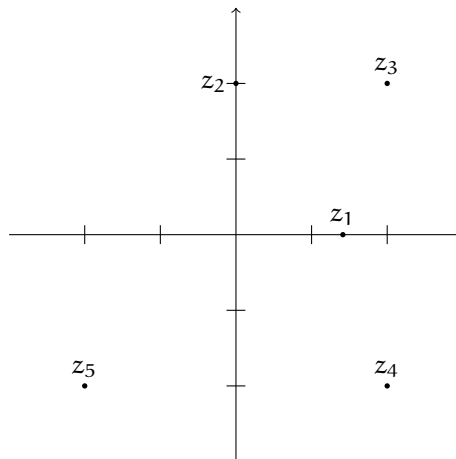


# Algèbre et Arithmétique 1

## Corrigé

### Exercice 1



### Exercice 2

$$(a) \frac{1}{2+i} = \frac{1-i}{(1+i)(1-i)} = \frac{1-i}{2};$$

$$(b) \frac{1+i}{1-i} = \frac{(1+i)^2}{(1-i)(1+i)} = \frac{1+2i-1}{2} = 2;$$

$$(c) (1+i)^4 = ((1+i)^2)^2 = (2i)^2 = -4;$$

$$(d) \left(\frac{1}{2} - i\frac{\sqrt{3}}{2}\right)^3 = \left(\frac{1}{2}\right)^3 - 3i\frac{\sqrt{3}}{2}\left(\frac{1}{2}\right)^2 + 3\left(\frac{i\sqrt{3}}{2}\right)^2 - \left(\frac{i\sqrt{3}}{2}\right)^3 = \frac{1}{8} - \frac{3i\sqrt{3}}{8} - \frac{9}{8} + \frac{3i\sqrt{3}}{8} = -1;$$

### Exercice 3

Soit  $z \in \mathbb{C}$ , alors

$$(a) z + 2i = iz - 1 \iff z(1-i) = -1-2i \iff z = -\frac{1+2i}{1-i} = \boxed{\frac{1-3i}{2}};$$

$$(b) (3+2i)(z-1) = i \iff (3+2i)z = 3+3i \iff z = \frac{3+3i}{3+2i} = \boxed{\frac{15+3i}{13}};$$

$$(c) (2-i)z + 1 = (3+2i)z - i \iff 1+i = (1+3i)z \iff z = \frac{1+i}{1+3i} = \boxed{\frac{2-i}{5}};$$

$$(d) (4-2i)z^2 = (1+5i)z \iff z = 0 \text{ ou } (4-2i)z = 1+5i \iff z = 0 \text{ ou } z = \frac{1+5i}{4-2i} = \boxed{\frac{-3+11i}{10}};$$

### Exercice 5

$$(a) \text{ On a } z^2 = -\frac{1}{2} + i\frac{\sqrt{3}}{2} \text{ et } z^3 = -1.$$

$$(b) \text{ On a donc } z^4 = z^3 z = -z \text{ et } z^5 = z^3 z^2 = -z^2 \text{ et } z^6 = (z^3)^2 = 1.$$

$$(c) \text{ Comme } z^6 = 1, \text{ l'inverse de } z \text{ est } z^{-1} = z^5.$$

(d) On a  $1 + i\sqrt{3} = 2z$ , donc

$$(1 + i\sqrt{3})^5 = (2z)^5 = 32z^5 = -32z^2 = 16 - 16i\sqrt{3}$$

(e) On en déduit

$$\begin{aligned}(1 + i\sqrt{3})^5 + (1 - i\sqrt{3})^5 &= 32 \\ (1 + i\sqrt{3})^5 - (1 - i\sqrt{3})^5 &= -32i\sqrt{3}\end{aligned}$$

### Exercice 6

Soit  $z = a + ib \in \mathbb{C}$  tel que  $|1 + iz| = |1 - iz|$ . Alors

$$|1 + i(a + ib)| = |1 - i(a + ib)|$$

Donc

$$|1 - b + ia| = |1 + b - ia|$$

Donc

$$(1 - b)^2 + a^2 = (1 + b)^2 + a^2$$

Donc

$$1 - 2b + b^2 = 1 + 2b + b^2$$

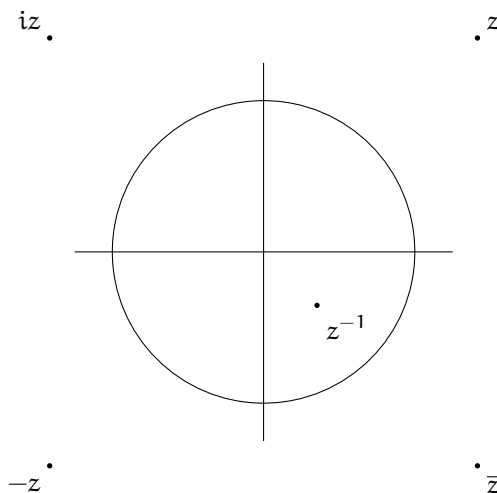
Donc

$$b = -b$$

Ainsi  $b = 0$  et  $z \in \mathbb{R}$ .

### Exercice 9

$$\begin{aligned}z &= 2e^{i\frac{\pi}{4}} \\ z^{-1} &= \frac{1}{2}e^{-i\frac{\pi}{4}} \\ -z &= 2e^{i\frac{5\pi}{4}} \\ iz &= 2e^{i\frac{3\pi}{4}}\end{aligned}$$



### Exercice 10

(a)  $1 = e^{i0}$ ;

(b)  $-1 = e^{i\pi}$ ;

(c)  $i = e^{i\frac{\pi}{2}}$ ;

(d)  $-i = e^{-i\frac{\pi}{2}}$ ;

(e)  $1 + i = \sqrt{2}e^{i\frac{\pi}{4}}$ ;

(f)  $1 - i = \sqrt{2}e^{-i\frac{\pi}{4}}$ ;

(g)  $-1 + i\sqrt{3} = 2e^{i\frac{2\pi}{3}}$ ;

(h)  $1 + i\sqrt{3} = 2e^{i\frac{\pi}{3}}$

**Exercice 11**

(a) On a

$$\begin{aligned}
 \cos^3(x) &= \left( \frac{e^{ix} + e^{-ix}}{2} \right)^3 \\
 &= \frac{e^{i3x} + 3e^{i2x}e^{-ix} + 3e^{ix}e^{-i2x} + e^{-i3x}}{8} \\
 &= \frac{(e^{i3x} + e^{-i3x}) + 3(e^{ix} + e^{-ix})}{8} \\
 &= \boxed{\frac{3}{4} \cos(x) + \frac{1}{4} \cos(3x)}
 \end{aligned}$$

(b) On a

$$\begin{aligned}
 \sin^3(x) &= \left( \frac{e^{ix} - ie^{-ix}}{2i} \right)^3 \\
 &= \frac{e^{i3x} - 3e^{i2x}e^{-ix} + 3e^{ix}e^{-i2x} - e^{-i3x}}{-8i} \\
 &= \frac{(e^{i3x} - e^{-i3x}) - 3(e^{ix} - e^{-ix})}{-8i} \\
 &= \boxed{\frac{3}{4} \sin(x) - \frac{1}{4} \sin(3x)}
 \end{aligned}$$

(c) On a

$$\begin{aligned}
 \sin^2(5x) &= \left( \frac{e^{i5x} - e^{-i5x}}{2i} \right)^2 \\
 &= \frac{e^{i10x} - 2e^{i5x}e^{-i5x} + e^{-i10x}}{-4} \\
 &= \frac{1}{2} - \frac{1}{2} \cos(10x)
 \end{aligned}$$

Donc

$$\begin{aligned}
 \cos(3x) \sin^2(5x) &= \frac{1}{2} \cos(3x) - \frac{1}{2} \cos(3x) \cos(10x) \\
 &= \frac{1}{2} \cos(3x) - \frac{1}{2} \left( \frac{e^{i3x} + e^{-i3x}}{2} \cdot \frac{e^{i10x} + e^{-i10x}}{2} \right) \\
 &= \frac{1}{2} \cos(3x) - \frac{1}{4} \left( \frac{e^{i13x} + e^{i7x} + e^{-i7x} + e^{-i13x}}{2} \right) \\
 &= \boxed{\frac{1}{2} \cos(3x) - \frac{1}{4} \cos(7x) - \frac{1}{4} \cos(13x)}
 \end{aligned}$$