**Algorithm** **Design** **and** **Analysis** **(Fall** **2022)**

**Final** **Exam** **(A)**

1. (25 points) Given a set of n points x1 , . . . ,xn in R1 , the objective is to use a minimum number of unit intervals (closed intervals with length 1, [t,t + 1]) to cover all the n points. A point x is covered by the interval [t,t+1] if x ∈ [t,t+1]. Design a polynomial time algorithm for deciding the minimum number of unit intervals needed to cover all the n points. Prove the correctness of your algorithm, and analyze its time complexity.

2. (25 points) Given an undirected edge-weighted graph G = (V,E), two vertices s,t ∈ V , and θ ∈ Z+ , design a polynomial time algorithm to decide if G contains a minimum spanning tree such that s and t are connected by the tree edges with weights at most θ . Prove the correctness of your algorithm, and analyze its time complexity. You can assume the edge weights are positive integers.

3. (25 points) Given a ground set U = {1, . . . ,n} and a collection of k subsets A = {A1 , . . . ,Ak }, a system of distinct representatives of A is a “representative”collection T of distinct elements from the sets in A. Specifically, we have |T| = k, and the k distinct elements in T can be ordered as u1 , . . . ,uk such that ui ∈ Ai for each i = 1, . . . ,k . For example, {A1 = {2, 8},A2 = {8},A3 = {4, 5},A4 = {2, 4, 8}} has a system of distinct representatives {2, 4, 5, 8} where 2 ∈ A1 , 4 ∈ A4 , 5 ∈ A3 , 8 ∈ A2 , while {A1 = {2, 8},A2 = {8},A3 = {4, 8},A4 = {2, 4, 8}} does not have a system of distinct representatives.

(a) (10 points) Design a polynomial time algorithm to decide if A has a system of

distinct representatives.

(b) (15 points) Given a ground set U = {1, . . . ,n} and two collections of k subsets A =

{A1 , . . . ,Ak } and B = {B1 , . . . ,Bk }, a common system of distinct representatives is a collection T of k elements that is a system of distinct representatives of both A and B. Design a polynomial time algorithm to decide if A and B have a common system of distinct representatives.

For each part, prove the correctness of your algorithm, and analyze its time complexity.

4. (25 points) Given a directed edge-weighted graph G = (V,E) (where the weights are integers and can be negative), two vertices s and t, and an integer k, the problem is to decide if there is a simple s-t path (an s-t path that does not visit a vertex more than once) with length exactly k .

(a) (10 points) Prove that this problem is NP-complete.

(b) (15 points) Suppose G is known to be a directed acyclic graph. Is this problem in

P or still NP-complete? Prove your answer.