Semiparametric Identification of SVAR Models with Zero Lower Bound *

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This version: May 1, 2025

Abstract: The US federal funds rate was frequently constrained to zero after the Great Recession, and the Federal Reserve has since turned to unconventional monetary policy tools. This paper uses a structural vector autoregression with a Zero Lower Bound (SVAR-ZLB) model to characterize the censored nominal interest rate and the effect of unconventional monetary policy. The existing literature relies on the assumption of Gaussian shocks and zero short-run effects of the unconventional monetary policy to identify this model, but this paper studies model identification without relying on these assumptions. In the case of empirically relevant non-Gaussian shocks, this paper proposes a generic semiparametric identification scheme to prove point identification, without relying on the parametric form of the shock distribution. The key problem of model identification that this paper solves is to deal semiparametrically with the non-linearity arising from censoring of the nominal interest rate at zero. An efficient Bayesian inference routine is designed to facilitate model estimation in practice. The empirical results suggest that the unconventional monetary policy has a small and transitory effect.

Keywords: Zero Lower Bound, structural vector autoregression, censored variable, shadow interest rate, independent component analysis, Heckman selection model, mixture of normals, Gibbs sampler.

*Email: `lidake930516@gmail.com`. I received helpful comments from Chris Sims, Mikkel Plagborg-Møller, Mark Watson, Bo Honoré, Ulrich Müller and Michal Kolesár. I am responsible for the remaining errors.

1 Introduction

Low nominal interest rates have been viewed as an important feature of the US economy in recent decades and among them the federal funds rate has even become frequently constrained at zero after the Great Recession. The new feature of close-to-zero nominal interest rates has raised a challenge for central banks, which can no longer easily implement conventional interest rate cuts and have needed to turn to untested unconventional monetary policy tools. The Zero Lower Bound (ZLB) has thus given rise to increased research interest in the field of macroeconomics and has resulted in a huge debate on the effect of unconventional monetary policies among researchers (Kuttner, 2018; Bernanke, 2020).

An alternative to imposing many assumptions of fully-specified dynamic stochastic general equilibrium (DSGE) models is to adapt the model-robust structural vector autoregression (SVAR) framework to the ZLB setting to model macroeconomic data in the ZLB periods. Although conventional SVAR models can only deal with unconstrained data, Mavroeidis (2021) first incorporated ZLB into SVAR models to formulate this SVAR-ZLB model, to characterize the censored nominal interest rate and the unconventional monetary policy. Relative to conventional SVAR models, the key novelty of the SVAR-ZLB model setup is the non-linearity arising from censoring of the nominal interest rate at zero from below. He also presented how to use the ZLB as an additional identification channel to point-identify the model. However, this point identification strategy assumes that the short-run effect of the unconventional monetary policy is zero, which might not be true in reality.[1] Furthermore, the assumption of Gaussian shocks is crucial in Mavroeidis (2021) identification strategy because of the model's non-linearity, which limits generalization to other shock distributions. It is still unclear whether the SVAR-ZLB model in general can be point-identified and how this can be done without relying on the exact parametric form of the shock distribution.

This paper proposes a generic semiparametric identification scheme for the SVAR-ZLB model under independent non-Gaussian shocks, providing point identification without relying on the parametric form of the shock distribution. First, this paper clearly explains the known result of no point identification with Gaussian shocks through the new lens of the likelihood function, thus demonstrating why there is an advantage in specifying non-Gaussian shocks while maintaining independence across shocks. Second, this paper develops a semiparametric

[1]For the sake of generality and convenience, in the rest of this paper, the SVAR-ZLB model by default refers to the model setup without zero-restrictions on the short-run effect of the unconventional monetary policy.

identification scheme to point-identify the SVAR-ZLB model under non-Gaussian shocks, with the exact shock distribution unknown. To identify the structural parameters, this work adjusts the technique from the existing literature on independent component analysis (ICA), because the usual ICA technique only deals with the linear case and cannot handle the non-linearity arising from censoring. Moreover, this paper also designs an efficient Bayesian inference routine with a Gibbs sampler to facilitate model estimation in practice. Following this inference routine, the empirical analysis shows that the effect of the unconventional monetary policy is relatively small and transitory.

Censoring and kink in simultaneous equations, as two important features of this SVAR-ZLB model, make model identification challenging, compared to conventional SVAR models. Although Mavroeidis (2021) figured out an identification strategy when shocks are Gaussian, his point identification argument relies on the assumption of zero short-run effect of the unconventional monetary policy, and is also heavily connected to the parametric form of the Gaussian distribution, as in parametric sample selection models. When relaxing this assumption on the unconventional monetary policy, Mavroeidis claimed the lack of point identification by counting the number of parameters and the first and second moments. In this paper, I set up the SVAR-ZLB model with endogenously switched regimes as in Aruoba et al. (2021b) and revisit the Gaussian setup from the perspective of likelihood. After a generic likelihood evaluation formula is provided, the known result of no point identification is interpreted through the lens of the joint Gaussian likelihood. Furthermore, this work utilizes the circular symmetry of the contour in the shock space to conclude that a rotation of the true SVAR-ZLB model will fit the data equally well. However, this is only a special case for Gaussian shocks, and specifying non-Gaussian shocks will break the circular contour and give a potential to point-identify the model.

The main contribution of this paper is a generic semiparametric point identification scheme for the SVAR-ZLB model in the case of independent non-Gaussian shocks, which does not need to rely on the exact parametric form of the shock distribution. This paper thus proves that, under some mild regularity conditions, the model can be point-identified once the Gaussian shocks are ruled out. The whole semiparametric identification scheme can be decomposed into three steps. First, this paper uses the non-Gaussianity in shocks and applies the ICA technique to identify the impact matrix, namely the structural parameters that represent the short-run effects of shocks. To handle the truncated-support problem arising from censoring in the model, I extend the conventional ICA technique by exploiting the structure of the Hessian matrix of the log density of the data, conditional on being away from the ZLB. Second, this paper reformulates the reduced form of the SVAR-ZLB model into a three-equation semiparametric Heckman selection model (Heckit) to identify the reduced-form coefficients for the lagged variables in the context of censoring. Third, this paper also points out the special link between the structural and reduced form of this SVAR-ZLB model across two endogenously switched regimes, which is crucial in identifying the kink in the model, namely the effect of the unconventional monetary policy.

Within this semiparametric identification scheme, the big challenge is how to apply ICA in the context of censoring. Although the SVAR-ICA literature has shown how to use independent non-Gaussian shocks for model identification (Lanne et al., 2017; Gourieroux et al., 2017), researchers have mostly used the Darmois-Skitovich theorem, which can only deal with the unconstrained linear case. However, for the SVAR-ZLB model, censoring at ZLB brings in non-linearity and gives a truncated support in the shock space, which makes the shocks dependent conditional on being uncensored. This poses a big challenge, because the theoretical foundation of the SVAR-ICA literature, namely the Darmois-Skitovich theorem, breaks down in this context of censoring. Nevertheless, building on the idea of Lin (1998), this paper examines ICA through Hessian matrices of log densities and adjusts the ICA technique to deal with the truncated support in the shock space. With the aid of Hessian matrices, ICA is discussed from the likelihood perspective to show that censoring does not hinder identification through non-Gaussianity. Furthermore, by examining ICA through Hessian matrices, this paper mathematically links the ICA technique to the identification-through-heteroskedasticity technique, which is a commonly used identification strategy in the conventional SVAR literature.

This semiparametric identification scheme suggests three different strategies for estimating the SVAR-ZLB model in practice. First, people can use this semiparametric identification scheme directly to estimate the SVAR-ZLB model. Without relying on the parametric form of the shock distribution, robust semiparametric estimators can be derived from this semiparametric identification scheme. Nevertheless, there is a cost for the unknown parametric form because the semiparametric estimator might not be efficient for a moderate sample size in practice and will require researchers to choose among several possible semiparametric estimators and associated tuning parameters. Second, if researchers know the parametric form of the shock distribution, they can simply implement maximum likelihood estimation, in which the point identification will be automatically guaranteed by this semiparametric identification scheme, for any non-Gaussian shock distribution under mild regularity conditions. Finally, researchers can also use a flexible parametric distribution, such as a mixture of normal distributions, to approximate the unknown shock distribution and then implement maximum likelihood estimation in this parametric setting.

To facilitate model estimation in practice, this paper designs an efficient Bayesian inference routine for researchers to use. First, the unknown shock distribution is approximated with a mixture of normal distributions, following the spirit of the Sieve approach. Then, the data augmentation technique and the conjugate priors in my Bayesian inference framework are used to propose a Gibbs sampler, which can make posterior draws on the parameters and the augmented data in an alternating way, and compute the Bayesian posterior densities efficiently. Moreover, this Bayesian inference routine allows researchers to take advantage of standard Bayesian tools such as imposing shrinkage on the autoregressive coefficients and expressing prior beliefs about sign or magnitude constraints on impulse responses. In a simulation study with a calibrated trivariate model with non-Gaussian shocks, this paper finds that, even under a relatively dispersed prior, this model can be precisely estimated if it is identified through non-Gaussianity, especially when the sample size is relatively large and the occurrence of ZLB is not rare.

This paper also runs an empirical analysis on three key US economic variables, using the efficient inference routine. By fitting the SVAR-ZLB model to the time series of the federal funds rate, the output growth and the inflation rate, the results demonstrate a big and persistent effect of conventional monetary policy but a small and transitory effect of the unconventional monetary policy, given the same magnitude of a monetary policy shock. For the estimated shock distribution, it tends to have a severe fat tail, which supports the non-Gaussianity assumption and helps with model identification. Furthermore, the shadow interest rate, which represents the desired nominal interest rate that central banks would like to select when ignoring the ZLB, is also imputed through the model and reveals a small magnitude across the ZLB periods.

Literature Review. There has been a long-standing debate about the effectiveness of unconventional monetary policies in the macroeconomics literature (Kuttner, 2018; Bernanke, 2020). Eggertsson and Woodford (2003) argued that quantitative easing would be ineffective if the expectation about the future enforcement of policies is not changed. However, Debortoli et al. (2020) provided evidence that ZLB is irrelevant to the economy using their empirical study and their theoretical models, as long as the unconventional monetary policy is well tuned to follow a shadow rate rule. Sims and Wu (2019) also concluded in their four-equation New Keynesian model that engaging in quantitative easing significantly mitigates the costs of a binding ZLB. In addition, Gertler and Karadi (2011) determined that the unconventional monetary policy can offset disruptions in private financial intermediaries and bring net benefits when the ZLB binds.

To flexibly model the ZLB data, this paper builds on Mavroeidis (2021) and Aruoba et al. (2021b) to set up a SVAR-ZLB model that has two endogenously switched regimes. The ZLB constraint for nominal interest rates has inspired the econometrics literature to combine SVAR models with ZLB. One obvious weakness of conventional SVAR models is that variables are unconstrained, and the zero nominal interest rate is treated as a regular unconstrained data point, which will lead to estimation bias as in standard Tobit models (Amemiya, 1984). Mavroeidis (2021) incorporated ZLB into SVAR models to characterize the censoring of the nominal interest rate and the different effects of conventional and unconventional monetary policies. In the SVAR-ZLB model, the zero nominal interest rate in fact implies a negative unobserved shadow interest rate, which represents the desired stance of central banks if there is no ZLB. Furthermore, the effect of the negative shadow interest rate is allowed to be different from that of the positive nominal interest rate, to show the distinction between unconventional and conventional monetary policy. Aruoba et al. (2021a) derived this econometric tool from approximating New Keynesian models and further generalized the setup. Aruoba et al. (2021b) also formalized two endogenously switched regimes in this SVAR-ZLB model, to characterize the different parameters for normal and ZLB periods. The regime switching is endogenous because the regime is directly defined through the nominal interest rate.

In contrast to the SVAR-ZLB literature, the novelty of this paper is its presentation of how to identify the SVAR-ZLB model without the assumption of the unconventional monetary policy and the parametric form of the shock distribution. Mavroeidis (2021) used the ZLB as an additional identification channel under Gaussian shocks to point-identify all the structural parameters, based on the assumption that the unconventional monetary policy has zero short-run effect. When relaxing this assumption on unconventional monetary policy, he claimed the lack of point identification, because the number of parameters he had is more than the number of the first and the second moments he could use.[2] It is still unclear how this lack of point identification can be understood from the likelihood perspective and how to link it to the identification problem in conventional SVAR models. More importantly,

[2] Mavroeidis (2021) referred to this lack of point identification as set identification, but we need to be aware that the set is identified solely from the coherency conditions of the DGP and the identified set is usually quite wide.

whether we can identify this model without restrictions on the unconventional monetary policy, even without knowing the exact shock distribution, is still a question.

The key technical challenge this paper solves is how to apply ICA in the context of censoring. Comon (1994) formalized the idea of ICA to untangle the independent components in a linear combination. Lanne et al. (2017) first introduced ICA to linear SVAR models to rigorously identify the structural parameters. To understand model identification through non-Gaussian shocks, Hyvarinen et al. (2001), Sims (2020) and Jarocinski (2021) depicted the star-shaped contour of independent non-Gaussian distributions and explained the intuition. Furthermore, Gourieroux et al. (2017) and Fiorentini and Sentana (2020) studied pseudo maximum likelihood estimators when the non-Gaussian shock distributions in SVAR models are misspecified. The linear SVAR-ICA approach has recently been used in several applied papers, with applications to the oil market (Braun, 2021) and the effects of monetary policies (Jarocinski, 2021). However, most of this SVAR-ICA literature (Lanne et al., 2017; Gourieroux et al., 2017) used the Darmois-Skitovich theorem to prove identification, which can only work for the unconstrained linear case and cannot handle the non-linearity from censoring. In particular, censoring generates a truncated joint support of independent shocks, and the shocks are no longer independent conditional on being uncensored. Unfortunately, the non-linearity from censoring cannot even be solved using the techniques in the nonlinear ICA literature (Gunsilius and Schennach, 2021), because there is no one-to-one mapping from the censored observations to the shocks. Nevertheless, Lin (1998) provided another perspective to examine linear ICA, i.e., applying ICA through Hessian matrices of log densities, which can be adjusted to work for the truncated joint support in this paper.

Because of the model's non-linearity from censoring, the proposed semiparametric identification scheme also exploits the semiparametric estimation of Heckman selection models, to handle the ZLB of the nominal interest rate and the regime-based selection of data. Amemiya (1984) described how to specify different sample selection models, including the three-equation Heckman selection model used in this paper. Once the sample selection model is specified, a semiparametric estimation method is needed. Powell (1984) gave a robust estimator, namely the censored least absolute deviation (CLAD) estimator, for the censored regression, which is used in this work to estimate the selection equation. Honore et al. (1997) developed a symmetric trimming technique to handle Heckman selection models when error terms are assumed to be symmetrically distributed. In Newey et al. (1990), they summarized different semiparametric estimation procedures using nonparametric regression techniques for Heckman selection models, including estimation based on kernel estimators (Robinson, 1988) and on series approximation (Cosslett, 1984), to estimate the outcome equation. Chamberlain (1986) further gave the asymptotic efficiency bound on the semiparametric estimation of Heckman selection models under different conditions.

Outline. The rest of this paper is organized as follows. Section 2 describes a bivariate setup of the SVAR-ZLB model. Section 3 interprets the lack of point identification under Gaussian shocks from the likelihood perspective. Section 4 proposes the semiparametric identification scheme for non-Gaussian shocks. Section 5 generalizes the identification argument to a multivariate setting. Section 6 discusses the efficient Bayesian inference routine in practice and gives a simulation study on the model estimation. Section 7 demonstrates the empirical results using the efficient inference routine.

2 Bivariate Model Setup

This section presents how to set up a SVAR-ZLB model to characterize ZLB of the nominal interest rate and the effect of unconventional monetary policy. A simple bivariate setting is used first in this section and Section 3 - Section 4 to simplify the illustration.[3] Section 2.1 motivates the two key features of a SVAR-ZLB model. Section 2.2 specifies the model in detail in the form of two different regimes.

2.1 Motivation for Censoring and Kink

I first use one motivation in the SVAR-ZLB literature to show the two key features we need to consider when involving ZLB into SVAR models. A simple linearized macroeconomic structure indicates that ZLB will generate the censoring and the kink in simultaneous equations, which are the two key features to be modeled in the SVAR-ZLB setup.

Mavroeidis (2021) offers a good motivation for how to model SVAR with ZLB, which is briefly illustrated here. If we consider the nominal interest rate r_t, the shadow interest rate r_t^*, the inflation rate π_t and log-deviation of the long-term bond quantity $b_{L,t}$. The shadow interest rate represents the desired nominal interest rate that central banks want to select when ignoring ZLB. A simple linearized macroeconomic structure has the following form

$$r_t^* = c_1 + \gamma_1 \pi_t + \sigma_1 \varepsilon_{1t} \tag{1}$$

[3] A generalized multivariate setting is discussed in Section 5.

$$\pi_t = c_2 + \gamma_2 r_t + \phi b_{L,t} + \sigma_2 \varepsilon_{2t} \tag{2}$$

$$b_{L,t} = \min\{\alpha r_t^*, 0\} \tag{3}$$

$$r_t = \max\{r_t^*, 0\} \tag{4}$$

where (1) - (4) represent the Taylor rule, the private sector equation, the quantitative easing and ZLB respectively.

After we plug (3) into (2), the private-sector equation will become

$$\pi_t = c_2 + \gamma_2 r_t + \gamma_2^* \min\{r_t^*, 0\} + \sigma_2 \varepsilon_{2t} \tag{5}$$

where we define $\gamma_2^* = \phi\alpha$. Note that there is a kink between γ_2 and γ_2^* in (5), which represent the two different effects of the interest rate when above or below zero, namely the effect of conventional monetary policy and the effect of unconventional monetary policy. Therefore, (1), (4) and (5) constitute a set of simultaneous equations with the censoring and the kink, to describe the macroeconomic variables that may potentially subject to the ZLB constraint.

2.2 Specify the SVAR-ZLB Model with Two Regimes

Now I formalize the setup of the SVAR-ZLB model which has the two above-mentioned features and specify the model in terms of two regimes to facilitate our discussion. In contrary to the exogenous regime switching in the SVAR literature (Sims and Zha, 2006), the two regimes in the SVAR-ZLB will switch endogenously based on the level of the nominal interest rate.

We first consider a simple bivariate SVAR-ZLB model and use the conventional notations in SVAR models, for the observables $y_t = (y_{1t}, y_{2t})'$, where y_{1t} is nominal interest rate and y_{2t} is a private-sector variable (e.g. the inflation rate). In addition, y_{1t}^* denotes the shadow interest rate which is the latent interest rate not being censored. To facilitate our specification on the SVAR-ZLB model, I follow the regime-switching idea in Aruoba et al. (2021b) to define two regimes using the regime indicator

$$s_t = \mathbf{1}\{y_{1t} > 0\} \tag{6}$$

where $s_t = 1$ means the standard regime (i.e. normal periods) and $s_t = 0$ means the ZLB regime (i.e. ZLB periods).

The data in two regimes are modeled differently. When in the standard regime (i.e.

$y_{1t} > 0, s_t = 1)$, we model the data as

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} y_{1t}^* \\ y_{2t} \end{bmatrix} = Bx_t + \varepsilon_t \tag{7}$$

whereas in ZLB periods (i.e. $y_{1t} = 0, s_t = 0$) we model the data as

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21}^* & A_{22} \end{bmatrix} \begin{bmatrix} y_{1t}^* \\ y_{2t} \end{bmatrix} = Bx_t + \varepsilon_t \tag{8}$$

The censoring at ZLB delivers the constrained nominal interest rate[4]

$$y_{1t} = \max\{y_{1t}^*, 0\} \tag{9}$$

For simplicity, we can denote $y_t^* = (y_{1t}^*, y_{2t})$ and further abbreviate the two impact matrices in (7) and (8) as

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \quad A^* = \begin{bmatrix} A_{11} & A_{12} \\ A_{21}^* & A_{22} \end{bmatrix} \tag{10}$$

A few things need to be clarified for the model specification (7) to (9). First, the lagged terms we use in (7) and (8) are only the lagged observables, i.e. $x_t = (y_{t-1}', y_{t-2}', \cdots, y_{t-p}', 1)'$.[5] Second, the structural shocks $\varepsilon_t = (\varepsilon_{1t}, \varepsilon_{2t})'$ are exogenous, not influenced by the censoring, and stay independent across time and across coordinates with their respective density being f_i, i.e. $\varepsilon_{it} \overset{\text{iid}}{\sim} f_i, i = 1, 2$. Thrid, as in the SVAR literature, this model is subject to a normalization issue and here we normalize the diagonal of A and A^* to be 1, i.e. $A_{11} = A_{22} = 1$.

Two minor issues regarding simultaneous equation models with two regimes as in (7) and (8) are continuity and coherency of the data generating process (DGP). This paper restricts the model to satisfy the continuity condition and the coherency condition, which are two minimum conditions on the DGP and commonly used in the SVAR-ZLB literature

[4]Even if y_{1t} is censored at $b \neq 0$, we can shift down the time series of y_{1t} by b and accordingly adjust the constant term in the true SVAR model. Thus, assuming censorship at 0 makes the modeling simple and without loss of generality. See appendix for the alternative modeling with $b \neq 0$.

[5]Including only the lagged observables and excluding the lagged shadow interest rate makes the model more tractable. Furthermore, Aruoba et al. (2021a) showed that their New Keynesian model with ZLB is well approximated by a SVAR-ZLB model with only the lagged observables included on the right-hand side.

(Mavroeidis, 2021; Aruoba et al., 2021b), in order to guarantee the equilibrium in the model to be continuous and unique. The definition of continuity and coherency in the SVAR-ZLB model is as follows:

- Continuity. There should be no jump in y_{2t} across the two regimes when $y^*_{1t} = 0$.
- Coherency. There exists a unique regime being coherent at time t. Either $y_{1t} > 0$ or $y_{1t} = 0$ can be true.

The continuity condition is satisfied naturally in my model setup, because the only change in parameters across the two regimes is the kink between A_{21} and A^*_{21}. For the coherency condition, I simply need to restrict the domain of A and A^* and force the determinants of these two impact matrices to be of the same sign, i.e.

$$\text{sign}(|A|) = \text{sign}(|A^*|) \tag{11}$$

A detailed discussion on these two minimum conditions can be found in Appendix A. With these two minimum conditions on the DGP, the model now will always yield one unique equilibrium, which is also continuous in terms of the shocks.

After specifying the model, we can now interpret the economic meanings in the SVAR-ZLB model. In (7) and (8), the first row represents the monetary policy equation and the second row represents the private-sector equation. For the parameters, the matrix for the lagged terms B remains the same across regimes, and the only change among the parameters across regimes is the kink between A_{21} and A^*_{21}, which captures the change in the short-run effect of monetary policy from the conventional one A_{21} to the unconventional one A^*_{21}. The long-run effects of monetary policy will also be different when we feed different short-run effects into the model. For the variables, both the lagged terms x_t and the exogenous shocks ε_t are unchanged across regimes, and the only change happens in the kink of $(y^*_{1t}, y_{2t})'$.

To facilitate our discussion of the model identification later in this paper, I also use the regime-contingent notations as in Aruoba et al. (2021b) to rewrite the model. We can combine the two regimes in (7) and (8) into one regime-contingent model

$$A(s_t)y_t(s_t) = Bx_t + \varepsilon_t \tag{12}$$

$$y^*_t = y_t(s_t) \tag{13}$$

where the regime-contingent impact matrix is

$$A(s_t) = s_t \cdot A + (1 - s_t) \cdot A^* \tag{14}$$

and $y_t(s) = (y_{1t}(s), y_{2t}(s))'$ is defined as the latent outcome generated using the parameters in regime s. Note that the DGP will choose y_t^* to be one of the two sets of latent outcomes $y_t(1)$ and $y_t(0)$, and which latent outcome gets chosen will recursively depend on the sign of the chosen y_{1t}^* through (9). Thus, in contrast to the regime-switching SVAR literature (Sims and Zha, 2006), the regime s_t here is endogenously switching based on the endogenous variable y_{1t}.[6]

After building the regime-contingent structural-form equations in (12), we can also derive the regime-contingent reduced-form equations,

$$y_t(s_t) = A(s_t)^{-1} B x_t + A(s_t)^{-1} \varepsilon_t = \beta(s_t) x_t + u_t(s_t) \tag{15}$$

where $\beta(s_t) = A(s_t)^{-1}B$ are the reduced-form lag coefficients and intercepts, $u_t(s_t) = G(s_t)\varepsilon_t$ are the reduced-form errors and $G(s_t) = A(s_t)^{-1}$ translating the structural shocks to the reduced-form errors. With the similar notation rule as in (14), I denote $\beta = \beta(1)$, $\beta^* = \beta(0)$, $G = G(1)$ and $G^* = G(0)$ to simplify the illustration of the proof later.

3 Lack of Point Identification under Gaussian Shocks

This section interprets why we cannot achieve point identification of the SVAR-ZLB model under Gaussian shocks, from the likelihood perspective. Section 3.1 first provides the generic formulas to evaluate the likelihood of the SVAR-ZLB model for any shock distribution. Section 3.2 points out that the key reason of no point identification in the Gaussian case is the circular contour which makes the likelihood unchanged after a rotation of the SVAR-ZLB model. Section 3.3 illustrates the advantage of specifying more realistic non-Gaussian shocks for the purpose of point identification.

[6] With the coherency condition, $y_{1t}(1) > 0$ implies $y_{1t}(0) > 0$ and vice versa. Thus, there exists one unique regime that can be coherently chosen in the DGP.

3.1 Likelihood Evaluation for Any Shock Distribution

For any shock distribution, there are two ways to generically evaluate the likelihood of the SVAR-ZLB model. One way is to directly use observables and evaluate in the y-space, and another way is to transform observables to shocks and evaluate in the ε-space. The first way is useful to compute the likelihood in practice, whereas the second way is important to understand the identification problem with Gaussian or non-Gaussian shocks.

We first discuss how to evaluate the likelihood directly in the y-space. For simplicity, we only focus how to compute the likelihood of $y_t|x_t$ in this subsection. We denote the joint distribution of the two shocks as $\mathbf{f}(\varepsilon_t) = f_1(\varepsilon_{1t}) \cdot f_2(\varepsilon_{2t})$. One the one hand, the uncensored observation y_t in the standard regime $s_t = 1$ gives

$$p(y_t, s_t = 1|x_t) = |A| \cdot \mathbf{f}(Ay_t - Bx_t) \tag{16}$$

On the other hand, the censored observation $y_t = (0, y_{2t})'$ in the ZLB regime $s_t = 0$ implies a negative shadow interest rate y_{1t}^* and involves integral in the tail,

$$p(y_t, s_t = 0|x_t) = \int_{-\infty}^{0} |A^*| \cdot \mathbf{f}(A^*(y_{1t}^*, y_{2t})' - Bx_t) dy_{1t}^* \tag{17}$$

Therefore, the full likelihood of $y_t|x_t$ is comprised of (16) and (17),

$$p(y_t|x_t) = s_t \cdot p(y_t, s_t = 1|x_t) + (1 - s_t) \cdot p(y_t, s_t = 0|x_t) \tag{18}$$

Besides the evaluation in the y-space, we can also evaluate the likelihood in the ε-space. For the standard regime $s_t = 1$, we can easily impute the shocks $\varepsilon_t = Ay_t - Bx_t$ and plug it into the shock distribution $\mathbf{f}(\varepsilon_t)$. For the ZLB regime, we will need to impute the pseudo shocks $\varepsilon_t^0 = A^*(0, y_{2t})' - Bx_t$, and integrate out the true shocks subject to a linear constraint. The details of likelihood evaluation in the ε-space can be found in Lemma B.1.

The likelihood evaluation formulas in the y-space and in the ε-space are both useful in this paper. Although these two formulas are equivalent as shown in Lemma B.1, the evaluation in the y-space is easier to compute the likelihood in practice in Section 6 and Section 7, whereas the evaluation in the ε-space is more important for us to understand the identification problem with Gaussian or non-Gaussian shocks in Section 3.2 and Section 3.3.

3.2 Rotation of Likelihood Function for Gaussian Shocks

When we take the common assumption that the shocks are Gaussian, the SVAR-ZLB model will not be point-identified, because the circular contour of the Gaussian distribution enables us to preserve the likelihood value when we rotate this model, even in the case of censoring. A rotation of the SVAR-ZLB model means the rotation of everything in the ε-space when we evaluate the likelihood.

The existing literature (Mavroeidis, 2021) relied on an additional assumption to achieve the point identification of this model under Gaussian shocks, i.e. the short-run effect of unconventional monetary policy is exactly zero.[7] However, this assumption might not hold in reality and the identification strategy based on this assumption can be fragile. When we relax this assumption and want to estimate the short-run effect of unconventional monetary policy, the literature concluded the lack of point identification (a.k.a. under-identification) in this model, by counting the number of first and second moments and unknown parameters. This paper tries to interpret this under-identification problem from the new perspective of likelihood.

I first use two figures to revisit the likelihood evaluation in y-space and ε-space for the Gaussian shocks. First for the y-space, as shown in the left panel of Figure 1, (y_{1t}^*, y_{2t}) follows a joint Gaussian distribution whose contour is colored in red when $s_t = 1$, and follows another joint Gaussian distribution whose contour is colored in blue when $s_t = 0$. One uncensored observation for $s_t = 1$ is shown as a red point, and another censored observation for $s_t = 0$ is shown as a blue point. Due to the censoring, the observation for $s_t = 0$ always appears on the ZLB boundary line (colored in green). Moreover, the likelihood calculation for the censored observation involves integrating out the tail along the integral line (denoted as a blue arrow). Second we can translate the uncensored and the censored observation, the ZLB boundary line and the integral line into the ε-space, which are depicted in the right panel of Figure 1 using the same legend scheme. Note that the contour in the ε-space is circular, because of the independent standard Gaussian distribution of the shocks.[8] The green line represents the set of shocks that are marginally binding at ZLB and the blue arrow represent those possible true shocks that can match the observed y_{2t} and imply a negative shadow

[7]For the model setup as in Section 2, this assumption will even imply the long-run effect of unconventional monetary policy is also zero.

[8]To simplify the discussion in this section, we temporarily change to a new normalization scheme where the variance of these Gaussian shocks are normalized to be 1, instead of forcing the diagonal of A and A^* to be 1. The details of the mapping between these two normalization scheme can be found in Lemma B.2.

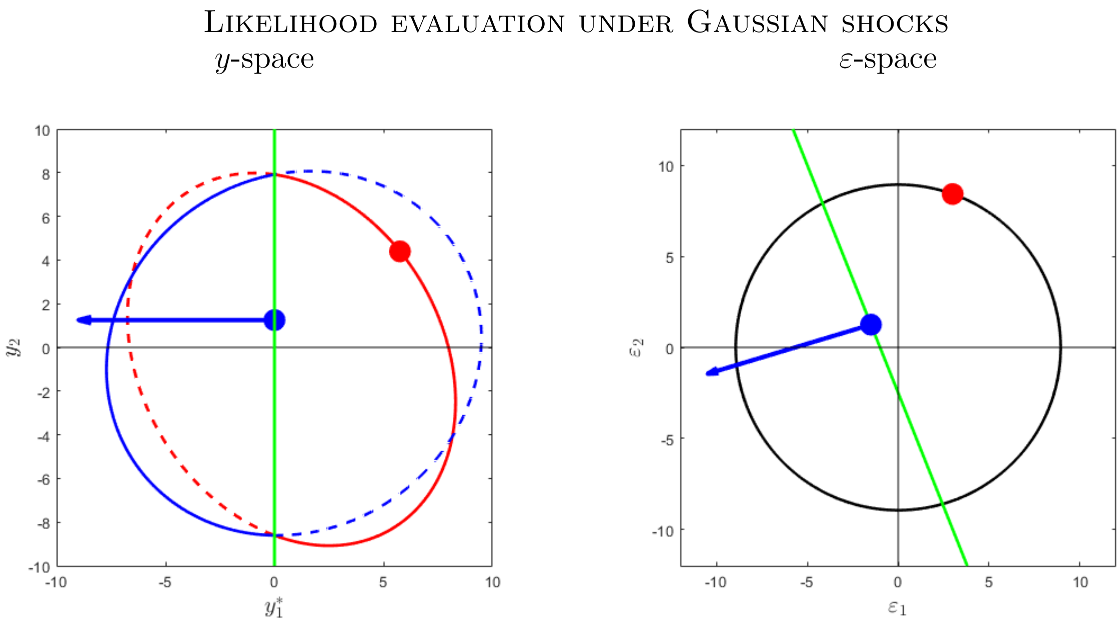

Figure 1: The observation is depicted as a red point for the standard regime and as a blue point for the ZLB regime. The joint distribution is depicted as a red ellipse for the standard regime and as a blue ellipse for the ZLB regime. The contour of the shocks, distributed as independent standard Gaussian, is plotted as a black circle. The green line is the ZLB boundary line, and the blue arrow is the integral line towards the tail.

interest rate.[9]

The circular contour for the Gaussian distribution in the ε-space sheds light on why we cannot achieve point identification of the model under Gaussian shocks. If we are able to rotate everything in the right panel of Figure 1, including the uncensored and the censored observation, the ZLB boundary line and the integral line, by the same angle, we will actually change all the structural parameters of the SVAR-ZLB model while preserving the likelihood value due to the circular symmetry of the Gaussian distribution. From this likelihood perspective, we can link the under-identification problem of SVAR-ZLB models to the well-known under-identification problem of conventional SVAR models, if shocks are assumed to be Gaussian.[10] However, in order to make A and A^* only differ in the bottom-left

[9]For the detailed formulas of the ZLB boundary line and the integral line in the right panel of Figure 1, see the remark of Lemma B.1.

[10]There is a small difference when we rotate the SVAR-ZLB model, that is some rotations will be ruled out because of violating the coherency condition (11). All the rotations that make the model satisfy the coherency condition yield the identified set of the structural parameters. We need to be cautious that the identified set can often be quite wide in this under-identification problem.

corner, the challenge is that we cannot simply rotate A^* but we still need to rotate the integral line. Fortunately, I show in Lemma B.3 that we can follow a specific mapping rule to transform A^* to both keep the structural connection with A and get the integral line rotated by the same angle.

Theorem 1. *Under Gaussian shocks, all the structural parameters (A, A^*_{21}, B) in the SVAR-ZLB model will not be point-identified.*

See proof in Appendix B.

Theorem 1 formally proves that there is no way to achieve point identification of the SVAR-ZLB model simply because of the rotation in the circular contour.[11] In contrast with Mavroeidis (2021), instead of assuming $A^*_{21} = 0$ to achieve point identification, I remove this assumption in order to directly estimate the short-run effect of unconventional monetary policy and end up with no point identification. If the shocks are indeed Gaussian, the only way to point-identify this model is to make further assumptions. For example, one can follow Mavroeidis (2021) to make assumptions on unconventional monetary policy, or follow the SVAR literature to impose zero-restrictions on A, or even assume stochastic volatility of shocks.[12] Moreover, the lack of point identification can also appear when shocks follow any elliptical distribution with a diagonal dispersion matrix, where the contour is again circular and the same rotation logic can be applied. But it is noteworthy that the Gaussian distribution is the only elliptical distribution that generate independent shocks and have a diagonal dispersion (covariance) matrix.

3.3 Advantage of Specifying non-Gaussian shocks

If we get rid of the assumption that shocks are exactly Gaussian and take a more realistic stance on the non-Gaussian shocks, this will bring a big advantage on identifying the SVAR-ZLB model. The lack of point identification in Section 3.2 is a problem special to the case when shocks are exactly Gaussian, which is an unrealistic assumption compared to many

[11] Mavroeidis (2021) proved the under-identification when relaxing the assumption of A^*_{21}, by counting the number of the available moments and the parameters, but it is unclear whether we have exhausted all the useful moments. This paper discusses the under-identification from the likelihood perspective and formally rules out the identification through any other potentially useful moments.

[12] Aruoba et al. (2021b) did not use the assumption $A^*_{21} = 0$ but assumed stochastic volatility of shocks. Hence their model is identified through heteroskedasticity, instead of through the help of ZLB. Moreover, in Section 4.1 of their paper, they also tried a model under homoskedastic shocks, which I believe to be under-identified due to Theorem 1 of my paper.

findings in the applied research favoring non-Gaussian shocks. In addition, the SVAR-ICA literature provides solid foundation for using non-Gaussian shocks to identify conventional SVAR models and can be adjusted to help identify SVAR-ZLB models.

There are many findings in the SVAR literature that support the non-Gaussianity of shocks. Among them, Brunnermeier et al. (2021) used a regime-switching SVAR model to fit the US time series and found that the t-distribution fits the shocks better than the Gaussian distribution. The big shocks in their model tend to appear in one coordinate at a time, which makes the independent t-distribution a good specification for the shocks. Many recent empirical papers generate empirical findings with their shocks and their impulse responses directly identified trhough non-Gaussianity (Braun, 2021; Jarocinski, 2021).

The SVAR-ICA literature has also pointed out how to use non-Gaussian shocks to identify SVAR in the linear unconstrained case. For independent non-Gaussian shocks, Lanne et al. (2017) relies on the Darmois-Skitovich Theorem to identify the SVAR model. Gourieroux et al. (2017) further proved that this identification strategy is fairly robust even when we slightly misspecify the shock distribution. Sims (2020) examined this identification strategy through the non-circular contour of the independent t-distribution, which shows a big advantage of specifying non-Gaussian shocks.

However, the SVAR-ICA literature does not know how to identify the model if we have non-linearity arising from the censoring at ZLB, and this paper tries to extend this literature to deal with the censoring problem. The key challenge is that the unconditionally independent shocks are no longer independent conditional on being not censored, due to the truncated support in the ε-space as in Figure 1, and thus the conventional wisdom in the SVAR-ICA literature cannot be directly applied. Following the spirit of Sims (2020), I use Figure 2, which is analog to Figure 1 in Section 3.2, to illustrate the likelihood of the SVAR-ZLB model under the independent t-distribution. If the shock is distributed as t-distribution with 3 degrees of freedom, the contour in the ε-space is now star-shaped and pointing out along each axis. Although we only have a truncated support on the right of the ZLB boundary line, that still intuitively seems to rule out the rotation problem in Section 3.2, since the star-shaped contour is not likely to preserve the likelihood after we rotate everything in the ε-space.[13] Nevertheless, we still need to have a rigorous identification scheme to prove that the truncated support with the non-circular contour indeed brings the identification of the SVAR-ZLB model for a generic non-Gaussian shock distribution, which is listed in Section 4.

[13]Note that the likelihood will remain the same if we rotate exactly by a multiple of 90 degrees, which is just changing the label and the sign of shocks.

Likelihood evaluation under t-distributed shocks.

y-space ε-space

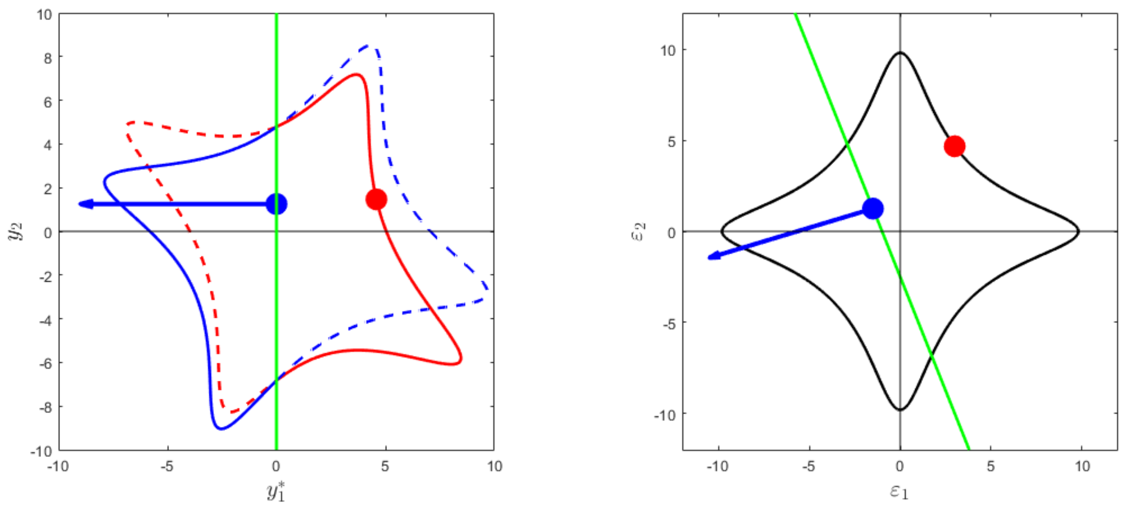

Figure 2: The observation is depicted as a red point for the standard regime and as a blue point for the ZLB regime. The joint distribution is depicted as a red ellipse for the standard regime and as a blue ellipse for the ZLB regime. The contour of the shocks, distributed as independent t_3 distribution, is plotted as a black curve. The green line is the ZLB boundary line, and the blue arrow is the integral line towards the tail.

4 Point Identification under non-Gaussian Shocks

This section proposes a semiparametric identification scheme for SVAR-ZLB models under non-Gaussian shocks, without relying on the parametric form of the shock distribution. The identification scheme is decomposed into three parts. First, Section 4.1 discusses how to adjust the ICA techniques through Hessian matrices of log densities to deal with the non-linearity problem arising from the censoring in the model. Second, Section 4.2 reformulates the SVAR-ZLB model as a three-equation Heckman selection model. Third, Section 4.3 uses the special link between the structural form and the reduced form of the model.

4.1 Apply Independent Component Analysis through Hessian Matrices

The first step is to identify the impact matrix A through non-Gaussian shocks. This paper adjusts the ICA argument through Hessian matrices and thus deal with the truncated support due to the censoring in the model. In addition, when we discuss ICA from the lens of Hessian matrices, the identification technique through non-Gaussian shocks becomes mathematically related to identification-through-heteroskedasticity.

Now we consider how to identify A when the shock distribution f_i is non-Gaussian with an unknown parametric form. The key challenge that prevents us from directly applying ICA is the truncated support in the ε-space, i.e. $\beta_{1\cdot}x_t + G_{1\cdot}\varepsilon_t \geq 0$, which makes $(\varepsilon_{1t}, \varepsilon_{2t})$ no longer independent. However, by borrowing ideas from Lin (1998), I achieve the identification of A using Hessian matrices of the log-likelihood function, at two interior points of this truncated support.

Theorem 2. *Suppose*

- f_i *has a full support on* $\mathbf{R}$
- $\log f_i \in C^2$
- $(\log f_i)''$ *is not constant (i.e.* f_i *is non-Gaussian)*

then A *is point-identified using only the uncensored data given* x_t*, i.e. using* $y_t|x_t, y_{1t} > 0$*.*

Proof. The log-likelihood of (y_{1t}, y_{2t}) conditional on x_t and $y_{1t} > 0$ can be written as,

$$\log p(y_t|x_t, y_{1t} > 0) = \log|A| + \log \mathbf{f}(\varepsilon_t) - \log \Pr(y_{1t} > 0|x_t) \tag{19}$$

where $p(y_t|x_t, y_{1t} > 0)$ is the conditional distribution in the interior of the truncated support, $\mathbf{f}(\varepsilon_t)$ is the unconditional shock distribution, $\Pr(y_{1t} > 0|x_t)$ is the probability of being uncensored.

Then we take second-order derivatives with respect to y_t on both sides of (19) and get

$$H_y(y_t) = A' H_\varepsilon(\varepsilon_t) A \tag{20}$$

where $H_y(y_t) = \frac{\partial^2 \log p(y_t|x_t, y_{1t}>0)}{\partial y_t \partial y_t'}$, and $H_\varepsilon(\varepsilon_t) = \frac{\partial^2 \log \mathbf{f}(\varepsilon_t)}{\partial \varepsilon_t \partial \varepsilon_t'}$.

The formerly independent shocks imply that $\log \mathbf{f}(\varepsilon_t) = \log f_1(\varepsilon_{1t}) + \log f_2(\varepsilon_{2t})$, and we have a diagonal matrix for $H_\varepsilon(\varepsilon_t)$:

$$H_\varepsilon(\varepsilon_t) = \begin{bmatrix} (\log f_1)''(\varepsilon_{1t}) & 0 \\ 0 & (\log f_2)''(\varepsilon_{2t}) \end{bmatrix} \tag{21}$$

Given the diagonal structure in $H_\varepsilon(\varepsilon_t)$, we can follow the technique similar to identification through heteroskedasticity. First, we pick two points in the interior of the truncated support, namely y_t and $\tilde{y}_t$ in the y-space, or equivalently ε and $\tilde{\varepsilon}_t$ in the ε-space. Then we compute the eigenvalue decomposition of the following term,

$$H_y(y_t)^{-1} H_y(\tilde{y}_t) = A^{-1} H_\varepsilon(\varepsilon_t)^{-1} H_\varepsilon(\tilde{\varepsilon}_t) A \tag{22}$$

where the left-hand side is the observed Hessian matrices, and the right-hand side is in the form of eigenvalue decomposition. The eigenvectors we collect from (22) gives the identification of A.

Note that we will not get a unique eigenvalue decomposition, if $H_\varepsilon(\varepsilon_t)$ is proportional to $H_\varepsilon(\tilde{\varepsilon}_t)$. However, if we get into this knife-edge situation, we can always get around it by slightly change one point we have picked. For example, we can keep the point $\tilde{\varepsilon}_t$ and then change the first coordinate of the point ε_t by $\Delta\varepsilon_{1t}$. Then we get the Hessian matrix at the new point $(\varepsilon_{1t} + \Delta\varepsilon_{1t}, \varepsilon_{2t})$ as

$$H_\varepsilon(\varepsilon_{1t} + \Delta\varepsilon_{1t}, \varepsilon_{2t}) = \begin{bmatrix} (\log f_1)''(\varepsilon_{1t} + \Delta\varepsilon_{1t}) & 0 \\ 0 & (\log f_2)''(\varepsilon_{2t}) \end{bmatrix} \tag{23}$$

which will no longer be proportional to $H_\varepsilon(\tilde{\varepsilon}_t)$. □

Theorem 2 proves the semiparametric identification of A using the property of the likelihood function in the model, but we do not rely on a specific parametric form of the shock distribution. Therefore, if the mild regularity conditions in Theorem 2 hold, the non-Gaussian shocks in the SVAR-ZLB model, which brings the non-circular contour with a truncated support, are enough to identify the impact matrix A in the SVAR-ZLB model. Since we only require two points in the truncated support to prove the identification of A, using all the points in the truncated support will even give over-identification of A.

4.2 Reformulate as the Heckman Selection Model

The next step is to identify the reduced-form parameters. To show how to estimate the reduced-form parameters β and β^* in the case of the censoring and the kink, I reformulate the SVAR-ZLB model as a three-equation Heckman selection model (Amemiya, 1984).

We can rewrite the regime-contingent SVAR model in the form of a three-equation Heckman selection model,

$$y_{1t}(1) = \beta_{1\cdot}x_t + u_{1t}(1) \tag{24}$$

$$y_{2t}(1) = \beta_{2\cdot}x_t + u_{2t}(1) \tag{25}$$

$$y_{2t}(0) = \beta^*_{2\cdot}x_t + u_{2t}(0) \tag{26}$$

where $y_{1t}(1), y_{2t}(1), y_{2t}(0)$ are latent variables as in conventional Heckman selection models. In the selection equation (24), we only observe $y_{1t} = \max\{y_{1t}(1), 0\}$. In the observation equation (25), $y_{2t} = y_{2t}(1)$ is observed when $y_{1t}(1) > 0$. However, in contrary to conventional Heckman selection models, when $y_{1t}(1) < 0$, we can still observe $y_{2t} = y_{2t}(0)$ from another observation equation (26).

Theorem 3. *If the following regularity conditions*

- *shock distribution f_i is symmetric around 0*
- *both control functions $H_1(z) = E[u_{2t}(1)|u_{1t}(1) > z]$ and $H_0(z) = E[u_{2t}(0)|u_{1t}(1) \leq z]$ are not linear in z*
- *$E[x_t x_t']$ has full rank*

hold, then we can identify all the reduced-form parameters in the above-mentioned Heckman selection model (24) - (26), namely $\beta_{1\cdot}$, $\beta_{2\cdot}$ and $\beta^*_{2\cdot}$.

Proof. To identify the reduced-form parameters $\beta_{1\cdot}$, $\beta_{2\cdot}$ and $\beta^*_{2\cdot}$, we simply need to use the censored least absolute estimator (Powell, 1984) and the semiparametric estimator for Heckman selection models (Newey et al., 1990):

- Given (24), we rely on the symmetry distribution of $u_{1t}(1)$ and the censored least absolute estimator to identify $\beta_{1\cdot}$ through quantile regression,

$$\text{med}(y_{1t}|x_t) = \max\{\beta_{1\cdot}x_t, 0\} \tag{27}$$

- Given (24) and (25), we can identify $\beta_{2\cdot}$ through the semiparametric estimator for conventional Heckman selection models,

$$\begin{aligned} &E[y_{2t}|x_t, y_{1t} > 0] = E[y_{2t}(1)|x_t, y_{1t}(1) > 0] \\ =&\beta_{2\cdot}x_t + E[u_{2t}(1)|u_{1t}(1) > -\beta_{1\cdot}x_t] = \beta_{2\cdot}x_t + H_1(-\beta_{1\cdot}x_t) \end{aligned} \tag{28}$$

- Given (24) and (26), we can semiparametrically identify $\beta^*_{2\cdot}$ in the similar way,

$$\begin{aligned} &E[y_{2t}|x_t, y_{1t} = 0] = E[y_{2t}(0)|x_t, y_{1t}(1) \leq 0] \\ =&\beta^*_{2\cdot}x_t + E[u_{2t}(0)|u_{1t}(1) \leq -\beta_{1\cdot}x_t] = \beta^*_{2\cdot}x_t + H_0(-\beta_{1\cdot}x_t) \end{aligned} \tag{29}$$

□

Theorem 3 gives a routine for estimating the reduced-form parameters in the SVAR-ZLB model.[14] Because of the censoring, the OLS estimates of the reduced-form parameters will be biased, and an estimator involving the censoring and the sample selection is necessary to generate unbiased estimates. In this three-equation Heckman selection model, we simply run the censored absolute deviation estimator for the selection equation (24), and run the semiparametric Heckit estimator twice, for the observation equation (25) and (26) separately. The estimators we use in this section again do not rely on the parametric form of the shock distribution.

4.3 Use the Link between Structural Form and Reduced Form

The last step of this identification scheme is to use the special link between the structural form and the reduced form in this SVAR-ZLB model to identify the remaining parameters, including B, $\beta^*_{1\cdot}$ and A^*_{21}.

[14]Note that $\beta^*_{1\cdot}$ does not appear in this three-equation Heckman selection model, and thus cannot be directly identified in this section. However, we can identify $\beta^*_{1\cdot}$ in Section 4.3.

After we semiparametrically identify A, $\beta_{1\cdot}$, $\beta_{2\cdot}$ and $\beta^*_{2\cdot}$ in Section 4.1 and Section 4.2, the remaining task is to identify B, $\beta^*_{1\cdot}$ and A^*_{21}. First, it is straightforward now to identify B, because we have $\beta = (\beta'_{1\cdot}, \beta'_{2\cdot})'$ and $B = A\beta$. Second, for identifying $\beta^*_{1\cdot}$ and A^*_{21}, we can exploit the special link between the structural form and the reduced form of this SVAR-ZLB model. As mentioned in Section 2.2, although the reduced forms across the two regimes are largely different, the structural forms across the two regimes only differ between A_{21} and A^*_{21}, which leads to the identification of $\beta^*_{1\cdot}$ and the over-identification of A^*_{21}.

Theorem 4. *Given A, B, β and $\beta^*_{2\cdot}$, if*

- $A_{11} \neq 0$
- *there is at least one non-zero element in $\beta_{1\cdot}$*

then $\beta^*_{1\cdot}$ and A^*_{21} will be identified.

Proof. To identify $\beta^*_{1\cdot}$ and A^*_{21}, we rely on the special link between the structural form and the reduced form in the ZLB regime,

$$\begin{bmatrix} A_{11} & A_{12} \\ A^*_{21} & A_{22} \end{bmatrix} \begin{bmatrix} \beta^*_{1\cdot} \\ \beta^*_{2\cdot} \end{bmatrix} = \begin{bmatrix} B_{1\cdot} \\ B_{2\cdot} \end{bmatrix} \tag{30}$$

Notice that in (30), the only unknown parameters at this stage are $\beta^*_{1\cdot}$ and A^*_{21}. In the first place, we can directly identify $\beta^*_{1\cdot}$ from the first row of (30),

$$\beta^*_{1\cdot} = \frac{1}{A_{11}}(B_{1\cdot} - A_{12}\beta^*_{2\cdot}) \tag{31}$$

Then we can always find at least one non-zero element in $\beta^*_{1\cdot}$, if $\beta_{1\cdot}$ has at least one non-zero element, because they are proportional to each other as shown in (32)

$$\begin{aligned} \beta^*_{1\cdot} = G^*_{1\cdot}B &= \frac{1}{|A^*|}(A_{22}, -A_{12}) \cdot B \\ =&\frac{|A|}{|A^*|}\frac{1}{|A|}(A_{22}, -A_{12}) \cdot B = \frac{|A|}{|A^*|}G_{1\cdot}B = \frac{|A|}{|A^*|}\beta_{1\cdot} \end{aligned} \tag{32}$$

Now without loss of generality, suppose there is a non-zero element in the first column of $\beta^*_{1\cdot}$, i.e. $\beta^*_{11} \neq 0$. Then we can directly identify A^*_{21} from the second row of (30),

$$A^*_{21} = \frac{1}{\beta^*_{11}}(B_{21} - A_{22}\beta^*_{21}) \tag{33}$$

□

4.4 Takeaway from the semiparametric Identification Scheme

After proving the identification of the SVAR-ZLB model using this semiparametric identification scheme, I now discuss how it sheds light on the estimation of SVAR-ZLB models in practice. We will consider two different cases: when we know or when we do not know the parametric form of the non-Gaussian shock distribution.

If we do not know the parametric form of the non-Gaussian shock distribution, as we mentioned through Section 4.1 - Section 4.3, we can use this semiparametric identification scheme to directly estimate the SVAR-ZLB model. In this case, we will need to nonparametrically estimate the Hessian matrices of the log-density of y_t and then semiparametrically estimate the three-equation Heckman selection model. Both the bandwidth for estimators and the points to be selected for Hessian matrices will be a important practical choice that researchers need to make.[15] As in other nonparametric estimation, the cost of not knowing the parametric form might be the loss of efficiency in practice.

However, if we turn out to know the parametric form of the shock distribution, we can directly run Maximum Likelihood Estimation (MLE) for this SVAR-ZLB model. Specifying the non-Gaussian shock distribution, or at least flexibly approximating the non-Gaussian shock distribution, is a more practical option when we have a limited amount of data, as in most of the macroeconometrics literature. Although the likelihood function tend to behave differently when we specify different shock distributions, the point identification in the MLE is always guaranteed by this semiparametric identification scheme, which works for a generic non-Gaussian distribution.

With the above-mentioned two cases combined together, a more practical way of flexibly estimating the SVAR-ZLB model is to approximate the shock distribution with a mixture of normals, in the spirit of Sieve estimation. When we look at real data, we might not be able to determine which particular non-Gaussian shock distribution fits the data best. Nevertheless, the mixture-of-normal distribution is a flexible parametric distribution that can universally approximate any smooth shock distribution, when the number of mixture components goes to infinity (Goodfellow et al., 2016). In the limit, when the number of mixture components is infinity, we are back to the semiparametric estimation, in a way similar to using infinite

[15] The form of eigenvalue distribution will still hold if we average over two sets of points throughout (20) - (22). One can choose two sets of points that are likely to have significantly different eigenvalues to efficiently estimate A in practice.

basis functions in Sieve regression. In practice, using a data-driven method to determine the finite number of mixture components, we can closely approximate the unknown shock distribution with this mixture-of-normal parametric distribution and implement the model estimation in the parametric setup. A practical and efficient Bayesian inference routine is discussed in Section 6.

5 Generalize to Multivariate SVAR-ZLB Models

This section generalizes the bivariate model setup to the multivariate model setup and restate the identification arguments for the multivariate SVAR-ZLB model. In this section, I denote the n-dimensional observable vector as $y_t = (y_{1t}, y_{2t}')'$, where y_{1t} is the nominal interest rate and y_{2t} is a $(n-1)$-by-1 vector of private-sector variables, and I still use y_{1t}^* to represent the shadow interest rate.

The multivariate SVAR-ZLB model can similarly be written as a multivariate SVAR model with endogenously switched regimes, as in Section 2,

$$A(s_t)y_t(s_t) = Bx_t + \varepsilon_t \tag{34}$$

$$y_t^* = y_t(s_t) \tag{35}$$

$$y_{1t} = \max\{y_{1t}^*, 0\} \tag{36}$$

$$s_t = \mathbf{1}\{y_{1t} > 0\} \tag{37}$$

where s_t is the regime indicator, $y_t(s) = (y_{1t}(s), y_{2t}(s)')'$ is the latent outcome when fixed at the parameters in regime s, and $y_t^* = (y_{1t}^*, y_{2t}')'$ represents the shadow interest rate coupled with the observed private-sector variables. The lagged terms $x_t = (y_{t-1}', y_{t-2}', \cdots, y_{t-p}', 1)'$. The shocks are independent across time and across coordinates, and each follow the distribution f_i. The impact matrix across the two regimes have the following structure

$$A(1) = A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \quad A(0) = A^* = \begin{bmatrix} A_{11} & A_{12} \\ A_{21}^* & A_{22} \end{bmatrix} \tag{38}$$

where the diagonal of A and A^* are normalized to be 1. We also impose the domain restriction on A and A^* as in (11) to guarantee the coherency. Note that in (38), A_{11} is a scalar, whereas A_{12}', A_{21}, A_{21}^* are $(n-1)$-by-1 vectors and A_{22} is a $(n-1)$-by-$(n-1)$ matrix.

Now we can restate the identification arguments that we show in Section 3 and Section 4.

Theorem 5. *Under Gaussian shocks, all the structural parameters, namely A, A^*_{21}, B in* (34) *-* (38)*, will not be point-identified.*

See proof in Appendix B.

Theorem 6. *Under non-Gaussian shocks, if all the regularity conditions in Theorem 2 - Theorem 4 hold, all the structural parameters, namely A, A^*_{21}, B in* (34) *-* (38)*, will be point-identified.*

See proof in Appendix B.

All the identification arguments in Section 3 and Section 4 are still valid, once we adjust the notations for the multivariate setup. First, Theorem 5 shows that we can still interpret the lack of point identification through the circular (or spherical) contour in the Gaussian likelihood. Second, Theorem 6 indicates that the generic semiparametric identification scheme still works for the multivariate case.

6 Bayesian Inference Routine and Simulation Study

This section proposes an efficient Bayesian inference routine for the SVAR-ZLB model and runs a simulation study to show whether the model is precisely estimated in practice. Section 6.1 discusses the advantage of Bayesian inference on this model when we use the data augmentation technique. Section 6.2 presents the setup and the result in the simulation study.

6.1 Bayesian Inference with a Gibbs Sampler

Using the data-augmentation and a Gibbs sampler in a Bayesian inference framework, we can easily deal with the censored nominal interest rate and handle the non-Gaussian shock distribution, which we approximate as a mixture of normals.

In practice, even if we do not know the parametric form of the shock distribution, we can still flexibly approximate that with a mixture-of-normal distribution, as mentioned in Section 4.4. For each shock ε_{it}, its distribution f_i is now approximated as

$$f_i \approx \sum_{k=1}^{\bar{K}} \pi_{ik} \mathcal{N}(\mu_{ik}, \sigma^2_{ik}) \tag{39}$$

where $k = 1, 2, \cdots, \bar{K}$ is the index of the mixture components. The mixture components are all independent across coordinates, to guarantee the independence across shocks. The number of components $\bar{K}$ needs to be specified in advance, but we can compare different specifications of $\bar{K}$ using Bayes factors after we run the Bayesian inference procedure. As a good starting point, this paper uses 4 mixture components to characterize the non-Gaussian shock distribution.[16]

When doing Bayesian inference, one big advantage we can rely on is the data augmentation. In the SVAR-ZLB model with a mixture-of-normal shock distribution, two latent variables will be extremely valuable for evaluating the likelihood if we can observe them in the data. The first one is Z_{it}, the component indicator in the mixture-of-normal distribution for the shock ε_{it}, which tell us which component is operative for generating that particular shock. Anther one is y_{1t}^*, the shadow interest rate, which is unconstrained interest rate before the censoring. In the Bayesian framework, the line between variables and parameters becomes vague. Thus, it is easy to treat Z_{it} and y_{1t}^* as the augmented data, and make posterior draws jointly on the augmented data and the model parameters in (7) and (8).

The lag order and the prior also need to be determined beforehand. First, the lag order can be determined by fitting a vanilla VAR model to the data and evaluating some well-known information criteria, such as Akaike information criterion (AIC). Second, the prior we choose is conjugate to the posterior for the structural parameters.[17] For example, the prior on B on μ_{ik} are independent Gaussian, the prior on $\{\pi_{ik}\}_{k=1}^{\bar{K}}$ is Dirichlet independent across i, and the prior on σ_{ik}^2 is independent inverse-Gamma. The details of the imposed prior for Bayesian inference can be found in C.2.

The likelihood can be evaluated with or without the augmented data. On the one hand, when we do not involve the augmented data, (16) and (17) can yield the likelihood, where a one-dimensional numerical integral is recommended to compute (17) in practice.[18] This likelihood evaluation can help us find the posterior peak of all the parameters, without including the large dimension of the augmented data. On the other hand, when we have the augmented data, the likelihood with the observed data and the augmented data given x_t will

[16]The comparison of Bayes factors with different specifications of $\bar{K}$ will be added in the future version of this paper.

[17]The prior is conjugate to all the parameters except A and A^* in the Gibbs sampler.

[18]For the mixture-of-normal shock distribution, (17) can be computed analytically. However, the computation complexity will increase exponentially when we have more mixture components. Thus, using a one-dimensional numerical integral is more generic and robust in computation.

be

$$p(y_t, y_{1t}^*, \{Z_{it}\}_{i=1}^n | x_t) = \left(\prod_{i=1}^n p(Z_{it}|x_t) \right) p(y_t, y_{1t}^* | x_t, \{Z_{it}\}_{i=1}^n) \tag{40}$$

where

$$\begin{aligned} p(Z_{it} = k|x_t) &= \pi_{ik} \\ p(y_t, y_{1t}^*, s_t = 1|x_t, \{Z_{it}\}_{i=1}^n) &= \mathbf{1}\{y_{1t}^* = y_{1t}\} p(y_t, s_t = 1|x_t, \{Z_{it}\}_{i=1}^n) \\ p(y_t, y_{1t}^*, s_t = 0|x_t, \{Z_{it}\}_{i=1}^n) &= \mathbf{1}\{y_{1t}^* < 0\} p((y_{1t}^*, y_{2t}), s_t = 0|x_t, \{Z_{it}\}_{i=1}^n) \end{aligned}$$

Note that the last two formulas are simply evaluating the likelihood of an unconstrained linear SVAR model with Gaussian shocks.

After we combine the prior with the likelihood, a Gibbs sampler can be applied. The Gibbs sampler utilizes the tractable posterior of a subset of parameters conditioning on other parameters and samples from the posterior alternatingly for the different sets of parameters. In particular, each time we condition on other parameters and make posterior draws on a subset of the parameters, in the order: (i) Z_{it}; (ii) π_{ik}; (iii) B and μ_{ik}; (iv) σ_{ik}^2; (v) A and A^*; (vi) y_{1t}^*. To compute the posterior density, we use the Gibbs sampler to generate 10,000 draws from the posterior. The details of the Gibbs sampler can be found in Appendix C.

6.2 Simulation Study

I specify a trivariate SVAR-ZLB model, calibrated towards the DGP of real data, with a mixture-of-normal shock distribution. I then run Bayesian inference based on Section 6.1 to see if a point-identified model will be precisely estimated in practice.

The simulation DGP is calibrated towards the real data in Section 7. In particular, I calibrate A, A^*, B to match the persistence of the real data and the simultaneous equation structure. I also use the calibrated mixture-of-normal distribution to mimic the non-Gaussian distribution in the data, where the number of mixture components is 4. In addition, the lag order of the simulation DGP is also calibrated to the lag structure of the real data, which is set to 4 and treated as known in the simulation. I then start the simulation at the sample mean of the real data and then simulate for 421 periods, which has exactly the same length of my real data. Since the nominal interest rate turns out to be relatively persistent, the simulated ZLB occurrence can be too high or too low compared to the real data. Thus, I tune the random seed in the simulation to pick one realization of the simulated path that

Quantiles of Posterior Density in Simulation

	5%	16%	50%	84%	95%	True
A_{21}	-0.17	-0.11	-0.00	0.12	0.18	-0.12
A_{31}	-0.17	-0.12	-0.04	0.04	0.09	0.01
A_{12}	-0.24	-0.18	-0.02	0.13	0.20	0.03
A_{32}	-0.02	0.01	0.10	0.19	0.29	0.02
A_{13}	-0.28	-0.24	-0.09	0.05	0.13	-0.17
A_{23}	-0.42	-0.29	-0.16	-0.05	0.01	-0.02
A^*_{21}	-0.52	-0.36	-0.15	-0.01	0.06	-0.27
A^*_{31}	-0.18	-0.11	0.03	0.18	0.30	-0.02

Table 1: The first five columns represent the quantiles of the posterior density of each parameter. The last column represents the true parameter value in the simulation. The diagonal of A and A^* is normalized to be 1.

match the fraction of ZLB periods in the real data, which is about 20% of the sample period. The quantiles of the posterior density for A and A^* are reported in Table 1.[19]

The credible intervals demonstrate that this model, which is pointed identified through non-Gaussian shocks, can be precisely estimated in practice. As shown in Table 1, the posterior density of each parameter is around the true parameter value. The 68% credible intervals (from 16%-quantile to 84%-quantile) and the 90% credible intervals (from 5%-quantile to 95%-quantile) all have the reasonable widths and still well capture the true values. With this realistic simulation setup, we show that this model can be precisely estimated, given the sample size is reasonably large and the occurrence of ZLB periods is not extremely rare.

7 Empirical Results

This section uses the proposed efficient Bayesian inference routine to estimate the SVAR-ZLB model using real data. By fitting the model to three key US economic variables, we find that the effect of unconventional monetary policy is small and short-lived. Moreover, the fat-tail of the shocks in the estimated model support the non-Gaussianity of shocks in reality. The implied shadow interest rate stays small in magnitude in the ZLB periods, except a sudden decline at the beginning of the Great Recession.

[19] The credible intervals in the simulation study for all the other parameters will be listed in an online supplement appendix in the future version.

This paper uses three monthly US time series to run the empirical analysis. The monthly dataset includes the federal fund rate, the output growth, and the inflation rate.[20] The sample period I choose is 1985:Jan - 2020:Jan, which is usually considered as a moderate period without too much variation in the shock variance.[21] The time series of these three US macroeconomic variables are plotted in Figure 3. The ZLB periods span from 2008:Dec to 2015:Dec.[22] The Bayesian inference setup is the same as in Section 6.2, where the lag order is set to be 4 based on AIC, the number of mixture components is set to be 4 (symmetric on both sides of zero), and the conjugate prior is imposed.

When we estimate the model using the efficient Bayesian inference routine, we find strong evidence for the non-Gaussianity of the shocks. Using the posterior modal estimates of the parameters, we can characterize the shock distribution with a QQ plot as in Figure 4. Each shock exhibits a fat-tail in the distribution, which would help the model identification as mentioned in Section 3.3.[23] Moreover, the estimated mixture-of-normal shock distribution tends to have a high probability on one component with a small variance, while having a low probability on other components with a big variance. The normal mean of these mixture components all tend to be close to zero in the posterior.

The posterior quantiles for all the parameters are reported in Table 2.[24] We notice that the off-diagonal elements in A and A^* tend to be intermediate, which coincides with the intermediate correlation across the three variables. The short-run effects of both conventional and unconventional monetary policy tend to be small, and not significantly from zero. The monetary equation shows a persistent lag structure in B, which explains the persistence in the federal funds rate. All the shocks have at least one mixture component with a big variance, which contributes to the fat tail in the shock distribution and helps the model

[20]The output growth is defined as log difference of industrial production, and the inflation rate is defined as the log difference of PCE price level. The data source is Fred. See the data labels of "FEDFUNDS", "INDPRO" and "PCEPI" for details.

[21]Starting from the moderate period in the empirical study is usually a challenge for identifying the monetary policy shocks, since the results are found to be different from the ones using the data back to 1970s (Ramey, 2016). However, this paper tries to rely on the non-Gaussianity of shocks to identify the monetary policy shocks in the moderate period.

[22]In the ZLB periods, the effective federal funds rate is clearly constrained at zero, but not exactly zero. To fit the sharp censoring at zero in the model, this paper around all the small values from 0 to 25 basis points to zero. Researchers can try to model this small movements from 0 to 25 basis points in future work.

[23]Specifying a mixture-of-normal distribution does not necessarily imply a fat-tail. The fat tail is an estimate through the data.

[24]The credible intervals in the empirical analysis for all the other parameters will be listed in an online supplement appendix in the future version.

Time Series of Macroeconomic Variables

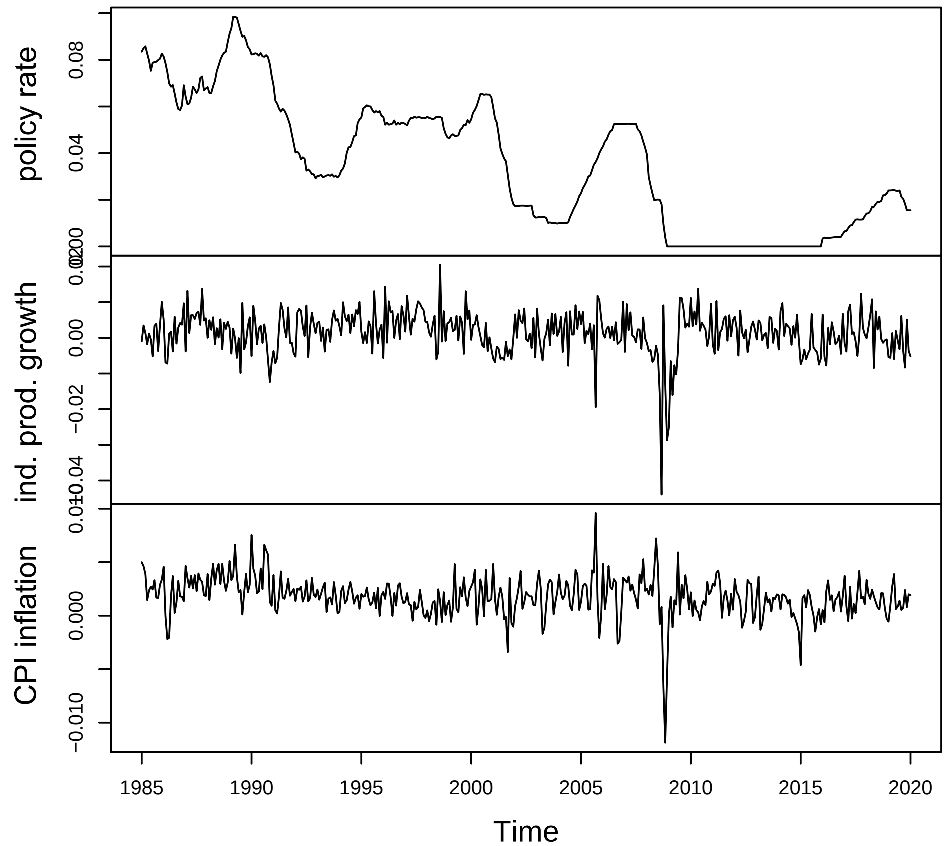

Figure 3: The sample period is 1985:Jan to 2020:Jan. For the variables, "policy rate" means the federal funds rate, "ind. prod. growth" means the output growth measured via industrial production, "CPI inflation" means the inflation rate measured via the PCE price level. The unit is percentage points.

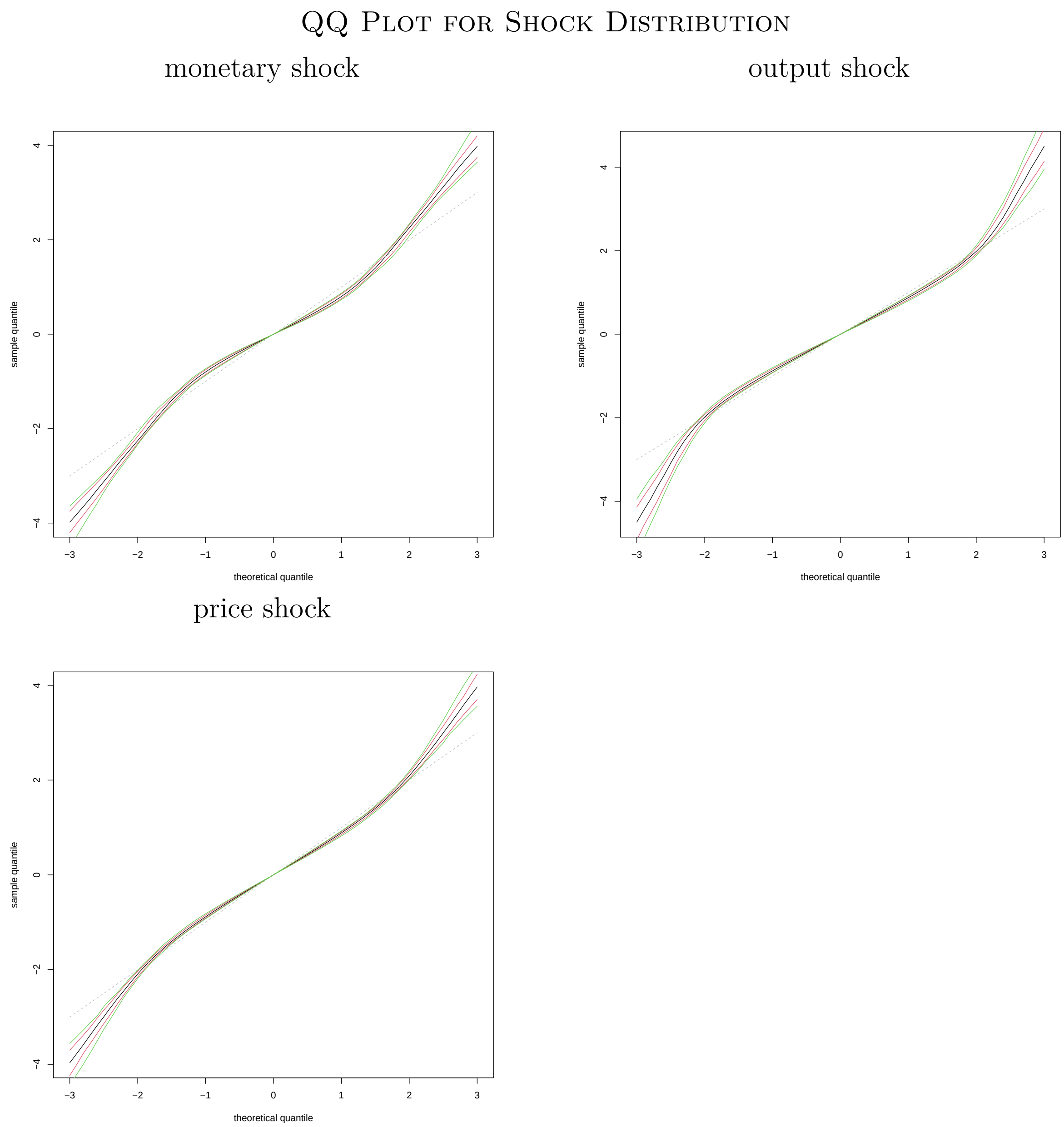

Figure 4: The horizontal axis is the quantiles of Gaussian distribution, and the vertical axis is the quantiles of the shock distribution.

Quantiles of Posterior Density in Empirical Analysis

	5%	16%	50%	84%	95%
A_{21}	-0.14	-0.11	-0.06	-0.03	-0.00
A_{31}	-0.05	-0.03	0.02	0.07	0.09
A_{12}	-0.12	-0.08	-0.03	0.02	0.06
A_{32}	-0.12	-0.05	0.05	0.15	0.24
A_{13}	-0.29	-0.25	-0.16	-0.08	-0.03
A_{23}	-0.31	-0.21	-0.07	0.04	0.17
A^*_{21}	-0.33	-0.26	-0.15	-0.00	0.10
A^*_{31}	-0.13	-0.05	0.07	0.19	0.28

Table 2: The first five columns represent the quantiles of the posterior density of each parameter. The diagonal of A and A^* is normalized to be 1.

identification.

Given the posterior density of all the parameters, we can also plot the impulse responses of the three economic variables, in the standard regime and the ZLB regime respectively. In Figure 5, we use the parameters in the standard regime and trace out the impulse responses, where as in Figure 6 we use the parameters in the ZLB regime to achieve the same goal. The three shocks, by examining their effects on the economic variables, can be labeled as the monetary policy shock, the demand shock, and the supply shock. Figure 5 shows the impulse response to a one-standard-deviation shock when we stay in the standard regime. The monetary policy shock increases the nominal interest rate and has a persistent effect, and the negative response in output growth and inflation rate is also similar to conventional wisdom.[25] In contrary, Figure 6 presents the similarly defined responses in the ZLB regime. Now the monetary policy shock can not persistently affect the shadow interest rate since the increase in the shadow interest rate cannot pass to the nominal interest rate in the next period. Thus, the long-run effect of the unconventional monetary policy is quite limited.[26] The empirical results suggest the unconventional monetary policy has a small and short-lived effect, compared to the conventional monetary policy.

We can now easily discuss the posterior distribution of the shadow interest rate, as shown in Figure 7. By using the data augmentation technique in the efficient Bayesian inference routine, it is easy to draw from the posterior of y^*_{1t} in the ZLB periods. The overall magnitude

[25] We also see a little evidence of the price puzzle, which is a common issue in the monetary SVAR literature (Ramey, 2016).

[26] A possible way to allow a persistent effect in the shadow interest rate is to add lagged shadow interest rates in the model. Further research is useful to consider the persistence of the shadow interest rate.

Impulse Response in Standard Regime

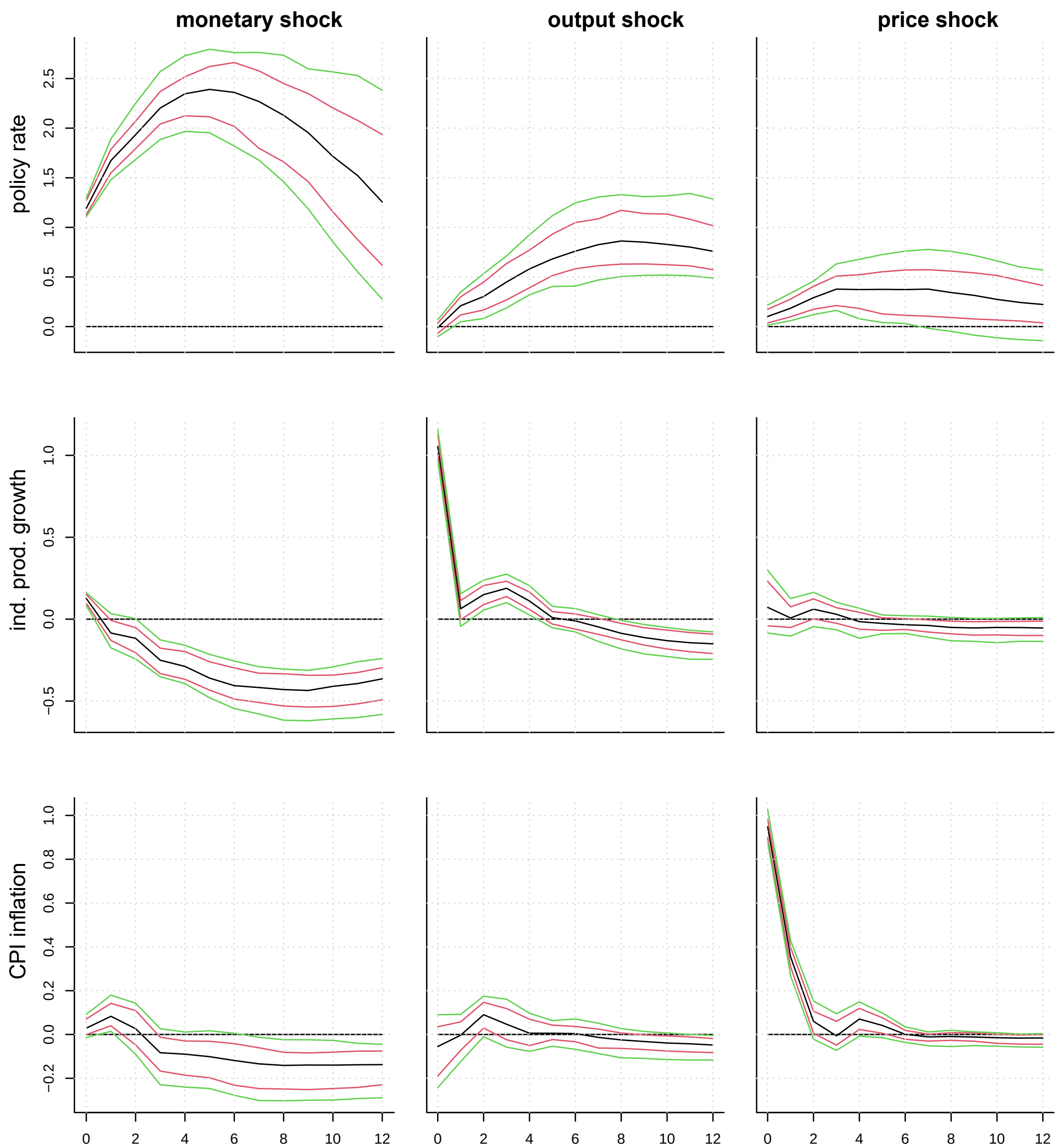

Figure 5: Impulse response w.r.t. one-standard deviation shock. Each row represent one variable, and each column represent one shock. Note the first row represents the response of the nominal interest rate. The 68% and 90 % credible bands are plotted in red and green. The horizontal axis spans from 0 to 12 months after the shock occurrence.

Impulse Response in ZLB Regime

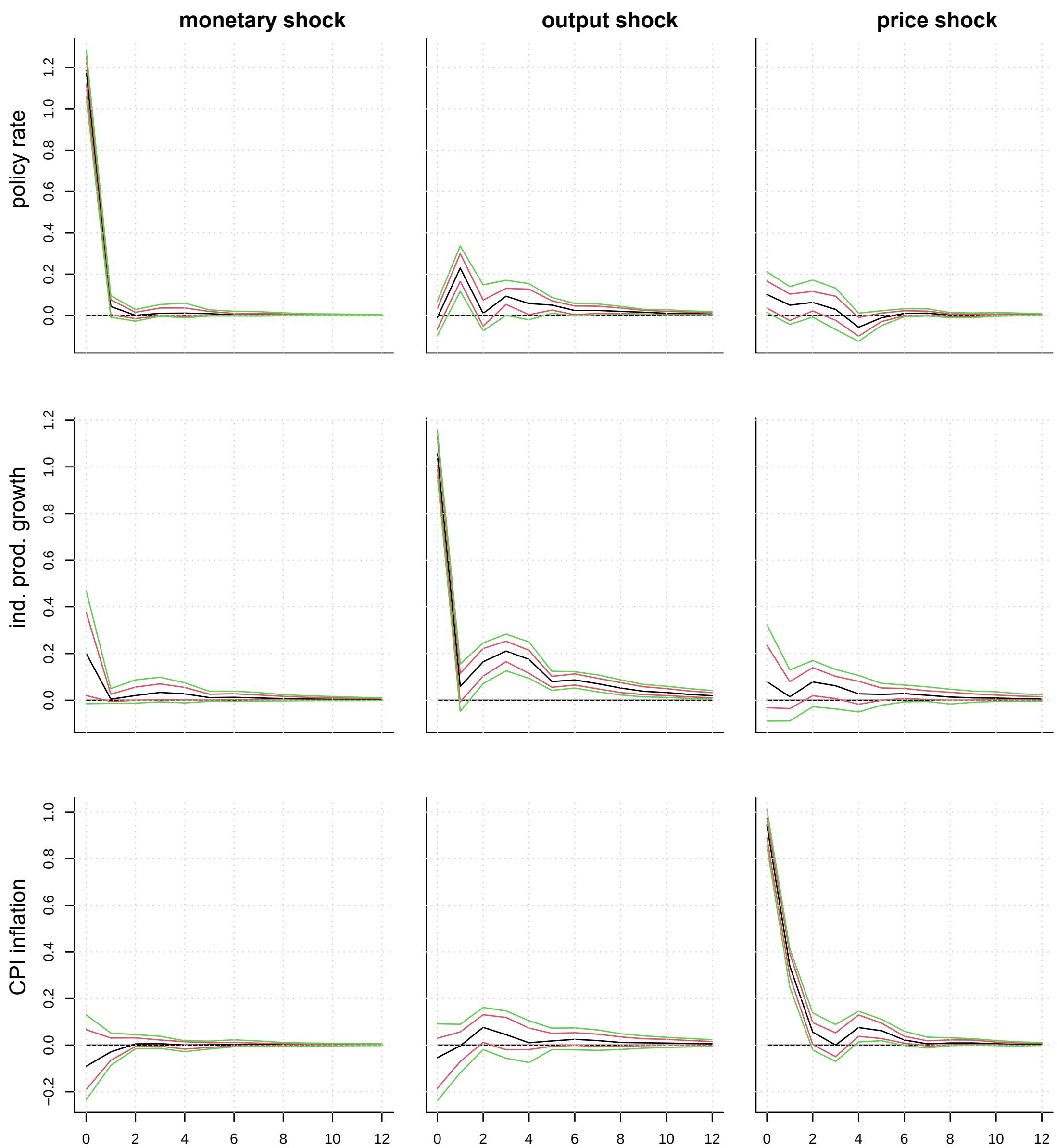

Figure 6: Impulse response w.r.t. one-standard deviation shock. Each row represent one variable, and each column represent one shock. Note the first row represents the response of the shadow interest rate. The black line is the posterior mean, the red line is the 68% credible interval and the green line is the 90% credible interval. The horizontal axis spans from 0 to 12 months after the shock occurrence.

Figure 7: Negative shadow interest rate between 2008:Dec and 2015:Dec. The gray line is the nominal interest rate. The black line, the red line, and the green line are the posterior mean, the 68% credible interval and the 90% credible interval.

of the shadow interest rate is small, meaning the desired interest rate in central banks was not largely deviating from zero in the ZLB periods, except the sudden decline at the beginning of the Great Recession.

8 Conclusion

This paper uses a SVAR-ZLB model to characterize the censored nominal interest rate and the effect of unconventional monetary policy, and considers the model identification when we do not assume the short-run effect of unconventional monetary policy is zero. I first interpret why we cannot achieve point identification of this SVAR-ZLB model if the shocks are Gaussian, from the new likelihood perspective. The Gaussian shocks have a circular contour in the likelihood function, so any rotated version of the model will fit the censored and the uncensored data equally well. It turns out this lack of point identification is only a problem for the Gaussian distribution. If we remove the Gaussian assumption and specify the empirically relevant non-Gaussian distribution for the shocks, the non-Gaussian shocks will help point-identify the model. Then I propose a generic semiparametric identification scheme, following independent component analysis and Heckman selection models, to prove the point identification. Using ICA to identify the structural parameters will encounter a non-linearity problem from the censoring, and I adjust the ICA techniques through Hessian matrices of log densities to deal with this problem. Moreover, I design an efficient Bayesian inference routine to facilitate the model estimation in practice. The empirical results show that the unconventional monetary policy has a small and transitory effect.

There are also interesting questions that need to be answered in the future research. First, when we specify the non-Gaussian shock distribution, we might ask if we still have consistency in MLE when we slightly misspecify the non-Gaussian distribution. For example, we might know the shock is t-distributed but we might get a wrong estimate for the degree of freedom. Second, although we have the model being point-identified, we might wonder if the model has weak identification issues in finite samples. It is still unknown whether the model identification from ICA and semiparametric Heckman selection models is efficient in finite samples.

Appendix A Coherency Conditions on DGP

We assume the coherency of DGP, which implies the continuity of variables w.r.t. shocks and the existence of a unique equilibrium, for any shock value. The two coherency conditions we impose on the DGP are continuity and uniqueness. As in Aruoba et al. (2021b), we need continuity as a necessary condition to achieve coherency, and the continuity and the uniqueness together are necessary and sufficient conditions for coherency. We now explain how we can guarantee these two conditions in the SVAR-ZLB model.

For continuity, it is guaranteed by the model setup where the two regimes only differ in A_{21} and A^*_{21}. First, A_{11} cannot change across regimes, because that will make the definition of the shadow interest rate vague and the effect of unconventional monetary policy will be not properly defined. For example, if we make A_{11} half as much as before in the ZLB regime, then the accordingly defined shadow interest rate will be twice as large as before. Thus the effect of unconventional monetary policy will also be half as much as before. Second, A_{12}, A_{22} and B cannot be contingent on the regime, because the contingency in these parameters will make the y_{2t} not continuous across the two regimes when $y^*_{1t} = 0$. Notice that, in the marginal case of $y^*_{1t} = 0$, the value of y_{2t} follows

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} 0 \\ y_{2t} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A^*_{21} & A_{22} \end{bmatrix} \begin{bmatrix} 0 \\ y_{2t} \end{bmatrix} = \begin{bmatrix} A_{12} \\ A_{22} \end{bmatrix} y_{2t} = Bx_t + \varepsilon_t \tag{A.1}$$

Hence, the same value of y_{2t} will realize no matter we are using the standard regime or the ZLB regime, when $y^*_{1t} = 0$. With the only change arising in A_{21} and A^*_{21}, we can conclude that, given x_t, y^*_{1t} and y_{2t} will continuously change with ε_t.

As for uniqueness, we need the domain restriction, i.e. $|A|/|A^*| > 0$, to guarantee the existence of a unique equilibrium. We can apply (Gouriéroux et al. (1980), Theorem 1), to get the necessary and sufficient condition in my model setup, which is $\text{sign}(|A|) = \text{sign}(|A^*|)$.

Appendix B Proofs

B.1 Proofs of Lemmas

Lemma B.1. *For any continuous shock distribution f_i, the likelihood evaluation formulas in the y-space and in the ε-space are equivalent. In particular, for the standard regime, we* *can show that (16) is equivalent to imputing the shocks $\varepsilon_t = Ay_t - Bx_t$ and evaluating*

$$p(y_t, s_t = 1|x_t) = |A| \cdot \mathbf{f}(\varepsilon_t) \tag{B.1}$$

whereas for the ZLB regime, we can show that (17) is equivalent to imputing the pseudo shocks $\varepsilon_t^0 = A^*(0, y_{2t})' - Bx_t$, and integrating out the true shocks subject to a linear constraint

$$p(y_t, s_t = 0|x_t) = \int_{-\infty}^{\varepsilon_{1t}^0} f(\varepsilon_{1t}) f\left(\varepsilon_{2t}^0 - \frac{G_{21}^*}{G_{22}^*}(\varepsilon_{1t} - \varepsilon_{1t}^0)\right) \frac{1}{G_{22}^*} d\varepsilon_{1t} \tag{B.2}$$

Similarly, the full likelihood evaluated in the ε-space is comprised of (B.1) and (B.2), as written in (18).

Proof. The equivalence between (16) is equivalent to (B.1) can be easily seen once we plug $\varepsilon_t = Ay_t - Bx_t$.

To prove the equivalence between (17) is equivalent to (B.2), we can start from the y-space:

$$\begin{aligned}
&\int_{-\infty}^{0} |A^*| f_1(A_{1\cdot}(y_{1t}^*, y_{2t})' - B_{1\cdot}x_t) f_2(A_{2\cdot}^*(y_{1t}^*, y_{2t})' - B_{2\cdot}x_t) dy_{1t}^* \\
&= \int_{-\infty}^{0} f(A_{1\cdot}(y_{1t}^*, y_{2t})' - B_{1\cdot}x_t) \cdot \\
&\quad f\left(A_{22}y_{2t} - B_{2\cdot}x_t + \frac{A_{21}^*}{A_{11}}[(A_{11}y_{1t}^* + A_{12}y_{2t} - B_{1\cdot}x_t) - (A_{12}y_{2t} - B_{1\cdot}x_t)]\right) \frac{|A^*|}{A_{11}} A_{11} dy_{1t}^* \\
&= \int_{-\infty}^{\varepsilon_{1t}^0} f(\varepsilon_{1t}) f\left(\varepsilon_{2t}^0 + \frac{A_{21}^*}{A_{11}}(\varepsilon_{1t} - \varepsilon_{1t}^0)\right) \frac{|A^*|}{A_{11}} d\varepsilon_{1t} \\
&= \int_{-\infty}^{\varepsilon_{1t}^0} f(\varepsilon_{1t}) f\left(\varepsilon_{2t}^0 - \frac{G_{21}^*}{G_{22}^*}(\varepsilon_{1t} - \varepsilon_{1t}^0)\right) \frac{1}{G_{22}^*} d\varepsilon_{1t}
\end{aligned} \tag{B.3}$$

□

Remark. From Lemma B.1, we can explicitly characterize the different lines in the right panel of Figure 1. The boundary line (l_b) represents $0 = \beta_{1\cdot}x_t + G_{1\cdot}\varepsilon_t$. The integral line ($l_g$) represents $y_{2t} = \beta_{2\cdot}^* + G_{2\cdot}^*\varepsilon_t$. Note that the pseudo shocks by definition are also on the integral line (l_g), i.e. $y_{2t} = \beta_{2\cdot}^*x_t + G_{2\cdot}^*\varepsilon_t^0$, and thus the integral line (l_g) can be equivalently written as $\varepsilon_{2t} = \varepsilon_{2t}^0 - \frac{G_{21}^*}{G_{22}^*}(\varepsilon_{1t} - \varepsilon_{1t}^0)$, which shows up in the last step of (B.3).

Lemma B.2. *The model can be normalized as with $\mathbb{A}, \mathbb{A}^*, \mathbb{B}$, with the shock η_{it} whose variance is 1.*

Proof. The standard regime can be written by dividing by σ_1 and σ_2 for the two rows separately,

$$\begin{bmatrix} A_{11}/\sigma_1 & A_{12}/\sigma_1 \\ A_{21}/\sigma_2 & A_{22}/\sigma_2 \end{bmatrix} \begin{bmatrix} y^*_{1t} \\ y_{2t} \end{bmatrix} = \begin{bmatrix} B_{1\cdot}/\sigma_1 \\ B_{2\cdot}/\sigma_2 \end{bmatrix} x_t + \begin{bmatrix} \varepsilon_{1t}/\sigma_1 \\ \varepsilon_{2t}/\sigma_2 \end{bmatrix} \tag{B.4}$$

whereas in the ZLB regime,

$$\begin{bmatrix} A_{11}/\sigma_1 & A_{12}/\sigma_1 \\ A^*_{21}/\sigma_2 & A_{22}/\sigma_2 \end{bmatrix} \begin{bmatrix} y^*_{1t} \\ y_{2t} \end{bmatrix} = \begin{bmatrix} B_{1\cdot}/\sigma_1 \\ B_{2\cdot}/\sigma_2 \end{bmatrix} x_t + \begin{bmatrix} \varepsilon_{1t}/\sigma_1 \\ \varepsilon_{2t}/\sigma_2 \end{bmatrix} \tag{B.5}$$

Hence, we can define the re-normalized parameters to simplify the notation

$$\mathbb{A} = \begin{bmatrix} A_{11}/\sigma_1 & A_{12}/\sigma_1 \\ A_{21}/\sigma_2 & A_{22}/\sigma_2 \end{bmatrix} \tag{B.6}$$

$$\mathbb{A}^* = \begin{bmatrix} A_{11}/\sigma_1 & A_{12}/\sigma_1 \\ A^*_{21}/\sigma_2 & A_{22}/\sigma_2 \end{bmatrix} \tag{B.7}$$

$$\mathbb{B} = \begin{bmatrix} B_{1\cdot}/\sigma_1 \\ B_{2\cdot}/\sigma_2 \end{bmatrix} \tag{B.8}$$

$$\eta_t = \begin{bmatrix} \varepsilon_{1t}/\sigma_1 \\ \varepsilon_{2t}/\sigma_2 \end{bmatrix} \tag{B.9}$$

We can see (B.4) and (B.5) is an equivalent way to specify the SVAR-ZLB model, except the shock variance is now 1.

The structure of this SVAR-ZLB model still requires that $\mathbb{A}$ and $\mathbb{A}^*$ have the same element except the bottom-left corner. $\square$

Lemma B.3. *Given the true value of $\mathbb{A}$, $\mathbb{A}^*$, and a rotation matrix R, there exists a unique $\tilde{\mathbb{A}}^*$ such that the model setup is still valid and $\tilde{\mathbb{G}}^*_{2\cdot} = \mathbb{G}^*_{2\cdot}R'$. The mapping rule $R^* : (\mathbb{A}, \mathbb{A}^*, R) \to \tilde{\mathbb{A}}^*$ is stated as*

$$\tilde{\mathbb{A}}^* = \begin{bmatrix} (R\mathbb{A})_{11} & (R\mathbb{A})_{12} \\ \frac{(R\mathbb{A})_{11}}{(R\mathbb{A}^*)_{11}}(R\mathbb{A}^*)_{21} & (R\mathbb{A})_{22} \end{bmatrix} \tag{B.10}$$

Proof. With A and R, we can first define the rotated matrix $\tilde{A} = RA$. For the new value of $\tilde{A}^*$, we first need to force $\tilde{A}^*$ and $\tilde{A}$ have the same elements except for the bottom-left corner, to preserve the structure of SVAR-ZLB model. Thus $\tilde{A}^*{}_{i,j} = (R \cdot A)_{i,j}$, for

$(i,j)=(1,1),(1,2),(2,2)$. Now the model setup is preserved. We further need to find a new value $\tilde{A}^*_{21}$ in $\tilde{A}^*$, such that $\tilde{G}^*_{2\cdot} = G^*_{2\cdot}R'$, where $\tilde{G}^* = \tilde{A}^{*-1}$.

We can first show (B.10) is a sufficient condition for $\tilde{G}^*_{2\cdot} = G^*_{2\cdot}R'$. By using the $\tilde{A}^*$ generated by the mapping (B.10), we have

$$
\begin{aligned}
\tilde{G}^*_{2\cdot} &= \begin{bmatrix} \tilde{G}^*_{21} & \tilde{G}^*_{22} \end{bmatrix} \\
&= \frac{1}{\tilde{A}^*_{11}\tilde{A}^*_{22} - \tilde{A}^*_{12}\tilde{A}^*_{21}} \begin{bmatrix} -\tilde{A}^*_{21} & \tilde{A}^*_{11} \end{bmatrix} \\
&= \frac{1}{(RA)_{11}(RA)_{22} - (RA)_{12}\frac{(RA)_{11}}{(RA^*)_{11}}(RA^*)_{21}} \begin{bmatrix} -\frac{(RA)_{11}}{(RA^*)_{11}}(RA^*)_{21} & (RA)_{11} \end{bmatrix} \\
&= \frac{1}{\frac{(RA)_{11}}{(RA^*)_{11}}(RA^*)_{11}(RA^*)_{22} - (RA^*)_{12}\frac{(RA)_{11}}{(RA^*)_{11}}(RA^*)_{21}} \begin{bmatrix} -\frac{(RA)_{11}}{(RA^*)_{11}}(RA^*)_{21} & \frac{(RA)_{11}}{(RA^*)_{11}}(RA^*)_{11} \end{bmatrix} \\
&= \frac{1}{(RA^*)_{11}(RA^*)_{22} - (RA^*)_{12}(RA^*)_{21}} \begin{bmatrix} -(RA^*)_{21} & (RA^*)_{11} \end{bmatrix} \\
&= \frac{1}{|RA^*|} \begin{bmatrix} -(RA^*)_{21} & (RA^*)_{11} \end{bmatrix} \\
&= e_2'(RA^*)^{-1} \\
&= e_2'(G^*R') \\
&= G^*_{2\cdot}R'
\end{aligned} \tag{B.11}
$$

Then, let's prove that (B.10) is a necessary condition for $\tilde{G}^*_{2\cdot} = G^*_{2\cdot}R'$. Given $\tilde{A}^*{}_{i,j} = (R \cdot A)_{i,j}$, for $(i,j)=(1,1),(1,2),(2,2)$, we need $\tilde{A}^*_{21}$ to satisfy

$$
e_2' \begin{bmatrix} (RA)_{11} & (RA)_{12} \\ \tilde{A}^*_{21} & (RA)_{22} \end{bmatrix}^{-1} = G^*_{2\cdot}R' \tag{B.12}
$$

where $\tilde{A}^*_{21}$ is the only unknown and the number of equations is two. Hence, there are at most one $\tilde{A}^*_{21}$ to make (B.12) hold. □

Remark: It is noteworthy that in this SVAR model with ZLB, we also consider the case when we rotate the true A and B matrix by θ to get a new set of parameter values, namely $\tilde{A} = R \cdot A$ and $\tilde{B} = R \cdot B$. Since the level curve of joint standard normal is a circle. If we can rotate everything in the shock space by θ, we will preserve the likelihood. First the boundary line will rotate by θ, because the boundary line is now $\tilde{G}_{1\cdot}\tilde{m} + \tilde{G}_{1\cdot}\tilde{\varepsilon}_t = 0$. Second, the uncensored observation will rotate by θ, i.e. $\tilde{\varepsilon}_t = R \cdot \varepsilon_t$. Third, the censored observation will also rotate, because they can be treated as the marginal cases in the standard regime. Finally, I will prove that we can rotate the integral line by θ.

B.2 Proof of Theorems

Proof of Theorem 1.

Proof. Under the true parameter values, we are going to use the following representation

$$\mathbb{A}y_t(1) = \mathbb{B}x_t + \eta_t \tag{B.13}$$

$$\mathbb{A}^* y_t(0) = \mathbb{B}x_t + \eta_t \tag{B.14}$$

Note that $\eta_t \overset{\text{iid}}{\sim} \mathcal{N}(0, I)$, and thus given x_t we have

$$\begin{bmatrix} y_{1t}(1) \\ y_{2t}(1) \end{bmatrix} \sim \mathcal{N}(\beta x_t, \mathbb{G}\mathbb{G}') \tag{B.15}$$

$$\begin{bmatrix} y_{1t}(1) \\ y_{2t}(0) \end{bmatrix} \sim \mathcal{N}\left(\begin{bmatrix} \beta_{1\cdot} \\ \beta^*_{2\cdot} \end{bmatrix} x_t, \begin{bmatrix} \mathbb{G}_{1\cdot} \\ \mathbb{G}^*_{2\cdot} \end{bmatrix} \begin{bmatrix} \mathbb{G}_{1\cdot} \\ \mathbb{G}^*_{2\cdot} \end{bmatrix}' \right) \tag{B.16}$$

To evaluate the likelihood in the standard regime,

$$p(y_t, s_t = 1|x_t) = p(y_t(1)|x_t) = \phi(y_t(1); \beta x_t, \mathbb{G}\mathbb{G}') \tag{B.17}$$

whereas the likelihood in the ZLB regime,

$$\begin{aligned} p(y_t, s_t = 0|x_t) =& p(y_{1t}(1) < 0, y_{2t}(0)|x_t) \\ =& p(y_{2t}(0)|x_t) \cdot p(y_{1t}(1) < 0|y_{2t}(0), x_t) \\ =& \phi(y_{2t}(0); \beta^*_{2\cdot} x_t, \mathbb{G}^*_{2\cdot}\mathbb{G}^{*'}_{2\cdot}) \cdot \\ & \Phi(0; \beta_{1\cdot} x_t + \Omega_{12}\Omega_{22}^{-1}(y_{2t}(0) - \beta^*_{2\cdot} x_t), \Omega_{11} - \Omega_{12}\Omega_{22}^{-1}\Omega_{21}) \end{aligned} \tag{B.18}$$

where $\phi(\cdot; m, \tau^2)$ means Gaussian density with mean m and variance τ^2, and $\Phi(\cdot; m, \tau^2)$ means the corresponding Gaussian distribution function, and

$$\Omega = \begin{bmatrix} \mathbb{G}_{1\cdot} \\ \mathbb{G}^*_{2\cdot} \end{bmatrix} \begin{bmatrix} \mathbb{G}_{1\cdot} \\ \mathbb{G}^*_{2\cdot} \end{bmatrix}'$$

Now if we rotate the true parameter values to get $\tilde{\mathbb{A}} = R\mathbb{A}$, $\tilde{\mathbb{B}} = R\mathbb{B}$, and $\tilde{\mathbb{A}}^* = R^*(\mathbb{A}^*; \mathbb{A}, R)$. Then the model representation is

$$\tilde{\mathbb{A}} y_t(1) = \tilde{\mathbb{B}} x_t + \tilde{\eta}_t = \tilde{\mathbb{A}}^* y_t(0) \tag{B.19}$$

Note that now $\tilde{\mathbb{G}} = \mathbb{G}R'$, $\tilde{\mathbb{G}}^*_{2\cdot} = \mathbb{G}^*_{2t}R'$, and thus

$$\tilde{\mathbb{G}}\tilde{\mathbb{G}}' = \mathbb{G}\mathbb{G}' \tag{B.20}$$

$$\begin{bmatrix} \tilde{\mathbb{G}}_{1\cdot} \\ \tilde{\mathbb{G}}^*_{2\cdot} \end{bmatrix} \begin{bmatrix} \tilde{\mathbb{G}}_{1\cdot} \\ \tilde{\mathbb{G}}^*_{2\cdot} \end{bmatrix}' = \begin{bmatrix} \mathbb{G}_{1\cdot} \\ \mathbb{G}^*_{2\cdot} \end{bmatrix} \begin{bmatrix} \mathbb{G}_{1\cdot} \\ \mathbb{G}^*_{2\cdot} \end{bmatrix}' \tag{B.21}$$

$$\tilde{\mathbb{G}}\tilde{\mathbb{B}} = \mathbb{G}\mathbb{B} = \beta \tag{B.22}$$

$$\tilde{\mathbb{G}}^*_{2\cdot}\tilde{\mathbb{B}} = \mathbb{G}^*_{2\cdot}\mathbb{B} = \beta^*_{2\cdot} \tag{B.23}$$

So the distribution of $y_t(1)$ and $(y_{1t}(1), y_{2t}(0))$ is the same as (B.15) (B.16). Hence the likelihood value is the same after we use the rotated version of parameters. □

Proof of Theorem 5.

Proof. The proof is mostly the same as the proof in Theorem 1, because we can simply change the notations from the bivariate case to the multivariate case to complete most of the proof in Theorem 1. However, the only part that we cannot be trivially complete when we change the notations is the proof in Lemma B.3.

We can still use the mapping rule (B.10) and show it is a sufficient condition for $\tilde{G}^*_{2\cdot} = G^*_{2\cdot}R'$ in this multivariate case. By using the matrix inverse formula a 2-by-2 block matrix and the mapping rule, we get

$$\begin{aligned}
\tilde{G}^*_{2\cdot} &= \left[-\left(\tilde{A}^*_{22} - \tilde{A}^*_{21}\tilde{A}^{*-1}_{11}\tilde{A}^*_{12}\right)^{-1}\tilde{A}^*_{21}\tilde{A}^{*-1}_{11}, \quad \left(\tilde{A}^*_{22} - \tilde{A}^*_{21}\tilde{A}^{*-1}_{11}\tilde{A}^*_{12}\right)^{-1}\right] \\
&= \left[-\left((RA)_{22} - \tfrac{(RA)_{11}}{(RA^*)_{11}}(RA^*)_{21}(RA)^{-1}_{11}(RA)_{12}\right)^{-1}\tfrac{(RA)_{11}}{(RA^*)_{11}}(RA^*)_{21}(RA)^{-1}_{11}, \quad \left((RA)_{22} - \tfrac{(RA)_{11}}{(RA^*)_{11}}(RA^*)_{21}(RA)^{-1}_{11}(RA)_{12}\right)^{-1}\right] \\
&= \left[-\left((RA^*)_{22} - (RA^*)_{21}(RA^*)^{-1}_{11}(RA^*)_{12}\right)^{-1}(RA^*)_{21}(RA^*)^{-1}_{11}, \quad \left((RA^*)_{22} - (RA^*)_{21}(RA^*)^{-1}_{11}(RA^*)_{12}\right)^{-1}\right] \\
&= e_2'(RA^*)^{-1} \\
&= e_2'(G^*R') \\
&= G^*_{2\cdot}R'
\end{aligned} \tag{B.24}$$

Then, we can still prove that (B.10) is a necessary condition for $\tilde{G}^*_{2\cdot} = G^*_{2\cdot}R'$. Given

$\tilde{A}^*{}_{i,j} = (R \cdot A)_{i,j}$, for $(i, j) = (1,1), (1,2), (2,2)$, we again need $\tilde{A}^*_{21}$ to satisfy

$$e_2' \begin{bmatrix} (RA)_{11} & (RA)_{12} \\ \tilde{A}^*_{21} & (RA)_{22} \end{bmatrix}^{-1} = G^*_{2\cdot} R' \tag{B.25}$$

where the $(n-1)$ vector $\tilde{A}^*_{21}$ is the only unknown and the number of equations is $n(n-1)$. Hence, there are at most one $\tilde{A}^*_{21}$ to make (B.12) hold. □

Proof Theorem 6.

Proof. The proof is mostly the same as the proof in Theorem 2 - Theorem 4, because we can simply change the notations from the bivariate case to the multivariate case to complete most of the proof in Theorem 2 - Theorem 4. It is noteworthy that in the multivariate case, (25) and (26) now both represent $(n-1)$ equations, but the semiparametric Heckit estimator will still work for each one of these $(n-1)$ equations. □

Appendix C Gibbs Sampling Scheme

C.1 Symmetry in Mixture of Normal Distributions

To easily identify all the parameters including the constant term, I impose symmetry in the mixture of normal distributions. In particular, for the $\bar{K}$ components, the first half of the components will have positive component mean, and the second half will have the same component variance but the opposite component mean compared to the first half. Technically speaking, we impose restrictions on μ_{ik} and σ^2_{ik} so that

$$\begin{aligned} \mu_{i+\bar{K}/2,k} &= -\mu_{ik} \\ \sigma^2_{i+\bar{K}/2,k} &= \sigma^2_{ik} \\ \pi_{i+\bar{K}/2,k} &= \pi_{ik} \\ \forall i = 1, 2, \cdots, \bar{K}/2 \end{aligned} \tag{C.1}$$

C.2 Prior

We impose conjugate priors on all the parameters. The details of the prior is stated below.[27]

[27] The dataset is scaled up or down to make the vanilla VAR(p) model generate residual variance around 1. This will help all the following prior make sense.

Prior on A and A^*. Since the diagonal of A and A^* is normalized to be one, we can simply put independent prior for each off-diagonal element in A and A^*. For each (i, j) where $i \neq j$, the Gaussian prior is $A_{ij} \sim \mathcal{N}(0,1)$ and $A^*_{ij} \sim \mathcal{N}(0,1)$.

Prior on B. We can simply put independent prior for each element in B. For the lagged coefficient in equation i of variable j at lag l, the Gaussian prior is $B_{ijl} \sim \mathcal{N}(\mathbf{1}\{i=j\}, 2)$. For the constant term in B, the Gaussian prior is $B_{i0} \sim \mathcal{N}(0, 20^2)$.

Prior on μ_{ik}. We can simply put independent prior for each element μ_{ik} as $\mu_{ik} \sim \mathcal{N}(0,1), k = 1, 2, \cdots, \bar{K}/2$.

Prior on σ^2_{ik}. We can simply put independent prior for each element σ^2_{ik} as $\sigma^2_{ik} \sim \text{IG}(5,2), k = 1, 2, \cdots, \bar{K}/2$, where $\text{IG}(5,2)$ means inverse-gamma with shape 5 and scale 2.

Prior on π_{ik}. For the i-th shock, the component probability needs to sum up to 1, i.e. $\sum_{k=1}^{\bar{K}} \pi_{ik} = 1$. Because we impose the symmetry, we need $\sum_{k=1}^{\bar{K}/2} 2\pi_{ik} = 1$, and thus we can simply put a Dirichlet prior on $(2\pi_{i1}, 2\pi_{i2}, \cdots, 2\pi_{i,\bar{K}/2})$ and get $(\pi_{i1}, \pi_{i2}, \cdots, \pi_{i,\bar{K}/2}) \propto \text{Dir}(1.5)$, where $\text{Dir}(1.5)$ means Dirichlet prior with all the shape parameters equal to 1.5.

C.3 Gibbs Sampler

Now we denote all the parameters as the set $\theta = \{A, A^*, B, \pi_{ik}, \mu_{ik}, \sigma^2_{ik}\}$, and denote $\theta \backslash \theta_1$ as all the parameters except the subset θ_1.

With the augmented data, the likelihood can be evaluated by (40), and the posterior density with the augmented data will be

$$p(\theta, Z_{it}, y^*_{1t} | y_t) \propto p(\theta) \cdot p(Z_{it}|\theta) \cdot p(y_t, y^*_{1t}|\theta, Z_{it}) \tag{C.2}$$

We can now propose a Gibbs sampler to draw all the parameters θ, and the augmented data Z_{it}, y^*_{1t} alternatingly,[28]

- For $p(Z_{it}|\theta, y^*_{1t}, y_t)$:
  draw Z_{it} from the multinomial distribution, since $p(Z_{it} = k|\theta, y^*_{1t}, y_t) \propto \pi_k \phi(\varepsilon_{it}; \mu_{ik}, \sigma^2_{ik})$

[28] A detailed mathematical formula for the Gibbs sampling will be available in the online supplement appendix in the future version.

- For $p(\pi_{ik}|\theta\backslash\pi_{ik}, Z_{it}, y_{1t}^*, y_t)$:
  draw π_{ik} from the Dirichlet distribution, since $p(\pi_{ik}|\theta\backslash\pi_{ik}, Z_{it}, y_{1t}^*, y_t) \propto \pi_{ik}^{1.5-1+\sum_t \mathbf{1}\{Z_{it}=k\}}$

- For $p(B, \mu_{ik}|\theta\backslash\{B, \mu_{ik}\}, y_{1t}^*, y_t)$:
  set up dummy variables $\delta_{ikt} = \mathbf{1}\{Z_{it} = k\}$ for equation i, and then draw B and all the elements μ_{ik} jointly from the Gaussian posterior derived from WLS, where the weight for equation i at time t is $1/\sigma_{i,Z_{it}}$

- For $p(\sigma_{ik}^2|\theta\backslash\sigma_{ik}^2, y_{1t}^*, y_t)$:
  draw σ_{ik}^2 from the inverse-Gamma distribution, since
  $p(\sigma_{ik}^2|\theta\backslash\sigma_{ik}^2,y_{1t}^*,y_t)\propto(\sigma^2)^{-5-1-\frac{1}{2}\sum_t \mathbf{1}\{Z_{it}=k\}}\exp(-\frac{1}{\sigma^2}[2+0.5\sum_t(\varepsilon_{it}-\mu_{i,Z_{it}}\cdot\mathbf{1}\{Z_{it}=k\})^2])$

- For $p(A, A^*|\theta\backslash\{A, A^*\}, y_{1t}^*, y_t)$:
  draw A and A^* jointly using the Metropolis algorithm[29]

- For $p(y_{1t}^*|\theta, Z_{it}, y_t)$:
  draw y_{1t}^* from the truncated normal distribution, since we only need to consider the static correlation between y_{1t}^* and y_{2t} in the ZLB regime,

$$\begin{bmatrix} y_{1t}^* \\ y_{2t} \end{bmatrix} \sim \mathcal{N}\left(\begin{bmatrix} \beta_{1t}^* x_t + \mu_{1,Z_{1t}} \\ \beta_{2t}^* x_t + \mu_{2,Z_{2t}} \end{bmatrix}, G^* \begin{bmatrix} \sigma_{1,Z_{1t}}^2 & \\ & \sigma_{2,Z_{2t}}^2 \end{bmatrix} G^{*-1} \right)$$

  from which we can derive the truncated normal distribution of $y_{1t}^*|y_{1t}^* \leq 0, y_{2t}$

[29]To achieve better posterior sampling convergence, this paper draws each element of A and A^* individually conditional on other elements using the Metropolis algorithm, to explore the non-smooth posterior of A and A^*.

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