Divide and Conquer

Sorting & Inversions

Sorting Problem

- **Input:** A set of n integers
  - $x_1, x_2, x_3, \ldots, x_n$
- **Output:** The same set of n integers in ascending order.

How many sorting algorithms you know?

Insertion Sort

- **Input:** A set of n integers
  - $x_1, x_2, x_3, \ldots, x_n$
- **Output:** The same set of n integers in ascending order.
- Plan 1:
  - Try **Insertion Sort:** fix the output one by one.
  - How fast is it?

1	10	5	26	3	4	16	4	2

Insertion Sort

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1								

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1	10							

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1	10	5	26	3	4	16	4	2

1	5	10						

Insertion Sort

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1	10	5	26	3	4	16	4	2

1	5	10	26					

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1	10	5	26	3	4	16	4	2

1	3	5	10	26				

Insertion Sort

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1	10	5	26	3	4	16	4	2

1	3	4	5	10	26			

Insertion Sort

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1	3	4	5	10	16	26		

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1	10	5	26	3	4	16	4	2

1	3	4	4	5	10	16	26	

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1	2	3	4	4	5	10	16	26

Insertion Sort

- **Input:** A set of n integers
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- Plan 1:
  - Try **Insertion Sort:** fix the output one by one.
  - How fast is it?

How many number we should insert?

1	10	5	26	3	4	16	4	2

1	2	3	4	4	5	10	16	26

How long it takes to insert one number?

Insertion Sort

- **Input:** A set of n integers
  - $x_1, x_2, x_3, \dots, x_n$
- **Output:** The same set of n integers in ascending order.
- Plan 1:
  - Try **Insertion Sort:** fix the output one by one.
  - How fast is it?

n numbers

1	10	5	26	3	4	16	4	2

1	2	3	4	4	5	10	16	26

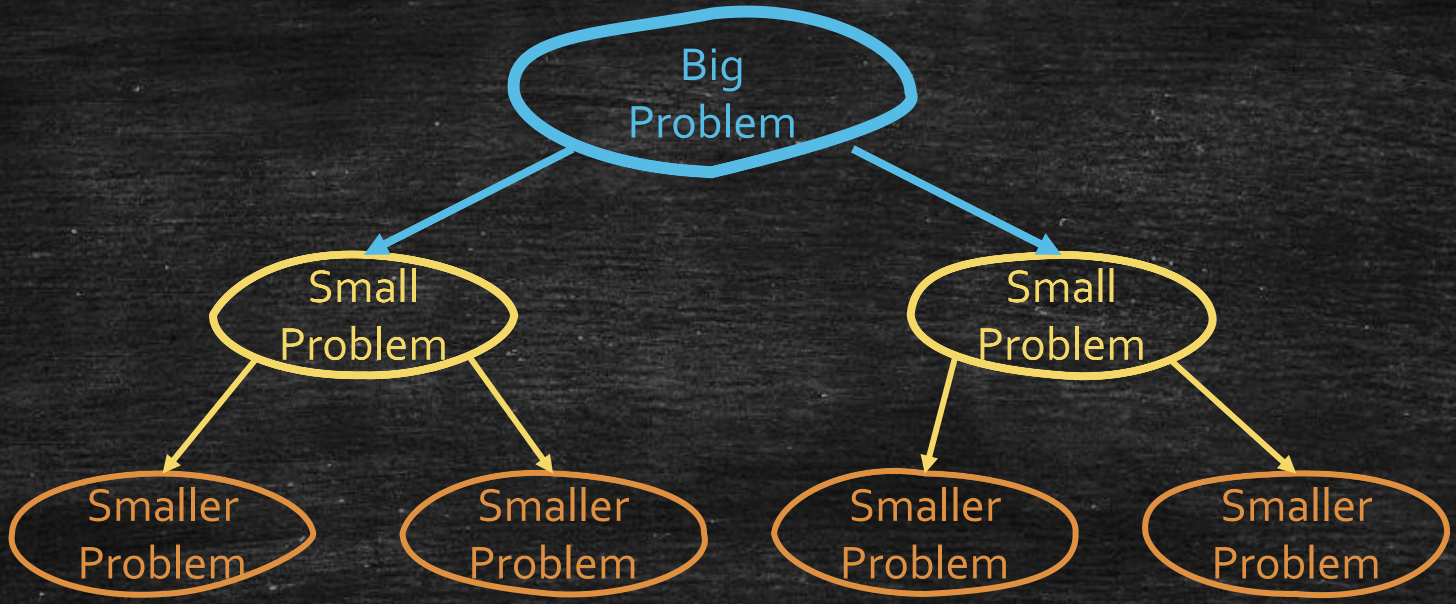
At most n operations

Don't forget the correctness!

- Is it **correct**?
- How to **prove** it!
  - Using Induction?

Prove correctness by induction.

- Base Case
  - If we only insert one integer, it is sorted.
- Induction
  - Assume the list is sorted after the $k-1$-th insertion.
  - Prove the list is sorted after the k-th insertion.
  - Assume the integer j is inserted at location i.
    - $j \geq a_i \geq a_{\leq i}$.
    - $j < a_{i+1} \leq a_{\geq i+1}$.



Ok! Let’s move to divide and conquer!

Divide and Conquer

- Recall the divide and conquer
- Divide
  - Divide the problem into small size subproblems.
- Recurse
  - Solve the small size subproblems.
- Combine
  - Combine the output of small size subproblems to get the answer of the original problem.
- Basic solver
  - If the problem size is small enough, we should solve it directly.

Merge sort

- **Input:** A set of n integers
  - $x_1, x_2, x_3, \dots, x_n$
- **Output:** The same set of n integers in ascending order.
- Plan 2: Divide and Conquer (Merge Sort)
  - Divide: Dive the input into two subsets:
    - $x_1, x_2, \dots, x_{n/2}$; $x_{n/2+1}, x_{n/2+2}, \dots, x_n$
  - Recurse: Sort two subsets (smaller size problems).
    - Let $y_1, y_2, \dots, y_{n/2}$; $y_{n/2+1}, y_{n/2+2}, \dots, y_n$ be the output (sorted list).
  - Combine: Merge to sorted list to one long sorted list.

Merge Sort

1	10	5	26	3	4	16	4	2

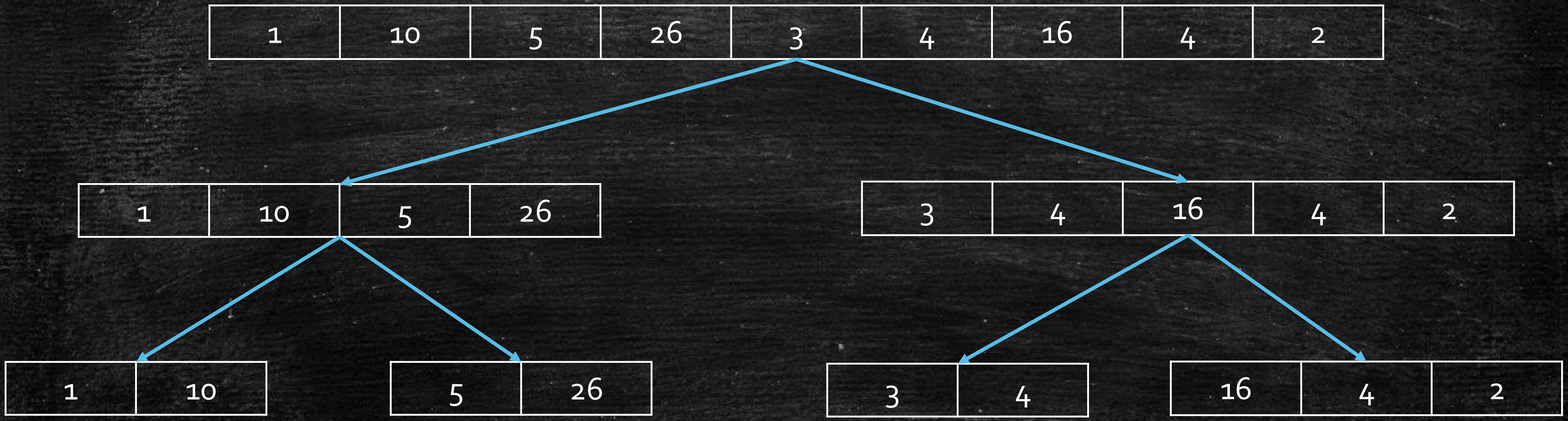
Merge Sort

1	10	5	26	3	4	16	4	2

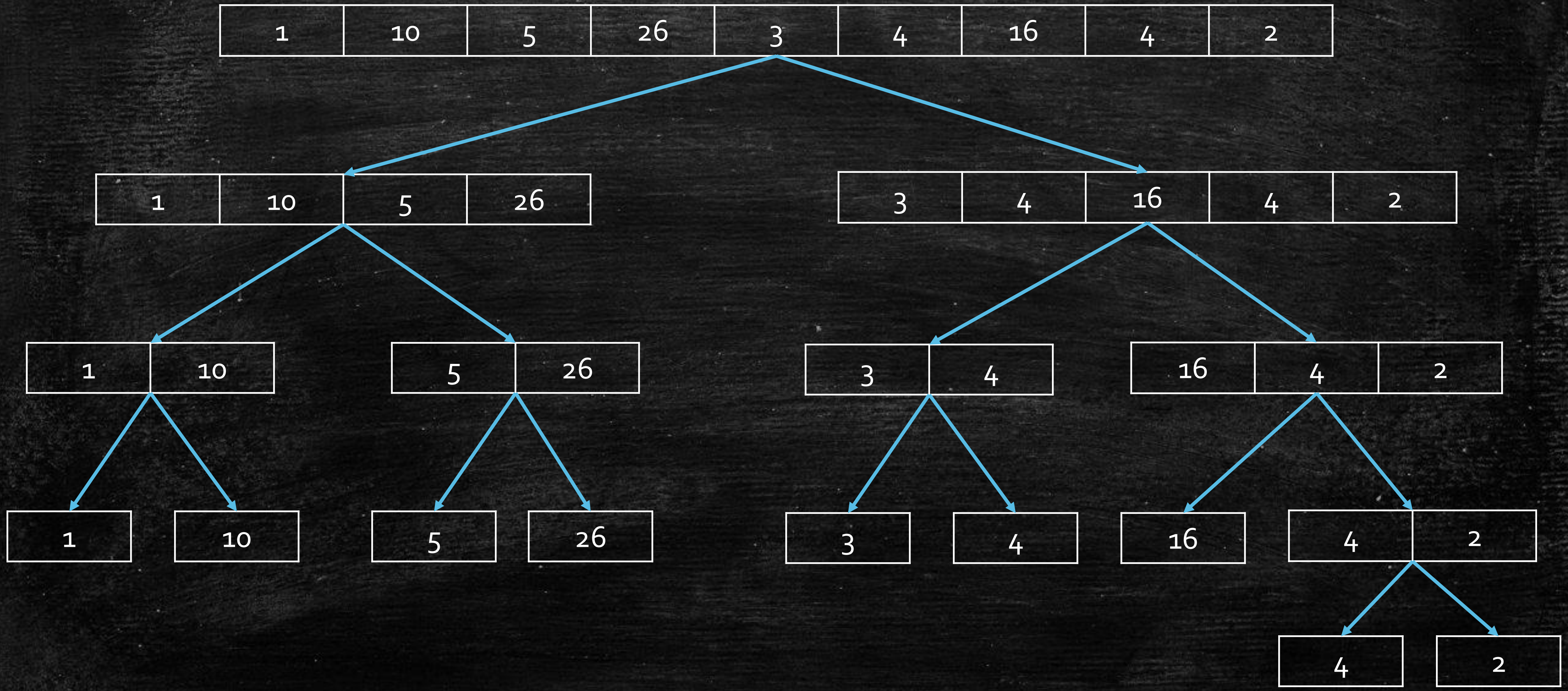
1	10	5	26

3	4	16	4	2

Merge Sort



Merge Sort



1 10 5 26 3 4 16 4 2
1 10 5 26
3 4 16 4 2
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10
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4 2
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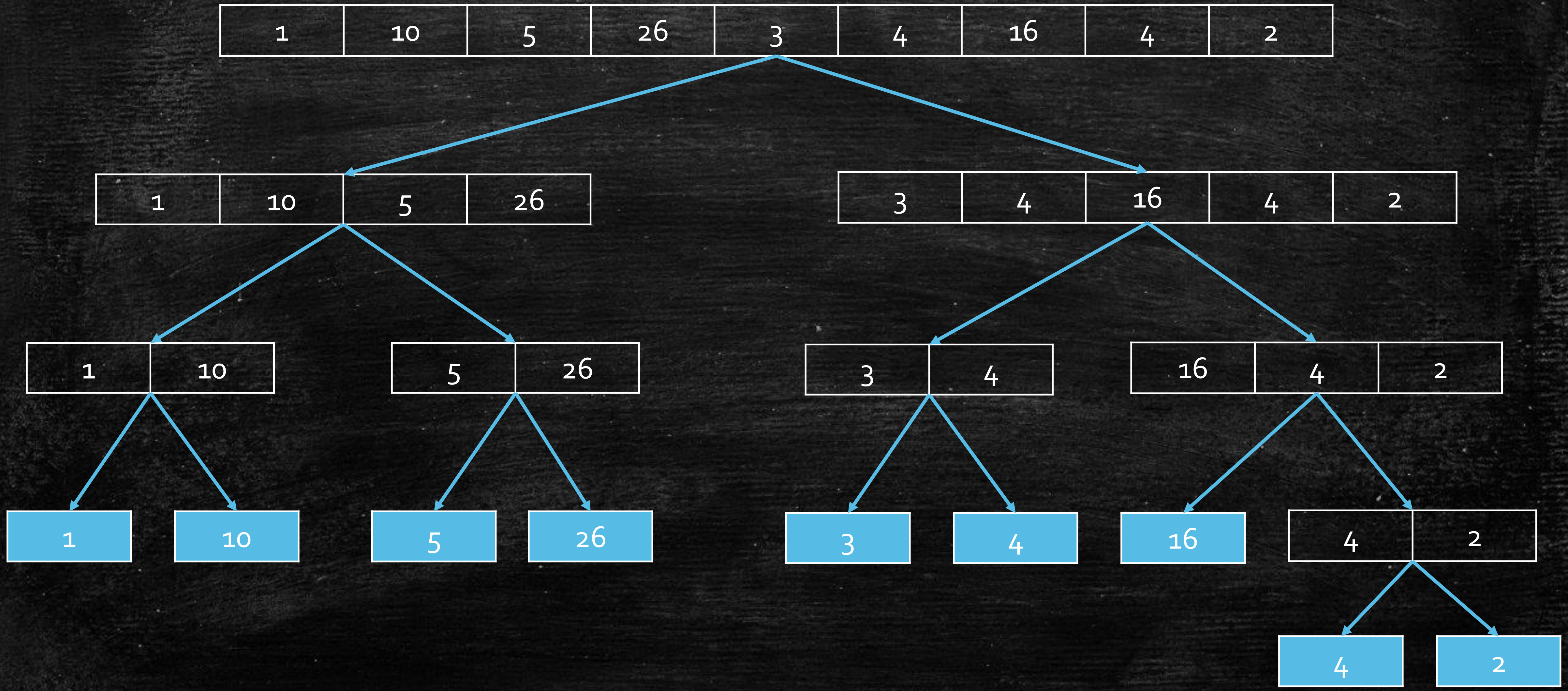
2

Merge Sort



1 | 10 | 5 | 26 | 3 | 4 | 16 | 4 | 2
1 | 10 | 5 | 26
3 | 4 | 16 | 4 | 2
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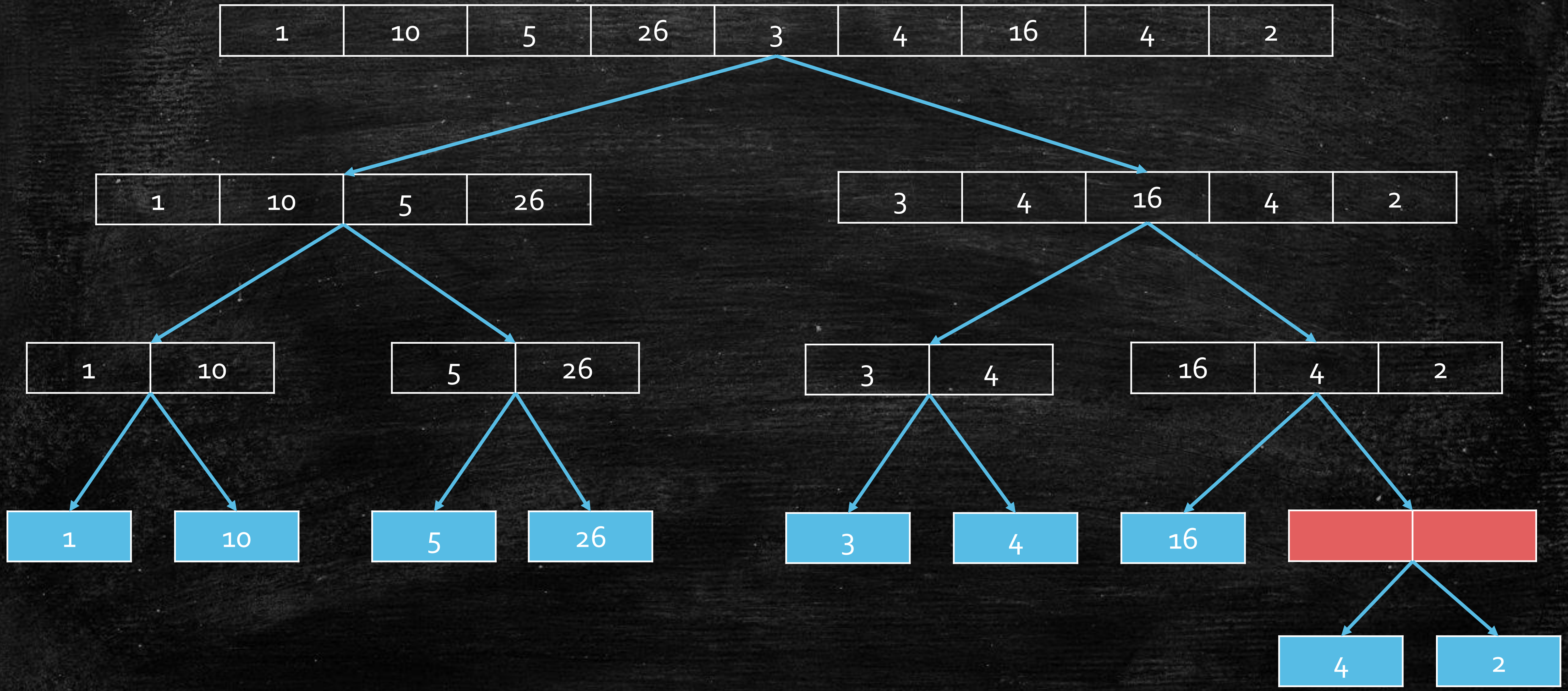
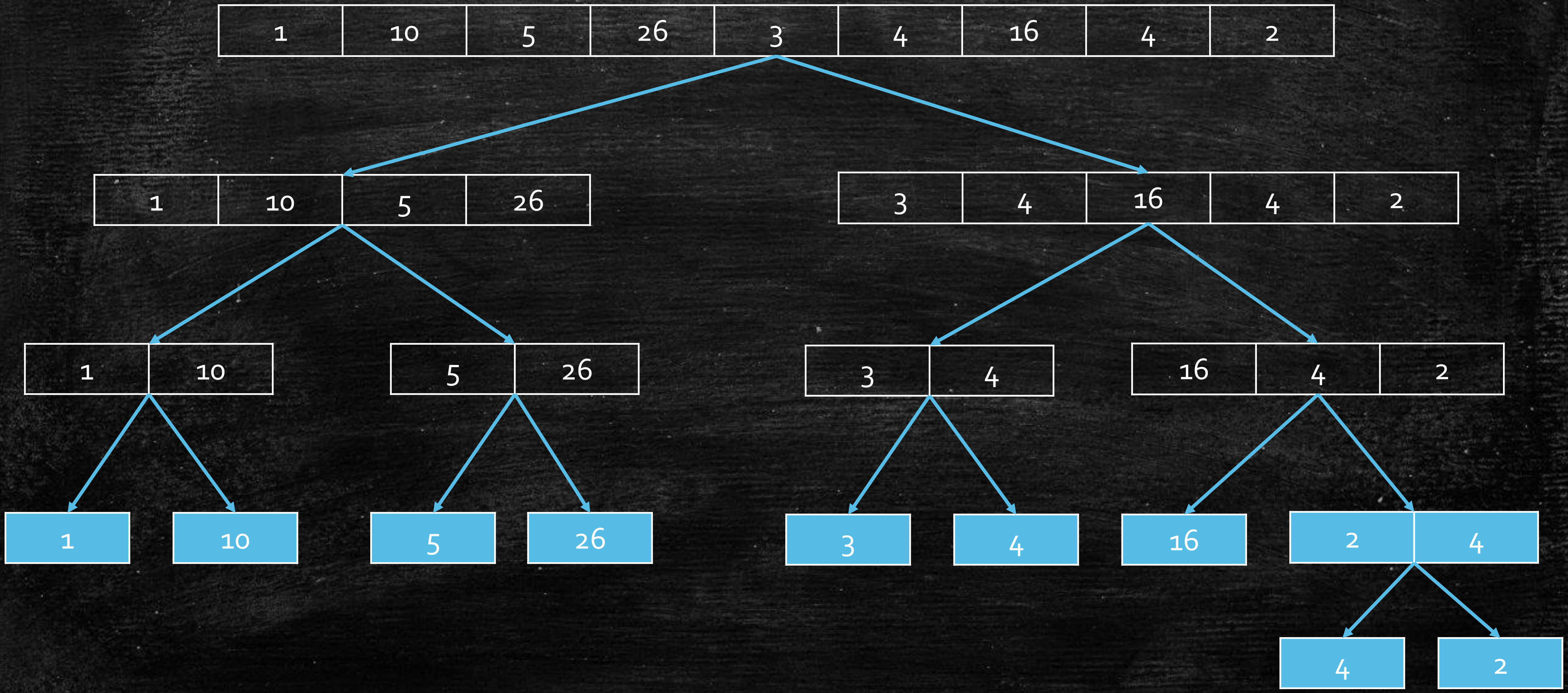
2

Merge Sort



Merge Sort



1 10 5 26 3 4 16 4 2
1 10 5 26
3 4 16 4 2
1 10
5 26
3 4
16 4 2
1
10
5
26
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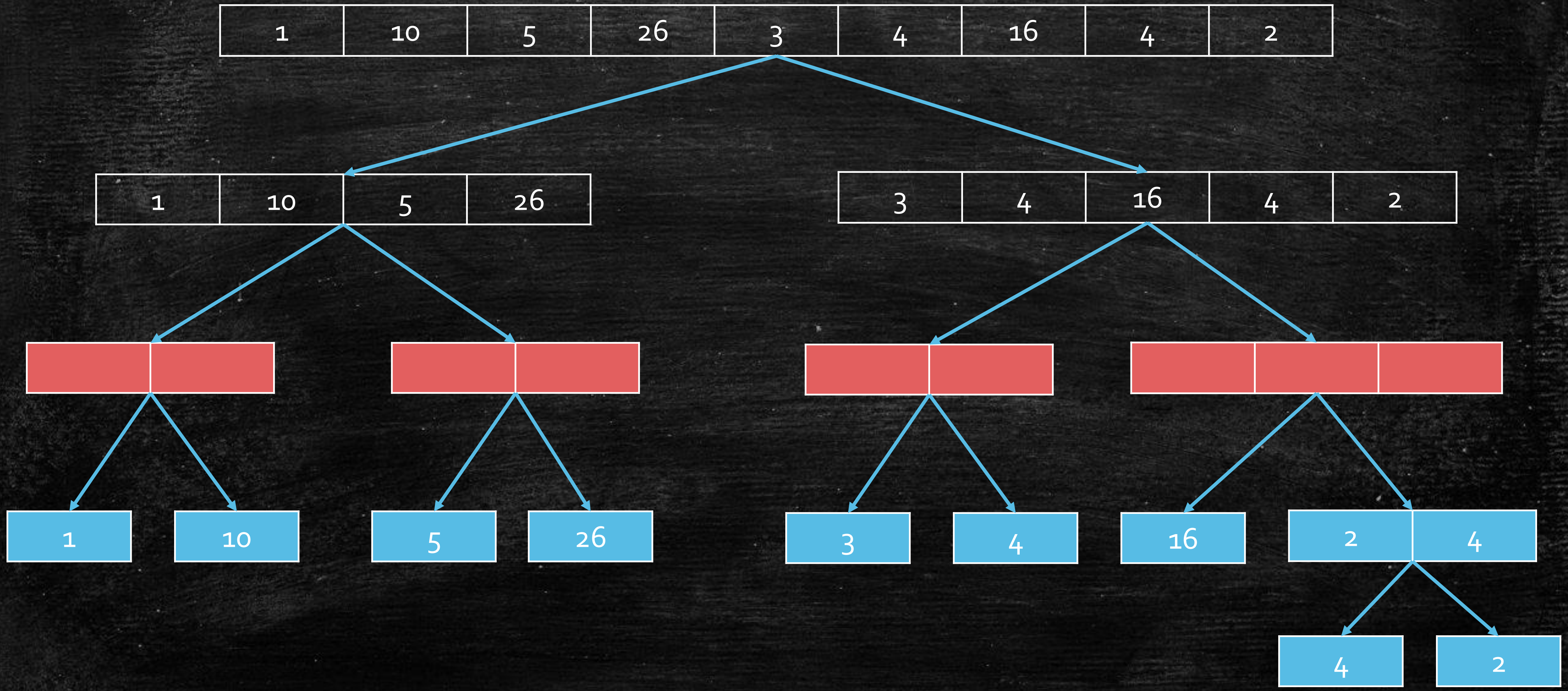
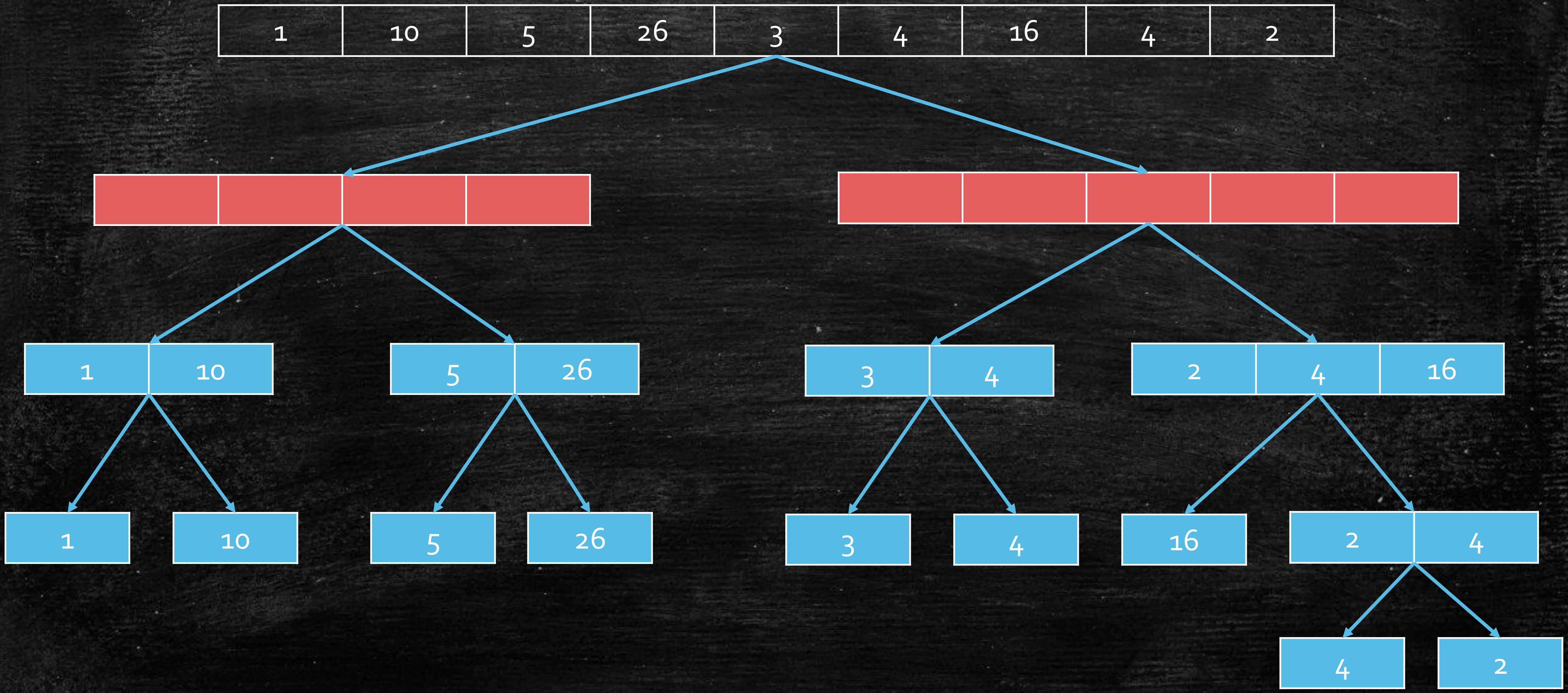
16
2 4
4
2

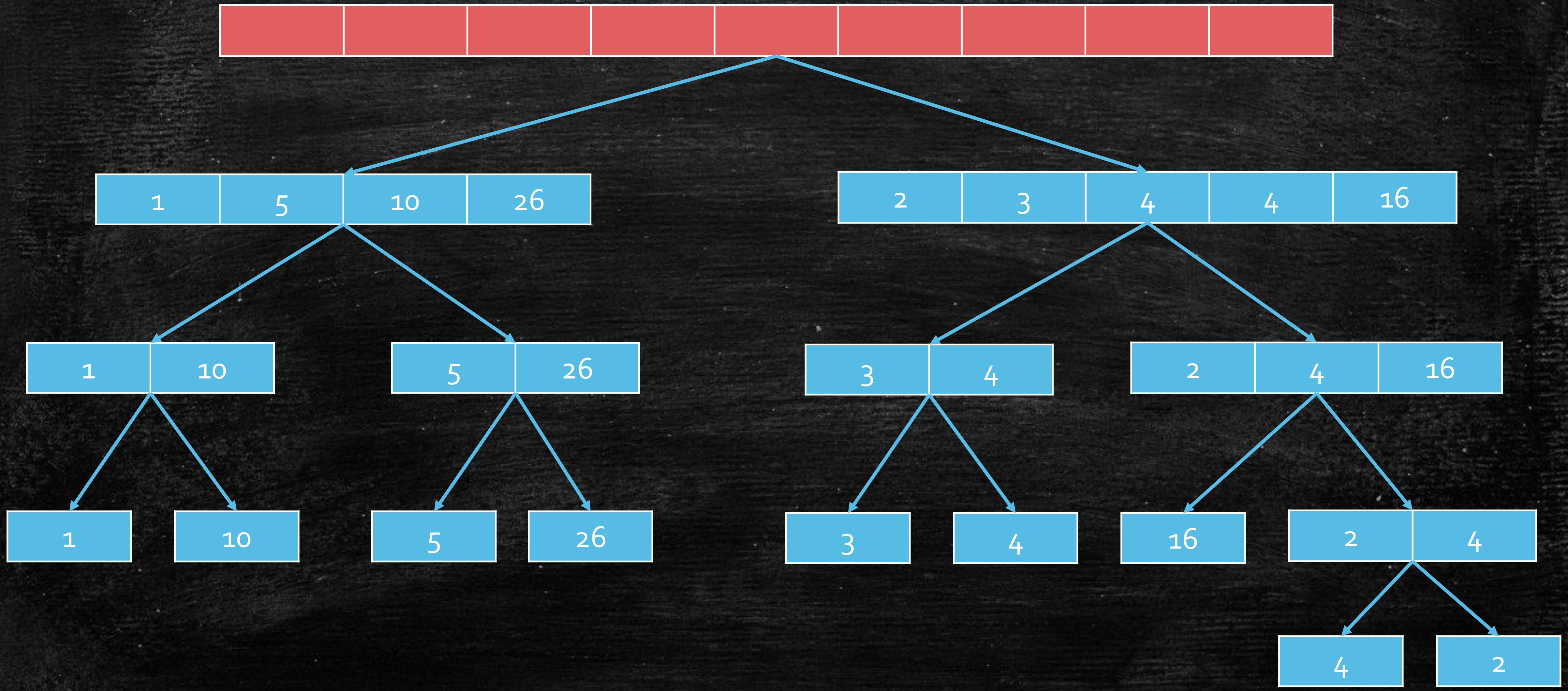
Merge Sort



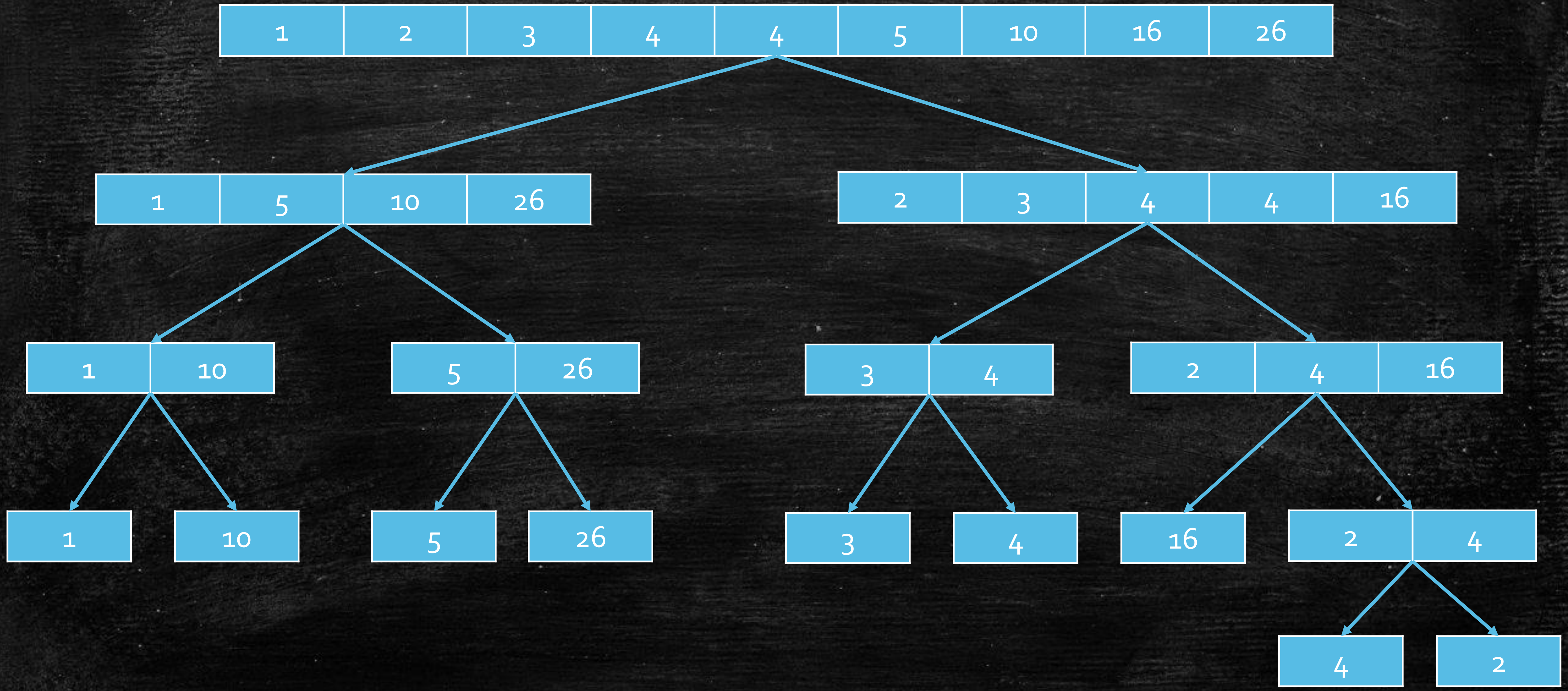
Merge Sort



Merge Sort



Merge Sort



1 2 3 4 4 5 10 16 26
1 5 10 26
2 3 4 4 16
1 10
5 26
3 4
2 4 16
1
10
5
26
3
4
16
2 4
4
2

What is the next?

The remaining questions

- How to merge two sorted lists?
- How fast we can make it?

Merge two sorted lists

- **Input:** two sorted lists $A = a_1, a_2, \dots, a_n$, $B = b_1, b_2, \dots, b_m$
- **Output:** a sorted list C
- Straightforward Plan
  - Insert each b into A iteratively.
  - Time:
    - m integer to insert.
    - n times for each insertion.
  - Totally: $O(mn)$
  - Merge Sort:
    - $T(n) = 2T\left(\frac{n}{2}\right) + O(n^2/4)$

Do we really need so many comparisons?

Merge two sorted lists

- **Input:** two sorted lists $A = a_1, a_2, \dots, a_n$, $B = b_1, b_2, \dots, b_m$
- **Output:** a sorted list C
- Smarter Plan
  - Maintain 2 pointers $i = 1, j = 1$
  - Repeat
    - Append $\min\{a_i, b_j\}$ to C
    - If a_i is smaller, then move i to $i + 1$; If b_j is smaller, then move j to $j + 1$.
    - Break if $i > n$ or $j > m$
  - Append the reminder of the non-empty list to C

Example

1	5	10	26

2	3	4	4	16

- Plan
  - Maintain 2 pointers $i = 1, j = 1$
  - Repeat
    - Append $\min\{a_i, b_j\}$ to C
    - If a_i is smaller, then move i to $i + 1$; If b_j is smaller, then move j to $j + 1$.
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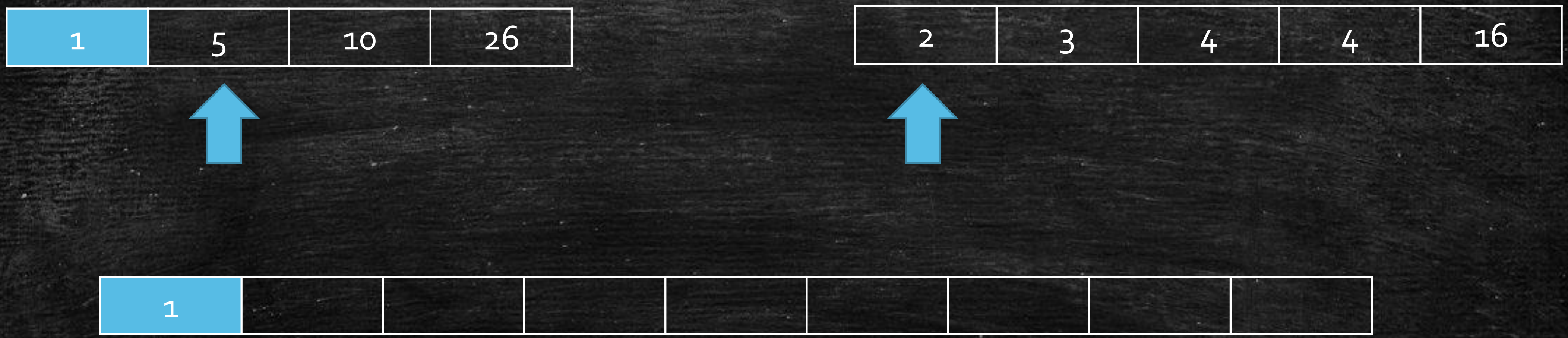

- Plan
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  - Repeat
    - Append $\min\{a_i, b_j\}$ to C
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Example

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  - Maintain 2 pointers $i = 1, j = 1$
  - Repeat
    - Append $\min\{a_i, b_j\}$ to C
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    - Break if $i > n$ or $j > m$
  - Append the reminder of the non-empty list to C

Example

1	5	10	26

2	3	4	4	16

1	2	3						

- Plan
  - Maintain 2 pointers $i = 1, j = 1$
  - Repeat
    - Append $\min\{a_i, b_j\}$ to C
    - If a_i is smaller, then move i to $i + 1$; If b_j is smaller, then move j to $j + 1$.
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Example

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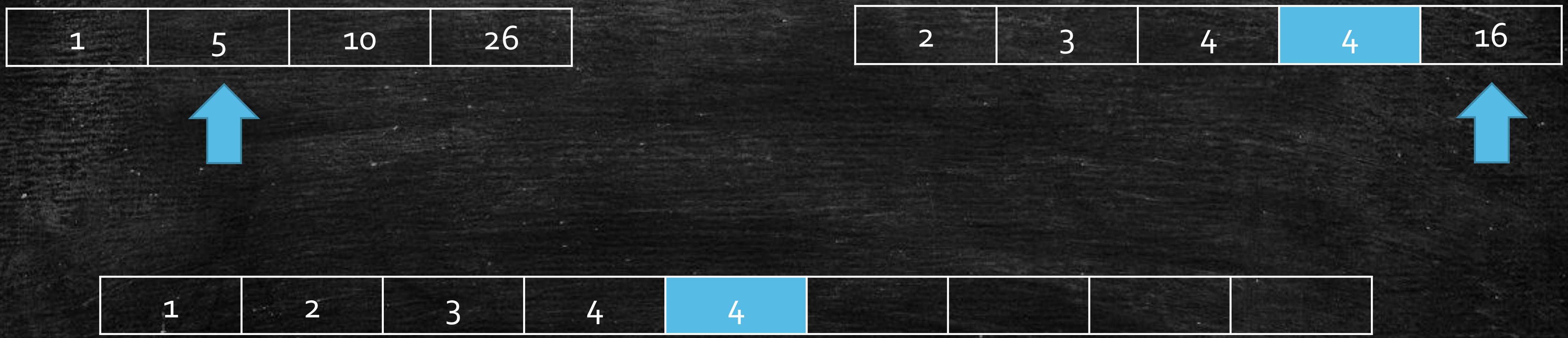

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  - Append the reminder of the non-empty list to C

Example

1	5	10	26

2	3	4	4	16

1	2	3	4	4	5			

- Plan
  - Maintain 2 pointers $i = 1, j = 1$
  - Repeat
    - Append $\min\{a_i, b_j\}$ to C
    - If a_i is smaller, then move i to $i + 1$; If b_j is smaller, then move j to $j + 1$.
    - Break if $i > n$ or $j > m$
  - Append the reminder of the non-empty list to C.

Example

1	5	10	26

2	3	4	4	16

1	2	3	4	4	5	10		

- Plan
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  - Repeat
    - Append $\min\{a_i, b_j\}$ to C
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Example

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2	3	4	4	16

1	2	3	4	4	5	10	16	

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Example

1	5	10	26

2	3	4	4	16

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- Plan
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  - Repeat
    - Append $\min\{a_i, b_j\}$ to C
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    - Break if $i > n$ or $j > m$
  - Append the reminder of the non-empty list to C.

Is it correct?

Prove by induction.

Is Merge Sort correct?

Prove by induction.

How fast is the algorithm?

- Plan
  - Maintain 2 pointers $i = 1, j = 1$
  - Repeat
    - Append $\min\{a_i, b_j\}$ to C
    - If a_i is smaller, then move i to $i + 1$; If b_j is smaller, then move j to $j + 1$.
    - Break if $i > n$ or $j > m$
  - Append the reminder of the non-empty list to C
- Analysis 1
  - i at most move n times.
  - j at most move m times for one i move.
  - $O(nm)$?

How fast is the algorithm?

- Analysis 1
  - i at most move n times.
  - j at most move m times for one i movement.
  - $O(nm)$?
- Analysis 2
  - How many time it takes to append one number to C?
  - 1 number → 1 comparison!
  - Output $m + n$ numbers → $m + n$ comparisons!
  - A **linear time** algorithm!
  - $\boldsymbol{O(n+m)}$

Charging Argument!

Finally…

- What is the running time of merge sort?
  - Equip the linear time combining into merge sort.
  - Assume merge sort runs $T(n)$ time to sort a size n list.
  - What is $T(n)$?
  - $T(n) = 2T\left(\frac{n}{2}\right) + O(n)$ $\boxed{\frac{n}{2} + \frac{n}{2}}$
- I believe you know it is $O(n \log n)$! But, why?

$$T(n) = T\left(\frac{n}{2}\right) + O(n)$$

$$T(n) = T\left(\frac{n}{2}\right) + O(n^2)$$

$$T(n) = 4T\left(\frac{n}{5}\right) + O(n^3)$$

An efficient tool to calculate these expressions

$$T(n) = 3T\left(\frac{n}{2}\right) + O(n^4)$$

Master Theorem

- Master Theorem
  - If $T(n) = aT\left(\frac{n}{b}\right) + O(n^d)$
  - $T(n) = \begin{cases} O(n^d) & a < b^d \\ O(n^{\log_b a}) & a > b^d \\ O(n^d \log n) & a = b^d \end{cases}$

How to understand it?

Understand the parameters

- Master Theorem
  - If $T(n) = \underbrace{aT\left(\frac{n}{b}\right) + O(n^d)}$

    Divide into a problems

  - $T(n) = \begin{cases} O(n^d) & a < b^d \\ O(n^{\log_b a}) & a > b^d \\ O(n^d \log n) & a = b^d \end{cases}$

Understand the parameters

- Master Theorem
  - If $T(n) = \mathbf{a}T\left(\frac{n}{\mathbf{b}}\right) + O(n^{\mathbf{d}})$

    Divide into a problems

    Subproblem size: n/b

  - $T(n) = \begin{cases} O(n^d) & a < b^d \\ O(n^{\log_b a}) & a > b^d \\ O(n^d \log n) & a = b^d \end{cases}$

Understand the parameters

- Master Theorem
  - If $T(n) = aT\left(\frac{n}{b}\right) + O(n^d)$
  - $T(n) = \begin{cases} O(n^d) & a < b^d \\ O(n^{\log_b a}) & a > b^d \\ O(n^d \log n) & a = b^d \end{cases}$

Combining cost: $O(n^d)$

Divide into a problems

Subproblem size: n/b

Running time of merge sort

- Recall
  - Merge sort divides the problem to **two** $\boldsymbol{n/2}$**-size** problems.
  - Merging two $n/2$-size sorted lists takes $\boldsymbol{O(n)}$ times.
  - $T(n) = 2T\left(\frac{n}{2}\right) + O(n)$
- Use Master Theorem
  - If $T(n) = \boldsymbol{a}T\left(\frac{n}{\boldsymbol{b}}\right) + O\left(n^{\boldsymbol{d}}\right)$, then $T(n) = \begin{cases} O\left(n^d\right) & a < b^d \\ O\left(n^{\log_b a}\right) & a > b^d. \\ O\left(n^d \log n\right) & a = b^d \end{cases}$
  - $a = 2, b = 2, d = 1$
  - $T(n) = O(n \log n)$

Understand the formula & proof

Level 0 n 1 problem

Level 1 $\frac{n}{b}$ $\frac{n}{b}$ $\frac{n}{b}$ a problems

Level 2 $\frac{n}{b^2}$ $\frac{n}{b^2}$ …… $\frac{n}{b^2}$ $\frac{n}{b^2}$ a^2 problems

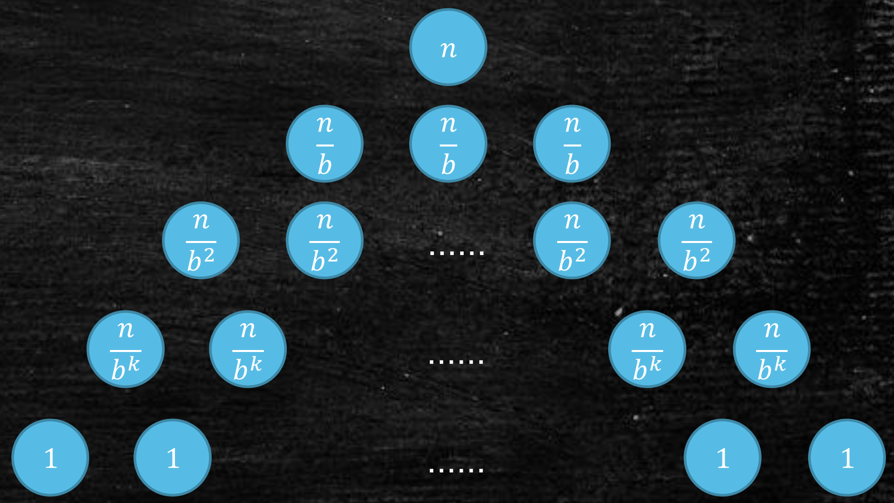
Level k $\frac{n}{b^k}$ $\frac{n}{b^k}$ …… $\frac{n}{b^k}$ $\frac{n}{b^k}$ a^k problems

Level $\log_b n$ 1 1 …… 1 1 $a^{\log_b n}$ problems

Understand the formula & proof

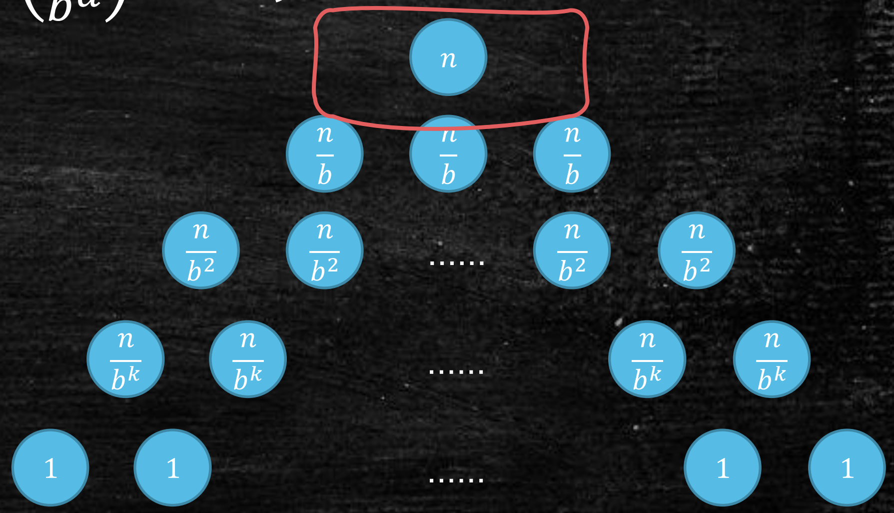
- The running time of solving all size-1 problem is
  - $a^{\log_b n} \cdot O(1) = O(n^{\log_b a})$
- The total running time is
  - $O(n^d) + a \cdot O\left(\left(\frac{n}{b}\right)^d\right) + \cdots + a^k \cdot O\left(\left(\frac{n}{b^k}\right)^d\right) + \cdots + a^{\log_b n} \cdot O(1)$
- Simplification
  - $O(n^d) \cdot (1 + \frac{a}{b^d} + \cdots + \left(\frac{a}{b^d}\right)^k + \cdots + \left(\frac{a}{b^d}\right)^{\log_b n})$

$\boxed{n = b^{\log_b n}}$



Case 1: $a < b^d$

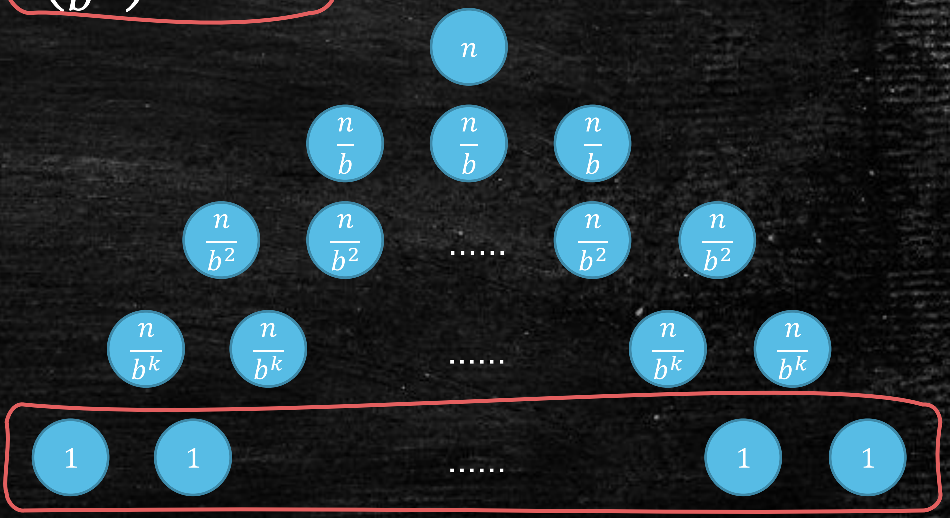
- $T(n) = \begin{cases} O(n^d) & a < b^d \\ O(n^{\log_b a}) & a > b^d \\ O(n^d \log n) & a = b^d \end{cases}$
- $T(n) = O(n^d) \cdot \left(1 + \frac{a}{b^d} + \cdots + \left(\frac{a}{b^d}\right)^k + \cdots + \left(\frac{a}{b^d}\right)^{\log_b n}\right)$
- $a < b^d \rightarrow \frac{a}{b^d} < 1$
- $T(n) = O(n^d)$



Case 2: $a > b^d$

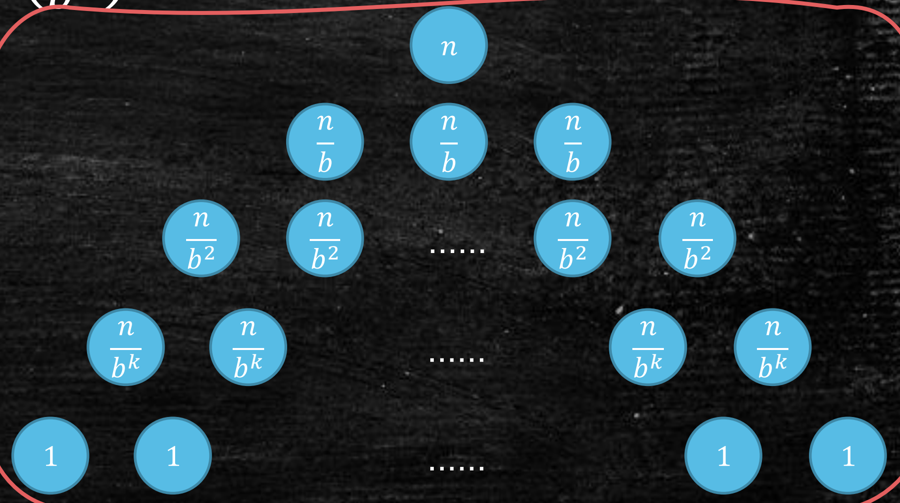
- $T(n) = \begin{cases} O(n^d) & a < b^d \\ O(n^{\log_b a}) & a > b^d \\ O(n^d \log n) & a = b^d \end{cases}$
- $T(n) = O(n^d) \cdot (1 + \frac{a}{b^d} + \cdots + \left(\frac{a}{b^d}\right)^k + \cdots + \boxed{\left(\frac{a}{b^d}\right)^{\log_b n}})$
- $a > b^d \rightarrow \frac{a}{b^d} > 1$
- The last term dominates the sum
- $T(n) = O\left(n^d \left(\frac{a}{b^d}\right)^{\log_b n}\right) = O\left(n^d \frac{a^{\log_b n}}{n^d}\right)$

$= O(a^{\log_b n}) = O(n^{\log_b a})$



Case 3: $a = b^d$

- $T(n) = \begin{cases} O(n^d) & a < b^d \\ O(n^{\log_b a}) & a > b^d \\ O(n^d \log n) & a = b^d \end{cases}$
- $T(n) = O(n^d) \cdot \left(1 + \frac{a}{b^d} + \cdots + \left(\frac{a}{b^d}\right)^k + \cdots + \left(\frac{a}{b^d}\right)^{\log_b n}\right)$
- $a = b^d \rightarrow \frac{a}{b^d} = 1$
- $T(n) = O(n^d \log_b n)$



Divide and Conquer

Counting Inversions

Counting Inversions

- **Input:** A list of n integers
  - $x_1, x_2, x_3, \dots, x_n$
- **Output:** number of **inversions**
- Application
  - You rank n songs.
  - Music site consults database to find people with **similar** tastes.
  - What is **similar**?
    - My rank: $1, 2, 3, 4, 5, \dots, n$
    - Your rank: $x_1, x_2, x_3, \dots, x_n$
    - Songs i, j are **inverted** if $i < j$ but $x_i > x_j$.
    - Similar metric: number of inversions.

Counting Inversions vs Merge Sort

Counting Inversions

- **Input:** A list of n integers
  - $x_1, x_2, x_3, \ldots, x_n$
- **Output:** number of **inversions**
- Plan 1: Brute-force
  - $\boldsymbol{O(n^2)}$

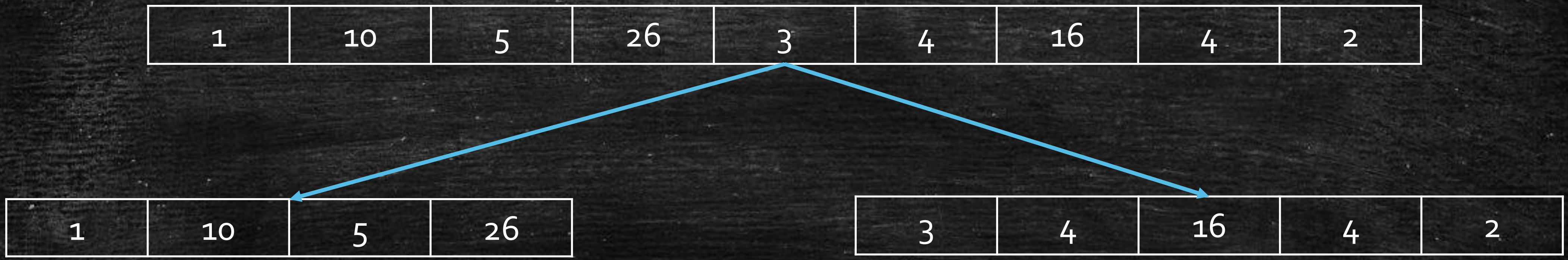
Counting Inversions

- **Input:** A list of n integers
  - $x_1, x_2, x_3, \ldots, x_n$
- **Output:** number of **inversions**
- Plan 2: Divide and Conquer (Merge Sort Style)
  - Divide: Dive the input into two subsets:
    - $x_1, x_2, \ldots, x_{n/2}$; $x_{n/2+1}, x_{n/2+2}, \ldots, x_n$
  - Recurse: count inversions in the two subsets.
    - Let c_1, c_2 be the two numbers.
  - Combine: Return the total number of inversions.
    - Count the inversions across subsets, to be c_3.
    - Return $c_1 + c_2 + c_3$.

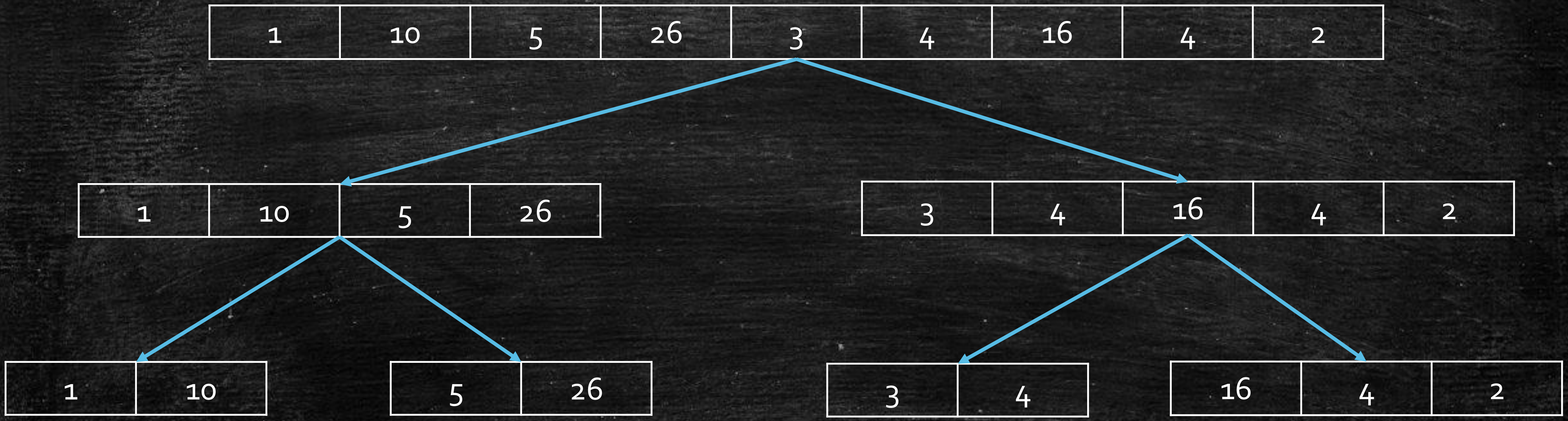
Counting Inversions

1	10	5	26	3	4	16	4	2

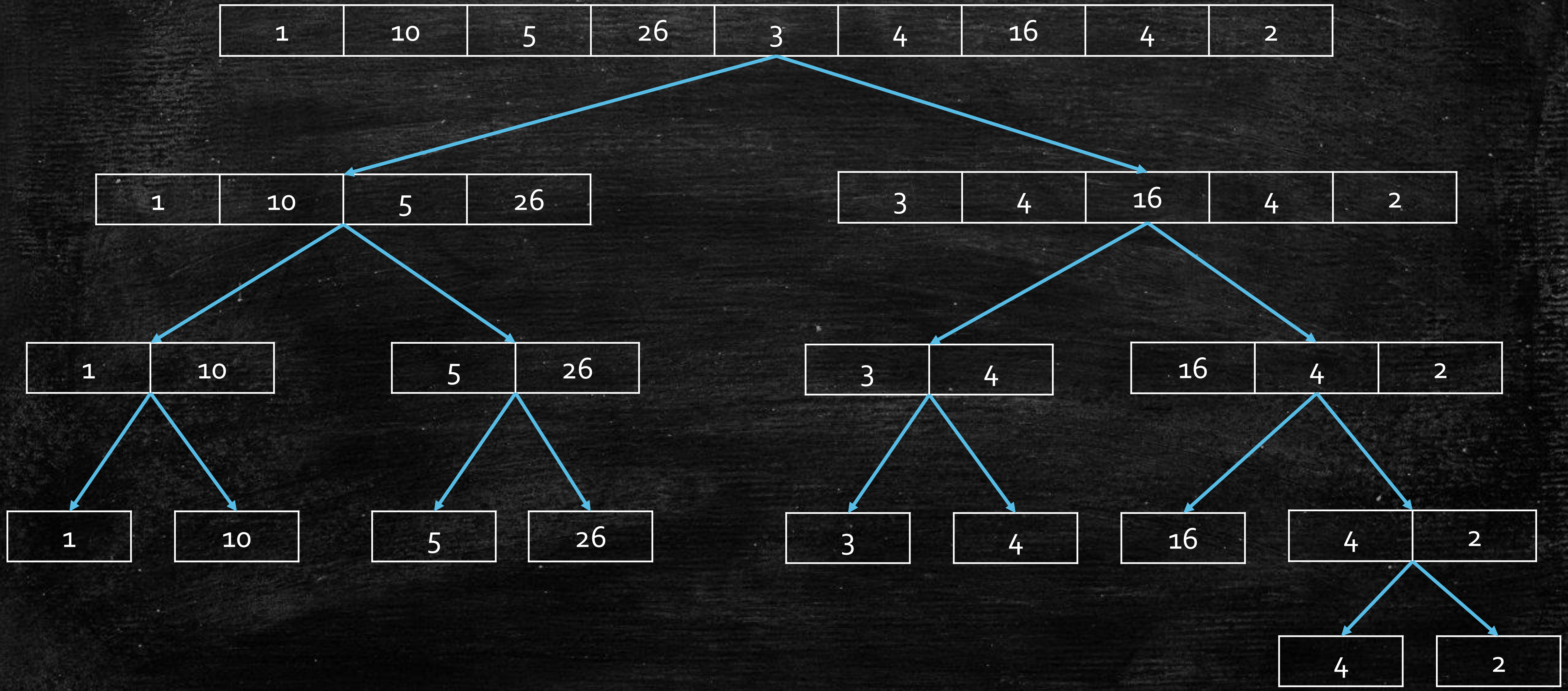
Counting Inversions



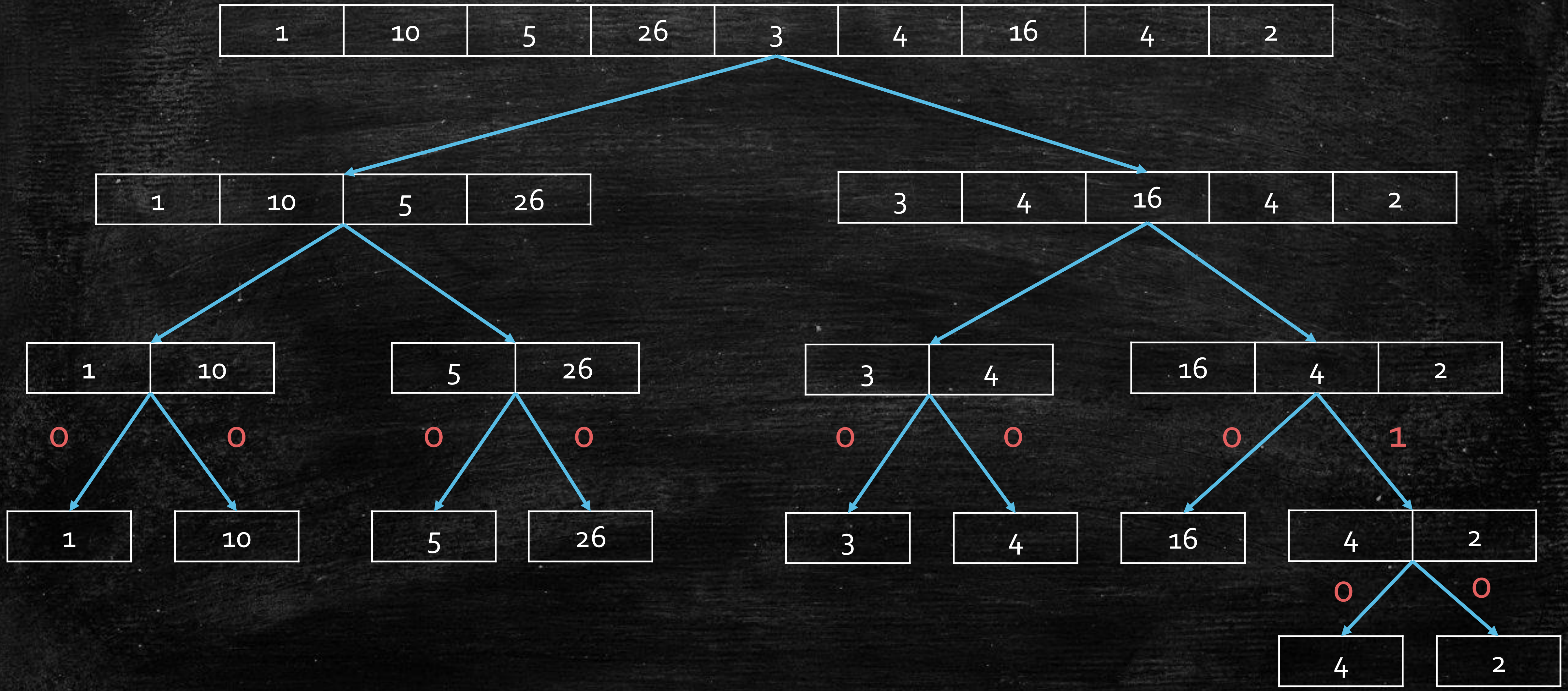
Counting Inversions



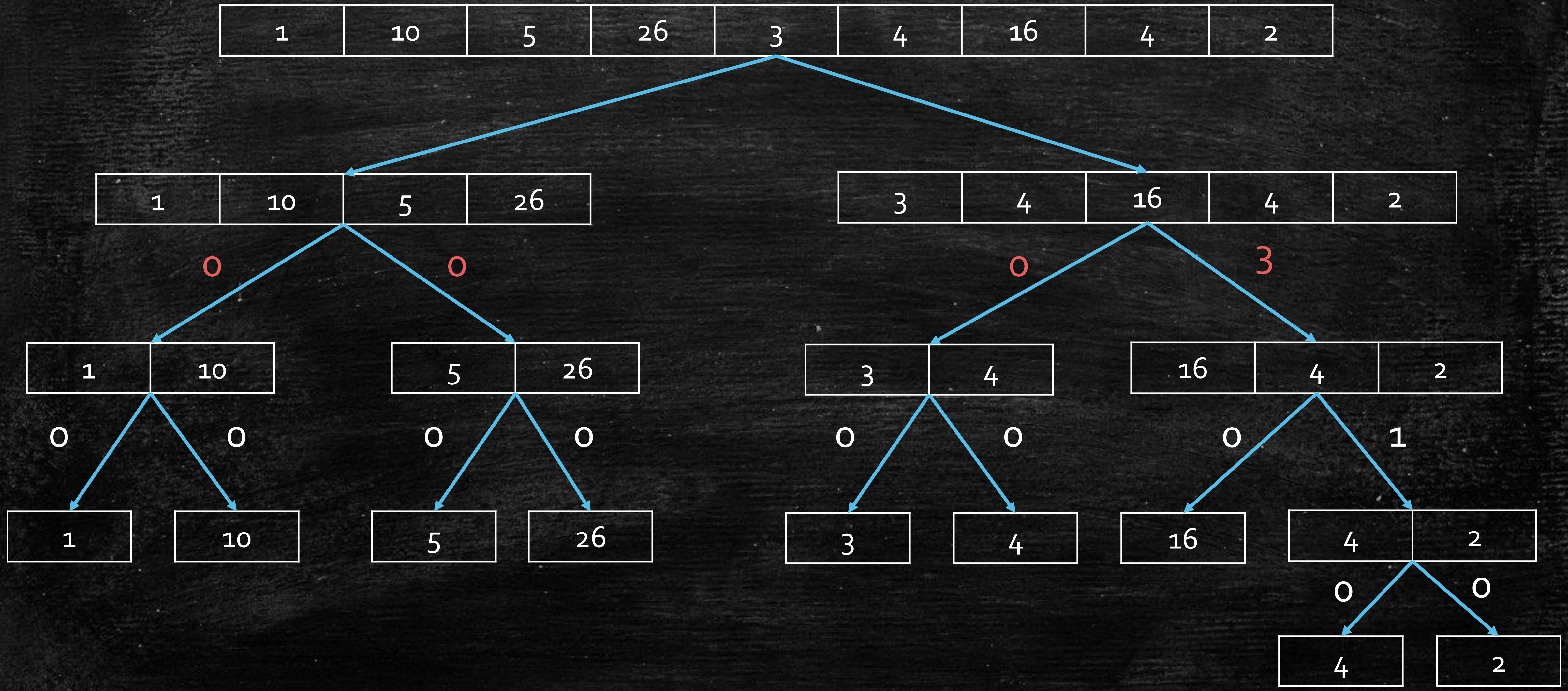
Counting Inversions



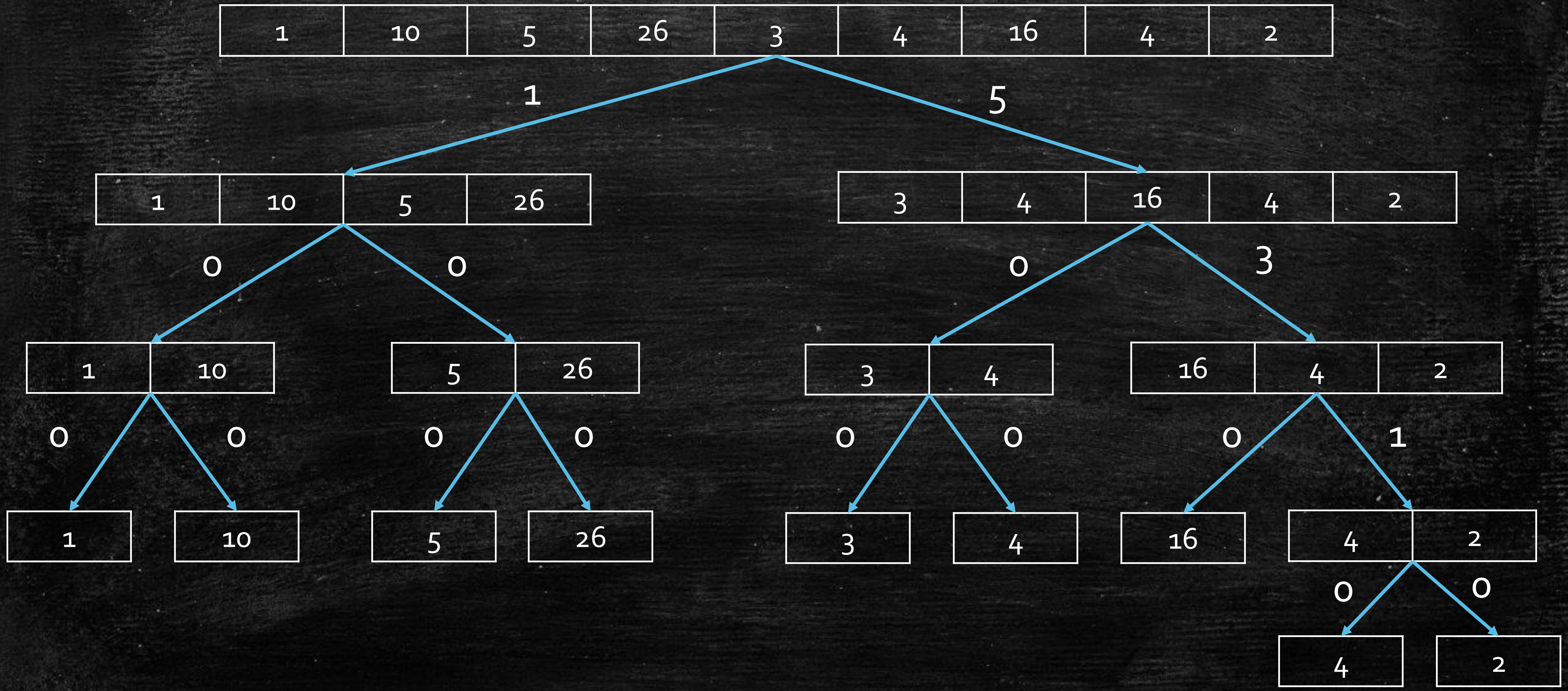
Counting Inversions



Counting Inversions



Counting Inversions 19



Count inversions across two lists

- **Input:** two lists $A = a_1, a_2, \dots, a_n,\ B = b_1, b_2, \dots, b_m$
- **Output:** number of inversions across two lists
- Plan
  - For each a_i in A
    - Count the number of $b_j < \mathrm{a_i}$ in B.
    - Add the number into total number of inversions.
  - Return the total number.

Example

1	10	5	26

3	4	16	4	2

Counter = 0

- Plan
  - For each a_i in A
    - Count the number of $b_j < \mathrm{a_i}$ in B.
    - Add the number into total number of inversions.
  - Return the total number.

Example

1	10	5	26

3	4	16	4	2

Counter = 4

- Plan
  - For each a_i in A
    - Count the number of $b_j < \mathrm{a_i}$ in B.
    - Add the number into total number of inversions.
  - Return the total number.

Example

1	10	5	26

3	4	16	4	2

Counter = 8

- Plan
  - For each a_i in A
    - Count the number of $b_j < \mathrm{a_i}$ in B.
    - Add the number into total number of inversions.
  - Return the total number.

Example

1	10	5	26

3	4	16	4	2

Counter = 13

- Plan
  - For each a_i in A
    - Count the number of $b_j < \mathrm{a_i}$ in B.
    - Add the number into total number of inversions.
  - Return the total number.

Example

1	10	5	26

3	4	16	4	2

Counter = 13

- Plan
  - For each a_i in A
    - Count the number of $b_j < \mathrm{a_i}$ in B.
    - Add the number into total number of inversions.
  - Return the total number.

How long it takes?

- Analysis
  - Each a_i, scan the whole list B.
  - It takes $O(nm)$.
- How to improve?
  - It become easier when the two lists are sorted!
  - Why not do **merging** and **counting** together!

Merge & Count

- Plan
  - Maintain 2 pointers $i = 1, j = 1$, and a counter $c = 0$
  - Repeat
    - Append $\min\{a_i, b_j\}$ to C
    - If a_i is smaller, then move i to $i + 1$; If b_j is smaller, then move j to $j + 1$.
    - If we move i to $i + 1$, then $c = c + j - 1$.
    - Break if $i > n$ or $j > m$
  - Append the reminder of the non-empty list to C
  - If $i \leq n$, $c = c + m \cdot (n - i + 1)$

$a_i > b_{j'}, \forall j' < j$

$a_{i'} > b_{j'}, \forall j' \leq m, \forall i' \geq i$

Example

1	5	10	26

2	3	4	4	16

- Plan
  - Maintain 2 pointers $i = 1, j = 1$, and a counter $c = 0$
  - Repeat
    - Append $\min\{a_i, b_j\}$ to C
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  - If $i \leq n$, $c = c + m \cdot (n - i + 1)$

Example

1	5	10	26

2	3	4	4	16

c+=0 c+=4

1	2	3	4	4	5			

- Plan
  - Maintain 2 pointers $i = 1, j = 1$, and a counter $c = 0$
  - Repeat
    - Append $\min\{a_i, b_j\}$ to C
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  - Maintain 2 pointers $i = 1, j = 1$, and a counter $c = 0$
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Example

1	5	10	26

c+=0 c+=4 c+=4

2	3	4	4	16

1	2	3	4	4	5	10	16	

- Plan
  - Maintain 2 pointers $i = 1, j = 1$, and a counter $c = 0$
  - Repeat
    - Append $\min\{a_i, b_j\}$ to C
    - If a_i is smaller, then move i to $i + 1$; If b_j is smaller, then move j to $j + 1$.
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  - Append the reminder of the non-empty list to C
  - If $i \leq n$, $c = c + m \cdot (n - i + 1)$

Example

1	5	10	26
c+=0	c+=4	c+=4	c+=5

2	3	4	4	16

1	2	3	4	4	5	10	16	26

- Plan
  - Maintain 2 pointers $i = 1, j = 1$, and a counter $c = 0$
  - Repeat
    - Append $\min\{a_i, b_j\}$ to C
    - If a_i is smaller, then move i to $i + 1$; If b_j is smaller, then move j to $j + 1$.
    - If we move i to $i + 1$, then $c = c + j - 1$.
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  - Append the reminder of the non-empty list to C
  - If $i \leq n$, $c = c + m \cdot (n - i + 1)$

Try to prove to yourself the correctness.

How fast is it now?

- The same as Merging two sorted listed.
- Counting Inversion is as fast as Merge Sort.
- $T(n) = 2T\left(\frac{n}{2}\right) + O(n) = O(n \log n)$

Today's goal

- Learn Insertion Sort and Merge Sort
- Learn how to Count Inversions with Merge Sort
- Learn to prove the correctness of them
- Learn to analyze their running time
  - **Charging Argument**
- Learn the Master Theorem