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Evaluation of Multidimensional Linear Zeta Functions

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It is shown how to evaluate functions of the form

$$\sum_{m_1,\ldots,m_N=1}^{\infty} (c_1 m_1 + c_2 m_2 + \cdots + c_N m_N)^{-s}$$

in terms of the Riemann zeta-function.

Various mathematical and physical problems require the evaluation of complicated multiple summation ζ-functions, or other closely related functions. For this purpose one can hardly do better than to express the complicated function in terms of more elementary functions (of the same type), defined by simple series whose analytic continuation throughout the complex plane is well understood. A prototypical formula of this type is [1]

$$\sum_{m,n=-\infty}^{\infty}{}' (m^2+n^2)^{-s} = 4\zeta(s)\beta(s), \tag{1}$$

where the prime means that $n=m=0$ is excluded from the sum. Of course, $\zeta(s)$ is the Riemann ζ-function [2], while $\beta(s) = \sum_0^\infty (-)^n (2n+1)^{-s}$ defines, for $\operatorname{Re} s > 0$, a function whose properties throughout the s-plane are well known [3]. Thus Eq. (1) fully reveals the properties of the two-dimensional Epstein ζ-function [4] on the left. A discrete collection of comparable formulae [5] for ζ-functions in $N=2, 4, 6$, and 8 dimensions are known. The proof [5] of Eq. (1) and these other formulae involves theta-function identities which are themselves quite unique. Thus it is not known if Eq. (1) can be extended in any reasonably simple fashion to the sum $\sum_{mn} (am^2+bn^2)^{-s}$ with unequal coefficients $a \neq b$. The same statement applies to all the other discrete results on Epstein ζ-functions in $N=2, 4, 6$, and 8 dimensions, which is what we mean by characterizing these results as "discrete."

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In Ref. [6], parallel results on a much simpler class of multidimensional sums were obtained. Representative of these results is the formula

$$\sum_{m_a=1}^{\infty} (m_1 + \cdots + m_N)^{-s} = \frac{1}{(N-1)!} \sum_{k=0}^{N-1} C_k^N \zeta(s-k), \tag{2}$$

where the constants C_k^N are defined by

$$(m-1)(m-2)\cdots(m-N+1) = \sum_{k=0}^{N-1} C_k^N m^k. \tag{3}$$

Equation (2) continues the function defined by the left-hand side of (2) for $\operatorname{Re} s > N$ throughout the s-plane, just as Eq. (1) continues the Epstein ζ-function defined by the left-hand numerical series for $\operatorname{Re} s > 1$ throughout the s-plane. Unlike the Epstein case, however, Eq. (2) can readily be generalized to unequal coefficients, and in a variety of other ways. Evidently, none of these more general formulae are known, and we wish to present some of them here.

The following theorem will be established.

THEOREM. *The N-dimensional sum*

$$L_N(s \mid c_2, ..., c_N \mid M_1, ..., M_N)$$
$$\equiv \sum_{m_1, ..., m_N = 1}^{\infty} m_1^{M_1} ... m_N^{M_N} [m_1 + c_2 m_2 + \cdots + c_N m_N]^{-s}, \tag{4a}$$

where $\operatorname{Re} s > N + M_1 + \cdots + M_N$, $0 < c_a < 2$, $M_1, M_2, ..., M_N$ *are non-negative integers, can be evaluated explicitly in terms of the Riemann* ζ*-function:*

$$L_N(s \mid c_a \mid M_a) = \sum_{b_2, ..., b_N = 0}^{\infty} \binom{-s}{b_2 \cdots b_N} (c_2 - 1)^{b_2} \cdots (c_N - 1)^{b_N}$$
$$\times \sum_{p=0}^{M_1 + \cdots + M_N + b_2 + \cdots + b_N + N - 2} C_p(M_1, M_2 + b_2, ..., M_N + b_N)$$
$$\times \zeta(s + p + 1 - N - M_1 - \cdots - M_N) \qquad \text{all } s. \tag{4b}$$

Here $\binom{-s}{b_2 ... b_N}$ are the multinomial coefficients, and the constants $C_p(M_1, M_2 + b_2, ...)$ are defined by Eq. (10) below. Note that the Riemann ζ-function $\zeta(s + p + 1 - N - M_1 - \cdots - M_N)$ does not involve the multinomial summation indices $b_2, ..., b_N$ in Eq. (4b), and consequently some further simplification through evaluation or rearrangement of the multinomial summation may be possible. More will be said about this later.

Before we prove the theorem, let us make two comments about it. First, the proof is rather elementary, consisting mainly of judicious use of the binomial or multinomial expansion and Euler–MacLaurin sum rules. Second, the functions $L_N(s|c_a|0)$ with $M_a=0$ have the following physical significance: $L_N(s|c_a|0)$ is the ζ-function associated with (i.e. constructed from the eigenvalues of) the Hamiltonian operator of a system of N noninteracting harmonic oscillators. Some simple properties of these ζ-functions were obtained in Ref. [6]. Much more general results will be derived here.

The only real complications we encounter are of a combinatorial nature. To expose the idea of the proof and to introduce notation, let us consider the case $N=2$. In this case we have

$$\begin{aligned} L_2 &= \sum_{m_{1,2}=1}^{\infty} m_1^{M_1} m_2^{M_2} (m_1+c_2 m_2)^{-s} \\ &= \sum_{m_{1,2}=1}^{\infty} m_1^{M_1} m_2^{M_2} [m_1+m_2+(c_2-1)\,m_2]^{-s} \\ &= \sum_{m_{1,2}=1}^{\infty} m_1^{M_1} m_2^{M_2} (m_1+m_2)^{-s} \\ &\quad \times \sum_{b=0}^{\infty} \binom{-s}{b} \left[\frac{(c_2-1)\,m_2}{m_1+m_2}\right]^b \qquad \operatorname{Re} s > 2+M_1+M_2,\ 0<c_2<2. \end{aligned} \tag{5}$$

Here the binomial expansion has been used and is absolutely convergent for $0<c_2<2$. If we (temporarily) hold $\operatorname{Re} s > 2+M_1+M_2$, then the sum over m_1, m_2 is also absolutely convergent, and hence can be commuted through the binomial summation $\sum_b$, yielding

$$L_2 = \sum_{b=0}^{\infty} \binom{-s}{b} (c_2-1)^b \sum_{m_{1,2}=1}^{\infty} m_1^{M_1} m_2^{M_2+b} (m_1+m_2)^{-s-b}. \tag{6}$$

To evaluate $\sum_{m_1,m_2}$ here we employ the Euler–MacLaurin sum rule [6, 7]

$$\begin{aligned} g_{M_1 M_2}(k) &= \sum_{\substack{m_{1,2}\geqslant 1 \\ (m_1+m_2=k)}} m_1^{M_1} m_2^{M_2} \\ &= \sum_{m_1=1}^{k-1} m_1^{M_1} (k-m_1)^{M_2} \\ &= k^{M_1+M_2+1} \frac{M_1!\,M_2!}{(M_1+M_2+1)!} + \sum_{r=0}^{M_1} \binom{M_1}{r} (-)^{M_1+r} k^r \\ &\quad \times \left\{\frac{1}{2} B_2 \binom{M_1+M_2-r}{1} k^{M_1+M_2-r-1} \right. \end{aligned}$$

$$+\frac{1}{4}B_4\binom{M_1+M_2-r}{3}k^{M_1+M_2-r-3}$$

$$+\frac{1}{6}B_6\binom{M_1+M_2-r}{5}k^{M_1+M_2-r-5}+\dots\Big\}$$

$$\equiv\sum_{p=0}^{M_1+M_2}C_p(M_1,M_2)\,k^{M_1+M_2+1-p}\qquad M_1+M_2>0, \tag{7}$$

where the curly bracket terminates at the smallest positive power of k (either k or k^2). $g_{M_1M_2}(k)$ is a polynomial in k of degree M_1+M_2+1, as indicated in the final equality. Inserting the final equality in Eq. (6) we have

$$\begin{aligned}L_2(s|c_2|M_1,M_2)&=\sum_{b=0}^{\infty}\binom{-s}{b}(c_2-1)^b\sum_{k=2}^{\infty}k^{-s-b}g_{M_1,M_2+b}(k)\\&=\sum_{b=0}^{\infty}\binom{-s}{b}(c_2-1)^b\sum_{p=0}^{M_1+M_2+b}C_p(M_1,M_2+b)\\&\quad\times\zeta(s+p-1-M_1-M_2).\end{aligned}\tag{8}$$

Note that in the final equality, to evaluate the sum over k as the Riemann ζ-function $\zeta(s+p-1-M_1-M_2)$, this sum must be extended to include $k=1$ (because $k=m_1+m_2$ initially runs from 2 to ∞). The extension to $k=1$ is trivial and changes nothing because $g_{M_1M_2}$ $(k=1)=0$, as can be verified from Eq. (7). Thus the proof of the Theorem for $N=2$ is complete.

It is quite obvious how to generalize this calculation to arbitrary N. Equations (5), (6) are replaced by

$$\begin{aligned}L_N&=\sum_{m_a=1}^{\infty}m_1^{M_1}\cdots m_N^{M_N}(m_1+c_2m_2+\cdots+c_Nm_N)^{-s}\\&=\sum_{m_a=1}^{\infty}m_1^{M_1}\cdots m_N^{M_N}[m_1+\cdots+m_N+(c_2-1)\,m_2+\cdots+(c_N-1)m_N]^{-s}\\&=\sum_{b_2,\dots,b_N=0}^{\infty}\binom{-s}{b_2\cdots b_N}(c_2-1)^{b_2}\cdots(c_N-1)^{b_N}\\&\quad\times\sum_{m_a=1}^{\infty}\frac{m_1^{M_1}m_2^{M_2+b_2}\cdots m_N^{M_N+b_N}}{(m_1+\cdots+m_N)^{s+b_2+\cdots+b_N}}\\&\qquad\mathrm{Re}\,s>N+M_1+\cdots+M_N,\qquad 0<c_a<2.\end{aligned}\tag{9}$$

Here the multinomial expansion is used, and is absolutely convergent for $0<c_a<2$. Moreover, the sum over $m_1,\dots,m_N$ is absolutely convergent for $\mathrm{Re}\,s>N+M_1+\dots$, and has been commuted through the multinomial summation. It remains to evaluate the sum over $m_1, ..., m_N$. For this we need the generalization of the Euler–MacLaurin sum rule (7) to arbitrary N:

$$g_{M_1 \cdots M_N}(k) \equiv \sum_{\substack{m_1, ..., m_N \geqslant 1 \\ (m_1 + \cdots + m_N = k)}} m_1^{M_1} \cdots m_N^{M_N}$$

$$= \sum_{p=0}^{M_1 + \cdots + M_N + N - 2} C_p(M_1, ..., M_N)\, k^{M_1 + \cdots + M_N + N - 1 - p}. \quad (10)$$

This is a polynomial in k of degree $M_1 + \cdots + M_N + N - 1$, as the final line indicates. The coefficients $C_p(M_1, ..., M_N)$ here can be obtained by repeated application of Eq. (7). For example,

$$g_{M_1 M_2 M_3}(k) = \sum_{\substack{m_{1,2,3} \geqslant 1 \\ (m_1 + m_2 + m_3 = k)}} m_1^{M_1} m_2^{M_2} m_3^{M_3}$$

$$= \sum_{\substack{m_3 \geqslant 1 \\ n \geqslant 2 \\ (n + m_3 = k)}} m_3^{M_3} \sum_{m_1 = 1}^{n-1} m_1^{M_1} (n - m_1)^{M_2}$$

$$= \sum_{m_3 = 1}^{k-1} m_3^{M_3} g_{M_1 M_2}(k - m_3)$$

$$= \sum_{p_1 = 0}^{M_1 + M_2} C_{p_1}(M_1, M_2)$$

$$\times \sum_{p_2 = 0}^{M_1 + M_2 + M_3 + 1 - p_1} C_{p_2}(M_3, M_1 + M_2 + 1 - p_1)$$

$$\times k^{M_1 + M_2 + M_3 + 2 - p_1 - p_2}. \quad (11)$$

Equation (10) in Eq. (9) immediately yields the theorem (4b). One detail which must be mentioned is that $k = m_1 + \cdots + m_N \geqslant N$ in Eq. (9), but k must be summed from 1 to ∞ to yield the Riemann ζ-function $\zeta(s + p + 1 - N - M_1 - \cdots - M_N)$ in Eq. (4b). The terms $k = 1, 2, ..., N - 1$ are trivially added and change nothing because

$$g_{M_1 \ldots M_N}(k) = 0, \qquad k = 1, 2, ..., N - 1. \quad (12)$$

This completes the proof of the Theorem. In the remainder of this paper we discuss modifications of the theorem, which could be formulated as additional theorems, but which for brevity we merely illustrate by means of examples.

Comment 1. There is a trivial way to extend the range of the parameters c_a in the theorem to values outside the interval $(0, 2)$. Consider $N=2$ again, and note that L_2 in Eq. (5) has the property

$$L_2(s \mid c_2 \mid M_1, M_2) = c_2^{-s} L_2(s \mid 1/c_2 \mid M_2, M_1). \tag{13}$$

Equation (8) evaluates the right-hand side for $\frac{1}{2} < c_2 < \infty$.
Similarly, for $N=3$,

$$\begin{aligned} &L_3(s \mid c_2, c_3 \mid M_1, M_2, M_3) \\ &\qquad = c_2^{-s} L_3(s \mid 1/c_2, c_3/c_2 \mid M_2, M_1, M_3), \end{aligned} \tag{14}$$

which displays the pattern for general N.

Comment 2. Consider the case $N=2$ with $c_2=2$ and $M_{1,2}=0$. One finds with some elementary rearrangements

$$\begin{aligned} \sum_{m,n=1}^{\infty} (m+2n)^{-s} &= \sum_{r=1}^{\infty} \frac{1}{r^s} N(r=m+2n) \\ &= \left[\frac{1}{3^s}+\frac{1}{4^s}\right] + 2\left[\frac{1}{5^s}+\frac{1}{6^s}\right] + 3\left[\frac{1}{7^s}+\frac{1}{8^s}\right] + \cdots \\ &= \frac{1}{2}\left\{1+\frac{2}{2^s}+\frac{3}{3^s}+\frac{4}{4^s}+\cdots\right\} \\ &\quad -\frac{1}{2}(1+2^{-s})\left\{1+\frac{1}{2^s}+\frac{1}{3^s}+\frac{1}{4^s}+\cdots\right\} \\ &= \frac{1}{2}\zeta(s-1) - \frac{1}{2}(1+2^{-s})\,\zeta(s). \end{aligned} \tag{15}$$

Note that $N(r=m+2n)$ is the number of ways the integer r can be given as the sum of positive integers m and $2n$. There are many other ways to derive Eq. (15). Combining Eqs. (15), (13), and (7) (or (19) below) one obtains the nontrivial looking identity

$$\left[\frac{2}{s-1}(2^{s-1}-1)-2^{s-1}\right]\zeta(s-1)$$

$$=\sum_{p=1}^{\infty}\binom{-s}{p}(-)^p(2^s-2^{-p})\,\zeta(-p)\zeta(s+p). \qquad (16)$$

This is not an obvious identity, and consequently, by working only with Eqs. (8) and (13), one would not be led to the simple result (15). One must conclude that, at least for integral values of the parameters $c_2, ..., c_N$ in the general function $L_N(s\,|\,c_2, ..., c_N\,|\,M_1, ..., M_N)$, there are simple special case formulae waiting to be found. Returning again to the $N=2$ example, we quote the following result for $c_2=\text{integer}=d>2$;

$$\sum_{m,n=1}^{\infty}(m+dn)^{-s}$$

$$=\sum_{r=1}^{\infty}\frac{1}{r^s}N(r=m+dn)$$

$$=\left[\frac{1}{(d+1)^s}+\cdots+\frac{1}{(2d)^s}\right]+2\left[\frac{1}{(2d+1)^s}+\cdots+\frac{1}{(3d)^s}\right]+\cdots$$

$$=\frac{1}{d}\zeta(s-1)-\frac{1}{d}\left[1+\frac{d-1}{d^s}\right]\zeta(s)$$

$$-\frac{1}{d^{s+1}}\left\{\zeta\left(s,\frac{2}{d}\right)+2\zeta\left(s,\frac{3}{d}\right)+\cdots+(d-2)\zeta\left(s,\frac{d-1}{d}\right)\right\}. \qquad (17)$$

Here $\zeta(s,a)$ is the well-known Hurwitz ζ-function [3] $\zeta(s,a)=\sum_0^{\infty}(n+a)^{-s}$.

Comment 3. The ζ-function $\zeta(s+p-1-M_1-M_2)$ in Eq. (8) does not depend on the binomial summation index b. Thus one could hope to evaluate this sum in closed form and obtain a simpler expression for L_2. The same remark applies to the more general formula (4b). Here we give an example in which this will be accomplished: namely the calculation of L_2 with $M_{1,2}=0$. This calculation begins with the formula [6]

$$\sum_{m_{1,2}=1}^{\infty} \frac{m_2^b}{(m_1+m_2)^{s+b}} = \sum_{k=1}^{\infty} \frac{1}{k^{s+b}} \sum_{m_2=1}^{k-1} m_2^b$$
$$= \frac{1}{b+1}\zeta(s-1) - \frac{1}{2}(1+\delta_{b0})\zeta(s)$$
$$- \sum_{p=1}^{b-1} \binom{b}{p} \zeta(-p)\,\zeta(s+p), \tag{18}$$

where $\zeta(-2n)=0$, $\zeta(1-2n)=-B_{2n}/2n$ for $n=1, 2, \ldots$, and B_{2n} are the Bernoulli numbers. This formula in Eq. (6) gives

$$\sum_{m_{1,2}=1}^{\infty} (m_1+c_2m_2)^{-s} \qquad 0<c_2<2 \qquad \text{all } s$$
$$=(c_2-1)^{-1}(s-1)^{-1}(1-c_2^{1-s})\,\zeta(s-1)-\tfrac{1}{2}(1+c_2^{-s})\,\zeta(s)$$
$$-\sum_{p=1}^{\infty}\binom{-s}{p}(c_2-1)^p\,(c_2^{-s-p}-1)\zeta(-p)\zeta(s+p). \tag{19}$$

Here the sum over b has been evaluated with the help of the binomial expansion.

Comment 4. All of the preceeding results have counterparts with one or more alternating signs under the sum. All one needs to do is modify the Euler–MacLaurin sum rules to incorporate this alternating sign. A simple way to do this was given in Refs. [6, 7]. For example,

$$\sum_{m=1}^{k-1} (-)^{m+1} m^a = (-)^k k^a + \eta(-a) + (-)^{k+1}$$
$$\times \sum_{r=0}^{a} \binom{a}{r} k^{a-r} \eta(-r), \tag{20}$$

where $\eta(s)$ is the "Riemann η-function"

$$\eta(s) \equiv \sum_{m=1}^{\infty} (-)^{m+1} \frac{1}{m^s} \qquad \text{Re}\, s>0$$
$$=(1-2^{1-s})\,\zeta(s) \qquad \text{all } s.$$

There are three ways to put alternating signs into Eq. (19). For $0<c_2<2$,

$$\sum_{m,n=1}^{\infty} (-)^{n+1} (m+c_2 n)^{-s}$$
$$= \sum_{a=0}^{\infty} \binom{-s}{a} (c_2-1)^a$$
$$\times \left\{ \eta(-a)\zeta(s+a) - \eta(s) + \sum_{r=0}^{a} \binom{a}{r} \eta(-r)\eta(s+r) \right\}, \tag{21}$$

$$\sum_{m,n=1}^{\infty} (-)^{m+1} (m+c_2 n)^{-s}$$
$$= \sum_{a=0}^{\infty} \binom{-s}{a} (c_2-1)^a$$
$$\times \left\{ -\eta(-a)\eta(s+a) + \zeta(s) - \sum_{r=0}^{a} \binom{a}{r} \eta(-r)\zeta(s+r) \right\}, \tag{22}$$

$$\sum_{m,n=1}^{\infty} (-)^{m+n} (m+c_2 n)^{-s}$$
$$= \sum_{a=0}^{\infty} \binom{-s}{a} (c_2-1)^a$$
$$\times \left\{ -\frac{1}{a+1} \eta(s-1) + \frac{1}{2}(1+\delta_{a0})\,\eta(s) + \sum_{n=1}^{a-1} \binom{a}{n} \zeta(-n)\eta(s+n) \right\}, \tag{23}$$

where the sum over a can be evaluated as in Eq. (19) with the help of the binomial theorem.

Comment 5. A final remark concerns the numbers $M_1, \ldots, M_N$ in Eq. (4) and in many other formulae in this paper. These numbers are only allowed to have integral values. The reason for this is that for integral M_a the Euler–MacLaurin sum rules (7), (10), (11), ... yield polynomials in k, and hence lead to a manageable calculation. When the M_a are nonintegral, the Euler–MacLaurin summation yields an infinite series in k which is generally an asymptotic series.

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