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In this paper, the two-dimensional one-body Casimir effect is analyzed on the example of square-shaped nanocells. In the classical one-dimensional two-body Casimir effect a Casimir force appears between two plates as a difference of electromagnetic pressures of zero-point quantum-vacuum oscillation on different sides of each of the plates. The plates are pushed forwards each other by external quantum-vacuum oscillation fields, which in classical configuration exceed internal quantum-vacuum oscillation fields. It is possible to try to create a difference of electromagnetic pressures of quantum-vacuum oscillation on different sides of a single plate due to the difference of the geometry of vacuum resonators on different sides of the plate. For this purpose, it is necessary to grow nanocells on one of surfaces of a smooth metallic plate. As a result, it has been found that the formula for the force per unit and for the energy density difference per unit area is very similar to the formula of the classical Casimir effect, except for the value of the proportionality coefficient.

Keywords: Two-dimensional one body Casimir effect, nanocells, nanohoneycomb, Casimir thrust

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The introduction of half-quanta in the context of black-body radiation by Planck in 1911 was fundamental for discovering the Casimir effect presented by Casimir in his seminal paper. This is one of the most direct manifestations of a quantum and relativistic phenomenon caused by the zero-point oscillation of quantized fields.

The Casimir effect in its simplest form is the force between a pair of neutral, parallel conducting plates resulting from the modification of the electromagnetic vacuum properties caused by the change in boundary conditions.

The calculation of the Casimir force is a particularly complicated theoretical problem. Remarkably, for closed configurations, i.e., when the Casimir effect manifests for one body instead of two, the Casimir force can be not only attractive but repulsive as well. As it has been shown by Boyer, the latter is true for an ideal metal spherical shell.

Antipin A. V. expects the appearance of the driving force/thrust (due to the Casimir effect on one body), as a result of the difference in the impact of virtual particles (photons) on external and internal reflecting surfaces of pyramidal, conical or V-shaped objects.

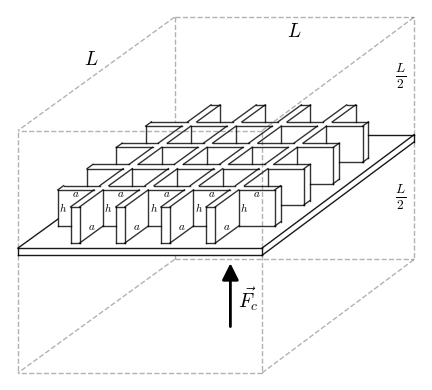
This work is dedicated to one-body Casimir effect for open configuration of perfectly conducting nanohoneycomb.

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Let us consider a cubic cavity of volume bounded by perfectly conducting walls where perfectly conducting square plate with side is placed in this cavity parallel to the face, and let the distance between the plate and the face be sufficiently large, , for example.

One side of this perfectly conducting square plate is a smooth plane and another is covered with perfectly conducting square-shaped honeycombs with a square side .

On both sides of the plate the expressions $\sum\hbar\omega\big/2$ where the summation extends over all possible resonance frequencies of the cavity (a large cavity: between smooth plane and face) and the cavity (a small cavity, one honeycomb cell: between the bottom of the honeycomb and the opposite face) are divergent and devoid of physical meaning but the difference between these sums on the opposite sides of the plate, $\left(\sum\,\,\hbar\omega\right)\_{I}\big/{\left(2\,V\_{I}\right)} - \left(\sum\,\,\hbar\omega\right)\_{II}\big/{\left(2\,V\_{II}\right)}$, will be shown to have a well-defined value and this value will be interpreted as the interaction between the plate and the both remote faces.



[fig:honeycomb\_box\_L]

The possible oscillations inside cavities defined by

, , (a large cavity between smooth plane and face)

and

, , (a small cavity, one honeycomb cell)

have wave vectors

, , (a large cavity between smooth plane and face),

and

, , (a small cavity, one honeycomb cell),

where . , are positive integers;

.

Let us write the expression for the sum of zero-point energy in general form

.

Two standing waves correspond to every , , , but in case when one of the is zero, there is only one wave. That is of no importance in case of one honeycomb cell cavity for , since for very large we may regard as continuous variable, replacing summation over with integration. Thus, for a small cavity consisting of one honeycomb, we find

.

Considering we can find the specific energy density , where :

, , where the notation is meant to indicate that the term with and has to be multiplied by $1\big/2$. Thus, for a small cavity consisting of one honeycomb, we have

. That is of no importance in case of a large cavity for , , since for very large we may regard , as continuous variables. Thus, for a large cavity between smooth plane and face we find .

For very large the last summation may also be replaced by an integral and, therefore, it can be seen that energy of a large cavity is given by

, , , ,

Now for the specific energy density for a large cavity, where we can write the following sequence of transformations:

, , , . And finally for a large cavity between smooth plane and face we can formulate the energy density as following ,

but for a small cavity, one honeycomb cell we have .

Therefore, it is obvious that the interaction energy is determined by the following energy density difference:

2

The expression obtained is clearly infinite, and to proceed with the calculation, it is convenient to introduce a regulator.

In order to obtain a finite result, it is necessary to multiply the integrands by a regularization function which is unity for but tends to zero sufficiently rapidly for . Where may be defined by . The physical meaning is obvious, our plate is hardly an obstacle for very short waves (X-rays e.g.) and, therefore, the zero-point energy of these waves will not be influenced by the position of the plate.

The purpose of regulator is to make the expression finite, and influence of its specific type will be removed by a limit transition in the end.

Introducing the variable , , we have:

2

If and we have so , where the cutting frequency is .

Introducing function

,

we can write

And at least, introducing

we have

Integration of with example of regulator function is presented in Appendix A.

To receive a way of calculating , the Euler-Maclaurin 2D formula could be considered.

**Euler-Maclaurin 2D formula**

According to A. Bikyalis we apply the Euler - Maclaurin formula twice on and on . Since we are dealing with a very complex mathematical problem of integrating the function, which often oscillates and suffers discontinuities at the points of each integer value of the argument due to the presence of the multiplier , hereafter, for the formula

,

we will use the fact that the remainder can also be expressed in the form

.

We can see that it consists of 4 parts:

the integral ,

the half sum ,

the sum of Bernoulli polynomials

,

and the remainder

.

When applying it to twice on and on we should have the following summands which can be represented as the table:

Taking into account that the function is symmetric on its and arguments, so the two-dimensional Euler–Maclaurin matrix presented above is symmetric too.

Taking value of parameter we have:

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# A way of calculating \delta\left(E/V\right)

Let us consider the expression

Firstly, we can see, that

Therefore, we have

On the other hand, we have found that

$\sum\limits\_{n\_x=0}^{\infty} \sum\limits\_{n\_y=0}^{\infty} G\left(n\_x, n\_y\right) -\int\limits\_{0}^{\infty} \int\limits\_{0}^{\infty} G\left(n\_x, n\_y\right)\,d{n\_x}\,d{n\_y} = \\ \begin{array}{cccc} \, & \int\limits\_{n\_y}^{} \underset{n\_x}{H\_{\sum}}\,G & + \int\limits\_{n\_y}^{}{\sum\limits\_{n\_x}^{}}^{B}\,G & + \int\limits\_{n\_y}^{}\,\underset{n\_x}{R\_{p}}\,G \\ +\underset{n\_y}{H\_{\sum}}\,\int\limits\_{n\_x}^{}\,G & + \underset{n\_y}{H\_{\sum}}\,\underset{n\_x}{H\_{\sum}}\,G & + \underset{n\_y}{H\_{\sum}}\,{\sum\limits\_{n\_x}^{}}^{B}\,G & + \underset{n\_y}{H\_{\sum}}\,\underset{n\_x}{R\_{p}}\,G \\ + {\sum\limits\_{n\_y}^{}}^{B}\,\int\limits\_{n\_x}^{}\,G & + {\sum\limits\_{n\_y}^{}}^{B}\,\underset{n\_x}{H\_{\sum}}\,G & + {\sum\limits\_{n\_y}^{}}^{B}\,{\sum\limits\_{n\_x}^{}}^{B}\,G & + {\sum\limits\_{n\_y}^{}}^{B}\,\underset{n\_x}{R\_{p}}\,G \\ + \underset{n\_y}{R\_{p}}\,\int\limits\_{n\_x}^{}\,G & + \underset{n\_y}{R\_{p}}\,\underset{n\_x}{H\_{\sum}}\,G & + \underset{n\_y}{R\_{p}}\,{\sum\limits\_{n\_x}^{}}^{B}\,G & + \underset{n\_y}{R\_{p}}\,\underset{n\_x}{R\_{p}}\,G \end{array}$,

where , and are

.

Now using the 1D Euler–Maclaurin formula in the form ([eq:7]) we can see that

will be equal to

Now we can find the expression ([eq:12]) by using the following summation

It is easy to see that the sum of all summands without remainder

is 0. And, therefore, any possible non zero result of expression ([eq:12]) should be attributed to the remainder

Or using symmetric properties of , we can rewrite the resulting formula

with the following summands:

;

;

;

;

Considering that should be because derivative with respect to , for is , and gives in summation, we can simplify the result in the form

or in the detailed form

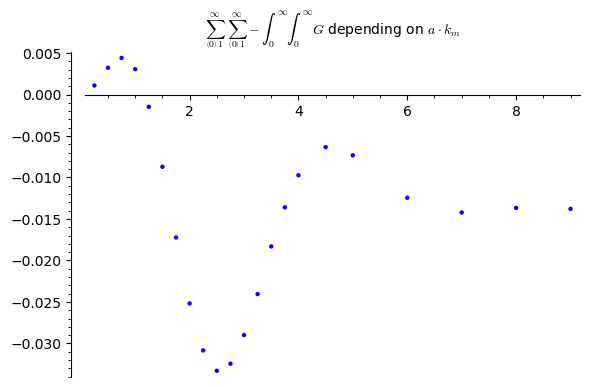
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# How does \delta\left(E/V\right) depend on a k\_m?

Casimir in his original work has provided his formula in assumption that . But now we can investigate how the expression ([eq:12]) will depends on

|  |  |  |  |
| --- | --- | --- | --- |
|  |  |  |  |
| 0.25 | 0.00108989 | 4.99669e-07 | 2 |
| 0.5 | 0.00322377 | 7.99836e-07 | 6 |
| 0.75 | 0.00440392 | 8.72173e-07 | 11 |
| 1.0 | 0.00304837 | 8.49556e-07 | 17 |
| 1.25 | -0.00148115 | 1.77116e-06 | 18 |
| 1.5 | -0.00870823 | 3.67271e-06 | 18 |
| 1.75 | -0.0172279 | 6.80405e-06 | 18 |
| 2.0 | -0.0251746 | 9.98944e-06 | 19 |
| 2.25 | -0.0308341 | 9.37033e-06 | 23 |
| 2.5 | -0.0332867 | 1.42819e-05 | 23 |
| 2.75 | -0.0324363 | 2.09102e-05 | 23 |
| 3.0 | -0.0289864 | 2.96151e-05 | 23 |
| 3.25 | -0.02404795 | 4.07897e-05 | 23 |
| 3.5 | -0.0183092 | 8.56118e-06 | 45 |
| 3.75 | -0.0135918 | 2.185368e-05 | 36 |
| 4 | -0.00973151 | 1.46105e-05 | 45 |
| 4.5 | -0.00634270 | 1.93298e-05 | 48 |
| 5 | -0.00732334 | 2.32423e-05 | 52 |
| 6 | -0.0124360 | 2.37328e-05 | 66 |
| 7 | -0.0142094 | 3.85332e-05 | 69 |
| 8 | -0.0136610 | 3.29375e-05 | 87 |
| 9 | -0.0137827 | 3.58598e-05 | 99 |

Thus, for the energy density difference per we find



[fig:G\_on\_a\_km]

According to our calculation we can see that

where is a function depended on material properties with well defined limit at . So

.

For the energy density difference per (in the limit as ) we find that

,

where is a square side of honeycombs measured in microns.

Can this difference of specific energy density be interpreted as the cause of the force applied to perfectly conducting honeycomb on a plate? For example, my investigations of the configuration used by Casimir have shown that for the geometric configuration of two perfectly conducting plates . But what can be said about honeycomb configuration? Research of this question presented in appendixes B and C shows that .

# 

Therefore, the following conclusions can be drawn: there is a force applied to perfectly conducting honeycombs on a plate as a result of the difference in specific energy density on its different sides. This force depends on the material of the plate. This force depends on the cutoff frequency of the honeycomb plate material at least. This force can be interpreted as the pressure of the zero-point electromagnetic oscillations.

Although the effect is small, an experimental confirmation seems not unfeasible and might be of a certain interest.

Tuo Qu, Fang Liu, Yuechai Lin, Yidong Huang have reported the production of gold nano-honeycomb with a size of about 2 microns. According to the formula presented in this work that honeycomb should have Casimir energy density difference about , that is 8.55 dynes per square meter of the panel, which is quite an acceptable value for practical use of the expected effect for satellite orbit correction.

It is important to point out, that according to the proposed method, the is calculated not for the total surface area of the honeycombs, but for a part of the panel occupied by cavities (minus the area of the honeycomb walls).

That does not mean, the honeycomb walls should be made as thin as possible, because with a decrease in the wall thickness, the value of will also decrease.

For simplicity of calculations, square-shaped nanocells have been analyzed, but the obtained result can be used, with some corrections unknown so far, to estimate the Casimir effect in honeycombs of a different shape (hexagonal, cylindrical, etc.). Such honeycombs are simpler to be manufactured, but require much more complex calculations to measure the Casimir effect exactly. In addition, the bottom of the honeycomb is not necessarily flat, but spherically concave, for example, which does not fundamentally affect the magnitude of the thrust.

In addition, the appearance of Casimir thrust is expected not only in metal honeycombs, but in dielectric honeycombs as well, because the Casimir effect in dielectrics is well known.

It has been assumed that the height of the walls of the honeycombs is infinite, while the real height is finite.

At the same time, by using the Antipin approach (see Appendix D) it can be shown that the effect should be observed at a finite height of the ribs, although the dependence of the effect on the height may be the subject of further research.

To answer Hrvoje, who considers the Casimir effect as not a consequence of the existence of virtual quantum photons, but as manifestation of the London-Van der Waals dispersion forces, I would like to note that setting up an experiment to measure the thrust produced by nanocells grown on metal plate could serve as a critical experiment to find out which point of view on the nature of the Casimir force corresponds to reality.

# 

Let us use the regularization function in the form

.

Starting from ([eq:4]) and by introducing a variable and , we have

.

And using this variable, we can make the following substitution

.

And now we can rewrite the integral ([eq:5]) in form

changing the integration variable from to

because in this form integral can be taken analytically. So, we have the following integrand

and the following limits of integration by : , .

Let us use the Abel substitution:

and the following limits of integration by : ,

Let us denote dependency of from

and derivatives

.

Now we can rewrite the integrand, making it depending on

Let us extract coefficient near from the denominator.

Now let us move the above coefficient up to the numerator. So, the new numerator will be

Accordingly, the new denominator will be

Now we should convert this denominator to the following form

So, we have the following equation

and its solution

,

.

After the conversion determined above, the integrand can be presented as

Let us check determinant by using the expression of and found above.

The determinant is negative and the integral can be easily calculated:

# 

Let us consider a rectangular resonator with size .

For the electric mode

,

we have the following solution

and

with

,

by using , we have

Field energy density $\left(\int \frac{E\_x^2+E\_y^2+E\_z^2}{8 \pi}dV\right)\big/{V}$ is

Full energy density $\left(\int \frac{|\vec{E}|^2}{8 \pi}dV + \int \frac{|\vec{H}|^2}{8 \pi}dV\right)\big/{V}$ is

Electromagnetic pressure $\left({\int \frac {H\_x^2+H\_y^2}{8 \pi} dS}\right)\big/{S}$ on plate is

Their relation is

2

Considering solution with wave propagation in -direction, we have which gives

and

.

In this case the relation of electromagnetic pressure per field energy density is equal to ,

.

Considering solution with wave propagation in -direction, we have which gives

.

In this case the relation of electromagnetic pressure per field energy density is

.

Considering solution with wave propagation in -direction, , as in -direction

For the magnetic mode

,

we have the following solution

and

with

by using , we have

.

Magnetic field energy density $\left(\int \frac{H\_x^2+H\_y^2+H\_z^2}{8 \pi}dV\right)\big/{V}$ is

.

Full energy density $\left(\int \frac{|\vec{E}|^2}{8 \pi}dV + \int \frac{|\vec{H}|^2}{8 \pi}dV\right)\big/{V}$ is

.

Electromagnetic pressure $\left({\int \frac {H\_x^2+H\_y^2}{8 \pi} dS}\right)\big/{S}$ on plate is

.

Their relation is

.

Considering solution with wave propagation in -direction, we have which gives

and

In this case the relation of electromagnetic pressure per field energy density is

.

Considering solution with wave propagation in -direction, we have which gives

.

In this case the relation of electromagnetic pressure per field energy density is

.

Considering solution with wave propagation in -direction, as in -direction

So, we can see that task of calculating electromagnetic force in the nanohoneycomb configuration is quite easy, because

.

We can see that if we decrease then also decreases and that leads to

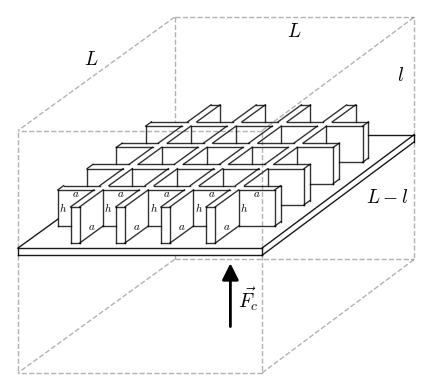
.

On the other hand, the same result can be shown by using

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Let us consider a cubic cavity of volume bounded by perfectly conducting walls where perfectly conducting square plate with side is placed in this cavity parallel to the face and let the distance between the plate and the face be sufficiently large, , for example.

One side of this perfectly conducting square plate is a smooth plane and another is covered by perfectly conducting square-shaped honeycombs with a square side .



[fig:honeycomb\_box\_H]

On both sides of the plate the expressions $1\big/2\sum\,\hbar\omega$ where the summation extends over all possible resonance frequencies of the cavity (a large cavity between smooth plane and face) and the cavity (a small cavity, one honeycomb cell) are divergent and devoid of physical meaning, but it will be shown that for the both opposite sides the derivative ${d\left<0|\hat{\mathcal{H}}|0\right>}\big/{dl}$ of the vacuum Hamiltonian of the whole system for these sums, $\left<0|\hat{\mathcal{H}}|0\right> = 1\big/2\,\left(\sum\,\,\hbar\omega\right)\_{I} + 1\big/2\,\left(\sum\,\,\hbar\omega\right)\_{II}$, has a well-defined value and this value will be interpreted as the interaction between the plate and the both faces.

The possible oscillations of the cavities defined by

, , (a large cavity between smooth plane and face)

and

, , (a small cavity, one honeycomb cell)

have the wave vectors

, , (a large cavity between smooth plane and face),

and

, , (a small cavity, one honeycomb cell),

where . , are positive integers;

.

Let us write the expression for the sum of zero-point energy in general form

.

Two standing waves correspond to every , , , but in case when one of the is zero, there is only one wave. That is of no importance in case of one honeycomb cell cavity for , since for very large we may regard as continuous variable, replacing summation over with integration. Thus, for a small cavity consisting of one honeycomb, we find

.

Considering we can find the specific energy density , where :

,

.

That is of no importance in case of large cavity for , since for very large we may regard , as continuous variables. Thus, for large cavity between smooth plane and face we find

.

For very large the last summation may be replaced by an integral and, therefore, it can be seen that energy of a large cavity is given by

where , , .

Now for the specific (per area) energy density for a large cavity, where we can write the following sequence of transformations:

, , , . Therefore, it can be seen that specific (per area) vacuum Hamiltonian $\frac{\left<0|\hat{\mathcal{H}}|0\right>}{S}$ of the whole system is given by

$$\begin{array}{lr} \begin{array}{l} \frac{\left<0|\hat{\mathcal{H}}|0\right>}{S} = \frac{\hbar\,c}{a^2\,\pi} \cdot\\ \cdot \Bigg\{l\sum\limits\_{n\_x=(0)1}^{\infty}\sum\limits\_{n\_y=(0)1}^{\infty}\left[\,\int\limits\_{0}^{\infty}\sqrt{n\_x^2\frac{\pi^2}{a^2}+n\_y^2\frac{\pi^2}{a^2}+k\_z^2}\,dk\_z\right] + \\ (L-l)\int\limits\_{0}^{\infty}\int\limits\_{0}^{\infty}\left[\,\int\limits\_{0}^{\infty}\sqrt{k\_x^2+k\_y^2+k\_z^2}\,dk\_z\right]\,\left(\frac{a}{\pi}dk\_x\right)\,\left(\frac{a}{\pi}dk\_y\right)\Bigg\} \end{array}\end{array}$$

and interaction $\frac{F}{S} = \frac{d}{dl} \,\frac{\left<0|\hat{\mathcal{H}}|0\right>}{S}$ is

.

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Then, the formula for the force acting on a perfectly conducting honeycomb on a plate obtained by using Hamiltonian mechanics approach is the same as the formula ([eq:2]) obtained for the difference of specific energy density on its different sides .

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Antipin gives an estimated calculation of the driving force of the V-shaped plate by using the Casimir formula, “with the most general and natural approximations known as PFA (Proximity Force Approximation), or PAA (Pairwise Additive Approximation), calculation method."

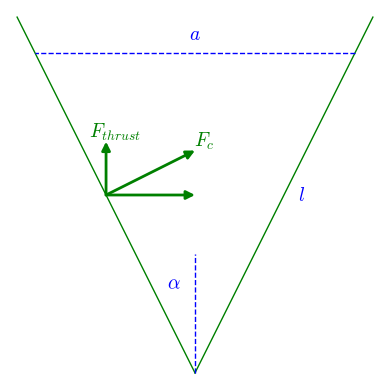
Casimir’s interaction energy is given by

.

Casimir’s force is

.

Driving force of the metal V-shaped plate is



[fig:Antipin’s V-shaped plate]

.

Let us make a substitution

.

The following substitution can be made:

.

Thus, the formula for the thrust for the V-shaped plate is obtained basing on the length of its sides.

Antipin indicates that the minimal value of is limited by the “cutoff level”, which is determined technologically:

* by the accuracy of plate manufacturing (their roughness, degree of flatness), as well as
* by the minimum wavelength of photons that can effectively reflect the substance from which the V-shaped plate is made (by value).

Investigating the dependence of the coefficient in the angle thrust formula, which depends on the half-angle , it can be seen that for a given length of the V-shape sides, it is more efficient to make the angle as small as possible. However, for the purposes of this work (studying the possibility of obtaining thrust by using nanocells), it is important to note that for a V-shaped plate with a right angle , the coefficient $\left({cos\, \alpha}\right)\big/\left({\left(sin\, \alpha\right)^4}\right) = 2\sqrt{2}$. Thus, by composing a honeycomb structure from many rectangular V-shaped plates, it can be shown that the thrust of the panel consisting of rectangular honeycombs is not zero.

Indeed, a rectangular honeycomb with a cell size of and with the same edge height equal to can be imagined as a combination of four V-shaped plates where a half-angle is equal to . The thrust of every V-shaped plate

directed along the bisector of each angle must be multiplied by $sin\left({\pi}/{4}\right)={\sqrt{2}}\big/{2}$ and, when multiplied by 4, the thrust of such a honeycomb cell will be equal to

So, the formula for the specific thrust of cells obtained by using the PFA (Proximity Force Approximation) or PAA (Pairwise Additive Approximation) method

is to some extent similar to the formula for the magnitude of the two-dimensional Casimir effect on honeycombs presented in the first part of this work. At least the value of the exponent in the denominator is the same

It should be noted, that this formula is obtained without taking into account the finiteness of the cell edge height (i.e., in the approximation of the infinite edge height), in contrast to the PFA version of the formula for which the edge height is assumed to be equal to the cell width.

The approximate agreement of these formulas indicates that the effect should also be observed at finite height of the ribs, although the dependence of the effect on this height may be the object of further research.

Theoretically it is possible to achieve a greater value of thrust with a panel produced from V-shaped plates, but producing panels from honeycombs seems to be technologically simpler than from V-shaped plates.

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