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Comment on the preceding paper by T. H. Boyer

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It is shown that the proposed new noncovariant way of calculating the electromagnetic energy and momentum of a classical moving charged sphere is neither conceptually simpler nor physically acceptable. The fundamental question is whether electromagnetic interactions can be separated from nonelectromagnetic ones in a Poincaré-invariant way. This question is answered in the affirmative.

Theorists have been pondering the structure of the electron long before quantum mechanics came into existence, and in fact already in prerelativistic times. The classical models of the electron proposed by Abraham and Lorentz near the turn of the century considered the possibility that the electron is of purely electromagnetic nature. That hope has long since been abandoned. However, while the lack of a full understanding of the stability and self-energy of the electron persists to the present day, much of it is now well understood.[1]

The preceding paper by Boyer[2] raises some questions that have been settled some time ago: whether the electromagnetic energy and momentum of the Coulomb field surrounding a charged particle are or are not the components of a four-vector. Since only uniformly moving charges are considered, radiation fields do not enter the discussion. The uniformly moving charge then carries only the (generalized) Coulomb field which is the Lorentz-transformed static Coulomb field.

In that paper (in the following referred to as CME) it is proposed to define the electromagnetic energy and momentum densities of the (generalized) *Coulomb* field as (we use $c=1$ and Gaussian units as in CME)

$$U=\frac{1}{8\pi}(E^2+B^2) \text{ and } \vec{S}=\frac{1}{4\pi}\vec{E}\times\vec{B} \tag{1}$$

in *every* inertial frame such that the total energy and momentum due to that field are

$$P_e^0=\int d^3xU \text{ and } \vec{P}_e=\int d^3x\vec{S}\ . \tag{2}$$

For Coulomb fields these do not transform as the components of a four-vector. Equations (1) and (2) are known to be valid for radiation fields where they do transform as components of a four-vector; they are also used in nonrelativistic macroscopic Maxwell theory for open systems; and they date from prerelativistic days.

Two claims are made in CME: (a) that these equations are demanded by a consideration of the cohesive forces that are needed to prevent a charged sphere from expanding, and (b) that these equations lead to a conceptually simpler theory than the expressions that are used to define a *four-vector* of Coulomb energy and momentum.[3,4] The present paper proposes to show that both these claims are unwarranted.

The model of this classical charged particle is a sphere of radius a, mass m, and uniformly distributed surface charge e. As a free object it is a *closed* system that has a total energy P^0 and momentum $\vec{P}$ which transform as components of a four-vector. If the entire particle were expressible by means of a field and an associated energy tensor $\Theta^{\mu\nu}$ such a tensor would necessarily have to satisfy

$$\partial_\alpha\Theta^{\alpha\mu}=0 \tag{3}$$

since the system is closed. The momentum defined by

$$P^\mu=\int_\sigma d^3\sigma_\alpha\Theta^{\alpha\mu}(x) \tag{4}$$

would therefore be independent of the choice of the spacelike surface σ.

The particle however is not purely electromagnetic but contains an electromagnetic component (the Coulomb field) and a nonelectromagnetic one. We shall accept the usual assumption that these two components are additive in the energy tensors,

$$\Theta^{\mu\nu}=\Theta_e^{\mu\nu}+\Theta_n^{\mu\nu}\ . \tag{5}$$

Neither of the two components of $\Theta^{\mu\nu}$ are separately conserved,

$$\partial_\alpha\Theta_e^{\alpha\mu}=-\partial_\alpha\Theta_n^{\alpha\mu}\neq 0\ . \tag{6}$$

This means that the decomposition of P^μ into P_e^μ and P_n^μ,

$$P^\mu = \int_\sigma d^3\sigma_\alpha \Theta_e^{\alpha\mu} + \int_\sigma d^3\sigma_\alpha \Theta_n^{\alpha\mu}$$
$$\equiv P_e^\mu + P_n^\mu \ , \tag{7}$$

involves two surface integrals which are *not separately* independent of σ. But the sum *is* independent of σ as long as the *same* σ is chosen in both integrals. Of special interest will be two choices of σ:

(I) σ is the surface $t=\text{const}$ in the inertial system of the observer. Each observer has his own surface σ.

(II) σ is the surface $t_R=\text{const}$ in the (inertial) rest system of the particle. All observers agree to use that surface in (7). It must be emphasized that the separation (7) of the momentum into an electromagnetic and nonelectromagnetic part is not an observable separation but serves the convenience of the theory. It corresponds to the separation of the observed mass into an electromagnetic and a nonelectromagnetic part.

A macroscopic charged sphere would be described by an energy tensor only for its electromagnetic fields. The nonelectromagnetic component would be described by a force density $f^\mu(x)$. In CME this component is broken up into two parts: a "mechanical" part leading to a momentum P_m^μ and a part describing the cohesive forces that prevent the charged sphere from expanding. The part f_{coh}^μ is just the force density that provides the Poincaré stresses.[5] It is not an *external* force as stated in CME (end of its second paragraph) and is in fact not physically separable from the rest of the nonelectromagnetic components in a classical macroscopic body. Without its inclusion the *physical* charged particle cannot possibly be proven to have a momentum (4) that is a four-vector.

If the momentum P_m^μ is separated and assumed to be a four-vector by itself (as done in CME) the local conservation law (3) is reduced to

$$\partial_\alpha \Theta_e^{\alpha\mu} + f_{\text{coh}}^\mu = 0 \ . \tag{8}$$

For the sake of argument we follow the scenario of CME where the charged sphere is produced by contraction of an infinite sphere but apply the stabilizing f_{coh}^μ at all times, contracting adiabatically to $r=a$. This ensures a closed system giving

$$P^\mu = P_m^\mu + P_e^\mu + P_{\text{coh}}^\mu \ , \tag{9}$$

$$P_e^\mu = \int_\sigma d^3\sigma_\alpha \Theta_e^{\alpha\mu} \ , \quad P_{\text{coh}}^\mu = -\int_{V_{4,\sigma}} d^4x f_{\text{coh}}^\mu \ . \tag{10}$$

The integral for P_{coh}^μ is to be taken over a four-dimensional volume that extends from the spacelike surface σ into $-\infty$.[6] As emphasized before, the two σ in P_e^μ and P_{coh}^μ must be the same.

If one now makes the choice (I) as is proposed in CME, one obtains for the system S in which the particle moves with velocity $\vec{v}$ the following. The electromagnetic momentum is as in CME

$$P_e^0 = \int d^3x U = \gamma m_e (1 + \tfrac{1}{3}\vec{v}^2) \ ,$$
$$\vec{P}_e = \int d^3x \vec{S} = \tfrac{4}{3}\gamma m_e \vec{v} \ , \tag{I) (11}$$

where[7]

$$m_e = \int d^3x_R U_R = \frac{e^2}{2a} \ . \tag{12}$$

Thus, P_e^μ is *not* a four-vector. This is not surprising since each observer S chooses a different σ, and P_e^μ depends on σ.

In the rest frame f_{coh}^μ is[8] (we shall drop the subscript coh on f_{coh}^μ from here on)

$$f_R^0 = 0, \quad \vec{f}_R = -2\pi\sigma^2 \hat{r}\delta(r-a) \ ,$$
$$\sigma = \frac{e}{4\pi a^2} \tag{13}$$

as also given in CME. The evaluation of P_{coh}^μ in (10) is now done by transforming from S back to S_R. If L is the Lorentz transformation that maps S_R to S then we have

$$P_{\text{coh}}^0 = -\int_{V_{4,\sigma}} d^4x f^0$$
$$= -\int_{V_4, L^{-1}\sigma} d^4x_R \gamma (f_R^0 - \vec{v}\cdot\vec{f}_R)$$
$$= -\tfrac{1}{3} m_e \gamma \vec{v}^2 \ ,$$
$$\vec{P}_{\text{coh}} = -\int_{V_{4,\sigma}} d^4x \vec{f}$$
$$= -\int_{V_4, L^{-1}\sigma} d^4x_R (\gamma \vec{f}_R^{\parallel} + \vec{f}_R^{\perp} - \gamma \vec{v} f_R^0)$$
$$= -\tfrac{1}{3} m_e \gamma \vec{v} \ .$$

In the notation $P^\mu = (P^0, \vec{P})$, therefore,

$$P_{\text{coh}}^\mu = -\tfrac{1}{3} m_e \gamma (\vec{v}^2, \vec{v}) \quad \text{(I)} \tag{14}$$

and with (11)

$$P_e^\mu + P_{\text{coh}}^\mu = \gamma m_e (1, \vec{v}) = m_e v^\mu \ . \quad \text{(I)} \tag{15}$$

This is the result obtained in CME in slightly different form. It is argued there that the fact that the non–four-vector of cohesive momentum thus obtained just compensates the non–four-vector of electromagnetic momentum and yields a sum which *is* a four-vector justifies the choice of surface (I). Our presentation shows that this choice (I) is one of the arbitrary choices one can make, each choice giving a different unobservable separation of $m_e v^\mu$ into P_e^μ and $P_{\rm coh}^\mu$.

We now repeat this calculation for the choice (II), where P_e^μ involves an integration over $\sigma = L\sigma_R$:

$$P_e^0 = \int_{L\sigma_R} d^3\sigma_\alpha \Theta_e^{\alpha 0} = \int d^3x_R \gamma(U - \vec{v}\cdot\vec{S}) = \gamma m_e \ ,$$

$$P_e^k = \int d^3\sigma_\alpha \Theta_e^{\alpha k} = \int d^3x_R \gamma(S^k + v_l \Theta_e^{lk}) = \gamma m_e v^k \ .$$

Since this calculation exists only in either nonrelativistic approximation[9] or in a didactic context in rather complicated form,[10] the details are shown in the Appendix. They were unfortunately omitted in Ref. 3, p. 90.

The above components lead to

$$P_e^\mu = m_e v^\mu \ , \quad \text{(II)} \qquad (16)$$

which is a four-vector. This is also not surprising since all observers use the same reference plane σ_R and integrate over it as they see it.

We next evaluate $P_{\rm coh}^\mu$ using the choice (II). With (13)

$$\begin{aligned} P_{\rm coh}^0 &= -\int_{V_4, L\sigma_R} d^4x f^0 \\ &= -\int_{V_4,\sigma_R} d^4x_R \gamma(f_R^0 - \vec{v}\cdot\vec{f}_R) = 0 \ , \\ \vec{P}_{\rm coh} &= -\int_{V_4, L\sigma_R} d^4x \, \vec{f} \\ &= -\int_{V_4,\sigma_R} d^4x_R (\gamma \vec{f}_R^{\,\parallel} + \vec{f}_R^{\,\perp} - \gamma\vec{v} f_R^0) \\ &= 0 \ ; \end{aligned}$$

the integrals vanish term by term so that

$$P_{\rm coh}^\mu = 0 \qquad (17)$$

and the cohesive forces contribute nothing to the momentum and energy of the particle. They give (trivially) a four-vector. The choice II permits one to ignore them in most cases.

From this calculation one concludes that the choice (II) is the simpler choice both conceptually and formally. In addition there are very good reasons to have an electromagnetic energy-momentum *four-vector* (16) rather than the expressions (11). These are basically that one wants to be able to formulate a theory of electromagnetic interactions in a Poincaré-invariant way. This is not entirely possible on the *classical* level because of the cohesive forces which are necessarily nonelectromagnetic. But at least they do not need to spoil the four-vector character of P_e^μ. Indeed, both in the Lorentz-Dirac theory and in quantum electrodynamics one uses four-vector momenta. All relativistic quantum field theories also give *Poincaré-invariant separations* between electromagnetic and nonelectromagnetic interactions.

Finally, one should note that a more careful study of P_e^μ on the *quantum-mechanical* level shows that in the point limit $a \to 0$, P_e^μ *vanishes* because m_e vanishes.[11] In that point limit therefore P_e^μ is trivially a four-vector.

Note added. In a note added to his paper Professor Boyer makes a third claim in favor of the choice I for the specification of σ in electromagnetic momenta: the choice II "is completely inappropriate for composite systems, such as colliding point charges." I shall demonstrate in the following that this claim is also incorrect by deriving the covariant integral Poynting theorem.

Consider a closed system of N point charges in mutual electromagnetic interaction. The electromagnetic energy tensor $\Theta_e^{\mu\nu}$ satisfies

$$\partial_\alpha \Theta_e^{\alpha\mu} = F^{\mu\alpha} j_\alpha \ , \qquad (18)$$

where j^μ is the sum of all point-charge current four-vector densities

$$\begin{aligned} j^\mu(x) &= \sum_{a=1}^{N} j_a^\mu(x) \ , \\ j_a^\mu(x) &= e_a \int_{-\infty}^{\infty} \delta_4(x - z_a) v_a^\mu d\tau \ ; \end{aligned} \qquad (19)$$

$z_a(\tau)$ and $v_a(\tau) = \dot{z}_a(\tau)$ are the position and velocity four-vectors of particle a. Equation (18) is the local form of Poynting's theorem.

One can integrate both sides of (18) over a four-dimensional volume between two spacelike planes σ_1 and σ_2, later than σ_1. Using Gauss's theorem (Ref. 3, p. 281),

$$\int \epsilon_\sigma d^3\sigma_\alpha \Theta_e^{\alpha\mu} = \int d^4x F^{\mu\alpha} j_\alpha \ .$$

Following the choice (I) for σ one has $d^4x = d^3x\, dt$. If the surfaces σ_1 and σ_2 are separated infinitesimally,

$$\frac{d}{dt} \int \epsilon_\sigma d^3\sigma_\alpha \Theta_e^{\alpha\mu} = \int d^3x F^{\mu\alpha} j_\alpha \ . \qquad (20)$$

Neither side of this equation is a four-vector. If, on the other hand, one uses choice II one has the invariant factorization $d^4x = d^3\sigma d\tau$ where

$$d^3\sigma = -d^3\sigma_\mu \hat{P}^\mu \ ,$$

$\hat{P}^\mu$ being the unit vector in the direction of the total momentum of the closed system. One now obtains

$$\frac{d}{d\tau}\int \epsilon_\sigma d^3\sigma_\alpha \Theta_e^{\alpha\mu} = \int d^3\sigma F^{\mu\alpha} j_\alpha$$
$$= \sum_{a=1}^{N} e_a F^{\mu\nu}(z_a) v_{a\nu} \,. \qquad (21)$$

Both sides of this equation are four-vectors. Both (20) and (21) are integral forms of Poynting's theorem. Professor Boyer is willing to accept only the form (20). The manifestly covariant form, however, is (21). It is only the latter which is appropriate for a manifestly covariant formulation of the theory.

The explicit form of the left-hand sides of (20) and (21) are as follows: (20) gives the familiar result with

$$d^3\sigma^\mu = (d^3x, dt d^2\vec{\sigma})$$

in the notation used in (14) and (15):

$$\frac{d}{dt}\int d^3x \Theta_e^{0\mu} + \int d^2\sigma_k \Theta_e^{k\mu} \,. \qquad (22)$$

The left side of (21) gives, with the covariant separation

$$(P_\perp^{\alpha\beta} = \eta^{\alpha\beta} + \hat{P}^\alpha \hat{P}^\beta) \,,$$
$$d^3\sigma_{||}^\mu = d^3\sigma \hat{P}^\mu \,, \quad d^3\sigma_\perp^\mu = d\tau\, d^3\sigma_\beta P_\perp^{\beta\mu} \,,$$

the less familiar covariant result

$$-\frac{d}{d\tau}\int d^3\sigma \hat{P}_\alpha \Theta_e^{\alpha\mu} + \int d^2\sigma^\beta P_{\beta\alpha}^\perp \Theta_e^{\alpha\mu} \,. \qquad (23)$$

It reduces to (22) in the rest frame of the system. (21) and (23) give the integral Poynting theorem in covariant form.

APPENDIX

A well-known Lorentz boost from the rest frame transforms $\vec{E}_R$, $\vec{B}_R = 0$ into

$$\vec{E}^{||} = \vec{E}_R^{||} \,, \quad \vec{E}^\perp = \gamma \vec{E}_R^\perp \,,$$
$$\vec{B}^{||} = 0 \,, \quad \vec{B}^\perp = \gamma \vec{v} \times \vec{E}_R^\perp \,. \qquad (A1)$$

The superscripts $||$ and $\perp$ refer to vectors parallel and perpendicular to $\vec{v}$. Since

$$E_R^{\perp 2} = E_R{}^2 - E_R^{||2} \,,$$
$$(\vec{v} \times \vec{E}_R)^2 = \vec{v}^2 (E_R{}^2 - E_R^{||2}) \,,$$
$$E_R^{||2} = \tfrac{1}{3} E_R{}^2 \,,$$

we have

$$8\pi U = E^2 + B^2 = E_R^{||2} + \gamma^2 E_R^{\perp 2} + \gamma^2 (\vec{v} \times \vec{E}_R)^2$$
$$= E_R{}^2 \gamma^2 (1 + \tfrac{1}{3}\vec{v}^2) \,.$$

If we define

$$m_e = \frac{1}{8\pi}\int d^3x_R E_R{}^2 \,, \qquad (A2)$$

then

$$\int U d^3x_R = m_e \gamma^2 (1 + \tfrac{1}{3}\vec{v}^2) \,. \qquad (A3)$$

Similarly,

$$4\pi\vec{S} = (\vec{E}_R^{||} + \gamma \vec{E}_R^\perp) \times (\vec{v} \times \vec{E}_R^\perp)\gamma$$
$$= \gamma^2 \vec{v} E_R^{\perp 2} - \gamma |\vec{v}| E_R^{||} \vec{E}_R^\perp \,.$$

The last term integrates to zero so that

$$\int \vec{S} d^3x_R = \tfrac{4}{3}\gamma^2 m_e \vec{v} \,. \qquad (A4)$$

Finally, using the result obtained for U, the dyadic $\overleftrightarrow{\Theta}$ gives

$$4\pi\overleftrightarrow{\Theta}\cdot\vec{v} = [\vec{E}\vec{E} + \vec{B}\vec{B} - \tfrac{1}{2}\overleftrightarrow{1}(E^2 + B^2)]\cdot\vec{v}$$
$$= (\vec{E}_R^{||} + \gamma\vec{E}_R^\perp) E_R^{||} |\vec{v}| - \tfrac{1}{2}\vec{v} E_R{}^2 \gamma^2 (1 + \tfrac{1}{3}\vec{v}^2) \,.$$

The $\vec{E}_R^\perp E_R^{||}$ term integrates again to zero. The remainder is

$$\vec{v}[\tfrac{1}{3}E_R{}^2 - \tfrac{1}{2}E_R{}^2\gamma^2(1 + \tfrac{1}{3}\vec{v}^2)]$$
$$= -\vec{v} E_R{}^2 (\tfrac{1}{6}\gamma^2 + \tfrac{1}{2}\gamma^2\vec{v}^2) \,.$$

Thus,

$$\int \overleftrightarrow{\Theta}\cdot\vec{v} d^3x_R = -m_e \vec{v}\gamma^2(\tfrac{1}{3} + \vec{v}^2) \,. \qquad (A5)$$

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[1]For a review see F. Rohrlich, in *The Physicist's Conception of Nature*, edited by J. Mehra (Reidel, Dordrecht-Holland, 1973); C. Teitelboim, D. Villarroel, and Ch. G. Van Weert, Riv. Nuovo Cimento 3, 1 (1980).

[2]T. H. Boyer, preceding paper, Phys. Rev. D. 25, 3246 (1982). This paper will be referred to as CME.

[3]F. Rohrlich, *Classical Charged Particles* (Addison-Wesley, Reading, Mass., 1965).

[4]J. D. Jackson, *Classical Electrodynamics*, 2nd ed. (Wiley, New York, 1975).

[5]H. Poincaré, Rend. Circ. Mat. Palermo 21, 129 (1906); see also Ref. 4, p. 792 ff.

[6]If there existed a $\Theta_{\rm coh}^{\mu\nu}$ so that $f_{\rm coh}^\mu = \partial_\alpha \Theta_{\rm coh}^{\alpha\mu}$ as in field theory, then

$$-\int d^4x f_{\rm coh}^\mu = \int d^3\sigma_\alpha \Theta_{\rm coh}^{\alpha\mu}$$

since $\Theta_{\rm coh}^{\mu\nu} = 0$ on the surface at $t = -\infty$.

[7]For the sake of clarity we use a subscript R to indicate

the rest frame, S_R. Quantities without that subscript refer to S in which the particle moves uniformly.

[8]See for example Ref. 3, Eq. (6-5) where there is a factor $\frac{1}{2}$ missing.

[9]F. Rohrlich, Am. J. Phys. 28, 639 (1960).

[10]R. Benumof, Am. J. Phys. 39, 392 (1971).

[11]H. Grotch, E. Kazes, F. Rohrlich, and D. H. Sharp, Acta Phys. Austriaca, 54, 31 (1982); E. J. Moniz and D. H. Sharp, Phys. Rev. D 15, 2850 (1977).