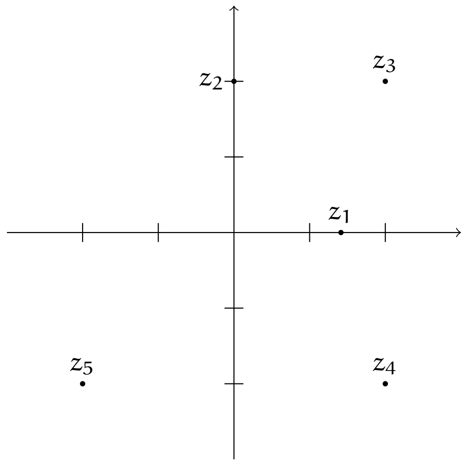
Algèbre et Arithmétique 1

Corrigé

Exercice 1



Exercice 2

(a) $\dfrac{1}{2+\mathrm{i}} = \dfrac{1-\mathrm{i}}{(1+\mathrm{i})(1-\mathrm{i})} = \dfrac{1-\mathrm{i}}{2}$;

(b) $\dfrac{1+\mathrm{i}}{1-\mathrm{i}} = \dfrac{(1+\mathrm{i})^2}{(1-\mathrm{i})(1+\mathrm{i})} = \dfrac{1+2\mathrm{i}-1}{2} = 2$;

(c) $(1+\mathrm{i})^4 = ((1+\mathrm{i})^2)^2 = (2\mathrm{i})^2 = -4$;

(d) $\left(\dfrac{1}{2} - \mathrm{i}\dfrac{\sqrt{3}}{2}\right)^3 = \left(\dfrac{1}{2}\right)^3 - 3\mathrm{i}\dfrac{\sqrt{3}}{2}\left(\dfrac{1}{2}\right)^2 + 3\left(\dfrac{\mathrm{i}\sqrt{3}}{2}\right)^2 - \left(\dfrac{\mathrm{i}\sqrt{3}}{2}\right)^3 = \dfrac{1}{8} - \dfrac{3\mathrm{i}\sqrt{3}}{8} - \dfrac{9}{8} + \dfrac{3\mathrm{i}\sqrt{3}}{8} = -1$;

Exercice 3

Soit $z \in \mathbf{C}$, alors

(a) $z + 2\mathrm{i} = \mathrm{i}z - 1 \iff z(1-\mathrm{i}) = -1 - 2\mathrm{i} \iff z = -\dfrac{1+2\mathrm{i}}{1-\mathrm{i}} = \boxed{\dfrac{1-3\mathrm{i}}{2}}$;

(b) $(3+2\mathrm{i})(z-1) = \mathrm{i} \iff (3+2\mathrm{i})z = 3 + 3\mathrm{i} \iff z = \dfrac{3+3\mathrm{i}}{3+2\mathrm{i}} = \boxed{\dfrac{15+3\mathrm{i}}{13}}$;

(c) $(2-\mathrm{i})z + 1 = (3+2\mathrm{i})z - \mathrm{i} \iff 1 + \mathrm{i} = (1+3\mathrm{i})z \iff z = \dfrac{1+\mathrm{i}}{1+3\mathrm{i}} = \boxed{\dfrac{2-\mathrm{i}}{5}}$;

(d) $(4-2\mathrm{i})z^2 = (1+5\mathrm{i})z \iff z = 0$ ou $(4-2\mathrm{i})z = 1 + 5\mathrm{i} \iff z = 0$ ou $z = \dfrac{1+5\mathrm{i}}{4-2\mathrm{i}} = \boxed{\dfrac{-3+11\mathrm{i}}{10}}$;

Exercice 5

(a) On a $z^2 = -\dfrac{1}{2} + \mathrm{i}\dfrac{\sqrt{3}}{2}$ et $z^3 = -1$.

(b) On a donc $z^4 = z^3 z = -z$ et $z^5 = z^3 z^2 = -z^2$ et $z^6 = (z^3)^2 = 1$.

(c) Comme $z^6 = 1$, l'inverse de z est $z^{-1} = z^5$.

(d) On a $1+i\sqrt{3}=2z$, donc

$$(1+i\sqrt{3})^5=(2z)^5=32z^5=-32z^2=16-16i\sqrt{3}$$

(e) On en déduit

$$(1+i\sqrt{3})^5+(1-i\sqrt{3})^5=32$$
$$(1+i\sqrt{3})^5-(1-i\sqrt{3})^5=-32i\sqrt{3}$$

Exercice 6

Soit $z=a+ib\in\mathbf{C}$ tel que $|1+iz|=|1-iz|$. Alors

$$|1+i(a+ib)|=|1-i(a+ib)|$$

Donc

$$|1-b+ia|=|1+b-ia|$$

Donc

$$(1-b)^2+a^2=(1+b)^2+a^2$$

Donc

$$1-2b+b^2=1+2b+b^2$$

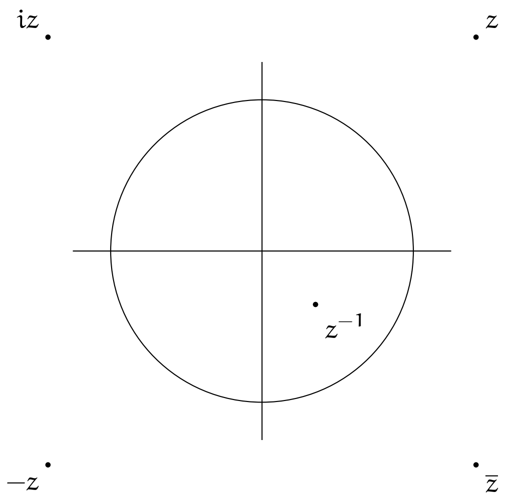
Donc

$$b=-b$$

Ainsi $b=0$ et $z\in\mathbf{R}$.

Exercice 9

$$z=2e^{i\frac{\pi}{4}}$$
$$z^{-1}=\frac{1}{2}e^{-i\frac{\pi}{4}}$$
$$-z=2e^{i\frac{-3\pi}{4}}$$
$$iz=2e^{i\frac{3\pi}{4}}$$



Exercice 10

(a) $1=e^{i0}$;

(b) $-1=e^{i\pi}$;

(c) $i=e^{i\frac{\pi}{2}}$;

(d) $-i=e^{-i\frac{\pi}{2}}$;

(e) $1+i=\sqrt{2}e^{i\frac{\pi}{4}}$;

(f) $1-i=\sqrt{2}e^{-i\frac{\pi}{4}}$;

(g) $-1+i\sqrt{3}=2e^{i\frac{2\pi}{3}}$;

(h) $1+i\sqrt{3}=2e^{i\frac{\pi}{3}}$

Exercice 11

(a) On a

$$\begin{aligned}
\cos^3(x) &= \left(\frac{e^{ix}+e^{-ix}}{2}\right)^3 \\
&= \frac{e^{i3x}+3e^{i2x}e^{-ix}+3e^{ix}e^{-i2x}+e^{-i3x}}{8} \\
&= \frac{(e^{i3x}+e^{-i3x})+3(e^{ix}+e^{-ix})}{8} \\
&= \boxed{\frac{3}{4}\cos(x)+\frac{1}{4}\cos(3x)}
\end{aligned}$$

(b) On a

$$\begin{aligned}
\sin^3(x) &= \left(\frac{e^{ix}-e^{-ix}}{2i}\right)^3 \\
&= \frac{e^{i3x}-3e^{i2x}e^{-ix}+3e^{ix}e^{-i2x}-e^{-i3x}}{-8i} \\
&= \frac{(e^{i3x}-e^{-i3x})-3(e^{ix}-e^{-ix})}{-8i} \\
&= \boxed{\frac{3}{4}\sin(x)-\frac{1}{4}\sin(x)}
\end{aligned}$$

(c) On a

$$\begin{aligned}
\sin^2(5x) &= \left(\frac{e^{i5x}-e^{-i5x}}{2i}\right)^2 \\
&= \frac{e^{i10x}-2e^{i5x}e^{-i5x}+e^{-i10x}}{-4} \\
&= \frac{1}{2}-\frac{1}{2}\cos(10x)
\end{aligned}$$

Donc

$$\begin{aligned}
\cos(3x)\sin^2(5x) &= \frac{1}{2}\cos(3x)-\frac{1}{2}\cos(3x)\cos(10x) \\
&= \frac{1}{2}\cos(3x)-\frac{1}{2}\left(\frac{e^{i3x}+e^{-i3x}}{2}\cdot\frac{e^{i10x}+e^{-i10x}}{2}\right) \\
&= \frac{1}{2}\cos(3x)-\frac{1}{4}\left(\frac{e^{i13x}+e^{i7x}+e^{-i7x}+e^{-i13x}}{2}\right) \\
&= \boxed{\frac{1}{2}\cos(3x)-\frac{1}{4}\cos(7x)-\frac{1}{4}\cos(13x)}
\end{aligned}$$

(d) On a

$$\cos^3(3x) = \frac{3}{4}\cos(3x)+\frac{1}{4}\cos(9x)$$

et

$$\begin{aligned}
\cos^2(x)\sin(2x) &= \left(\frac{e^{ix}+e^{-ix}}{2}\right)^2\left(\frac{e^{i2x}-e^{-i2x}}{2i}\right)\\
&= \frac{\left(e^{i2x}+2+e^{-i2x}\right)\left(e^{i2x}-e^{-i2x}\right)}{8i}\\
&= \frac{e^{i4x}+2e^{i2x}+1-1-2e^{-i2x}-e^{-i4x}}{8i}\\
&= \frac{e^{i4x}-e^{-i4x}}{8i}+2\frac{e^{i2x}-e^{-i2x}}{8i}\\
&= \frac{1}{2}\sin(2x)+\frac{1}{4}\sin(4x)
\end{aligned}$$

Ainsi,

$$\cos^2(x)\sin(2x)+\cos^3(3x) = \boxed{\frac{1}{2}\sin(2x)+\frac{3}{4}\cos(3x)+\frac{1}{4}\sin(4x)+\frac{1}{4}\cos(9x)}$$

Exercice 12

On a

$$\frac{e^{i\frac{\pi}{3}}}{e^{i\frac{\pi}{4}}} = e^{i\left(\frac{\pi}{3}-\frac{\pi}{4}\right)} = e^{i\frac{\pi}{12}}$$

Donc

$$\cos(\pi/12)+i\sin(\pi/12) = e^{i\frac{\pi}{12}} = e^{i\frac{\pi}{3}}e^{-i\frac{\pi}{4}} = (\cos(\pi/3)+i\sin(\pi 3))(\cos(\pi/4)-i\sin(\pi/4))$$

En développant on obtient

$$\begin{aligned}
\cos(\pi/12) &= \cos(\pi/3)\cos(\pi/4)+\sin(\pi/3)\sin(\pi/4)\\
&= \frac{1}{2}\frac{\sqrt{2}}{2}+\frac{\sqrt{3}}{2}\frac{\sqrt{2}}{2}\\
&= \frac{\sqrt{6}+\sqrt{2}}{4}
\end{aligned}$$

et

$$\begin{aligned}
\sin(\pi/12) &= -\sin(\pi/4)\cos(\pi/3)+\sin(\pi/3)\cos(\pi/4)\\
&= -\frac{\sqrt{2}}{2}\frac{1}{2}+\frac{\sqrt{3}}{2}\frac{\sqrt{2}}{2}\\
&= \frac{\sqrt{6}-\sqrt{2}}{4}
\end{aligned}$$

Exercice 13

(a) $(1+i)^9 = (\sqrt{2}e^{i\frac{\pi}{4}})^9 = 16\sqrt{2}e^{i\frac{9\pi}{4}} = 16\sqrt{2}e^{i\frac{\pi}{4}}$;

(b) $(1-i)^7 = (\sqrt{2}e^{-i\frac{\pi}{4}})^7 = 8\sqrt{2}e^{-i\frac{7\pi}{4}} = 8\sqrt{2}e^{i\frac{\pi}{4}}$;

(c) $\dfrac{(1+i)^9}{(1-i)^7} = \dfrac{16\sqrt{2}e^{i\frac{\pi}{4}}}{8\sqrt{2}e^{i\frac{\pi}{4}}} = 2$;

Exercice 14

(a) On a

$$1+e^{ia} = e^{i\frac{a}{2}}\left(e^{-i\frac{a}{2}}+e^{i\frac{a}{2}}\right) = 2e^{i\frac{a}{2}}\frac{e^{i\frac{a}{2}}+e^{-i\frac{a}{2}}}{2} = 2\cos(a/2)e^{i\frac{a}{2}}$$

Ce nombre est bien sous forme exponentielle car $\left|\frac{a}{2}\right| \leqslant \frac{\pi}{2}$ donc $\cos(a/2) \geqslant 0$.