Signal Processing 128 (2016) 123–130

Contents lists available at ScienceDirect

Signal Processing

journal homepage: www.elsevier.com/locate/sigpro

Short communication

# Fourth-order cumulants-based sparse representation approach for DOA estimation in MIMO radar with unknown mutual coupling

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ARTICLE INFO

*Article history:*
Received 11 December 2015
Received in revised form
17 February 2016
Accepted 27 March 2016
Available online 31 March 2016

*Keywords:*
Multiple-input multiple-output radar
Direction of arrival estimation
Sparse representation
Mutual coupling
Fourth-order cumulants

ABSTRACT

In this paper, a sparse representation approach based on fourth-order cumulants (FOC) is proposed for direction of arrival (DOA) estimation in monostatic multiple-input multiple-output (MIMO) radar with unknown mutual coupling. For applying the sparse representation theory successfully, exploiting the special banded symmetric Toeplitz structure of mutual coupling matrices (MCM) in both transmit array and receive array, the unknown MCM in received data can be turned into a diagonal one to eliminate the mutual coupling. Then based on the new received data, a reduced dimensional transformation matrix is formulated, and the proposed method further constructs a FOC matrix with special formation, which reduce the computational complexity of sparse signal reconstruction. Finally a reweighted $l_1$-norm constraint minimization sparse representation framework is designed, and the DOAs can be obtained by finding the non-zero rows in the recovered matrix. Owing to utilizing the fourth-order cumulants and reweighted sparse representation framework, compared with ESPRIT-Like, FOC-MUSIC and $l_1$-SVD algorithms, the proposed method performs well in both white and colored Gaussian noise conditions, meanwhile it has higher angular resolution and better angle estimation performance. Simulation results verify the effectiveness and advantages of the proposed method.



## 1. Introduction

Constituted of a novel array configuration and the transmitted orthogonal waveforms, MIMO radar has attracted increasing attention because of the fact that it has a large number of potential advantages over conventional phased-array radar [1], such as higher resolution and better parameter identifiability. MIMO radar can be divided into two categories, one is known as statistical MIMO radar, and the other is colocated MIMO radar [2]. The former is large antennas spaced to obtain the spatial gain, whereas the latter is closed antennas spaced to form a large aperture of virtual array, which obtains higher spatial resolution and more degrees of freedom. Colocated MIMO radar includes bistatic MIMO radar and monostatic MIMO radar. In bistatic MIMO radar, transmit array and receive array are mutually separated away but they are closed to each other in the monostatic counterpart. In this paper, we investigate angle estimation problem in monostatic MIMO radar.

Angle estimation plays an important role in array signal processing and radar applications. Aiming at the DOA estimation in MIMO radar, a host of literatures have appeared, which mainly exploit the resemblance of signal model between MIMO radar and conventional array. The most representative algorithms are subspace-based angle estimation methods, which are based on angle searching or rotational invariance technique. More specifically, they include MUSIC, Capon, ESPRIT and so forth. Both MUSIC and Capon algorithms have tremendous computational complexity and are detrimental for real-time signal processing. ESPRIT algorithm has high computation efficiency and better angle estimation performance. The derived algorithms include RD-Capon, RD-ESPRIT and Unitary ESPRIT, which provide lower calculation burden than ESPRIT. In [3], the transmit beamspace energy focusing technique is proposed with the input signal-to-noise ratio (SNR) gain maximized. Unfortunately, when mutual coupling is considered, subspace-based methods mentioned above encounter performance degradation and even fail to work [4]. To eliminate the influence of mutual coupling, some methods have been developed. A MUSIC-Like [5] algorithm is presented with the angles being estimated by spatial peak search, and a Root-MUSIC-based [6] algorithm is developed with lower computational complexity. In [7], with unknown mutual coupling an ESPRIT-Like algorithm is proposed with good angle estimation performance, meanwhile it avoids the spatial peak search.

However, during the DOA estimation in array signal processing, it often experiences the circumstance of spatially Gaussian colored noises rather than white noises, and the algorithms mentioned

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http://dx.doi.org/10.1016/j.sigpro.2016.03.019
0165-1684/

above suffer further angle estimation performance reduction. The application of high order cumulants can solve this problem, because the fourth-order cumulants can suppress Gaussian noises of arbitrary covariance [8]. Based on this specialty, some algorithms are proposed such as FOC-MUSIC [9], FOC-ESPRIT [10] and so on. In [11], with mutual coupling an improved FOC-MUSIC algorithm is presented, which provides good performance with both white and colored Gaussian noises.

Recently, the emerging sparse representation field has attracted increasing attention in the areas of signal analysis, and it can be applied for DOA estimation by finding the sparsest representation of the data. All simulation results show that compared with conventional subspace-based algorithms, sparse representation-based approaches have many noteworthy advantages and adapt better to challenging circumstances, in addition, they provide higher angle resolution and rely less on the prior information of incident signal number. A few algorithms have been developed for angle estimation, for instance, $l_1$-SVD (singular valve decomposition) algorithm [12], FOCUSS [13], $l_1$-SRACV [14] and W-$l_1$-SRACV [15] algorithms. W-$l_1$-SRACV has a super resolution by designing a weighted $l_1$-norm minimization framework with array covariance vectors. It is worth noting that estimation performance degradation occurs when mutual coupling is considered or noises are colored Gaussian in the algorithms mentioned above. In [16], a revised $l_1$-SVD algorithm has been proposed for angle estimation in passive array, and the unknown mutual coupling is eliminated. In [17], with mutual coupling a outstanding sparse representation algorithm is proposed, which involves single measurement vector problem and refined maximum likelihood estimation procedure, however, it is invalid with colored Gaussian noise.

In this paper, a fourth-order cumulants-based sparse representation approach for DOA estimation in monostatic MIMO radar with unknown mutual coupling is proposed. Firstly, the new method exploits the special banded symmetric Toeplitz structure of MCM in both transmit and receive arrays to make the unknown MCM in received data change into a diagonal one, which eliminates the effect of mutual coupling. Secondly, for reducing the computational complexity, a reduced dimensional transformation matrix is formulated to reduce the dimension of steering matrix by the linear transformation. Thirdly, the proposed method constructs a FOC matrix with special formation, which further reduces the calculation burden of sparse signal reconstruction. Finally, a reweighted $l_1$-norm constraint minimization sparse representation framework is designed, and the DOAs can be estimated by finding the non-zero rows in the recovered matrix. Owing to utilizing the FOC and reweighted sparse representation scheme, the proposed method provides precise DOA estimation, and it has higher angular resolution and better angle estimation performance than conventional algorithms for both white and colored Gaussian noises.

*Notation*: $(\cdot)^{\mathrm{H}}$, $(\cdot)^{\mathrm{T}}$, $(\cdot)^{-1}$ and $(\cdot)^{*}$ denote conjugate-transpose , transpose, inverse and conjugate, respectively. $\otimes$ and $\odot$ denote the Kronecker and the Khatri–Rao product operators, respectively. $\mathbf{I}_K$ denotes a $K \times K$ dimensional unit matrix. $\|\cdot\|_1$ and $\|\cdot\|_F$ denote the $l_1$ norm and Frobenius norm, respectively. $\mathrm{vec}(\cdot)$ denotes the vectorization operator.

## 2. Data model

Consider a narrowband monostatic MIMO radar system equipped with a transmit array of $M$ antennas and a receive array of $N$ antennas, and the arrays are both half-wavelength spaced uniform linear arrays (ULAs). Transmit and receive arrays are closely located in monostatic MIMO radar, so that for a target modeled as a point scatterer in the far-field, both the spatial angles can be seen as the same (i.e. direction of arrival (DOA)) by the arrays. The transmit array utilizes the antennas to transmit $M$ different narrowband orthogonal waveforms, which have the identical bandwidth and center frequency. Assume that the number of the targets is $P$, and let $\theta_p$ denote the DOA of the $p$th target. Under the circumstance that there are $K = k + 1$ nonzero mutual coupling coefficients and $\min\{M, N\} > 2k$, the effect of mutual coupling in both transmit array and receive array is considered. Consequently, the output of the matched filters at the receiver at a certain snapshot can be expressed as

$$\mathbf{x}(t) = \bar{\mathbf{A}}\mathbf{s}(t) + \mathbf{n}(t) \tag{1}$$

where $\mathbf{x}(t) \in \mathbb{C}^{MN\times 1}$, $\mathbf{s}(t) = [s_1(t), s_2(t), \ldots, s_P(t)]^{\mathrm{T}} \in \mathbb{C}^{P\times 1}$ is the non-Gaussian signal, $\mathbf{n}(t) \in \mathbb{C}^{MN\times 1}$ is the white or colored Gaussian noise vector with zero-mean. $\mathbf{a}_t(\theta_p) = [1, e^{j\pi \sin(\theta_p)}, e^{j\pi 2\sin(\theta_p)}, \ldots, e^{j\pi(M-1)\sin(\theta_p)}]^{\mathrm{T}}$, $p = 1, 2, \ldots, P$ and $\mathbf{a}_r(\theta_p) = [1, e^{j\pi \sin(\theta_p)}, e^{j\pi 2\sin(\theta_p)}, \ldots, e^{j\pi(N-1)\sin(\theta_p)}]^{\mathrm{T}}$, $p = 1, 2, \ldots, P$ are transmit and receive steering vectors, respectively. $\bar{\mathbf{A}} = [\mathbf{C}_t\mathbf{a}_t(\theta_1) \otimes \mathbf{C}_r\mathbf{a}_r(\theta_1), \ldots, \mathbf{C}_t\mathbf{a}_t(\theta_P) \otimes \mathbf{C}_r\mathbf{a}_r(\theta_P)] \in \mathbb{C}^{MN\times P}$, where $\mathbf{C}_t$ and $\mathbf{C}_r$ are mutual coupling matrices of the transmit array and the receive array, which are banded symmetric Toeplitz matrices and can be given by

$$\begin{aligned}\mathbf{C}_t &= \mathrm{Toeplitz}\{[1, c_{t1}, \ldots, c_{tk}, 0, \ldots, 0]\} \in \mathbb{C}^{M\times M}\\ \mathbf{C}_r &= \mathrm{Toeplitz}\{[1, c_{r1}, \ldots, c_{rk}, 0, \ldots, 0]\} \in \mathbb{C}^{N\times N}\end{aligned} \tag{2}$$

where $c_{ij}$ $(i = t, r; j = 0, 1, \ldots, k)$ is the non-zero mutual coupling coefficient. The behavior of mutual coupling is affected by many factors, however, the mutual coupling coefficients hold the property that they are factors concerned with the distance between two elements and satisfy $0 < |c_{ik}| < \cdots < |c_{i1}| < |c_{i0}| = 1$. It is worth mentioning that the magnitude of the complex coupling coefficients decreases quite fast as the distance increases [18], consequently, just a few mutual coupling coefficients are non-zero ones, moreover, the effect of mutual coupling corresponding to $c_{ij}$ attenuates quickly with $j$ increasing.

On the basis of the construction features of $\bar{\mathbf{A}}$ and the property of Kronecker product operator, the receive data in MIMO radar with unknown mutual coupling can be rewritten as

$$\begin{aligned}\mathbf{x}(t) &= (\mathbf{C}_t \otimes \mathbf{C}_r)[\mathbf{a}_t(\theta_1) \otimes \mathbf{a}_r(\theta_1), \ldots, \mathbf{a}_t(\theta_P) \otimes \mathbf{a}_r(\theta_P)]\mathbf{s}(t) + \mathbf{n}(t)\\ &= \mathbf{CAs}(t) + \mathbf{n}(t)\end{aligned} \tag{3}$$

where $\mathbf{C} = (\mathbf{C}_t \otimes \mathbf{C}_r) \in \mathbb{C}^{MN\times MN}$, $\mathbf{A} = [\mathbf{a}(\theta_1), \ldots, \mathbf{a}(\theta_P)] \in \mathbb{C}^{MN\times P}$ with the steering vector $\mathbf{a}(\theta_p) = \mathbf{a}_t(\theta_p) \otimes \mathbf{a}_r(\theta_p)$, $p = 1, 2, \ldots, P$. Some assumptions about signals and noises are brought in:

1. The signals are stationary non-Gaussian with zero-mean and independent with each other.
2. The noises are Gaussian with zero-mean and maybe colored and spatially correlated.
3. The noises and the signals are mutually statistically independent.

According to the assumptions shown above and the definition of cumulants, with constructing a FOC matrix $\mathbf{C}_x$ by the form in [11], the traditional fourth-order cumulant matrix of $\mathbf{x}$ with $J$ snapshots is given by

$$\begin{aligned}\mathbf{C}_x &= \sum_{p=1}^{P} [(\mathbf{Ca}(\theta_p)) \otimes (\mathbf{Ca}(\theta_p))^*]\beta_p[(\mathbf{Ca}(\theta_p)) \otimes (\mathbf{Ca}(\theta_p))^*]^{\mathrm{H}}\\ &= [(\mathbf{CA}) \odot (\mathbf{CA})^*]\mathbf{C}_s[(\mathbf{CA}) \odot (\mathbf{CA})^*]^{\mathrm{H}}\end{aligned} \tag{4}$$

where $\mathbf{C}_x \in \mathbb{C}^{M^2N^2\times M^2N^2}$, $\mathbf{C}_s = \mathrm{diag}(\beta_1, \beta_2, \ldots, \beta_P)$ with $\beta_p$ being the fourth-order cumulant of signal $s_p$. Obviously, in its current form of Eq. (4), the sparse representation manner cannot be applied successfully because of the existence of unknown mutual coupling matrix $\mathbf{C}$. As a result, we have to adopt a different approach such as jointly optimizing $\mathbf{C}$ and the sparse matrix. However, it is a non-convex optimization problem, which cannot be solved in polynomial time. To avoid solving such a complex optimization problem, an efficient sparse representation framework based on fourth-order cumulants is proposed for DOA estimation in MIMO radar with unknown mutual coupling.

## 3. FOC-based sparse representation approach for DOA estimation in the presence of mutual coupling

In this section, the details of removing the effect of unknown mutual coupling in both transmit and receive arrays for DOA estimation in monostatic MIMO radar are introduced. In terms of $\mathbf{C}_t\mathbf{a}_t(\theta_p)$, $p = 1, 2, \ldots, P$, exploiting the banded symmetric Toeplitz structure characteristic of the mutual coupling matrix $\mathbf{C}_t$ in Eq. (2), the selection matrix $\mathbf{J}_1 = [\mathbf{0}_{\bar{M}\times k}\ \mathbf{I}_{\bar{M}}\ \mathbf{0}_{\bar{M}\times k}] \in \mathbb{C}^{\bar{M}\times M}$ is defined to choose the central $\bar{M} = M - 2k$ rows of $\mathbf{C}_t$. Let $z = e^{j\pi \sin(\theta_p)}$, we have

$$\mathbf{J}_1\mathbf{C}_t\mathbf{a}_t(\theta_p) = \begin{pmatrix} c_{tk} + c_{t(k-1)}z + \cdots + z^k + c_{t1}z^{k+1} + \cdots + c_{tk}z^{2k} \\ c_{tk}z + c_{t(k-1)}z^2 + \cdots + z^{k+1} + \cdots + c_{tk}z^{k+k+1} \\ \vdots \\ c_{tk}z^{\bar{M}-2} + \cdots + z^{\bar{M}+k-2} + \cdots + c_{tk}z^{\bar{M}+k+k-2} \\ c_{tk}z^{\bar{M}-1} + \cdots + z^{\bar{M}+k-1} + \cdots + c_{tk}z^{\bar{M}+k+k-1} \end{pmatrix}$$

$$= (c_{tk} + c_{t(k-1)}z + \cdots + z^k + \cdots + c_{tk}z^{k+k}) \begin{pmatrix} 1 \\ z \\ z^2 \\ \vdots \\ z^{\bar{M}-1} \end{pmatrix} = c_t(\theta_p)\tilde{\mathbf{a}}_t(\theta_p),$$

$$p = 1, 2, \ldots, P \tag{5}$$

where $c_t(\theta_p) = z^k[1 + \sum_{i=1}^{k} c_{ti}(z^i + z^{-i})]$ is a scalar and $\tilde{\mathbf{a}}_t(\theta_p) = (1\ z\ z^2\ \cdots\ z^{\bar{M}-1})^{\mathrm{T}}$. Analogously, another selection matrix is defined as $\mathbf{J}_2 = [\mathbf{0}_{\bar{N}\times k}\ \mathbf{I}_{\bar{N}}\ \mathbf{0}_{\bar{N}\times k}] \in \mathbb{C}^{\bar{N}\times N}$ with $\bar{N} = N - 2k$, similar to Eq. (5), it directly results in

$$\mathbf{J}_2\mathbf{C}_r\mathbf{a}_r(\theta_p) = c_r(\theta_p)\tilde{\mathbf{a}}_r(\theta_p) \tag{6}$$

where $c_r(\theta_p) = z^k[1 + \sum_{i=1}^{k} c_{ri}(z^i + z^{-i})]$, $\tilde{\mathbf{a}}_r(\theta_p) = (1\ z\ \cdots\ z^{\bar{N}-1})^{\mathrm{T}}$, and hence we further have

$$(\mathbf{J}_1 \otimes \mathbf{J}_2)\mathbf{C}\mathbf{a}(\theta_p) = (\mathbf{J}_1 \otimes \mathbf{J}_2)[\mathbf{C}_t\mathbf{a}_t(\theta_p) \otimes \mathbf{C}_r\mathbf{a}_r(\theta_p)] = [\mathbf{J}_1\mathbf{C}_t\mathbf{a}_t(\theta_p)] \otimes [\mathbf{J}_2\mathbf{C}_r\mathbf{a}_r(\theta_p)] = c(\theta_p)\tilde{\mathbf{a}}(\theta_p), \quad p = 1, 2, \ldots, P \tag{7}$$

in which $c(\theta_p) = c_t(\theta_p)c_r(\theta_p)$ and $\tilde{\mathbf{a}}(\theta_p) = \tilde{\mathbf{a}}_t(\theta_p) \otimes \tilde{\mathbf{a}}_r(\theta_p)$. Let $\mathbf{J} = \mathbf{J}_1 \otimes \mathbf{J}_2 \in \mathbb{C}^{\bar{M}\bar{N}\times MN}$, multiplying $\mathbf{J}$ on the left of $\mathbf{x}(t)$ it can be derived that

$$\mathbf{y}(t) = \mathbf{J}[\mathbf{C}\mathbf{A}\mathbf{s}(t) + \mathbf{n}(t)] = \tilde{\mathbf{A}}\mathbf{D}\mathbf{s}(t) + \tilde{\mathbf{n}}(t) \tag{8}$$

where $\mathbf{y}(t)$ is the new received data vector, diagonal matrix $\mathbf{D} = \mathrm{diag}[|c(\theta_1)|, \ldots, |c(\theta_P)|]$ contains the information of unknown mutual coupling, $\tilde{\mathbf{A}} = [\tilde{\mathbf{a}}(\theta_1), \ldots, \tilde{\mathbf{a}}(\theta_P)] \in \mathbb{C}^{\bar{M}\bar{N}\times P}$ and $\tilde{\mathbf{n}}(t) = \mathbf{J}\mathbf{n}(t)$ are the new steering matrix and noise, respectively. Since the mutual coupling matrix $\mathbf{C}$ turns into a diagonal one, there exists no influence on the steering matrix any more, namely, the effect of mutual coupling is compensated in Eq. (8).

In order to reduce the burden in the following computation of fourth-order cumulants, a reduced dimensional transformation is formulated based on the unique construction of monostatic MIMO radar. The column of $\tilde{\mathbf{A}}$ is $\tilde{\mathbf{a}}_t(\theta_p) \otimes \tilde{\mathbf{a}}_r(\theta_p) \in \mathbb{C}^{\bar{M}\bar{N}\times 1}$, $p = 1, 2, \ldots, P$, and can be written as

$$\tilde{\mathbf{a}}(\theta_p) = (1, \ldots, z^{\bar{N}-1}; z, \ldots, z^{\bar{N}}; \ldots; z^{\bar{M}-1}, \ldots, z^{\bar{M}+\bar{N}-2})^{\mathrm{T}} \tag{9}$$

It can be discovered that many repeated terms are contained in $\tilde{\mathbf{a}}(\theta_p)$, which satisfy

$$\tilde{\mathbf{a}}(\theta_p) = \mathbf{G}\mathbf{b}(\theta_p) \tag{10}$$

in which the detailed expressions of $\mathbf{G}$ and $\mathbf{b}(\theta_p)$ are

$$\mathbf{G} = [\mathbf{L}_0^{\mathrm{T}}, \mathbf{L}_1^{\mathrm{T}}, \ldots, \mathbf{L}_{\bar{M}-1}^{\mathrm{T}}]^{\mathrm{T}}$$
$$\mathbf{b}(\theta_p) = [1, \exp(j\pi \sin\theta_p), \ldots, \exp(j\pi(\bar{M} + \bar{N} - 2)\sin\theta_p)]^{\mathrm{T}} \tag{11}$$

where $\mathbf{b}(\theta_p) \in \mathbb{C}^{(\bar{M}+\bar{N}-1)\times 1}$ and $\mathbf{G} \in \mathbb{C}^{\bar{M}\bar{N}\times(\bar{M}+\bar{N}-1)}$ with $\mathbf{L}_m = [\mathbf{0}_{\bar{N}\times m}, \mathbf{I}_{\bar{N}}, \mathbf{0}_{\bar{N}\times(\bar{M}-m-1)}] \in \mathbb{C}^{\bar{N}\times(\bar{M}+\bar{N}-1)}$, $m = 0, 1, \ldots, \bar{M} - 1$ . The repeated terms will result in a mass of redundant information in the following operation of fourth-order cumulants, and then the tremendous calculation burden in sparse signal reconstruction, which may even cause the invalidation for DOA estimation in sparse representation algorithms. To solve this problem, a reduced dimensional transformation is constructed as follows:

$$\mathbf{J}_3 = (\mathbf{G}^{\mathrm{H}}\mathbf{G})^{(-\frac{1}{2})}\mathbf{G}^{\mathrm{H}} \tag{12}$$

Accordingly, multiplying the reduced dimensional transformation matrix $\mathbf{J}_3$ with the data vector $\mathbf{y}(t)$, we have

$$\tilde{\mathbf{y}}(t) = (\mathbf{G}^{\mathrm{H}}\mathbf{G})^{(\frac{1}{2})}\mathbf{B}\mathbf{D}\mathbf{s}(t) + \mathbf{J}_3\tilde{\mathbf{n}}(t) = \mathbf{F}\mathbf{B}\mathbf{D}\mathbf{s}(t) + \bar{\tilde{\mathbf{n}}}(t) \tag{13}$$

where $\mathbf{B} = [\mathbf{b}(\theta_1), \mathbf{b}(\theta_2), \ldots, \mathbf{b}(\theta_P)] \in \mathbb{C}^{(\bar{M}+\bar{N}-1)\times P}$, $\tilde{\mathbf{y}}(t)$ and $\bar{\tilde{\mathbf{n}}}(t)$ are the new data vector and noise after compensating the mutual coupling errors and wiping off the redundant rows, respectively. At the same time, $\mathbf{F}$ is calculated as

$$\mathbf{F} = \mathrm{diag}[1, \sqrt{2}, \ldots, \underbrace{\delta, \ldots, \delta}_{|\bar{M}-\bar{N}|+1}, \ldots, \sqrt{2}, 1] \tag{14}$$

where $\delta = \min(\sqrt{\bar{M}}, \sqrt{\bar{N}})$. It is worth mentioning that the reduced dimensional transformation not only remarkably reduces the dimension of the received data matrix from $\bar{M}\bar{N} \times J$ to $(\bar{M} + \bar{N} - 1) \times J$, but also avoids the additional spatial colored noise.

After eliminating the effect of unknown mutual coupling and reducing the dimension of received data, the fourth-order cumulant $c_{\tilde{y}}(k_1, k_2, k_3, k_4)$ of data $\tilde{\mathbf{y}}$ is defined as

$$\begin{aligned}\mathrm{cum}\{\tilde{\mathbf{y}}_{k_1}, \tilde{\mathbf{y}}_{k_2}^*, \tilde{\mathbf{y}}_{k_3}, \tilde{\mathbf{y}}_{k_4}^*\} &= \mathrm{E}(\tilde{\mathbf{y}}_{k_1}\tilde{\mathbf{y}}_{k_2}^*\tilde{\mathbf{y}}_{k_3}\tilde{\mathbf{y}}_{k_4}^*) - \mathrm{E}(\tilde{\mathbf{y}}_{k_1}\tilde{\mathbf{y}}_{k_2}^*)\mathrm{E}(\tilde{\mathbf{y}}_{k_3}\tilde{\mathbf{y}}_{k_4}^*) - \mathrm{E}(\tilde{\mathbf{y}}_{k_1}\tilde{\mathbf{y}}_{k_3})\mathrm{E}(\tilde{\mathbf{y}}_{k_2}^*\tilde{\mathbf{y}}_{k_4}^*) \\ &\quad - \mathrm{E}(\tilde{\mathbf{y}}_{k_1}\tilde{\mathbf{y}}_{k_4}^*)\mathrm{E}(\tilde{\mathbf{y}}_{k_2}^*\tilde{\mathbf{y}}_{k_3})\end{aligned} \tag{15}$$

where $\tilde{\mathbf{y}}_{k_1}$ is the $k_1$th element in $\tilde{\mathbf{y}}$ and the indices $k_2$, $k_3$ and $k_4$ are similarly defined, $1 \le k_1, k_2, k_3, k_4 \le \bar{M} + \bar{N} - 1$. In fact each item in the estimation of $c_{\tilde{y}}(k_1, k_2, k_3, k_4)$ is obtained from

$$\mathrm{E}(\tilde{\mathbf{y}}_{k_1}\tilde{\mathbf{y}}_{k_2}^*\tilde{\mathbf{y}}_{k_3}\tilde{\mathbf{y}}_{k_4}^*) = \frac{1}{J}\sum_{t=1}^{J}\tilde{\mathbf{y}}_{k_1}(t)\tilde{\mathbf{y}}_{k_2}^*(t)\tilde{\mathbf{y}}_{k_3}(t)\tilde{\mathbf{y}}_{k_4}^*(t)$$
$$\mathrm{E}(\tilde{\mathbf{y}}_{k_1}\tilde{\mathbf{y}}_{k_2}^*) = \frac{1}{J}\sum_{t=1}^{J}\tilde{\mathbf{y}}_{k_1}(t)\tilde{\mathbf{y}}_{k_2}^*(t) \tag{16}$$

According to Eq. (15), we design the fourth-order cumulant matrix $\mathbf{C}_{\tilde{y}}$ as follows:

$$\mathbf{C}_{\tilde{y}}(k_2, k_1) = \mathrm{cum}\{\tilde{\mathbf{y}}_{k_2}, \tilde{\mathbf{y}}_{k_1}^*, \tilde{\mathbf{y}}_{k_1}, \tilde{\mathbf{y}}_{k_1}^*\} \tag{17}$$

where $\mathbf{C}_{\tilde{y}}(k_2, k_1)$ stands for the element in $\mathbf{C}_{\tilde{y}}$ whose row and column are $k_2$ and $k_1$, respectively. Based on the property of the fourth-order cumulants and the assumptions that the signals $s_i$ $(i = 1, 2, \ldots, P)$ are independent with each other and so forth, it can be derived that

$$\mathbf{C}_{\bar{y}}(k_2, k_1) = \sum_{p=1}^{P} \mathbf{Fb}_{k_2}(\theta_p)\mathbf{Fb}^*_{k_1}(\theta_p)\mathbf{Fb}_{k_1}(\theta_p)\mathbf{Fb}^*_{k_1}(\theta_p)|c(\theta_p)|^4\beta_p \tag{18}$$

where $\mathbf{b}_{k_2}(\theta_p)$ and $\mathbf{b}_{k_1}(\theta_p)$ are the $k_2$th and the $k_1$th elements in the vector $\mathbf{b}(\theta_p)$, respectively, and $\beta_p = \mathrm{cum}(s_p, s_p^*, s_p, s_p^*)$ is the fourth-order cumulant of signal $s_p$, $p = 1, 2, \dots, P$. With constructing a FOC matrix $\mathbf{C}_{\bar{y}}$, the $(\bar{M} + \bar{N} - 1)^2$ values are organized compatibly. Since it consists of all possible permutations of two indices $k_1$ and $k_2$, $\mathbf{C}_{\bar{y}}$ can be derived as

$$\mathbf{C}_{\bar{y}} = \sum_{k_2=1}^{\bar{M}+\bar{N}-1}\sum_{k_1=1}^{\bar{M}+\bar{N}-1}\sum_{p=1}^{P} \mathbf{Fb}_{k_2}(\theta_p)\mathbf{Fb}^*_{k_1}(\theta_p)|c(\theta_p)|^4\beta_p|\mathbf{Fb}_{k_1}(\theta_p)|^2 = \mathbf{FBD}_{mc}\mathbf{C}_s\mathbf{S}_{FB}^{\mathrm{H}} \tag{19}$$

in which $\mathbf{C}_{\bar{y}} \in \mathbb{C}^{(\bar{M}+\bar{N}-1)\times(\bar{M}+\bar{N}-1)}$, diagonal matrixes $\mathbf{D}_{mc}$ and $\mathbf{C}_s$ are expressed as $\mathbf{D}_{mc} = \mathrm{diag}(|c(\theta_1)|^4, \dots, |c(\theta_P)|^4)$ and $\mathbf{C}_s = \mathrm{diag}(\beta_1, \beta_2, \dots, \beta_P)$. Meanwhile, the element of the $i$th row and $j$th column in $\mathbf{S}_{FB}$ is

$$\mathbf{S}_{FB}(i, j) = |[\mathbf{Fb}(\theta_j)]_i|^2[\mathbf{Fb}(\theta_j)]_i \tag{20}$$

where $1 \le i \le (\bar{M} + \bar{N} - 1)$, $1 \le j \le P$, $[\mathbf{Fb}(\theta_j)]_i$ is the $i$th value entry in $\mathbf{Fb}(\theta_j)$. In order to further reduce the computational complexity of the following sparse representation scheme for DOA estimation, the FOC observation $\mathbf{C}_{\bar{y}}$ can be translated from a $(\bar{M} + \bar{N} - 1) \times (\bar{M} + \bar{N} - 1)$ dimensional matrix into a $(\bar{M} + \bar{N} - 1) \times P$ one, that is

$$\tilde{\mathbf{C}}_{\bar{y}} = \mathbf{FBD}_{mc}\mathbf{C}_s\mathbf{S}_{FB}^{\mathrm{H}}\mathbf{V}_s \tag{21}$$

where $\tilde{\mathbf{C}}_{\bar{y}} = \mathbf{C}_{\bar{y}}\mathbf{V}_s$ contains the vast majority of the signal power, which can be used to estimate the DOAs accurately instead of $\mathbf{C}_{\bar{y}}$. The components of $\mathbf{V}_s$ are the right singular vectors corresponding to the $P$ largest singular values. In addition, the left and the right singular vectors make up $\mathbf{U}$ and $\mathbf{V}$, respectively, which satisfy $\mathbf{C}_{\bar{y}} = \mathbf{U}\boldsymbol{\Lambda}\mathbf{V}^{\mathrm{H}}$ with $\boldsymbol{\Lambda} = \mathrm{diag}(\omega_1, \omega_2, \dots, \omega_{\bar{M}+\bar{N}-1})$ being the diagonal matrix, and $\omega_1 \ge \omega_2 \ge \cdots \ge \omega_{\bar{M}+\bar{N}-1}$ are singular values.

Let $\mathbf{T} = \mathbf{D}_{mc}\mathbf{C}_s\mathbf{S}_{FB}^{\mathrm{H}}\mathbf{V}_s$, $\mathbf{D}_{mc}$ and $\mathbf{C}_s$ are both diagonal matrices. When certain rows of original recovered matrix model without mutual coupling are zero vectors, the corresponding rows in $\mathbf{T}$ are zero ones inevitably even if the components of $\mathbf{D}_{mc}$ and $\mathbf{S}_{FB}^{\mathrm{H}}$ are entirely unknown. It indicates that the introduction of $\mathbf{D}_{mc}$, $\mathbf{S}_{FB}^{\mathrm{H}}$ and $\mathbf{V}_s$ never change the sparse solution, and hence in its current condition the sparse representation manner can be applied successfully to estimate DOAs in Eq. (21) after eliminating the mutual coupling by the linear transformation in Eq. (8).

To use the sparse representation scheme to estimate DOA, let $\{\hat{\theta}_l\}_{l=1}^{L}$ be the discretized sampling grids of all DOAs of interest. The number of the potential DOAs will be much greater than the number of the targets, i.e. $L \gg P$. Then a dictionary $\tilde{\mathbf{B}}_{\hat{\theta}} = \mathbf{F}[\mathbf{b}(\hat{\theta}_1), \mathbf{b}(\hat{\theta}_2), \dots, \mathbf{b}(\hat{\theta}_L)] \in \mathbb{C}^{(\bar{M}+\bar{N}-1)\times L}$ is constructed for the potential DOAs. Let $\mathbf{T}_{\hat{\theta}}$ satisfy $\tilde{\mathbf{C}}_{\bar{y}} = \tilde{\mathbf{B}}_{\hat{\theta}}\mathbf{T}_{\hat{\theta}}$, when there are true targets on certain points in spatial domain, the corresponding row vectors in $\mathbf{T}_{\hat{\theta}}$ are non-zero vectors and the rest zero ones. As a consequence, $\mathbf{T}_{\hat{\theta}}$ have the same row support with the primary matrix $\mathbf{T}$, and we are able to exchange the problem of DOA estimation for the problem of finding out the locations in dictionary that correspond to the nonzero rows in $\mathbf{T}_{\hat{\theta}}$. For measuring the smallest number of the non-zero rows, a direct sparse metric is the $l_0$-norm penalty. However, the $l_0$-norm minimization is a non-convex and NP-hard problem, thereby cannot be solved. Referencing to [19], the $l_1$-norm penalty can be adopted to solve this problem reasonably. To obtain the sparse matrix $\mathbf{T}_{\hat{\theta}}$, a $l_1$-norm constrained minimization problem can be considered as

$$\min_{\mathbf{T}_{\hat{\theta}}} \| \mathbf{T}_{\hat{\theta}}^{(l_2)} \|_1, \quad \text{s. t.} \ \| \hat{\tilde{\mathbf{C}}}_{\bar{y}} - \tilde{\mathbf{B}}_{\hat{\theta}}\mathbf{T}_{\hat{\theta}} \|_F^2 \le \eta \tag{22}$$

where $[\mathbf{T}_{\hat{\theta}}^{(l_2)}]_i = \| \mathbf{T}_{\hat{\theta}}(i, :) \|_2$ and $\mathbf{T}_{\hat{\theta}}^{(l_2)} \in \mathbb{C}^{L\times 1}$. $\eta$ is a regularization parameter, by collecting $J$ snapshots $\hat{\tilde{\mathbf{C}}}_{\bar{y}}$ is the sample estimation of true value $\tilde{\mathbf{C}}_{\bar{y}}$, and $\hat{\tilde{\mathbf{C}}}_{\bar{y}}$ is obtained from Eqs. (8), (13), (16) and (21).

To enhance the $l_1$-norm penalty for better approximating to the $l_0$-norm one, a reweighted $l_1$-norm minimization sparse representation scheme is presented as follows. Dividing the over-complete dictionary $\tilde{\mathbf{B}}_{\hat{\theta}}$ into two parts by the columns, the first one is assumed to consist of the potential $P$ steering vectors that correspond to the true DOAs of targets, and the other is composed of the remaining steering vectors corresponding to the zero rows in the recovered matrix $\mathbf{T}_{\hat{\theta}}$. Thus, $\tilde{\mathbf{B}}_{\hat{\theta}}$ can be expressed as $\tilde{\mathbf{B}}_{\hat{\theta}} = [\tilde{\mathbf{B}}_{\hat{\theta}}^1, \tilde{\mathbf{B}}_{\hat{\theta}}^2]$. As a result, selecting the $(\bar{M} + \bar{N} - 1 - P)$ columns in $\mathbf{U}$ from the $(P + 1)$th column to the $(\bar{M} + \bar{N} - 1)$th one as the noise subspace $\mathbf{U}_n$, $\tilde{\mathbf{B}}_{\hat{\theta}}^{\mathrm{H}}\mathbf{U}_n$ can be written as

$$\tilde{\mathbf{B}}_{\hat{\theta}}^{\mathrm{H}}\mathbf{U}_n = \begin{bmatrix} (\tilde{\mathbf{B}}_{\hat{\theta}}^1)^{\mathrm{H}}\mathbf{U}_n \\ (\tilde{\mathbf{B}}_{\hat{\theta}}^2)^{\mathrm{H}}\mathbf{U}_n \end{bmatrix} = \begin{bmatrix} \mathbf{W}_1 \\ \mathbf{W}_2 \end{bmatrix} \tag{23}$$

in which each item in $\mathbf{W}_1$ has been proved that it tends to zero when $J \to \infty$, therefore based on this property [20] a reweighted matrix is designed as

$$\mathbf{W}_r = \mathrm{diag}[(\mathbf{W}_1^{(l_2)})^{\mathrm{T}}, (\mathbf{W}_2^{(l_2)})^{\mathrm{T}}]/\max(\mathbf{W}_2^{(l_2)}) \tag{24}$$

where $\mathbf{W}_1^{(l_2)}(i)/\max(\mathbf{W}_2^{(l_2)})$ is smaller than $\mathbf{W}_2^{(l_2)}(j)/\max(\mathbf{W}_2^{(l_2)})$, and $0 < \mathbf{W}_2^{(l_2)}(j)/\max(\mathbf{W}_2^{(l_2)}) < 1$, at the same time, $\mathbf{W}_1^{(l_2)}(i)/\max(\mathbf{W}_2^{(l_2)}) \to 0$ with the snapshots $J \to \infty$, in which $\mathbf{W}_1^{(l_2)}(i)$ and $\mathbf{W}_2^{(l_2)}(j)$ are the $i$th and the $j$th entries in the vectors $\mathbf{W}_1^{(l_2)}$ and $\mathbf{W}_2^{(l_2)}$, respectively. $\mathbf{W}_r$ can accomplish the task that large weights punish the elements who are more likely to be zeros in the sparse vector $\mathbf{T}_{\hat{\theta}}^{(l_2)}$, in contrast, small weights reserve the larger entries. In other words, the introduction of the reweighted matrix can enhance the sparse solution, and improve the accuracy of DOA estimation. Consequently, the sparse representation framework transforms into

$$\min_{\mathbf{T}_{\hat{\theta}}} \| \mathbf{W}_r\mathbf{T}_{\hat{\theta}}^{(l_2)} \|_1, \quad \text{s. t.} \ \| \hat{\tilde{\mathbf{C}}}_{\bar{y}} - \tilde{\mathbf{B}}_{\hat{\theta}}\mathbf{T}_{\hat{\theta}} \|_F \le \sqrt{\eta} \tag{25}$$

where $\eta$ is a regularization parameter that sets the error amount, and choosing the value of $\eta$ plays an important role in the final DOA estimation. Let $\Delta\tilde{\mathbf{C}}_{\bar{y}} = \hat{\tilde{\mathbf{C}}}_{\bar{y}} - \tilde{\mathbf{B}}_{\hat{\theta}}\mathbf{T}_{\hat{\theta}}$ be the FOC estimation errors, in order to obtain the distribution property of $\mathrm{vec}(\Delta\tilde{\mathbf{C}}_{\bar{y}})$, we firstly derive the distribution of $\mathrm{vec}(\Delta\mathbf{C}_{\bar{y}})$, and $\Delta\mathbf{C}_{\bar{y}} = \hat{\mathbf{C}}_{\bar{y}} - \mathbf{C}_{\bar{y}}$ with $\hat{\mathbf{C}}_{\bar{y}}$ being the sample estimation of true value $\mathbf{C}_{\bar{y}}$, thus $\Delta\tilde{\mathbf{C}}_{\bar{y}} = \Delta\mathbf{C}_{\bar{y}}\mathbf{V}_s$. On the basis of the element construction of $\hat{\mathbf{C}}_{\bar{y}}$, $\mathrm{vec}(\hat{\mathbf{C}}_{\bar{y}})$ can be considered as a set of fourth-order cumulant estimation values of $\bar{\mathbf{y}}$. In accordance with [8], for a stationary process $\bar{\mathbf{y}}$, all items $\hat{c}_{\bar{y}}(k_2, k_1, k_1, k_1)$ in the FOC estimation $\mathrm{vec}(\hat{\mathbf{C}}_{\bar{y}})$ hold the property that $\sqrt{J}[\hat{c}_{\bar{y}}(k_2, k_1, k_1, k_1) - c_{\bar{y}}(k_2, k_1, k_1, k_1)]$ satisfy the asymptotic normal distribution. Additionally, there is the fact that $\lim_{J\to\infty}[\hat{c}_{\bar{y}}(k_2, k_1, k_1, k_1)] = c_{\bar{y}}(k_2, k_1, k_1, k_1)$, therefore in the case of multiple snapshots, $\mathrm{vec}(\Delta\mathbf{C}_{\bar{y}})$ is asymptotic normal with zero-mean, that is

$$\mathrm{vec}(\Delta\mathbf{C}_{\bar{y}}) \sim \mathrm{AsN}(0, \mathbf{V}) \tag{26}$$

where $\mathrm{AsN}(\mu, \mathbf{V})$ represents the asymptotic normal distribution with mean $\mu$, and $\mathbf{V} \in \mathbb{C}^{(\bar{M}+\bar{N}-1)^2\times(\bar{M}+\bar{N}-1)^2}$ is the covariance matrix. Moreover, the covariance of FOC estimation errors is given by [8]

$$\begin{aligned}
&\mathrm{cov}\{\hat{c}_{\tilde{y}}(\tau), \hat{c}_{\tilde{y}}(\rho)\} \\
&= \frac{1}{J}\{Q_{44} - \sum_{v=1}^{3}[Q_{42}(\tau;\, k_1', k_{v1}')m_2(k_{v2}', k_{v3}') + Q_{42}(\tau;\, k_{v2}', k_{v3}') \\
&\quad m_2(k_{v1}') + Q_{42}(\rho;\, k_1, k_{v1})m_2(k_{v2}, k_{v3}) + Q_{42}(\rho;\, k_{v2}, k_{v3}) \\
&\quad m_2(k_{v1})] + \sum_{v,u=1}^{3}[Q_{22}(k_1, k_{v1};\, k_1', k_{u1}')m_2(k_{v2}, k_{v3}) \\
&\quad m_2(k_{u2}', k_{u3}') + Q_{22}(k_1, k_{v1};\, k_{u2}', k_{u3}')m_2(k_{v2}, k_{v3})m_2(k_{u1}') \\
&\quad + Q_{22}(k_{v2}, k_{v3};\, k_{u2}', k_{u3}')m_2(k_{v1})m_2(k_{u1}')]\}
\end{aligned} \tag{27}$$

in which $\tau = k_1, k_2, k_3, k_4$, $\rho = k_1', k_2', k_3', k_4'$, the abbreviations are $Q_{44} = Q_{44}(\tau, \rho)$, $m_2(\lambda) = m_2(k_1, \lambda)$ and $m_2(\lambda') = m_2(k_1', \lambda')$. The estimation can be obtained from $Q_{kn}= \lim_{J\to\infty} J\mathrm{cov}\ \{\hat{m}_k, \hat{m}_n\} = \sum_{\xi=-\infty}^{\infty}\mathrm{cov}\{f_k(t), f_{k'}(t+\xi)\}$ with $f_k(t)= \mathbf{y}_{k_1}(t)\mathbf{y}_{k_2}^*(t)\cdots\mathbf{y}_{k_k}(t),\ k \le 4$ , similarly, $f_{k'}(t) = \mathbf{y}_{k_1'}(t)\mathbf{y}_{k_2'}^*(t)\cdots\mathbf{y}_{k_n'}(t),\ n \le 4$. $k_{11} = k_2, k_{12} = k_3, k_{13} = k_4;\ k_{21} = k_3, k_{22} = k_2, k_{23} = k_4;\ k_{31} = k_4, k_{32}= k_3, k_{33} = k_2$ and the same to $k'$. Meanwhile, Eq. (16) is used to obtain the $k$th-order sample moment $\hat{m}_k$. When $k_2 = k_3 = k_4$ and $k_2' = k_3' = k_4'$, $\mathbf{V}$ is composed of $\mathbf{V}(i, j) = \mathrm{cov}\{\hat{c}_{\tilde{y}}(k_1, k_2, k_2, k_2), \hat{c}_{\tilde{y}}\ (k_1', k_2', k_2', k_2')\}$ with $\hat{c}_{\tilde{y}}$ coming from $\hat{\mathbf{C}}_{\tilde{y}}$. The relationships between $i$ and $k_1$, $k_2$ are derived as

$$\begin{aligned}
k_1 &= [i/(\bar{M} + \bar{N} - 1)]_{rem} \\
k_2 &= [i/(\bar{M} + \bar{N} - 1)]_{ceil}
\end{aligned} \tag{28}$$

where $[\cdot]_{rem}$ denotes the remainder of the value $i/(\bar{M} + \bar{N} - 1)$, and $[\cdot]_{ceil}$ represents the smallest integer that is not less than the value $i/(\bar{M} + \bar{N} - 1)$. The relationships between $j$ and $k_1'$, $k_2'$ coincide with Eq. (28). Due to $\Delta\tilde{\mathbf{C}}_{\tilde{y}} = \Delta\mathbf{C}_{\tilde{y}}\mathbf{V}_s$, exploiting the property of vector quantization operator, it can be derived that

$$\mathrm{vec}(\Delta\tilde{\mathbf{C}}_{\tilde{y}}) = [(\mathbf{V}_s)^{\mathrm{T}} \otimes \mathbf{I}_{\bar{M}+\bar{N}-1}]\mathrm{vec}(\Delta\mathbf{C}_{\tilde{y}}) = \mathbf{E}_v\mathrm{vec}(\Delta\mathbf{C}_{\tilde{y}}) \tag{29}$$

in which $\mathbf{E}_v \in \mathbb{C}^{(\bar{M}+\bar{N}-1)P\times(\bar{M}+\bar{N}-1)^2}$. Based on Eq. (29) and the invariance speciality of the linear transformation in asymptotic normal distribution of $\mathrm{vec}(\Delta\mathbf{C}_{\tilde{y}})$ [21], the conclusions about mean $\tilde{\mu}$ and covariance matrix $\tilde{\mathbf{V}}$ in $\mathrm{vec}(\Delta\tilde{\mathbf{C}}_{\tilde{y}})$ are inferred as

$$\mathrm{vec}(\Delta\tilde{\mathbf{C}}_{\tilde{y}}) \sim \mathrm{AsN}(0, \mathbf{E}_v\mathbf{V}(\mathbf{E}_v)^{\mathrm{H}}) \tag{30}$$

where $\tilde{\mathbf{V}} \in \mathbb{C}^{(\bar{M}+\bar{N}-1)P\times(\bar{M}+\bar{N}-1)P}$. That is, $\mathrm{vec}(\Delta\tilde{\mathbf{C}}_{\tilde{y}})$ satisfies asymptotic normal distribution with zero-mean and covariance matrix $\mathbf{E}_v\mathbf{V}(\mathbf{E}_v)^{\mathrm{H}}$.

As a consequence, according to [12], in the proposed method, the regularization parameter $\eta$ is chosen as the upper limit value of asymptotic chi-square distribution with $(\bar{M} + \bar{N} - 1)P$ degrees of freedom and a high probability $1 - \varepsilon$ confidence interval upon variance uniformization of the FOC estimation errors, and $\varepsilon = 0.001$ is enough. With Matlab software, the function $chi2inv(1 - \varepsilon, (\bar{M} + \bar{N} - 1)P)$ can be used to calculate $\eta$. On the other hand, Eq. (25) can be calculated by SOC (second order cone) programming software packages such as SeDuMi [22] and CVX [23]. Finally, the DOA estimates are obtained by plotting $\mathbf{T}_{\hat{\theta}}^{(l_2)}$, solved from Eq. (25).

## 4. Related remarks

**Remark 1.** The computational complexity is analyzed as follows. The major burden of the proposed method is focused on the calculation of fourth-order cumulants and solving the sparse solution of recovered matrix in Eq. (25) through SOC programming. The former requires only $O[(\bar{M} + \bar{N} - 1)^2J]$ on account of the specially designed FOC formation, and the latter needs $O(L_1^3P)$, where $L_1$ is the total number of the discretized sampling grids, thus the main computational complexity of the proposed method is $O[(\bar{M} + \bar{N} - 1)^2J + L_1^3P]$. However, it costs $O[\bar{M}^4\bar{N}^4J + \bar{M}^2\bar{N}^2L_1(\bar{M}^2\bar{N}^2 - P)]$ calculation in FOC-MUSIC algorithm in [11] for all the indices $(k_1, k_2, k_3, k_4)$ being permutated in the matrix $\mathbf{C}_x$. The applications of the reduced dimensional transformation and the advantageous formation of FOC matrix make the calculation burden reduce greatly, and then the computational complexity of the proposed method is more reasonable than FOC-MUSIC algorithm.

**Remark 2.** The maximum number of targets that can be detected is one of the aspects in the performance analyses of proposed method. After compensating the errors of mutual coupling and utilizing the reduced dimensional transformation, the dimension of steering matrix is reduced from $MN \times P$ to $(\bar{M} + \bar{N} - 1) \times P$. In this case, any set of $\bar{M} + \bar{N} - 1$ columns in the complete dictionary $\tilde{\mathbf{B}}_{\hat{\theta}}$ are independent, which derives that the smallest integer number of columns that are linearly dependent in the complete dictionary $\tilde{\mathbf{B}}_{\hat{\theta}}$ is $\bar{M} + \bar{N}$, and it can be expressed as $\mathrm{Spark}(\tilde{\mathbf{B}}_{\hat{\theta}}) = \bar{M} + \bar{N}$. According to [24], under the sparse representation framework with multiple measurement vector (MMV) problem, a sufficient condition to uniquely determine the $P$-sparse vectors in $\tilde{\mathbf{C}}_{\tilde{y}} = \tilde{\mathbf{B}}_{\hat{\theta}}\mathbf{T}_{\hat{\theta}}$ is $P < [\mathrm{Spark}(\tilde{\mathbf{B}}_{\hat{\theta}}) - 1 + \mathrm{rank}(\tilde{\mathbf{C}}_{\tilde{y}})]/2$, under the circumstance of $\mathrm{rank}(\tilde{\mathbf{C}}_{\tilde{y}}) = P$, it yields that $P < \mathrm{Spark}(\tilde{\mathbf{B}}_{\hat{\theta}}) - 1$, namely, the maximal number of identified targets is $\bar{M} + \bar{N} - 2$ in the proposed method.

**Remark 3.** The chosen value of the number of targets $P$ is important, for the reason that either adding spurious peaks or missing actual sources may cause the deviations of the subspace and the signal power, which further lead to the deviations of the FOC observation matrix $\tilde{\mathbf{C}}_{\tilde{y}}$ and the reweighted matrix $\mathbf{W}_r$. When $P$ is unknown in real scenarios, several methods can estimate it effectively, such as Akaike information criterion (AIC) and minimum description length (MDL) [25]. In this paper, the value of $P$ is assumed to be known.

**Remark 4.** According to the specialty of mutual coupling coefficients, just a few values are suitable for choosing the value of $k$ when form the selection matrices $\mathbf{J}_1$ and $\mathbf{J}_2$. In fact, if the chosen value of $k$ is a little less than the real value, most of the mutual coupling is compensated and the remanent mutual coupling may cause performance degradation; on the contrary, if the chosen value of $k$ is slightly larger than the real value, it leads to the losses of array aperture. According to [26], up to now there are no references and methods that provide theoretical guidelines on how to directly and accurately obtain the real number of non-zero mutual coupling coefficients.

## 5. Simulation results

In this section, compared with the ESPRIT-Like algorithm [7], FOC-MUSIC algorithm [11] and $l_1$-SVD algorithm in [16], some simulation results are provided to illustrate the effectiveness and advantages of the proposed method. The root mean square error (RMSE) of angle estimation is defined as

$$(1/P)\sum_{p=1}^{P}\sqrt{(1/Q)\sum_{i=1}^{Q}(\hat{\theta}_{p,i} - \theta_p)^2} \tag{31}$$

where $\hat{\theta}_{p,i}$ is the estimator of the true value of DOA $\theta_p$ for the $i$th Monte Carlo trial, $Q$ is the number of the Monte Carlo trials, which is chosen as $Q$=500 in the following simulations. A narrowband monostatic MIMO radar system composed of $M$ transmit antennas and $N$ receive antennas is considered, whose arrays are both half-wavelength spaced ULAs. Furthermore, the received data in simulations are binary phase shift keying (BPSK) signals mixed with the Gaussian noise and the effect of mutual coupling between array elements in both transmitter and receiver, and SNR is defined as SNR=$10\log_{10}(\|\mathbf{CAS}\|_F^2/\|\mathbf{N}\|_F^2)$. For this MIMO radar system, $M=N=7$ is considered, and the exact value of $k$ is chosen in all analyzed methods when eliminate the mutual coupling. The colored noise involved in the corresponding simulations is complex Gaussian noise with zero-mean, whose covariance matrix is generated by $\mathbf{R}(k_1,k_2)=\sigma_n^2\rho_r^{|k_1-k_2|}e^{j\pi(k_1-k_2)/2}$, where $\mathbf{R}(k_1,k_2)$ is the element of the $k_1$th row and $k_2$th column in $\mathbf{R}$; $\sigma_n^2$ is power level that can be adjusted to give the desired SNR, and $\sigma_n^2=1$ is adopted in the simulations; $\rho_r$ is regression coefficient which adjusts the spatial correlation between noises, $0<\rho_r\le 1$. Besides, the discretized sampling grids are uniform with 0.05° from −90° to 90° for the proposed method as well as the FOC-MUSIC and the $l_1$-SVD algorithms.

Fig. 1 depicts the spatial spectrum of the proposed method for the number of nonzero mutual coupling coefficients being $K$=2 and $K$=3 with SNR=−10 dB and SNR=0 dB, respectively, where the noise is white Gaussian, $J$=2000 and the DOAs of three uncorrelated targets are $\theta_1=-20°$, $\theta_2=0°$ and $\theta_3=20°$. Two mutual coupling cases are considered: 1 $[c_{t0},c_{t1}]=[1,0.0174+j0.0377]$ in the transmit array and $[c_{r0},c_{r1}]=[1,0.0521-j0.1029]$ in the receive array when $K$=2, which are also used in the following figures; 2 $[c_{t0},c_{t1},c_{t2}]=[1,0.6+j0.2,0.02+j0.1]$ and $[c_{r0},c_{r1},c_{r2}]=[1,0.5+j0.3,0.01+j0.2]$ when $K$=3. As can be seen in Fig. 1, the spatial peaks of the proposed method are quite sharp and the sidelobe suppression is very low, even with $k$ increasing and SNR decreasing. It is indicated that the proposed method can provide superior estimation performance.

Fig. 2 shows the RMSE of DOA estimation versus SNR in different methods with Gaussian white noise, when there are three uncorrelated targets located at $\theta_1=-20°$, $\theta_2=0°$ and $\theta_3=20°$, at the same time, $J$=4000, and mutual coupling case 1 is considered. From Fig. 2, it can be observed that the $l_1$-SVD algorithm gets better performance than others in low SNR region. However, the proposed method provides the best angle estimation performance than $l_1$-SVD, ESPRIT-Like and FOC-MUSIC when the SNR exceeds about 0 dB, as a result of the utilization of the designed form of fourth-order cumulant matrix and reweighted sparse representation framework.

Fig. 3 illustrates the RMSE of DOA estimation versus SNR for Gaussian colored noise, $\rho_r=0.7$ and the other terms and conditions keep the same with Fig. 2. Compared with $l_1$-SVD, ESPRIT-Like and FOC-MUSIC algorithms, it is obvious in Fig. 3 that the estimation performance of the proposed method is the best in all SNR region. Owing to the special property of fourth-order cumulants, the colored noises are suppressed successfully by the proposed method, while they cause performance degradation in $l_1$-SVD and ESPRIT-Like, especially in low SNR region.

Fig. 4 demonstrates the RMSE of DOA estimation versus the correlation level of Gaussian colored noise, where SNR=0 dB, $\rho_r$ varies from $\rho_r=0$ to $\rho_r=1$ and the other conditions keep the same with Fig. 2. It is clear in Fig. 4 that for the proposed method and FOC-MUSIC algorithm, the estimation performance does not depend on the level of correlation of the colored noise. In addition, the estimation superiority of the proposed method over other analyzed methods is increasingly prominent with the increase of the noise correlation.

Figs. 5 and 6 show the RMSE versus angle separation with mutual coupling case 1, where $J$=4000, Gaussian white noise and SNR=5 dB are considered in Fig. 5, while Gaussian colored noise with $\rho_r=0.7$ and SNR=0 dB are considered in Fig. 6. There are two uncorrelated targets in each method, whose DOAs are $\theta_1=0°$ and $\theta_2=0°+\Delta\theta$, respectively, with $\Delta\theta$ varying from 2° to 14°. It can be seen from Fig. 5 that the estimation performance of the proposed method and ESPRIT-Like is approximate within a certain value of $\Delta\theta$, which is better than both the FOC-MUSIC algorithm and the $l_1$-SVD algorithm. When the angle separation is over about 6°, the proposed method performs the best in all algorithms. In Fig. 6, the proposed method holds the best angle estimation performance for closely spaced targets, which means that the proposed method has the highest spatial angular resolution.

Figs. 7 and 8 show the RMSE of DOA estimation in different methods versus snapshots with mutual coupling case 1. In Fig. 7 the noise is Gaussian white and SNR=5 dB, while in Fig. 8 it is

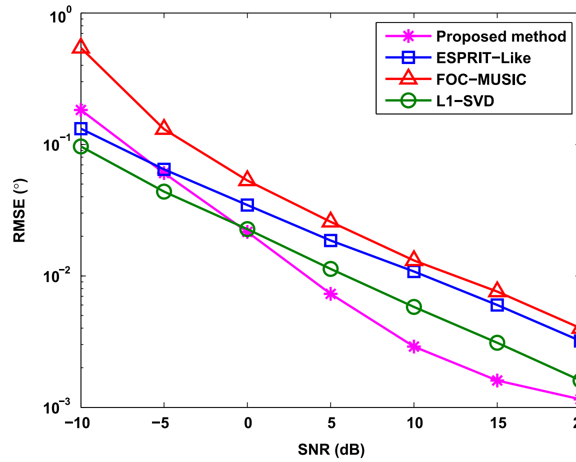


**Fig. 2.** RMSE versus SNR with three targets for white noise.

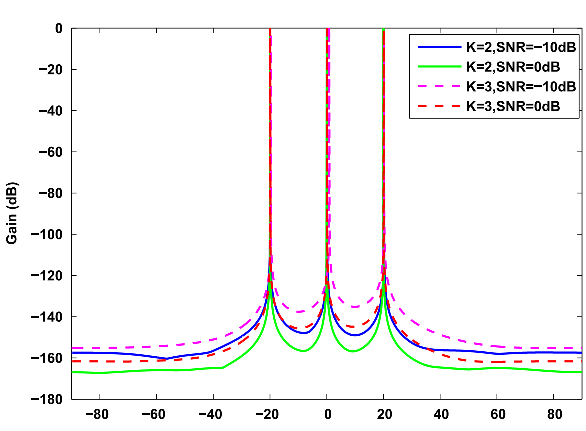


**Fig. 1.** The spatial spectrum of the proposed method for $K$=2 and $K$=3.

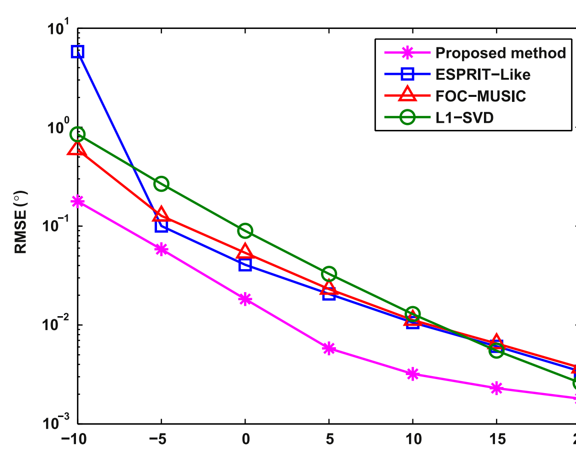


**Fig. 3.** RMSE versus SNR with three targets for colored noise.

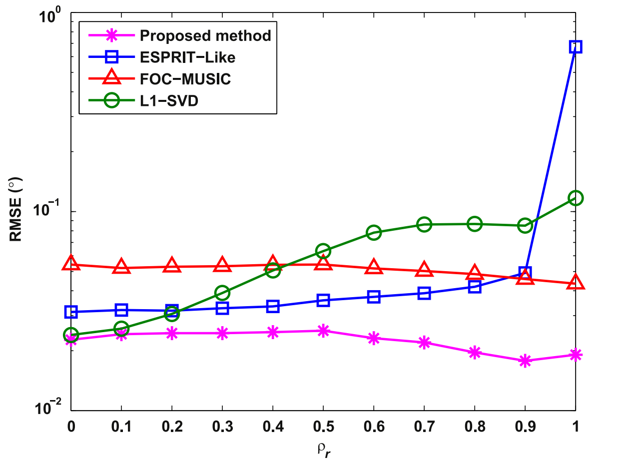


**Fig. 4.** RMSE versus correlation level of colored noise with three targets and SNR = 0 dB.

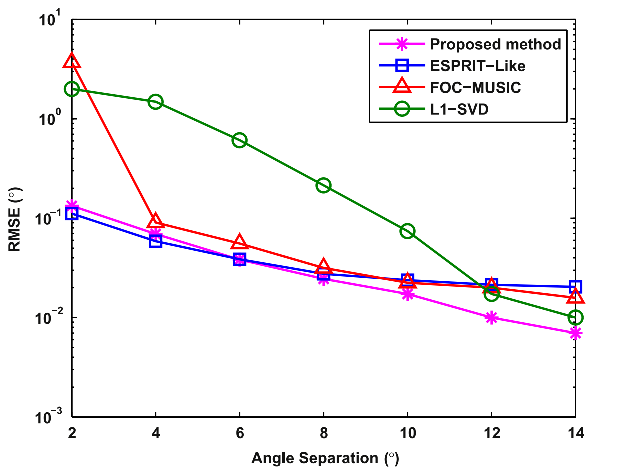


**Fig. 5.** RMSE versus angle separation with two targets for white noise.

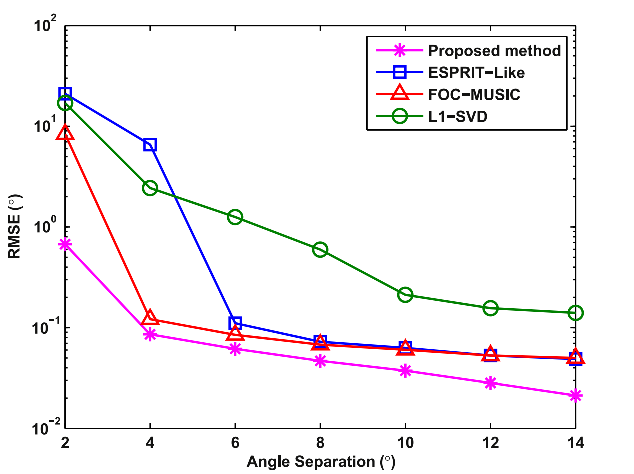


**Fig. 6.** RMSE versus angle separation with two targets for colored noise.

Gaussian colored with $\rho_r = 0.7$ and SNR = 0 dB. Three uncorrelated targets with different DOAs as $\theta_1 = -11.5°$, $\theta_2 = 0°$ and $\theta_3 = 11.5°$ are used. From Figs. 7 and 8, it can be observed that the proposed method provides excellent estimation performance in all snapshot region for both Gaussian white noise and colored noise, moreover, the estimation superiority of the proposed method over ESPRIT-Like, FOC-MUSIC and $l_1$-SVD algorithms is increasingly prominent

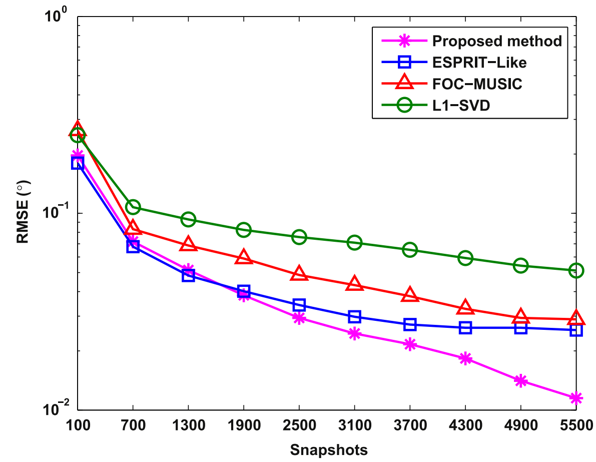


**Fig. 7.** RMSE versus snapshots with three targets for white noise.

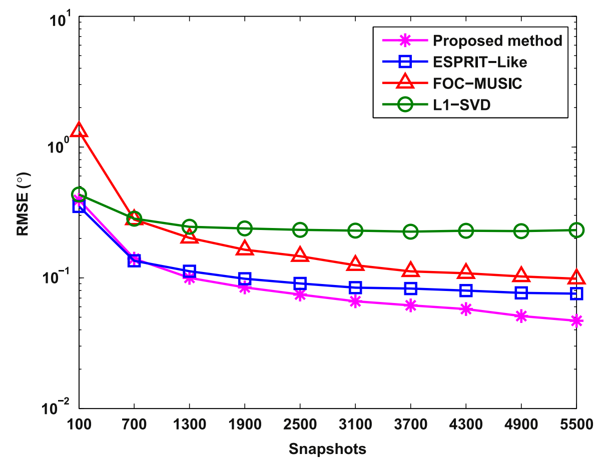


**Fig. 8.** RMSE versus snapshots with three targets for colored noise.

with the addition of $J$.

Figs. 9 and 10 demonstrate the target resolution probability of different methods versus SNR for Gaussian white noise and Gaussian colored noise with $\rho_r = 0.7$, respectively, where $J$ = 2000, mutual coupling case 1 is considered. The DOAs of targets are $\theta_1 = -11°$, $\theta_2 = 0°$ and $\theta_3 = 11°$. Additionally, they can be regarded successfully detected when all of the absolute DOA estimation errors of the three targets are within 0.1°. In the figures, all methods provide 100% target resolution probability with enough high SNR, however, the resolution probability of each method starts to drop at a certain point, which is defined as SNR threshold. It is clear in both Figs. 9 and 10 that the proposed method possesses lower SNR threshold than ESPRIT-Like, FOC-MUSIC and $l_1$-SVD, meanwhile they indicate that no matter whether the noise is Gaussian white or colored, the proposed method provides higher target resolution probability.

## 6. Conclusion

In this paper, we have proposed a sparse representation method based on fourth-order cumulants for DOA estimation in monostatic MIMO radar with unknown mutual coupling. In the proposed method, after eliminating the mutual coupling by the linear transformation, the reduced dimensional transformation matrix is formulated to reduce the complexity. Then making full use of the advantageous formation of the FOC matrix, the DOAs are

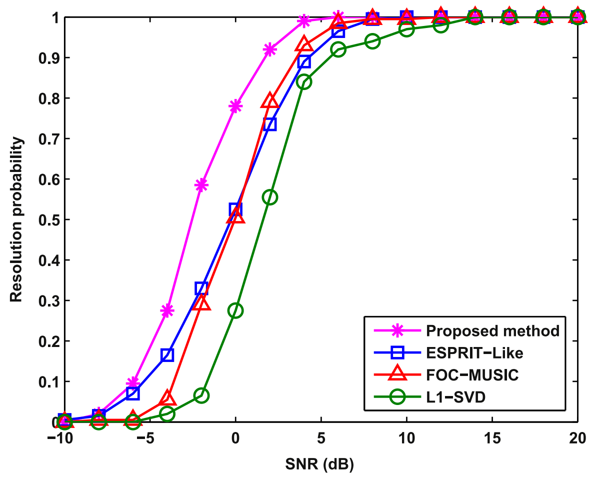


**Fig. 9.** Target resolution probability versus SNR with three targets for white noise.

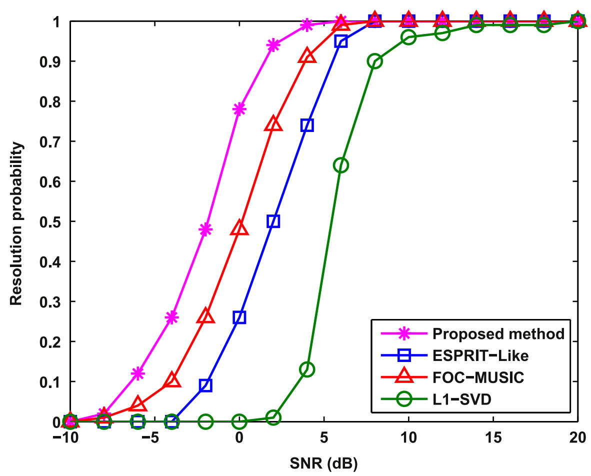


**Fig. 10.** Target resolution probability versus SNR with three targets for colored noise.

obtained by solving the designed reweighted $l_1$-norm constraint minimization sparse representation framework. The computational complexity of the proposed method is analyzed, and the simulation results demonstrate that the proposed method provides better angle estimation performance and higher spatial resolution than ESPRIT-Like, FOC-MUSIC and $l_1$-SVD algorithms for both white and colored Gaussian noise conditions.

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