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| DOI: |  | ISSN (Print) |

|  |
| --- |
| Original article |
| In print article |
| [https://doi.org/](https://doi.org/\mydoi) |
| UDC 576.3/7.002:531/534 |
| PACS numbers: 81.35. +k |

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Received on April 17, 2023. Reviewed on May 10, 2023.

Accepted for publication on May 22, 2023.

By analogy with the classical one-dimensional two-body Casimir effect, when on each of the plates there acts a pushing in Casimir force as a difference of electromagnetic pressures of zero-point quantum vacuum oscillations on different sides of each of the plates, due to the fact that the geometry of the vacuum resonator in the inner and outer regions is different, it is possible to try to create a difference of electromagnetic pressures of quantum vacuum fluctuations on different sides of only one plate due to the difference of the geometry of vacuum resonators on both sides of the plate. For this purpose on one of surfaces of a smooth metallic plate it is necessary to grow nanocells.

In this paper, the two-dimensional one-body Casimir effect is analysed on the example of square-shaped nanocells. As a result, it was found that the formula for the force per unit and for the energy density difference per unit area is very similar to the formula of the classical Casimir Effect, except for the value of the proportionality coefficient.

Two dimensional one body Casimir effect, nanocells, nanohoneycomb, Casimir thrust, Casimir driving force

**In cites**: . . . [https://doi.org/](https://doi.org/\mydoi).

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Introduction by Planck in 1911 of half-quanta in the context of black-body radiation become a precondition of the Casimir effect discovering which was made in the seminal paper by Casimir . This phenomena is one of the most direct manifestations of a quantum and relativistic phenomenon caused by the zero-point oscillations of quantized fields.

The Casimir effect in its simplest form is the pushing in of a pair of neutral, parallel conducting plates resulting from the modification of the electromagnetic vacuum by the boundaries.

The calculation of the Casimir force is a particularly complicated theoretical problem. Remarkably, for closed configurations, i.e. for the Casimir effect for one body instead of two, the Casimir force can be not only attractive but also repulsive. As was shown by Boyer , the latter is true for an ideal metal spherical shell.

Antipin A. V. expects appearance of the driving force (due to the one body Casimir effect) as a result of the difference of impact of the virtual particles (photons) on the external and internal reflective surfaces of pyramidal, conical or corner shape.

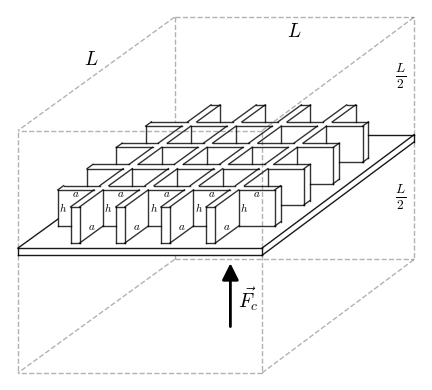
This work is dedicated to one body Casimir effect for opened configuration of perfectly conducting nano honeycomb.

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Let us consider a cubic cavity of volume bounded by perfectly conducting walls and let a perfectly conducting square plate with side be placed in this cavity parallel to the face and let us imagine the situation in which this plate is at a very large, say distance a from the face.

One side of this perfectly conducting square plate is a pure plane and other is covered with perfectly conducting square-shaped honeycombs with a square side .

On both sides of the the plate expressions $\sum\hbar\omega\big/2$ where the summation extends over all possible resonance frequencies of the cavity (large cavity: between pure plane and face) and the cavity (a small cavity, one honeycomb cell: between the bottom of the honeycomb and the opposite face) are divergent and devoid of physical meaning but the difference between these sums on the opposite sides of the plate, $\left(\sum\,\,\hbar\omega\right)\_{I}\big/{\left(2\,V\_{I}\right)} - \left(\sum\,\,\hbar\omega\right)\_{II}\big/{\left(2\,V\_{II}\right)}$, will be shown to have a well defined value and this value will be interpreted as the interaction between the plate and the both remote faces.



[fig:honeycomb\_box\_L]

The possible vibrations inside cavities defined by

, , (large cavity between pure plane and face)

and

, , (small cavity, one honeycomb cell)

have wave vectors

, , (large cavity between pure plane and face),

and

, , (small cavity, one honeycomb cell),

where . , are positive integers;

.

Let us write the expression for the sum of zero-point energy in general form

To every , , correspond two standing waves unless one of the is zero, when there is only one.

In case of one honeycomb cell cavity for this is without importance since for very large we may regard as continuous variable, replacing summation over with integration. Thus, for a small cavity consisting of one honeycomb, we find

.

In consideration of we can find the specific energy density , where :

, , where the notation is meant to indicate that the term with and has to be multiplied by $1\big/2$. Thus, for a small cavity consisting of one honeycomb, we have

. And in case of large cavity for , this is without importance since for very large we may regard , as continuous variables. Thus, for large cavity between pure plane and face we find .

For very large also this last summation may be replaced by an integral and it is therefore easily seen that energy of large cavity is given by

, , , ,

Now for the specific energy density for large cavity, where we can write the following sequence of transformations:

, , , . And finally for large cavity between pure plane and face we can write the following energy density formula ,

but for small cavity, one honeycomb cell we have .

Therefore, it is easy to see that the interaction energy is determined by the following energy density difference:

2

This expression is clearly infinite, and to proceed with the calculation, it is convenient to introduce a regulator.

In order to obtain a finite result it is necessary to multiply the integrands by a regularization function which is unity for but tends to zero sufficiently rapidly for . Where may be defined by . The physical meaning is obvious: for very short waves (X~rays e.g.) our plate is hardly an obstacle at all and therefore the zero point energy of these waves will not be influenced by the position of this plate.

The regulator will serve to make the expression finite, and in the end influence of its specific type will be removed by a limit transition.

Introducing the variable , , we have:

2

If and we have so where the cutting frequency is

Introducing function:

we can write:

And at least, introducing

We have

Integration of with example of regulator function see in Appendix A.

Now aiming to receive a way of calculating we should consider

# 

According to A.Bikyalis we just apply Euler - Maclaurin formula twice on and on . Starting from the following form of this formula:

,

or better since we are dealing with a mathematically very complex problem of integration that is rather difficult, often oscillating and suffering discontinuities at the points of each integer value of the argument due to the presence of the multiplier of the function, we will use the fact that the remainder term can also be expressed in the form:

we see that it consists of 4 parts:

integral ,

half sum ,

sum of Bernoulli polynomials

,

and the remainder

.

When applying it to twice on and on we should have the following summands which can be represented in the table form:

And taking into account that we have function which is symmetric on its and arguments, so above two dimentional Euler–Maclaurin marix is symmetric too.

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Taking value of parameter we have:

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# A way of calculating \delta\left(E/V\right)

Let’s consider the expression

First we see, that

So we have

On the other hand we have found that

$\sum\limits\_{n\_x=0}^{\infty} \sum\limits\_{n\_y=0}^{\infty} G\left(n\_x, n\_y\right) -\int\limits\_{0}^{\infty} \int\limits\_{0}^{\infty} G\left(n\_x, n\_y\right)\,d{n\_x}\,d{n\_y} = \\
\begin{array}{cccc} \, & \int\limits\_{n\_y}^{} \underset{n\_x}{H\_{\sum}}\,G & + \int\limits\_{n\_y}^{}{\sum\limits\_{n\_x}^{}}^{B}\,G & + \int\limits\_{n\_y}^{}\,\underset{n\_x}{R\_{p}}\,G \\
+\underset{n\_y}{H\_{\sum}}\,\int\limits\_{n\_x}^{}\,G & + \underset{n\_y}{H\_{\sum}}\,\underset{n\_x}{H\_{\sum}}\,G & + \underset{n\_y}{H\_{\sum}}\,{\sum\limits\_{n\_x}^{}}^{B}\,G & + \underset{n\_y}{H\_{\sum}}\,\underset{n\_x}{R\_{p}}\,G \\
+ {\sum\limits\_{n\_y}^{}}^{B}\,\int\limits\_{n\_x}^{}\,G & + {\sum\limits\_{n\_y}^{}}^{B}\,\underset{n\_x}{H\_{\sum}}\,G & + {\sum\limits\_{n\_y}^{}}^{B}\,{\sum\limits\_{n\_x}^{}}^{B}\,G & + {\sum\limits\_{n\_y}^{}}^{B}\,\underset{n\_x}{R\_{p}}\,G \\
+ \underset{n\_y}{R\_{p}}\,\int\limits\_{n\_x}^{}\,G & + \underset{n\_y}{R\_{p}}\,\underset{n\_x}{H\_{\sum}}\,G & + \underset{n\_y}{R\_{p}}\,{\sum\limits\_{n\_x}^{}}^{B}\,G & + \underset{n\_y}{R\_{p}}\,\underset{n\_x}{R\_{p}}\,G \end{array}$

where , and are:

Now using 1D Euler–Maclaurin formula in the form ([eq:7]) we can see that:

will be equal to

So we can find expression ([eq:12]) using the following summation:

It is easy to see that all terms without remainder part:

are 0 in sum. And so any possible non zero result of expression ([eq:12]) should be related with remainder part:

Or using symmetric properties of we can rewrite result formula:

with the following terms:

Considering should be , because derivative with respect to when gives , and gives in summation, we can simplify result in form:

or in detailed form:

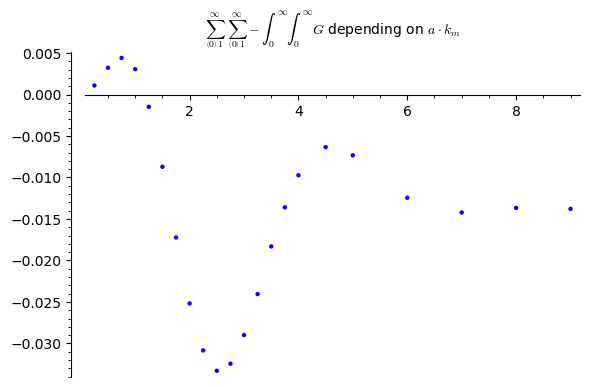
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# How \delta\left(E/V\right) depends on a k\_m ?

Casimir in his original work has provided his formula in assumption when as . But now we can investigate how the expression ([eq:12]) will depends on

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|  |  |  |  |
| 0.25 | 0.00108989 | 4.99669e-07 | 2 |
| 0.5 | 0.00322377 | 7.99836e-07 | 6 |
| 0.75 | 0.00440392 | 8.72173e-07 | 11 |
| 1.0 | 0.00304837 | 8.49556e-07 | 17 |
| 1.25 | -0.00148115 | 1.77116e-06 | 18 |
| 1.5 | -0.00870823 | 3.67271e-06 | 18 |
| 1.75 | -0.0172279 | 6.80405e-06 | 18 |
| 2.0 | -0.0251746 | 9.98944e-06 | 19 |
| 2.25 | -0.0308341 | 9.37033e-06 | 23 |
| 2.5 | -0.0332867 | 1.42819e-05 | 23 |
| 2.75 | -0.0324363 | 2.09102e-05 | 23 |
| 3.0 | -0.0289864 | 2.96151e-05 | 23 |
| 3.25 | -0.02404795 | 4.07897e-05 | 23 |
| 3.5 | -0.0183092 | 8.56118e-06 | 45 |
| 3.75 | -0.0135918 | 2.185368e-05 | 36 |
| 4 | -0.00973151 | 1.46105e-05 | 45 |
| 4.5 | -0.00634270 | 1.93298e-05 | 48 |
| 5 | -0.00732334 | 2.32423e-05 | 52 |
| 6 | -0.0124360 | 2.37328e-05 | 66 |
| 7 | -0.0142094 | 3.85332e-05 | 69 |
| 8 | -0.0136610 | 3.29375e-05 | 87 |
| 9 | -0.0137827 | 3.58598e-05 | 99 |

Thus for the energy density difference per we find



[fig:G\_on\_a\_km]

According to our calculation we can see that

Where is some function depended on material properties with well defined limit at . So

For the energy density difference per (in the limit as ) we find

where is a square side of honeycombs measured in microns.

Can this difference of specific energy density be interpreted as the cause of the force applied to perfectly conducting honeycomb on a plate? For example my investigations of the original Casimir’s configuration shown that in geometric configuration of two perfectly conducting plates . But what about honeycomb configuration? Investigation of this question in appendixes B and C shows that .

# 

We are thus led to the following conclusions. There is a force applied to perfectly conducting honeycombs on a plate as a result of the difference in specific energy density on its different sides. This force is dependent of the material of the plate. This force depends at least on the cutoff frequency of the honeycomb plate material. This force can be interpreted as the pressure of the zero point electromagnetic oscillations.

Although the effect is small, an experimental confirmation seems not unfeasable and might be of a certain interest.

Tuo Qu, Fang Liu, Yuechai Lin, Yidong Huang reported about the production of gold nano-honeycomb with a size of about 2 microns.

According to the formula received in this work such honeycomb should have the difference Casimir energy densities about , that is 8.55 dynes per square meter of the panel, which is already quite an acceptable value for the practical use of the expected effect for correcting satellite orbits.

The only thing to point out is that, according to my method of calculation, the resulting formula gives calculated not for the total surface area of the honeycombs, but for part of the panel’s area occupied by recesses (minus the part of the honeycomb area occupied by walls).

This does not mean the need to strive to make very thin walls of the honeycomb, because with a decrease in the wall thickness, the value of will also decrease.

For simplicity of calculations, square-shaped nanocells were analyzed, but the result obtained can be used, with some so far unknown correction, to estimate the Casimir effect in honeycombs of a different shape, for example, hexagonal, cylindrical, etc. honeycomb shape would be more complex. In addition, the bottom of the honeycomb may not necessarily be flat, but, for example, spherically concave, which should also not fundamentally affect the magnitude of the thrust.

In addition, Casimir thrust is expected not only in metal honeycombs, but also in dielectric honeycombs, because the Casimir effect in dielectrics is well known.

In the work, an implicit assumption was used about the infinite value of the height of the walls of the honeycombs, while the real height of the walls will be finite.

At the same time, using the Antipin approach (see Appendix D), it can be shown that the effect should also be observed at a finite height of the ribs, although the dependence of the effect on this height may be the subject of further research.

Regarding criticism: Hrvoje consider the Casimir effect not as a consequence of the existence of virtual quantum photons, but as only a manifestation of the London-van der Waals dispersion forces.

I want to note that setting up an experiment to measure the thrust force produced by nanocells grown on metal plate could serve as a kind of critical experiment to find out which of the points of view on the nature of the Casimir forces corresponds to reality.

# 

2 Let’s use the regularization function in the form

Starting from ([eq:4]) and by introducing a variable and we have

And using this variable we can make the following substitution

And now we can rewrite integral ([eq:5]) in form:

changing the integration variable from to

Because in this form that integral can be taken analytically. So we have the following integrand:

and the following limits of integration by : ,

Let’s use Abel substitution:

and the following limits of integration by : ,

Let’s write dependency of from

and derivatives:

Now we can rewrite the integrand making it depending from

Let’s extract coefficient near from the denominator

Now let’s move the above coefficient up to the numerator. So new numerator will be:

And, accordingly, the new denominator will be:

Now we should convert this denominator to the following form:

So we have the following equation

and its solution

,

After above determined conversion the integrand can be written as:

Let’s check determinant using above found expression of and .

Determinant is negative. So the integral can be easily taken:

# 

2

Let’s consider rectangular resonator with size .

**In electric mode**

we have the following solution

and

with

using we have

Field energy density $\left(\int \frac{E\_x^2+E\_y^2+E\_z^2}{8 \pi}dV\right)\big/{V}$ is

Full energy density $\left(\int \frac{|\vec{E}|^2}{8 \pi}dV + \int \frac{|\vec{H}|^2}{8 \pi}dV\right)\big/{V}$ is

Electromagnetic pressure $\left({\int \frac {H\_x^2+H\_y^2}{8 \pi} dS}\right)\big/{S}$ on plate is

Their relation is

2

Considering solution with wave propagation in direction we have which give:

and

Relation of electromagnetic pressure per field energy density in this case is equal to

Considering solution with wave propagation in direction we have which give:

Relation of electromagnetic pressure per field energy density in this case is

Considering solution with wave propagation in direction is similar to it in direction

**In magnetic mode**

we have the following solution

and

with

using we have

Magnetic field energy density $\left(\int \frac{H\_x^2+H\_y^2+H\_z^2}{8 \pi}dV\right)\big/{V}$ is

Full energy density $\left(\int \frac{|\vec{E}|^2}{8 \pi}dV + \int \frac{|\vec{H}|^2}{8 \pi}dV\right)\big/{V}$ is

Electromagnetic pressure $\left({\int \frac {H\_x^2+H\_y^2}{8 \pi} dS}\right)\big/{S}$ on plate is

Their relation is

Considering solution with wave propagation in direction we have which give:

and

Relation of electromagnetic pressure per field energy density in this case is

Considering solution with wave propagation in direction we have which give:

Relation of electromagnetic pressure per field energy density in this case is

Considering solution with wave propagation in direction is similar to it in direction

So we see that task of calculation of electromagnetic force in the nano honeycomb configuration is quit easy, because we can see that

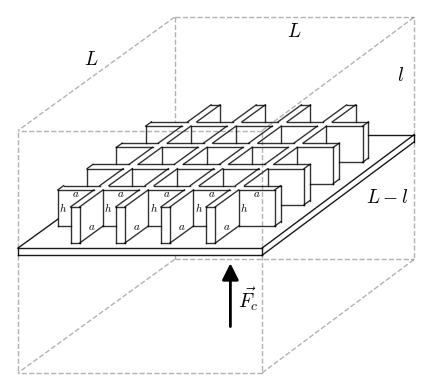
Also we see that if we decrease then we have decreased . So this leads to

On the other hand I can show the same result using

# 

Let us consider a cubic cavity of volume bounded by perfectly conducting walls and let a perfectly conducting square plate with side be placed in this cavity parallel to the face and let us consider the situation in which this plate is at a very large, say distance a from the face.

One side of this perfectly conducting square plate is a pure plane and other is covered by perfectly conducting honeycomb.



[fig:honeycomb\_box\_H]

On both sided of the the plate expressions $1\big/2\sum\,\hbar\omega$ where the summation extends over all possible resonance frequencies of the cavity (large cavity between pure plane and face) and the cavity (small cavity, one honeycomb cell) are divergent and devoid of physical meaning but the derivative ${d\left<0|\hat{\mathcal{H}}|0\right>}\big/{dl}$ of the whole system’s vacuums Hamiltonian for these sums for the both opposite sides, $\left<0|\hat{\mathcal{H}}|0\right> = 1\big/2\,\left(\sum\,\,\hbar\omega\right)\_{I} + 1\big/2\,\left(\sum\,\,\hbar\omega\right)\_{II}$, will be shown to have a well defined value and this value will be interpreted as the interaction between the plate and the both faces.

The possible vibrations of the cavities defined by

, , (large cavity between pure plane and face)

and

, , (small cavity, one honeycomb cell)

have wave vectors

, , (large cavity between pure plane and face),

and

, , (small cavity, one honeycomb cell),

where . , are positive integers;

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Let us write the expression for the sum of zero-point energy in general form

To every , , correspond two standing waves unless one of the is zero, when there is only one.

In case of one honeycomb cell cavity for this is without importance since for very large we may regard as continuous variable, replacing summation over with integration. Thus, for a small cavity consisting of one honeycomb, we find

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In consideration of we can find the specific energy density , where :

,

.

And in case of large cavity for , this is without importance since for very large we may regard , as continuous variables. Thus, for large cavity between pure plane and face we find

.

For very large also this last summation may be replaced by an integral and it is therefore easily seen that energy of large cavity is given by

where , , .

Now for the specific (per area) energy density for large cavity, where we can write the following sequence of transformations:

, , , . It is therefore easily seen that specific (per area) vacuum Hamiltonian $\frac{\left<0|\hat{\mathcal{H}}|0\right>}{S}$ of the whole system is given by

$$\begin{array}{lr}
\begin{array}{l}
\frac{\left<0|\hat{\mathcal{H}}|0\right>}{S} = \frac{\hbar\,c}{a^2\,\pi} \cdot\\
\cdot \Bigg\{l\sum\limits\_{n\_x=(0)1}^{\infty}\sum\limits\_{n\_y=(0)1}^{\infty}\left[\,\int\limits\_{0}^{\infty}\sqrt{n\_x^2\frac{\pi^2}{a^2}+n\_y^2\frac{\pi^2}{a^2}+k\_z^2}\,dk\_z\right] + \\
(L-l)\int\limits\_{0}^{\infty}\int\limits\_{0}^{\infty}\left[\,\int\limits\_{0}^{\infty}\sqrt{k\_x^2+k\_y^2+k\_z^2}\,dk\_z\right]\,\left(\frac{a}{\pi}dk\_x\right)\,\left(\frac{a}{\pi}dk\_y\right)\Bigg\}
\end{array}\end{array}$$

and interaction $\frac{F}{S} = \frac{d}{dl} \,\frac{\left<0|\hat{\mathcal{H}}|0\right>}{S}$ is:

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So using Hamiltonian mechanics approach I see that formula for the force to perfectly conducting honeycomb on a plate is the same as formula ([eq:2]) which is received for the difference of specific energy density on its different sides .

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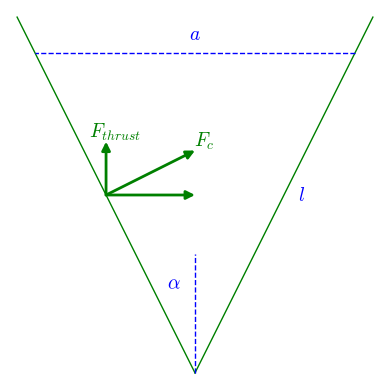
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Antipin gives an estimated calculation of the thrust force of the “corner” using the Casimir formula, “with the most general and natural approximations known as PFA (Proximity Force Approximation), or PAA (Pairwise Additive Approximation), calculation method , ".

Casimir’s interaction energy is given by

Casimir’s force

Driving force of the metal corner



[fig:Antipins\_angle]

Let’s make a substitution

You can make a substitution

Thus, the formula for the thrust of the corner is obtained basing on the length of its wings.

Antipin indicates that the value of is limited from below by the cutoff level, which is determined technologically:

* by the accuracy of manufacturing of the plates (their roughness, degree of flatness), as well as
* by the the minimum wavelength of photons that can effectively reflect the substance from which the corner is made (by value).

Investigating the dependence of the coefficient in the angle thrust formula, which depends on the opening half angle , it can be seen that for a given length of the angle wings, it is more profitable to make the minimum possible opening angle. However, for the purposes of this work (study of the possibility of obtaining thrust using nanocells), it is important to note that for a corner with a right angle , the coefficient $\left({cos\, \alpha}\right)\big/\left({\left(sin\, \alpha\right)^4}\right) = 2\sqrt{2}$. Thus, composing a structure in the form of honeycombs from many rectangular corners, it is shown that the thrust of the panel consisting of rectangular honeycombs is not zero.

Indeed, a rectangular honeycomb with a cell size of and with the same edge height equal to can be imagined as a combination of four corners with a half opening angle equal to . Driving force of the every corner

directed along the bisector of each angle must be multiplied by $sin\left({\pi}/{4}\right)={\sqrt{2}}\big/{2}$ and, when multiplied by 4, the thrust of one square opened cell will be equal to

So, the formula for the specific thrust of cells, obtained using the PFA (Proximity Force Approximation) method, or PAA (Pairwise Additive Approximation)

to some extent similar to the formula for the magnitude of the two-dimensional Casimir effect on honeycombs, produced in the first part of this work: at least the value of the exponent in the denominator is the same.

It should be noted here that this formula is obtained without taking into account the finiteness of the cell edge height (in other words, in the approximation of the infinite edge height), in contrast to the PFA version of the formula for which the edge height is assumed to be equal to the cell width.

The approximate correspondence of these formulas indicates that the effect should also be observed at a finite height of the ribs, although the dependence of the effect on this height may be the object of further research.

Theoretically it is possible to achieve a greater value of thrust with a panel of acute angled corners, but technologically the production of panels from honeycombs seems to be simpler than the production of panels from V-shaped corners.

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