Divide and Conquer

Selection

Selection Problem
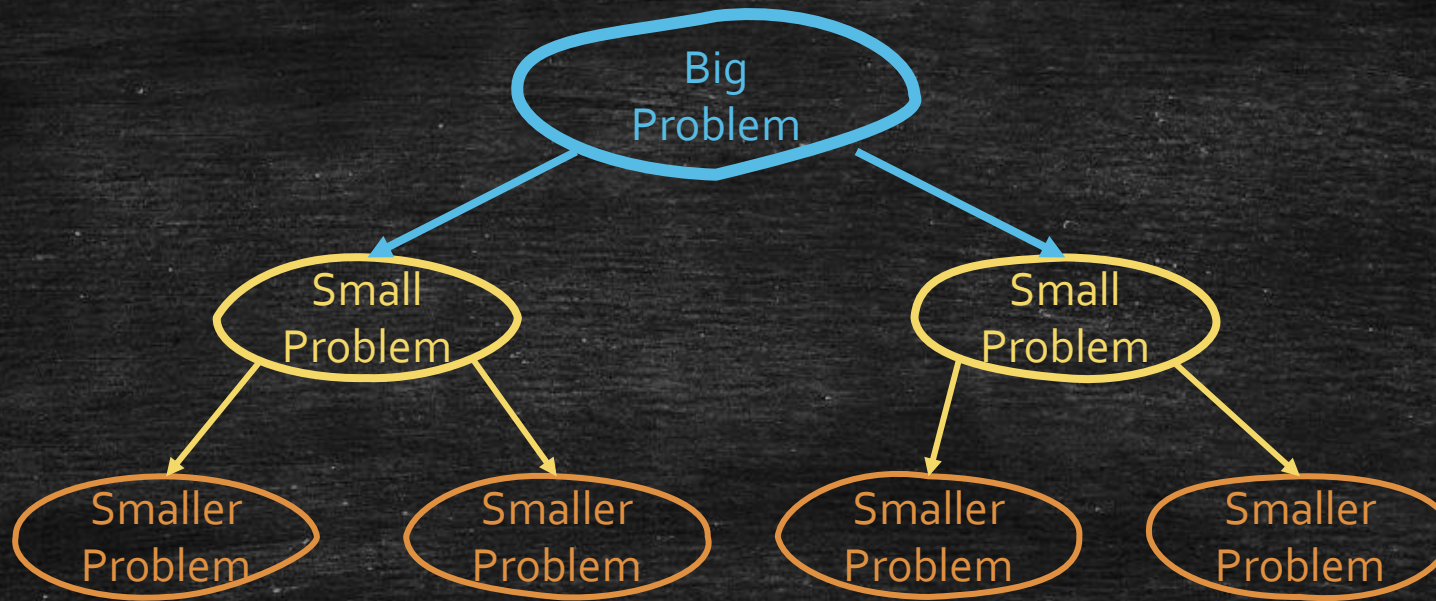
- **Input:** A set S of n integers $x_1, x_2, \ldots, x_n$ and an integer k
- **Output:** The k-th smallest integer x^* among $x_1, x_2, \ldots, x_n$

One-by-one Selection

- **Input:** A set S of n integers $x_1, x_2, \ldots, x_n$ and an integer k

- **Output:** The k-th smallest integer x^* among $x_1, x_2, \ldots, x_n$

- Plan 1
  - Select the smallest integer. $O(n)$
  - Select the 2nd smallest integer. $O(n-1)$
  - ...
  - Select the k-th smallest integer. $O(n-k+1)$

  - Total running time $\boldsymbol{O(nk)}$

Sorting

- **Input:** A set S of n integers $x_1, x_2, \ldots, x_n$ and an integer k

- **Output:** The k-th smallest integer x^* among

- Plan 2
  - $x_1, x_2, \ldots, x_n$ Sort the integers in ascending order. $O(n \log n)$
  - Output the k-th integer. $O(1)$
  - Total running time $O(n \log n)$

Ok! Let's move to divide and conquer!

Divide and Conquer

- **Input:** A set S of n integers $x_1, x_2, \ldots, x_n$ and an integer k

- **Output:** The k-th smallest integer x^* among $x_1, x_2, \ldots, x_n$

- Plan 2: Divide and Conquer
  - Divide:
    - Pick an arbitrary value v among $x_1, x_2, x_3, \ldots$.
    - Divide $x_1, x_2, x_3, \ldots$ into three subsets:
      - L : $x < v$
      - M : $x = v$
      - R : $x > v$
  - Recurse: find x^* in the subset contains x^*.
  - Combine: we already have x^*!

Divide

- Choose $v = 3$.

1	2	5	3	1	3	4	0

- What is L , M, and R?

1	2	5	3	1	3	4	0

- L:

1	2	1	0

- M:

3	3

- R:

5	4

Divide

- L:

1	2	1	0

- M:

3	3

- R:

5	4

Recurse

- Roughly sorted list

Recurse

- How to find x^* in L,M,R?
  - Recall x^* is the k-th smallest integer in S.

- L:

1	2	1	0

  - x^* is the k-th integer in L

- M:

3	3

  - $x^* = 3$

- R:

5	4

  - x^* is the $(k - |\text{L}| - |\text{M}|)$-th integer in R

Example: $k = 4$

1	2	5	3	1	3	4	0

| 1 | 2 | 1 | 0 | | 3 | 3 | | 5 | 4 |

Example: $k = 4$

1	2	5	3	1	3	4	0

1	2	1	0

Example: $k = 4$

| 1 | 2 | 5 | 3 | 1 | 3 | 4 | 0 |

| 1 | 2 | 1 | 0 |

| 0 | | 1 | 1 | | 2 |

Example: $k = 4$

1	2	5	3	1	3	4	0

1	2	1	0

2

Example: $k = 4$

1	2	5	3	1	3	4	0

1	2	1	0

2

2

Output 2

Formalize

Function Select(S,k)

- Divide:
  - Pick an arbitrary value v among $x_1, x_2, x_3, \dots$.
  - Divide $x_1, x_2, x_3, \dots$ into three subsets:
    - L : $x < v$,
    - M : $x = v$,
    - R : $x > v$.

- Recurse:
  - Recurse the subset contains x^*.
    - If $k \leq |L|$, output Select(L,k);
    - If $|L| < k \leq |L| + |M|$, output v;
    - If $|L| + |M| < k$, output Select(R, $k - |L| - |M|$).

Running Time

We want to know $T(n)$

Function Select(S,k)

- **Divide:**
  - Pick an arbitrary value v among $x_1, x_2, x_3, \ldots$.
  - Divide $x_1, x_2, x_3, \ldots$ into three subsets:
    - L : $x < v$,
    - M : $x = v$,
    - R : $x > v$.

  Divide: $O(n)$

- **Recurse:**

  $T(|L|)$

  - Recurse the subset contains x^*.
    - If $k \leq |L|$, output Select(L,k);
    - If $|L| < k \leq |L| + |M|$, output v;   **$O(1)$**
    - If $|L| + |M| < k$, output Select($R, k - |L| - |M|$).   **$T(|R|)$**

Running Time

- $T(n) \leq O(n) + max\{T(|L|), T(|R|)\}$

  $\downarrow$

  $\leq O(n) + T(n-1)$

  $\leq O(n) + \underbrace{O(n-1) + T(n-2)} \leq \cdots$

  $= O(n) + O(n-1) + O(n-2) + \cdots + O(1) = \mathbf{O(n^2)}$

- Very Bad!
  - One-by-one: $O(nk)$
  - Sorting: $O(n \log n)$

Fact

- $|L| + |M| + |R| = |S| = n$

- $|M| \geq 1$

- $|L|, |R| \leq n-1$

Is it really that bad?

- Yes, the unluckiest case:
  - $k = 1$
  - Each time, v is the largest integer.
  - $T(n) = O(n) + T(n-1) = O(n) + O(n-1) + T(n-2) = \cdots = O(n^2)$

- What if we are **super lucky**?
  - Each time, v is in the middle.
  - $T(n) = T\left(\frac{n}{2}\right) + O(n) = T\left(\frac{n}{4}\right) + O\left(\frac{n}{2}\right) + O(n) = \cdots = O(n)$.

- What if we are lucky?
  - Each time, v is in the middle range $[\frac{1}{3}n, \frac{2}{3}n]$.
  - $T(n) = T\left(\frac{2}{3}n\right) + O(n) = T\left(\frac{4}{9}n\right) + O\left(\frac{2}{3}n\right) + O(n) = \cdots = O(n)$.

- Idea: to make us reasonably lucky in average by randomness.

What is the next?

- Improving the running time **with** randomness.

- Improving the running time **without** randomness.

Using Randomness!

Function Select(S,k)

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- **Recurse:**
  - Recurse the subset contains x^*.
    - If $k \leq |L|$, output Select(L,k);
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Using Randomness!

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  - Recurse the subset contains x^*.
    - If $k \leq |L|$, output Select(L, k);
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Quick Selection

Function Select(S,k)

- Divide:
  - Pick a **random** value v among $x_1, x_2, x_3, \ldots$.
  - Divide $x_1, x_2, x_3, \ldots$ into three subsets:
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- Recurse:
  - Recurse the subset contains x^*.
    - If $k \leq |L|$, output Select(L,k);
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When we are lucky
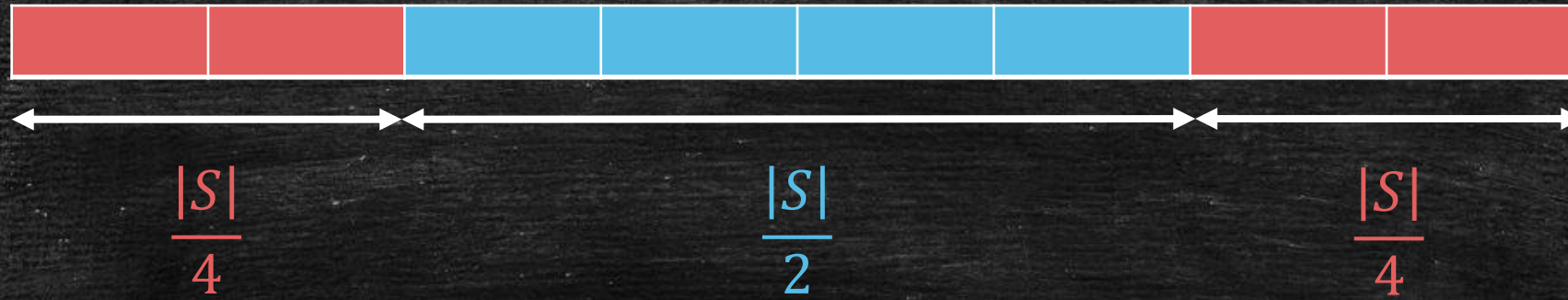
- 🟦 : Lucky pivot area

- 🟥 : Bad pivot area

- Fact 1: With $\frac{1}{2}$ probability, we are lucky!

- Fact 2: If we are always lucky, $T(n) = T\left(\frac{3n}{4}\right) + O(n) = O(n)$

Analysis

- $\tau(n)$: Time we reduce n to $\frac{3n}{4}$

- $T(n) = \tau(n) + T(\frac{3n}{4})$

- $E[\tau(n)]$: The expected time we reduce n to $\frac{3n}{4}$

- $E[T(n)] = E\left[\tau(n) + T\left(\frac{3n}{4}\right)\right]$

  $\qquad = E[\tau(n)] + E\left[T\left(\frac{3n}{4}\right)\right]$

- $E[\tau(n)] = O(n)$

- $E[T(n)] = O(n) + E\left[T\left(\frac{3n}{4}\right)\right] = \boldsymbol{O(n)}$

Fact

Since we are lucky with probably $\frac{1}{2}$, so the expected number of rounds it takes to become lucky is 2.

Evaluate Random Algorithm by Expected Running Time!

Other Viewpoints

- Worst Case Running Time
  - $O(n^2)$

- The Probability it runs in $O(n)$?

What if we do not want randomness?

Throw Randomness!

Function Select(S,k)

- **Divide:**
  - Pick a **random** value v among $x_1, x_2, x_3, \dots$.
  - Divide $x_1, x_2, x_3, \dots$ into three subsets:
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  - Recurse the subset contains x^*.
    - If $k \leq |L|$, output Select(L,k);
    - If $|L| < k \leq |L| + |M|$, output v;
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Throw Randomness!

Function Select(S, k)

- **Divide:**
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  - Divide $x_1, x_2, x_3, \dots$ into three subsets:
    - L : $x < v$,
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    - R : $x > v$.

- **Recurse:**
  - Recurse the subset contains x^*.
    - If $k \leq |L|$, output Select(L, k);
    - If $|L| < k \leq |L| + |M|$, output v;
    - If $|L| + |M| < k$, output Select($R, \text{k} - |\text{L}| - |\text{M}|$).

Median of medians (1973)

Blum, M.; Floyd, R. W.; Pratt, V. R.; Rivest, R. L.; Tarjan, R. E.

Function Select(S,k)

- Divide:
  - Pick a **good pivot** value v among $x_1, x_2, x_3, \ldots$.
  - Divide $x_1, x_2, x_3, \ldots$ into three subsets:
    - L : $x < v$,
    - M : $x = v$,
    - R : $x > v$.

- Recurse:
  - Recurse the subset contains x^*.
    - If $k \leq |L|$, output Select(L,k);
    - If $|L| < k \leq |L| + |M|$, output v;
    - If $|L| + |M| < k$, output Select(R, $\mathrm{k} - |L| - |M|$).

Trade-off

- The time of finding a good pivot.

- The quality of the pivot.

- We can find the median by sorting the array, but it takes too much time.

- We can find an arbitrary pivot in $O(1)$ but it may be too bad.

- Look at the recursive running time:
  - $T(n) = T(c \cdot n) + findPivot + O(n).$

How to select a good pivot?

- Partition S into subsets with size 5.

12	23	53	84	90	32	4	5	32	63	13	14	8	2	42

5   5   5

How to select a good pivot?

- Partition S into subsets with size 5.

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32	4	5	32	63

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How to select a good pivot?

- Partition S into subsets with size 5.

| 12 | 23 | 53 | 84 | 90 |

| 32 | 4 | 5 | 32 | 63 |

| 13 | 14 | 8 | 2 | 42 |

- Find the medians of them: v_1, v_2, v_3
  - 53, 32, 13

How to select a good pivot?

- Partition S into subsets with size 5.

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- Find the medians of them: v_1, v_2, v_3
  - 53, 32, 13

- Fix v to be the median of v_1, v_2, v_3
  - $v = 32$

How long it takes?

- Partition S into subsets with size 5.

| 12 | 23 | 53 | 84 | 90 |

$O(n)$

| 32 | 4 | 5 | 32 | 63 |

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- Find the medians of them: v_1, v_2, v_3
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$O(n)$

- Fix v to be the median of v_1, v_2, v_3
  - $v = 32$

$T(n/5)$

Why it is good?

- It should be in the middle range

- Why? Two questions
  - How many integers should be no greater than v?
  - How many integers should be no less than v?

Answer them step by step

- Partition S into subsets

| 12 | 23 | 53 | 84 | 90 |

| 32 | 4 | 5 | 32 | 63 |

| 13 | 14 | 8 | 2 | 42 |

Smaller than v.

Larger than v.

- Answer
  - We have $\frac{n}{5}$ groups, so $\frac{n}{5}$ medians.
  - v is no greater than $n/10$ medians, no less than $n/10$ medians.
  - Each median is no greater than 2 integers, no less than 2 integers.
  - v is no greater than $\frac{3n}{10}$ integers, no less than $3n/10$ integers.

The running time

Function Select(S,k)

- Divide:
  - Pick a **good pivot** value v among $x_1, x_2, x_3, \dots$.
  $$T\left(\frac{n}{5}\right) + O(n)$$
  - Divide $x_1, x_2, x_3, \dots$ into three subsets:
    - L : $x < v$,
    $$O(n)$$
    - M : $x = v$,
    - R : $x > v$.

- Recurse:
  - Recurse the subset contains x^*.
  $$T(|L|) \leq T\left(n - \frac{3}{10}n\right)$$
    - If $k \leq |L|$, output Select(L,k);
    - If $|L| < k \leq |L| + |M|$, output v;
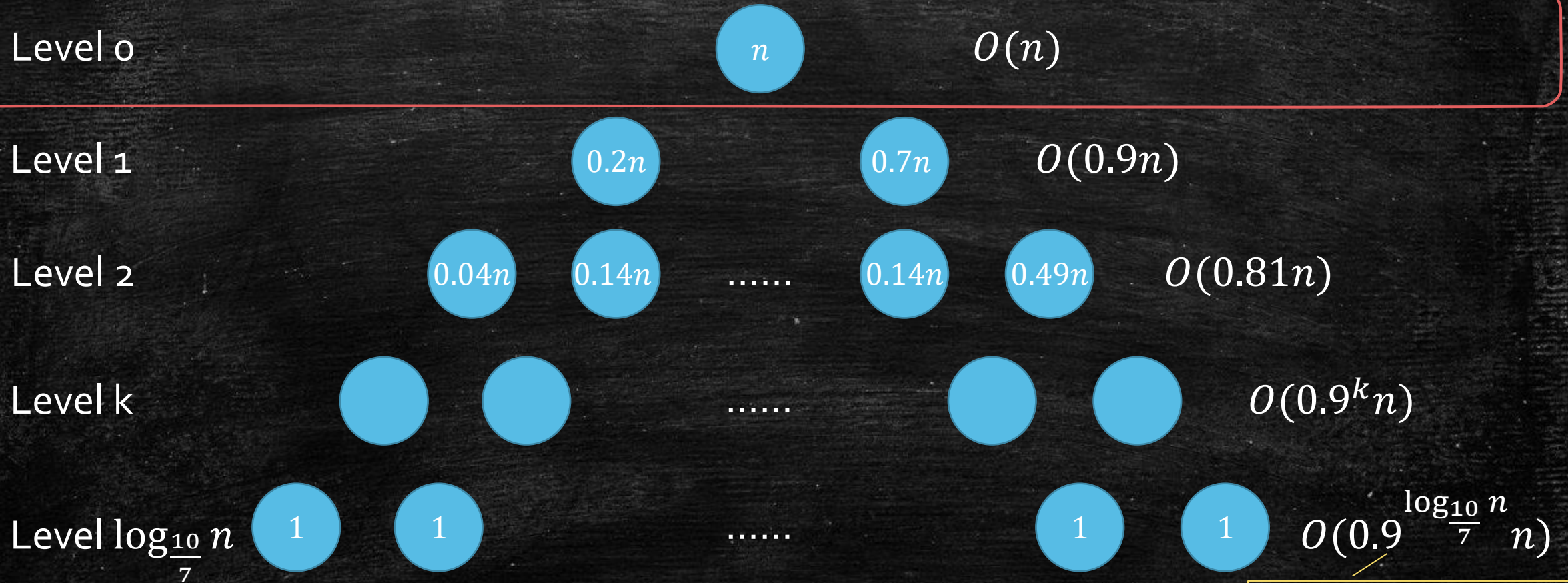   - If $|L| + |M| < k$, output Select(R, $k - |L| - |M|$).
    $$T(|R|) \leq T\left(n - \frac{3}{10}n\right)$$

Guess time!

$$T(n) = T(0.2n) + T(0.7n) + O(n)$$

Observation: Comparing to Master Theorem

Level 0 n $O(n)$

Level 1 $0.2n$ $0.7n$ $O(0.9n)$

Level 2 $0.04n$ $0.14n$ ...... $0.14n$ $0.49n$ $O(0.81n)$

Level k   ...... $O(0.9^k n)$

Level $\log_{\frac{10}{7}} n$ 1 1 ...... 1 1 $O(0.9^{\log_{\frac{10}{7}} n} n)$

We allow some problem with size ≤ 1.

Make a guess

$$T(n) = T(0.2n) + T(0.7n) + O(n)$$

- Guess: $T(n) \leq Bn$!

- Try to prove it inductively
  - Basic step: $T(1) = 1 \leq Bn$
  - Inductive step:
    $$T(n) \leq T(0.2\text{n}) + T(0.7\text{n}) + Cn$$
    $$\leq 0.9Bn + Cn$$
    $$\leq Bn$$

- We have $T(n) \leq 10Cn = O(n)$

Assume
$O(n) \leq Cn$

It holds when
$B \geq 10C$

Remember not to use induction with Big O notations!

One more Question

What if we group them by 2,3,4,5,...?

Today's goal

- Learn the quick selection algorithm
- Learn to make it polynomial by randomness (in expectation) **analytically**
- Learn to make it polynomial by median of medians **analytically**
- Remember to try to group by 2,3,4,5,6...