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Frequency domain identification of complex sinusoids in the presence of additive noise

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Abstract: This paper describes a new approach for identifying the parameters of complex sinusoids from a finite number of measurements, in presence of additive and uncorrelated white noise. The proposed approach deals with frequency domain data and as a major feature, it enables the estimation to be frequency selective. In many aspects the new method resembles the well–known ESPRIT subspace algorithm, originally developed in the time domain. However, the sub–band frequency selective feature allows a reduction of the computations and can improve the quality of the estimates. The properties of the proposed method are analyzed by means of Monte Carlo simulations and its performance is compared with those of other estimation algorithms.

Keywords: System identification; Sinusoidal models; Discrete Fourier Transform; ESPRIT algorithm.

1. INTRODUCTION

Spectral analysis is a well studied topic due to its vast number of applications like, for example, speech analysis, radar and sonar systems, vibration monitoring, astronomy, geophysics, seismology and MR spectroscopy.

In the spectral analysis literature (Marple, 1987; Kay, 1988; Stoica and Moses, 1997) two different methodologies are usually described. The first methodology contains the classical nonparametric approaches, involving periodogram and correlogram methods. The second methodology contains the parametric approaches, also called model–based. One of the distinguishing properties resides in the fact that nonparametric methods treat frequency domain data, while parametric methods are commonly developed in the time domain.

Among the second methods, the subspace based estimators such as MUSIC (MUltiple SIgnal Classification) and ESPRIT (Estimation of Signal Parameters by Rotational Invariance Techniques) have been recognized to provide very accurate frequency estimates (Therrien, 1992; Stoica and Moses, 1997).

The important theoretical results provided by (Pintelon et al., 1997), with reference to input–output models, and by (McKelvey, 2000, 2002), for state–space models, have allowed to directly implement in the frequency domain many parametric approaches originally developed for time domain data (Pintelon and Schoukens, 2012).

With reference to state–space models, motivated by the work of (McKelvey and Viberg, 2001), a series of papers have been published where a frequency subspace–based method, denoted F–ESPRIT, has been proposed and analyzed, see (Gunnarsson and McKelvey, 2007) and the references therein.

In a similar way, starting from input–output representations, in this paper a new frequency subspace–based approach is proposed for the identification of complex sinusoids, in the presence of additive and uncorrelated white noise. This new method resembles in many aspects the well–known ESPRIT algorithm.

The subspace-based approach described in this paper and the F–ESPRIT method share some common aspects. In fact both methods solve the estimation problem by mapping it into a new formulation, where the original data become N–periodic. However, the F–ESPRIT method works with the Hankel matrices of the data. On the contrary, the proposed approach works with the covariance matrices of the data. As a major feature, both methods enable the estimation to be frequency selective, i.e. the user has the possibility to take into account some *a priori* information, by selecting the data just from the frequency sub–bands in which the signal harmonics are known to reside. As illustrated by the numerical examples, this property has a twofold effect. It allows a reduction of the computational time and improves the quality of the estimates, especially under low Signal to Noise Ratio (SNR) conditions.

The organization of the paper is as follows. Section 2 defines the problem of identifying complex sinusoids buried in white measurement noise. Section 3 introduces a novel frequency domain description of the system model. In Section 4 an equivalent state–space representation is developed. Sections 5 describes the new frequency domain subspace–based algorithm, while in Section 6 the effectiveness of the proposed method and its frequency selective features are verified by means of Monte Carlo simulations. In particular, the performance of the new method is compared with those of the ESPRIT and F–ESPRIT algorithms. It is shown that the proposed method and the F–ESPRIT algorithm have very similar behaviors and are both characterized by high frequency resolution properties. Finally, some concluding remarks are reported in Section 7 and the future developments for a statistical analysis of the proposed method are briefly illustrated.

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2. STATEMENT OF THE PROBLEM

Consider the following one–dimensional, discrete time model for n complex sinusoids buried in measurement noise

$$x(t) = \sum_{i=1}^{n} \rho_i \, \mathrm{e}^{\gamma_i t} \qquad t = 0, 1, 2 \ldots \tag{1}$$

$$y(t) = x(t) + v(t), \tag{2}$$

where $\gamma_i = -\beta_i + j\omega_i$ contains the unknown damping and frequency parameters, $\rho_i = |\rho_i| \mathrm{e}^{j\varphi_i}$ is the unknown complex gain and $v(t)$ is the complex noise.

The following assumptions are imposed.

A1. The number n is *a priori* known and all the components are present, $\rho_i \neq 0 \; \forall i$.
A2. The n frequencies are distinct, i.e. $\omega_i \neq \omega_l$ for $i \neq l$, with $\omega_i \in (-\pi, \pi]$.
A2. The additive noise $v(t)$ is a zero–mean, complex–valued circular white noise, with *unknown* variance σ_v^*, uncorrelated with $x(t)$.

Thus, for $v(t)$ it results

$$E\{v(t)\} = 0 \tag{3}$$

$$E\{v(t)\, v(s)^H\} = \sigma_v^* \text{ for } t = s \tag{4}$$

$$E\{v(t)\, v(s)^H\} = 0 \quad \text{for } t \neq s, \tag{5}$$

where $E\{\cdot\}$ denotes expectation and $(\cdot)^H$ is the Hermitian operation. In the following we will face the identification problem in the frequency domain, using the Discrete Fourier Transform (DFT) of the signals.

For a generic signal $\{s(t)\}_{t=0}^{N-1}$, observed at N equidistant time instants, the Discrete Fourier Transform (DFT) is defined as

$$S(\omega_k) = \frac{1}{\sqrt{N}} \sum_{t=0}^{N-1} s(t)\, \mathrm{e}^{-j\omega_k t}, \tag{6}$$

where $\omega_k = 2\pi k/N$, $k = 0, \ldots, N-1$.

In the frequency domain, the problem under investigation can be stated as follows.

Problem 1. According to (6), let $Y(\omega_k)$ be the DFT of the noisy measurements $\{y(t)\}_{t=0}^{N-1}$ generated by the system (1)–(2). Given $Y(\omega_k)$ for $k = 0, \ldots, N-1$, estimate the signal parameters ω_i, β_i, ρ_i $(i = 1, \ldots, n)$ and, possibly, the noise variance σ_v^*.

Remark 1. The main focus will be on the non–linear problem of estimating the parameters $\gamma_i \, (i = 1, \ldots, n)$. Once the parameters $\gamma_i = -\beta_i + j\omega_i$ are known, the parameters ρ_i can be recovered from the DFTs of $y(t)$ by using linear regression (Martin, 1994). See also Appendix A.

3. FREQUENCY DOMAIN SETUP

In this section a new frequency domain description for the noisy model (1)–(2) is introduced.

Define $\lambda_i = \mathrm{e}^{\gamma_i}$, with $i = 1, \ldots, n$. It can be easily verified that $(1 - \lambda_i \, z^{-1})$ is an annihilating polynomial for the generic i-th signal $x_i(t) = \rho_i \, \mathrm{e}^{\gamma_i t}$ in (1), i.e.

$$(1 - \lambda_i \, z^{-1}) x_i(t) = 0, \tag{7}$$

where z^{-1} denotes the backward shift operator. By extending this observation for all the components of $x(t)$, we obtain the following homogeneous AR equation

$$A(z^{-1}) x(t) = 0, \tag{8}$$

where

$$A(z^{-1}) = \prod_{i=1}^{n} (1 - \lambda_i \, z^{-1}) \tag{9}$$

$$= 1 + \alpha_1 \, z^{-1} + \cdots + \alpha_n \, z^{-n} \tag{10}$$

and α_i $(i = 1, \ldots, n)$ are complex parameters.

Similarly to (6), let $X(\omega_k)$ be the DFT of the noise–free sequence $\{x(t)\}_{t=0}^{N-1}$ defined in (1).

It is a well–known fact (Pintelon et al., 1997) that for finite N, even in absence of noise, the DFT $X(\omega_k)$ does no longer satisfy the relation $A(\mathrm{e}^{-j\omega_k})\, X(\omega_k) = 0$, which would be the frequency domain correspondence of (8). On the contrary, it exactly satisfies an extended relation that includes also a transient term

$$A(\mathrm{e}^{-j\omega_k})\, X(\omega_k) = T(\mathrm{e}^{-j\omega_k}), \tag{11}$$

where $T(z^{-1})$ is a polynomial of order $n-1$

$$T(z^{-1}) = \tau_0 + \tau_1 \, z^{-1} + \cdots + \tau_{n-1} \, z^{-n+1}. \tag{12}$$

By considering the whole number of frequencies, eq. (11) can be rewritten in a matrix form. For this purpose, introduce the parameter vectors

$$\theta_\alpha = [1 \; \alpha_1 \ldots \alpha_n]^T \tag{13}$$

$$\theta_\tau = [\tau_0 \; \tau_1 \ldots \tau_{n-1}]^T. \tag{14}$$

and define the following vector Θ, with dimension

$$p = 2n + 1\,, \tag{15}$$

containing the whole set of parameters

$$\Theta = [\; \theta_\alpha^T \; -\theta_\tau^T \;]^T. \tag{16}$$

Remark 2. The estimation of the parameter vector Θ allows to recover all the coefficients of relation (11). It must be observed that this relation is always exactly satisfied, also for a finite number of data and the vector θ_τ is identically zero only if $\{x(t)\}_{t=0}^{N-1}$ is an N–periodic sequence. From this consideration it can be easily deduced that, in general, omitting the estimate of the term θ_τ can affect the estimate of the system parameters θ_α, especially in case of short and non periodic data sequences. This fact will be highlighted by a numerical example in Section 6.

Remark 3. The idea to take into account the effect of θ_τ is applied in many contexts, see e.g. (Pintelon and Schoukens, 2012). In some recent papers, this technique has been applied also for the identification of dynamic errors–in-variables models (Soverini and Söderström, 2014, 2015). In particular, the effect of the term θ_τ on the estimate of θ_α has been recently analyzed in (Söderström and Soverini, 2017), with reference to the maximum likelihood criterion.

In absence of noise, the parameter vector Θ (16) can be recovered by means of the following procedure. Define the row vectors

$$\Omega_{n+1}(\omega_k) = [1 \;\; \mathrm{e}^{-j\omega_k} \; \ldots \; \mathrm{e}^{-j(n-1)\omega_k} \;\; \mathrm{e}^{-jn\omega_k}\,] \tag{17}$$

$$\Omega_n(\omega_k) = [1 \;\; \mathrm{e}^{-j\omega_k} \; \ldots \; \mathrm{e}^{-j(n-1)\omega_k}\,], \tag{18}$$

whose entries are constructed with multiple frequencies of ω_k, and construct the following matrices

$$\Psi_{n+1} = \begin{bmatrix} \Omega_{n+1}(\omega_0) \\ \vdots \\ \Omega_{n+1}(\omega_{N-1}) \end{bmatrix} \qquad \Psi_n = \begin{bmatrix} \Omega_n(\omega_0) \\ \vdots \\ \Omega_n(\omega_{N-1}) \end{bmatrix}. \tag{19}$$

of dimension $N \times (n+1)$ and $N \times n$, respectively.

With the DFT samples $X(\omega_k)$ construct the following $N \times N$ diagonal matrix

$$V_X^{diag} = \mathrm{diag}\,[X(\omega_0),\, X(\omega_1), \ldots, X(\omega_{N-1})] \tag{20}$$

and compute the $N \times (n+1)$ matrix

$$\Pi_X = V_X^{diag}\, \Psi_{n+1}. \tag{21}$$

Then, construct the $N \times p$ matrix

$$\Phi_X = [\, \Pi_X \mid \Psi_n \,]. \tag{22}$$

Thus, eq. (11) for all k can be rewritten as

$$\Phi_X\, \Theta = 0. \tag{23}$$

This condition can be written also by using the $p \times p$ matrix

$$\Sigma_X = \frac{1}{N}(\Phi_X^H\, \Phi_X). \tag{24}$$

It results in

$$\Sigma_X\, \Theta = 0. \tag{25}$$

Remark 4. Since $x(t)$ satisfies the relation (8), the relation (11) cannot be satisfied by a polynomial $A(z^{-1})$ with order lower than n. Therefore, the matrix Σ_X in (24) is positive semidefinite, with only one null eigenvalue, i.e.

$$\Sigma_X \geq 0 \qquad \dim\ker\,\Sigma_X = 1. \tag{26}$$

In presence of noise, the previous procedure can be modified as follows. With the DFT samples $Y(\omega_k)$ construct the $N \times N$ diagonal matrix

$$V_Y^{diag} = \mathrm{diag}\,[Y(\omega_0),\, Y(\omega_1), \ldots, Y(\omega_{N-1})] \tag{27}$$

and compute the matrix

$$\Pi_Y = V_Y^{diag}\, \Psi_{n+1}. \tag{28}$$

Then, construct the $N \times p$ matrix

$$\Phi_Y = [\, \Pi_Y \mid \Psi_n \,] \tag{29}$$

and compute the $p \times p$ positive definite matrix

$$\Sigma_Y = \frac{1}{N}\big(\Phi_Y^H\, \Phi_Y\big). \tag{30}$$

Because of Assumptions A3, when $N \to \infty$, we obtain

$$\Sigma_Y = \Sigma_X + \tilde{\Sigma}^*, \tag{31}$$

where

$$\tilde{\Sigma}^* = \begin{bmatrix} \sigma_v^*\, I_{n+1} & 0 \\ 0 & 0_n \end{bmatrix} \tag{32}$$

and 0_n is the null square matrix of dimension n.

From (25) and (31), the parameter vector Θ, defined in (16), can be obtained as the kernel of $(\Sigma_Y - \tilde{\Sigma}^*)$, i.e. it results

$$\big(\Sigma_Y - \tilde{\Sigma}^*\big)\, \Theta = 0, \tag{33}$$

where the first entry of Θ must be normalized to 1.

By considering the particular structure of the matrices Σ_Y and $\tilde{\Sigma}^*$, the parameter vector θ_α can be obtained by solving a problem with reduced dimensions. For this purpose, partition matrix Σ_Y, defined in (30), as follows

$$\Sigma_Y = \begin{bmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{bmatrix}, \tag{34}$$

where Σ_{11} is a square matrix of dimension $n+1$ and Σ_{22} is a square matrix of dimension n. In similar way, let us partition the matrix $\tilde{\Sigma}^*$ in (32).

Relation (33) can be expanded as follows

$$\Big(\Sigma_{11} - \sigma_v^*\, I_{n+1}\Big)\, \theta_\alpha + \Sigma_{12}\theta_\tau = 0 \tag{35}$$

$$\Sigma_{21}\theta_\alpha + \Sigma_{22}\theta_\tau = 0. \tag{36}$$

Considering the last n equations (36) leads to

$$\theta_\tau = -\Sigma_{22}^{-1}\, \Sigma_{21}\theta_\alpha. \tag{37}$$

The expression (37) can thus be substituted into (35) and the following problem of dimension $n+1$ can be defined

$$\Big(\Sigma - \sigma_v^*\, I_{n+1}\Big)\, \theta_\alpha = 0, \tag{38}$$

where

$$\Sigma = \Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}. \tag{39}$$

Remark 5. It is worth observing that the equation (38) has been obtained starting from the DFTs of non periodic data, however the effect of the transient term $T(z^{-1})$ that was introduced in eq. (11) has been now eliminated. In this way, the original problem (33) has been mapped into the new problem (38), that can be thought as generated starting from the DFTs of a N–periodic sequence $x(t)$. This aspect will be further explained in the next section.

Remark 6. Note that matrix (39) has dimension $n+1$, thus the problem (38) can be considered as the frequency counterpart of the well–known Pisarenko method, see e.g. (Stoica and Moses, 1997).

Remark 7. The described procedure can be applied also when only a subset of the whole frequency range is used, i.e. $\omega_k \in W = [\omega_I,\, \omega_F]$, with $I \geq 0$ and $F \leq N-1$, on condition that the number of frequencies $L = F - I + 1$ is large enough. The subset $W = [\omega_I,\, \omega_F]$ must be chosen by the user on the basis of *a priori* knowledge of the sinusoidal signal $x(t)$ and allows a frequency selective estimate of its model parameters.

4. EQUIVALENT STATE SPACE MODEL

The complex signal $y(t)$, defined in (1)–(2), can be recursively computed with the following n–dimensional state-space model (McKelvey and Viberg, 2001)

$$z(t+1) = A\, z(t) \qquad z(0) = z_0 \tag{40}$$

$$x(t) = C\, z(t) \tag{41}$$

$$y(t) = x(t) + v(t), \tag{42}$$

where

$$A = \mathrm{diag}\,[\, \lambda_1,\, \lambda_2, \ldots, \lambda_n \,] \tag{43}$$

$$C = [\,\underbrace{1,\, 1, \ldots, 1}_{n}\,] \tag{44}$$

$$z_0 = [\, \rho_1,\, \rho_2, \ldots, \rho_n \,]^T \tag{45}$$

and

$$\lambda_i = \mathrm{e}^{\gamma_i} \qquad i = 1, \ldots, n. \tag{46}$$

The vector $z(t) \in C^n$ is the state vector, the matrix $A \in C^{n\times n}$ is the state transition matrix and its eigenvalues are equal to λ_i. The triplet $(A,\, C,\, z_0)$ is a state space realization of $x(t)$. This realization is not unique. In fact, given an arbitrary non–singular matrix $T \in C^{n\times n}$, then the triplet $(T^{-1}A\,T,\, C\,T,\, T^{-1}z_0)$ is another valid realization of $x(t)$. All the state transition matrices for the different realizations share the mathematical property of similarity, and hence they share the same eigenvalues.

According to (6), let $Z(\omega_k)$ be the DFT of the state vector $z(t)$. By introducing the term $W_N = \mathrm{e}^{j\frac{2\pi}{N}}$, the state–space model (40)–(41) can be written in the frequency domain as follows

$$W_N^k\, Z(\omega_k) = A\, Z(\omega_k) + T\, \frac{1}{\sqrt{N}}\, W_N^k \tag{47}$$

$$X(\omega_k) = C\, Z(\omega_k), \tag{48}$$

where the vector $T = z(0) - z(N)$ represents the transient term due to the difference between the initial state $z(0)$ and the final state $z(N) = A^N\, z(0)$.

Remark 8. Relations (47)–(48) constitute the frequency domain state–space representation of $x(t)$. This model is equivalent to the polynomial representation (11). If the signal $x(t)$ is N–periodic, then $z(N) = z(0)$ and the vector T in (47) is null; equivalently, the polynomial $T(z^{-1})$ in (11) is null.

Remark 9. On the basis of the observations stated in Remark 4, we can say that the computation of the matrix Σ (39) allows to map the original problem, defined by (33), into a new and simpler problem, defined by (38), where the effect of the transient term $T(z^{-1})$ has been adjusted for. From the state–space point of view, we can say that an equivalent problem can be defined starting from the equations (40)–(45), when the vector T is null ($z(N) = z(0)$), i.e. in case of N–periodic data.

This reasoning can be generalized for orders greater than n. By considering row vectors $\Omega_{m+1}(\omega_k)$, $\Omega_m(\omega_k)$ in (17)–(18) with $m > n$, we can repeat the procedure of Section 3, substituting n with m. We can compute the matrix Σ_Y as in (30). Asymptotically ($N \to \infty$) we can define again a problem of type (33), where the dimension of the kernel Θ is now equal to $m - n + 1$. However, this fact does not prevent to partition the new matrix Σ_Y according to (34), where the matrix Σ_{11} has now dimension $m + 1$. We can proceed similarly to (35)–(37) and finally we can construct the matrix Σ of type (39), whose dimension is now $m + 1$. The new reduced problem defined by (38) has now a kernel θ_α with dimension $m - n + 1$. Since the considerations of Remarks 4 and 8 still hold, the $(m + 1)$–dimensional matrix Σ can be thought as produced starting from a state–space representation (40)–(45), in case of N–periodic data.

The matrix Σ is the frequency analogue of the Toeplitz covariance matrix of $y(t)$, defined by model (40)–(42). By following the approach described in (Rao and Arun, 1992), we can factorize the matrix Σ as follows

$$\Sigma = \mathcal{O}\,\Gamma, \tag{49}$$

where

$$\mathcal{O} = \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^m \end{bmatrix} = \begin{bmatrix} 1 & 1 & \dots & 1 \\ \lambda_1 & \lambda_2 & \dots & \lambda_n \\ {\lambda_1}^2 & {\lambda_2}^2 & \dots & {\lambda_n}^2 \\ \vdots & \vdots & \ddots & \vdots \\ {\lambda_1}^m & {\lambda_2}^m & \dots & {\lambda_n}^m \end{bmatrix} \tag{50}$$

$$\Gamma = \left[PC^T,\, A^{-1}PC^T,\, A^{-2}PC^T,\, \dots,\, A^{-m}PC^T \right] \tag{51}$$

and

$$P = \frac{1}{N} \sum_{k=0}^{N-1} Z(\omega_k)\, Z(\omega_k)^H. \tag{52}$$

The matrix P is the frequency analogue of the sample covariance matrix of the state vector $z(t)$ appearing in (40). Since the n frequencies are distinct (see assumption A2), the model (40)–(41) is minimal and consequently the matrices $\mathcal{O}$ and Γ have full rank equal to n. Since $\mathcal{O}$ has n columns and Γ has n rows, if $m > n$ then both $\mathcal{O}$ and Γ are of rank n. In the noise–free case, when $v(t) = 0$, also the rank of Σ is equal to the model order n.

5. SUBSPACE BASED ALGORITHM

In the following we will briefly describe an algorithm for determining the eigenvalues $\lambda_i = \mathrm{e}^{\gamma_i}$ of A ($i = 1, \dots, n$), starting from the knowledge of the matrix Σ. This algorithm can be considered as the frequency analogue of the well–known ESPRIT method (Roy and Kailath, 1989).

Define $m > n$ and compute the $(2m + 1)$–dimensional matrix Σ_Y in (30). Determine the $(m + 1)$–dimensional matrix Σ, according to equation (39). For this matrix, the properties (49)–(51) hold. Compute the singular value decomposition

$$\Sigma = U\, \Lambda\, U^H, \tag{53}$$

where matrix $\Lambda = \mathrm{diag}\,[\lambda_1, \dots, \lambda_m, \lambda_{m+1}]$ contains the eigenvalues, arranged in decreasing order $\lambda_1 \geq \lambda_2 \cdots \geq \lambda_{m+1}$, and the columns of U are the corresponding normalized eigenvectors.

Remark 10. In absence of noise, the last $(m - n)$ eigenvalues in Λ are equal to zero. In the presence of noise, all the eigenvalues in Λ are different from zero. Moreover the values of the last $(m - n)$ eigenvalues in Λ asymptotically ($N \to \infty$) coincide with the noise variance σ_v^*. This consideration is the basis of many subspace–based estimation methods, that make reference to the concept of signal and noise subspaces (Stoica and Moses, 1997; Therrien, 1992).

Let

$$\mathcal{O}_1 = [\, I_m\; 0\,]\, \mathcal{O} \tag{54}$$

$$\mathcal{O}_2 = [\, 0\; I_m\,]\, \mathcal{O}, \tag{55}$$

where I_m is the identity matrix of dimension $m \times m$. Note that $\mathcal{O}_1$ ($\mathcal{O}_2$) is obtained from $\mathcal{O}$ by deleting the last (first) row. Clearly,

$$\mathcal{O}_2 = \mathcal{O}_1\, A. \tag{56}$$

Let $\{u_1, \dots, u_n\}$ and $\{u_{n+1}, \dots, u_{m+1}\}$ denote the orthonormal eigenvectors of Σ, associated with $\{\lambda_1, \dots, \lambda_n\}$ and $\{\lambda_{n+1}, \dots, \lambda_{m+1}\}$, respectively. Let $B = [u_1, \dots, u_n]$ be the matrix containing the first n columns of matrix U in (53). Similarly to (54), define

$$B_1 = [\, I_m\; 0\,]\, B \tag{57}$$

$$B_2 = [\, 0\; I_m\,]\, B. \tag{58}$$

It can be proved (Stoica and Söderström, 1991) that

$$B_2 = B_1\, \phi, \tag{59}$$

where

$$\phi = D^{-1} A\, D \tag{60}$$

and D is a nonsingular matrix, such that

$$B = \mathcal{O}\, D. \tag{61}$$

Hence, the matrices A and ϕ are linked by the similarity relation (60) and share the same eigenvalues.

Since the matrices B_1 and B_2 have full rank (equal to n), the matrix ϕ can be consistently estimated in the least squares sense from (59), with the expression

$$\hat{\phi}_{LS} = (B_1^H B_1)^{-1} B_1^H B_2. \tag{62}$$

Here, and in the following, the symbol $\hat{}$ denotes an estimate.

As an alternative to (62), a total least squares estimate of ϕ can be obtained from (59), as follows. Compute the eigenvalue decomposition of the matrix $[B_1\ B_2]$

$$\begin{bmatrix} B_1^H \\ B_2^H \end{bmatrix} [B_1 \quad B_2] = V\, S\, V^H. \tag{63}$$

Partition matrix V in four $n \times n$ submatrices

$$V = \begin{bmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{bmatrix}. \tag{64}$$

Then, it results in

$$\hat{\phi}_{TLS} = -V_{12} V_{22}^{-1}. \tag{65}$$

Since the matrices ϕ and A are similar, see (60), the eigenvalues of the matrix A in (43) can be estimated by computing the eigenvalues $\hat{\lambda}_i$ of the matrix $\hat{\phi}$ in (62) or (65).

From (46), the frequencies ω_i can be found as

$$\hat{\omega}_i = \angle \hat{\lambda}_i \qquad i = 1, \ldots, n. \tag{66}$$

Computing

$$\hat{\gamma}_i = \log_e(\hat{\lambda}_i) \qquad i = 1, \ldots, n \tag{67}$$

it is possible to extract

$$\hat{\beta}_i = -\mathrm{Re}\,[\hat{\gamma}_i]. \tag{68}$$

It has been proved that the estimates obtained using the LS and the TLS methods have the same asymptotic accuracy (Rao and Hari, 1989).

6. NUMERICAL EXAMPLES

In this section, the effectiveness of the proposed method is tested by means of numerical simulations.

Example 1. The proposed example mimics a Magnetic Resonance data analysis. A model of type (1) with $n = 5$ components has been considered, simulating a 5–peak spectrum (Chen et al., 1994). The true frequency and damping parameters are reported in the first rows of Table 1 and Table 2. For completeness, the complex gains (not estimated) are

$$\rho_1 = 6.1\,\mathrm{e}^{j0.083\pi} \qquad \rho_2 = 9.9\,\mathrm{e}^{j0.083\pi} \qquad \rho_3 = 6.0\,\mathrm{e}^{j0.083\pi}$$
$$\rho_4 = 2.8\,\mathrm{e}^{j0.083\pi} \qquad \rho_5 = 17.0\,\mathrm{e}^{j0.083\pi}$$

The number of samples is $N = 1000$ and the sampling frequency is $f_s = 5$ kHz, so that the frequency resolution results in $df = f_s/N = 5$ Hz. A Monte Carlo simulation of 100 independent runs has been performed by adding to the noise–free sequences $x(t)$ different white noise realizations with variance $\sigma_v^* = 0.4275$, corresponding to a Signal to Noise Ratio (SNR) of 10 dB. The FFT spectrum of a typical noisy data sequence is presented in Fig. 1.

Tables 1 and 2 report the empirical means of the estimates of the system parameters ω_i and β_i, together with the corresponding standard deviations, obtained with the proposed frequency domain algorithm, denoted in the tables as TLS-ESPRIT-FD. The tables report also the results obtained with the classical time domain TLS-ESPRIT algorithm, denoted as TLS-ESPRIT-TD, and with the frequency domain F-ESPRIT algorithm proposed in (McKelvey and Viberg, 2001). For the TLS-ESPRIT-FD and the F-ESPRIT algorithms the value of $m > n$ has been fixed to $m = 20$. The last row of Table 1 reports also the estimates of the frequency parameters ω_i, obtained with the proposed algorithm when the transient term $T(\mathrm{e}^{-j\omega_k})$ in (11) is not considered. The corresponding row for the damping parameters β_i in Table 2 is not reported, since the obtained estimates are completely wrong.

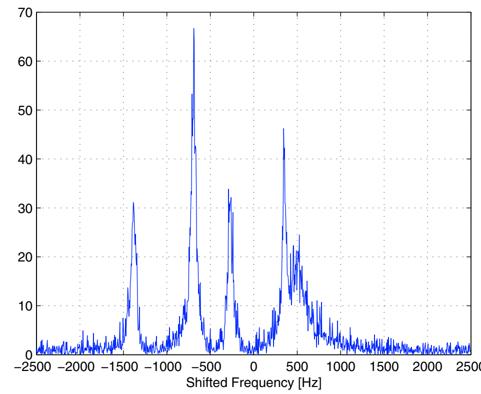

Fig. 1. FFT spectrum of a typical noisy data sequence.

Tables 1 and 2 show that the described identification method yields very good results, that coincide with those obtained by means of the corresponding time domain version, as well with those obtained by the F-ESPRIT algorithm. As pointed out in Remark 2, the beneficial effect of including the transient term is well illustrated by the worse estimates reported in the last row of Table 1.

Example 2. In this example we study the spectral resolution properties of the TLS-ESPRIT-FD algorithm, and compare the obtained results with those given by the TLS-ESPRIT-TD and F-ESPRIT algorithms. The original example has been proposed in (Jakobsson and Stoica, 2000). The model consists of $n = 13$ pure sinusoidal signals, with three dominant spectral lines and ten smaller spectral lines. All the spectral lines have a phase offset of $\pi/4$. The simulated model of type (1) is characterized by

$$\begin{aligned} &\rho_i = 1 \quad \text{for } i = 1, 2, 3 \\ &\rho_4 = 0.3 \\ &\rho_i = 0.1 \quad \text{for } i = 5, \ldots, 13 \\ &\varphi_i = 0.25\,\pi \quad \text{for } i = 1, \ldots, 13 \end{aligned}$$

The damping parameters are $\beta_i = 0$ ($i = 1, \ldots, 13$) and the frequency parameters ω_i are

$$0.0625, 0.0725, 0.25, 0, 28, 0.33, 0.35,$$
$$0.37, 0.39, 0.41, 0.43, 0.45, 0.47, 0.49\,.$$

The number of samples is $N = 100$ and the sampling frequency is $f_s = 1$ Hz, so that the frequency resolution results in $df = f_s/N = 0.01$ Hz.

The SNR for the k–th spectral line is defined as

$$\mathrm{SNR}_k = 10 \log_{10} \frac{\rho_k^2}{\sigma_v^*} \quad [\mathrm{dB}]$$

The aim of the experiments is to resolve the first two spectral lines at the frequencies $\omega_1 = 0.0625$ and $\omega_2 = 0.0725$ when the measurements are affected by noise.

For this purpose, a Monte Carlo simulation of 100 independent runs has been performed by adding to the noise–free sequences $x(t)$ different white noise realizations with variance $\sigma_v^* = 0.0316$, corresponding to a $\mathrm{SNR}_1 = 15$ dB.

Table 1. True and estimated values of the frequency parameters ω_i

	ω_1	ω_2	ω_3	ω_4	ω_5
True values	-1379	-685	-271	353	478
TLS − ESPRIT − FD	-1379.4 ± 1.9313	-685.2 ± 1.5269	-271.0 ± 1.9062	352.7 ± 5.4685	478.9 ± 8.2490
F − ESPRIT	-1379.4 ± 1.9589	-685.2 ± 1.5366	-271.0 ± 1.9305	352.7 ± 6.0561	479.0 ± 8.5704
TLS − ESPRIT − TD	-1379.4 ± 1.9587	-685.2 ± 1.5356	-271.0 ± 1.9332	352.7 ± 6.0505	479.0 ± 8.5412
TLS − ESPRIT − FD − NT	-1385.3 ± 2.7891	-694.9 ± 1.8369	-273.7 ± 2.2713	380.3 ± 3.6867	547.0 ± 4.9791

Table 2. True and estimated values of the damping parameters β_i

	β_1	β_2	β_3	β_4	β_5
True values	208	256	197	117	808
TLS − ESPRIT − FD	207.4163 ± 9.3354	254.3927 ± 7.8947	197.2370 ± 10.3361	120.0101 ± 22.0046	811.7624 ± 49.3890
F − ESPRIT	207.4071 ± 9.5336	254.4057 ± 7.7556	197.2158 ± 10.4217	120.1904 ± 22.9233	811.9025 ± 50.5558
TLS − ESPRIT − TD	207.3822 ± 9.5432	254.4200 ± 7.7677	197.2045 ± 10.4174	120.1061 ± 22.8714	811.9764 ± 50.5543

Table 3 reports the empirical means of the estimates of the parameters ω_i, β_i $(i = 1, 2)$ together with the corresponding standard deviations, obtained with the TLS-ESPRIT-FD algorithm. The table reports also the results obtained with the TLS-ESPRIT-TD and F-ESPRIT algorithms. For the TLS-ESPRIT-FD and the F-ESPRIT algorithms the value of m has been fixed to $m = 50$.

The last column of the table reports the mean value (in ms) of the time requested to carry out a single run of the Monte Carlo simulation. This value strongly depends on the specific features of the PC used for the simulations and, moreover, it may slightly change in different Monte Carlo sessions. However, it provides the correct order of magnitude of the computational efficiency of the algorithms and allows to make a comparison of their performances. Of course, in this respect, particular attention must be given to the coding.

Table 3 shows that, in this case, all the algorithms have similar, very good performances. As far as the computational efficiency is concerned, it can be observed that the frequency domain algorithms are slightly slower than the corresponding time domain implementation. This aspect is highly compensated by the fact that in the frequency domain the filtering operations can be implemented in a straightforward way, with great benefits for the identification, as shown in the next example.

A second Monte Carlo simulation of 100 independent runs has been performed with a noise variance $\sigma_v^* = 0.1$, corresponding to a $\mathrm{SNR}_1 = 10$ dB.

The results are reported in Table 4. From the first three lines of the table, it can be observed that, in this case, when the whole frequency range $[0 \;\; 0.5]$ is used for the identification, all the algorithms have unsatisfactory performances and fail the determination of the frequencies ω_i $(i = 1, 2)$.

However, the algorithms TLS-ESPRIT-FD and F-ESPRIT yield good estimates when the identification is performed by using only the data belonging to a specific frequency window defined by the user, $W = [\omega_I, \omega_F]$, as described in Remark 7. In particular, for the considered example, the two lines ω_1 and ω_2 have been identified by using only the $L = 20$ frequencies in the window $W = [0 \;\; 0.2]$.

The last two lines of Table 4 report the results obtained with TLS-ESPRIT-FD and F-ESPRIT. The value of m has been fixed to $m = 10$. In this case both methods give very good parameter estimates. Moreover, from the last column, it can be noted that the computational efficiency of the frequency domain algorithms is now improved, since only $L < N$ data are used for the identification.

7. CONCLUSIONS

In this paper a novel frequency subspace–based approach has been proposed for the identification of complex sinusoids affected by additive white noise. Its estimation properties have been tested and compared by means of Monte Carlo simulations. The numerical results have confirmed the good performance of the new method. It can be easily observed that the steps of the proposed frequency ESPRIT algorithm formally coincide with those of the corresponding time domain version. Thus, a possible open problem is to investigate how the statistical results of the classical ESPRIT method, e.g. (Stoica and Söderström, 1991), can be extended to the frequency domain. Another interesting question is how these results can be modified when only a subset of the whole frequency range is used for the identification.

Appendix A. PROOF OF REMARK 1

Because of assumption A2, the DFT of (2) is

$$Y(\omega_k) = X(\omega_k) + V(\omega_k), \tag{A.1}$$

where $V(\omega_k)$ is complex white noise.

According to (6), the DFT of (1) can be written as

$$X(\omega_k) = \frac{1}{\sqrt{N}} \sum_{t=0}^{N-1} x(t)\, \mathrm{e}^{-j\omega_k t} \tag{A.2}$$

$$= \frac{1}{\sqrt{N}} \sum_{t=0}^{N-1} \Big[\sum_{i=1}^{n} \rho_i\, \mathrm{e}^{\gamma_i t} \Big] \mathrm{e}^{-j\omega_k t} \tag{A.3}$$

$$= \frac{1}{\sqrt{N}} \sum_{t=0}^{N-1} \sum_{i=1}^{n} \rho_i\, \mathrm{e}^{(\beta_i + j(\omega_i - \omega_k))t}. \tag{A.4}$$

Table 3. True and estimated values of the parameters ω_i and β_i $(i = 1, 2)$ – SNR=15 dB

	ω_1	ω_2	β_1	β_2	Time (ms)
True values	0.0625	0.0725	0.0	0.0	
TLS – ESPRIT – FD	0.0625 ± 0.0002	0.0725 ± 0.0002	$-0.2205 * 10^{-3} \pm 0.0015$	$0.1591 * 10^{-3} \pm 0.0015$	20.6
F – ESPRIT	0.0625 ± 0.0002	0.0725 ± 0.0002	$-0.2155 * 10^{-3} \pm 0.0015$	$0.1876 * 10^{-3} \pm 0.0014$	20.3
TLS – ESPRIT – TD	0.0625 ± 0.0002	0.0725 ± 0.0002	$-0.2390 * 10^{-3} \pm 0.0014$	$0.1788 * 10^{-3} \pm 0.0014$	16.6

Table 4. True and estimated values of the parameters ω_i and β_i $(i = 1, 2)$ – SNR=10 dB

	ω_1	ω_2	β_1	β_2	Time (ms)
True values	0.0625	0.0725	0.0	0.0	
TLS – ESPRIT – FD	-0.2112 ± 0.1932	-0.0104 ± 0.1287	-0.0002 ± 0.0185	-0.0019 ± 0.0115	18.8
F – ESPRIT	-0.2087 ± 0.1889	-0.0087 ± 0.1285	0.0007 ± 0.0195	-0.0008 ± 0.0115	18.6
TLS – ESPRIT – TD	-0.2113 ± 0.1930	-0.0102 ± 0.1285	0.0010 ± 0.0240	-0.0035 ± 0.0158	16.0
TLS – ESPRIT – FD [0 – 0.20]	0.0642 ± 0.0019	0.0768 ± 0.0031	$0.0429 * 10^{-3} \pm 0.0097$	$0.1555 * 10^{-3} \pm 0.0108$	10.0
F – ESPRIT [0 – 0.20]	0.0646 ± 0.0018	0.0785 ± 0.0040	$0.2045 * 10^{-3} \pm 0.0096$	$-0.7789 * 10^{-3} \pm 0.0111$	9.4

Recalling that

$$\sum_{t=0}^{N-1} \mathrm{e}^{\alpha t} = \frac{1 - \mathrm{e}^{\alpha N}}{1 - \mathrm{e}^{\alpha}}, \tag{A.5}$$

we obtain

$$X(\omega_k) = \frac{1}{\sqrt{N}} \sum_{i=1}^{n} \rho_i \, \frac{1 - \mathrm{e}^{(\beta_i + j(\omega_i - \omega_k))N}}{1 - \mathrm{e}^{(\beta_i + j(\omega_i - \omega_k))}}. \tag{A.6}$$

If, in (A.6), the parameters β_i and ω_i $(i = 1, \dots, n)$ are known, the relations (A.1) for $k = 0, \dots, N-1$ constitute a set of linear equation in the unknowns ρ_i, where the measurements $Y(\omega_k)$ are affected by the white noise $V(\omega_k)$. These equations can be solved with classical LS techniques.

REFERENCES

Chen, H., Van Huffel, S., Decanniere, C. and Vanhecke, P. (1994). A signal-enhancement algorithm for the quantification of NMR data in the time domain. *Journal of Magnetic Resonance A*, 109(1), 46–55.

Ljung, L. (1999). *System identification–Theory for the user*. (2nd ed.), Upper Saddle River, NJ, USA: Prentice–Hall.

Gunnarsson, J. and McKelvey, T. (2007). Reducing noise sensitivity in F-ESPRIT using weighting matrices. *EUSIPCO 2007*, Poznan, Poland, 783–787.

Kay, S. M. (1988). *Modern Spectral Estimation*, Prentice–Hall, Englewood Cliffs, New Jersey.

Jakobsson, A. and Stoica, P. (2000). Combining Capon and APES for estimation of spectral lines. *Circuits, Systems and Signal Processing*, 19(2), 159–169.

Marple, L. (1987). *Digital Spectral Analysis with Applications*, Prentice–Hall, Englewood Cliffs, New Jersey.

Martin, Y.L. (1994). A global approach to accurate and automatic quantitative analysis of NMR spectra by complex least–squares curve fitting. *Journal of Magnetic Resonance A*, 111, 1–10.

McKelvey, T. (2000). Frequency domain identification. *Proc. of the 12th IFAC Symposium on System Identification*, Plenary Paper. Santa Barbara, California USA.

McKelvey, T. (2002). Frequency domain identification methods. *Circuits Systems Signal Processing*, 21(1), 39–55.

McKelvey, T. and Viberg, M. (2001). A robust frequency domain subspace algorithm for multi–component harmonic retrieval. *Proc. of the 35th Asilomar Conference on Signals, Systems and Computers*, Pacific Grove, CA, 1288–1292.

Pintelon, R. and Schoukens, J. (2012). *System identification: a frequency domain approach* (2nd ed.). NY: IEEE Press.

Pintelon, R., Schoukens, J. and Vandersteen G. (1997). Frequency domain system identification using arbitrary signals. *IEEE Transactions on Automatic Control*, 42, 1717–1720.

Rao, B.D. and Arun, K.S. (1992). Model based processing of signals: a state space approach. *Proc. of the IEEE*, 80(2), 283–309.

Rao, B.D. and Hari, K.S. (1989). Performance analysis of ESPRIT and TAM in determining the Direction of Arrival of plane waves in noise. *IEEE Trans. on Acoustic, Speech and Signal Processing*, 37(12), 1990–1995.

Roy, R. and Kailath, T. (1989). ESPRIT–Estimation of signal parameters via rotational invariance techniques. *IEEE Transactions on Acoustics, Speech and Signal Processing*, 37, 984–995.

Söderström, T. and Soverini, U. (2017). Errors–in–variables identification using maximum likelihood estimation in the frequency domain. *Automatica*, 79, 131–143.

Söderström, T. and Stoica, P. (1989). *System Identification*, Prentice Hall, Cambridge, UK.

Soverini, U. and Söderström, T. (2014). Frequency domain maximum likelihood identification of noisy input–output models. *Proc. of the 19–th IFAC World Conference*, Cape Town, South Africa, 4625–4630.

Soverini, U. and Söderström, T. (2015). Frequency domain EIV identification combining the Frisch Scheme and Yule–Walker equations. *Proc. of the 14–th European Control Conference*, Linz, Austria, 2038–2043.

Stoica, P. and Moses, R. (1997). *Introduction to Spectral Analysis*, Prentice–Hall, Upper Saddle River, New Jersey, 1997.

Stoica, P. and Söderström, T. (1991). Statistical analysis of MUSIC and Subspace Rotation estimates of sinusoidal frequencies. *IEEE Trans. Signal Processing*, 39(8), 1836–1847.

Therrien, C.W. (1992). *Discrete random signals ans statistical signal processing*, Prentice–Hall, Englewood Cliffs, New Jersey, 1992.