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Optimally weighted ESPRIT using uniform circular arrays

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Abstract

We study a weighted ESPRIT method for estimating the direction cosine (of 2-D angles of incoming wave arrivals) based on uniform circular arrays. An optimum weighting matrix is found and shown to satisfy a generalized centro-Hermitian relationship. As a result, an optimally weighted ESPRIT method is formulated. For a single low (altitude) angle source, the method provides more than 3 dB reduction in estimation variance.

Keywords: Angle of arrival estimation; Uniform circular arrays; Optimum weighting matrix; Optimally weighted ESPRIT

1. Introduction

Two-dimensional angles refer to the elevation angle (denoted by θ) and azimuth angle (denoted by ϕ) defined in the spherical coordinate system. Two-dimensional angle estimation finds

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applications traditionally in radar processing and later in cellular communications [1]. During the last two decades, a tremendous amount of effort had been devoted to this problem and led to many approaches being developed, with some typical ones given in [2–4]. The majority of those approaches require two-dimensional uniform arrays of various shapes: circular, rectangular and cross. A circular array has a much simpler configuration than a rectangular array but enjoys a minimum Cramer-Rao bound (CRB) [4] among the three. Recently, circular arrays have gained an increased popularity in beamforming for mobile communications [5,6].

Based on uniform circular arrays (UCAs), the ESPRIT and MUSIC methods were proposed in [2] for 2-D angle estimation. The ESPRIT method exploits the rotation invariance property of the signal subspace of a forward–backward averaged covariance matrix and presents a closed-form direction cosine solution. Direction cosine is defined as a complex variable $\mu = \sin\theta \mathrm{e}^{\mathrm{j}\phi}$, from which 2-D angles can be determined. Compared with the MUSIC method, the ESPRIT method is attractive in computation but suffers a poor performance. To improve the ESPRIT estimation accuracy, we consider an optimally weighted ESPRIT (OW-ESPRIT) method in this paper. The OW-ESPRIT method reduces the direction cosine estimation variance by choosing an optimum weighting matrix for each direction cosine and still retains the computational advantage over the MUSIC method.

Note that it does not appear likely to develop the OW-ESPRIT methods for 2-D angles individually. To our knowledge, there have been no such techniques reported before. Choice of direction cosine is justified by the fact that its estimation error provides a meaningful measure for joint 2-D angle errors. The novel contribution of this paper includes a simplification to the estimation error expression of direction-cosine originally derived in [7] and revelation of insights into the optimum weighting matrix. Our OW-ESPRIT can be viewed as an extension of the method in [8] to 2-D angle case.

The paper is organized as follows. In Section 2, the data model is reviewed. Section 3 formulates the weighted ESPRIT and elaborates on how to find the optimum weight. The performance investigation of the OW-ESPRIT is presented in Section 4. Section 5 contains conclusions.

2. Data model

A UCA is sitting in the x–y plane, with the radius r and the center at the origin of the Cartesian coordinate system. The array consists of N identical and omni-directional (in both elevation and azimuth ranges) sensors. The nth sensor is located at $[x_n, y_n, z_n] = [r\cos(\frac{2\pi}{N}n), r\sin(\frac{2\pi}{N}n), 0]$. K narrow-band waves with the same wavelength λ propagate towards the array from sources with the directions described by the unknown 2-D angles $(\theta_1, \phi_1), (\theta_2, \phi_2), \ldots, (\theta_K, \phi_K)$ where $0° \leqslant \phi_i \leqslant 360°$. Such a planar circular array is unable to tell apart the sources above or below the plane of the array (see the measurement model (1) in [2]). The range of the elevation angle is therefore restricted to be within one half of the space, i.e., $0° \leqslant \theta_i \leqslant 90°$or $90° \leqslant \theta_i \leqslant 180°$. The sources are far from the array and consequently the wavefronts reaching the array are almost planar. To develop the ESPRIT [2] (as well as the weighted ESPRIT (W-ESPRIT) in this paper), the original measurement of N data (taken at the array) has to undergo a phase mode excitation beamforming transformation. The size of the beamformer is $2M + 1$ where $M \approx 2\pi r/\lambda$ [9]. Furthermore, if the number of sensors on the array satisfies $N > 2M + 6$, with the center of the

UCA chosen as the phase reference point, the beamformed measurement at the tth sampling instant, is represented by (refer to [2] for details):

$$\mathbf{z}(t) = \mathbf{A}\mathbf{s}(t) + \mathbf{w}(t), \quad t = 1, \ldots, T, \tag{1}$$

where

$$\mathbf{A} = [\mathbf{J}(\xi_1)\mathbf{v}(\phi_1), \mathbf{J}(\xi_2)\mathbf{v}(\phi_2), \ldots, \mathbf{J}(\xi_K)\mathbf{v}(\phi_K)], \quad \xi_i = (2\pi r/\lambda)\sin\theta_i,$$

$$\mathbf{J}(\xi) = \mathrm{diag}[J_M(\xi), \ldots, J_1(\xi), J_0(\xi), J_1(\xi), \ldots, J_M(\xi)] \quad \text{(diagonal matrix)},$$

$$\mathbf{v}(\phi) = [\mathrm{e}^{-\mathrm{j}M\phi}, \ldots, \mathrm{e}^{-\mathrm{j}\phi}, 1, \mathrm{e}^{\mathrm{j}\phi}, \ldots, \mathrm{e}^{\mathrm{j}M\phi}]^{\mathrm{T}}, \quad \mathbf{s}(t) = [s_1(t), \ldots, s_K(t)]^{\mathrm{T}},$$

$$\mathbf{w}(t) = [w_1(t), \ldots, w_{2M+1}(t)]^{\mathrm{T}}$$

and T denotes the (total) number of measurements.

In the expressions following (1), $J_p(\xi)$ denotes the Bessel function of the first kind of order p, $s_i(t)$ is the random (baseband) wave signal arriving at the center of the array from the ith source and $w_m(t)$ signifies the thermal noise embedded in the measurement. M corresponds to the maximum phase mode (of the incoming wave) excitable by the array. A larger N helps to better suppresses modes $m > M$ as shown in [2]. To obtain more data in (1), one has to increase the array radius, and consequently the number of sensors on the array. More data, in general, would lead to a better estimation accuracy at an increased computational burden.

Assume that (1) the signal vector $\mathbf{s}(t)$ is Gaussian with mean zero and covariance $\mathbf{R}_{\mathrm{s}} = E\{\mathbf{s}(t)\mathbf{s}^{\mathrm{H}}(t)\}$ and (2) the noise vector $\mathbf{w}(t)$ contains independent and identically distributed Gaussian random variables with mean zero and unknown variance σ^2. The superscript H denotes the conjugate and transpose operation. Signal and noise variables are further assumed to be independent between themselves as well as over different measurement instants. Then the covariance matrix associated with the transformed measurement, will be given by

$$\mathbf{R} = \Re e[E\{\mathbf{W}^{\mathrm{H}}\mathbf{z}(t)\mathbf{z}^{\mathrm{H}}(t)\mathbf{W}\}] = \mathbf{W}^{\mathrm{H}}\mathbf{A}\Re e[\mathbf{R}_{\mathrm{s}}]\mathbf{A}^{\mathrm{H}}\mathbf{W} + \sigma^2\mathbf{I}_{2M+1}, \tag{2}$$

where

$$\mathbf{W} = \frac{1}{\sqrt{2M+1}}[\mathbf{v}(\alpha_{-M}), \ldots, \mathbf{v}(\alpha_{-1}), \mathbf{v}(\alpha_0), \mathbf{v}(\alpha_1), \ldots, \mathbf{v}(\alpha_M)] \quad \text{with} \quad \alpha_i = \frac{2\pi}{2M+1}i,$$
$$i \in [-M, M].$$

The real operation applied to the matrix in (2) turns out to be equivalent [2] to the forward–backward averaging. This operation is employed to improve estimation robustness against wave correlation (which can be caused for example by multipath propagation) as well as to reduce computational overhead for the eigen-decomposition of $\mathbf{R}$.

Let $\mathbf{U}_{\mathrm{s}}$ contain the K principal *real* eigenvectors of $\mathbf{R}$. $\mathbf{U}_{\mathrm{s}}$ should span the same column space as $\mathbf{W}^{\mathrm{H}}\mathbf{A}$, for the ESPRIT-type methods to work. This requirement embraces the well-known subspace rotation invariance property many approaches hinge upon, and can be met if $\Re e[\mathbf{R}_{\mathrm{s}}]$ is of full rank and $\mathbf{A}$ has a full column rank. In practice, one can only obtain the finite sample covariance $\hat{\mathbf{R}} = \frac{1}{T}\sum_{t=1}^{\mathrm{T}}\Re e[\mathbf{W}^{\mathrm{H}}\mathbf{z}(t)\mathbf{z}^{\mathrm{H}}(t)\mathbf{W}]$ and $\mathbf{U}_{\mathrm{s}}$ has to be replaced by its estimate $\hat{\mathbf{U}}_{\mathrm{s}}$ which contains the K (*real*) principal unitary eigenvectors of $\hat{\mathbf{R}}$. Based on the assumptions made on the signal and noise, it can be shown that $\hat{\mathbf{U}}_{\mathrm{s}}$ is large sample consistent, i.e., $\hat{\mathbf{U}}_{\mathrm{s}} \stackrel{T\to\infty}{\longrightarrow} \mathbf{U}_{\mathrm{s}}$.

3. Weighted ESPRIT and optimum weight

Let $\mathbf{C}_0 = \mathrm{diag}\{(-1)^M, \ldots, (-1)^1, 1^0, 1^1, \ldots, 1^M\}$, $\mathbf{\Delta}^{(-1)}$ be a submatrix of $\mathbf{I}_{2M+1}$ with the last two rows deleted, $\mathbf{\Delta}^{(1)}$ a submatrix of $\mathbf{I}_{2M+1}$ with the first two rows deleted, and $\mathbf{\Delta}^{(0)}$ a submatrix of $\mathbf{I}_{2M+1}$ with the first and last rows deleted, and finally $\hat{\mathbf{E}} = [\mathbf{\Delta}^{(-1)}\mathbf{C}_0\mathbf{W}\hat{\mathbf{U}}_\mathrm{s}, \mathbf{\Delta}^{(1)}\mathbf{C}_0\mathbf{W}\hat{\mathbf{U}}_\mathrm{s}]_{(2M-1)\times(2K)}$. One can construct a $2K \times K$ matrix

$$(\hat{\mathbf{E}}^\mathrm{H}\mathbf{Q}\hat{\mathbf{E}})^{-1}\hat{\mathbf{E}}^\mathrm{H}\mathbf{Q}\mathbf{\Gamma}\hat{\mathbf{U}}_\mathrm{s}^{(0)}, \tag{3}$$

where $\mathbf{\Gamma} = \mathrm{diag}[-(M-1), \ldots, -1, 0, 1, \ldots, M-1] \cdot \lambda/(\pi r)$, $\hat{\mathbf{U}}_\mathrm{s}^{(0)} = \mathbf{\Delta}^{(0)}\mathbf{C}_0\mathbf{W}\hat{\mathbf{U}}_\mathrm{s}$ and $\mathbf{Q}$ is a user-chosen weighting matrix.

It is known [2] that

$$\mathbf{\Gamma}\mathbf{U}_\mathrm{s}^{(0)} = \mathbf{E}\begin{bmatrix}\mathbf{T}^{-1}\mathbf{\Phi}\mathbf{T}\\ \mathbf{T}^{-1}\mathbf{\Phi}^*\mathbf{T}\end{bmatrix},$$

where $\mathbf{E}$ and $\mathbf{U}_\mathrm{s}^{(0)}$ are the noiseless counterparts of $\hat{\mathbf{E}}$ and $\hat{\mathbf{U}}_\mathrm{s}^{(0)}$ constructed from $\mathbf{U}_\mathrm{s}$, $\mathbf{T}$ is a $K \times K$ real non-singular matrix, $\mathbf{\Phi}$ is a $K \times K$ diagonal matrix with the diagonal elements equal to direction cosines, and the superscript $*$ denotes the conjugation only (without transpose). Then one can have

$$(\mathbf{E}^\mathrm{H}\mathbf{Q}\mathbf{E})^{-1}\mathbf{E}^\mathrm{H}\mathbf{Q}\mathbf{\Gamma}\mathbf{U}_\mathrm{s}^{(0)} = \begin{bmatrix}\mathbf{T}^{-1}\mathbf{\Phi}\mathbf{T}\\ \mathbf{T}^{-1}\mathbf{\Phi}^*\mathbf{T}\end{bmatrix} = \begin{bmatrix}\mathbf{T}^{-1}\mathbf{\Phi}\mathbf{T}\\ (\mathbf{T}^{-1}\mathbf{\Phi}\mathbf{T})^*\end{bmatrix}$$

because $\mathbf{T}$ is real. This result illustrates that the upper half and the lower half of $(\mathbf{E}^\mathrm{H}\mathbf{Q}\mathbf{E})^{-1}\mathbf{E}^\mathrm{H}\mathbf{Q}\mathbf{\Gamma}\mathbf{U}_\mathrm{s}^{(0)}$ are mutually conjugate and the eigenvalues of the upper half directly correspond to the direction cosines, as long as $\mathbf{Q}$ is chosen such that $\mathbf{E}^\mathrm{H}\mathbf{Q}\mathbf{E}$ is invertible. This property also holds for the large sample case in (3), if

$$\mathbf{D}\tilde{\mathbf{I}}\mathbf{Q}\tilde{\mathbf{I}}\mathbf{D} = \mathbf{Q}^*, \tag{4}$$

where $\mathbf{D} = \mathrm{diag}\{(-1)^{M-2}, \ldots, (-1), (-1)^0, 1^1, \ldots, 1^M\}$ and $\tilde{\mathbf{I}}$ is a reverse permutation matrix of appropriate dimensions. To prove this claim, we have used the relations: $\tilde{\mathbf{I}}_{2M+1}\mathbf{W} = \mathbf{W}^*$, $\tilde{\mathbf{I}}_{2M-1}\mathbf{\Gamma} = -\mathbf{\Gamma}\tilde{\mathbf{I}}_{2M-1}, \tilde{\mathbf{I}}_{2M-1}\mathbf{\Delta}^{(i)} = \mathbf{\Delta}^{(-i)}\tilde{\mathbf{I}}_{2M+1}$ and $\mathbf{D}\tilde{\mathbf{I}}_{2M-1}\mathbf{\Delta}^{(i)}\mathbf{C}_0 = (-1)^{i+1}\mathbf{\Delta}^{(-i)}\mathbf{C}_0\tilde{\mathbf{I}}_{2M+1}$. The relationship (4) can be regarded as generalized *centro-Hermitianity*. (Relationship of $\tilde{\mathbf{I}}\mathbf{Q}\tilde{\mathbf{I}} = \mathbf{Q}^*$ is called *centro-Hermitianity*.) In the sequel, we only consider generalized centro-Hermitian weighting matrix. Hence, the W-ESPRIT method estimates direction cosines as the eigenvalues of the upper half $K \times K$ submatrix of the matrix (3). Since $\hat{\mathbf{U}}_\mathrm{s} \overset{T\to\infty}{\to} \mathbf{U}_\mathrm{s}$, the W-ESPRIT direction cosine estimates are also large sample consistent.

Define $\boldsymbol{\alpha}_i^\mathrm{H} = \boldsymbol{\rho}_i^\mathrm{T}\mathbf{J}_\Psi(\mathbf{E}^\mathrm{H}\mathbf{Q}\mathbf{E})^{-1}\mathbf{E}^\mathrm{H}\mathbf{Q}\mathbf{G}_i$ where $\mathbf{E} = [\mathbf{\Delta}^{(-1)}\mathbf{C}_0\mathbf{W}\mathbf{U}_\mathrm{s}, \mathbf{\Delta}^{(1)}\mathbf{C}_0\mathbf{W}\mathbf{U}_\mathrm{s}], \mathbf{J}_\Psi = [\mathbf{I}_K, \mathbf{0}_K]_{K\times 2K}$, $\mathbf{G}_i = \mathbf{\Gamma}\mathbf{\Delta}^{(0)} - \mu_i\mathbf{\Delta}^{(-1)} - \mu_i^*\mathbf{\Delta}^{(1)}$, and $\boldsymbol{\rho}_i^\mathrm{T}$ = the ith row of (the real matrix) $(\mathbf{A}^\mathrm{H}\mathbf{A})^{-1}\mathbf{A}^\mathrm{H}\mathbf{W}\mathbf{U}_\mathrm{s}$. Following steps in [8], it can be shown that the W-ESPRIT direction cosine estimates have an asymptotic ($T \gg 1$) error, $\Delta\mu_i^{\text{W-ESP}} = \boldsymbol{\alpha}_i^\mathrm{H}\mathbf{C}_0\mathbf{W} \cdot (\hat{\mathbf{U}}_\mathrm{s} - \mathbf{U}_\mathrm{s}) \cdot \boldsymbol{\beta}_i$ where $\Delta\mu_i^{\text{W-ESP}} = \hat{\mu}_i^{\text{W-ESP}} - \mu_i$, $\hat{\mu}_i^{\text{W-ESP}}$ is the direction cosine estimate given by the W-ESPRIT method and $\boldsymbol{\beta}_i$ = the ith column of (the *real* matrix) $[(\mathbf{A}^\mathrm{H}\mathbf{A})^{-1}\mathbf{A}^\mathrm{H}\mathbf{W}\mathbf{U}_\mathrm{s}]^{-1}$. This result is an extension of (18) in [7] for non-identity $\mathbf{Q}$, but presented using a simpler $\boldsymbol{\alpha}_i^\mathrm{H}$ (denoted by $\boldsymbol{\alpha}_i^\mathrm{T}$ therein). The first and second order statistics of the estimation errors for the W-ESPRIT method can then be established below.

Theorem 1. *If T is sufficiently large, the W-ESPRIT direction cosine estimates are unbiased and the estimation errors have the variances*

$$\mathrm{var}(\Delta\mu_i^{\text{W-ESP}}) = \frac{\sigma^2}{2T}\cdot\boldsymbol{\alpha}_i^{\mathrm{H}}\boldsymbol{\alpha}_i\cdot(\Re e[\mathbf{R}_s])^{-1} + \sigma^2(\Re e[\mathbf{R}_s])^{-2}.$$

By using the eigenvector statistics in [7] and the orthogonal property of $\boldsymbol{\alpha}_i^{\mathrm{H}}\mathbf{C}_0\mathbf{W}\mathbf{U}_s = \mathbf{0}$, one can obtain that $\mathrm{var}(\Delta\mu_i^{\text{W-ESP}}) = \frac{\sigma^2}{2T}\cdot\boldsymbol{\alpha}_i^{\mathrm{H}}\boldsymbol{\alpha}_i\cdot(\boldsymbol{\eta}_i^{\mathrm{T}}\mathbf{W}^{\mathrm{H}}\mathbf{A}\Re e[\mathbf{R}_s]\mathbf{A}^{\mathrm{H}}\mathbf{W}\boldsymbol{\eta}_i + \sigma^2\boldsymbol{\eta}_i^{\mathrm{T}}\boldsymbol{\eta}_i)$ where $\boldsymbol{\eta}_i = \mathbf{U}_s(\boldsymbol{\Lambda}_s - \sigma^2\mathbf{I}_K)^{-1}\boldsymbol{\beta}_i$ with $\boldsymbol{\Lambda}_s = \mathrm{diag}[\lambda_1,\ldots,\lambda_K]$ containing the K largest eigenvalues of $\mathbf{R}$. Result then follows since $\boldsymbol{\eta}_i^{\mathrm{T}}\mathbf{W}^{\mathrm{H}}\mathbf{A}\Re e[\mathbf{R}_s]\mathbf{A}^{\mathrm{H}}\mathbf{W}\boldsymbol{\eta}_i + \sigma^2\boldsymbol{\eta}_i^{\mathrm{T}}\boldsymbol{\eta}_i$ reduces to $(\Re e[\mathbf{R}_s])^{-1} + \sigma^2(\Re e[\mathbf{R}_s])^{-2}$. $\boldsymbol{\alpha}_i^{\mathrm{H}}$ can be further simplified to be equal to the ith row of $\mathbf{J}_\Psi(\mathbf{E}_{\mathbf{A}}^{\mathrm{H}}\mathbf{Q}\mathbf{E}_{\mathbf{A}})^{-1}\mathbf{E}_{\mathbf{A}}^{\mathrm{H}}\mathbf{Q}\mathbf{G}_i$ where $\mathbf{E}_{\mathbf{A}} = [\boldsymbol{\Delta}^{(-1)}\mathbf{C}_0\mathbf{A}, \boldsymbol{\Delta}^{(1)}\mathbf{C}_0\mathbf{A}]$. These simplifications completely eliminate eigenvectors in the variance expression and thereby alleviates computational load for variance calculation.

We like to remark that the orthogonal property of $\boldsymbol{\alpha}_i^{\mathrm{H}}\mathbf{C}_0\mathbf{W}\mathbf{U}_s = \mathbf{0}$ has not been observed and utilized in deriving estimation error (12) of [7] for the ESPRIT method. Without using this property, the resulting variance expression for the W-ESPRIT method can not be presented in its current form allowing for the determination of the optimum weighting matrix.

From the variance expression in Theorem 1, it can be shown (see the proof given in Appendix C of [8]), that the optimum weighting matrix is $\mathbf{Q}_{\mathrm{opt}}^i = [\mathbf{G}_i\mathbf{G}_i^{\mathrm{H}}]^{-1}$ which leads to a minimum $\mathrm{var}(\Delta\mu_i^{\text{W-ESP}})$ for the direction cosine μ_i, among other choices of the weighting matrix $\mathbf{Q}$. Note that $\mathbf{Q}_{\mathrm{opt}}^i$ meets the requirement (4). The W-ESPRIT method with $\mathbf{Q} = \mathbf{Q}_{\mathrm{opt}}^i$ is termed as the OW-ESPRIT method.

Sparse structure of $\mathbf{G}_i$ can be exploited to reduce the load for computing the matrix in (3). Let $\boldsymbol{\Phi}_k^i = \mathrm{diag}[1, \mathrm{e}^{\mathrm{j}\phi_i}, \ldots, \mathrm{e}^{\mathrm{j}(k-1)\phi_i}]$ and $\overline{\mathbf{G}}_i = (\boldsymbol{\Gamma}/\sin\theta_i)\boldsymbol{\Delta}^{(0)} - \boldsymbol{\Delta}^{(-1)} - \boldsymbol{\Delta}^{(1)}$. Then $\mathbf{G}_i = \sin\theta_i\boldsymbol{\Phi}_{2M+1}^i\times\overline{\mathbf{G}}_i\boldsymbol{\Phi}_{2M+1}^{i*}$ and $\mathbf{Q}_{\mathrm{opt}}^i = (\mathbf{G}_i\mathbf{G}_i^{\mathrm{H}})^{-1} = (\sin\theta_i)^2\boldsymbol{\Phi}_{2M-1}^i(\overline{\mathbf{G}}_i\overline{\mathbf{G}}_i^{\mathrm{T}})^{-1}\boldsymbol{\Phi}_{2M-1}^{i*}$. Defining a square matrix $\tilde{\mathbf{G}}_i = \{\overline{\mathbf{G}}_i$ without the last two columns$\}$, $\tilde{\mathbf{G}}_i^{-1}$ will be an upper triangular matrix and the computation of $\tilde{\mathbf{G}}_i^{-1}$ from $\tilde{\mathbf{G}}_i$ requires $(2M-1)(M-1)$ flops (one flop is roughly one multiplication plus one addition). Applying the matrix inversion formula (D.1) in [8] twice, one can have

$$(\overline{\mathbf{G}}_i\overline{\mathbf{G}}_i^{\mathrm{T}})^{-s1} = \tilde{\mathbf{G}}_i^{-\mathrm{T}}(\mathbf{I} - \mathbf{u}_1\mathbf{u}_1^{\mathrm{T}} - \mathbf{u}_2\mathbf{u}_2^{\mathrm{T}})\tilde{\mathbf{G}}_i^{-1} \tag{5}$$

where $\mathbf{u}_1$ and $\mathbf{u}_2$ are vectors computed from $\overline{\mathbf{G}}_i$ in $8M$ flops. Making use of (5), the matrix in (3) can be computed in flops of the order M^2. In contrast, by direct inversion, the optimum weighting matrix alone needs flops proportional to M^3. The computational saving can be significant if M is large.

When $\theta_i = 0°$, the ith wavefront is perpendicular to the plane of the array, and ϕ_i (μ_i as well) becomes unidentifiable (in other words, no contribution due to the ith incoming wave is represented in the transformed measurement (1)). In this case, the CRB for μ_i is undefined either. $\mathbf{Q}_{\mathrm{opt}}^i$ is ill conditioned when $\theta_i \to 0°$. A numerical instability for computing $\mathbf{Q}_{\mathrm{opt}}^i$ was encountered using MATLAB 6.1 package for elevation angle less than 3°. In simulations, elevation angle was therefore chosen from the range between 3° and 90° for the OW-ESPRIT method.

4. Performance investigation

The OW-ESPRIT method is now compared with the ESPRIT method, via numerical computation and simulation. A UCA of $N = 17$ sensors was used to receive incoming wave(s). The radius r of the array was chosen to be equal to λ and, as a result of a condition on M presented in Section 2, we chose $M = 6$. $T = 500$ snapshots of measurements were recorded. Noise variance was fixed at $\sigma^2 = 0.1$. In all the figures to follow, the theoretic and experimental variances of the ESPRIT and OW-ESPRIT estimation errors are plotted. Experimental variances were calculated from 200 simulation runs. The MUSIC theoretical variances and the CRBs in [2] are also included for comparison. Note that in [2], only the MUSIC variances and the CRBs for 2-D angles ξ_i and ϕ_i are given. For direction cosines, the MUSIC variances and the CRBs can be computed from $(\xi_i^2\mathrm{var}(\Delta\phi_i) + \mathrm{var}(\Delta\xi_i))(\lambda/2\pi r)^2$ and $(\xi_i^2\mathrm{CRB}(\phi_i) + \mathrm{CRB}(\xi_i))(\lambda/2\pi r)^2$, $i = 1, 2, \ldots, K$, respectively.

$\mathbf{Q}^i_{\mathrm{opt}}$ depends on the *unknown* direction cosine of the ith source, and in simulations was replaced by its asymptotic ($T \gg 1$) yet consistent approximate

$$\hat{\mathbf{Q}}^i_{\mathrm{opt}} = \left[\left(\boldsymbol{\Gamma}\boldsymbol{\Delta}^{(0)} - \hat{\mu}_i^0\boldsymbol{\Delta}^{(-1)} - \hat{\mu}_i 0^*\boldsymbol{\Delta}^{(1)}\right)\left(\boldsymbol{\Delta}^{(0)^{\mathrm{T}}}\boldsymbol{\Gamma}^{\mathrm{T}} - \hat{\mu}_i^0\boldsymbol{\Delta}^{(-1)^{\mathrm{T}}} - \hat{\mu}_i 0^*\boldsymbol{\Delta}^{(1)^{\mathrm{T}}}\right)\right]^{-1},$$

where $\hat{\mu}_i^0$ is a currently available estimate of μ_i, and can be obtained, e.g., via the ESPRIT method. The OW-ESPRIT direction cosine estimate $\hat{\mu}_i^{\text{OW-ESP}}$ is then chosen as the eigenvalue of the matrix in (3) (with $\mathbf{Q} = \hat{\mathbf{Q}}^i_{\mathrm{opt}}$) which is closest to $\hat{\mu}_i^0$.

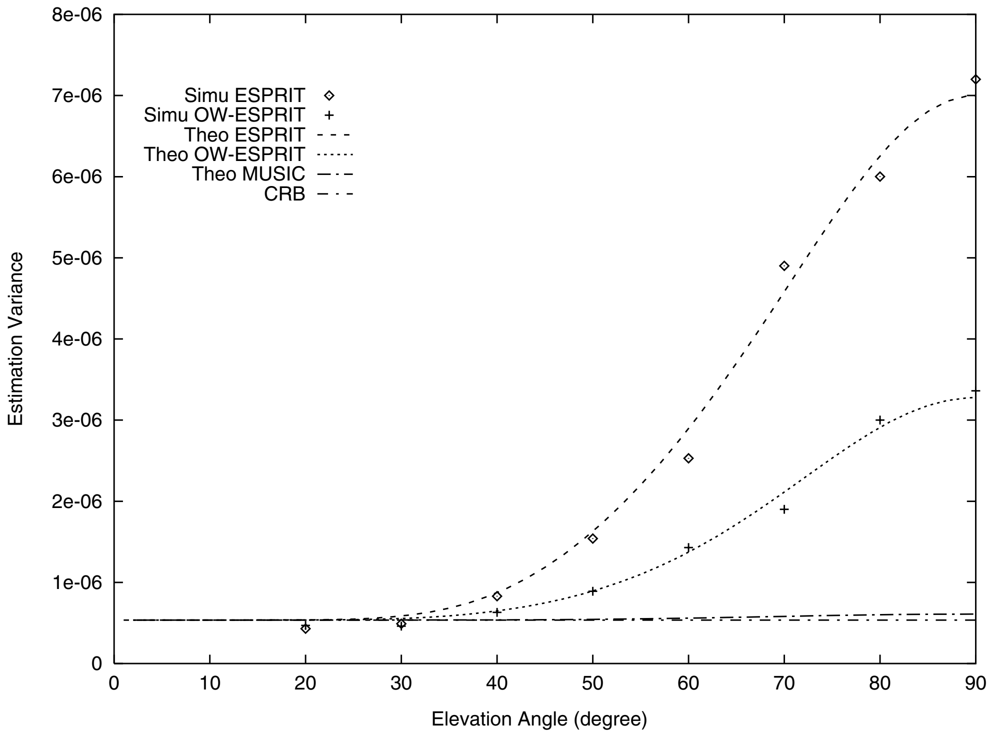

Fig. 1. Estimation variance of μ_1 versus θ_1 for single wave. $\phi_1 = 0°$ and SNR = 10 dB.

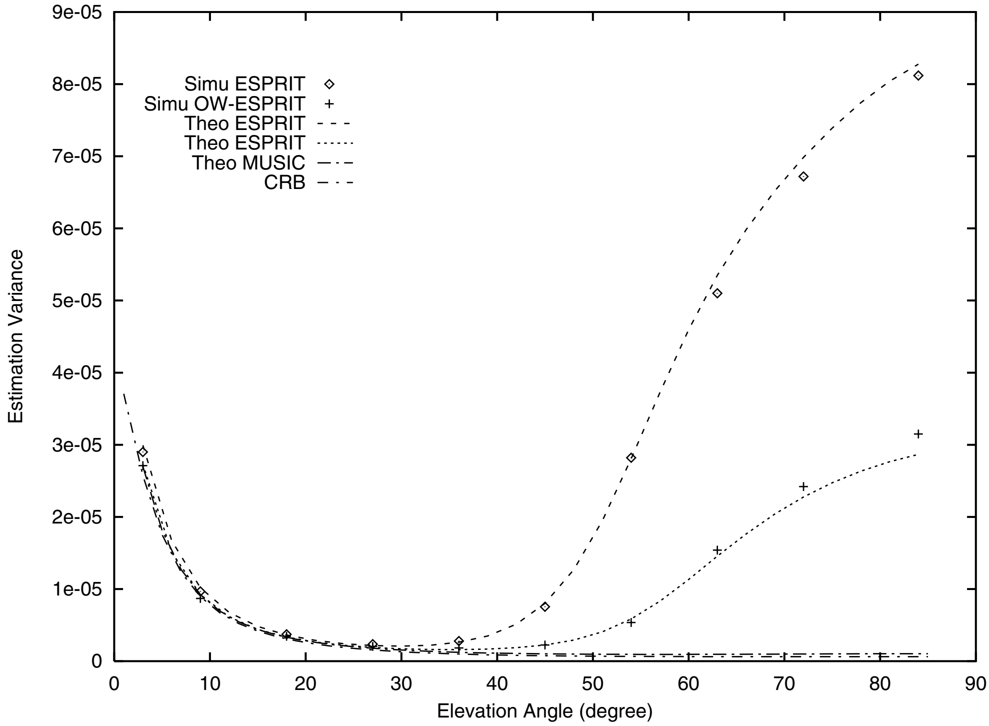

Fig. 2. Estimation variance of μ_1 versus θ_1 for two waves. $\phi_1 = 0°$, $\phi_2 = 40°$, $\theta_2 = \theta_1 + 4°$ and SNR = 10 dB.

Example 1. One source with the source covariance matrix $\mathbf{R}_s \stackrel{\text{def}}{=} E\{\mathbf{s}(t)\mathbf{s}^{\mathrm{H}}(t)\} = 1$ (SNR $\stackrel{\text{def}}{=} E\{s_1(t)s_1^*(t)\}/\sigma^2 = 10$ dB), is considered. Using our variance expression and the expression of $\mathbf{Q}^i_{\text{opt}}$, it can be shown that, in this single source case, the choice of ϕ_1 does not affect the theoretical variances of μ_1 for the ESPRIT, OW-ESPRIT and MUSIC methods, and CRB. (The CRB is even independent of θ_1.) Hence we set $\phi_1 = 0°$ and varied θ_1 from 3° to 90°. Variances of $\Delta\mu_1^{\text{ESP}}$ and $\Delta\mu_1^{\text{OW-ESP}}$ versus θ_1 are shown in Fig. 1. The improvement provided by the OW-ESPRIT method becomes significant for large elevation (low altitude) angle source, as high as 3 dB.

Example 2. Two highly correlated[1] sources, with

$$\mathbf{R}_s = \begin{bmatrix} 1 & 0.9e^{j\pi/4} \\ 0.9e^{-j\pi/4} & 1 \end{bmatrix} \quad \left(\text{SNR} = E\{s_1(t)s_1^*(t)\}/\sigma^2 = E\{s_2(t)s_2^*(t)\}/\sigma^2 = 10 \text{ dB as well}\right),$$

are considered. We set $\phi_1 = 0°$, $\phi_2 = 40°$, $\theta_2 = \theta_1 + 4°$, and varied θ_1 from 3° to 86°. Variances of $\Delta\mu_1^{\text{ESP}}$ and $\Delta\mu_1^{\text{OW-ESP}}$ versus θ_1 are shown in Fig. 2. Variances of $\Delta\mu_2^{\text{ESP}}$ and $\Delta\mu_2^{\text{OW-ESP}}$ follow a sim-

[1] The OW-ESPRIT method also works for two coherent sources provided that the source correlation is not purely real, thanks to the real operation used in (2).

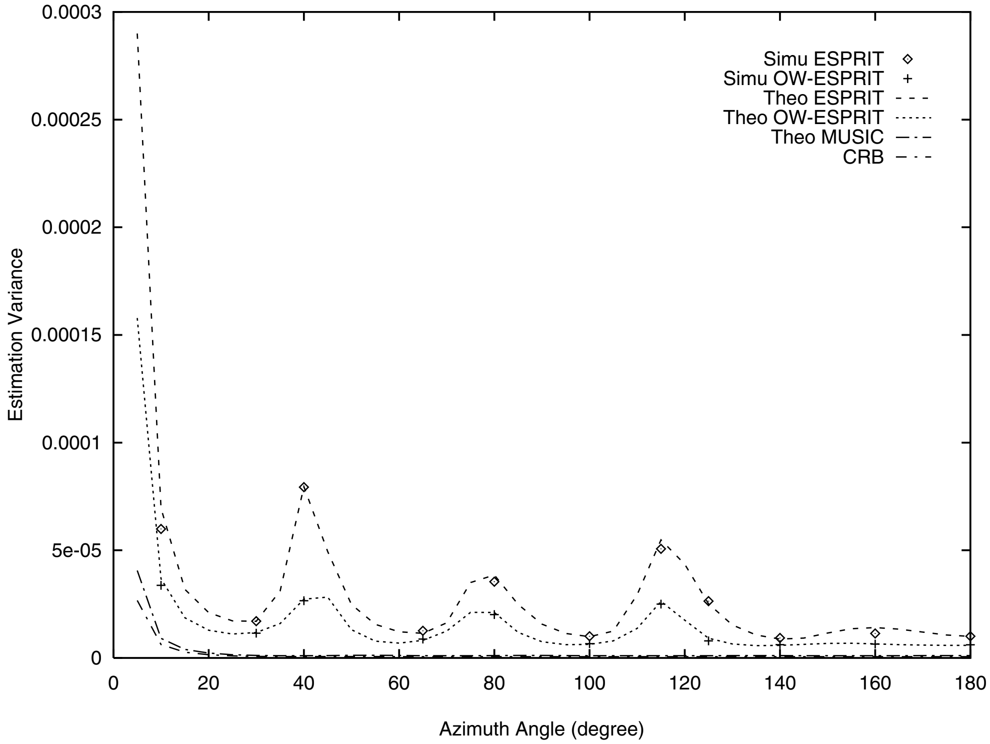

Fig. 3. Estimation variance of μ_1 versus ϕ_1 for two waves. $\phi_2 = 0°$, $\theta_1 = 80°$, $\theta_2 = 84°$ and SNR = 10 dB.

ilar pattern and are not included. Again, a significantly improved performance (>5 dB) of the OW-ESPRIT method is observed for low altitude angle sources. Nonetheless, this substantial improvement can not be always achieved for low altitude angle sources, as shown in Fig. 3. Angle setting (for Fig. 3) was: $\theta_1 = 80°$, $\theta_2 = 84°$, $\phi_2 = 0°$, and ϕ_1 being varied from 0° to 360°. Variances of $\Delta\mu_1^{\mathrm{ESP}}$ and $\Delta\mu_1^{\mathrm{OW\text{-}ESP}}$ versus ϕ_1 are shown in Fig. 3. For some azimuth angles, the improvement is just 1.5 dB.

Finally observe that the OW-ESPRIT method is still inferior to the MUSIC method, in all the cases considered. The computationally demanding MUSIC method may be needed if more accurate estimates are expected. A fast version of the MUSIC method via iterative polynomial rooting is available in [10].

5. Conclusions

An optimally weighted ESPRIT method for estimating direction cosine of 2-D angles has been proposed. As shown both theoretically and experimentally, the method provides more accurate direction cosine estimates than the existing (non-weighted) ESPRIT method. A significant improvement can be achieved by the method, for a single low-altitude source within the entire

360° azimuthal range. Potential applications are early warning of a low-flying intruder and tracking of a mobile far away from the base station.

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