$$\underset{f}{\arg\min} \sum_{i=1}^{n} (y_i - f(x_i))^2 + \lambda \int (f''(t))^2 \, dt \qquad \lambda > 0$$

→ Solution $\hat{f} \in \left\{ \text{natural cubic spline with knots at } x_1 \dots x_n \right\}$

$$\hat{f} \in \text{span}(g_1, g_2, \dots, g_K) \quad \text{for some } K \in \mathbb{N}$$

→ Given a basis, $\hat{f}$ can be computed in closed form

→ We can search among f's of the form

$$f(t) = \sum_{i=1}^{K} g_i(t)\, a_i \,, \qquad a_i \in \mathbb{R},\ 1 \leq i \leq K$$

$$\Rightarrow (f''(t))^2 = \sum_{i=1}^{K} \sum_{j=1}^{K} a_i a_j \, g_i''(t)\, g_j''(t)$$

Call $\underline{G}_i \in \mathbb{R}^K$ $\quad \underline{G}_i = \begin{pmatrix} g_1(x_i) \\ g_2(x_i) \\ g_3(x_i) \\ \vdots \\ g_K(x_i) \end{pmatrix}$

$$\boxed{\underset{\underline{a} \in \mathbb{R}^K}{\arg\min} \sum_{i=1}^{n} (Y_i - \underline{a}^t \underline{G_i})^2 + \lambda \, \underline{a}^t H \underline{a}}$$

$$H \in \mathbb{R}^{K \times K}, \quad H_{ij} = \int g_i''(t)\, g_j''(t)\, dt$$

$$\Rightarrow \boxed{\hat{\underline{a}} = (G^t G + \lambda H)^{-1} G^t \underline{y}}$$

$$G \in \mathbb{R}^{n \times K} \qquad G = \begin{pmatrix} - \underline{G_1}^t - \\ \vdots \\ - \underline{G_n}^t - \end{pmatrix}$$

$\Rightarrow$ it may be worth the effort choosing a basis that makes H easy to compute