Comments, Notes and Problem Solutions for Terry Tao's Analytic Number Theory Notes

1 Summing monotone functions

Exercise 4. *For non-negative numbers $k, l \geq 0$, show that*

$$\sum_{n \leq x} \log^k n \log^l \frac{x}{n} = x P_{k,l}(\log x) + O_{k,l}(\log^{k+l}(2+x)) \tag{1}$$

for all $x \geq 0$ and some polynomal $P_{k,l}(t)$ with leading term $l! t^k$.

Proof. We prove Exercise 4 by induction on l. Furthermore, we will show that

$$P_{k,l}(t) = \sum_{i=0}^{k} (-1)^i (l+i)! \binom{k}{i} t^{k-i}. \tag{2}$$

We begin with the case where $l = 0$. By Lemma 2, we have

$$\sum_{n \leq x} \log^k n = \int_1^x \log^k t \, dt + O_k(\log^k(2+x)) \tag{3}$$

for $x \geq 0$. Using integration by parts, for any $k \geq 0$, we have

$$\int_1^x \log^k t \, dt = t \log^k t \Big|_1^x - k \int_1^x \log^{k-1} t \, dt. \tag{4}$$

It now follows by induction on (4) that

$$\begin{aligned}
\int_1^x \log^k t \, dt &= x \log^k x - kx \log^{k-1} x + k(k-1) x \log^{k-1} x + \cdots + (-1)^k k!(x-1) \\
&= x\big(\log^k x - k \log^{k-1} x + \cdots + (-1)^k k!\big) + (-1)^{k+1} k! \\
&= x \sum_{i=0}^{k} (-1)^i \frac{k!}{(k-i)!} \log^{k-j} x + (-1)^{k+1} k! \\
&= x P_{k,0}(\log x) + O_k(1).
\end{aligned} \tag{5}$$

Combining (3) with (5) shows that (1) holds in this case.

Now we assume that (1) holds for some value $l-1 \geq 0$. Then, by the induction hypothesis, we have

$$\begin{aligned}\sum_{n\leq x} \log^k n \log^l \frac{x}{n} &= \sum_{n\leq x}(\log^k n)(\log x - \log n)^l \\ &= \sum_{n\leq x}(\log^k n)(\log x - \log n)^{l-1}(\log x - \log n) \\ &= x(\log x \cdot P_{k,l-1}(\log x) - P_{k+1,l-1}(\log x)) + O_{k,l}(\log^{k+l}(2+x)) \qquad (6)\end{aligned}$$

for $x \geq 0$. From (6), we see that in order to complete the proof, we must show that

$$t \cdot P_{k,l-1}(t) - P_{k+1,l-1}(t) = P_{k,l}(t).$$

From (2), we have

$$\begin{aligned}t \cdot P_{k,l-1}(t) - P_{k+1,l-1}(t) &= \sum_{i=0}^{k}(-1)^i(l-1+i)!\binom{k}{i}t^{k+1-i} - \sum_{i=0}^{k+1}(-1)^i(l-1+i)!\binom{k+1}{i}t^{k+1-i} \\ &= \sum_{i=0}^{k}(-1)^i(l-1+i)!\left(\binom{k}{i} - \binom{k+1}{i}\right)t^{k+1-i} + (-1)^{k+1}(l+k)! \\ &= \sum_{i=1}^{k}(-1)^{i+1}(l-1+i)!\binom{k}{i-1}t^{k+1-i} + (-1)^{k+1}(l+k)! \\ &= \sum_{i=0}^{k-1}(-1)^i(l+i)!\binom{k}{i}t^{k-i} + (-1)^{k+1}(l+k)! \\ &= P_{k,l}(t).\end{aligned}$$

Therefore (1) holds. □

Exercise 6. *Let $f : \mathbb{N} \to \mathbb{C}$, $F : \mathbb{R}^+ \to \mathbb{C}$ and $g : \mathbb{R}^+ \to \mathbb{R}^+$ be functions such that $g(x) \to 0$ as $x \to \infty$. Then the following are equivalent:*

(i) One has

$$\sum_{y\leq n<x} f(n) = F(x) - F(y) + O(g(x) + g(y))$$

for all $1 \leq y < x$.

(ii) There exists a constant $c \in \mathbb{C}$ such that

$$\sum_{n<x} f(n) = c + F(x) + O(g(x))$$

for all $x \geq 1$. In particular, $c = -F(1) + O(g(1))$.

The quantity c in (ii) is unique; it is also real-valued if f and F are real-valued.

If in addition $F(x) \to 0$ as $x \to 0$, then when (ii) holds, $\sum_{n=1}^{\infty} f(n)$ converges conditionally to c.