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Multidimensional Euler and Poisson summation formulas

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The well-known Euler-MacLaurin summation formula

$$\sum_{a \leq k \leq b} f(k) - \tfrac{1}{2}(f(a)+f(b)) = \int_a^b f(x)\,dx + \tfrac{1}{12}(f'(b)-f'(a)) - \int_a^b f''(x)G(x)\,dx \tag{1}$$

with integers a, b and Euler's function

$$G(x) = \frac{1}{2\pi^2} \sum_{n=1}^{\infty} \frac{\cos(2\pi n x)}{n^2} \tag{2}$$

is a useful tool to discuss special summation problems. It is not difficult to deduce from (1) and (2) the identity

$$\sum_{a \leq k \leq b} f(k) - \tfrac{1}{2}(f(a)+f(b)) = \sum_{n=-\infty}^{+\infty} \int_a^b f(x) e^{2\pi i n x}\,dx \tag{3}$$

which is known as the Poisson summation formula.

One of us presented in [11] an algorithm which can be regarded as a generalization of the Euler summation formula to twodimensional problems.

The main concept of this formula was the observation, that the Euler function $G(x)$ is closely related to Green's function for the second order differential operator $(d/dx)^2$ and the boundary condition of periodicity.

In multidimensional problems the second order differential operator may be taken as a linear second order differential operator with constant coefficients. In this paper we base our summation formulas on the Laplace-operator and admit arbitrary lattices. The theory thus contains all summation formulas which are based on second order elliptic differential operators with constant coefficients.

We first give a short survey of the concepts used to describe q-dimensional lattices. Then we define the simplest elliptic operators in $\mathbf{R}^q$ and Green's function to the "boundary condition" of periodicity, which we call lattice function.

The "Euler summation" formula then follows by a standard application of Green's integral theorem. For the application of the summation formula a study of its different terms is essential. We confine our applications to spherical summations and discuss these cases in detail. We thus obtain new theorems on multidimensional convergence.

By use of expansion theorems for the lattice function an analogue of the Poisson summation formula can be obtained.

A generalization of the Hardy–Landau identity concludes the paper.

1. Definitions and notations

$\mathbf{R}^q$ denotes the q-dimensional (real) Euclidean space. We consistently write $x=(x_1,\ldots,x_q)$, $y=(y_1,\ldots,y_q),\ldots$ for the elements of $\mathbf{R}^q$. Inner product and norm are defined as usual by

$$xy := \sum_{i=1}^{q} x_i y_i, \qquad x^2 := xx, \qquad |x| := \sqrt{xx}. \tag{1}$$

To describe problems of spherical symmetry polar coordinates

$$x = r\xi \tag{2}$$

with $r=|x|$ and $\xi \in R^q$; $\xi^2=1$ are introduced. The unit sphere $x^2=1$ is denoted by Ω_q, its surface measure by

$$\omega_q = \int_{\Omega_q} dS = 2\frac{\pi^{q/2}}{\Gamma(q/2)}. \tag{3}$$

Let Λ be a non-degenerate lattice in $\mathbf{R}^q$ and $g^1,\ldots,g^q$ a basis of this lattice. We may assume that this basis is a reduced basis in the sense of the Geometry of Numbers [1] though we never use this property explicitly.

There exists an inverse lattice Λ^{-1}, generated by a basis $h^1,\ldots,h^q$ which satisfies

$$h^i g^k = \delta^{ik}. \tag{4}$$

If it seems relevant to state the dimension of the lattice we write Λ_q instead of Λ.

We then define the fundamental region $F = F(\Lambda)$ as the set of all $x \in \mathbf{R}^q$ with

$$-\tfrac{1}{2} \leq xh^i < \tfrac{1}{2} \tag{5}$$

for $i = 1, \ldots, q$. The volume $D = D(\Lambda)$ of the fundamental region is given by

$$D = \int_F dV = |g^1, \ldots, g^q| > 0. \tag{6}$$

Suppose next that $\mathfrak{G}$ is a regular region, i.e., a region with boundary $\partial\mathfrak{G}$ for which Green's theorem is valid. We shall say that f belongs to the class $C^{(m)}(\bar{\mathfrak{G}})$ if it has derivatives of order m in $\mathfrak{G}$ which are continuous in $\bar{\mathfrak{G}} := \mathfrak{G} \cup \partial\mathfrak{G}$.

Denote by

$$\Delta = \nabla\nabla = \left(\frac{\partial}{\partial x_1}\right)^2 + \cdots + \left(\frac{\partial}{\partial x_q}\right)^2 \tag{7}$$

the Laplace-operator and by

$$\frac{\partial}{\partial n} = n\nabla \tag{8}$$

the operator of the normal derivative. Then, for each complex number λ Green's integral theorem yields

$$\begin{aligned} &\int_{\mathfrak{G}} \{\Phi(x)(\Delta + \lambda)\Psi(x) - \Psi(x)(\Delta + \lambda)\Phi(x)\}\, dV \\ &= \int_{\partial\mathfrak{G}} \left\{\Phi(x)\frac{\partial}{\partial n}\Psi(x) - \Psi(x)\frac{\partial}{\partial n}\Phi(x)\right\} dS. \end{aligned} \tag{9}$$

Let us write as usual $e(ax) := e^{2\pi i(a_1x_1 + \cdots + a_qx_q)}$ for a, $x \in \mathbf{R}^q$. Then the functions ϕ_h defined by

$$\phi_h(x) := \frac{1}{\sqrt{D}}\, e(hx) \qquad (h \in \Lambda^{-1}, x \in \mathbf{R}^q) \tag{10}$$

are periodic with respect to the lattice Λ, i.e.,

$$\phi_h(x + g) = \phi_h(x) \quad \text{for all} \quad g \in \Lambda. \tag{11}$$

A simple calculation yields

$$\Delta_x \phi_h(x) = -4\pi^2 h^2 \phi_h(x). \tag{12}$$

We shall say that λ is an eigenvalue of the lattice Λ with respect to the operator Δ if there is a nontrivial solution U of the differential equation

$$(\Delta + \lambda) U(x) = 0 \tag{13}$$

which satisfies $U(x+g) = U(x)$ for all $g \in \Lambda$.

The function U then is called eigenfunction of the lattice with regard to the eigenvalue λ and the operator Δ.

Since the solutions of (13) are analytic, application of the standard multidimensional Fourier analysis shows that the functions $\phi_h(x)$ defined in (10) are the only eigenfunctions. Thus the numbers

$$\lambda_h := 4\pi^2 h^2; \qquad h \in \Lambda^{-1} \tag{14}$$

are the eigenvalues of the lattice with regard to the operator Δ. The set of all eigenvalues is the spectrum $S(\Lambda)$. It is easy to see that the system $\{\phi_h(x) | h \in \Lambda^{-1}\}$ is orthonormal in the sense of

$$\int_F \phi_h(x) \phi_{h'}(x)\, dV = \delta_{hh'} = \begin{cases} 1 & \text{for} \quad h = h' \\ 0 & \text{for} \quad h \neq h'. \end{cases} \tag{15}$$

2. The lattice function

We next define a function associated to the lattice Λ which we call the lattice function. This function is essentially Green's function (in an enlarged sense) to the operator Δ and the "boundary condition" of periodicity with regard to Λ.

Although the procedure to prove the existence and the main properties of the lattice function is nearly standard, a short and direct approach shall be presented here.

DEFINITION 1. A function $G(\lambda; x)$ is called lattice function of the lattice Λ with respect to the operator Δ and the parameter $\lambda \in \mathbf{C}$ if it has the following properties:

i) $G(\lambda; x)$ is twice continuously differentiable for $x \notin \Lambda$ with

$$(\Delta + \lambda) G(\lambda; x) = 0$$

if $\lambda \notin S(\Lambda)$
and

$$(\Delta + \lambda)G(\lambda; x) = \frac{1}{\sqrt{D}} \sum_{\lambda_h = \lambda} \phi_h(x)$$

if $\lambda = \lambda_h \in S(\Lambda)$ where the summation is to be taken over all $h \in \Lambda^{-1}$ with

$$\lambda_h = 4\pi^2 h^2 = \lambda.$$

ii) For all $x \notin \Lambda$ and $g \in \Lambda$

$$G(\lambda; x + g) = G(\lambda; x)$$

is satisfied.

iii) In the neighbourhood of the origin the estimates

$$G(\lambda; x) - \frac{1}{\sigma_q} \frac{1}{|x|^{q-2}} = 0\left(\frac{1}{|x|^{q-3}}\right)$$

$$\nabla G(\lambda; x) - \frac{1}{\sigma_q} \nabla \frac{1}{|x|^{q-2}} = 0\left(\frac{1}{|x|^{q-2}}\right)$$

for $q \geq 3$ and

$$G(\lambda; x) + \frac{1}{\sigma_2} \ln|x| = 0(1)$$

$$\nabla G(\lambda; x) + \frac{1}{\sigma_2} \nabla \ln|x| = 0(1)$$

for $q = 2$ are valid, where

$$\sigma_q := (q-2)\omega_q \quad \text{if} \quad q \geq 3$$

and

$$\sigma_2 := 2\pi.$$

iv) For all

$$\lambda = \lambda_h \in S(\Lambda)$$

and all ϕ_h with $4\pi^2 h^2 = \lambda_h$

$$\int_F G(\lambda; x)\phi_h(x)\, dV = 0$$

is valid.

We summarize the main results of the classical theory. All these results may be derived directly by means of Green's integral theorem using the boundary condition of periodicity and the special form of the singularity of the lattice function.

1. $G(\lambda; x)$ is uniquely determined.
2. Let ϕ_h be an eigenfunction to the eigenvalue λ_h and suppose $\lambda^* \notin S(\Lambda)$, then ϕ_h is a solution of the integral equation

$$\phi_h(x) = (\lambda_h - \lambda^*) \int_F G(\lambda^*; x-z)\phi_h(z)\, dV(z).$$

3. Suppose λ^* is not an eigenvalue, then the identity

$$G(\lambda; x) = G(\lambda^*, x) - \frac{1}{\sqrt{D}} \sum_{\lambda_h = \lambda} \frac{1}{\lambda - \lambda^*} \phi_h(x)$$
$$+ (\lambda - \lambda^*) \int_F G(\lambda^*, x-z) G(\lambda; z)\, dV(z)$$

holds for $x \in F, |x| \neq 0$ and all λ where the sum $\sum_{\lambda_h = \lambda}$ occurs only if λ is an eigenvalue.

Following Hilbert's approach to the theory of Green's function [7] we prove the existence of the lattice function by first giving an explicit representation of the lattice function to the parameter $\lambda^* = -1$. We then use the theory of integral equations with singular kernel to deduce the full theory of the lattice function.

We start with the discussion of the function

$$\phi(q; r) = c_q \int_1^\infty \exp(-rt)(t^2 - 1)^{(q-3)/2}\, dt \tag{1}$$

with

$$c_q := \frac{1}{\omega_q \Gamma(q-1)}. \tag{2}$$

For the dimensions $q=2$ and 3 we get

$$\phi(2;r)=\frac{1}{2\pi}K_0(r) \quad \text{and} \quad \phi(3;r)=\frac{1}{4\pi}\frac{\exp(-r)}{r} \tag{3}$$

where K_0 is the Kelvin function of order zero. The functions $\phi(q;r)$ are proportional to the Kelvin functions of order zero and dimension q. (see [13, §6]).

Here the constants c_q are chosen such that the functions $\phi(q;|x|)$ have the characteristic singularities of the lattice functions as given in Definition 1, (iii)

$$\left.\begin{aligned} &\phi(q;r)-\frac{1}{\sigma_q}\frac{1}{r^{q-2}}=0\left(\frac{1}{r^{q-3}}\right) \\ &\frac{d}{dr}\left(\phi(q;r)-\frac{1}{\sigma_q}\frac{1}{r^{q-2}}\right)=0\left(\frac{1}{r^{q-2}}\right). \end{aligned}\right\} \quad (r\to 0)$$

For $r\to\infty$ we have the estimates

$$\phi(q;r)=0(r^{(1-q)/2}\exp(-r)) \tag{4}$$

The differential equation

$$\left(r^{1-q}\frac{d}{dr}r^{q-1}\frac{d}{dr}-1\right)\phi(q;r)=0 \tag{5}$$

is equivalent to

$$(\Delta-1)\phi(q;|x|)=0 \tag{6}$$

for $|x|\neq 0$.

For $|x|\neq 0$ the functions $\Phi(q;|x|)$ are analytic and can be estimated by

$$|\phi(q;|x|)|=0(|x|^{(1-q)/2}\exp(-x)) \tag{7}$$

for $|x|>1$.

The series

$$\sum_{g\in\Lambda}\Phi(q;|x+g|)=\lim_{N\to\infty}\sum_{|g|\leq N}\Phi(q;|x+g|) \tag{8}$$

therefore converges for all $x\notin\Lambda$.

The limit possesses all the defining properties of the lattice function to the parameter $\lambda^* = -1$. Therefore we have, with $\lambda^* = -1$,

$$G(\lambda^*; x) = \sum_{g \in \Lambda} \Phi(q; |x+g|). \tag{9}$$

We now define $G(\lambda; x)$, λ not an eigenvalue, as the solution of the integral equation

$$G(\lambda; x) = G(-1; x) + (\lambda+1) \int_F G(-1; x-z) G(\lambda; z)\, dV(z). \tag{10}$$

The factor $\lambda+1$ is an eigenvalue of the kernel $G(-1; x-z)$ if and only if $\lambda \in S(\Lambda)$.

The existence of $G(\lambda; x)$ thus follows from the theory of integral equations. This identity also establishes the close relation between the lattice function and the resolvent of the kernel $G(-1; x-z)$.

In the case of an eigenvalue λ we consider

$$G(\lambda; x) = \tilde{G}(-1; x) + (\lambda+1) \int_F G(-1; x-z) G(\lambda; z)\, dV(z) \tag{11}$$

with

$$\tilde{G}(-1; x) := G(-1; x) - \frac{1}{\sqrt{D}} \sum_{\lambda_h = \lambda} \frac{1}{\lambda+1} \phi_h(x). \tag{12}$$

We remember that the functions ϕ_h form a set of orthonormal eigenfunctions. Since the summation (12) is to be extended over all h with

$$\lambda = 4\pi^2 h^2 = \lambda_h \tag{13}$$

we get

$$\int_F \tilde{G}(-1; x) \phi_h(x)\, dV = 0. \tag{14}$$

for all h with $\lambda_h = \lambda$.

The integral equation (11) therefore has a solution which is uniquely determined if we require that

$$\int_F G(\lambda; x)\phi_h(x)\, dV = 0 \tag{15}$$

for all eigenfunctions ϕ_h to the eigenvalue λ.

The theory of Green's function yields several results which we state here. Suppose that f is a bounded function defined in $\mathbf{R}^q$ which satisfies

$$f(x+g) = f(x) \tag{16}$$

for all $g \in \Lambda$. Let this function satisfy a Hölder-condition at the point x_0. Then

$$U(x) = \int_F G(\lambda; x-z) f(z)\, dV(z) \tag{17}$$

is differentiable everywhere and twice differentiable at x_0 with

$$(\Delta + \lambda)U(x) = -f(x) + \frac{1}{\sqrt{D}} \sum_{\lambda_h = \lambda} \int_F f(x)\phi_h(x)\, dV \tag{18}$$

at this point.

We next define the iterated lattice functions.

DEFINITION 2. The functions

$$G^{(1)}(\lambda; x) = G(\lambda; x)$$

$$G^{(m)}(\lambda; x) = \int_F G^{(m-1)}(\lambda; z) G(\lambda; x-z)\, dV(z) \quad \text{for} \quad m = 2, 3, \ldots$$

are called the m-th iterated lattice functions of the lattice Λ with respect to the parameter λ and the operator Δ.

Obviously, for all $x \notin \Lambda$ and $g \in \Lambda$

$$G^{(m)}(\lambda; x+g) = G^{(m)}(\lambda; x) \tag{19}$$

is satisfied.

In analogy to techniques of potential theory it may be shown that the differential equation

$$(\Delta+\lambda)G^{(m)}(\lambda;x)=-G^{(m-1)}(\lambda;x) \quad \text{for} \quad m=2,3,\ldots \tag{20}$$

is valid for all $x\notin \Lambda$.

For all $m>q/2$ the m-th iterated lattice functions $G^{(m)}(\lambda;x)$ are continuous in $\mathbf{R}^q$. The eigenfunction expansion, which is equivalent to the Fourier series expansion in this case, yields

$$\frac{1}{\sqrt{D}}\sum_{\lambda_h\neq\lambda}\frac{\phi_h(x)}{(\lambda_h-\lambda)^m} \tag{21}$$

which is absolutely and uniformly convergent for all $m>q/2$. The representation theorem thus gives

LEMMA 1. *For $m>q/2$ the m-th iterated lattice function is continuous and can be represented by the series*

$$G^{(m)}(\lambda;x)=\frac{1}{\sqrt{D}}\sum_{\lambda_h\neq\lambda}\frac{\phi_h(x)}{(\lambda_h-\lambda)^m},$$

which is absolutely and uniformly convergent for all x.

3. Euler Summation Formula

Suppose that $\mathfrak{G}\subset\mathbf{R}^q$ is a regular region with continuously differentiable boundary surface $\partial\mathfrak{G}$. Let f be a function with continuous first and second derivatives in $\bar{\mathfrak{G}}:=\mathfrak{G}\cup\partial\mathfrak{G}$. Then Green's integral theorem gives with arbitrary parameter λ

$$\begin{aligned}\int_{\substack{x\in\mathfrak{G}\\|x-g|\geq\varepsilon}} &\{f(x)(\Delta+\lambda)G(\lambda;x)-G(\lambda;x)(\Delta+\lambda)f(x)\}\,dV \\ &=\int_{\substack{x\in\partial\mathfrak{G}\\|x-g|\geq\varepsilon}}\left\{f(x)\frac{\partial}{\partial n}G(\lambda;x)-G(\lambda;x)\frac{\partial}{\partial n}f(x)\right\}dS \\ &\quad-\sum_{g\in\bar{\mathfrak{G}}}\int_{\substack{|x-g|=\varepsilon\\x\in\bar{\mathfrak{G}}}}\left\{f(x)\frac{\partial}{\partial n}G(\lambda;x)-G(\lambda;x)\frac{\partial}{\partial n}f(x)\right\}dS\end{aligned} \tag{1}$$

for sufficiently small $\varepsilon>0$.

Condition (i) of Definition 1 yields

$$\int_{\substack{x\in\mathfrak{G}\\|x-g|\geq\epsilon}} f(x)(\Delta+\lambda)G(\lambda;x)\,dV=\frac{1}{\sqrt{D}}\sum_{\substack{\lambda_h=\lambda\\h\in\Lambda^{-1}}}\int_{\substack{x\in\mathfrak{G}\\|x-g|\geq\epsilon}} f(x)\phi_h(x)\,dV. \tag{2}$$

Hence, on passing to the limit $\varepsilon\to 0$ and observing the characteristic singularity of the lattice function, standard methods of potential theory enable us to get the following identity:

THEOREM 1. *Let $\mathfrak{G}\subset\mathbf{R}^q$ be a regular region with continuously differentiable boundary surface $\partial\mathfrak{G}$. Let f be a twice continuously differentiable function in $\bar{\mathfrak{G}}=\mathfrak{G}\cup\partial\mathfrak{G}$. Then for $\lambda\notin S(\Lambda)$ we have*

$$\begin{aligned}\sum_{g\in\mathfrak{G}}{}' f(g):&=\sum_{g\in\bar{\mathfrak{G}}} f(g)-\tfrac{1}{2}\sum_{g\in\partial\mathfrak{G}} f(g)\\ &=-\int_{\partial\mathfrak{G}}\left\{f(x)\frac{\partial}{\partial n}G(\lambda;x)-G(\lambda;x)\frac{\partial}{\partial n}f(x)\right\}dS\\ &\quad-\int_{\mathfrak{G}} G(\lambda;x)(\Delta+\lambda)f(x)\,dV\end{aligned}$$

For $\lambda\in S(\Lambda)$ we have

$$\begin{aligned}\sum_{g\in\mathfrak{G}}{}' f(g)=&-\int_{\partial\mathfrak{G}}\left\{f(x)\frac{\partial}{\partial n}G(\lambda;x)-G(\lambda;x)\frac{\partial}{\partial n}f(x)\right\}dS\\ &-\int_{\mathfrak{G}} G(\lambda;x)(\Delta+\lambda)f(x)\,dV\\ &+\frac{1}{\sqrt{D}}\sum_{\lambda_h=\lambda}\int_{\mathfrak{G}} f(x)\phi_h(x)\,dV,\end{aligned}$$

where the sum $\sum_{\lambda_h=\lambda}$ is to be taken over all normalized engenfunctions ϕ_h to the eigenvalue λ and $\partial/\partial n$ denotes the derivative in the direction of the outer normal n.

The difference between the two cases shows that we must expect a kind of resonance phenomena in multidimensional summation problems. Several examples will later bring out this structure more clearly. From an algorithmic point of view we observe, that these formulas express a sum in terms of integrals over $\mathfrak{G}$ and its boundary $\partial\mathfrak{G}$ involving derivatives of f up to the second order.

We now integrate by parts in analogy to the algorithm of the Euler-MacLaurin summation formula. Provided the function f is sufficiently smooth, we get from Green's theorem by means of the iterated lattice function

$$\int_{\mathfrak{G}} G^{(k)}(\lambda; x)(\Delta+\lambda)^k f(x)\,dV \tag{3}$$

$$= -\int_{\mathfrak{G}} G^{(k+1)}(\lambda; x)(\Delta+\lambda)^{k+1} f(x)\,dV$$

$$-\int_{\partial\mathfrak{G}} \left\{\frac{\partial}{\partial n} G^{(k+1)}(\lambda; x)(\Delta+\lambda)^k f(x) - G^{(k+1)}(\lambda; x)\frac{\partial}{\partial n}[(\Delta+\lambda)^k f(x)]\right\} dS.$$

Consequently we find

$$\int_{\mathfrak{G}} G(\lambda; x)(\Delta+\lambda) f(x)\,dV = (-1)^{m-1}\int_{\mathfrak{G}} G^{(m)}(\lambda; x)(\Delta+\lambda)^m f(x)\,dV \tag{4}$$

$$+\sum_{k=1}^{m-1}(-1)^k\int_{\partial\mathfrak{G}} \left\{\frac{\partial}{\partial n} G^{(k+1)}(\lambda; x)(\Delta+\lambda)^k f(x) - G^{(k+1)}(\lambda; x)\frac{\partial}{\partial n}(\Delta+\lambda)^k f(x)\right\} dS.$$

We formulate this extension of Theorem 1 as

THEOREM 2. *Let $\mathfrak{G}\subset \mathbf{R}^q$ be a regular region with continuously differentiable boundary surface $\partial\mathfrak{G}$. Suppose $f\in C^{(2m)}(\bar{\mathfrak{G}})$. Then for any number λ*

$$\sum_{g\in\bar{\mathfrak{G}}}{}' f(g) = \sum_{k=0}^{m-1}(-1)^{k+1}\int_{\partial\mathfrak{G}} \left\{[(\Delta+\lambda)^k f(x)]\frac{\partial}{\partial n} G^{(k+1)}(\lambda; x)\right\} dS$$

$$-\sum_{k=0}^{m-1}(-1)^{k+1}\int_{\partial\mathfrak{G}} \left\{G^{(k+1)}(\lambda; x)\frac{\partial}{\partial n}[(\Delta+\lambda)^k f(x)]\right\} dS$$

$$+(-1)^m\int_{\mathfrak{G}} G^{(m)}(\lambda; x)(\Delta+\lambda)^m f(x)\,dV$$

$$+\frac{1}{\sqrt{D}}\sum_{\lambda_h=\lambda}\int_{\mathfrak{G}} f(x)\phi_h(x)\,dV$$

where $\sum_{\lambda_h=\lambda}$ only occurs if $\lambda\in S(\Lambda)$.

The close relation of these identities to the Euler–MacLaurin formulas is best seen by specializing the parameters λ. In the case $\lambda=0$ we have the eigenfunction

$\phi_0 = 1/\sqrt{D}$ and the summation formula becomes

$$\begin{aligned}\sum_{g\in\bar{\mathfrak{G}}}' f(g) &= \sum_{g\in\bar{\mathfrak{G}}} f(g) - \tfrac{1}{2} \sum_{g\in\partial\mathfrak{G}} f(g) \\ &= \int_{\partial\mathfrak{G}} \left\{\left(\frac{\partial}{\partial n} f(x)\right) G(0; x) - f(x)\left(\frac{\partial}{\partial n} G(0; x)\right)\right\} dS \\ &\quad - \int_{\mathfrak{G}} G(0; x)\Delta f(x)\, dV \\ &\quad + \frac{1}{\sqrt{D}} \int_{\mathfrak{G}} f(x)\, dV\end{aligned} \tag{5}$$

in the simplest case.

It seems that the extension of the well-known Euler–MacLaurin formula to parameters $\lambda \neq 0$ has never been used though it is not difficult to see that the lattice function to the lattice $\mathbf{Z}$ e.g. is

$$G(\lambda; x) = \sum{}' \frac{e^{2\pi i n x}}{4\pi^2 n^2 - \lambda} \tag{6}$$

where the summation $\sum'$ is to be extended over all integers n with

$$4\pi^2 n^2 - \lambda \neq 0. \tag{7}$$

This function is known explicitly [16].

The generalization of the Euler–MacLaurin formula based on this function and the operator

$$\frac{d^2}{dx^2} + \lambda \tag{8}$$

opens a new perspective to subtle questions of convergence, because it is closely related to alternating or oscillating properties of the elements of a series, which is not absolutely convergent. For one-dimensional problems of convergence direct techniques are well known, which are particularly suited for this type of series.

In multidimensional summation problems the situation is different because the concept of the alternating or oscillating series is not directly applicable. Here the operator $\Delta + \lambda$ may be used to adapt the summation formula to oscillating properties of the elements of the series. This aspect will be of great adventage in deriving special identities. The "Hardy–Landau" identities are examples of this technique.

4. Spherical summations

We now apply the multidimensional Euler summation formula to spherical sums

$$\sum_{|g|\le N} f(g) \tag{1}$$

which provide interesting examples particularly when N tends towards infinity.

Taking $\mathfrak{G}$ as the region $|x|\le N$ Theorem 2 leads to

COROLLARY 1. *Let f be a $2m$-times continuously differentiable function in $|x|\le N$. Then for each $\lambda\in\mathbf{C}$*

$$\begin{aligned}\sum_{|g|\le N} f(g) &= \frac{1}{\sqrt{D}}\sum_{\lambda_h=\lambda}\int_{|x|\le N} f(x)\phi_h(x)\,dV\\ &\quad+(-1)^m\int_{|x|\le N} G^{(m)}(\lambda;x)[(\Delta+\lambda)^m f(x)]\,dV\\ &\quad+R_m(N)\end{aligned}$$

with

$$\begin{aligned}R_m(N):=&\frac{1}{2}\sum_{|g|=N} f(g)\\ &+\sum_{k=0}^{m-1}(-1)^{k+1}\int_{|x|=N}\frac{\partial}{\partial n}G^{(k+1)}(\lambda;x)[(\Delta+\lambda)^k f(x)]\,dS\\ &-\sum_{k=0}^{m-1}(-1)^{k+1}\int_{|x|=N}G^{(k+1)}(\lambda;x)\frac{\partial}{\partial n}[(\Delta+\lambda)^k f(x)]\,dS.\end{aligned}$$

Questions of convergence as $N\to\infty$ require estimates of the integrals for this limit.

The following result plays an important part in this respect.

THEOREM 3. *For every lattice Λ and every complex number λ the estimates*

$$\int_{|x|=N}|G(\lambda;x)|\,dS=0(N^{q-1})$$

and

$$\int_{|x|=N} \left|\frac{\partial}{\partial n} G(\lambda; x)\right| dS = 0(N^{q-1})$$

hold uniformly with respect to N.

Since the integrands are absolutely integrable on $|x| = N$ and the total measure of the sphere $|x| = N$ is $\omega_q N^{q-1}$ results of this type are to be expected.

The detailed proof is best carried out by using multidimensional polar coordinates.

We set

$$\begin{aligned} x &= r\xi, \qquad \xi^2 = 1, \qquad r = |x| \\ y &= \rho\eta, \qquad \eta^2 = 1, \qquad \rho = |y| \end{aligned} \tag{2}$$

and regard (r, ξ) and (ρ, η) as "polar coordinates" of x and y. We shall need an estimate for the scalar product $\xi\eta$ if x and y are confined to

$$|x - y| \leq \delta \tag{3}$$

with (sufficiently small) fixed positive δ.

We get from (3) the inequality

$$r^2 + \rho^2 - 2r\rho\xi\eta \leq \delta^2 \tag{4}$$

which is equivalent to

$$\left(1 - \frac{\delta^2}{r^2}\right) \leq -\left(\frac{\rho}{r} - \xi\eta\right)^2 + (\xi\eta)^2 \leq (\xi\eta)^2 \tag{5}$$

We thus obtain

LEMMA 2. *For x fixed with $|x| \geqslant \delta$ the inequality $|x - y| \leq \delta$ implies in polar coordinates*

$$\xi\eta \geqslant \sqrt{1 - \frac{\delta^2}{r^2}}.$$

We now prove

LEMMA 3. *Suppose that δ is fixed with $0<\delta<1$. Then the estimate*

$$\int_{\substack{|x|=N\\|x-y|\leq\delta}} \frac{1}{|x-y|^k}\, dS(x) \leq \frac{\omega_{q-1}}{(q-1)-k} \frac{\delta^{q-1-k}}{\sqrt{1-\dfrac{\delta^2}{N^2}}}$$

is valid for all $N\geq 1$ and all k with $0\leq k<q-1$.

In polar coordinates the surface element of $|x|=N$ can be written as

$$dS(x)=N^{q-1}d\omega(\xi) \tag{6}$$

where $d\omega(\xi)$ is the surface element of the unit sphere Ω. By Lemma 2 we see that

$$\int_{\substack{|x|=N\\|x-y|\leq\delta}} \frac{dS(x)}{|x-y|^k} \leq \int_{\substack{\xi\eta\geq\sqrt{1-(\delta^2/N^2)}\\|\xi|=1}} \frac{N^{(q-1)-k}d\omega(\xi)}{[1+s^2-2s(\xi\eta)]^{k/2}} \tag{7}$$

with the abbreviation $s=\rho/N$.

We express the last integral by means of the Funk–Hecke-formula [13, §2] and obtain

$$\omega_{q-1}N^{(q-1)-k}\int_{\sqrt{1-(\delta^2/N^2)}}^{1} \frac{(1-t^2)^{(q-3)/2}}{[1+s^2-2st]^{k/2}}\, dt. \tag{8}$$

We observe the inequality

$$1+s^2-2st=(s-t)^2+1-t^2\geq 1-t^2 \tag{9}$$

and find as a majorant for (8)

$$\omega_{q-1}N^{(q-1)-k}\int_{\sqrt{1-(\delta^2/N^2)}}^{1} (1-t^2)^{(q-3-k)/2} \frac{t}{\sqrt{1-\dfrac{\delta^2}{N^2}}}\, dt \tag{10}$$

$$=\frac{\omega^{q-1}}{2((q-1)-k)} \frac{\delta^{q-1-k}}{\sqrt{1-\dfrac{\delta^2}{N^2}}}$$

which proves Lemma 3.

For two-dimensional problems we need an estimate of

$$\int_{\substack{|x|=N\\|x-y|\leq\delta}} |\ln|x-y||\, dS_2(x). \tag{11}$$

If δ is a positive number with $0<\delta<1$ we have for $|x-y|\leq\delta$

$$|\ln|x-y||=-\ln|x-y|=-\tfrac{1}{2}\ln(N^2+\rho^2-2N\rho\xi\eta). \tag{12}$$

According to Lemma 2 the condition $|x-y|\leq\delta$ implies

$$\xi\eta\geqslant\sqrt{1-\frac{\delta^2}{N^2}}. \tag{13}$$

Thus we can write with the above abbreviations

$$\int_{\substack{|x|=N\\|x-y|\leq\delta}} |\ln|x-y||\, dS(x)\leq -N\int_{\sqrt{1-(\delta^2/N^2)}}^{1} (1-t^2)^{-1/2}\ln[N^2+\rho^2-2N\rho t]\,dt. \tag{14}$$

We now use for $\sqrt{1-(\delta^2/N^2)}\leq t\leq 1$ the estimate

$$N^2+\rho^2-2N\rho t=(\rho-Nt)^2+N^2(1-t^2) \tag{15}$$

which gives

$$N^2+\rho^2-2N\rho t\geqslant N^2(1-t^2)\geqslant\delta^2. \tag{16}$$

Because $0<\delta<1$ the integral (11) is less than

$$-2N\ln\delta\int_{\sqrt{1-(\delta^2/N^2)}}^{1}\frac{t}{\sqrt{1-t^2}}\frac{dt}{\sqrt{1-\frac{\delta^2}{N^2}}}\leq-\frac{2N\ln\delta}{\sqrt{1-\frac{\delta^2}{N^2}}}\frac{\delta}{N}=-\frac{2\delta\ln\delta}{\sqrt{1-\frac{\delta^2}{N^2}}} \tag{17}$$

and we get

LEMMA 4.

$$\int_{\substack{|x|=N\\|x-y|\leq\rho}} |\ln|x-y||\, dS_2(x)\leq-\frac{2\delta\ln\delta}{\sqrt{1-\frac{\delta^2}{N^2}}}.$$

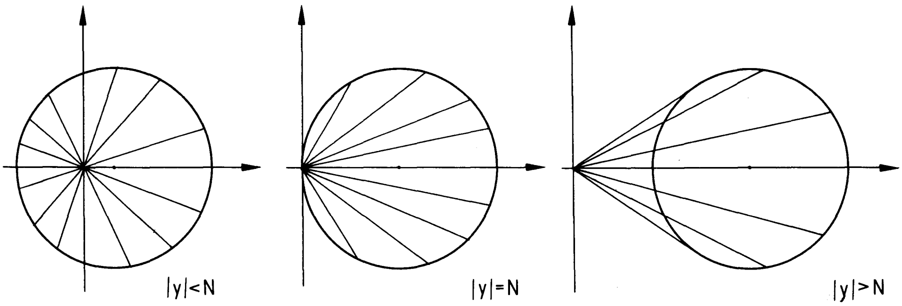

Figure 1.

We next evaluate integrals of the type

$$\int_{|x-y|=N} \frac{|xn|}{|x|^q}\, dS_q(x) \tag{18}$$

and distinguish the three cases $|y|<N$, $|y|=N$, and $|y|>N$. The integral (18) can be interpreted as the "q-dimensional angle" under which the sphere of radius N around the point y is seen from origin (see Figure 1). We use polar coordinates $x=r(\xi)\xi$ to represent the sphere $|x-y|=N$ by its projection on the unit sphere $\xi^2=1$. The three cases mentioned above require three different calculations.

Let n denote the normal and dS the surface element on the sphere $|x-y|=N$. We then use the identity

$$|xn|\, dS(x)=[r(\xi)]^q\, d\omega(\xi)$$

which relates the surface element dS at x to its projection $d\omega$ on Ω in ξ.

For fixed y and N we now introduce $g(\xi)$ as the number of positive solutions r of $|r\xi-y|=N$. We then get

$$\int_{|x-y|=N} \frac{|xn|}{|x|^q}\, dS_q(x)=\int_\Omega g(\xi)\, d\omega(\xi). \tag{19}$$

In the case $|y|<N$ we have exactly one positive radius r to every direction ξ such that

$$|r\xi-y|^2=(r-y\xi)^2+y^2-(y\xi)^2=N^2 \tag{20}$$

and consequently $g(\xi)=1$ for all ξ. If $|y|=N$ equation (20) has just one positive solution for $y\xi>0$ and no positive solution for $y\xi\leq 0$.

We thus obtain by integration

$$\int_{|x-y|=N} \frac{|xn|}{|x|^q}\, dS_q(x) = \begin{cases} \omega_q & \text{for} \quad 0<|y|<N \\ \frac{1}{2}\omega_q & \text{for} \quad |y|=N \end{cases} \tag{21}$$

If $|y|>N$ we have the two positive solutions

$$r_{1/2}(\xi) = y\xi \pm \sqrt{(y\xi)^2-(y-N^2)} \tag{22}$$

for $y\xi \geqslant \sqrt{(y^2-N^2)}$. Accordingly we get in this case

$$\int_{|x-y|=N} \frac{|xn|}{|x|^q}\, dS_q(x) = 2\int_{\eta\xi \geqslant \sqrt{1-(N^2/y^2)}} d\omega(\xi) \tag{23}$$

which yields the estimate

$$\int_{|x-y|=N} \frac{|xn|}{|x|^q}\, dS_q(x) = 2\omega_{q-1}\int_{\sqrt{1-(N^2/y^2)}}^{1} (1-t^2)^{(q-3)/2}\, dt < \omega_q \tag{24}$$

With (21) and (24) and a change of variables $x \curvearrowright x-y$, $y \curvearrowright -y$ we now get

LEMMA 5. *For all* $y \in \mathbf{R}^q$ *and* $N>0$ *we have*

$$\int_{|x|=N} \left| \frac{(x-y)n(x)}{|x-y|^q} \right| dS_q(x) \leq \omega_q.$$

We can now prove Theorem 3. We start by remembering that there is a positive constant $\mu=\mu(\Lambda)$ such that

$$|g-g'| \geqslant \mu \quad \text{for} \quad g, g' \in \Lambda \quad \text{and} \quad g \neq g'. \tag{25}$$

We set $\delta \leq \frac{1}{2}\operatorname{Min}(1, \mu)$. Then we use, that to each λ there is a constant c such that the estimates

$$\begin{aligned} |G(\lambda; x)| &\leq c\, |\ln|x-g|| \quad \text{for} \quad q=2 \\ |G(\lambda; x)| &\leq c\, |x-g|^{2-q} \quad \text{for} \quad q \geqslant 3 \end{aligned} \tag{26}$$

and

$$\left|\nabla_x G(\lambda; x) + \frac{1}{2\pi}\frac{x-g}{|x-g|^2}\right| \le \frac{c}{|x-g|} \qquad \text{for} \quad q=2$$

$$\left|\nabla_x G(\lambda; x) + \frac{1}{\omega_q}\frac{x-g}{|x-g|^q}\right| \le \frac{c}{|x-g|^{q-2}} \quad \text{for} \quad q \geqslant 3 \tag{27}$$

hold uniformly in the sphere $|x-g| \le \delta$.

We next define the distance of x to the lattice Λ as

$$D(x; \Lambda) := \operatorname*{Min}_{g \in \Lambda} |x-g| \tag{28}$$

With these notations we have

LEMMA 6. *There is a constant c depending on* λ *and* δ, *such that*

$$|G(\lambda; x)| \le c, \qquad |\nabla G(\lambda; x)| \le c$$

for all $x \in \mathbf{R}^q$ *with* $D(x; \Lambda) \geqslant \delta$.

Denote by $Z(N, \delta)$ the number of lattice points g with

$$N - \delta \le |g| \le N + \delta \tag{29}$$

We then have the well-known result [4]

LEMMA 7.

$$Z(N; \delta) = 0(N^{q-1})$$

for the fixed number δ *and* $N \to \infty$.

We can now discuss the integral

$$\int_{|x|=N} |G(\lambda; x)|\, dS(x) = \int_{\substack{|x|=N \\ D(x;\Lambda)\geqslant\delta}} |G(\lambda; x)|\, dS(x)$$

$$+ \int_{\substack{|x|=N \\ D(x;\Lambda)\le\delta}} |G(\lambda; x)|\, dS(x) \tag{30}$$

by using the different results for $D(x; \Lambda) \geqslant \delta$ and for $D(x; \Lambda) \leq \delta$. First we get

$$\int_{\substack{|x|=N \\ D(x;\Lambda)\geqslant\delta}} |G(\lambda; x)|\, dS(x) \leq C\omega_q N^{q-1} \tag{31}$$

by Lemma 6. Then we have for the dimensions $q \geqslant 3$

$$\int_{\substack{|x|=N \\ D(x;\Lambda)\leq\delta}} |G(\lambda; x)|\, dS_q(x) \leq Z(N; \delta) \int_{\substack{|x|=N \\ |x-g|\leq\rho}} \frac{C}{|x-g|^{q-2}}\, dS_q(x) \tag{32}$$

by (27). With Lemma 3 and Lemma 7 we then get

$$\int_{\substack{|x|=N \\ D(x;\Lambda)\leq\delta}} |G(\lambda; x)|\, dS_q(x) = 0(N^{q-1}) \tag{33}$$

for $q \geqslant 3$. This establishes the proof of the first part of Theorem 3 for $q \geqslant 3$. The case $q = 2$ is proved by the same arguments with Lemma 4 and Lemma 7.

From (28), Lemma 2, and Lemma 5 it follows that

$$\int_{\substack{|x|=N \\ |x-g|\leq\delta}} \left| \frac{\partial}{\partial n} G(\lambda; x) \right| dS_q(x) \tag{34}$$

is less than a positive constant for all N and $g \in \Lambda$. We can thus complete the proof of Theorem 3 by the estimates

$$\begin{aligned} &\int_{|x|=N} \left| \frac{\partial}{\partial n} G(\lambda; x) \right| dS_q(x) \\ &= \int_{\substack{|x|=N \\ D(x;\Lambda)\geqslant\delta}} \left| \frac{\partial}{\partial n} G(\lambda; x) \right| dS_q(x) + \int_{\substack{|x|=N \\ D(x;\Lambda)\leq\delta}} \left| \frac{\partial}{\partial n} G(\lambda; x) \right| dS_q(x) \qquad (35) \\ &= 0(N^{q-1}) + 0(Z(N; \delta)) = 0(N^{q-1}). \end{aligned}$$

Similar results are easily obtained for the iterated lattice functions $G^{(m)}(\lambda; x)$ introduced in Definition 2. Since each iteration reduces the order of the singularity by two, $G^{(m)}(\lambda; x)$ is continuous for $m > q/2$ and continuously differentiable for $m > (q/2)+1$. The estimates

$$\int_{|x|=N} |G^{(m)}(\lambda; x)|\, dS_q(x) = 0(N^{q-1})$$
$$\int_{|x|=N} \left|\frac{\partial}{\partial n} G^{(m)}(\lambda; x)\right| dS_q(x) = 0(N^{q-1}) \tag{36}$$

therefore are obvious for $m > (q/2)+1$. For the intermediate cases $1 < m < (q/2)+1$ we use Lemma 3 and estimate the sum

$$\int_{\substack{|x|=N\\ D(x;\Lambda)\geqslant\delta}} \cdots + \int_{\substack{|x|=N\\ D(x;\Lambda)\leq\delta}} \cdots = \int_{|x|=N} \cdots \tag{37}$$

as in (35). This leads to

THEOREM 4. *For every lattice Λ and every complex number λ and $m \geqslant 1$ the estimates*

$$\int_{|x|=N} |G^{(m)}(\lambda; x)|\, dS_q(x) = 0(N^{q-1})$$
$$\int_{|x|=N} \left|\frac{\partial}{\partial n} G^{(m)}(\lambda; x)\right| dS_q(x) = 0(N^{q-1})$$

hold uniformly with respect to N.

5. Convergence theorems

Suppose that f is in $C^{(2m)}(\mathbf{R}^q)$ with the following properties:

(i) *There is a number λ such that for each integer $k = 0, 1, \ldots, m-1$ the asymptotic relations*

$$(\Delta+\lambda)^k f(x) = o(|x|^{1-q})$$
$$\nabla(\Delta+\lambda)^k f(x) = o(|x|^{1-q})$$

are valid for $|x| \to \infty$.

(ii) *There is a positive real number ε such that*

$$(\Delta+\lambda)^m f(x) = 0(|x|^{-(q+\varepsilon)}).$$

According to the property (i) we can estimate the term $R_m(N)$ defined in Corollary 1.

In connection with Lemma 7 we have for $N \to \infty$

$$\frac{1}{2} \sum_{|g|=N} f(g) = o\left(N^{1-q} \sum_{|g|=N} 1\right) = o(1). \tag{1}$$

For the surface integrals it follows that

$$\begin{aligned}
&\sum_{k=0}^{m-1} (-1)^{k+1} \int_{|x|=N} (\Delta+\lambda)^k f(x) \frac{\partial}{\partial n} G^{(k+1)}(\lambda; x)\, dS \\
&\quad - \sum_{k=0}^{m-1} (-1)^{k+1} \int_{|x|=N} G^{(k+1)}(\lambda; x) \frac{\partial}{\partial n} (\Delta+\lambda)^k f(x)\, dS \\
&\quad = o\left(N^{1-q}\left[\sum_{k=0}^{m-1} \int_{|x|=N} \left\{|G^{(k+1)}(\lambda; x)| + \left|\frac{\partial}{\partial n} G^{(k+1)}(\lambda; x)\right|\right\} dS.\right.\right.
\end{aligned}$$

Hence, by virtue of Theorem 4, we obtain for $N \to \infty$

LEMMA 8.

$R_m(N) = o(1)$.

Moreover, according to condition (ii), we find with suitable positive constants B, C

$$\begin{aligned}
&\left|(-1)^m \int_{N' \le |x| \le N} G^{(m)}(\lambda; x)(\Delta+\lambda)^m f(x)\, dV\right| \\
&\quad \le B \int_{N'}^{N} \frac{r^{q-1}}{(1+r)^{q+\varepsilon}} \left(\int_{|x|=r} |G^{(m)}(\lambda; x)|\, dS\right) dr \\
&\quad \le BC \int_{N'}^{N} \frac{r^{q-1}}{(1+r)^{q+\varepsilon}}\, dr.
\end{aligned} \tag{3}$$

Thus the integral

$$(-1)^m \int_{\mathbf{R}^q} G^{(m)}(\lambda; x)(\Delta+\lambda)^m f(x)\, dV \tag{4}$$

is absolutely convergent.

Combining the results (1), Lemma 8, and (4) we get by means of the Euler summation formula

THEOREM 5. *Suppose that* $f \in C^{(2m)}(\mathbf{R}^q)$ *satisfies the asymptotic properties* (*i*) *and* (*ii*) *for a parameter* λ. *Then the limit*

$$\lim_{N\to\infty}\left(\sum_{|g|\le N} f(g) - \frac{1}{\sqrt{D}}\sum_{\lambda_h=\lambda}\int_{|x|\le N} f(x)\phi_h(x)\,dV\right)$$

exists and we have

$$\lim_{N\to\infty}\left(\sum_{|g|\le N} f(g) - \frac{1}{\sqrt{D}}\sum_{\lambda_h=\lambda}\int_{|x|\le N} f(x)\phi_h(x)\,dV\right)$$
$$=(-1)^m\int_{\mathbf{R}^q} G^{(m)}(\lambda;x)(\Delta+\lambda)^m f(x)\,dV$$

where the sum $\sum_{\lambda_h=\lambda}$ *is to be extended over all normalized eigenfunctions* ϕ_h *to the eigenvalue* λ.

The convergence of multidimensional (spherical) sums

$$\sum_{g\in\Lambda} f(g) = \lim_{N\to\infty}\sum_{|g|\le N} f(g) \tag{5}$$

therefore is closely connected with the spectrum $S(\Lambda)$ of the lattice Λ.

For the class of non-eigenvalues λ Theorem 5 immediately implies the convergence of the series (5). In order to ensure convergence of (5) for eigenvalues $\lambda \in S(\Lambda)$, however, we have to require the additional condition:
(iii) *for all lattice points* $h \in \Lambda^{-1}$ *with* $\lambda_h = 4\pi^2 h^2 = \lambda$ *the integrals*

$$\int_{\mathbf{R}^q} f(x)\phi_h(x)\,dV$$

exist in the (*spherical*) *sense*

$$\int_{\mathbf{R}^q}\cdots = \lim_{N\to\infty}\int_{|x|\le N}\cdots.$$

In order to simplify notations we introduce a class of function spaces which forms a natural setting for multidimensional convergence theorems.

DEFINITION 3. A $2m$-times continuously differentiable function f in $\mathbf{R}^q$ is called of class $C_\lambda^{(2m)}(\mathbf{R}^q)$ if it satisfies the conditions (i), (ii) and (iii).

With the concept of functions $f \in C_\lambda^{(2m)}(\mathbf{R}^q)$ Theorem 5 has the following extension:

THEOREM 6. *For each* $f \in C_\lambda^{(2m)}(\mathbf{R}^q)$ *the series*

$$\sum_{g \in \Lambda} f(g) = \lim_{N \to \infty} \sum_{|g| \le N} f(g)$$

is convergent. In particular, for each $\lambda \notin S(\Lambda)$ *we have*

$$\sum_{g \in \Lambda} f(g) = (-1)^m \int_{\mathbf{R}^q} G^{(m)}(\lambda; x)(\Delta + \lambda)^m f(x)\, dV$$

while for each $\lambda \in S(\Lambda)$

$$\sum_{g \in \Lambda} f(g) = \frac{1}{\sqrt{D}} \sum_{\lambda_h = \lambda} \int_{\mathbf{R}^q} f(x)\phi_h(x)\, dV$$
$$+ (-1)^m \int_{\mathbf{R}^q} G^{(m)}(\lambda; x)(\Delta + \lambda)^m f(x)\, dV.$$

6. Poisson summation formula

The convergence theorems now enable us to prove the multidimensional Poisson summation formula under assumptions which are weaker than the conditions used so far.

Because the m-th iterated lattice function $G^{(m)}(\lambda; x)$ has an absolutely and uniformly convergent Fourier-expansion for $m > q/2$ Lebesgue's theorem gives us

$$(-1)^m \int_{\mathbf{R}^q} G^{(m)}(\lambda; x)[(\Delta + \lambda)^m f(x)]\, dV \tag{1}$$
$$= \frac{1}{\sqrt{D}} \sum_{\lambda_h \neq \lambda} \frac{1}{(\lambda - \lambda_h)^m} \int_{\mathbf{R}^q} \phi_h(x)[(\Delta + \lambda)^m f(x)]\, dV$$

for all functions $f \in C_\lambda^{(2m)}(\mathbf{R}^q)$.

Repeated application of Green's theorem, in connection with Theorem 4, yields

$$\int_{\mathbf{R}^q} \phi_h(x)[(\Delta + \lambda)^m f(x)]\, dV$$
$$= \int_{\mathbf{R}^q} [(\Delta + \lambda)^m \phi_h(x)] f(x)\, dV$$
$$= (\lambda - \lambda_h)^m \int_{\mathbf{R}^q} \phi_h(x) f(x)\, dV. \tag{2}$$

Inserting (2) into (1) we thus obtain the *Poisson summation formula*

THEOREM 7. *Let f be of class $C_\Lambda^{(2m)}(\mathbf{R}^q)$ with $m>q/2$. Then*

$$\sum_{g\in\Lambda} f(g)=\frac{1}{\sqrt{D}}\sum_{h\in\Lambda^{-1}}\int_{\mathbf{R}^q} f(x)\phi_h(x)\,dV.$$

It should be noted that the Poisson summation formula developed here is based on spherical summation.

7. Extensions of the Hardy–Landau identity

We illustrate the multidimensional Poisson summation formula by examples of the Analytic Theory of Numbers.

In particular, we are interested in deriving simple extensions of the well-known two-dimensional Hardy–Landau identity [5], [6]

$$\frac{1}{D}\sum_{|h|\le R}{}' 1=R\sum_{g\in\Lambda}\frac{J_1(2\pi|g|R)}{|g|}. \tag{1}$$

We begin with the functions

$$f_R(x)=\frac{J_v(q;2\pi|x|R)}{|x|^v}H_n(q;x), \tag{2}$$

where $H_n(q;x)$ is a homogeneous harmonic polynomial of degree n in q dimensions and

$$J_v(q;r)=\left(\frac{r}{2}\right)^{(2-q)/2}\Gamma(q/2)J_{v+(q-2)/2}(r) \tag{3}$$

is the Bessel function of order v and dimension q [13, §3]. We restrict ourselves to the case

$$v>n+\frac{q-1}{2}. \tag{4}$$

Expanding $J_v(q; 2\pi rR)$ in powers of its argument we obtain

$$\frac{J_v(q; 2\pi rR)}{r^v} = \frac{(\pi R)^v \Gamma(\frac{q}{2})}{\Gamma(v+\frac{q}{2})}\left[1 - \frac{(\pi rR)^2}{1!(v+\frac{q}{2})} + - \cdots\right]. \tag{5}$$

For $r \to \infty$ standard estimates [17 §7.1] yield for any $n \in \mathbf{N}$

$$J_v(q; 2\pi rR) = J_n(q; 2\pi rR)\cos(v-n)\frac{\pi}{2} + J_{n+1}(q; 2\pi rR)\sin(v-n)\frac{\pi}{2} + 0(r^{-(q-1)/2}). \tag{6}$$

It is easy to see that

$$\left.\begin{aligned}(\Delta+\lambda)^k f_R(x) &= o(|x|^{1-q})\\ \nabla(\Delta+\lambda)^k f_R(x) &= o(|x|^{1-q})\end{aligned}\right\} \quad (|x|\to\infty) \tag{7}$$

hold for each non-negative integer k.

From (3) we get

$$(\Delta + 4\pi^2 R^2)\frac{J_v(q; 2\pi\,|x|\,R)}{|x|^v} = 0\left(\frac{J_{v+1}(q; 2\pi\,|x|\,R)}{|x|^{v+1}}\right). \tag{8}$$

Hence, it is obvious that

$$(\Delta+\lambda)f_R(x) = 0(|x|^{-(q+\varepsilon)}) \tag{9}$$

for $\lambda = 4\pi^2R^2$ and $\varepsilon = v - n - (q-1)/2$.

For $\lambda = 4\pi^2R^2$ Theorem 5 therefore gives

$$\lim_{N\to\infty}\left(\sum_{|g|\le N} f_R(g) - \frac{1}{\sqrt{D}}\sum_{|h|=R}\int_{|x|\le N} f_R(x)\phi_h(x)\,dV\right) = \frac{1}{\sqrt{D}}\sum_{|h|\ne R}\int_{\mathbf{R}^q} f_R(x)\phi_h(x)\,dV. \tag{10}$$

For all numbers $\lambda = 4\pi^2h^2$ with $\lambda \notin S(\Lambda)$ the function f_R belongs to a space $C_\lambda^{(2m)}(\mathbf{R}^q)$ with $2m > q$ and we obtain from Theorem 7

$$\lim_{N\to\infty}\sum_{|g|\le N} f_R(g) = \frac{1}{\sqrt{D}}\sum_{h\in\Lambda^{-1}}\int_{\mathbf{R}^q} f_R(x)\phi_h(x)\,dV. \tag{11}$$

With the abbreviation

$$I(q;h)=\frac{1}{\sqrt{D}}\int_{\mathbf{R}^q} f_R(x)\phi_h(x)\,dV \tag{12}$$

we find by introducing polar coordinates

$$I(q;h)=\frac{i^n}{D}\,\omega_q H_n\left(q;\frac{h}{|h|}\right)\int_0^\infty J_v(q;2\pi rR)J_n(q;2\pi\,|h|\,r)r^{n-v+q-1}\,dr. \tag{13}$$

Under the restriction (4) the integrals exist and their values are known [17 §13.4]

$$I(q;h)=\begin{cases}\dfrac{2i^n}{\omega_q D}\dfrac{\pi^{v-n}}{R^{2n-v+q}\Gamma(v-n)}\left(1-\left(\dfrac{|h|}{R}\right)^2\right)^{v-n-1}H_n(q;h) & \text{for}\quad |h|<R\\ 0 & \text{for}\quad |h|>R.\end{cases} \tag{14}$$

Thus, for all numbers R with $4\pi^2R^2\notin S(\Lambda)$ and $v>n+(q-1)/2$ the formula (11) can be rewritten in the form

$$\frac{2i^n\pi^{v-n}}{\omega_q DR^{2n-v+q}\Gamma(v-n)}\sum_{|h|<R}\left(1-\left(\frac{|h|}{R}\right)^2\right)^{v-n-1}H_n(q;h)$$
$$=\sum_{g\in\Lambda}\frac{J_v(q;2\pi\,|g|\,R)}{|g|^v}H_n(q;g). \tag{15}$$

For odd n both sides of this identity are zero for reasons of symmetry. It is interesting therefore only for even n.

For the critical values R with $4\pi^2R^2\in S(\Lambda)$ and $v>n+((q-1)/2)$ identity (10) yields

$$\lim_{N\to\infty}\left\{\sum_{|g|\le N}\frac{J_v(q;2\pi\,|g|\,R)}{|g|^v}H_n(q;g)\right.$$
$$\left.-\frac{1}{D}\sum_{|h|=R}\int_{|x|\le N}\frac{J_v(q;2\pi\,|x|\,R)}{|x|^v}H_n(q;x)e(hx)\,dV\right\}$$
$$=\frac{2i^n\pi^{v-n}}{\omega_q DR^{2n-v+q}\Gamma(v-n)}\sum_{|h|<R}\left(1-\left(\frac{|h|}{R}\right)^2\right)^{v-n-1}H_n(q;h). \tag{16}$$

The integrals $I(q; h)$ exist also for $|h| = R$ provided that one of the conditions

$$\begin{cases} v-n > \mathrm{Max}\,(1, (q-1)/2) \\ v-n > ((q-1)/2) \quad \text{and} \quad v-n \text{ an odd integer} \end{cases} \tag{17}$$

is satisfied [17 §13.4], and we find in this case

$$\frac{2i^n\pi^{v-n}}{\omega_q DR^{2n-v+q}\Gamma(v-n)} \sum_{|h|\le R} \left(1-\left(\frac{|h|}{R}\right)^2\right)^{v-n-1} H_n(q; h)$$
$$= \sum_{g\in\Lambda} \frac{J_v(q; 2\pi\,|g|\,R)}{|g|^v} H_n(q; g). \tag{18}$$

For $q \geqslant 3$ the conditions (17) imply that $v-n-1$ is always greater than zero and all our integrals converge. For $q=2$, however, we find a striking difference between the cases $4\pi^2R^2 \notin S(\Lambda)$ for $\frac{1}{2} < v-n < 1$.

The convergence of our series depends on the convergence of the integrals

$$\int_0^N J_v(2\pi rR) J_n(2\pi\,|h|\,r) r^{n-v+1}\,dr. \tag{19}$$

From (6) we deduce that the integrand is asymptotically equal to

$$r^{n-v+1} J_n(2\pi\,|h|\,r) J_n(2\pi Rr) \cos\,(v-n)\frac{\pi}{2}$$
$$+ r^{n-v+1} J_n(2\pi\,|h|\,r) J_{n+1}(2\pi Rr) \sin\,(v-n)\frac{\pi}{2} + 0(r^{n-v-2}) \tag{20}$$

and we see that for $|h| \neq R$ and $\frac{1}{2} < v-n < 1$ the above integral is convergent. For $|h| = R$, however, the integral tends towards $+\infty$ with $N \to \infty$ if v is in $\frac{1}{2} < v-n < 1$.

LEMMA 9. *For $q=2$, $\frac{1}{2} < v-n < 1$ and $4\pi^2R^2 \notin S(\Lambda)$ the limit*

$$\lim_{N\to\infty} \sum_{|g|\le N} \frac{J_v(2\pi\,|g|\,R)}{|g|^v} H_n(2; q)$$

exists and is equal to

$$\frac{i^n\pi^{v-n-1}}{DR^{2n-v+2}\Gamma(v-n)} \sum_{|h|<R} \left(1-\left(\frac{|h|}{R}\right)^2\right)^{v-n-1} H_n(2; h)$$

For $4\pi^2 R^2 = 4\pi^2 |h|^2 \in S(\Lambda)$ the limit does not exist in general, but we get for $N \to \infty$

$$\sum_{|g| \leq N} \frac{J_v(2\pi\,|g|\,R)}{|g|^v} H_n(2; g)$$

$$= 2\pi \frac{i^n}{D} \frac{1}{R^n} \sum_{|h|=R} H_n(2; h) \int_0^N J_v(2\pi r R) J_n(2\pi\,|h|\,r) r^{n-v+1}\,dr$$

$$+ \frac{i^n \pi^{v-n-1}}{D R^{2n-v+2} \Gamma(v-n)} \sum_{|h|<R} \left(1 - \left(\frac{|h|}{R}\right)^2\right)^{v-n-1} H_n(2; h) + o(1)$$

For $q = 2$ the harmonic homogeneous polynomials of degree n are linear combinations of the functions

$$(x_1 + i x_2)^n + (x_1 - i x_2)^n \quad \text{and} \quad i^{-1}[(x_1 + i x_2)^n - (x_1 - i x_2)^n]. \tag{21}$$

Taking Λ as the standard lattice $\mathbf{Z}^2$, and setting $H_n(2; x) := (x_1 + i x_2)^n$ we deduce with $n = 4k$ from Lemma 9 the following identities [2, §5]:

1.) $\frac{1}{2} < v - 4k < 1$; $4\pi R^2 \notin S(\mathbf{Z}^2)$:

$$\sum_{(n_1, n_2) \in \mathbf{Z}^2} \frac{J_v(2\pi(\sqrt{n_1^2 + n_2^2}) R)}{(\sqrt{n_1^2 + n_2^2})^v} (n_1 + i n_2)^{4k}$$

$$= \frac{\pi^{v-4k-1}}{R^{8k-v+2} \Gamma(v-4k)} \sum_{n_1^2 + n_2^2 < R^2} \left(1 - \frac{n_1^2 + n_2^2}{R^2}\right)^{v-4k-1} (n_1 + i n_2)^{4k}$$

2.) $\frac{1}{2} < v - 4k < 1$; $4\pi R^2 \in S(\mathbf{Z}^2)$:

$$\sum_{n_1^2 + n_2^2 \leq N^2} \frac{J_v(2\pi(\sqrt{n_1^2 + n_2^2}) R)}{(\sqrt{n_1^2 + n_2^2})^v} (n_1 + i n_2)^{4k}$$

$$= \frac{2\pi}{R^{4k}} \sum_{n_2^1 + n_2^2 = R^2} (n_1 + i n_2)^{4k} \int_0^N J_v(2\pi r R) J_{4k}(2\pi \sqrt{n_1^2 + n_2^2}\, r)^{4k-v+1}\,dr$$

$$+ \frac{\pi^{v-4k-1}}{R^{8k-v+2} \Gamma(v-4k)} \sum_{n_1^2 + n_2^2 < R^2} \left(1 - \frac{n_1^2 + n_2^2}{R^2}\right)^{v-4k-1} (n_1 + i n_2)^{4k}$$

$$+ o(1) \qquad (N \to \infty).$$

3.) $v-4k=1$:

$$\sum_{(n_1,n_2)\in\mathbf{Z}^2} \frac{J_{4k+1}(2\pi(\sqrt{n_1^2+n_2^2})R)}{(\sqrt{n_1^2+n_2^2})^{4k+1}} (n_1+in_2)^{4k}$$
$$=\frac{1}{R^{4k+1}} \sum_{n_1^2+n_2^2\le R^2}' (n_1+in_2)^{4k}.$$

These and similar identities have proved useful to describe subtle irregularities of the distribution of lattice points in circles [3], [14].

It is to be expected that the above identities lead to similar results in higher dimensions.

BIBLIOGRAPHY

[1] J. W. S. Cassels, *An Introduction to the Geometry of Numbers*, Springer-Verlag (1959).

[2] W. Freeden, *Eine Verallgemeinerung der Hardy–Landauschen Identität*, Manuscripta math. *24*, 205–216 (1978).

[3] W. Freeden, *Über gewichtete Gitterpunktsummen in kreisförmigen Bereichen*, Mitt. Math. Seminar Giessen, *132*, 1–22 (1978).

[4] C. F. Gauss, *De nexu inter multitudinem classicum, in quas formae binariae secundi gradus distribuuntur, earumque determinantem*, Werke, vol. 2, 269–291 (1863).

[5] G. H. Hardy, *On the expression of a number as the sum of two squares*, Quat. Journ. Math. (Oxford) *46*, 263–283 (1915).

[6] G. H. Hardy and E. Landau, *The lattice points of a circle*, Proceedings of the Royal Society, A, *105* 244–258 (1924).

[7] D. Hilbert, *Grundzüge einer allgemeinen Theorie der linearen Intergralgleichungen* (1924).

[8] V. K. Ivanow, *Higher-dimensional generalization of the Euler summation formula*, (Russian) Izv. Vysš, Učebn. Zaved. Mathematika, no. 6 (*37*), 72–80 (1963).

[9] L. J. Mordell, *Poisson's summation formula in several variables and some applications to the Theory of Numbers*, Cambr. Phil. Soc., *25*, 412–420.

[10] L. J. Mordell, *Poisson's summation formula and the Riemann Zeta function*, Journ. London Math. Soc., Vol. *4*, 285–296 (1929).

[11] Cl. Müller, *Eine Verallgemeinerung der Eulerschen Summenformel und ihre Anwendung auf Fragen der analytischen Zahlentheorie*, Abh. Math. Sem. Hamburg XIX, 41–62 (1954).

[12] Cl. Müller, *Eine Erweiterung der Hardyschen Identität*, Abh. Math. Sem. Hamburg XIX, 66–76 (1954).

[13] Cl. Müller, *Eine neuer Zugang zur Theorie der Bessel-funktionen in mehreren Dimensionen*, Ann. Acad. Scient. Fenn. *415* (1968).

[14] Cl. Müller, and A. Dressler, *Über eine gewichtete Mittelung der Gitterpunkte in der Ebene*, Journ. für die reine und angew. Mathematik, *252*, 82–87 (1972).

[15] A. Oppenheim, *Some identities in the Theory of Numbers*, Proc. London Math. Soc., Series 2, Vol. XXVI, 295–350 (1927).

[16] F. Oberhettinger, *Fourier expansions: A collection of formulas*, Academic press, New York and London (1973).

[17] G. N. Watson, *A treatise on the theory of Bessel functions*, Cambridge (1914).