3

The best alignment for $X[1{\sim}m-1]$ and $Y[1{\sim}n]$.

Plus one cost

X	?	?	?	?	x_m
Y	?	?	?	?	_

Do you find out the subproblems?

Subproblems

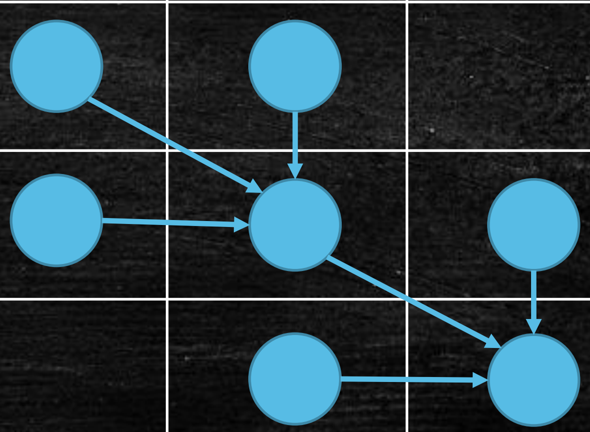
- $ED[\mathrm{i},\mathrm{j}]$: The edit distance between $X[1..i]$ and $Y[1..j]$.
- $ED[n,m]$: The edit distance between X and Y.
- How to solve $ED[i,j]$: min of three cases
  - $ED[i-1,j-1] + \mathbf{1}_{x_i \neq y_j}$
  - $ED[i,j-1] + 1$
  - $ED[i-1,j] + 1$

Is it DAG?

$ED[i,j]$	$j = 0$	$j = 1$	$j = 2$	$j = 3$	$j = 4$	$j = \cdots$	$j = n$
$i = 0$							
$i = 1$							
$i = 2$							
$i = 3$							
$i = \cdots$							
$i = m$							

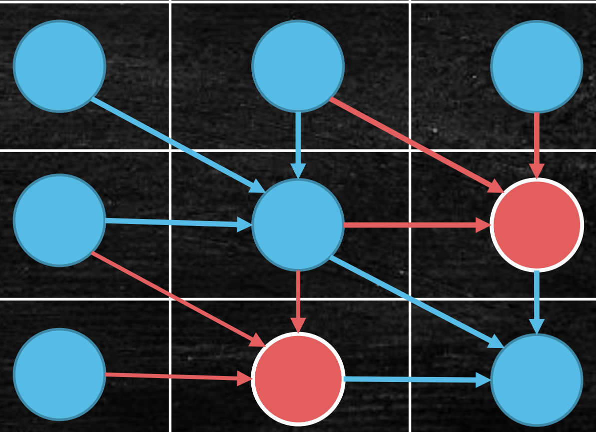
Is it DAG?

$ED[i,j]$	$j = 0$	$j = 1$	$j = 2$	$j = 3$	$j = 4$	$j = \cdots$	$j = n$
$i = 0$							
$i = 1$							
$i = 2$							
$i = 3$							
$i = \cdots$							
$i = m$							



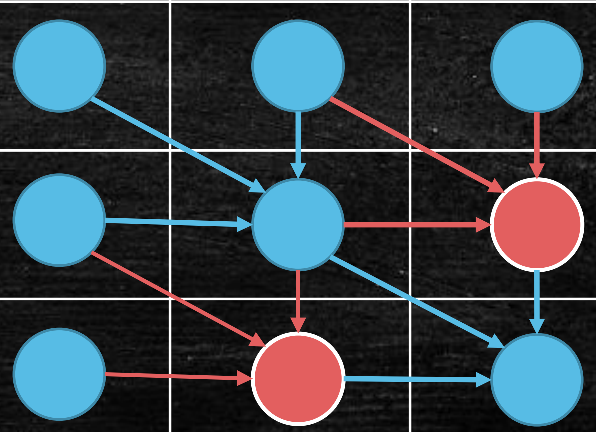
Is it DAG?

$ED[i,j]$	$j=0$	$j=1$	$j=2$	$j=3$	$j=4$	$j=\cdots$	$j=n$
$i=0$							
$i=1$							
$i=2$							
$i=3$							
$i=\cdots$							
$i=m$							



A topological order

$ED[i,j]$	$j = 0$	$j = 1$	$j = 2$	$j = 3$	$j = 4$	$j = \cdots$	$j = n$
$i = 0$							
$i = 1$							
$i = 2$							
$i = 3$							
$i = \cdots$							
$i = m$							



Start!

$ED[i,j]$	$j = 0$	$j = 1$	$j = 2$	$j = 3$	$j = 4$	$j = \cdots$	$j = n$
$i = 0$	0	1	2	3	4	5	6
$i = 1$	1						
$i = 2$	2						
$i = 3$	3						
$i = \cdots$	4						
$i = m$	5						

Start!

$ED[i,j]$	$j = 0$	$j = 1$	$j = 2$	$j = 3$	$j = 4$	$j = \cdots$	$j = n$
$i = 0$	0	1	2	3	4	5	6
$i = 1$	1						
$i = 2$	2						
$i = 3$	3						
$i = \cdots$	4						
$i = m$	5						

Start!

$ED[i,j]$	$j = 0$	$j = 1$	$j = 2$	$j = 3$	$j = 4$	$j = \cdots$	$j = n$
$i = 0$	0	1	2	3	4	5	6
$i = 1$	1						
$i = 2$	2						
$i = 3$	3						
$i = \cdots$	4						
$i = m$	5						

Start!

$ED[i,j]$	$j=0$	$j=1$	$j=2$	$j=3$	$j=4$	$j=\cdots$	$j=n$
$i=0$	0	1	2	3	4	5	6
$i=1$	1						
$i=2$	2						
$i=3$	3						
$i=\cdots$	4						
$i=m$	5						

Start!

$ED[i,j]$	$j = 0$	$j = 1$	$j = 2$	$j = 3$	$j = 4$	$j = \cdots$	$j = n$
$i = 0$	0	1	2	3	4	5	6
$i = 1$	1						
$i = 2$	2						
$i = 3$	3						
$i = \cdots$	4						
$i = m$	5						

Running Time?

$O(nm) \cdot O(1)$

$ED[i,j]$	$j=0$	$j=1$	$j=2$	$j=3$	$j=4$	$j=\cdots$	$j=n$
$i=0$	0	1	2	3	4	5	6
$i=1$	1						
$i=2$	2						
$i=3$	3						
$i=\cdots$	4						
$i=m$	5						

Knapsack Problems

- **Input:** n items with cost c_i and value v_i, and a capacity W.
- **Output:** Select a subset of items, with total cost at most W. The goal is to maximize the total value.

A nice greedy approach.

- Select the item with from larger **value-cost** ratio.

	Value	Cost
iPhone	8888	8888
Algorithm Book	10000	500
Laptop	8888	8500
Hermès	90000	100000

A nice greedy approach.

- Select the item with from larger **value-cost** ratio.

$W = 10000$

	Value	Cost
iPhone	8888	8888
Algorithm Book	10000	500
Laptop	8888	8500
Hermès	90000	100000

A nice greedy approach.

- Select the item with from larger **value-cost** ratio.

$W = 10000$

	Value	Cost
iPhone	8888	8888
Algorithm Book	10000	500
Laptop	8888	8500
Hermès	90000	100000

It looks quite intuitive, and it is correct now!

A nice greedy approach.

- Select the item with from larger **value-cost** ratio.

$W = 100000$

	Value	Cost
iPhone	8888	8888
Algorithm Book	10000	500
Laptop	8888	8500
Hermès	90000	100000

But when we become rich…

Problem: items are not divisible!

A nice greedy approach.

- Select the item with from larger **value-cost** ratio.

$W = 100000$

	Value	Cost
iPhone	8888	8888
Algorithm Book	10000	500
Laptop	8888	8500
Qie Gao	90000	100000

Buy 0.82112 portion of the "Qie Gao"

But when we become rich…

Problem: items are not indivisible!

What if items are indivisible?

- The Knapsack Problem is NP-Hard!
- Are we going to talk about approximation algorithms?
- No!
- Let's make a DP algorithm with reasonable running time!

Find out subproblems!

- What we always do before:
- $f[i]$: the maximum value we can get by using the first i items.

$f[i]$	5	10	13	16	21	30	?

Find out subproblems!

- What we always do before:
- $f[i]$: the maximum value we can get by using the first i items.

How to solve $f[i]$ by $f[j < i]$?

$f[i]$	5	10	13	16	21	30	?

Find out subproblems!

- What we always do before:
- $f[i]$: the maximum value we can get by using the first i items.

How to solve $f[i]$ by $f[j < i]$?

$f[i]$	5	10	13	16	21	30	?

We know $f[j]$ but we do not know how much budget it uses!

The space of subproblems is not large enough!

Find out subproblems!

- What we always do before:
- $f[i]$: the maximum value we can get by using the first i items.
- Use $g[i]$ to store how much budge $f[i]$ uses.

How to solve $f[i]$ by $f[j < i]$?

$f[i]$	5	10	13	16	21	30	?

We know $f[j]$ but we do not know how much budget it uses!

Find out subproblems!

- What we always do before:
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$f[i]$			13	16	21	30	?

It is greedy, and it is not optimal!

We know $f[j]$ but we do not know how much budget it uses!

Find out subproblems!

- What we always do before:
- $f[i]$: the maximum value we can get by using the first i items.
- ~~Use $g[i]$ to store how much budge $f[i]$ uses.~~

$f[i]$			13	16	21	30	?

How to solve $f[i]$ by $f[j < i]$?

It is greedy, and it is not optimal!

We know $f[j]$ but we do not know how much budget it uses!

Move back to recursion to check how much we need.

- Buy Hermès or not?
- Buy: Earn 90000, continue to buy "iPhone, Book, and Laptop" by $W - 100000$ budget.
- Not Buy: Earn 0, continue to buy "iPhone, Book, and Laptop" by W budget.

	Value	Cost
iPhone	8888	8888
Algorithm Book	10000	500
Laptop	8888	8500
Hermès	90000	100000

Find out subproblems!

- What we always do before:
- $f[i, \boldsymbol{w}]$: the maximum value we can get by using the first i items, and with $\boldsymbol{w}$ **budget**.
- ~~Use $g[i]$ to store how much budge $f[i]$ uses.~~

How to solve $f[i]$ by $f[j < i]$?

$f[i]$	5	10	13	16	21	30	?

We know $f[j]$ but we do not know how much budget it uses!

Find out subproblems!

- What we always do before:
- $f[i, \boldsymbol{w}]$: the maximum value we can get by using the first i items, and with $\boldsymbol{w}$ **budget**.

$f[i,w]$	0	1	2	3	4	5	6	…	W
0									
1									
2									
3						$f[i,w]$			
…									
n									$f[n,W]$

How to solve $f[i,w]$.

Solve $f[i, w]$

- What we always do before:
- $f[i, \boldsymbol{w}]$: the maximum value we can get by using the first i items, and with $\boldsymbol{w}$ **budget**.
- Two options for item i
  - **Buy it:** We can at most use $w - c_i$ budget before i.
  - **Not Buy it:** We can at most use w budget before i.
  - Solve $f[i, w] = \max\{f[i-1, w], f[i-1, w-c_i] + v_i\}$.

Check the topological order

- $f[i, \boldsymbol{w}]$: the maximum value we can get by using the first i items, and with $\boldsymbol{w}$ **budget**.
- $f[i, w] = \max\{f[i-1, w], f[i-1, w-c_i] + v_i\}$

$\boldsymbol{O(nW) \cdot O(1)}$

c_i

$f[i,w]$	0	1	2	3	4	5	6	…	W
0	0	0	0	0	0	0	0	0	0
1	0								
2	0								
3	0					$f[i,w]$			
…	0								
n	0								$f[n,W]$

Knapsack has many variants!

Surplus Supply

- **Input:** n items with cost c_i and value v_i, and a capacity W.
- **Output:** Select some items (each items can be selected more than once), with total cost at most W. The goal is to maximize the total value.

	Value	Cost
iPhone	8888	8888
Algorithm Book	10000	500
Laptop	8888	8500
Hermès	90000	100000

How to transfer subproblems now?

- Two options for item i
  - **Buy it:** We can at most use $w - c_i$ budget before i.
  - **Not Buy it:** We can at most use w budget before i.
  - Solve $f[i, w] = \max\{f[i-1, w], f[i-1, w-c_i] + v_i\}$.
- Problem!
  - We can buy multiple times!

A new subproblem transfer!

- Problem!
  - We can buy multiple times!
- New transfer
  - **Not Buy it anymore:** We can at most use w budget before i.
  - **Buy it:** We can at most use $w - c_i$ budget before i and i.
  - Solve $f[i, w] = \max\{f[i-1, w], \boldsymbol{f[i, w - c_i]} + v_i\}$.

Check the topological order

- $f[i, \mathbf{w}]$: the maximum value we can get by using the first i items, and with $\mathbf{w}$ **budget**.
- $f[i,w] = \max\{f[i-1,w], f[i,w-c_i]+v_i\}$

c_i

$f[i,w]$	0	1	2	3	4	5	6	…	W
0	0	0	0	0	0	0	0	0	0
1	0								
2	0								
3	0					$f[i,w]$			
…	0								
n	0								$f[n,W]$

Let us program it!

- $f[i, \boldsymbol{w}]$: the maximum value we can get by using the first i items, and with $\boldsymbol{w}$ **budget**.
- $f[i, w] = \max\{f[i-1, w], f[i, w - c_i] + v_i\}$

Knapsack with Surplus Supply $O(nW)$

function $knapsack(n)$
  $f[0,0] = f[0,1] = f[0,2] = \cdots = f[0, W] = 0$
  $f[0,0] = f[1,0] = f[2,0] = \cdots = f[n, 0] = 0$
  for w = 0 to W
    for $i = 1$ to n
      $f[i, w] = \max\{f[i-1, w], f[i, w - c_i] + v_i\}$
  return $f[n, W]$

We can make it simple

- $f[\boldsymbol{w}]$: the maximum value we can with $\boldsymbol{w}$ **budget**.
- $f[w] = \max_{i=1\sim n} \{f[w], f[w-c_i]+v_i\}$

Knapsack with Surplus Supply

function $knapsack(n)$
  $f[0] = f[1] = \cdots f[n] = 0$
  for $w = 0$ to W
    for $i = 1$ to n
      $f[w] = \max\{f[w], f[w-c_i]+v_i\}$
  return $f[n, W]$

$O(nW)$ but with less space.

They are not polynomial on the input size!

$O(nW)$ is not polynomial!

- Input: n items with cost c_i and value v_i, and a capacity W.
- Input size: the unit of bits to represent the input.
- $W = 2^N$ by using N bits
  - time complexity becomes $O(2^N)$.

Input of The Knapsack Problem

- **Input** of Knapsack
  - n,W
  - n values v_i and n costs c_i.
- Assume we have $O(N)$ bits, (**input size** = N)
- To present n values and costs, we need to use at least $O(n)$ bits. Hence, we at most present $n = O(N)$ with $O(N)$ bits.
- W can be $O(2^N)$
- So, the running time $O(nW)$ can become $O\left(N2^N\right)$, which is not polynomial!

Today's goal

- Learn what is DP.
- Learn how to **prove** DP's correctness.
- Learn the general **guideline** for designing DP Algorithms.
- Learn to apply the **guideline** on:
  - Fibonacci
  - Shortest Path on DAGs
  - Edit Distance
  - Knapsack