The Euler-MacLaurin Summation Formula, the Sampling Theorem, and Approximate Integration over the Real Axis

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Dedicated to Professor Alexander Ostrowski on the occasion of his 90th Birthday on 25 September 1983, in admiration and friendship.

Submitted by Walter Gautschi

ABSTRACT

The Euler-MacLaurin summation formula is used to deduce the Whittaker-Shannon sampling theorem for not necessarily band-limited functions, as well as to study numerical integration over the real axis. Concerning the latter, error estimates are determined in case the function to be integrated is smooth but not necessarily analytic. Two characteristic examples are given.

1. INTRODUCTION

One of the results that attracted Professor Ostrowski's particular attention in the many diverse fields in which he worked is the Euler-MacLaurin summation formula. Professor Ostrowski devoted three important papers [24–26] to this formula, dating back to 1969. The purpose of this note is to show that the Euler formula may be used to deduce the well-known Shannon sampling theorem as well as to study the approximation of $\int_{-\infty}^{\infty} f(x)\,dx$ by the Riemann sums $(1/W)\sum_{k=-\infty}^{\infty} f(k/W)$ for $W \to \infty$. As a matter of fact, the Euler formula will be applied to establish a basic lemma from which both the Shannon sampling theorem for not necessarily band-limited functions and the approximate integration result will follow.

The Shannon sampling series expansion, perhaps better known among mathematicians as Whittaker's cardinal series, has been studied by many different methods so far. There are methods based upon Fourier series expansions [3, 7], Parseval's formula [4, 29, 32], and the Poisson summation

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formula [1, 5], as well as upon Cauchy's residue formula [34, p. 114]. The Euler-formula approach (using Fourier-analytic tools) will add another to this list.

This note will not be concerned with quadrature formulae for $\int_a^b f(x)\,d\alpha(x)$ with $d\alpha(x)$ a finite Stieltjes measure on the finite or infinite interval $[a, b]$—a field that has been receiving considerable attention (see e.g. [11])—but with the case that $\alpha(x) = x$ and $[a, b]$ is the whole real axis. Our main emphasis will be placed upon this situation, in particular upon rather sharp error estimates in case the function to be integrated is smooth but not necessarily analytic. Truncation error estimates will also be considered.

The paper is divided up as follows. Section 2 contains the basic tools from Fourier analysis which will be needed. Section 3 deals with the Shannon sampling theorem and further results as an application of the Euler-MacLaurin formula. The first part of Section 4 treats the quadrature formula mentioned above, and the second handles the associated error estimates. Finally, two characteristic examples are discussed.

2. PRELIMINARIES

Let $C(\mathbb{R})$ denote the space of all complex-valued continuous and bounded functions defined on the real line $\mathbb{R}$, and $L^p(\mathbb{R})$, $1 \leqslant p < \infty$, the space of all measurable functions that are integrable to the pth power over $\mathbb{R}$, both endowed with the usual norms. Let $BV(\mathbb{R})$ be the space of functions which are of bounded variation over $\mathbb{R}$.

The Fourier transform $f^\wedge(v)$ of $f \in L^1(\mathbb{R})$ is defined by $f^\wedge(v) := (1/\sqrt{2\pi})\int_{\mathbb{R}} f(u)e^{-iuv}\,du$, $v \in \mathbb{R}$, and that of $f \in L^p(\mathbb{R})$, $1 < p \leqslant 2$, by $f^\wedge(v) = \mathrm{s}-\lim_{\rho\to\infty}(1/\sqrt{2\pi})\int_{-\rho}^{\rho} f(u)e^{-iuv}\,du$, the limit being understood in the $L^q(\mathbb{R})$ norm with $1/p + 1/q = 1$. If the Fourier transform $f^\wedge$ of $f \in L^p(\mathbb{R})$, $1 \leqslant p \leqslant 2$, belongs to $L^1(\mathbb{R})$, then

$$\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} f^\wedge(v)e^{ivx}\,dv = f(x) \qquad \text{a.e.} \tag{2.1}$$

Defining the convolution $f * g$ of $f \in L^p(\mathbb{R})$, $g \in L^1(\mathbb{R})$, $1 \leqslant p < \infty$, by

$$(f * g)(x) := \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} f(x-u)g(u)\,du,$$

then $f * g \in L^p(\mathbb{R})$, and one has for $1 \leqslant p \leqslant 2$ the convolution theorem

$$(f * g)^\wedge(v) = f^\wedge(v)g^\wedge(v) \qquad \text{a.e.} \tag{2.2}$$

The truncated Fourier inversion integral can be written as

$$\frac{1}{\sqrt{2\pi}}\int_{-\rho}^{\rho} f^{\wedge}(v)e^{ivx}\,dv = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} f(u)D_\rho(x-u)\,du, \tag{2.3}$$

where D_ρ denotes the Dirichlet kernel on $\mathbb{R}$, namely

$$D_\rho(u) := \sqrt{\frac{2}{\pi}}\,\frac{\sin \rho u}{u} \qquad (u \in \mathbb{R}) \tag{2.4}$$

with $D_\rho(0) = \rho\sqrt{2/\pi}$. One has by (2.1) for $f \in L^p(\mathbb{R})$, $1 \leqslant p \leqslant 2$, with $f^{\wedge} \in L^1(\mathbb{R})$

$$\lim_{\rho\to\infty} (f * D_\rho)(x) = f(x) \qquad \text{a.e.} \tag{2.5}$$

Moreover (2.5) can, according to Riemann's localization principle, be replaced by

$$\lim_{\rho\to\infty}\frac{1}{\sqrt{2\pi}}\int_a^b f(u)D_\rho(x-u)\,du = f(x) \qquad \text{a.e.,} \tag{2.6}$$

where $[a, b]$ is any finite interval having x as interior point. Note that (2.1), (2.5), and (2.6) hold in particular at each point of continuity of f.

3. THE EULER-MACLAURIN FORMULA AND THE SAMPLING THEOREM

Let us start off with the Euler-MacLaurin summation formula as it can be found in Ostrowski [24–26], namely

$$\sum_{k=n}^{m} f(k) = \int_n^m f(u)\,du + \tfrac{1}{2}[f(n) + f(m)]$$

$$+ \sum_{k=1}^{r}\frac{B_{2k}}{(2k)!}\left[f^{(2k-1)}(m) - f^{(2k-1)}(n)\right] + R_r, \tag{3.1}$$

$$R_r := \frac{-1}{(2r)!}\int_0^1 B_{2r}(u)\sum_{k=n}^{m-1} f^{(2r)}(k+u)\,du, \tag{3.2}$$

where $n < m$ ($m, n \in \{0, \pm 1, \pm 2, \ldots\}$, $r \in \{1,2,3,\ldots\}$), B_ν are the Bernoulli numbers, and $B_\nu(u)$ the Bernoulli polynomials.

Let $B_\nu^*(u)$ denote the periodic extension to $\mathbb{R}$ of $B_\nu(u)$ defined on $[0,1]$. Using the representation (cf. [16, §64])[1]

$$B_{2r}^*(u) = (-1)^{(r-1)} \frac{(2r)!}{(2\pi)^{2r}} \sum_{k=-\infty}^{\infty}{}' \frac{1}{k^{2r}} e^{-i2k\pi u} \qquad (u \in \mathbb{R}),$$

the remainder (3.2) can be rewritten as

$$R_r = (-1)^r \frac{1}{(2\pi)^{2r}} \sum_{k=-\infty}^{\infty}{}' \frac{1}{k^{2r}} \int_n^m f^{(2r)}(u) e^{-i2k\pi u}\, du. \tag{3.3}$$

As an application of the Euler formula let us first establish the basic lemma mentioned in Section 1.

LEMMA 1. *Let $f \in C(\mathbb{R}) \cap L^p(\mathbb{R})$, some $1 \leqslant p \leqslant 2$, be such that $f^\wedge \in L^1(\mathbb{R})$. For $D_{\pi W}(t)$ defined as in (2.4) and $W > 0$,*

$$\begin{aligned}\frac{1}{\sqrt{2\pi}\, W} & \sum_{k=-\infty}^{\infty} f\left(\frac{k}{W}\right) D_{\pi W}\left(t - \frac{k}{W}\right) \\ &= \frac{1}{\sqrt{2\pi}} \sum_{k=-\infty}^{\infty} e^{-i2k\pi W t} \int_{(2k-1)\pi W}^{(2k+1)\pi W} f^\wedge(v) e^{ivt}\, dv \qquad (t \in \mathbb{R}),\end{aligned} \tag{3.4}$$

the series being conditionally convergent.

Proof. One may assume $W = 1$ without loss of generality. Otherwise replace $f(\cdot)$ by $f(\cdot/W)$ and t by tW. Let us first consider the particular case that f and the derivative f'' both belong to $C(\mathbb{R}) \cap L^1(\mathbb{R})$. Then apply (3.1) for $r = 1$ with remainder term (3.3) to the function $g_t(u) := f(u) D_\pi(t-u)$, $t \in \mathbb{R}$ fixed. Since $g_t, g_t'' \in C(\mathbb{R}) \cap L^1(\mathbb{R})$ with $\lim_{j \to \pm\infty} g_t(j) = \lim_{j \to \pm\infty} g_t'(j) = 0$, one obtains for $n \to -\infty$, $m \to \infty$ that

$$\sum_{k=-\infty}^{\infty} g_t(k) = \int_{-\infty}^{\infty} g_t(u)\, du - \frac{1}{4\pi^2} \sum_{k=-\infty}^{\infty}{}' \frac{1}{k^2} \int_{-\infty}^{\infty} g_t''(u) e^{-i2k\pi u}\, du.$$

[1] The prime indicates that the term $k = 0$ is to be omitted.

Integrating by parts twice yields

$$\sum_{k=-\infty}^{\infty} g_t(k) = \sum_{k=-\infty}^{\infty} \int_{-\infty}^{\infty} g_t(u) e^{-i2k\pi u}\, du. \tag{3.5}$$

Using (2.3) with $\rho = \pi$ and the fact that $[f(\cdot)e^{-ih\cdot}]\hat{}(v) = f\hat{}(v+h)$, the integrals can be rewritten as

$$\int_{-\infty}^{\infty} g_t(u) e^{-i2k\pi u}\, du = \int_{-\pi}^{\pi} f\hat{}(v + 2k\pi) e^{ivt}\, dv$$

$$= e^{-i2k\pi t} \int_{(2k-1)\pi}^{(2k+1)\pi} f\hat{}(v) e^{ivt}\, dv.$$

Inserting this result into (3.5) proves the assertion (3.4) in the particular case.

In the general case consider the function

$$F_\rho(u) := (f * \chi_\rho)(u) \qquad (u \in \mathbb{R}, \quad \rho > 0),$$

χ_ρ being Fejér's kernel on $\mathbb{R}$, defined by

$$\chi_\rho(u) = \frac{1}{\sqrt{2\pi}\,\rho} \left[\frac{\sin \frac{1}{2}\rho u}{\frac{1}{2}u} \right]^2,$$

having the Fourier transform[2]

$$\chi_\rho\hat{}(v) = \left(1 - \frac{|v|}{\rho}\right)_+ \qquad (v \in \mathbb{R}, \quad \rho > 0).$$

One easily verifies that F_ρ satisfies the assumptions of the particular case. Hence (3.4) and (2.2) yield

$$\sum_{k=-\infty}^{\infty} F_\rho(k) D_\pi(t-k) = \sum_{k=-\infty}^{\infty} e^{-i2k\pi t} \int_{(2k-1)\pi}^{(2k+1)\pi} f\hat{}(v) \left(1 - \frac{|v|}{\rho}\right)_+ e^{ivt}\, dv. \tag{3.6}$$

[2]Here $f_+(x)$ is defined by $f(x)$ if $f(x) \geqslant 0$ and 0 if $f(x) < 0$.

The series on the right side, to be denoted by $S_\rho^R(t)$, is uniformly convergent with respect to $\rho > 0$, and its integrals are dominatedly convergent, since $f^\wedge \in L^1(\mathbb{R})$. Therefore

$$\lim_{\rho \to \infty} S_\rho^R(t) = \sum_{k=-\infty}^{\infty} e^{-i2k\pi t} \int_{(2k-1)\pi}^{(2k+1)\pi} f^\wedge(v) e^{ivt}\, dv. \tag{3.7}$$

Concerning the left side $S_\rho^L(t)$ of (3.6), one has by (2.2) and (2.1)

$$\begin{aligned} S_\rho^L(t) &= \sum_{k=-\infty}^{\infty} \left\{ \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f^\wedge(v)\left(1 - \frac{|v|}{\rho}\right)_+ e^{ikv}\, dv \right\} D_\pi(t-k) \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f^\wedge(v)\left(1 - \frac{|v|}{\rho}\right)_+ \left\{ \sum_{k=-\infty}^{\infty} D_\pi(t-k) e^{ikv} \right\} dv, \end{aligned}$$

the interchange of summation and integration being justified by

$$\left| \sum_{k=-n}^{n} D_\pi(t-k) e^{ikv} \right| \leqslant M \qquad (n \in \mathbb{N}),$$

where M depends only on t. Letting now $\rho \to \infty$, interchanging sums and integrals once more, and applying the inversion formula, (2.1) finally yields

$$\begin{aligned} \lim_{\rho \to \infty} S_\rho^L(t) &= \sum_{k=-\infty}^{\infty} \left\{ \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f^\wedge(v) e^{ikv}\, dv \right\} D_\pi(t-k) \\ &= \sum_{k=-\infty}^{\infty} f(k) D_\pi(t-k). \end{aligned} \tag{3.8}$$

So (3.4) follows from (3.6) for $\rho \to \infty$ in view of (3.7) and (3.8). ■

A generalization of the Shannon sampling theorem now follows as a simple application.

THEOREM 1. *Let $f \in C(\mathbb{R}) \cap L^p(\mathbb{R})$, some $1 \leqslant p \leqslant 2$, be such that $f^\wedge \in L^1(\mathbb{R})$. Then for $t \in \mathbb{R}$*

$$\left| \frac{1}{\sqrt{2\pi}\, W} \sum_{k=-\infty}^{\infty} f\left(\frac{k}{W}\right) D_{\pi W}\left(t - \frac{k}{W}\right) - f(t) \right| \leqslant \sqrt{\frac{2}{\pi}} \int_{|v| \geqslant \pi W} |f^\wedge(v)|\, dv; \tag{3.9}$$

in particular, uniformly in $t \in \mathbb{R}$,

$$\lim_{W \to \infty} \frac{1}{\sqrt{2\pi}\,W} \sum_{k=-\infty}^{\infty} f\left(\frac{k}{W}\right) D_{\pi W}\left(t - \frac{k}{W}\right) = f(t). \tag{3.10}$$

The proof follows immediately from (3.4) if one writes f via (2.1) in the form

$$f(t) = \frac{1}{\sqrt{2\pi}} \sum_{k=-\infty}^{\infty} \int_{(2k-1)\pi W}^{(2k+1)\pi W} f^{\wedge}(v) e^{ivt}\, dv \qquad (t \in \mathbb{R}).$$

The inequality (3.9) may be used to deduce associated error estimates by applying (4.10) below. See, for example, [7] and also [28].

If one restricts oneself to the case that $f^{\wedge}(v) = 0$ for $|v| \geqslant \pi W$, then one obtains the classical sampling theorem, also known as Whittaker's cardinal series representation.

Corollary 1. *Let* $f \in C(\mathbb{R}) \cap L^p(\mathbb{R})$, *some* $1 \leqslant p \leqslant 2$, *be such that* $f^{\wedge}(v) = 0$ *for almost all* $|v| \geqslant \pi W$. *Then*

$$\frac{1}{\sqrt{2\pi}\,W} \sum_{k=-\infty}^{\infty} f\left(\frac{k}{W}\right) D_{\pi W}\left(t - \frac{k}{W}\right) = f(t) \qquad (t \in \mathbb{R}), \tag{3.11}$$

the series being uniformly and absolutely convergent.

The absolute and uniform convergence follows from (cf. [33, p. 233])

$$\left\{ \frac{1}{W} \sum_{k=-\infty}^{\infty} \left| f\left(\frac{k}{W}\right) \right|^p \right\}^{1/p} \leqslant (1+\pi) \left\{ \int_{-\infty}^{\infty} |f(u)|^p\, du \right\}^{1/p} \qquad (1 \leqslant p < \infty).$$

Remark 1. The assumption $f^{\wedge}(v) = 0$ for $|v| \geqslant \pi W$ in Corollary 1 could equivalently be expressed by the assumption that f is an entire function of exponential type $\leqslant \pi W$. By using this setting Corollary 1 could be extended to the situation that $2 < p < \infty$ by approximating $f(t)$ by $f(t)(\sin \varepsilon t)/\varepsilon t$ for $\varepsilon \to 0+$ (cf. [27]).

4. APPROXIMATE INTEGRATION OVER THE REAL AXIS

4.1. *Theory*

Lemma 1 can also be used to deduce quadrature formulae for $\int_{-\infty}^{\infty} f(x)\, dx$ with error estimates. As far as the authors are aware, such formulae have been mainly studied in the past for functions that are holomorphic in a strip around the real axis; see e.g. [17, 18, 19, 21, 30, 31]. Instead, it will be assumed that $f \in C(\mathbb{R}) \cap L^1(\mathbb{R})$ either has a given rate of decay at infinity or is differentiable of a given order.

THEOREM 2. *Let $f \in C(\mathbb{R}) \cap L^1(\mathbb{R})$ be such that*

$$\sum_{k=-\infty}^{\infty} \left| f\left(\frac{k}{W} - u\right) \right| \tag{4.1}$$

is uniformly convergent in $u \in \mathbb{R}$ for each fixed $W > 0$. Then, for $W > 0$,

$$\frac{1}{W} \sum_{k=-\infty}^{\infty} f\left(\frac{k}{W}\right) - \int_{-\infty}^{\infty} f(u)\,du = \lim_{n \to \infty} \sqrt{2\pi} \sum_{k=-n}^{n}{}' \left(1 - \frac{|k|}{n}\right) f^{\wedge}(2k\pi W). \tag{4.2}$$

In particular, if the series $\sum_{k=-\infty}^{\infty} f^{\wedge}(2k\pi W)$ is convergent, then

$$\frac{1}{W} \sum_{k=-\infty}^{\infty} f\left(\frac{k}{W}\right) - \int_{-\infty}^{\infty} f(u)\,du = \sqrt{2\pi} \sum_{k=-\infty}^{\infty}{}' f^{\wedge}(2k\pi W) \qquad (W > 0). \tag{4.3}$$

Proof. Replacing f in (3.4) by $f * \chi_{\rho_n}$, $\rho_n := 2n\pi W$, yields [cf. (3.6)]

$$\frac{1}{W} \sum_{k=-\infty}^{\infty} (f * \chi_{\rho_n})\left(\frac{k}{W}\right) D_{\pi W}\left(t - \frac{k}{W}\right)$$
$$= \sum_{k=-\infty}^{\infty} \int_{(2k-1)\pi W}^{(2k+1)\pi W} f^{\wedge}(v) \left(1 - \frac{|v|}{\rho_n}\right)_{+} e^{i(v - 2k\pi W)t}\,dv, \tag{4.4}$$

where $\sum_{k=-\infty}^{\infty} |(f * \chi_{\rho_n})(k/W)| < \infty$ in view of the uniform convergence of (4.1). Furthermore, the series on the right side of (4.4) is finite. So one can integrate (4.4) term by term over $[-R, R]$ to deduce

$$\frac{1}{W} \sum_{k=-\infty}^{\infty} (f * \chi_{\rho_n})\left(\frac{k}{W}\right) \int_{-R}^{R} D_{\pi W}\left(t - \frac{k}{W}\right) dt$$
$$= 2 \sum_{k=-\infty}^{\infty} \int_{(2k-1)\pi W}^{(2k+1)\pi W} f^{\wedge}(v) \left(1 - \frac{|v|}{\rho_n}\right)_{+} \frac{\sin(v - 2k\pi W)R}{(v - 2k\pi W)}\,dv, \tag{4.5}$$

since

$$\int_{-R}^{R} e^{ixt}\,dt = 2\frac{\sin Rx}{x} \qquad (x \in \mathbb{R}).$$

Now $\int_{-R}^{R} D_{\pi W}(t - k/W)\,dt$ is bounded in absolute value by 6, and its limit for $R \to \infty$ equals $\sqrt{2\pi}$, so the limit of the left side of (4.5) for $R \to \infty$ equals

$$\frac{\sqrt{2\pi}}{W} \sum_{k=-\infty}^{\infty} (f * \chi_{\rho_n})\left(\frac{k}{W}\right). \tag{4.6}$$

Concerning the right side of (4.5), the limit for $R \to \infty$ equals

$$2\pi \sum_{k=-\infty}^{\infty} f^{\wedge}(2k\pi W)\left(1 - \frac{|2k\pi W|}{2n\pi W}\right)_{+}$$

in view of (2.6).

To establish (4.2) it remains to prove that (4.6) converges to $(\sqrt{2\pi}/W)\sum_{k=-\infty}^{\infty} f(k/W)$ for $n \to \infty$. Indeed, (4.6) equals

$$\frac{\sqrt{2\pi}}{W} \int_{-\infty}^{\infty} \left\{ \sum_{k=-\infty}^{\infty} f\left(\frac{k}{W} - u\right)\right\} \chi_{\rho_n}(u)\,du$$

in view of the uniform convergence of (4.1), and the latter integral converges to the sum in question because the term in curly braces is uniformly continuous and bounded (cf. [6, p. 122]). ■

Corollary 2.

(a) *If* $f \in C(\mathbb{R}) \cap L^1(\mathbb{R})$ *is such that* $f(t) = O(|t|^{-\gamma})$, $t \to \pm\infty$, *for some* $\gamma > 1$, *then, for* $W > 0$,

$$\left| \frac{1}{W} \sum_{k=-\infty}^{\infty} f\left(\frac{k}{W}\right) - \int_{-\infty}^{\infty} f(u)\,du \right| = \sqrt{2\pi} \left| \sum_{k=-\infty}^{\infty}{}' f^{\wedge}(2k\pi W) \right| \tag{4.7}$$

whenever the series on the right converges.

(b) *If* $f \in C(\mathbb{R}) \cap L^1(\mathbb{R}) \cap BV(\mathbb{R})$, *then* (4.7) *holds.*

Proof. If $f(t) = O(|t|^{-\gamma})$ for $\gamma > 1$, then the series (4.1) is uniformly convergent for $W > 0$. So part (a) follows by Theorem 2. Concerning (b), one knows that $f \in \mathrm{BV}(\mathbf{R})$ implies the uniform convergence of (4.1) (cf. [6, p. 124]) as well as $f^{\wedge}(v) = O(|v|^{-1})$ for $v \to \pm\infty$. Now the right side of (4.7) is $(C, 1)$-summable in view of (4.2). Since $f^{\wedge}(2k\pi W) = O(|k|^{-1})$, $k \to \pm\infty$, it is also convergent in view of the Hardy Tauberian theorem (cf. [14, p. 121]). So the result follows by Theorem 2. ■

REMARK 2. The reader may note that (3.5) or (4.3) is basically the well-known Poisson summation formula, which was actually introduced into number theory by Lejeune Dirichlet and is regarded by Bochner [2, p. 52] as a "broad-gauged duality formula which lies athwart most of analysis." In other words, in the course of Sections 3 and 4 we indirectly obtained the Poisson formula from the Euler-MacLaurin formula (via the sampling theorem). For the fact that an integration of the sampling theorem in its generalized form, basically (3.4), yields the Poisson formula; see [36]. Conversely, it would also be possible to study the sampling theorem as well as approximate integration as direct applications of the Poisson formula; see especially [1], and also [5, 35]. For a detailed investigation of the remainder term in the n-dimensional Euler formula, see [37]. It may also be used to deduce the multidimensional Poisson formula; see [22].

There exists a summation formula for the one-sided improper integral $\int_0^\infty f(u)\,du$. It is the summation formula of Abel-Plana; see [15, p. 274]. In [13, 20, 23] numerical integration for improper integrals of the first and the second kind is studied by using quite different methods.

4.2. *Error Estimates*

In order to give an estimate for the left-hand side in (4.7) it is useful to consider the modulus of continuity of $f \in L^1(\mathbf{R})$, defined by

$$\omega(\delta; f; L^1(\mathbf{R})) := \sup_{|h| \leqslant \delta} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} |f(x+h) - f(x)|\,dx.$$

COROLLARY 3. *Let $f \in C(\mathbf{R}) \cap L^1(\mathbf{R})$ be such that $f^{(r)} \in L^1(\mathbf{R})$ for some $r \in \{1, 2, 3, \ldots\}$. If $\omega(\delta; f^{(r)}; L^1(\mathbf{R}) \leqslant L\delta^\alpha$ for some $0 < \alpha \leqslant 1$, then, for $W > 0$,*

$$\left| \frac{1}{W} \sum_{k=-\infty}^{\infty} f\left(\frac{k}{W}\right) - \int_{-\infty}^{\infty} f(u)\,du \right| \leqslant \frac{M}{W^{r+\alpha}}, \tag{4.8}$$

where the constant M is given by

$$M := \frac{L\pi^\alpha}{(2\pi)^{r+\alpha}} \left(\sum_{k=1}^{\infty} \frac{1}{k^{r+\alpha}} \right) \leqslant \frac{L\pi^\alpha (r+\alpha)}{(2\pi)^{r+\alpha}(r+\alpha-1)}. \tag{4.9}$$

Concerning the proof, one has only to note that $f^{(r)} \in L^1(\mathbb{R})$ implies $f \in \mathrm{BV}(\mathbb{R})$, and to apply in Corollary 2(b) the estimate (cf. [6, (5.1.3), Proposition 5.1.14])

$$|f^{\wedge}(v)| = |(iv)^{-r}[f^{(r)}]^{\wedge}(v)| \leqslant \frac{|v|^{-r}}{2}\omega\left(\frac{\pi}{|v|}; f^{(r)}; L^1(\mathbb{R})\right) \qquad (v \neq 0). \quad (4.10)$$

Note that the rate of convergence in (4.8) is as good as one pleases provided the function is sufficiently smooth. This stands in contrast to the situation when $\int_a^b f(u)\,du$, $[a, b]$ being finite, is approximated by the corresponding Riemann sums; see [8]. For error estimates in case the function in question is holomorphic, one may consult [19, 21, 30, 31].

In practice one is also interested in the "truncation" error which arises when the infinite series in (4.8) is replaced by the truncated sum $\sum_{k=-N}^{N}$. This error is additional to that in (4.8). In this situation one has

COROLLARY 4. *Let f be given as in Corollary 3. Moreover, assume that $|f(t)| \leqslant M_f |t|^{-\gamma}$ for all $|t| \geqslant a > 0$ and some $\gamma > 1$. Then for $N > aW$ and M as given in* (4.9) *one has*

$$\left| \frac{1}{W} \sum_{k=-N}^{N} f\left(\frac{k}{W}\right) - \int_{-\infty}^{\infty} f(u)\,du \right| \leqslant \frac{M}{W^{r+\alpha}} + \frac{2M_f}{\gamma - 1}\left(\frac{W}{N}\right)^{\gamma-1}. \quad (4.11)$$

If $N = N(W)$ is chosen such that $W^{\sigma} \leqslant N(W)$ for $\sigma = (r + \alpha + \gamma - 1)/(\gamma - 1)$, then

$$\left| \frac{1}{W} \sum_{k=-N}^{N} f\left(\frac{k}{W}\right) - \int_{-\infty}^{\infty} f(u)\,du \right| = O\left(\frac{1}{W^{r+\alpha}}\right) \qquad (W \to \infty). \quad (4.12)$$

Furthermore, one has for $1/N = o(1/W)$, $W \to \infty$,

$$\lim_{W \to \infty} \frac{1}{W} \sum_{k=-N}^{N} f\left(\frac{k}{W}\right) = \int_{-\infty}^{\infty} f(u)\,du.$$

The proof follows from Corollary 3 and the estimate

$$\frac{1}{W}\left| \sum_{|k| \geqslant N+1} f\left(\frac{k}{W}\right) \right| \leqslant 2M_f W^{\gamma-1} \int_N^{\infty} u^{-\gamma}\,du = \frac{2M_f}{\gamma - 1}\left(\frac{W}{N}\right)^{\gamma-1}.$$

Truncation-error estimates using function-theory methods are to be found in [30, pp. 236–239].

Corollary 1 also enables one to deduce a quadrature formula for the product of two functions $f \in C(\mathbb{R}) \cap L^1(\mathbb{R})$ and $g \in L^1(\mathbb{R})$, where $f^\wedge(v) = 0$ for $|v| \geqslant \pi W$. Indeed, multiplying (3.11) by $g(t)$ and integrating termwise gives

COROLLARY 5. *Let $f \in C(\mathbb{R}) \cap L^p(\mathbb{R})$, some $1 \leqslant p \leqslant 2$, be such that $f^\wedge(v) = 0$ for $|v| \geqslant \pi W$. Then one has for each $g \in L^1(\mathbb{R})$*

$$\frac{1}{W} \sum_{k=-\infty}^{\infty} A_{k,W} f\left(\frac{k}{W}\right) = \int_{-\infty}^{\infty} f(t) g(t)\, dt, \tag{4.13}$$

where the "weights" $A_{k,W}$ are given by

$$A_{k,W} = (g * D_{\pi W})\left(\frac{k}{W}\right) \equiv \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(u) D_{\pi W}\left(u - \frac{k}{W}\right) du.$$

For the evaluation of integrals over a finite range of integration using Euler's formula see e.g. [38]. A number of investigations are concerned with evaluating $\int_{-\infty}^{\infty} f(u)\, d\alpha(u)$ for $\alpha \in \mathrm{BV}(\mathbb{R})$. These are intimately related to the determinacy of the moment problem for $d\alpha(u)$ on $\mathbb{R}$. In this respect one may consult [9], and the survey paper [11, especially pp. 117–118] and the literature cited there, as well as [10]. In [12] methods similar to ours are used to study $\int_{-\infty}^{\infty} f(u) e^{-u^2} du$.

4.3. *Applications*

Let us first consider a standard example, namely the approximate integration of the function $f_1(x) = 1/(1+x^2)$ over $\mathbb{R}$; it is analytic in $\{z = x + iy;\ x \in \mathbb{R},\ |y| < 1\}$ (cf. [21]). Since $f_1^\wedge(v) = \sqrt{\pi/2}\, e^{-|v|}$, $v \in \mathbb{R}$, an application of Corollary 2(a) yields

$$\left| \sum_{k=-\infty}^{\infty} \frac{W}{W^2 + k^2} - \int_{-\infty}^{\infty} \frac{du}{1+u^2} \right| = \pi \sum_{k=-\infty}^{\infty}{}' e^{-2|k|\pi W} = \frac{\pi e^{-\pi W}}{\sinh \pi W} \quad (W > 0). \tag{4.14}$$

Concerning the truncation error, one has for $W > 0$ and $N \gg W$

$$\left| \sum_{k=-N}^{N} \frac{W}{W^2 + k^2} - \int_{-\infty}^{\infty} \frac{du}{1+u^2} \right| \leqslant \frac{\pi e^{-\pi W}}{\sinh \pi W} + 2WN^{-1}.$$

Note that the error given in [21] in case of (4.14) is $\frac{11}{2}e^{-3W}/\sinh 3W$. Whereas this error was not aimed to be the best possible one, ours is.

As our second example, let us consider $f_2(x)=|x|/(1+|x|^3)$. It is easy to show that $f_2' \in L^1(\mathbb{R})$, and $\omega(\delta; f_2'; L^1(\mathbb{R}) \leqslant 8\delta$, $\delta>0$. This yields by Corollary 4

$$\left|\sum_{k=-N}^{N} \frac{|k|W}{W^3+|k|^3} - \int_{-\infty}^{\infty} \frac{|u|}{1+|u|^3}du\right| \leqslant \frac{4}{\pi}W^{-2}+2WN^{-1}.$$

Choosing $N^{-1}=O(W^{-3})$, $W\to\infty$, then the combined error is of order $O(W^{-2})$, $W\to\infty$.

It should be mentioned that f_2 does not have an analytic extension into the complex plane as f_1 does. So it cannot be handled by function-theoretic methods.

REFERENCES

1 R. P. Boas, Summation formulas and band-limited signals, *Tôhoku Math. J.* 24:121–125 (1972).

2 S. Bochner, The emergence of analysis in the Renaissance and after, in *History of Analysis* (R. J. Stanton and R. O. Wells, Jr., Eds.), Rice Univ. Studies, Vol. 64, Nos. 2, 3, 1978, pp. 11–56.

3 J. L. Brown, Jr., On the error in reconstructing a non-bandlimited function by means of the bandpass sampling theorem, *J. Math. Anal. Appl.* 18:75–84 (1967).

4 J. L. Brown, Jr., Sampling theorem for finite-energy signals, *IEEE Trans. Inform. Theory* IT-14:818–819 (1968).

5 P. L. Butzer, The Shannon sampling theorem and some of its generalizations; an overview, in *Proceedings of the International Conference on Constructive Function Theory (Varna, Bulgaria, June 1981)* (D. Vacov, Ed.), to appear.

6 P. L. Butzer and R. J. Nessel, *Fourier Analysis and Approximation*, Vol. I, Academic, New York, 1971.

7 P. L. Butzer and W. Splettstösser, A sampling theorem for duration-limited functions with error estimates, *Inform. and Control* 34:55–65 (1977).

8 C. K. Chui, Concerning rates of convergence of Riemann sums, *J. Approximation Theory* 4:279–287 (1971).

9 P. J. Davis and P. Rabinowitz, *Methods of Numerical Integration*, Academic, New York, 1975.

10 H. Esser, Mean convergence of the Lagrangean interpolation of improperly Riemann-Stieltjes integrable functions, in *Numerische Integration* (G. Hämmerlin, Ed.), ISNM Vol. 45, Birkhäuser, Basel, 1979, pp. 131–137.

11 W. Gautschi, A survey of Gauss-Christoffel quadrature formulae, in *E. B. Christoffel, The Influence of his Work on Mathematics and the Physical Sciences* (P. L. Butzer and F. Fehér, Eds.), Birkhäuser, Basel, 1981, pp. 72–147.

12 R. Gervais, O. I. Rahman, and G. Schmeisser, A quadrature formula of infinite order, to appear.
13 S. Haber and O. Shisha, Improper integrals, simple integrals, and numerical quadrature, *J. Approximation Theory* 11:1–15 (1974).
14 G. H. Hardy, *Divergent Series*, Clarenden, Oxford, 1949.
15 P. Henrici, *Applied and Computational Complex Analysis, Vol. I. Power Series — Integration — Conformal Mapping — Location of Zeros*, Wiley, New York, 1974.
16 K. Knopp, *Theory and Application of Infinite Series*, Hafner, New York, 1971.
17 R. Kress, Interpolation auf einem unendlichen Intervall, *Computing* 6:274–288 (1970).
18 R. Kress, On the general Hermite cardinal interpolation, *Math. Comp.* 26:925–933 (1972).
19 R. Kress, Zur Quadratur uneigentlicher Integrale bei analytischen Funktionen, *Computing* 13:267–277 (1974).
20 J. T. Lewis, C. F. Osgood, and O. Shisha, Infinite Riemann sums, the simple integral, and the dominated integral, in *General Inequalities I* (E. F. Beckenbach, Ed.), ISNM Vol. 41, Birkhäuser, Basel, 1978, pp. 233–242.
21 E. Martensen, Zur numerischen Auswertung uneigentlicher Integrale, *Z. Angew. Math. Mech.* 48:T83–T85 (1968).
22 C. Müller and W. Freeden, Multidimensional Euler and Poisson summation formulas, *Resultate Math.* 3:33–63 (1980).
23 C. F. Osgood and O. Shisha, Numerical quadrature of improper integrals and the dominated integral, *J. Approximation Theory* 20:139–152 (1977).
24 A. M. Ostrowski, Note on Poisson's treatment of the Euler-Maclaurin formula, *Comment. Math. Helv.* 44:202–206 (1969).
25 A. M. Ostrowski, Über das Restglied der Euler-Maclaurischen Formel, in *Abstract Spaces and Approximation* (P. L. Butzer and B.Sz.-Nagy, Eds.), ISNM Vol. 10, Birkhäuser, Basel, 1969, pp. 358–364.
26 A. M. Ostrowski, On the remainder term of the Euler-Maclaurin formula, *J. Reine Angew. Math.* 239/240:268–286 (1969).
27 W. Splettstösser, Sampling series approximation of continuous weak sense stationary processes, *Inform. Control*, to appear.
28 W. Splettstösser, R. L. Stens, and G. Wilmes, On Approximation by the interpolating series of G. Valiron, *Funct. Approximatio Comment. Math.* 11:39–56 (1981).
29 A. Steiner, Plancherel's theorem and the Shannon series derived simultaneously, *Amer. Math. Monthly* 87:193–197 (1980).
30 F. Stenger, Approximations via Whittaker's cardinal function, *J. Approximation Theory* 17:222–240 (1976).
31 F. Stenger, Numerical methods based on Whittaker cardinal, or sinc functions, *SIAM Rev.* 23:165–224 (1981).
32 R. L. Stens, Error estimates for sampling sums based on convolution integrals, *Inform. and Control* 45:37–47 (1980).
33 A. F. Timan, *Theory of Approximation of Functions of a Real Variable*, Pergamon, Oxford, 1963.

34 G. Wunsch, *Systemtheorie der Informationstechnik*, Akademische Verlagsgesellschaft Geest & Portig, Leipzig, 1971.
35 R. M. Young, *An Introduction to Nonharmonic Fourier Series*, Academic, New York, 1980.
36 P. L. Butzer and R. L. Stens, The Poisson summation formula, Whittaker's cardinal series and approximate integration, in *Proceedings of the II. Edmonton Conference on Approximation Theory. Edmonton, June 1982*, in preparation.
37 J. N. Lyness and B. J. J. McHugh, On the remainder term in the *N*-dimensional Euler-Maclaurin expansion, *Numer. Math.* 15:333–344 (1970).
38 J. N. Lyness and B. W. Ninham, Numerical quadrature and asymptotic expansions, *Math. Comp.* 21:162–178 (1967).

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