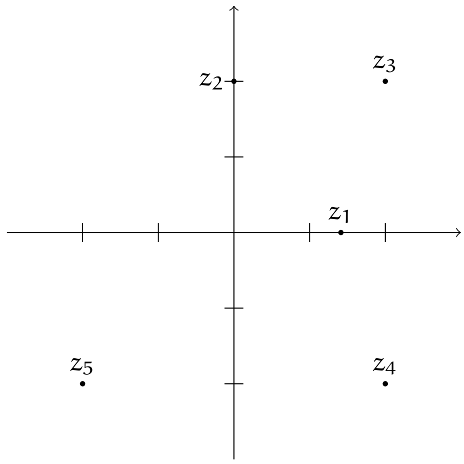
# Algèbre et Arithmétique 1

Corrigé

**Exercice 1**




**Exercice 2**

(a) $\dfrac{1}{2+i} = \dfrac{1-i}{(1+i)(1-i)} = \dfrac{1-i}{2}$;

(b) $\dfrac{1+i}{1-i} = \dfrac{(1+i)^2}{(1-i)(1+i)} = \dfrac{1+2i-1}{2} = 2$;

(c) $(1+i)^4 = ((1+i)^2)^2 = (2i)^2 = -4$;

(d) $\left(\dfrac{1}{2} - i\dfrac{\sqrt{3}}{2}\right)^3 = \left(\dfrac{1}{2}\right)^3 - 3i\dfrac{\sqrt{3}}{2}\left(\dfrac{1}{2}\right)^2 + 3\left(\dfrac{i\sqrt{3}}{2}\right)^2 - \left(\dfrac{i\sqrt{3}}{2}\right)^3 = \dfrac{1}{8} - \dfrac{3i\sqrt{3}}{8} - \dfrac{9}{8} + \dfrac{3i\sqrt{3}}{8} = -1$;

**Exercice 3**

Soit $z \in \mathbf{C}$, alors

(a) $z + 2i = iz - 1 \iff z(1-i) = -1-2i \iff z = -\dfrac{1+2i}{1-i} = \boxed{\dfrac{1-3i}{2}}$;

(b) $(3+2i)(z-1) = i \iff (3+2i)z = 3+3i \iff z = \dfrac{3+3i}{3+2i} = \boxed{\dfrac{15+3i}{13}}$;

(c) $(2-i)z + 1 = (3+2i)z - i \iff 1+i = (1+3i)z \iff z = \dfrac{1+i}{1+3i} = \boxed{\dfrac{2-i}{5}}$;

(d) $(4-2i)z^2 = (1+5i)z \iff z = 0 \text{ ou } (4-2i)z = 1+5i \iff z = 0 \text{ ou } z = \dfrac{1+5i}{4-2i} = \boxed{\dfrac{-3+11i}{10}}$;

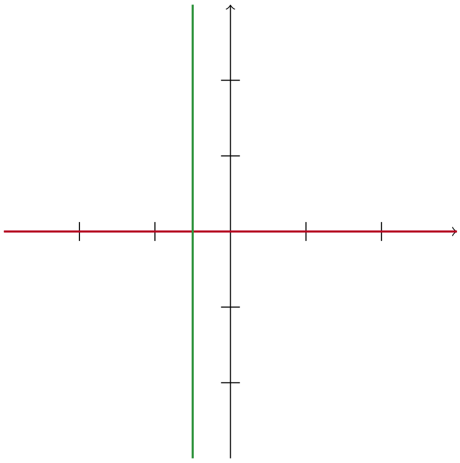
**Exercice 4**

Soit $z$ un nombre complexe. Soient $M_1$, $M_2$ et $M_4$ les points d'affixes $z$, $z^2$ et $z^4$. Alors

$$\begin{aligned} M_1, M_2 \text{ et } M_4 \text{ sont alignés} &\iff z^2 = z \text{ ou } \frac{z^4 - z^2}{z^2 - z} \in \mathbf{R} \\ &\iff z \in \{0, 1\} \text{ ou } \frac{(z^2 - z)(z^2 + z)}{z^2 - z} = z^2 + z \in \mathbf{R} \end{aligned}$$

On note $z = a + ib$, alors

$$\begin{aligned} z^2 + z \in \mathbf{R} &\iff (a+ib)^2 + a + ib \in \mathbf{R} \\ &\iff a^2 + i2ab - b^2 + a + ib \in \mathbf{R} \\ &\iff 2ab + b = 0 \\ &\iff b = 0 \text{ ou } a = -\frac{1}{2} \end{aligned}$$



**Exercice 5**

(a) On a $z^2 = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$ et $z^3 = -1$.

(b) On a donc $z^4 = z^3 z = -z$ et $z^5 = z^3 z^2 = -z^2$ et $z^6 = (z^3)^2 = 1$.

(c) Comme $z^6 = 1$, l'inverse de $z$ est $z^{-1} = z^5$.

(d) On a $1 + i\sqrt{3} = 2z$, donc

$$(1+i\sqrt{3})^5 = (2z)^5 = 32z^5 = -32z^2 = 16 - 16i\sqrt{3}$$

(e) On en déduit

$$(1+i\sqrt{3})^5 + (1-i\sqrt{3})^5 = 32$$
$$(1+i\sqrt{3})^5 - (1-i\sqrt{3})^5 = -32i\sqrt{3}$$

**Exercice 6**

Soit $z = a + ib \in \mathbf{C}$ tel que $|1 + iz| = |1 - iz|$. Alors

$$|1 + i(a+ib)| = |1 - i(a+ib)|$$

Donc

$$|1 - b + ia| = |1 + b - ia|$$

Donc

$$(1-b)^2 + a^2 = (1+b)^2 + a^2$$

Donc

$$1 - 2b + b^2 = 1 + 2b + b^2$$

Donc

$$b = -b$$

Ainsi $b = 0$ et $z \in \mathbf{R}$.

**Exercice 7**

On a

$$\begin{aligned}
\sum_{k=0}^{7}(1+i)^k &= \frac{1-(1+i)^8}{1-(1+i)} \\
&= \frac{1-16}{-i} \\
&= -15i
\end{aligned}$$

**Exercice 8**

Soit $z \in \mathbf{C}$. On a (par le changement d'indice $j = k+1$)

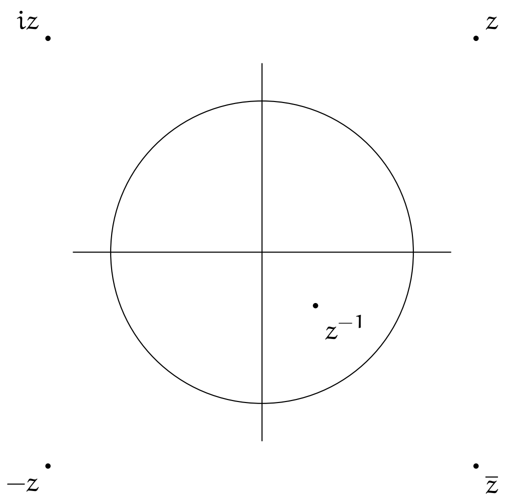
$$\begin{aligned}
(1-z)S_n = (1-z)\sum_{k=0}^{n} kz^k &= \sum_{k=0}^{n} kz^k - \sum_{k=0}^{n} kz^{k+1} \\
&= \sum_{k=1}^{n} kz^k - \sum_{j=1}^{n+1} (j-1)z^j \\
&= \sum_{k=1}^{n} z^k - nz^{n+1} \\
&= \frac{1-z^{n+1}}{1-z} - 1 - nz^{n+1} \\
&= \frac{1-z^{n+1}-(1-z)-n(1-z)z^{n+1}}{1-z} \\
&= \frac{nz^{n+2}-(n+1)z^{n+1}+z}{1-z}
\end{aligned}$$

Ainsi

$$S_n = \frac{nz^{n+2}-(n+1)z^{n+1}+z}{1-z}$$

**Exercice 9**

$$\begin{aligned}
z &= 2e^{i\frac{\pi}{4}} \\
z^{-1} &= \frac{1}{2}e^{-i\frac{\pi}{4}} \\
-z &= 2e^{i\frac{-3\pi}{4}} \\
iz &= 2e^{i\frac{3\pi}{4}}
\end{aligned}$$




**Exercice 10**

(a) $1 = e^{i0}$;

(b) $-1 = e^{i\pi}$;

(c) $i = e^{i\frac{\pi}{2}}$;

(d) $-i = e^{-i\frac{\pi}{2}}$;

(e) $1+i = \sqrt{2}e^{i\frac{\pi}{4}}$;

(f) $1-i = \sqrt{2}e^{-i\frac{\pi}{4}}$;

(g) $-1+i\sqrt{3} = 2e^{i\frac{2\pi}{3}}$;

(h) $1+i\sqrt{3} = 2e^{i\frac{\pi}{3}}$

**Exercice 11**

(a) On a

$$\begin{aligned}\cos^3(x) &= \left(\frac{e^{ix}+e^{-ix}}{2}\right)^3\\ &= \frac{e^{i3x}+3e^{i2x}e^{-ix}+3e^{ix}e^{-i2x}+e^{-i3x}}{8}\\ &= \frac{(e^{i3x}+e^{-i3x})+3(e^{ix}+e^{-ix})}{8}\\ &= \boxed{\frac{3}{4}\cos(x)+\frac{1}{4}\cos(3x)}\end{aligned}$$

(b) On a

$$\begin{aligned}\sin^3(x) &= \left(\frac{e^{ix}-e^{-ix}}{2i}\right)^3\\ &= \frac{e^{i3x}-3e^{i2x}e^{-ix}+3e^{ix}e^{-i2x}-e^{-i3x}}{-8i}\\ &= \frac{(e^{i3x}-e^{-i3x})-3(e^{ix}-e^{-ix})}{-8i}\\ &= \boxed{\frac{3}{4}\sin(x)-\frac{1}{4}\sin(x)}\end{aligned}$$

(c) On a

$$\begin{aligned}\sin^2(5x) &= \left(\frac{e^{i5x}-e^{-i5x}}{2i}\right)^2\\ &= \frac{e^{i10x}-2e^{i5x}e^{-i5x}+e^{-i10x}}{-4}\\ &= \frac{1}{2}-\frac{1}{2}\cos(10x)\end{aligned}$$

Donc

$$\begin{aligned}\cos(3x)\sin^2(5x) &= \frac{1}{2}\cos(3x)-\frac{1}{2}\cos(3x)\cos(10x)\\ &= \frac{1}{2}\cos(3x)-\frac{1}{2}\left(\frac{e^{i3x}+e^{-i3x}}{2}\cdot\frac{e^{i10x}+e^{-i10x}}{2}\right)\\ &= \frac{1}{2}\cos(3x)-\frac{1}{4}\left(\frac{e^{i13x}+e^{i7x}+e^{-i7x}+e^{-i13x}}{2}\right)\\ &= \boxed{\frac{1}{2}\cos(3x)-\frac{1}{4}\cos(7x)-\frac{1}{4}\cos(13x)}\end{aligned}$$

(d) On a

$$\cos^3(3x) = \frac{3}{4}\cos(3x)+\frac{1}{4}\cos(9x)$$

et

$$\begin{aligned}
\cos^2(x)\sin(2x) &= \left(\frac{e^{ix}+e^{-ix}}{2}\right)^2\left(\frac{e^{i2x}-e^{-i2x}}{2i}\right) \\
&= \frac{\left(e^{i2x}+2+e^{-i2x}\right)\left(e^{i2x}-e^{-i2x}\right)}{8i} \\
&= \frac{e^{i4x}+2e^{i2x}+1-1-2e^{-i2x}-e^{-i4x}}{8i} \\
&= \frac{e^{i4x}-e^{-i4x}}{8i}+2\frac{e^{i2x}-e^{-i2x}}{8i} \\
&= \frac{1}{2}\sin(2x)+\frac{1}{4}\sin(4x)
\end{aligned}$$

Ainsi,

$$\cos^2(x)\sin(2x)+\cos^3(3x) = \boxed{\frac{1}{2}\sin(2x)+\frac{3}{4}\cos(3x)+\frac{1}{4}\sin(4x)+\frac{1}{4}\cos(9x)}$$

**Exercice 12**

On a

$$\frac{e^{i\frac{\pi}{3}}}{e^{i\frac{\pi}{4}}} = e^{i\left(\frac{\pi}{3}-\frac{\pi}{4}\right)} = e^{i\frac{\pi}{12}}$$

Donc

$$\cos(\pi/12)+i\sin(\pi/12) = e^{i\frac{\pi}{12}} = e^{i\frac{\pi}{3}}e^{-i\frac{\pi}{4}} = (\cos(\pi/3)+i\sin(\pi3))(\cos(\pi/4)-i\sin(\pi/4))$$

En développant on obtient

$$\begin{aligned}
\cos(\pi/12) &= \cos(\pi/3)\cos(\pi/4)+\sin(\pi/3)\sin(\pi/4) \\
&= \frac{1}{2}\frac{\sqrt{2}}{2}+\frac{\sqrt{3}}{2}\frac{\sqrt{2}}{2} \\
&= \frac{\sqrt{6}+\sqrt{2}}{4}
\end{aligned}$$

et

$$\begin{aligned}
\sin(\pi/12) &= -\sin(\pi/4)\cos(\pi/3)+\sin(\pi/3)\cos(\pi/4) \\
&= -\frac{\sqrt{2}}{2}\frac{1}{2}+\frac{\sqrt{3}}{2}\frac{\sqrt{2}}{2} \\
&= \frac{\sqrt{6}-\sqrt{2}}{4}
\end{aligned}$$

**Exercice 13**

(a) $(1+i)^9 = (\sqrt{2}e^{i\frac{\pi}{4}})^9 = 16\sqrt{2}e^{i\frac{9\pi}{4}} = 16\sqrt{2}e^{i\frac{\pi}{4}}$;

(b) $(1-i)^7 = (\sqrt{2}e^{-i\frac{\pi}{4}})^7 = 8\sqrt{2}e^{-i\frac{7\pi}{4}} = 8\sqrt{2}e^{i\frac{\pi}{4}}$;

(c) $\dfrac{(1+i)^9}{(1-i)^7} = \dfrac{16\sqrt{2}e^{i\frac{\pi}{4}}}{8\sqrt{2}e^{i\frac{\pi}{4}}} = 2$;

**Exercice 14**

(a) On a

$$1+e^{ia} = e^{i\frac{a}{2}}\left(e^{-i\frac{a}{2}}+e^{i\frac{a}{2}}\right) = 2e^{i\frac{a}{2}}\frac{e^{i\frac{a}{2}}+e^{-i\frac{a}{2}}}{2} = 2\cos(a/2)e^{i\frac{a}{2}}$$

Ce nombre est bien sous forme exponentielle car $\left|\frac{a}{2}\right| \leqslant \frac{\pi}{2}$ donc $\cos(a/2) \geqslant 0$.

(b) On a

$$e^{ia}+e^{ib}=e^{i\frac{a+b}{2}}\left(e^{i\frac{b-a}{2}}+e^{-i\frac{b-a}{2}}\right)=2e^{i\frac{a+b}{2}}\frac{e^{i\frac{b-a}{2}}+e^{-i\frac{b-a}{2}}}{2}=2\cos((b-a)/2)e^{i\frac{a+b}{2}}$$

Ce nombre est bien sous forme exponentielle car $\left|\frac{b-a}{2}\right|\leqslant\frac{\pi}{2}$ donc $\cos((b-a)/2)\geqslant 0$.

**Exercice 15**

(a) Soit $x\not\equiv 0 \pmod{2\pi}$. Alors

$$\sum_{k=0}^{n}e^{ikx}=\frac{1-e^{i(n+1)x}}{1-e^{ix}}=\frac{e^{i\frac{n+1}{2}x}\left(e^{-i\frac{n+1}{2}x}-e^{i\frac{n+1}{2}x}\right)}{e^{i\frac{x}{2}}\left(e^{-i\frac{x}{2}}-e^{i\frac{x}{2}}\right)}=\frac{e^{i\frac{n+1}{2}x}}{e^{i\frac{x}{2}}}\times\frac{2\frac{e^{i\frac{n+1}{2}x}-e^{-i\frac{n+1}{2}x}}{2}}{2\frac{e^{i\frac{x}{2}}-e^{-i\frac{x}{2}}}{2}}=e^{i\frac{n}{2}x}\frac{\sin\left(\frac{n+1}{2}x\right)}{\sin\left(\frac{x}{2}\right)}$$

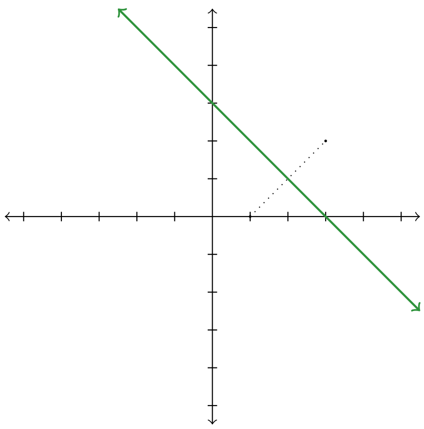
(b) On en déduit

$$\sum_{k=0}^{n}\cos(kx)=\sum_{k=0}^{n}\mathrm{Re}(e^{ikx})=\mathrm{Re}\left(\sum_{k=0}^{n}e^{ikx}\right)=\mathrm{Re}\left(e^{i\frac{n}{2}x}\frac{\sin\left(\frac{n+1}{2}x\right)}{\sin\left(\frac{x}{2}\right)}\right)=\frac{\sin\left(\frac{n+1}{2}x\right)}{\sin\left(\frac{x}{2}\right)}\cos\left(\frac{n}{2}x\right)$$

$$\sum_{k=0}^{n}\sin(kx)=\sum_{k=0}^{n}\mathrm{Im}(e^{ikx})=\mathrm{Im}\left(\sum_{k=0}^{n}e^{ikx}\right)=\mathrm{Im}\left(e^{i\frac{n}{2}x}\frac{\sin\left(\frac{n+1}{2}x\right)}{\sin\left(\frac{x}{2}\right)}\right)=\frac{\sin\left(\frac{n+1}{2}x\right)}{\sin\left(\frac{x}{2}\right)}\sin\left(\frac{n}{2}x\right)$$

**Exercice 16**

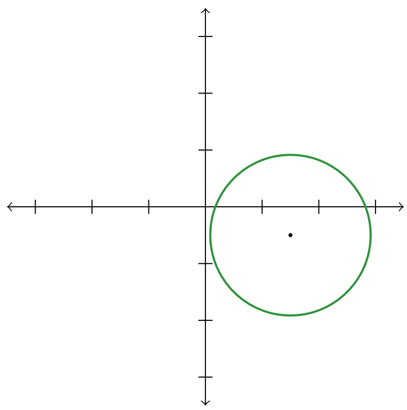
(a) Les points M d'affixe $z$ vérifiant $|z-1|=|z-3-2i|$ forment la médiatrice du segment $[1;3+2i]$.



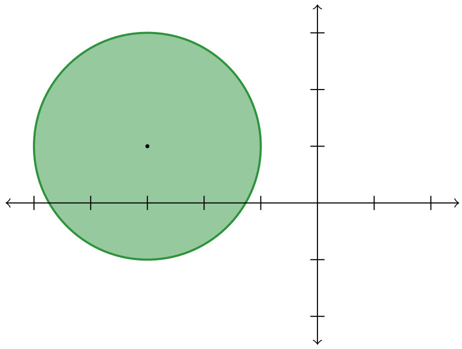
(b) Soit $z\in\mathbf{C}$, alors

$$\left|(1+i)z-2-i\right|=2\iff\left|z-\frac{2+i}{1+i}\right|=\frac{2}{|1+i|}\iff\left|z-\frac{3-i}{2}\right|=\sqrt{2}$$

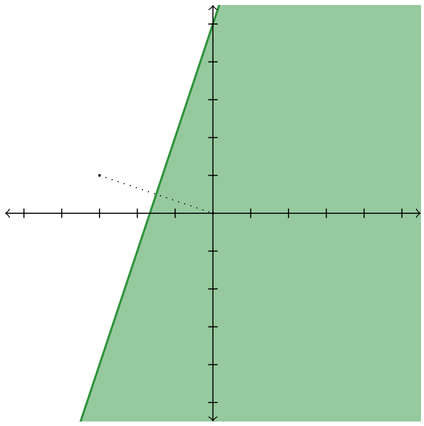
Les points M d'affixe $z$ vérifiant $|(1+i)z-2-i|=2$ forment donc le cercle de centre $O\left(\frac{3-i}{2}\right)$ et de rayon $\sqrt{2}$



(c) Les points M d'affixe $z$ vérifiant $|z-3+i| \leqslant 2$ forment le disque de centre $O(-3+i)$ et de rayon 2



(d) Les points M d'affixe $z$ vérifiant $|z+3-i| \geqslant |z|$ forment le demi-plan sous la médiatrice du segment $[0;-3+i]$



**Exercice 17**

(a) Soit $a+ib \in \mathbf{C}$, alors

$$\begin{aligned}(a+ib)^2 = i &\iff a^2-b^2+2abi = i\\ &\iff \left\{\begin{array}{rcl} a^2-b^2 &=& 0\\ 2ab &=& 1\\ a^2+b^2 &=& 1\end{array}\right.\\ &\iff \left\{\begin{array}{rcl} 2a^2 &=& 1\\ 2b^2 &=& 1\\ ab &>& 0\end{array}\right.\\ &\iff a=b=\pm\frac{\sqrt{2}}{2}\end{aligned}$$

Ainsi les racines carrées de $i$ sont $\frac{\sqrt{2}}{2}+i\frac{\sqrt{2}}{2}$ et $-\frac{\sqrt{2}}{2}-i\frac{\sqrt{2}}{2}$.

(b) Soit $a+ib \in \mathbf{C}$, alors

$$\begin{aligned}(a+ib)^2 = 5+12i &\iff a^2-b^2+2abi = 5+12i\\ &\iff \left\{\begin{array}{rcl} a^2-b^2 &=& 5\\ 2ab &=& 12\\ a^2+b^2 &=& 13\end{array}\right.\\ &\iff \left\{\begin{array}{rcl} 2a^2 &=& 18\\ 2b^2 &=& 8\\ ab &>& 0\end{array}\right.\\ &\iff a=\pm 3, \quad b=\pm 2\end{aligned}$$

Ainsi les racines carrées de $5+12i$ sont $3+2i$ et $-3-2i$.

(c) Soit $a+ib \in \mathbf{C}$, alors

$$\begin{aligned}
(a+ib)^2 = 1+4\sqrt{5}i &\iff a^2-b^2+2abi = 1+4\sqrt{5}i \\
&\iff \left\{\begin{array}{rcl} a^2-b^2 &=& 1 \\ 2ab &=& 4\sqrt{5} \\ a^2+b^2 &=& 9 \end{array}\right. \\
&\iff \left\{\begin{array}{rcl} 2a^2 &=& 10 \\ 2b^2 &=& 8 \\ ab &>& 0 \end{array}\right. \\
&\iff a = \pm\sqrt{5}, \quad b = \pm 2
\end{aligned}$$

Ainsi les racines carrées de $1+4\sqrt{5}i$ sont $\sqrt{5}+2i$ et $-\sqrt{5}-2i$.

(d) Soit $a+ib \in \mathbf{C}$, alors

$$\begin{aligned}
(a+ib)^2 = 1+i\sqrt{3} &\iff a^2-b^2+2abi = 1+i\sqrt{3} \\
&\iff \left\{\begin{array}{rcl} a^2-b^2 &=& 1 \\ 2ab &=& \sqrt{3} \\ a^2+b^2 &=& 2 \end{array}\right. \\
&\iff \left\{\begin{array}{rcl} 2a^2 &=& 3 \\ 2b^2 &=& 1 \\ ab &>& 0 \end{array}\right. \\
&\iff a = \pm\sqrt{\frac{3}{2}} = \frac{\sqrt{6}}{2}, \quad b = \pm\frac{\sqrt{2}}{2}
\end{aligned}$$

Ainsi les racines carrées de $1+i\sqrt{3}$ sont $\dfrac{\sqrt{6}}{2}+i\dfrac{\sqrt{2}}{2}$ et $-\dfrac{\sqrt{6}}{2}-i\dfrac{\sqrt{2}}{2}$.