Deviation of a system of nonreciprocally coupled harmonic oscillators from a conservative system

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(Received 30 May 2024; revised 13 December 2024; accepted 20 December 2024; published 16 January 2025)

Discrete systems of coupled linear mechanical oscillators with nonreciprocal interaction are a model for a variety of physical systems. In general, the presence of nonreciprocal interactions renders their dynamics nonconservative, but under certain conditions it remains conservative. In this paper we show which thermodynamic properties induced by nonreciprocity can be observed in conservative systems and which are specific to nonconservative systems. To this end, we formulate a criterion for identifying conservative systems and construct a measure to quantify the deviation from conservativity.

DOI: 10.1103/PhysRevE.111.014132

I. INTRODUCTION

Nonreciprocity in interactions is inherent to nonequilibrium systems [1–7], networks of neurons [8,9], social groups with conformist and contrarian members [10], directional interface growth phenomena [11–13], metamaterials [14–17], and active matter [18–27]. For example, interactions between charged colloidal or dust particles do not effectively satisfy Newton's third law (actio = reactio) [7,28]. This effective violation of the fundamental principle is explained by the transfer of energy and momentum from the nonequilibrium environment to the interacting particles.

In its turn, arising nonreciprocity at the level of separate interactions leads to symmetry breaking already at mesoscopic scale and apparent violation of laws of statistical physics and kinetics. In particular, nonreciprocity of interactions can lead to the failure of Onsager's reciprocal relations for kinetic coefficients [29], odd elasticity [16], nonreciprocal phase transitions [30–32], and the violation of the equipartitionbrk theorem [33–37].

In this paper, we focus on the study of discrete systems of coupled linear mechanical oscillators with nonreciprocal interaction in contact with Langevin thermostats. The temperatures of the thermostats acting on the individual degrees of freedom of the oscillators are not necessarily equal to each other. This "mathematical" model can serve to describe the dynamical properties of a wide range of "physical" systems, both natural, such as ordered structures in colloids [38] and complex plasmas [28,33–37], and artificial, such as nonreciprocal robotic metamaterials [14,39,40] and levitated nanoparticles, where nonreciprocity is maintained by an external control scheme (e.g., using optical feedback) [41–45].

It is known that nonreciprocal interaction in such systems leads to the appearance of properties that are not specific to reciprocal systems. In particular, nonreciprocal interaction affects the behavior of systems after long-term evolution, when the transient dynamics due to the initial conditions decays and the system reaches a steady state under the influence of a thermostat. Such thermodynamic properties of nonreciprocal systems include the appearance of negative heat fluxes [46,47] and a nonuniform stationary distribution of the mean kinetic energies of the oscillators, even at the same thermostat temperatures [28,33–37]. Based on these effects, a nonreciprocal setup can be used to build a "Maxwell demon" [48], a directional amplifier [49–52], and a nonreciprocal refrigerator [47,53,54].

Sometimes, the concept of nonreciprocity is identified with nonconservativity [46]. However, if one understands by conservative a system whose dynamics is derived from the Hamiltonian (total kinetic and potential energy), this is not so. Under certain conditions, the dynamics of systems with nonreciprocal interaction can still remain conservative [28,31]. In particular, for the considered discrete systems of coupled linear mechanical oscillators the conservativity condition is determined by the simultaneous symmetrizability of the force interaction matrix and the mass matrix [55,56].

In this work, we investigate in detail how the deviation from conservativity of a discrete system of coupled linear mechanical oscillators with nonreciprocal interaction affects its long-term properties when exposed to a system of Langevin thermostats. To this end, we analyze the changes in the transfer matrix, which relates the detailed heat fluxes entering the system of oscillators and the temperatures of thermostats, as a function of the degree to which the system deviates from conservativity. In particular, we relate the reciprocity and conservativity conditions of the oscillator system to the classification of the transport matrix in terms of the physics of non-Hermitian systems [57]. This allows us to determine which long-term thermodynamic properties can be observed already in conservative systems with asymmetric (nonreciprocal) interaction matrices, and which are specific exclusively for nonconservative systems.

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The rest of this paper is organized as follows. In Sec. II, we recall the known conditions for nonreciprocal systems of coupled linear mechanical oscillators to be conservative. In addition, we relate the conservativity condition to the possibility of describing a mechanical system as a Hermitian eigenvalue problem. In Sec. III, we discuss the possibility of detailed balance in nonreciprocal systems of oscillators coupled to a system of heat baths and introduce a scalar measure of nonconservativity. In Sec. IV, we consider effects arising in systems of nonreciprocally coupled mechanical oscillators after their long-term exposure to a system of Langevin thermostats. We discuss which of them can already be observed in conservative systems with nonreciprocal interactions, and which are exclusively specific to nonconservative systems characterized by a nonzero nonconservativity measure. In Sec. V, we discuss a possible experimental setup where theoretical findings obtained in this work can be directly demonstrated and validated. In Sec. VI, we draw conclusions.

II. CONSERVATIVITY

Here we summarize the known conditions for nonreciprocal systems of coupled linear mechanical oscillators to be conservative and link the conservativity condition to the possibility of describing a mechanical system as a Hermitian eigenvalue problem.

A. System with two degrees of freedom

To demonstrate the idea of analyzing the interaction matrix to determine whether a nonreciprocal dynamical system is conservative, consider two coupled harmonic oscillators. The equations of motion for this system are

$$m_1\ddot{q}_1 + f_{11}q_1 + f_{12}q_2 = 0, \tag{1}$$

$$m_2\ddot{q}_2 + f_{21}q_1 + f_{22}q_2 = 0, \tag{2}$$

where q_1, q_2 are the one-dimensional displacements of the oscillators with masses m_1, m_2, respectively; $f_{11}, f_{12}, f_{21}, f_{22}$ are the diagonal and off-diagonal elements of the interaction matrix. If the interaction matrix is symmetric, namely, $f_{12} = f_{21}$, we deal with a classical reciprocal system where Newton's third law is satisfied and its dynamics is governed by Hamilton's equations. In the case of an asymmetric interaction matrix $f_{12} \neq f_{21}$ the system becomes nonreciprocal. However, it can be mapped onto a reciprocally coupled one [46]. Indeed, dividing Eqs. (1) and (2) by $|f_{12}|$ and $|f_{21}|$, respectively, and introducing new masses $\tilde{m}_1 = \frac{m_1}{|f_{12}|}$, $\tilde{m}_2 = \frac{m_2}{|f_{21}|}$, we can write them in the following symmetric form:

$$\tilde{m}_1\ddot{q}_1 + \frac{f_{11}}{|f_{12}|}q_1 + \mathrm{sign}(f_{12})q_2 = 0, \tag{3}$$

$$\tilde{m}_2\ddot{q}_2 + \mathrm{sign}(f_{21})q_1 + \frac{f_{22}}{|f_{21}|}q_2 = 0. \tag{4}$$

Thus, the considered system with two degrees of freedom (DOF) being nonreciprocal is conservative for all interaction matrices in the region of its dynamic stability. However, its Hamiltonian is now written through effective masses and elements of the transformed interaction matrix.

B. System with arbitrary number of degrees of freedom

In contrast to systems with two DOF, systems with an arbitrary number of DOF are not conservative for all interaction matrices [31]. For simplicity, consider a system with three DOF and a nonreciprocal interaction matrix,

$$\hat{F} = \begin{bmatrix} f_{11} & f_{12} & f_{13} \\ f_{21} & f_{22} & f_{23} \\ f_{31} & f_{32} & f_{33} \end{bmatrix}. \tag{5}$$

In order to symmetrize this matrix, we scale elements of its rows by an appropriate transformation of the masses of the initial dynamical system. The first row is multiplied by $\frac{f_{31}}{f_{13}}$, and the second one by $\frac{f_{32}}{f_{23}}$. As a result, we get the following transformed interaction matrix:

$$\hat{F} = \begin{bmatrix} \frac{f_{31}}{f_{13}}f_{11} & \frac{f_{31}}{f_{13}}f_{12} & f_{31} \\ \frac{f_{32}}{f_{23}}f_{21} & \frac{f_{32}}{f_{23}}f_{22} & f_{32} \\ f_{31} & f_{32} & f_{33} \end{bmatrix}, \tag{6}$$

which is symmetric if $f_{21}f_{13}f_{32} = f_{23}f_{31}f_{12}$. Thus, unlike the case of a system with two degrees of freedom, a system with three or more degrees of freedom is conservative only under certain conditions imposed on the interaction matrix.

For the general case of the system with $n \geqslant 3$,

$$\hat{M}\ddot{\vec{q}}(t) + \hat{F}\vec{q}(t) = 0, \tag{7}$$

the conservativity condition is determined by the possibility to symmetrize simultaneously the interaction matrix $\hat{F}$ and the mass matrix $\hat{M}$ [55,56]. In the case of a diagonal mass matrix, this condition reduces to the following set of constraints:

$$d_i f_{ij} = d_j f_{ji}, \quad 1 \leqslant i < j \leqslant n, \tag{8}$$

where d_i are varied positive values. It can be shown using the same procedure as for the case $n = 3$. Symmetrization of the dynamical system leads to a transformation of the mass matrix and the interaction matrix, which results in their rows being scaled. This transformation is equivalent to multiplying these matrices on the left by the diagonal matrix $\hat{D}$ (with d_i element on the main diagonal), namely, $\hat{M} \to \hat{D}\hat{M}$ and $\hat{F} \to \hat{D}\hat{F}$. All elements of the matrix $\hat{D}$, symmetrizing the system, are assumed to be positive, because otherwise the dynamical system will be unstable.

C. Connection to the Hermitian eigenvalue problem

The considered discrete system of coupled linear mechanical oscillators can be reduced to the Schrödinger equation, which is usually used in the analysis of non-Hermitian systems [40]. For this purpose, we transform the original system of second-order differential equations (7) to a system of $2n$ first-order equations:

$$\frac{d\vec{q}}{dt} = \dot{\vec{q}}, \tag{9}$$

$$\frac{d\dot{\vec{q}}}{dt} = -\hat{M}^{-1}\hat{F}\vec{q}. \tag{10}$$

Thus, the original system can be rewritten in the form of the Schrödinger equation,

$$i\frac{d}{dt}\begin{bmatrix}\vec{q}\\ i\dot{\vec{q}}\end{bmatrix}=\begin{bmatrix}\hat{0} & \hat{I}\\ \hat{M}^{-1}\hat{F} & \hat{0}\end{bmatrix}\begin{bmatrix}\vec{q}\\ i\dot{\vec{q}}\end{bmatrix}, \tag{11}$$

where $\hat{I}$ is the identity matrix. In the case when the matrices $\hat{M}^{-1}$ and $\hat{F}$ are simultaneously reduced to symmetric form, i.e., the system is conservative, the product of the symmetrized matrices $\hat{M}^{-1}\hat{F}$ can be reduced to diagonal form $\hat{\Sigma}$ using the reversible operator $\hat{P}$:

$$\hat{M}^{-1}\hat{F}=\hat{P}\hat{\Sigma}\hat{P}^{-1}. \tag{12}$$

In the case where the matrices $\hat{M}$ and $\hat{F}$ are positively defined, i.e., in the case of a stable dynamical system, the matrix $\hat{D}$ is also positively defined. Hence, there exists a real matrix $\sqrt{\hat{D}}$. In this case, the system can be rewritten in the following symmetric form by introducing a new vector variable $\vec{u}$:

$$i\frac{d\vec{u}}{dt}=\begin{bmatrix}\hat{0} & \sqrt{\hat{D}}\\ \sqrt{\hat{D}} & \hat{0}\end{bmatrix}\vec{u}, \tag{13}$$

$$\vec{u}=\begin{bmatrix}\sqrt{\hat{D}}\hat{P}^{-1}\vec{q}\\ i\hat{P}^{-1}\dot{\vec{q}}\end{bmatrix}. \tag{14}$$

Thus, the considered nonreciprocal system in the case of conservativity reduces to the Hermitian eigenvalue problem. However, in the case when the initial system with an asymmetric interaction matrix is not conservative this is not fulfilled and the associated Schrödinger equation starts to be governed by a non-Hermitian Hamiltonian (here the Hamiltonian does not mean the total energy, but the matrix operator in the Schrödinger equation). This indicates that the conservativity condition of the considered system of oscillators can have a significant influence on its dynamical properties, as will be shown below.

It is easy to see that the original dynamical system even being nonreciprocal is symmetric to time and coordinate reversal (7), so the Hamiltonian of its associated Schrödinger equation will be $\mathcal{PT}$ symmetric [58]. This means that the eigenvalues must be real or appear as complex-conjugate pairs. At the same time, the solutions of the system of equations in the presence of complex-conjugate eigenvalues will grow exponentially, which corresponds to the instability of the initial dynamical system.

III. DEPARTURE FROM CONSERVATIVITY

The condition of simultaneous symmetrizability of the interaction matrix and the mass matrix stated in the previous section allows one to judge about the conservativity of the system of coupled linear mechanical oscillators from the mathematical point of view. However, in physical systems, whether they are natural or artificially created, the interaction forces are known only with limited accuracy [59]. Therefore, a method is needed that would allow one to judge about the conservativity of the system taking into account possible measurment errors.

For this reason, we have developed our criterion for identifying whether a nonreciprocal system is conservative. This criterion naturally allows us to introduce a measure of how far the nonreciprocal system under consideration is from a conservative one.

We start our analysis with coupling the dynamical system of an arbitrary number of nonreciprocal harmonic oscillators to the system of Langevin heat baths. Equations of motions of the considered system have the following matrix form:

$$\hat{M}\ddot{\vec{q}}(t)+\hat{F}\vec{q}(t)+\hat{L}\dot{\vec{q}}(t)=\vec{\eta}(t), \tag{15}$$

where $\hat{M}$ and $\hat{L}$ are the diagonal mass and friction matrices, t is time, and $\vec{\eta}$ is the vector of independent fluctuation zero-mean Gaussian forces, such that $\langle\eta_i(t)\eta_i(t')\rangle=2\gamma_i T_i\delta(t-t')$ and $\langle\eta_i(t)\eta_j(t')\rangle=0$. Here, γ_i are the diagonal elements of the matrix $\hat{L}$ (which is diagonal) and δ is the delta function. After Fourier transforming the equations of motion (15) they become

$$-\omega^2\hat{M}\vec{q}(\omega)+\hat{F}\vec{q}(\omega)-i\omega\hat{L}\vec{q}(\omega)=\vec{\eta}(\omega), \tag{16}$$

where ω is the coordinate in the frequency space. The inverse matrix $\hat{G}=(-\omega^2\hat{M}+\hat{F}-i\omega\hat{L})^{-1}$ is the Green's operator that provides the solution of the system of equations (15) [60]. Heat flows between thermostats and corresponding degrees of freedom J_i are expressed via elements of this Green's operator:

$$\begin{aligned}J_i &\overset{\text{def}}{=} \langle\dot{q}_i\eta_i-\gamma_i\dot{q}_i^2\rangle\\ &=\frac{1}{\pi}\sum_{k=1}^{n}\gamma_i\gamma_k\bigg(T_i\int_{-\infty}^{\infty}\omega^2[\hat{G}_{ik}\hat{G}_{ki}^*]d\omega-T_k\\ &\quad\times\int_{-\infty}^{\infty}\omega^2|\hat{G}_{ik}|^2d\omega\bigg).\end{aligned} \tag{17}$$

Derivation of this formula is given in the Supplemental Material [61].

A. Detailed balance

It is obvious that for systems with a symmetric (reciprocal) interaction matrix, the coefficients at temperatures (17) will be equal due to the symmetry of the Green's function matrix. In this case, the equality of the thermostat temperatures to each other leads to the zeroing of each of the heat flows from the thermostat to the ith degree of freedom of the system, i.e., to the detailed balance (DB), in its steady state. In the general case, the condition of DB, according to expression (17), forms a system of n linear equations with respect to n unknowns T_i. This system has an infinite number of solutions if its matrix is singular, or the single solution $T_i=0$ otherwise. For a conservative system of nonreciprocally coupled harmonic oscillators, this matrix is singular because there exists a transformation from an initially nonreciprocal system to a reciprocal one. However, the thermostat temperatures at DB are no longer necessarily equal to each other.

For the systems which are not conservative this matrix cannot be singular. In order to show this, let us consider the relations that are established between the time averages for all possible quadratic combinations of the variables $\dot{q}_i, q_i$:

$$\langle q_i \dot{q}_i \rangle = 0, \tag{18}$$

$$\langle q_i \dot{q}_j \rangle = -\langle q_j \dot{q}_i \rangle, \tag{19}$$

$$\sum_{k=1}^{n} f_{ik} \langle q_k \dot{q}_i \rangle + \gamma_i \langle \dot{q}_i^2 \rangle = \langle \dot{q}_i \eta_i \rangle, \tag{20}$$

$$\sum_{k=1}^{n} f_{ik} \langle q_k \dot{q}_j \rangle + \gamma_i \langle \dot{q}_i \dot{q}_j \rangle + \sum_{k=1}^{n} f_{jk} \langle q_k \dot{q}_i \rangle + \gamma_j \langle \dot{q}_j \dot{q}_i \rangle = 0, \tag{21}$$

$$\langle \dot{q}_i \dot{q}_j \rangle - \sum_{k=1}^{n} f_{ik} \langle q_k q_j \rangle - \gamma_i \langle \dot{q}_i q_j \rangle = 0. \tag{22}$$

Derivation of these relations can be found in the Supplemental Material [61]. The number of unknowns in this system of equations, namely, the quadratic terms on $\dot{q}_i, q_i$, is equal to the number of equations. That means they can be expressed explicitly through the nonzero correlators $\langle \dot{q}_i \eta_i \rangle$. When the heat fluxes from all the thermostats to the system are zero, Eqs. (20) take the following form:

$$\sum_{k=1}^{n} f_{ik} \langle q_k \dot{q}_i \rangle = 0, \tag{23}$$

which means that the right-hand part of the linear system of equations (18)–(22) is zero column. In this case of a homogeneous system, the existence of a nonzero solution is equivalent to the equality to zero of the determinant of its matrix. At first, it is possible if the matrix rows corresponding to Eqs. (23) are linearly dependent. It can be shown that this condition coincides with the condition for the linear dynamical system to be conservative [Eq. (8)]. At second, this matrix will be singular if its remaining rows corresponding to Eqs. (18) and (19) and Eqs. (21) and (22) are linearly dependent. However, if we assume their linear dependence, the matrix will also be singular in the case that the fluxes from the thermostats to the dynamical system are not equal to zero and, therefore, according to the Rouché-Capelli theorem, will have either zero solutions or an infinite number of solutions, which is impossible. Thus, the obtained contradiction proves that for systems of nonreciprocally coupled harmonic oscillators which are nonconservative, the DB is impossible.

B. Nonconservativity measure

This result makes it possible to construct a measure that naturally allows us to describe the deviation of a nonreciprocal system from a conservative one. Let us denote the coefficients that stand at temperatures in expression (17) as

$$s_{ij} = \begin{cases} \frac{1}{\pi} \sum_{k \neq i}^{n} \gamma \int_{-\infty}^{\infty} \omega^2 [\hat{G}_{ik} \hat{G}_{ki}^*] d\omega, & i = j \\ -\frac{1}{\pi} \gamma \int_{-\infty}^{\infty} \omega^2 |\hat{G}_{ij}|^2 d\omega, & i \neq j, \end{cases} \tag{24}$$

assuming that all γ_i are the same and equal to γ. Then the heat fluxes from the thermostat to the system are expressed through the transfer matrix $\hat{S}$, which is formed by the elements s_{ij} and the temperatures vector:

$$\vec{J} = \gamma \hat{S} \vec{T}. \tag{25}$$

We will minimize the sum of squares of heat fluxes, $\sqrt{\sum_{k \neq i}^{n} \frac{J_i^2}{\gamma^2}}$, by the temperatures vector with the constraint $\sum_{i=1}^{n} T_i^2 = 1$ and denote this minimum as ε:

$$\varepsilon = \min_{\sum_{i=1}^{n} T_i^2 = 1} \sqrt{\sum_{k \neq i}^{n} \frac{J_i^2}{\gamma^2}}. \tag{26}$$

It is important to note that the parameter ε is not defined based on arbitrarily prescribed thermostat temperatures T_i. Instead, ε is determined by identifying the set of thermostat temperatures that minimizes the sum of the squared heat fluxes in the system. This minimum is nothing more than the smallest singular value of the transfer matrix $\hat{S}$.

For the conservative system, the value of this minimum is zero and the minimum point describes the temperature distribution between the heat baths under DB. Thus, the condition of $\varepsilon = 0$ is the criterion of conservativity. Note that the term "conservativity" refers specifically to the underlying system of coupled oscillators without considering random and frictional forces, which inherently render the entire thermostat-oscillator system nonconservative. Therefore, the conservativity condition $\varepsilon = 0$ does not depend on the friction induced by the thermostat and is determined by the collective nature of the nonreciprocal interaction between oscillators. For nonconservative nonreciprocal systems, the value of this minimum describes the minimum possible value of the sum of squares of the heat fluxes between the thermostat and the system. Thus, the value of this minimum ε can be used as a measure of the deviation of the nonreciprocal system from the conservative one.

Let us now consider examples of estimating the deviation from conservativity of a nonreciprocal system. As it was shown above, a system of three DOF can be symmetrized up to one of the interactions. Therefore, we will consider without any loss of generality a symmetric system, varying its nonsymmetricity by changing one component of the interaction matrix f_{12}. Additionally, we consider that all masses are equal to unity, the thermostat friction coefficients are equal to 0.1, the diagonal elements of the interaction matrix are the same and equal to unity, and all but one of the nondiagonal elements of the interaction matrix are equal to 0.1. Such a system remains stable for values of f_{12} from -0.125 to 10.9 (without taking friction into account).

Figures 1(a)–1(c) show the dependence of the introduced nonconservativity measure on the values of f_{12}. To resolve all scales, the dependence is split into three separate figures, generally forming their entire range of available values of f_{12}. Figures 1(d)–1(f) show the dependence of the thermostat temperatures minimizing the heat flows (25) on the interaction coefficient f_{12}. In Fig. 1(b), the measure of nonreciprocity turns to zero, reaching a minimum at the point where the linear dynamical system is conservative. As the value of f_{12} increases relative to the point where the system is conservative, the measure of nonconservativity first reaches a maximum of 0.15, and declines to almost zero with further increase. This is because the system becomes by thermodynamic properties

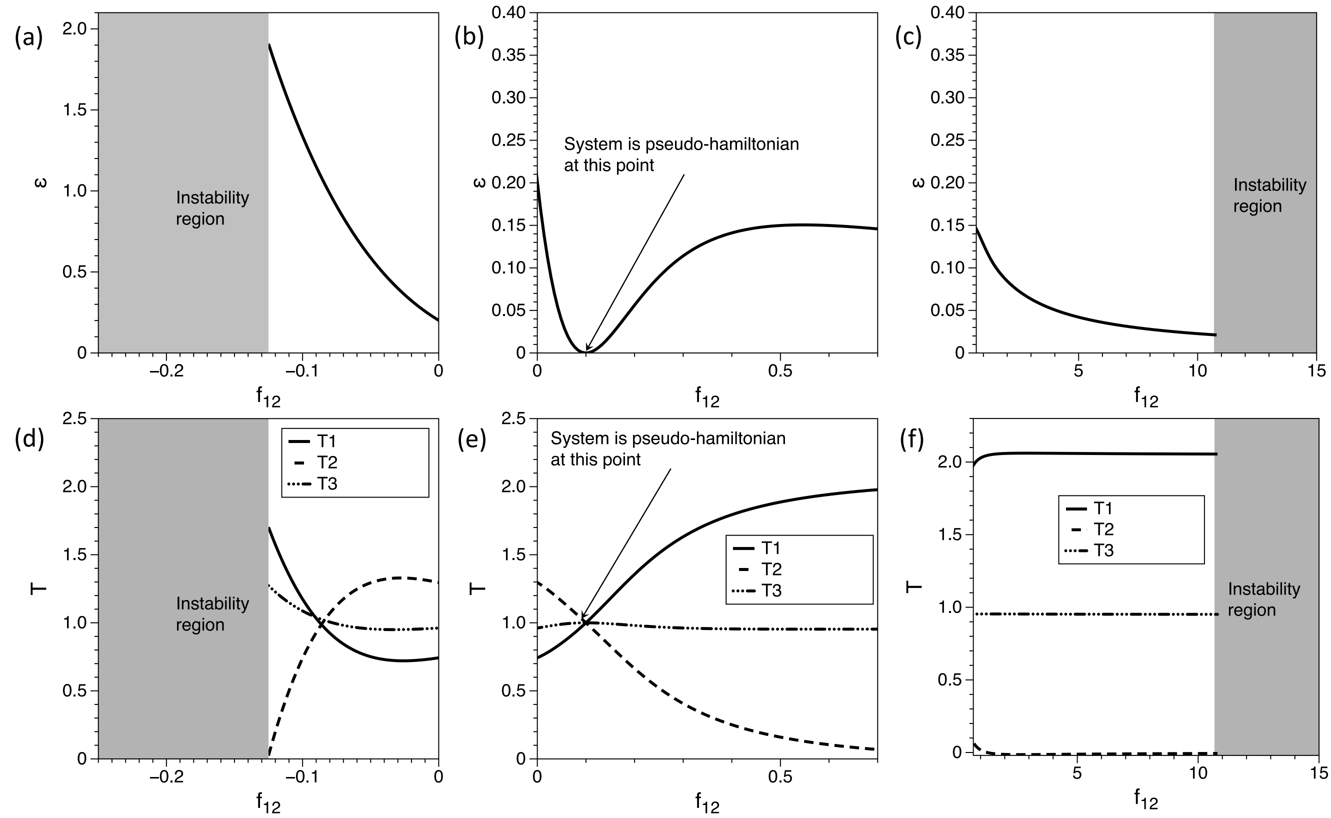

FIG. 1. [(a)–(c)] Dependence of the nonreciprocity measure on the interaction coefficient f_{12}. [(d)–(f)] Dependence of the thermostat temperatures minimizing the heat flows (25) on the interaction coefficient f_{12}. The gray color in the figures indicates the zone of dynamic instability of the system (without taking friction into acount).

close to the system with two DOF for which the system is conservative at any interaction matrices. When shifting left from the point where the system is conservative the measure of nonconservativity increases dramatically. It is interesting to note that the thermostat temperature corresponding to the second DOF turns to zero at the point near the left boundary of system's stability region. Approaching the right boundary of the stability region, the thermostat temperature corresponding to the second DOF crosses the value of zero and remains near it. We associate such behavior of temperatures of thermostats minimizing heat fluxes near the points of dynamical stability of the system with the appearance of poles of the Green's function in the region of complex frequencies.

An important question concerning the introduced measure of nonconservativity is what values of it should be considered large and what values should be considered small. This question is nontrivial and in this paper we only partially answer it. To answer it, we need to set some reference value against which we can judge the magnitude of the nonconservativeness measure. For example, such a reference could be tied to the value of the measure of nonconservativity at which properties uncharacteristic of conservative systems are manifested. These properties are discussed in the next section, in particular the relation between the characteristics of the transfer matrix $\hat{S}$ and the degree of nonconservativity of the system.

It is also important to note that the introduced measure of nonconservativity of the coupled oscillator system depends on the friction coefficients due to the presence of the thermostat. For certainty, one can consider the measure of nonconservativity for some fixed friction coefficient, as it was done for the considered example of the system with three DOF.

C. Loss of conservativity by long-range interactions

The conservativity condition imposes a set of relations on the nondiagonal elements of the interaction matrix. As a result, of the $n^2 - n$ nondiagonal elements, $(n^2 - n)/2 + n - 1$ can be chosen arbitrarily. The remaining $(n^2 - n)/2 - n + 1$ are determined by the conservativity conditions (8). For comparison, the symmetry condition of the interaction matrix allows half of the nondiagonal elements to vary arbitrarily, which is only $n - 1$ less than is obtained under the conservativity condition. Thus, as n grows, the probability that an arbitrary asymmetric interaction matrix is conservative decreases sharply. This is due to the fact that the number of elements of the interaction matrix grows quadratically with increasing n, while the number of additional varying elements with respect to the symmetric matrix grows linearly.

From this we could conclude that there are almost no conservative or weakly nonconservative nonreciprocal physical systems with a large number of degrees of freedom. However, this conclusion turns out to be erroneous. This is because, in practice, interaction matrices are often sparse and nonzero elements are concentrated along the main diagonal, due to the prevalence of short-range physical systems. For example, a one-dimensional chain of nonreciprocally coupled

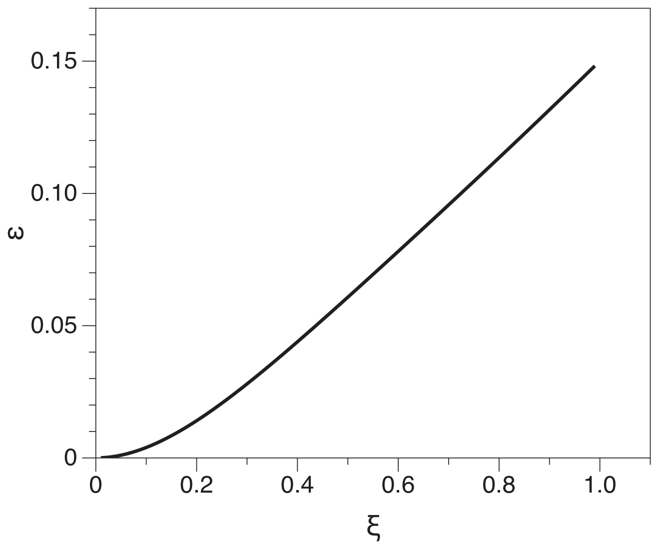

FIG. 2. Dependence of the nonreciprocity measure on the scaling parameter ξ for the linear chain of nonreciprocally coupled oscillators with four degrees of freedom.

mechanical oscillators interacting only with nearest neighbors with open boundary conditions is conservative according to Eq. (8) (in the case when $f_{ij}f_{ji} > 0$) independently of its size. Such a model is often used to describe various physical systems, including metamaterials [14,62], colloids [63], and complex plasmas [64].

However, with the appearance of long-range interactions the conservativity of the system disappears. As an example, we consider a system with four degrees of freedom with the following interaction matrix:

$$\hat{F} = \begin{bmatrix} f_{11} & f_{12} & \xi f_{13} & \xi f_{14} \\ f_{21} & f_{22} & f_{23} & \xi f_{24} \\ \xi f_{31} & f_{32} & f_{33} & f_{34} \\ \xi f_{41} & \xi f_{42} & f_{43} & f_{44} \end{bmatrix}, \tag{27}$$

where ξ is the scaling parameter, which regulates how significant the contribution of long-range interactions is. The value of the parameter $\xi = 0$ corresponds to a system with interaction only with nearest neighbors, and $\xi = 1$ to a long-range system. Figure 2 shows one of the possible dependencies of the introduced nonreciprocity measure ε of the considered nonreciprocal system on the scaling parameter ξ. For small values of ξ, ε grows quadratically from ξ, and for larger values this growth becomes linear. A characteristic feature of such dependencies is the growth of the nonreciprocity measure as the contribution of long-range interactions increases. Thus, nonreciprocal systems with a significant contribution of long-range interactions tend to be far from conservative.

IV. EFFECTS BEYOND NONRECIPROCITY AND NONCONSERVATIVITY

Here, we consider effects arising in systems of nonreciprocally coupled mechanical oscillators after their long-term exposition to a system of Langevin thermostats. We discuss which of them can be observed already in conservative systems with nonreciprocal interactions, and which are specific exclusively for nonconservative systems.

A. Equipartition theorem

One of the fundamental theorems of classical statistical mechanics is the equipartition theorem. One of the consequences of the equipartition theorem is the equality of time-averaged kinetic energies between the DOF of a conservative ergodic mechanical system. The system of coupled linear mechanical oscillators is not ergodic due to its integrability and impossibility of energy transfer between different modes [65]. At the same time, considering such a system in contact with a Langevin thermostat, which can reflect the placement of the system in a gas at a given temperature, the equipartition theorem comes into play. It allows us to conclude that, after a certain time, the mean kinetic energies of the coupled oscillators will equalize. They will also equalize with the temperature of the environment. Such equipartition will be fulfilled for systems with a symmetric interaction matrix, but its asymmetry will lead to a violation of the equipartition.

For conservative nonreciprocal systems we can formulate some generalization of the considered consequence of the equipartition theorem. For this purpose we will consider a system of coupled oscillators in contact with Langevin baths, whose temperatures are not necessarily equal to each other [Eq. (15)]. Since the system is conservative, its interaction matrix $\hat{F}$ can be reduced to a symmetric matrix by multiplying the system by a diagonal positive matrix $\hat{D}$. Substitution of the system temperatures $\hat{T} \to \hat{D}\hat{T}$ and masses $\hat{M} \to \hat{D}\hat{M}$ by effective ones leads the system to a symmetric form, for which the conclusions about the equipartition of kinetic energies are already valid. Thus, for conservative systems of coupled linear mechanical oscillators with asymmetric interaction matrix, some generalization of the equipartition theorem is fulfilled. According to this generalization, it is possible to select such a set of thermostat temperatures at which the average kinetic energies of the DOF will be equal to the temperatures of the associated thermostats. In this case, the relationship between thermostat temperatures as well as average kinetic energies will be determined by the ratio of the elements of the diagonal matrix $\hat{D}$ by which the dynamical system must be multiplied to obtain a symmetric interaction matrix.

In the case of nonreciprocal nonconservative systems such generalization of the equipartition theorem is impossible. That is, it is impossible to select such a set of thermostat temperatures at which the average kinetic energy of the degrees of freedom would equal the temperatures of the thermostats corresponding to them. In order to show that, it is necessary to use the previously obtained conclusion about the impossibility of a detailed balance in a nonconservative system and the fluctuation-dissipation theorem (FDT). Indeed, according to the FDT, the heat flux between an oscillator and its associated thermostat can be expressed through their temperature difference [33]:

$$J_i = \gamma_i\left(T_i - T_i^{\mathrm{eff}}\right), \tag{28}$$

$$T_i^{\mathrm{eff}} = \frac{1}{\pi}\sum_{k=1}^{n}\gamma_k T_k \int_{-\infty}^{\infty} \omega^2 |\hat{G}_{ik}|^2 d\omega, \tag{29}$$

where T_i^{eff} is the mean kinetic energy of the ith oscillator. Note that Eq. (28) is Eq. (17) with substituted term $\langle \dot{q}_i \eta_i \rangle = \gamma_i T_i$ according to the FDT. Therefore, the impossibility of selecting a set of thermostat temperatures in the case of a nonconservative system follows directly from the impossibility of zeroing all heat flows. The introduced measure of nonconservativity thus shows the deviation in a nonconservative system from the formulated generalization of the equipartition theorem.

A possible application of the results obtained here is related to the description of the kinetic energy distribution in systems of nonreciprocal nonlinear coupled oscillators. The point is that, at a certain degree of nonlinearity, dynamical chaos arises and the system loses its connection with the initial conditions [65–67]. Thus, in contrast to linear systems, the ergodic hypothesis is valid for them. We therefore expect that the generalization of the equipartition theorem for linear nonreciprocal systems formulated here will be useful in describing the properties of the long-term dynamics for a class of nonreciprocal nonlinear systems.

B. Non-Hermitian behavior of transfer matrix

The transfer matrix $\hat{S}$ links the heat flows between the oscillator system and the thermostat system. According to the FDT, expression (25) for the elements of the transfer matrix can be rewritten as follows:

$$s_{ij} = \delta_{ij} - \frac{1}{\pi}\gamma_j \int_{-\infty}^{\infty} \omega^2 |\hat{G}_{ij}|^2 d\omega, \tag{30}$$

where δ_{ij} is the Kronecker delta. Thus, the relationship between oscillators mean kinetic energies $\vec{T}^{\text{eff}}$ and thermostat temperatures $\vec{T}$ can be expressed through the $\hat{S}$ matrix,

$$\vec{T}^{\text{eff}} = (\hat{I} - \hat{S})\vec{T}, \tag{31}$$

where $\hat{I}$ is the identity matrix.

In the case when the interaction matrix is symmetric, the Green's function matrix is also symmetric, which leads to the fact that the matrices $(\hat{I} - \hat{S})$ and $\hat{S}$ are Hermitian (symmetric) and therefore have real eigenvalues. In the case when the initial interaction matrix is asymmetric but the system is conservative, multiplication by the diagonal matrix $\hat{D}$ reduces the system to a symmetric one, but the resulting thermostat temperatures in the final system, $\vec{T}_{fi}$, are related to the initial ones $\vec{T}_{in}$ by multiplication by the matrix $\vec{T}_{fi} = \hat{D}\vec{T}_{in}$. Denote the transition matrix of the initial system as $\hat{S}_{in}$, and of the symmetrized system as $\hat{S}_{fi}$ which is symmetric; then heat flow

$$\vec{J} = \hat{L}\hat{S}_{fi}\vec{T}_{fi} = \hat{L}\hat{S}_{fi}\hat{D}\vec{T}_{in} = \hat{L}\hat{S}_{in}\vec{T}_{in}, \tag{32}$$

and

$$\hat{S}_{in} = \hat{S}_{fi}\hat{D}. \tag{33}$$

Thus, the matrix $\hat{S}_{in}$ is the product of a symmetric matrix $\hat{S}_{fi}$ by a diagonal matrix $\hat{D}$. The following equality is satisfied for the matrix $\hat{S}_{in}$:

$$\hat{D}\hat{S}_{in} = \hat{S}_{in}^T\hat{D}, \tag{34}$$

where $\hat{D}$ is the Hermitian and positive defenite. Therefore, the transfer matrix of a conservative nonreciprocal system is quasi-Hermitian [57] and all its eigenvalues are real. This means that when the parameters of the dynamical system change in the region of its conservativity, there can be no imaginary eigenvalues and exceptional points (EPs) in the transfer matrix $\hat{S}$.

However, the situation will change in the region of nonconservativity, the indicator of which is a onzero measure of nonconservativity. In this region, the transfer matrix is no longer quasi-Hermitian, and hence the existence of imaginary eigenvalues, EPs, and nonorthogonality of eigenvectors is possible [57]. In Fig. 3, we have summarized the obtained relation between the non-Hermitian behavior of the transfer matrix and the conservativity and reciprocity characteristics of its corresponding dynamical system.

To demonstrate the possibility of EPs and imaginary eigenvalues in the transfer matrix, consider a one-dimensional chain of coupled oscillators with nonreciprocal interaction of nearest neighbors and semi-infinite boundary condition with the following interaction matrix:

$$\hat{F} = \begin{bmatrix} 1.0 & 0.14 & 0.06\xi \\ 0.06 & 1.0 & 0.14 \\ 0.14\xi & 0.06 & 1.0 \end{bmatrix}, \tag{35}$$

where ξ is the parameter varying from 0 to 1. $\xi = 0$ corresponds to the open boundary conditions and $\xi = 1$ to the periodic one. We also assume that all masses are equal to 1 and that the coefficients of friction are equal and equal to 0.1. In the case of the open boundary conditions such a nonreciprocal system will be conservative (without taking friction into account). However, in the case of the appearance of binding of two ends of the chain, the system loses its conservativity.

Figure 4 shows the dependences of the real and imaginary parts of the eigenvalues of the transfer matrix as a function of the parameter ξ. As can be seen, for small values of the parameter ξ all eigenvalues of the transfer matrix remain purely real (the region of small nonconservativity). However, at some value of ξ an EP is observed where two eigenvalues coalesce. As ξ increases further, the eigenvalues of the transfer matrix are characterized by nonzero imaginary parts.

It should be noted that the dynamical system itself in the whole range of variation of the parameter ξ is stable (taking into account friction); i.e., it does not have unbounded solutions. Thus, the long-term behavior of the oscillator system coupled to the heat baths is indeed determined by its Green's function and consequently by the transfer function.

As is known, the presence of exceptional points in the system can lead to a number of observable effects [57]. In particular, in the vicinity of EPs one can expect increased sensitivity of the long-term dynamics at large times when its parameters are changed.

The occurrence of EPs in the transfer matrix $\hat{S}$ in the region of nonconservativity gives a principal possibility to choose a reference value to characterize the smallness of the introduced nonreciprocity measure. For example, if for some classes of nonreciprocal systems EPs will appear only after some value of the nonconservativity measure, then this threshold value can be used as the reference value. However, this question requires further study and is not exhausted in this paper.

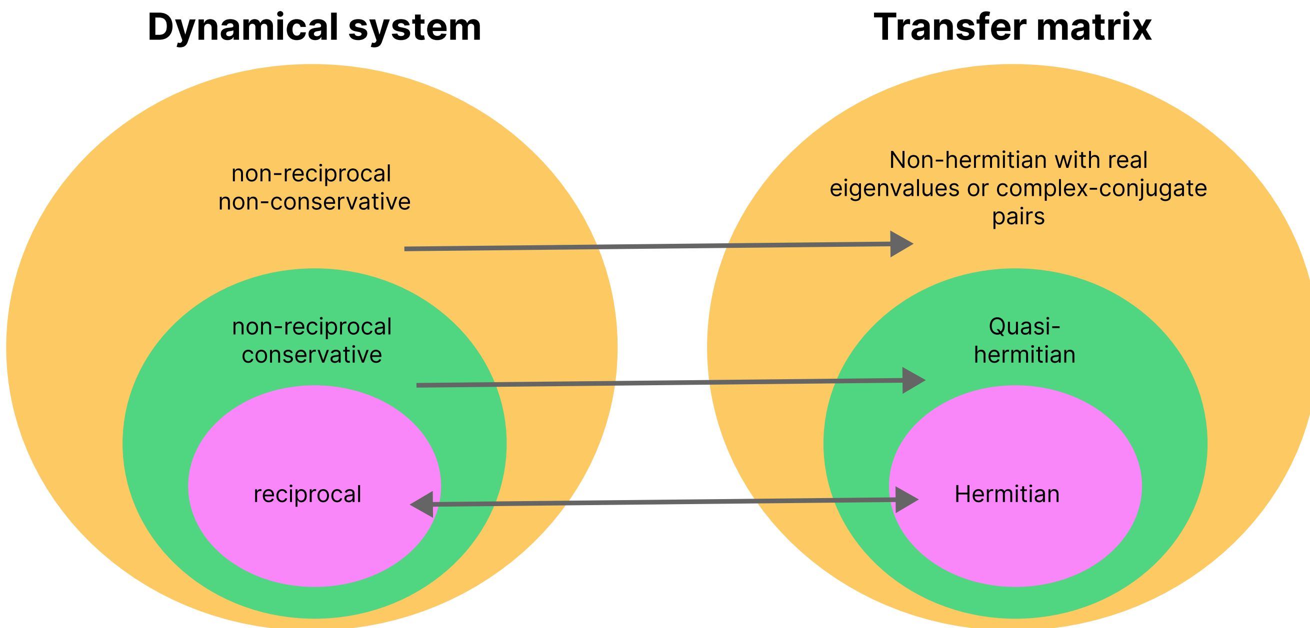

FIG. 3. Correspondence between the conservativity and reciprocity conditions of a dynamical system and the non-hermitian behavior of its transfer matrix.

C. Heat conductance

We will consider heat transfer between sets of heat reservoirs through a system of nonreciprocally coupled oscillators. In this case, we will assume that heat flowing in or out of a particular heat reservoir leads to a slow change in its temperature. Physically, such a system can be represented as a set of isolated gas systems having a certain temperature and in contact with the corresponding oscillator. Thus, heat can enter or leave the heat reservoir only through an action with a corresponding oscillator. The rate of temperature change in such a system is determined through the heat fluxes from the heat reservoir to the corresponding oscillator,

$$\frac{d\vec{T}}{dt} = \kappa \vec{J} = \kappa \gamma \hat{S} \vec{T}, \tag{36}$$

where κ is the kinetic coefficient that relates the rate of temperature change of the reservoirs to the corresponding heat fluxes. We have also assumed that all thermostat friction coefficients are equal to γ.

In case the system of oscillators is reciprocal, all the eigenvalues of the matrix $\hat{S}$ are real and nonpositive and the system relaxes to the thermodynamic equilibrium at which

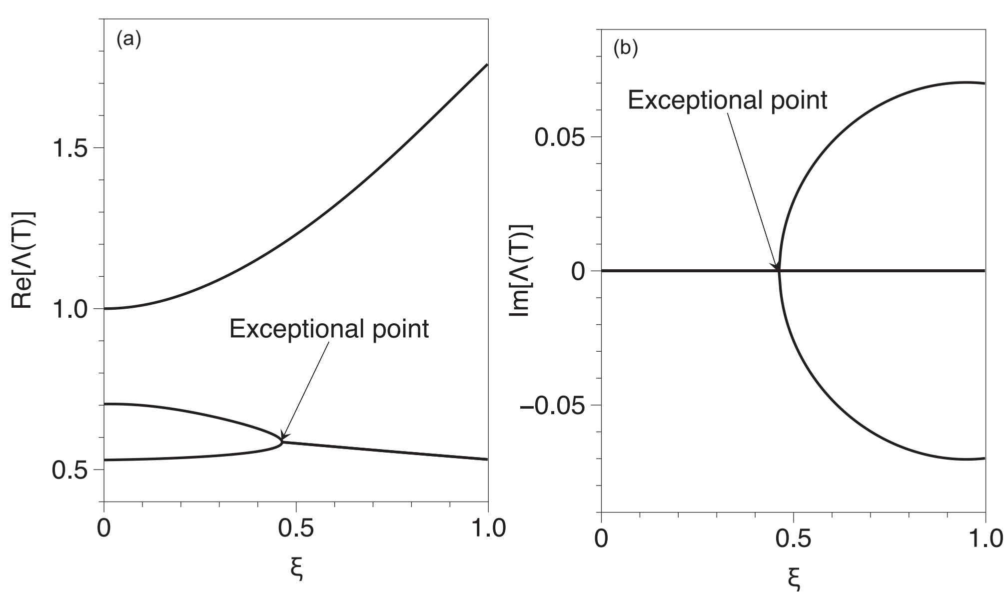

FIG. 4. Dependence of eigenvalues from the spectrum $\Lambda(\hat{S})$ of the transfer matrix on the degree of interaction between the two ends of a one-dimensional chain of coupled oscillators ξ: (a) the real part of eigenvalues and (b) the imaginary part.

all temperatures of the reservoirs become equal, i.e., lie in the kernel of the matrix $\hat{S}$. In the case of a nonreciprocal but conservative system of coupled oscillators the matrix $\hat{S}$ also has a maximal eigenvalue equal to zero and relaxes to some set of temperatures, which is determined by the generalized equipartition theorem discussed above. The nonuniformity of the steady-state temperature distribution leads to the possibility of reverse heat flow, which manifests itself in the simplest example of a system of oscillators with two degrees of freedom, which is always conservative. This makes it possible to use a conservative system of two coupled oscillators as a nonreciprocal refrigerator and a "Maxwell demon" [47].

However, in the case of a nonreciprocal and nonconservative system, the nature of the solution of Eq. (36) changes significantly. First, the matrix $\hat{S}$ is no longer degenerate and the maximum eigenvalue of the matrix $\hat{S}$ becomes greater than zero, leading to exponentially growing solutions. In order to avoid an exponential temperature rise in the system, heat can be additionally dissipated from the respective heat reservoirs. The second peculiarity which appears in the system of thermal baths connected by a nonconservative system of oscillators is the emergence of oscillatory solutions. This is due to the fact that, in the region of non-quasi-Hermiticity of the matrix $\hat{S}$, complex eigenvalues may appear. Thus, the coupling of thermal reservoir systems by means of a nonconservative system of oscillators can lead to an oscillatory character of their temperature changes, which is impossible in conservative nonreciprocal systems, in particular in the often considered system of nonreciprocal oscillators with two DOF. Experimental verification and investigation of such an effect can theoretically be realized, for example, in recently emerging systems where tunable nonreciprocal interaction between nanoparticles is achieved by optical feedback [41].

D. Non-Hermitian skin effect

Consider a system of $n = 10$ coupled linear mechanical oscillators with a nonreciprocal interaction matrix

$$\hat{F} = \begin{bmatrix} 1.0 & 0.16 & 0.0 & \cdots & 0.0 & 0.0 & 0.0 \\ 0.04 & 1.0 & 0.16 & \cdots & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.04 & 1.0 & \cdots & 0.0 & 0.0 & 0.0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0.0 & 0.0 & 0.0 & \cdots & 1.0 & 0.16 & 0.0 \\ 0.0 & 0.0 & 0.0 & \cdots & 0.4 & 1.0 & 0.16 \\ 0.0 & 0.0 & 0.0 & \cdots & 0.0 & 0.4 & 1.0 \end{bmatrix}. \tag{37}$$

This matrix corresponds to the Hamiltonian of the Hatano-Nelson model with tight binding under open boundary conditions (OBC) [62,68,69]. The eigenvectors of such a matrix are localized due to the non-Hermitian skin effect [70–74]. In order to understand the impact of this fact on the long-term properties under the influence of the Langevin thermostat system, we analyze the behavior of the eigenvectors of the matrix $\hat{I} - \hat{S}$. This matrix connects the average kinetic energies of the oscillators to the temperatures of the thermostats according to Eq. (31). The dependence of the component modulus of its eigenvectors on the system nodes is shown in Fig. 5 (friction coefficient $\gamma = 0.01$). According to this figure, localization of eigenvectors of the matrix $\hat{I} - \hat{S}$ at the left boundary of the system is observed. Thus, a non-Hermitian skin effect arises for the matrix $\hat{I} - \hat{S}$. It will result in direct amplification of the average kinetic energies of the oscillators when the system is coupled to a thermostatic environment [50].

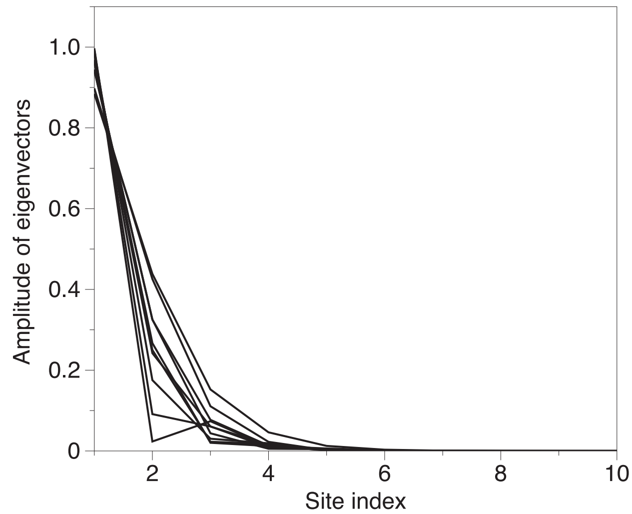

FIG. 5. Localization of eigenvectors of the transfer matrix $\hat{S}$ near the chain boundary.

Let us now discuss the connection of the arising non-Hermitian effect with the conditions of nonreciprocity and nonconservativity of the system of coupled linear mechanical oscillators. The considered model with interaction matrix is nonreciprocal but conservative. The elements d_i of the diagonal matrix $\hat{D}$, which symmetrizes the dynamical system, are expressed as follows:

$$d_i = \left(\frac{0.4}{0.16}\right)^{(i-1)}. \tag{38}$$

The transfer matrix $\hat{S}$ is the product of the symmetric matrix and the matrix $\hat{D}$. In the case if the matrix $\hat{D}$ were the identity matrix, i.e., we would be dealing with a reciprocal system, the non-Hermitian skin effect would not occur. However, multiplication of the symmetric matrix by a diagonal matrix $\hat{D}$ with exponential inhomogeneity of its elements leads to a sharp deviation of the matrix $\hat{S}$ from normality, which, according to [75], leads to the enhancement of the skin effect. Thus, the manifestation of the non-Hermitian skin effect in the long-term dynamics in contact with the thermostat is possible already in conservative systems with a nonreciprocal interaction matrix. At occurrence of interaction between two boundary oscillators the non-Hermitian skin effect decreases, and the measure of nonconservativity, on the contrary, increases. Therefore, one can suppose that deviation from conservativity of the system of coupled oscillators, on the contrary, leads to suppression of the non-Hermitian skin effect.

V. POTENTIAL EXPERIMENTAL VALIDATION AND REALIZATIONS

Recent experimental studies on heat flux and entropy production in systems driven by temperature gradients demonstrate the feasibility of experimentally probing the nonequilibrium dynamics of small coupled systems. In particular, systems such as two hydrodynamically coupled Brownian particles in an optical trap [42], resistors coupled via electric thermal noise [76], and single-electron boxes [77] with reservoirs at different temperatures have provided insights into fluctuation theorems, entropy production, and heat transport under steady nonequilibrium conditions. These experiments demonstrate the capability to precisely manipulate and measure energy exchange and thermodynamic properties of systems, providing a platform for testing theoretical predictions in real-world scenarios.

Validating theoretical predictions for nonreciprocal systems imposes specific requirements on the experimental setup. Such a system must allow for precise temperature control of its components and enable regulation of the degree of nonreciprocity in their interactions. We will discuss an approach to experimental verification exemplified by a promising and rapidly developing method involving the creation of tunable nonreciprocal interactions between optically levitated particles [41]. In such systems, the dynamics of particles can be described by a model of harmonically coupled oscillators connected to the system of Langevin heat baths in accordance with Eq. (15). Nonreciprocal coupling is achieved through carefully controlled phase-coherent optical tweezers, where the trapping lasers impart directionally dependent forces, thereby breaking the symmetry of the interaction. The frictional effects arise naturally from the collisions between the particles and the surrounding medium, introducing dissipative dynamics. An effective temperature can be imposed on one of the particles by sending a controlled Gaussian white noise to the acousto-optic deflector in such a way that the position of the corresponding trap is moved randomly along the direction where the particles are aligned [42,45].

Within such an experimental system, several phenomena described in the present theoretical work can be directly investigated. For instance, a system of three particles under two distinct scenarios—nonreciprocal and conservative versus nonreciprocal and nonconservative interactions—offers an opportunity to examine the generalized equipartition theorem outlined in Sec. IV A. In the first case, it is expected that it will be possible to select a set of thermostat temperatures (experimentally controlled by stochastic modulation of the traps) such that the average kinetic energies of the associated degrees of freedom match the effective temperatures of the thermostats. However, for a nonconservative nonreciprocal system, this balance cannot be achieved, which can be demonstrated experimentally with the help of the described setup.

Furthermore, the experimental setup potentially allows the investigation of the oscillatory relaxation of temperatures in a system of three nonreciprocally coupled oscillators, as described in Sec. IV C. This phenomenon arises when the transfer matrix connecting heat fluxes and temperatures contains complex eigenvalues. Experimentally, this can be achieved by choosing interaction parameters that induce such eigenvalues in the transfer matrix. After setting initial thermostat temperatures (experimentally controlled by stochastic modulation of the traps) and allowing the system to reach steady-state kinetic energies, the heat fluxes between each particle and its corresponding thermostat can be calculated from the particle trajectories using the definition in Eq. (17). By iteratively varying the intensity of the stochastic modulation of the traps based on the measured heat fluxes, the heat exchange between the thermal reservoirs can be modeled. The emergence of oscillatory behavior in the mean kinetic energies of the particles over time would confirm the theoretical predictions regarding the presence of exceptional points in the system.

Finally, the experimental platform enables the direct observation of temperature localization effects, as discussed in Sec. IV D. By designing the particle interactions to mimic the conditions described, one can induce the localization of the average kinetic energies near one edge of the particle chain. This phenomenon can be experimentally visualized and quantitatively analyzed, offering deeper insights into the interplay between nonreciprocal interactions and energy distribution in coupled systems.

Overall, the discussed experimental systems represent a highly adaptable and precise means of validating the theoretical results obtained in this work. The ability to control interaction strengths, impose effective temperatures, and measure heat fluxes with high accuracy opens up significant opportunities for advancing the understanding of nonreciprocal and nonconservative dynamics in coupled oscillator systems.

VI. CONCLUSIONS

In this paper, we have considered the influence of the conservativity and reciprocity condition on the long-term dynamics of a system of coupled linear oscillators in contact with a system of Langevin thermostats. For this purpose, we have used the Green's function formalism and developed a closed system of equations relating the time averages of all possible quadratic combinations of variables $\dot{q}_i, q_i$ [Eqs. (18)–(22)]. This approach allowed us to analyze the transport matrix $\hat{S}$ relating heat fluxes to thermostat temperatures in terms of the physics of non-Hermitian systems. We found that the conservativity condition has a determining influence on the long-term thermodynamic properties of such a system.

Here are our main findings:

Reciprocal conservative systems

(1) The transfer matrix $\hat{S}$ is singular Hermitian (symmetric) with maximum eigenvalue and minimum singular value equal to zero.

(2) When the isolated thermal reservoirs are connected to each other by a reciprocal system of coupled oscillators, the temperatures of the reservoirs come to the same steady-state value.

Nonreciprocal conservative systems

(1) The transfer matrix $\hat{S}$ is singular quasi-Hermitian with maximum eigenvalue and minimum singular value equal to zero.

(2) The stationary values of the mean kinetic energies of the oscillators are different when in contact with a thermostat system with the same temperature.

(3) Reverse heat fluxes and non-Hermitian skin-effect may occur.

(4) When isolated thermal reservoirs are connected to each other by a nonreciprocal conservative oscillator system,

the temperatures of the reservoirs arrive at an unequal steady-state value.

Nonreciprocal nonconservative systems

(1) The transfer matrix $\hat{S}$ is nonsingular and non-Hermitian. It may contain exceptional points and complex eigenvalues. The maximum eigenvalue and the minimum singular value are greater than zero.

(2) The stationary values of the mean kinetic energies of the oscillators are different when in contact with a thermostat system with the same temperature.

(3) Reverse heat fluxes and non-Hermitian skin-effect may occur.

(4) When isolated thermal reservoirs are connected to each other by a nonreciprocal nonconservative oscillator system, reservoir temperatures do not come to a stationary value and their evolution can have an oscillatory character.

Moreover, we discuss the experimental setup of optically levitated particles with tunable nonreciprocal interactions which can be incorporated to directly demonstrate and validate theoretical finding obtained in the present work.

To characterize nonconservative systems we have also developed a measure of nonreciprocity, which is equal to zero for conservative reciprocal and nonreciprocal systems and increases when the system deviates from conservativity. The introduced measure can be used to characterize and classify non-Hermitian properties of nonreciprocal systems. In addition, the nonreciprocity measure can be used to determine the conservativity of a wide class of nonreciprocal physical systems whose parameters are known with some error.

In summary, the results of our work can be useful for experimental and theoretical studies in the field of metamaterials, active matter, dusty plasma, colloidal systems, stochastic thermodynamics, and physics of non-Hermitian systems.

ACKNOWLEDGMENTS

The work of D.A.K. was supported by the Foundation for the Advancement of Theoretical Physics and Mathematics “BASIS”. The work of A.V.T. was partly supported by the HSE University Basic Research Program.

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