Semi-parametric Identification of SVAR Models with Zero Lower Bound *

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First version: November 1, 2021
This version: November 1, 2021

Abstract: The US federal funds rate was frequently constrained at zero after the Great Recession, and Federal Reserve has turned to unconventional monetary policy tools. This paper uses a model of structural vector autoregression with Zero Lower Bound (SVAR-ZLB) to characterize the censored nominal interest rate and the effect of unconventional monetary policy. The existing literature relies on the assumption of zero short-run effect of unconventional monetary policy to identify this model, but this paper examines point identification of this model without the assumption on unconventional monetary policy. First, in the case of classic Gaussian shocks, I explain the known result of no point identification from the new perspective of likelihood. However, in the case of empirically relevant non-Gaussian shocks, this paper proposes a generic semi-parametric identification scheme to prove point identification, without relying on the parametric form of the shock distribution. An efficient Bayesian inference routine is designed to facilitate the model estimation in practice.

Keywords: Zero Lower Bound, structural vector autoregression, censored variable, shadow interest rate, independent component analysis, Heckman selection model.

1 Introduction

Low nominal interest rates can be viewed as an important feature of the US economy in the recent decades and among them the federal funds rate even got constrained at zero frequently

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after the Great Recession. The new feature of close-to-zero nominal interest rates raised a challenge for central banks, which can no longer easily implement conventional interest rate cuts and need to turn to untested unconventional monetary policy tools. The Zero Lower Bound (ZLB) has thus given rise to a strong research interest in the macroeconomics literature and a huge debate among researchers on the effect of unconventional monetary policy (Kuttner, 2018; Bernanke, 2020).

Instead of making many assumptions on DSGE models as in the macroeconomics literature, the econometrics literature has tried another more flexible way of modeling the macroeconomic data in the ZLB periods, by relying on widely used structural vector autoregression (SVAR) models. Although conventional SVAR models can only deal with unconstrained data, Mavroeidis (2021) first incorporated ZLB into SVAR models to formulate this SVAR-ZLB model, in order to characterize the censored nominal interest rate and the unconventional monetary policy. He presented how to use the ZLB as an additional identification channel to point-identify the model. However, this point identification strategy is based on an assumption that the short-run effect of unconventional monetary policy is zero, which might not be true in reality.[1] Furthermore, the assumption of Gaussian shocks is crucial in Mavroeidis (2021) identification strategy because of the model non-linearity, which limits the generalization to other shock distributions. It is still a question whether we can point-identify the SVAR-ZLB model in general and how we can do it without relying on the exact parametric form of the shock distribution.

This paper proposes a generic semi-parametric identification scheme for the SVAR-ZLB model under independent non-Gaussian shocks, proving point identification without relying on the parametric form of the shock distribution. First, this paper clearly explains the known result of no point identification with Gaussian shocks through the new lens of the likelihood function, and thus offer the intuition why there is an advantage for specifying non-Gaussian shocks while maintaining the independence across shocks. Second, this paper rigorously develops a semi-parametric identification scheme to point-identify the SVAR-ZLB model under non-Gaussian shocks, with the exact shock distribution being unknown. To identify the structural parameters, I adjust the technique from the existing literature of independence component analysis (ICA), because the usual ICA technique only deals with the linear case and cannot handle the non-linearity arising from the censoring. Moreover, this paper also designs an efficient Bayesian inference routine with a Gibbs sampler to facilitate the model

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[1]For the sake of generality and convenience, in the rest of this paper, the SVAR-ZLB model by default refers to the model setup without the zero-restrictions on the effect of unconventional monetary policy.

estimation in practice.

The censoring and the kink in the simultaneous equations, as two important features of this SVAR-ZLB model, make the model identification challenging, compared to conventional SVAR models. Although Mavroeidis (2021) figured out an identification strategy when shocks are Gaussian, his point identification argument relies on the assumption of zero short-run effect of unconventional monetary policy, and is also heavily connected to the parametric form of the Gaussian distribution as in parametric sample selection models. When relaxing that assumption on unconventional monetary policy, Mavroeidis claimed the lack of point identification by counting the number of parameters and the number of the first and the second moments. In this paper, I set up the SVAR-ZLB model with endogenously switched regimes as in Aruoba et al. (2021) and revisits the Gaussian setup from the new perspective of likelihood. After a generic likelihood evaluation formula is provided, I interpret the known result of no point identification through the lens of the joint Gaussian likelihood, and utilize the circular symmetry of the contour in the shock space to conclude that a rotation of the true SVAR-ZLB model will fit the data equally well. However, this is only a special case for Gaussian shocks, and specifying non-Gaussian shocks will break the circular contour and give a potential to point-identify the model.

The main contribution of this paper is to rigorously propose a generic semi-parametric point identification scheme for the SVAR-ZLB model in the case of independent non-Gaussian shocks, without even relying on the exact parametric form of the shock distribution. This paper thus proved that, under some mild regularity conditions, the model is point-identified once the Gaussian shocks are ruled out. The whole semi-parametric identification scheme is decomposed into three steps. First, this paper uses the non-Gaussianity in shocks and applies the technique from independent component analysis (ICA) to identify the impact matrix, namely the structural parameters that represent the short-run effects of shocks. The technique from ICA is adjusted through Hessian matrices of log densities to handle the truncated-support problem arising from the censoring in the model. Second, this paper reformulates the reduced form of the SVAR-ZLB model into a three-equation semi-parametric Heckman selection model (Heckit) to identify the reduced-form coefficients for the lagged variables in the context of the censoring. Third, this paper also points out the special link between the structural form and the reduced form of this SVAR-ZLB model across two endogenously switched regimes, which is crucial to identify the kink of the model, namely the effect of unconventional monetary policy.

Within this semi-parametric identification scheme, the big challenge is how to apply ICA

in the context of censoring. Although the SVAR-ICA literature shows how to use independent non-Gaussian shocks for model identification (Lanne et al., 2017; Gourieroux et al., 2017), researchers mostly use the Darmois-Skitovich theorem, which can only deal with the unconstrained linear case. However, for the SVAR-ZLB model, the censoring at ZLB brings in non-linearity and gives a truncated support in the shock space, which makes the shocks no longer independent conditional on being uncensored. This poses a big challenge, because the theoretical foundation of the SVAR-ICA literature, namely the Darmois-Skitovich theorem, breaks down in this context of censoring. Nevertheless, building on the idea of Lin (1998), this paper examines ICA through Hessian matrices of log densities and adjusts the ICA technique to deal with the truncated support in the shock space. With the aid of Hessian matrices, I can discuss ICA from the likelihood perspective and show that the censoring does not hinder the identification through non-Gaussianity. Furthermore, by examining ICA through Hessian matrices, this paper mathematically links the ICA technique to the technique from identification-through-heteroskedasticity, one commonly used identification strategy in the conventional SVAR literature.

This semi-parametric identification scheme sheds light on three aspects of how to estimate the SVAR-ZLB model in practice. First, people can use this semi-parametric identification scheme directly to estimate the SVAR-ZLB model. Without relying on the parametric form of the shock distribution, robust semi-parametric estimators can be derived from this semi-parametric identification scheme. Nevertheless, there is a cost for the unknown parametric form, because the semi-parametric estimator might not be efficient in practice and will require researchers to consider many practical choices. Second, if researchers turn out to know the parametric form of the shock distribution, they can simply implement maximum likelihood estimation, in which the point identification will be automatically guaranteed by this semi-parametric identification scheme, for any non-Gaussian shock distribution under the mild regularity conditions. Finally, researchers can also use a flexible class of distributions to approximate the unknown shock distribution and then implement maximum likelihood estimation in this parametric setting.

To facilitate the model estimation in practice, this paper designs an efficient Bayesian inference routine for researchers to use. First, the unknown shock distribution is approximated with a mixture of normal distributions, following the spirit of Sieve approach. Then, in my Bayesian inference framework, I use the data augmentation technique and the conjugate priors to propose a Gibbs sampler, which can make posterior draws on different parameters and augmented data in an iterative way and compute the Bayesian posterior densities efficiently.

In a simulation study with a bivariate model with non-Gaussian shocks, this paper finds that the parameters are precisely estimated, especially when the sample size is relatively large and the occurrence of ZLB is not rare. The result in the simulation study shows that we can easily estimate this model when it is identified through non-Gaussianity.[2]

LITERATURE REVIEW.   There has been a long-time debate in the macroeconomics literature about the effectiveness of unconventional monetary policies (Kuttner, 2018; Bernanke, 2020). Eggertsson & Woodford (2003) argued that quantitative easing will be ineffective if the expectation about the future conduct of policy is not changed. However, Debortoli et al. (2020) provided evidence that ZLB is irrelevant for the economy using their empirical study and their theoretical models, as long as the unconventional monetary policy is well tuned to follow a shadow rate rule. Sims & Wu (2019) also concluded in their four-equation New Keynesian model that engaging in quantitative easing significantly mitigates the costs of a binding ZLB. In addition, Gertler & Karadi (2011) figured out that unconventional monetary policy can offset disruptions of private financial intermediaries and bring net benefits when ZLB binds.

To flexibly model the ZLB data, this paper builds on Mavroeidis (2021) and Aruoba et al. (2021) to set up a SVAR-ZLB model which has two endogenously switched regimes. The ZLB constraint for nominal interest rates has inspired the econometrics literature to combine SVAR models with ZLB. One obvious weakness of conventional SVAR models is that variables are unconstrained and the zero nominal interest rate is treated as a regular unconstrained data point, which will lead to estimation bias as in standard Tobit models (Amemiya, 1984). Mavroeidis (2021) incorporates ZLB into SVAR models to characterize the censoring of the nominal interest rate and the different effects of conventional and unconventional monetary policies. In the SVAR-ZLB model, the zero nominal interest rate in fact implies a negative unobserved shadow interest rate, which represents the desired stance of central banks if there is no ZLB. Furthermore, the effect of the negative shadow interest rate is allowed to be different from that of the positive nominal interest rate, to show the distinction between unconventional and conventional monetary policy. Aruoba et al. (2021) derived this econometric tool from approximating New Keynesian models and further generalized the setup. Aruoba et al. (2021) also formalized two endogenously switched regimes in this SVAR-ZLB model, to characterize the different parameters for normal periods and ZLB

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[2]A future version of this paper will add in an empirical study to run Bayesian estimation of this model for the real data.

periods. The regime switching is endogenous because the regime is directly defined through the nominal interest rate.

In contrast to the SVAR-ZLB literature, the novelty of this paper is presenting how to identify the SVAR-ZLB model without the assumption on unconventional monetary policy and without the parametric form of the shock distribution. Mavroeidis (2021) used the ZLB as an additional identification channel under Gaussian shocks to point-identify all the structural parameters, based on the assumption that unconventional monetary policy has zero short-run effect. When relaxing this assumption on unconventional monetary policy, he claimed the lack of point identification, because the number of parameters he had is more than the number of the first and the second moments he could use.[3] It is still unclear how to understand this lack of point identification from the likelihood perspective and how to link it to the identification problem in conventional SVAR models. More importantly, it is an open question if we can identify this model without restrictions on unconventional monetary policy, and even without knowing the exact shock distribution.

One big challenge this paper solves is how to apply ICA in the context of the censoring. Comon (1994) formalized the idea of ICA to untangle the independent components in a linear combination. Lanne et al. (2017) first introduced ICA to SVAR models to rigorously identify the structural parameters. Sims (2020) depicted the star-shaped contour of independent t-distributions to explain the identification through non-Gaussian shocks. Many papers recently discussed their empirical findings with non-Gaussian shocks (Braun, 2021; Brunnermeier et al., 2021; Jarocinski, 2021). However, most of this SVAR-ICA literature (Lanne et al., 2017; Gourieroux et al., 2017) uses the Darmois-Skitovich theorem to prove identification, which can only work for the unconstrained linear case and cannot handle the non-linearity from the censoring. In particular, the censoring generates a truncated joint support of independent shocks and the shocks are no longer independent conditional on being uncensored. Unfortunately, the non-linearity from the censoring cannot even be solved using the techniques in the nonlinear ICA literature (Gunsilius & Schennach, 2021), because there is no one-to-one mapping from the censored observations to the shocks. Nevertheless, Lin (1998) provided another perspective to examine linear ICA, i.e. applying ICA through Hessian matrices of log densities, which can be adjusted to work for the truncated joint support in this paper.

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[3]Mavroeidis (2021) refers to this lack of point identification as set identification, but we need to be aware that the set is identified solely from the coherency conditions of the DGP and the identified set is usually quite wide.

Because of the censoring, my proposed semi-parametric identification scheme also exploits the semi-parametric estimation of Heckman selection models. Amemiya (1984) described how to specify different sample selection models, including the three-equation Heckman selection model I use in this paper. Once the sample selection model is specified, the semi-parametric estimation is needed. Powell (1984) gave a robust estimator, namely censored least absolute deviation (CLAD) estimator, for the censored regression, which I use to estimate the selection equation. In Newey et al. (1990), they summarized different semi-parametric estimation procedures for Heckman selection models, including estimation based on kernel estimator (Robinson, 1988) and estimation based on series approximation (Cosslett, 1984), in order to estimate the outcome equation. Chamberlain (1986) further gave the asymptotic efficiency bound on the semi-parametric estimation of Heckman selection models.

OUTLINE. The rest of this paper is organized in the following way. Section 2 describes the bivariate setup of the SVAR-ZLB model. Section 3 interprets the lack of point identification under Gaussian shocks from the likelihood perspective. Section 4 proposes the semi-parametric identification scheme for non-Gaussian shocks. Section 5 generalizes the identification argument to the multivariate setting. Section 6 gives a simulation study to support my identification argument. Section 7 demonstrates the empirical result using my SVAR-ZLB model.

2   Bivariate Model Setup

This section presents how to set up a SVAR-ZLB model to characterize ZLB of the nominal interest rate and the effect of unconventional monetary policy. A simple bivariate setting is used first in this section and Section 3 - Section 4 to simplify the illustration.[4] Section 2.1 motivates the two key features of a SVAR-ZLB model. Section 2.2 specifies the model in detail in the form of two different regimes.

2.1   Motivation for Censoring and Kink

I first use one motivation in the SVAR-ZLB literature to show the two key features we need to consider when involving ZLB into SVAR models. A simple linearized macroeconomic struc-

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[4]A generalized multivariate setting is discussed in Section 5.

ture indicates that ZLB will generate the censoring and the kink in simultaneous equations, which are the two key features to be modeled in the SVAR-ZLB setup.

Mavroeidis (2021) offers a good motivation for how to model SVAR with ZLB, which is briefly illustrated here. If we consider the nominal interest rate r_t, the shadow interest rates r_t^*, the inflation rate π_t and log-deviation of the long-term bond quantity $b_{L,t}$. A simple linearized macroeconomic structure has the following form

$$r_t^* = c_1 + \gamma_1 \pi_t + \sigma_1 \varepsilon_{1t} \tag{1}$$

$$\pi_t = c_2 + \gamma_2 r_t + \phi b_{L,t} + \sigma_2 \varepsilon_{2t} \tag{2}$$

$$b_{L,t} = \min\{\alpha r_t^*, 0\} \tag{3}$$

$$r_t = \max\{r_t^*, 0\} \tag{4}$$

where (1) - (4) represent the Taylor rule, the private sector equation, the quantitative easing and ZLB respectively.

After we plug (3) into (2), the private-sector equation will become

$$\pi_t = c_2 + \gamma_2 r_t + \gamma_2^* \min\{r_t^*, 0\} + \sigma_2 \varepsilon_{2t} \tag{5}$$

where we define $\gamma_2^* = \phi\alpha$. Note that there is a kink between γ_2 and γ_2^* in (5), which represent the two different effects of the interest rate when above or below zero, namely the effect of conventional monetary policy and the effect of unconventional monetary policy. Therefore, (1), (4) and (5) constitute a set of simultaneous equations with the censoring and the kink, to describe the macroeconomic variables that may potentially subject to the ZLB constraint.

2.2 Specify the SVAR-ZLB Model with Two Regimes

Now I formalize the setup of the SVAR-ZLB model which has the two above-mentioned features and specify the model in terms of two regimes to facilitate our discussion. In contrary to the exogenous regime switching in the SVAR literature (Sims & Zha, 2006), the two regimes in the SVAR-ZLB will switch endogenously based on the level of the nominal interest rate.

We first consider a simple bivariate SVAR-ZLB model for the observables $y_t = (y_{1t}, y_{2t})'$, where y_{1t} is nominal interest rate and y_{2t} is a private-sector variable (e.g. the inflation rate). In addition, y_{1t}^* denotes the shadow interest rate which is the latent interest rate not being

censored. When in normal periods ($y_{1t} > 0$) we specify

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} y_{1t}^* \\ y_{2t} \end{bmatrix} = Bx_t + \varepsilon_t \tag{6}$$

whereas in ZLB periods ($y_{1t} = 0$) we specify

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21}^* & A_{22} \end{bmatrix} \begin{bmatrix} y_{1t}^* \\ y_{2t} \end{bmatrix} = Bx_t + \varepsilon_t \tag{7}$$

and the censoring at ZLB always delivers[5]

$$y_{1t} = \max\{y_{1t}^*, 0\} \tag{8}$$

The lagged terms we use are the lagged observables, i.e. $x_t = (y_{t-1}', y_{t-2}', \cdots, y_{t-p}', 1)'$. The structural shocks $\varepsilon_t = (\varepsilon_{1t}, \varepsilon_{2t})'$ are exogenous and independent across time and across coordinates with their respective density f_i, i.e. $\varepsilon_{it} \overset{\text{iid}}{\sim} f_i, i = 1, 2$. In addition, the variance of the shocks are normalized to 1.

After specifying the model, we can now interpret the economic meaning in the SVAR-ZLB model. In (6) and (7), the first row represents the monetary policy equation and the second row represents the private-sector equation. Note that from (6) to (7), the matrix for the lagged terms B remains the same[6], and the only change among the parameters is the kink between A_{21} and A_{21}^*, which captures the change in the effect of monetary policy from the conventional one A_{21}/A_{22} to the unconventional one A_{21}^*/A_{22}. For the variables, both the lagged terms x_t and the exogenous shocks ε_t are unchanged from (6) to (7), and the only change happens in $(y_{1t}^*, y_{2t})'$.

To facilitate our discussion on the SVAR-ZLB model, I use the regime-switching idea in Aruoba et al. (2021) to formulate two regimes. I denote the regime indicator

$$s_t = \mathbf{1}\{y_{1t} > 0\} \tag{9}$$

where $s_t = 1$ means the standard regime (6) and $s_t = 0$ means the ZLB regime (7). Therefore,

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[5]Even if y_{1t} is censored at $b \neq 0$, we can shift down the time series of y_{1t} by b and accordingly adjust the constant term in the true SVAR model. Thus, assuming censorship at 0 makes the modeling simple and without loss of generality. See appendix for the alternative modeling with $b \neq 0$.

[6]The assumption that B is the same in the normal periods and the ZLB periods comes from the coherency condition.

the two impact matrices under these two regimes can be denoted as A and A^*, where

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \quad A^* = \begin{bmatrix} A_{11} & A_{12} \\ A_{21}^* & A_{22} \end{bmatrix} \tag{10}$$

and the regime-contingent impact matrix is written as $A(s_t)$, where[7]

$$A(s_t) = s_t \cdot A + (1 - s_t) \cdot A^* \tag{11}$$

With the idea of regime switching and regime-contingent parameters, we can combine the two regimes into one regime-contingent SVAR model. Now we rewrite (6) and (7) as

$$A(s_t) \begin{bmatrix} y_{1t}(s_t) \\ y_{2t}(s_t) \end{bmatrix} = Bx_t + \varepsilon_t \tag{12}$$

where s_t is the endogenously switched regime based on (9) and $y_t(s) = (y_{1t}(s), y_{2t}(s))'$ is defined as the latent outcome when fixing at regime s The shadow interest rate and the observed private-sector variable (y_{1t}^*, y_{2t}) will choose one of the two latent outcomes based on the the endogenously switched regime s_t,

$$(y_{1t}^*, y_{2t})' = (y_{1t}(s_t), y_{2t}(s_t))' \tag{13}$$

Thus, (8), (9), (12) and (13) constitute the full specification of my SVAR-ZLB model, in terms of the two endogenously switched regimes.

To make this data generating process crystally clear, we consider the following two cases. First, at time t, we suppose the nature starts with $s_t = 1$, and (12) yields $y_{1t}(1) > 0$, which implies $y_{1t} = y_{1t}^* > 0$ and $s_t = 1$ is the coherent regime. Second, if the nature again starts with $s_t = 1$ but gets $y_{1t}(1) \leq 0$ which implies $y_{1t} = 0$ and $y_{1t}^* \leq 0$, the nature will restart with the coherent regime $s_t = 0$ and again gets $y_{1t}(0) \leq 0$, which again yields $y_{1t} = 0$ and $y_{1t}^* \leq 0$ but the values of y_{1t}^* might have changed.[8]

Built on the state-contingent structural-form equations, we can derive the state-contingent

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[7] In the rest of this paper, I will follow this notation rule to denote the parameters specific to the standard regime as θ, the ones specific to the ZLB regime as θ^*, and use $\theta(s_t)$ to represent regime-contingency.

[8] With the coherency conditions, I will show that $y_{1t}(1) > 0$ implies $y_{1t}(0) > 0$ and vice versa. Thus, there exists one and only coherent regime in the data generating process.

reduced-form equations,

$$y_t(s_t) = A(s_t)^{-1}Bx_t + A(s_t)^{-1}\varepsilon_t = \beta(s_t)x_t + u_t(s_t) \tag{14}$$

where $\beta(s_t) = A(s_t)^{-1}B$ are the reduced-form parameters, $u_t(s_t) = G(s_t)\varepsilon_t$ are the reduced-form errors and $G(s_t) = A(s_t)^{-1}$. With the same notation rule, I denote $\beta = \beta(1)$, $\beta^* = \beta(0)$, $G = G(1)$ and $G^* = G(0)$ to simplify the illustration.

COHERENCY CONDITION ON DGP. This paper follows the SVAR-ZLB literature to impose the coherency condition on the DGP to guarantee the coherency of the regimes, i.e. one and only one regime is coherent at time t (Mavroeidis, 2021; Aruoba et al., 2021). The coherency condition consists of two requirements:

- CONTINUITY. There should be no jump in y_{2t} across the two regimes when $y_{1t}^* = 0$

- UNIQUENESS. One and only one of the two events, $y_{1t}(1) > 0$ and $y_{1t}(0) \leq 0$, will be true.

In my SVAR-ZLB setup, the continuity requirement is satisfied naturally because the only parameter change across the regimes is the kink between A_{21} and A_{21}^*. As for the uniqueness requirement, we simply need to force the determinants of the two impact matrices have the same sign, i.e. $|A|/|A^*| > 0$.[9] Thus, the coherency condition generates one domain restriction, $|A|/|A^*| > 0$, on the parameters.

3 Lack of Point Identification under Gaussian Shocks

This section interprets why we cannot achieve point identification of the SVAR-ZLB model under Gaussian shocks, from the likelihood perspective. Section 3.1 first provides the generic formulas to evaluate the likelihood of the SVAR-ZLB model for any shock distribution. Section 3.2 points out that the key reason of no point identification in the Gaussian case is the circular contour which makes the likelihood unchanged after a rotation of the SVAR-ZLB model. Section 3.3 illustrates the advantage of specifying more realistic non-Gaussian shocks for the purpose of point identification.

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[9]See Appendix for proof.

3.1 Likelihood Evaluation for Any Shock Distribution

For any shock distribution, there are two ways to generically evaluate the likelihood of the SVAR-ZLB model. One way is to directly use observables and evaluate in the y-space, and another way is to transform observables to shocks and evaluate in the ε-space. The first way is useful to compute the likelihood in practice, whereas the second way is important to understand the identification problem with Gaussian or non-Gaussian shocks.

We first discuss how to evaluate the likelihood directly in the y-space. For simplicity, we only focus how to compute the likelihood of $y_t|x_t$ in this subsection. We denote the joint distribution of the two shocks as $\mathbf{f}(\varepsilon_t) = f_1(\varepsilon_{1t}) \cdot f_2(\varepsilon_{2t})$. One the one hand, the uncensored observation y_t in the standard regime $s_t = 1$ gives

$$p(y_t, s_t = 1|x_t) = s_t \cdot |A| \cdot \mathbf{f}(Ay_t - Bx_t) \tag{15}$$

On the other hand, the censored observation $y_t = (0, y_{2t})'$ in the ZLB regime $s_t = 0$ implies a negative shadow interest rate y_{1t}^* and involves integral in the tail,

$$p(y_t, s_t = 0|x_t) = (1 - s_t) \cdot \int_{-\infty}^{0} |A^*| \cdot \mathbf{f}(A^*(y_{1t}^*, y_{2t})' - Bx_t) dy_{1t}^* \tag{16}$$

Therefore, the full likelihood of $y_t|x_t$ is comprised of (15) and (16),

$$p(y_t|x_t) = p(y_t, s_t = 1|x_t) + p(y_t, s_t = 0|x_t) \tag{17}$$

Besides the evaluation in the y-space, we can also evaluate the likelihood in the ε-space. If we start from the standard regime $s_t = 1$ and impute the shocks $\varepsilon_t = Ay_t - Bx_t$, the likelihood in the standard regime will be

$$p(y_t, s_t = 1|x_t) = s_t \cdot |A| \cdot \mathbf{f}(\varepsilon_t) \tag{18}$$

However, if we now turn to the ZLB regime $s_t = 0$, we can use the censored observation $y_t = (0, y_{2t})'$ to impute the pseudo shocks $\varepsilon_t^0 = A^* y_t - Bx_t$, and the true shocks will be subject to a linear constraint $y_{2t} = \beta_{2\cdot}^* x_t + G_{2\cdot}^* \varepsilon_t$. Then the likelihood in the ZLB regime will be

$$p(y_t, s_t = 0|x_t) = (1 - s_t) \cdot \int_{-\infty}^{\varepsilon_{1t}^0} f_1(\varepsilon_{1t}) f_2 \left(\frac{1}{G_{22}^*} (y_{2t} - \beta_{2\cdot}^* x_t - G_{21}^* \varepsilon_{1t}) \right) \frac{1}{G_{22}^*} d\varepsilon_{1t} \tag{19}$$

where the last $\frac{1}{G_{22}^*}$ term is the Jacobian term for the mixed distribution. Similarly, the full likelihood evaluated in the ε-space is comprised of (18) and (19), as written in (17).

The likelihood evaluation formulas in the y-space and in the ε-space are both useful in this paper. Although these two formulas are equivalent as shown in Lemma B.1, the evaluation in the y-space is easier to compute the likelihood in practice in Section 7, whereas the evaluation in the ε-space is more important for us to understand the identification problem with Gaussian or non-Gaussian shocks in Section 3.2 and Section 3.3.

3.2 Rotation of Likelihood Function for Gaussian Shocks

Mavroeidis[10]

When we take the ubiquitous assumption that the shocks are Gaussian, the SVAR-ZLB model will not be point-identified, because the circular contour of the Gaussian distribution enables us to preserve the likelihood value when we rotate this model, even in the case of censoring. A rotation of the SVAR-ZLB model means the rotation of everything in the ε-space when we evaluate the likelihood.

I first use two figures to revisit the likelihood evaluation in y-space and ε-space for the Gaussian shocks. First for the y-space, as shown in the left panel of Figure 1, $y_t(s_t)$ follows a joint Gaussian (the red ellipse) when $s_t = 1$ and follows another joint Gaussian (the blue ellipse) when $s_t = 0$. One observation for $s_t = 1$ is shown as the red point, and another observation for $s_t = 0$ is shown as the blue point. Due to the censoring, the observation for $s_t = 0$ always appears on the ZLB boundary line, which is colored in green. Moreover, the likelihood calculation for the censored observation involves the integral in the tail, which is illustrated by the blue arrow. Second we translate the uncensored and the censored observation, the ZLB boundary line and the integral line into the ε-space, which are depicted in the right panel of Figure 1 using the same color scheme. Note that the contour in the ε-space is circular, because of the joint standard Gaussian distribution. The green line represents the set of shocks that are marginally binding at ZLB and the blue arrow represent those possible shocks that can match the observed y_{2t} and imply a negative shadow interest rate.[11]

The circular contour for the Gaussian distribution in the ε-space sheds light on why

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[10] For the model setup as in Section 2, this assumption will even imply the long-run effect of unconventional monetary policy is also zero.

[11] For the detailed formulas of the ZLB boundary line and the integral line, see appendix.

LIKELIHOOD EVALUATION UNDER GAUSSIAN SHOCKS
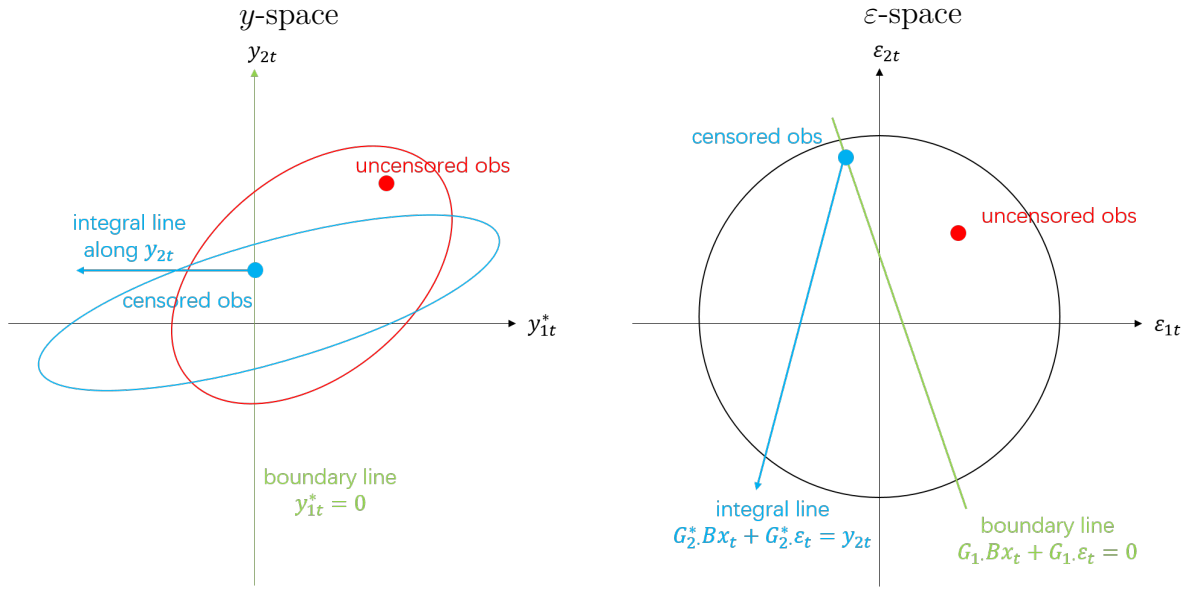
y-space — ε-space

Figure 1: The observation is depicted as a red point for the standard regime and as a blue point for the ZLB regime. The joint distribution is depicted as a red ellipse for the standard regime and as a blue ellipse for the ZLB regime. The green line is the ZLB boundary line, and the blue arrow is the integral line towards the tail.

we cannot achieve point identification of the model under Gaussian shocks. If we are able to rotate everything in the right panel of Figure 1, including the uncensored and the censored observation, the ZLB boundary line and the integral line, by the same angle, we will actually change all the structural parameters of the SVAR-ZLB model while preserving the likelihood value due to the circular symmetry of the Gaussian distribution. From this likelihood perspective, we can link the identification problem of SVAR-ZLB models to the well-known identification problem of conventional SVAR models, if shocks are assumed to be Gaussian.[12] However, in order to make A and A^* only differ in the bottom-left corner, the challenge is that we cannot simply rotate A^* but we still need to rotate the integral line. Fortunately, I show in Lemma B.2 that we can follow a specific decision rule to transform A^* to both keep the structural connection with A and get the integral line rotated by the same angle.

Theorem 1. *Under Gaussian shocks, all the structural parameters (A, A^*_{21}, B) in the SVAR-*

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[12]There is a small difference when we rotate the SVAR-ZLB model, that is some rotations will be ruled out because of violating the coherency condition. All the rotations that make the model satisfy the coherency condition yield the identified set of the structural parameters.

ZLB model will not be point-identified.

See proof in Appendix B.

Theorem 1 formally proves that there is no way to achieve point identification of the SVAR-ZLB model simply because of the rotation in the circular contour.[13] In contrast with Mavroeidis (2021), instead of assuming $A_{21}^* = 0$ to achieve point identification, I remove that dubious assumption in order to directly estimate the short-run effect of unconventional monetary policy and end up with no point identification. If the shocks are indeed Gaussian, the only way to point-identify this model is to make further assumptions. For example, one can follow Mavroeidis (2021) to make assumptions on unconventional monetary policy, or follow the SVAR literature to impose zero-restrictions on A or assume stochastic volatility of shocks. Moreover, the lack of point identification can also appear when shocks follow any elliptical distribution with a diagonal covariance matrix, where the contour is again circular and the same rotation logic can be applied.

3.3 Advantage of Specifying non-Gaussian shocks

If we get rid of the assumption that shocks are exactly Gaussian and take a more realistic stance on the non-Gaussian shocks, this will bring a big advantage on identifying the SVAR-ZLB model. The lack of point identification in Section 3.2 is a problem special to the case when shocks are exactly Gaussian, which is an unrealistic assumption compared to many findings in the applied research favoring non-Gaussian shocks. In addition, the SVAR-ICA literature provides solid foundation for using non-Gaussian shocks to identify conventional SVAR models and can be adjusted to help identify SVAR-ZLB models.

There are many findings in the SVAR literature that support the non-Gaussianity of shocks. Among them, Brunnermeier et al. (2021) used a regime-switching SVAR model to fit the US time series and found that the t-distribution fits the shocks better than the Gaussian distribution. After they identify their structural model through heteroskedasticity, their imputed shocks show a fat-tailed distribution. Besides, the big shocks in their model tend to appear in one coordinate at a time. This evidence makes the independent t-distribution a good specification for the shocks.

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[13]Mavroeidis (2021) proved the under-identification when relaxing the assumption of A_{21}^*, by counting the number of the available moments and the parameters, but it is unclear whether we have exhausted all the useful moments. This paper discusses the under-identification from the likelihood perspective and formally rules out the identification through any other potentially useful moments.

The SVAR-ICA literature has also pointed out how to use non-Gaussian shocks to identify conventional SVAR models. For independent non-Gaussian shocks, Lanne et al. (2017) relies on the Darmois-Skitovich Theorem to identify the SVAR model. Gourieroux et al. (2017) further proved that this identification strategy is fairly robust even when we slightly misspecify the shock distribution. Sims (2020) examined this identification strategy through the non-circular contour of the independent t-distribution, which shows a big advantage of specifying non-Gaussian shocks.

However, the SVAR-ICA literature does not know how to identify the model if we have a truncated support due to the censoring at ZLB, and this paper tries to extend this literature to deal with the truncated-support problem. The key challenge is that the formerly independent shocks are no longer independent after the censoring, and thus the conventional wisdom in the SVAR-ICA literature cannot be directly applied. Following the spirit of Sims (2020), I use Figure 2, which is analog to Figure 1 in Section 3.2, to illustrate the likelihood of the SVAR-ZLB model under the independent t-distribution. Note that the contour in the ε-space is now star-shaped and pointing out along each axis. Although we only have a truncated support on the right of the ZLB boundary line, that still intuitively seems to rule out the rotation problem in Section 3.2, since the star-shaped contour is not likely to preserve the likelihood after we rotate everything in the ε-space.[14] Nevertheless, we still need to have a rigorous identification scheme to prove that the truncated support with the non-circular contour indeed brings the identification of the SVAR-ZLB model for a generic non-Gaussian shock distribution, which is listed in Section 4.

4 Point Identification under non-Gaussian Shocks

This section proposes a semi-parametric identification scheme for SVAR-ZLB models under non-Gaussian shocks, without relying on the parametric form of the shock distribution. The identification scheme is decomposed into three parts. First, Section 4.1 discusses how to adjust the ICA argument through Hessian matrices to deal with the truncated-support problem in the model. Second, Section 4.2 reformulates the SVAR-ZLB model as a three-equation Heckman selection model. Third, Section 4.3 uses the special link between the structural form and the reduced form of the model.

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[14]Note that the likelihood will remain the same if we rotate exactly by a multiple of 90 degrees, which is just changing the label and the sign of shocks.
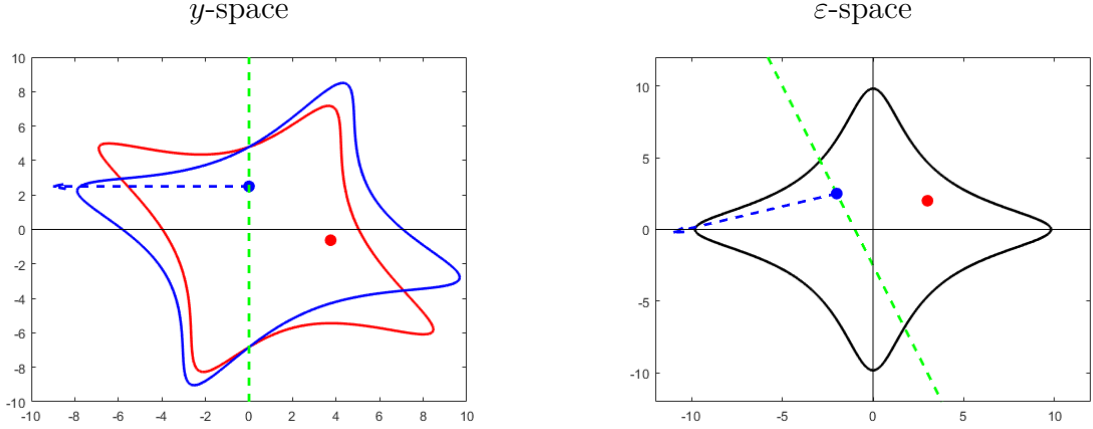
LIKELIHOOD EVALUATION UNDER T-DISTRIBUTED SHOCKS.

y-space ε-space

Figure 2: The observation is depicted as a red point for the standard regime and as a blue point for the ZLB regime. The joint distribution is depicted as a red ellipse for the standard regime and as a blue ellipse for the ZLB regime. The green line is the ZLB boundary line, and the blue arrow is the integral line towards the tail.

4.1 Apply Independent Component Analysis through Hessian Matrices

The first step is to identify the impact matrix A through non-Gaussian shocks. This paper adjusts the ICA argument through Hessian matrices and thus deal with the truncated support due to the censoring in the model. In addition, when we discuss ICA from the lens of Hessian matrices, the identification technique through non-Gaussian shocks becomes closely related to identification-through-heteroskedasticity.

Now we consider how to identify A when the shock distribution f_i is non-Gaussian with an unknown parametric form. The key challenge that prevents us from directly applying ICA is the truncated support in the ε-space, i.e. $\beta_1.x_t + G_1.\varepsilon_t \geq 0$, which makes $(\varepsilon_{1t}, \varepsilon_{2t})$ no longer independent. However, by borrowing ideas from Lin (1998), I achieve the identification of A using Hessian matrices of the log-likelihood function, at two interior points of this truncated support.

Theorem 2. *Suppose*

- $\log f \in C^2$

- $(\log f)''$ *is not constant (i.e. f is non-Gaussian)*

then A is identified using only the uncensored data given x_t, i.e. using $y_t | x_t, y_{1t} > 0$.

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Proof. The log-likelihood of (y_{1t}, y_{2t}) conditional on x_t and $y_{1t} > 0$ can be written as,

$$\log p(y_t | x_t, y_{1t} > 0) = \log |A| + \log \mathbf{f}(\varepsilon_t) - \log \Pr(y_{1t} > 0 | x_t) \tag{20}$$

where $p(y_t | x_t, y_{1t} > 0)$ is the conditional distribution in the interior of the truncated support, $\mathbf{f}(\varepsilon_t)$ is the unconditional shock distribution, $\Pr(y_{1t} > 0 | x_t)$ is the probability of being uncensored.

Then we take second-order derivatives with respect to y_t on both sides of (20) and get

$$H_y(y_t) = A' H_\varepsilon(\varepsilon_t) A \tag{21}$$

where $H_y(y_t) = \frac{\partial^2 \log p(y_t | x_t, y_{1t} > 0)}{\partial y_t \partial y_t'}$, and $H_\varepsilon(\varepsilon_t) = \frac{\partial^2 \log \mathbf{f}(\varepsilon_t)}{\partial \varepsilon_t \partial \varepsilon_t'}$.

The formerly independent shocks imply that $\log \mathbf{f}(\varepsilon_t) = \log f_1(\varepsilon_{1t}) + \log f_2(\varepsilon_{2t})$, and we have a diagonal matrix for $H_\varepsilon(\varepsilon_t)$:

$$H_\varepsilon(\varepsilon_t) = \begin{bmatrix} (\log f_1)''(\varepsilon_{1t}) & 0 \\ 0 & (\log f_2)''(\varepsilon_{2t}) \end{bmatrix} \tag{22}$$

Given the diagonal structure in $H_\varepsilon(\varepsilon_t)$, we can follow the technique similar to identification through heteroskedasticity. First, we pick two points in the interior of the truncated support, namely y_t and $\tilde{y}_t$ in the y-space, or equivalently ε and $\tilde{\varepsilon}_t$ in the ε-space. Then we compute the eigenvalue decomposition of the following term,

$$H_y(y_t)^{-1} H_y(\tilde{y}_t) = A^{-1} H_\varepsilon(\varepsilon_t)^{-1} H_\varepsilon(\tilde{\varepsilon}_t) A \tag{23}$$

where the left-hand side is the observed Hessian matrices, and the right-hand side is in the form of eigenvalue decomposition. The eigenvectors we collect from (23) gives the identification of A.

Note that we will not get a unique eigenvalue decomposition, if $H_\varepsilon(\varepsilon_t)$ is proportional to $H_\varepsilon(\tilde{\varepsilon}_t)$. However, if we get into this knife-edge situation, we can always get around it by slightly change one point we have picked. For example, we can keep the point $\tilde{\varepsilon}_t$ and then change the first coordinate of the point ε_t by $\Delta \varepsilon_{1t}$. Then we get the Hessian matrix at the new point $(\varepsilon_{1t} + \Delta \varepsilon_{1t}, \varepsilon_{2t})$ as

$$H_\varepsilon(\varepsilon_{1t} + \Delta \varepsilon_{1t}, \varepsilon_{2t}) = \begin{bmatrix} (\log f_1)''(\varepsilon_{1t} + \Delta \varepsilon_{1t}) & 0 \\ 0 & (\log f_2)''(\varepsilon_{2t}) \end{bmatrix} \tag{24}$$

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which will no longer be proportional to $H_\varepsilon(\tilde{\varepsilon}_t)$. □

Theorem 2 proves the semi-parametric identification of A using the property of the likelihood function in the model, but we do not rely on a specific parametric form of the shock distribution. Therefore, if the mild regularity conditions in Theorem 2 hold, the non-Gaussian shocks in the SVAR-ZLB model, which brings the non-circular contour with a truncated support, are enough to identify the impact matrix A in the SVAR-ZLB model. Since we only require two points in the truncated support to prove the identification of A, using all the points in the truncated support will even give over-identification of A.

4.2   Reformulate as the Heckman Selection Model

The next step is to identify the reduced-form parameters. To show how to estimate the reduced-form parameters β and β^* in the case of the censoring and the kink, I reformulate the SVAR-ZLB model as a three-equation Heckman selection model (Amemiya, 1984).

We can rewrite the regime-contingent SVAR model in the form of a three-equation Heckman selection model,

$$y_{1t}(1) = \beta_1.x_t + u_{1t}(1) \tag{25}$$

$$y_{2t}(1) = \beta_2.x_t + u_{2t}(1) \tag{26}$$

$$y_{2t}(0) = \beta_2^*.x_t + u_{2t}(0) \tag{27}$$

where $y_{1t}(1), y_{2t}(1), y_{2t}(0)$ are latent variables as in conventional Heckman selection models. In the selection equation (25), we only observe $y_{1t} = \max\{y_{1t}(1), 0\}$. In the observation equation (26), $y_{2t} = y_{2t}(1)$ is observed when $y_{1t}(1) > 0$. However, in contrary to conventional Heckman selection models, when $y_{1t}(1) < 0$, we can still observe $y_{2t} = y_{2t}(0)$ from another observation equation (27).

Theorem 3. *If the following regularity conditions*

- *shock distribution f_i is symmetric around 0*

- *both control functions $H_1(z) = E[u_{2t}(1)|u_{1t}(1) > z]$ and $H_0(z) = E[u_{2t}(0)|u_{1t}(1) \leq z]$ are not linear in z*

- *$E[x_t x_t']$ has full rank*

hold, then we can identify all the reduced-form parameters in the above-mentioned Heckman selection model (25) - (27), *namely* $\beta_1.$, $\beta_2.$ *and* $\beta_2^*.$.

Proof. To identify the reduced-form parameters $\beta_1.$, $\beta_2.$ and $\beta_2^*.$, we simply need to use the censored least absolute estimator (Powell, 1984) and the semi-parametric estimator for Heckman selection models (Newey et al., 1990):

- Given (25), we rely on the symmetry distribution of $u_{1t}(1)$ and the censored least absolute estimator to identify $\beta_1.$ through quantile regression,

$$\text{med}(y_{1t}|x_t) = \max\{\beta_1.x_t, 0\} \tag{28}$$

- Given (25) and (26), we can identify $\beta_2.$ through the semi-parametric estimator for conventional Heckman selection models,

$$E[y_{2t}|x_t, y_{1t} > 0] = E[y_{2t}(1)|x_t, y_{1t}(1) > 0]$$
$$= \beta_2.x_t + E[u_{2t}(1)|u_{1t}(1) > -\beta_1.x_t] = \beta_2.x_t + H_1(-\beta_1.x_t) \tag{29}$$

- Given (25) and (27), we can semi-parametrically identify $\beta_2^*.$ in the similar way,

$$E[y_{2t}|x_t, y_{1t} = 0] = E[y_{2t}(0)|x_t, y_{1t}(1) \leq 0]$$
$$= \beta_2^*.x_t + E[u_{2t}(0)|u_{1t}(1) \leq -\beta_1.x_t] = \beta_2^*.x_t + H_0(-\beta_1.x_t) \tag{30}$$

$\square$

Theorem 3 gives a routine for estimating the reduced-form parameters in the SVAR-ZLB model.[15] Because of the censoring, the OLS estimates of the reduced-form parameters will be biased, and an estimator involving the censoring and the sample selection is necessary to generate unbiased estimates. In this three-equation Heckman selection model, we simply run the censored absolute deviation estimator for the selection equation (25), and run the semi-parametric Heckit estimator twice, for the observation equation (26) and (27) separately. The estimators we use in this section again do not rely on the parametric form of the shock distribution.

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[15]Note that $\beta_1^*.$ does not appear in this three-equation Heckman selection model, and thus cannot be directly identified in this section. However, we can identify $\beta_1^*.$ in Section 4.3.

4.3 Use the Link between Structural Form and Reduced Form

The last step of this identification scheme is to use the special link between the structural form and the reduced form in this SVAR-ZLB model to identify the remaining parameters, including B, $\beta_{1.}^*$ and A_{21}^*.

After we semi-parametrically identify A, $\beta_{1.}$, $\beta_{2.}$ and $\beta_{2.}^*$ in Section 4.1 and Section 4.2, the remaining task is to identify B, $\beta_{1.}^*$ and A_{21}^*. First, it is straightforward now to identify B, because we have $\beta = (\beta_{1.}', \beta_{2.}')'$ and $B = A\beta$. Second, for identifying $\beta_{1.}^*$ and A_{21}^*, we can exploit the special link between the structural form and the reduced form of this SVAR-ZLB model. As mentioned in Section 2.2, although the reduced forms across the two regimes are largely different, the structural forms across the two regimes only differ between A_{21} and A_{21}^*, which leads to the identification of $\beta_{1.}^*$ and the over-identification of A_{21}^*.

Theorem 4. *Given A, B, β and $\beta_{2.}^*$, if*

- $A_{11} \neq 0$

- *there is at least one non-zero element in $\beta_{1.}$*

then $\beta_{1.}^*$ and A_{21}^* will be identified.

Proof. To identify $\beta_{1.}^*$ and A_{21}^*, we rely on the special link between the structural form and the reduced form in the ZLB regime,

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21}^* & A_{22} \end{bmatrix} \begin{bmatrix} \beta_{1.}^* \\ \beta_{2.}^* \end{bmatrix} = \begin{bmatrix} B_{1.} \\ B_{2.} \end{bmatrix} \tag{31}$$

Notice that in (31), the only unknown parameters at this stage are $\beta_{1.}^*$ and A_{21}^*. In the first place, we can directly identify $\beta_{1.}^*$ from the first row of (31),

$$\beta_{1.}^* = \frac{1}{A_{11}}(B_{1.} - A_{12}\beta_{2.}^*) \tag{32}$$

Then we can always find at least one non-zero element in $\beta_{1.}^*$, if $\beta_{1.}$ has at least one non-zero element, because they are proportional to each other as shown in (33)

$$\beta_{1.}^* = G_{1.}^* B = \frac{1}{|A^*|}(A_{22}, -A_{12}) \cdot B$$
$$= \frac{|A|}{|A^*|} \frac{1}{|A|}(A_{22}, -A_{12}) \cdot B = \frac{|A|}{|A^*|}G_{1.}B = \frac{|A|}{|A^*|}\beta_{1.} \tag{33}$$

Now without loss of generality, suppose there is a non-zero element in the first column of $\beta_{1.}^*$, i.e. $\beta_{11}^* \neq 0$. Then we can directly identify A_{21}^* from the second row of (31),

$$A_{21}^* = \frac{1}{\beta_{11}^*}(B_{21} - A_{22}\beta_{21}^*) \tag{34}$$

$\square$

4.4   Takeaway from the Semi-parametric Identification Scheme

After proving the identification of the SVAR-ZLB model using this semi-parametric identification scheme, I now discuss how it sheds light on the estimation of SVAR-ZLB models in practice. We will consider two different cases: when we know or when we do not know the parametric form of the non-Gaussian shock distribution.

If we do not know the parametric form of the non-Gaussian shock distribution, as we mentioned through Section 4.1 - Section 4.3, we can use this semi-parametric identification scheme to directly estimate the SVAR-ZLB model. In this case, we will need to nonparametrically estimate the Hessian matrices of the log-density of y_t and then semi-parametrically estimate the three-equation Heckman selection model. Both the bandwidth for estimators and the points to be selected for Hessian matrices will be a important practical choice that researchers need to make.[16] As in other nonparametric estimation, the cost of not knowing the parametric form might be the loss of efficiency in practice.

However, if we turn out to know the parametric form of the shock distribution, we can directly run Maximum Likelihood Estimation (MLE) for this SVAR-ZLB model. Specifying the non-Gaussian shock distribution, or at least flexibly approximating the non-Gaussian shock distribution, is a more practical option when we have a limited amount of data, as in most of the macroeconometrics literature. Although the likelihood function tend to behave differently when we specify different shock distributions, the point identification in the MLE is always guaranteed by this semi-parametric identification scheme, which works for a generic non-Gaussian distribution.

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[16]The form of eigenvalue distribution will still hold if we average over two sets of points throughout (21) - (23). One can choose two sets of points that are likely to have significantly different eigenvalues to efficiently estimate A in practice.

5   Generalize to Multivariate SVAR-ZLB Models

This section generalizes the bivariate model setup to the multivariate model setup and restate the identification arguments for the multivariate SVAR-ZLB model. In this section, I denote the n-dimensional observable vector as $y_t = (y_{1t}, y'_{2t})'$, where y_{1t} is the nominal interest rate and y_{2t} is a $(n-1)$-by-1 vector of private-sector variables, and I still use y^*_{1t} to represent the shadow interest rate.

The multivariate SVAR-ZLB model can similarly be written as a multivariate SVAR model with endogenously switched regimes, as in Section 2,

$$A(s_t) \begin{bmatrix} y_{1t}(s_t) \\ y_{2t}(s_t) \end{bmatrix} = Bx_t + \varepsilon_t \tag{35}$$

$$(y^*_{1t}, y'_{2t}) = (y_{1t}(s_t), y_{2t}(s_t)') \tag{36}$$

$$y_{1t} = \max\{y^*_{1t}, 0\} \tag{37}$$

$$s_t = \mathbf{1}\{y_{1t} > 0\} \tag{38}$$

where s_t is the regime indicator, $y_t(s) = (y_{1t}(s), y_{2t}(s)')'$ is the latent outcome when fixed at regime s, and the impact matrix across the two regimes have the following structure

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \quad A^* = \begin{bmatrix} A_{11} & A_{12} \\ A^*_{21} & A_{22} \end{bmatrix} \tag{39}$$

Note that in (39), A_{11} is a scalar, whereas A'_{12}, A_{21}, A^*_{21} are $(n-1)$-by-1 vectors and A_{22} is a $(n-1)$-by-$(n-1)$ matrix.

Now we can restate the identification arguments that we show in Section 3 and Section 4.

Theorem 5. *Under Gaussian shocks, all the structural parameters (A, A^*_{21}, B) in (35) - (39) will not be point-identified.*

See proof in Appendix B.

Theorem 6. *Under non-Gaussian shocks, if all the regularity conditions in Theorem 2 - Theorem 4 hold, all the structural parameters (A, A^*_{21}, B) in (35) - (39) will be point-identified.*

See proof in Appendix B.

All the identification arguments in Section 3 and Section 4 are still valid, once we adjust the notations for the multivariate setup. First, Theorem 5 shows that we can still interpret

the lack of point identification through the circular (or spherical) contour in the Gaussian likelihood. Second, Theorem 6 indicates that the generic semi-parametric identification scheme still works for the multivariate case.

6 Simulation Study

This section designs an efficient Bayesian estimation routine with a Gibbs sampler and runs a simulation study to check whether we can achieve point identification in practice. I specify a trivariate SVAR-ZLB model with t-distributed shocks as the true model, and run Bayesian estimation to see if the posterior densities of the structural parameters get close to the true values in a relatively large sample.

(To be added)

7 Empirical Results

8 Conclusion

This paper uses a SVAR-ZLB model to characterize the censored nominal interest rate and the effect of unconventional monetary policy, and considers the model identification when we do not assume the short-run effect of unconventional monetary policy is zero. I first interpret why we cannot achieve point identification of this SVAR-ZLB model if the shocks are Gaussian, from the likelihood perspective. The Gaussian shocks have a circular contour in the likelihood function, so any rotated version of the model will fit the censored and the uncensored data equally well. It turns out this lack of point identification is only a problem for the exact Gaussian distribution. If we remove the Gaussian assumption and specify the more realistic non-Gaussian distribution for the shocks, the non-Gaussian shocks will help point-identify the model. Then I propose a generic semi-parametric identification scheme, following independent component analysis and Heckman selection models, to prove the point identification. Using ICA to identify the structural parameters will encounter a problem from the truncated support in the shock space, and I apply ICA through Hessian matrices to deal with this problem. I run a simulation study as evidence to support my identification argument.

There are also interesting questions that need to be answered in the future research. First, when we specify the non-Gaussian shock distribution, we might ask if we still have

consistency in MLE when we slightly misspecify the non-Gaussian distribution. For example, we might know the shock is t-distributed but we might get a wrong estimate for the degree of freedom. Second, although we have the model being point-identified, we might wonder if the model has weak identification issues in finite samples. It is still unknown whether the model identification from ICA and semi-parametric Heckman selection models is efficient in finite samples.

Appendix A Coherency Condition

(To be added)

Appendix B Proofs

Lemma B.1. *For any continuous shock distribution f_i, the likelihood evaluation formulas in the y-space and in the ε-space are equivalent, i.e. (15) is equivalent to (18), (16) is equivalent to (19).*

Proof. Starting from the ε-space evaluation, we have

$$\int_{-\infty}^{\varepsilon_{1t}^0} f(\varepsilon_{1t}) f\left(\frac{y_{2t} - G_{2.}^* m - G_{21}^* \varepsilon_{1t}}{G_{22}^*}\right) \frac{1}{G_{22}^*} d\varepsilon_{1t}$$

$$= \int_{-\infty}^{\varepsilon_{1t}^0} f(\varepsilon_{1t}) f\left(\varepsilon_{2t}^0 + \frac{A_{21}^*}{A_{11}}(\varepsilon_{1t} - \varepsilon_{1t}^0)\right) \frac{|A^*|}{A_{11}} d\varepsilon_{1t}$$

$$= \int_{-\infty}^{0} f(A_{1.} y_t(0) - m_1) \cdot$$

$$f\left(A_{22} y_{2t}(0) - m_2 + \frac{A_{21}^*}{A_{11}}(A_{11} y_{1t}(0) + A_{12} y_{2t}(0) - m_1 - A_{12} y_{2t}(0) + m_1)\right) \frac{|A^*|}{A_{11}} A_{11} dy_{1t}(0)$$

$$= \int_{-\infty}^{0} |A^*| f(A_{1.} y_t(0) - m_1) f(A_{2.}^* y_t(0) - m_2) dy_{1t}(0)$$

$$= \int_{-\infty}^{0} |A^*| f(A_{1.} (y_{1t}^*, y_{2t})' - m_1) f(A_{2.}^* (y_{1t}^*, y_{2t})' - m_2) dy_{1t}^*$$

$\square$

Lemma B.2. *Given the true value of A, A_{21}^*, and a rotation matrix R, there exists a unique value $\tilde{A}_{21}^*$ such that $\tilde{G}_{2.}^* = G_{2.}^* R'$.*

Proof. In conventional SVAR models, we realized that rotation of the A matrix will not change the likelihood under Gaussian shocks. In this SVAR model with ZLB, we also consider the case when we rotate the true A and B matrix by θ to get a new set of parameter values, namely $\tilde{A} = R \cdot A$ and $\tilde{B} = R \cdot B$. Since the level curve of joint standard normal is a circle. If we can rotate everything in the shock space by θ, we will preserve the likelihood. First the boundary line will rotate by θ, because the boundary line is now $\tilde{G}_{1.} \tilde{m} + \tilde{G}_{1.} \tilde{\varepsilon}_t = 0$. Second, the uncensored observation will rotate by θ, i.e. $\tilde{\varepsilon}_t = R \cdot \varepsilon_t$. Third, the censored observation will also rotate, because they can be treated as the marginal cases in the standard regime.

Finally, I will prove that we can rotate the integral line by θ. Specifically, we need to find a new value $\tilde{A}_{21}^*$ such that the second row of G^* gets rotated by θ, namely $\tilde{G}_{2.}^* = G_{2.}^* R'$. It is noteworthy that after we find a new value $\tilde{A}_{21}^*$, we still have

$$\tilde{A} = R \cdot A = \begin{bmatrix} \tilde{A}_{11} & \tilde{A}_{12} \\ \tilde{A}_{21} & \tilde{A}_{22} \end{bmatrix}, \quad \tilde{A}^* = \begin{bmatrix} \tilde{A}_{11} & \tilde{A}_{12} \\ \tilde{A}_{21}^* & \tilde{A}_{22} \end{bmatrix}$$

and $\tilde{G}^* = \tilde{A}^{*-1}$.

First, let's prove it's a necessary condition:

$$\tilde{G}_{2.}^* = G_{2.}^* R'$$

$$\Rightarrow \begin{bmatrix} \tilde{G}_{21}^* & \tilde{G}_{22}^* \end{bmatrix} = \begin{bmatrix} G_{21}^* & G_{22}^* \end{bmatrix} \cdot R'$$

$$\Rightarrow \frac{1}{|\tilde{A}^*|} \begin{bmatrix} -\tilde{A}_{21}^* & \tilde{A}_{11} \end{bmatrix} = \frac{1}{|A^*|} \begin{bmatrix} -A_{21}^* & A_{11} \end{bmatrix} \cdot R'$$

$$\Rightarrow \frac{\tilde{A}_{11}}{|\tilde{A}^*|} = \frac{1}{|A^*|} \begin{bmatrix} -A_{21}^* & A_{11} \end{bmatrix} \cdot R'_{2.}$$

$$\Rightarrow \tilde{A}_{21}^* = \frac{e_1' R A e_1}{e_1' R A^* e_1} e_2' R A^* e_1$$

Then, let's prove it's a sufficient condition:

$$\tilde{G}_{21}^* = \frac{-\tilde{A}_2 1^*}{\tilde{A}_{11} \tilde{A}_{22} - \tilde{A}_{12} \tilde{A}_{21}^*}$$

$$= \frac{-\frac{e_1' R A e_1}{e_1' R A^* e_1} e_2' R A^* e_1}{\frac{e_1' R A e_1}{e_1' R A^* e_1} e_1' R A^* e_1 \tilde{A}_{22} - \tilde{A}_{12} \frac{e_1' R A e_1}{e_1' R A^* e_1} e_2' R A^* e_1}$$

$$= -\frac{1}{|A^*|} e_2' R A^* e_1$$

$$= e_2' (RA)^{-1} e_1$$

$$= e_2' G^* R' e_1$$

$$= \begin{bmatrix} G_{21}^* & G_{22}^* \end{bmatrix} \cdot R'_{1.}$$

$\square$

Proof for Theorem 1

Appendix C Gibbs Sampling Scheme

shock distribution: each shock ε_{it} follows a mixture of normals as $\sum_{k=1}^{K} \pi_{i,k} \mathcal{N}(\mu_{i,k}, \sigma_{i,k}^2)$

parameter: $\theta = \{A, A_{21}^*, B, \pi_{i,k}, \mu_{i,k}, \sigma_{i,k}^2\}$

data augmentation: mixture indicator $Z_{it} = 1, 2, ..., K$, shadow interest rate y_{1t}^*

prior: Gaussian on A, A_{21}^*, B, μ_{ik}, Dirichlet on π_{ik}, inverse-Gamma on σ_{ik}^2

full posterior with augmented data:

$$p(\theta, Z_{it}, y_{1t}^* | y_t) \propto p(A, A_{21}^*) p(B) p(\pi_{i,k}) p(\mu_{i,k}) p(\sigma_{i,k}^2) \cdot$$
$$p(Z_{it} | \pi_{ik}) p(y_t, y_{1t}^* | A, A_{21}^*, B, \pi_{i,k}, \mu_{i,k}, \sigma_{i,k}^2, Z_{it}) \qquad \text{(C.1)}$$

Gibbs sampling scheme:

- $p(Z_{it} | A, A_{21}^*, B, \pi_{i,k}, \mu_{i,k}, \sigma_{i,k}^2, y_{1t}^*, y_t)$: draw Z_{it} from the multinomial distribution

- $p(\pi_{i,k} | A, A_{21}^*, B, \mu_{i,k}, \sigma_{i,k}^2, Z_{it}, y_{1t}^*, y_t)$: draw $\pi_{i,k}$ from the Dirichlet distribution

- $p(B, \mu_{i,k} | A, A_{21}^*, \pi_{i,k}, \sigma_{i,k}^2, Z_{it}, y_{1t}^*, y_t)$: draw $B, \mu_{i,k}$ from the Gaussian posterior in GLS

- $p(\sigma_{i,k}^2 | A, A_{21}^*, B, \pi_{i,k}, \mu_{i,k}, Z_{it}, y_{1t}^*, y_t)$: draw $\sigma_{i,k}^2$ from the inverse-Gamma distribution

- $p(A, A_{21}^* | B, \pi_{i,k}, \mu_{i,k}, \sigma_{i,k}^2, Z_{it}, y_{1t}^*, y_t)$: draw A, A_{21}^* using the Metropolis algorithm

- $p(y_{1t}^* | A, A_{21}^*, B, \pi_{i,k}, \mu_{i,k}, \sigma_{i,k}^2, Z_{it}, y_t)$: draw y_{1t}^* from the truncated Gaussian distribution

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