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Higher order direction finding from rectangular cumulant matrices: The rectangular 2q-MUSIC algorithms☆

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ABSTRACT

Fourth-order (FO) high resolution direction finding methods such as 4-MUSIC have been developed for more than two decades for non-Gaussian sources mainly to overcome the limitations of second order (SO) high resolution methods such as MUSIC. In order to increase the performance of 4-MUSIC in the context of multiple sources, the MUSIC method has recently been extended to an arbitrary even order 2q ($q \geq 2$), for square arrangements of the 2qth-order data statistics, giving rise to the 2q-MUSIC algorithm. To further improve the performance of 2q-MUSIC, the purpose of this paper is to extend the latter to rectangular arrangements of the data statistics, giving rise to rectangular 2q-MUSIC algorithms. Two kinds of rectangular arrangements, corresponding to redundant and non-redundant arrangements are considered. In particular, it is shown that rectangular arrangements of the higher order (HO) data statistics achieve a trade-off between performance and maximal number of sources to be estimated. These rectangular arrangements also lead to a complexity reduction for a given level of performance, which is still increased by non-redundant arrangements of the statistics. These results, completely new, open new perspectives in HO array processing.

1. Introduction

In some application domains of direction finding, such as military applications (e.g., surveillance drones or antenna arrays for tactile communication), sensor arrays must be of small size and are composed of a small number of sensors. To obtain good performances in terms of accuracy, resolution, and identifiability even for small arrays, HO high resolution direction finding methods are of great interest. FO methods [1–5] have been developed for more than two decades for non-Gaussian sources, mainly to overcome the limitations of SO high resolution methods [6–10] in the context of multiple sources. Among these FO methods, the FO extension of the MUSIC (or 2-MUSIC) method [6,9], called 4-MUSIC [3], is the most popular. FO methods are asymptotically robust to the presence of a Gaussian noise whose spatial coherence is unknown. Moreover, despite of their higher variance [11], they generate a virtual increase of both the effective aperture and the number of sensors of the array. This gives rise to the FO Virtual Array (VA) concept presented in [12,13] and allows both an increasing resolution and the processing of more sources than sensors [1,4,12,13].

To further increase the performance of 4-MUSIC in the context of multiple sources, the MUSIC method has recently been extended to an arbitrary even order 2q ($q \geq 1$). This has given rise to the so-called 2q-MUSIC algorithm [14], which is based on *square arrangements* of the 2qth-order data statistics. It has been shown in [14] that the performance of 2q-MUSIC, in terms of resolution, robustness to modelling errors and number of sources to be processed, increases as q increases. This performance increase is directly linked to a virtual increase of both the effective aperture and the number of sensors, N, of the array. This introduces the HO VA concept presented in [15], which is a 2qth-order extension of the FO VA concept. In particular, it has been pointed out in [15] that, for $q \geq 2$, the number of virtual sensors of the VA depends on the way the 2qth-order data statistics are arranged in the exploited ($N^q \times N^q$) 2qth-order statistical matrix. This shows that there exists an optimal square arrangement of these statistics, which leads to a VA with $O(N^q)$ virtual sensors. The 2q-MUSIC method can therefore identify up to $O(N^q)$ statistically independent sources [15]. This identifiability issue has been complemented in [16] for the particular case of an array with electromagnetic vector-sensors.

It has been shown recently in [17] that for *particular non uniform*

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linear arrays of N identical sensors, called $2q$-level *nested arrays*, arranging the $2q$th-order data statistics in a $(N^{2q}\times 1)$ *vector*, $\mathbf{c}_{2q,x}$, instead of a square $(N^q\times N^q)$ matrix $\mathbf{C}_{2q,x}$, gives rise to a $2q$th-order VA corresponding to a uniform linear array (ULA) of $O(N^{2q})$ virtual sensors. It is then possible to use a spatial smoothing subspace based algorithm, introduced in [18] for q=1, to estimate the DOAs of $O(N^{2q})$ statistically independent sources, instead of $O(N^q)$, from the "covariance like" matrix $\mathbf{c}_{2q,x}\mathbf{c}_{2q,x}^{\mathrm{H}}$, where $^{\mathrm{H}}$ means transpose and conjugate. An extension of this concept of a 1D nested array to a 2D nested array is presented in [19] for q=1 while the associated 2D spatial smoothing algorithm for direction finding is described in [20]. This result proves that for specific array geometries and a judiciously chosen direction finding algorithm, it is possible to increase the number of sources to be processed by arranging the $2q$th-order data statistics in a vector instead of a square matrix.

In this paper, we answer the more general question whether it may be useful in practice to consider *arbitrary rectangular arrangements* of the $2q$th-order data statistics instead of square ones for $2q$th-order direction finding or array processing methods from an *arbitrary array* of sensors and not specific ones only. To this end, we extend, for arbitrary arrays of sensors, both the HO VA concept and the $2q$-MUSIC algorithm to arbitrary rectangular arrangements of the $2q$th-order data statistics, giving rise to rectangular $2q$-MUSIC algorithms. Two kinds of rectangular arrangements, corresponding to redundant and non-redundant arrangements, are considered in this paper, giving rise to redundant and non-redundant rectangular $2q$-MUSIC algorithms. Non-redundant arrangements correspond to arrangements for which the redundancy contained in the $2q$th-order data statistics has been removed before processing, to decrease the complexity and to improve the performance. Several non-redundant arrangements are possible and are considered and compared in this paper. Note that one of them has already been proposed recently in [21] but for square arrangements of the statistics and specific arrays corresponding to ULAs and uniform rectangular arrays (URAs). It is shown in this paper that rectangular arrangements of the HO data statistics allow to achieve a compromise between performance and maximal number of sources to be estimated. Moreover, these rectangular arrangements also lead to a complexity reduction for a given performance level, which is still improved by non-redundant arrangements of the data statistics. These new results allow the development of new powerful methods for HO array processing.

After an introduction of our notation, assumptions and data statistics in section 2, rectangular arrangements of the $2q$th-order data statistics are presented in section 3. The rectangular $2q$-MUSIC algorithm is then described in section 4 whereas its main properties and performance are analyzed and illustrated in section 5 for statistically independent sources through the extension of the HO VA concept to rectangular arrangements. Extensions of the results of sections 3 to 5 to non-redundant rectangular arrangements of the $2q$th-order data statistics are presented and illustrated in section 6. Finally section 7 concludes this paper.

2. Assumptions, notations, and data statistics

2.1. Assumptions and notations

We consider an array of N narrow-band (NB) identical sensors and we call $\mathbf{x}(t)$ the vector of complex amplitudes of the signals at the output of these sensors. Each sensor is assumed to receive the contribution of P zero-mean stationary NB sources corrupted by a noise. We assume that the P sources can be divided into G groups, with P_g sources in the group g, such that the sources in each group are statistically dependent, but not perfectly coherent, while sources belonging to different groups are statistically independent. In particular, G=P corresponds to P statistically independent sources whereas G=1 corresponds to the case where all the sources are dependent. Of course, the P_g parameters are such that $P=\sum_{g=1}^{G}P_g$. Under these assumptions, the observation vector can be written as follows

$$\mathbf{x}(t)=\sum_{p=1}^{P}\mathbf{a}(\theta_p,\phi_p)s_p(t)+\mathbf{n}(t)=\mathbf{A}\mathbf{s}(t)+\mathbf{n}(t)=\sum_{g=1}^{G}\mathbf{A}_g\mathbf{s}_g(t)+\mathbf{n}(t)$$
$$=\sum_{g=1}^{G}\mathbf{x}_g(t)+\mathbf{n}(t) \qquad (1)$$

where $\mathbf{n}(t)$ is the noise vector, assumed zero-mean, stationary and Gaussian, $s_p(t)$ is the complex amplitude of source p, $\mathbf{s}(t)$, independent of $\mathbf{n}(t)$, is the vector whose components are the $s_p(t)$, θ_p and ϕ_p are the azimuth and the elevation angles of source p, and $\mathbf{A}$ is the $(N\times P)$ matrix of the source steering vectors $\mathbf{a}(\theta_p,\phi_p)$ $(1\le p\le P)$, which contains in particular the information about the DOAs of the sources. Furthermore, $\mathbf{A}_g$ is the $(N\times P_g)$ submatrix of $\mathbf{A}$ corresponding to the gth group of sources, $\mathbf{s}_g(t)$ is the corresponding $(P_g\times 1)$ subvector of $\mathbf{s}(t)$ and $\mathbf{x}_g(t)=\mathbf{A}_g\mathbf{s}_g(t)$. In particular, in the absence of coupling between sensors and assuming a plane wave propagation, component n of the vector $\mathbf{a}(\theta_p,\phi_p)$, denoted by $a_n(\theta_p,\phi_p)$, can be written as [22]

$$a_n(\theta_p,\phi_p)=\exp\left(j\frac{2\pi}{\lambda}\mathbf{k}(\theta_p,\phi_p)^{\mathrm{T}}\mathbf{p}_n\right). \qquad (2)$$

Here, λ is the wavelength, $\mathbf{p}_n=[\mathbf{x}_n,\mathbf{y}_n,\mathbf{z}_n]^{\mathrm{T}}$ is the vector whose components are the coordinates of sensor n of the array and $\mathbf{k}(\theta_p,\phi_p)=[\cos(\theta_p)\cos(\phi_p),\ \sin(\theta_p)\cos(\phi_p),\ \sin(\phi_p)]^{\mathrm{T}}$ is the wave vector of source p.

2.2. Statistics of the data

The HO methods discussed in this paper exploit the information contained in the $2q$th-order circular cumulants of the data, $\mathrm{cum}\ [x_{i_1}(t),\ldots,x_{i_q}(t),\ x_{i_{q+1}}(t)^*,\ldots,x_{i_{2q}}(t)^*]$ $(1\le i_j\le N)$ $(1\le j\le 2q)$, where * means complex conjugate. In situations of practical interest, the $2q$th-order statistics of the data are not known a priori and have to be estimated from K data samples, $\mathbf{x}(k)=\mathbf{x}(kT_e)$, $1\le k\le K$, where T_e is the sample period, using empirical estimators presented in [14,15].

3. Rectangular arrangements of the $2q$th order data statistics

In this paper, we arrange the data $2q$th-order circular cumulants into rectangular matrices. To this end, we introduce two arbitrary integers, v and l, such that $0\le v\le 2q$ and $\sup(0,\ v-q)\le l\le \inf(v,\ q)$ (see hereafter for the justification of these bounds). The integer v controls the size $(N^v\times N^{2q-v})$ of the rectangular $2q$th-order cumulant matrix. The integer l controls the way the data $2q$th-order circular cumulants are arranged in the rectangular cumulant matrix of a given size. For given values of q and v, let us arrange the $2q$-tuple, $(i_1,\ldots,i_q,\ i_{q+1},\ldots,i_{2q})$, of indices i_j $(1\le j\le 2q)$ $(1\le i_j\le N)$, into one v-tuple and one $(2q-v)$-tuple, indexed by v and l, and defined by $(i_1,\ i_2,\ldots,i_l,\ i_{q+1},\ldots,i_{q+v-l})$ and $(i_{q+v-l+1},\ldots,i_{2q},\ i_{l+1},\ldots,i_q)$. As the indices i_j vary from 1 to N, the v-tuple and the $(2q-v)$-tuple take N^v and N^{2q-v} values, respectively. Numbering, in a natural way, the N^v values of the v-tuple and the N^{2q-v} values of the $(2q-v)$-tuple by the integers $I^{(v,l)}$ and $J^{(v,l)}$ such that $1\le I^{(v,l)}\le N^v$ and $1\le J^{(v,l)}\le N^{2q-v}$, we obtain

$$I^{(v,l)}=\sum_{j=1}^{l}N^{v-j}(i_j-1)+\sum_{j=1}^{v-l}N^{v-l-j}(i_{q+j}-1)+1 \qquad (3)$$

$$J^{(v,l)}=\sum_{j=1}^{q-v+l}N^{2q-v-j}(i_{q+v-l+j}-1)+\sum_{j=1}^{q-l}N^{q-l-j}(i_{l+j}-1)+1. \qquad (4)$$

Using the permutation invariance property of the cumulants, we deduce that $\mathrm{cum}\ [x_{i_1}(t),\ldots,x_{i_q}(t),\ x_{i_{q+1}}(t)^*,\ldots,x_{i_{2q}}(t)^*]=\mathrm{cum}[x_{i_1}(t),\ldots,x_{i_l}(t),\ x_{i_{q+1}}(t)^*,\ldots,x_{i_{q+v-l}}(t)^*,\ x_{i_{q+v-l+1}}(t)^*,\ldots,x_{i_{2q}}(t)^*,\ x_{i_{l+1}}(t),\ldots,x_{i_q}(t)]$. Assuming that the latter quantity is the element $[I^{(v,l)},\ J^{(v,l)}]$ of the rectangular cumulant matrix denoted as $\mathbf{C}_{2q,x}^{(v,l)}$, it is easy to verify, using the definition of the Kronecker product and the assumptions of section 2.1, that the $(N^v \times N^{2q-v})$ matrix $\mathbf{C}_{2q,x}^{(v,l)}$ can be written as

$$\mathbf{C}_{2q,x}^{(v,l)} = \sum_{g=1}^{G} [\mathbf{A}_g^{\otimes l} \otimes \mathbf{A}_g^{*\otimes(v-l)}]\mathbf{C}_{2q,s_g}^{(v,l)}[\mathbf{A}_g^{\otimes(q-v+l)} \otimes \mathbf{A}_g^{*\otimes(q-l)}]^{\mathrm{H}} + \eta_2 \mathbf{V}(v, l)\delta(q-1). \tag{5}$$

Here, $\mathbf{C}_{2q,s_g}^{(v,l)}$ is the $(P_g^v \times P_g^{2q-v})$ matrix of the $2q$th-order circular cumulants of $\mathbf{s}_g(t)$, η_2 is the mean power of the noise per sensor, $\mathbf{V}(v, l)$, defined for q=1 only, is the $(N^v \times N^{2-v})$ $(0 \le v \le 2)$ normalized rectangular spatial coherence matrix of the noise such that the total input power of the noise is $N\eta_2$, $\delta(\cdot)$ is the Kronecker symbol, $\otimes$ is the Kronecker product and $\mathbf{A}_g^{\otimes l}$ is the $(N^l \times P_g^l)$ matrix defined by $\mathbf{A}_g^{\otimes l} = \mathbf{A}_g \otimes \mathbf{A}_g \otimes \ldots \otimes \mathbf{A}_g$ with a number of Kronecker products equal to $l-1$. The rectangular cumulant matrix $\mathbf{C}_{2q,x}^{(v,l)}$ is thus described by 3 parameters q, v and l which characterize the statistics order, the size of the cumulant matrix and the arrangement of the statistics into this matrix, respectively. Of course, the integer l must be such that $0 \le l \le v$, $0 \le l \le q$ and $0 \le v - l \le q$, which finally gives $\sup(0, v-q) \le l \le \inf(v, q)$. In particular, for $v = q$, $\mathbf{C}_{2q,x}^{(v,l)}$ reduces to the $(N^q \times N^q)$ square matrix $\mathbf{C}_{2q,x}^{(q,l)}$ used and denoted by $\mathbf{C}_{2q,x}(l)$ in [14] with $0 \le l \le q$. In the particular case of statistically independent sources, expression (5) reduces to

$$\mathbf{C}_{2q,x}^{(v,l)} = \sum_{p=1}^{P} c_{2q,s_p}\, \mathbf{a}_{q,p,1}^{(v,l)} \mathbf{a}_{q,p,2}^{(v,l)\mathrm{H}} + \eta_2 \mathbf{V}(v, l)\delta(q-1). \tag{6}$$

Here, $c_{2q,s_p} = \mathrm{cum}[s_{i_1}(t), \ldots, s_{i_q}(t), s_{i_{q+1}}(t)^*, \ldots, s_{i_{2q}}(t)^*]$, with $i_j = p$ $(1 \le j \le 2q)$, is the $2q$th-order circular autocumulant of $s_p(t)$. Using the shorthand notation $\mathbf{a}_p$ for $\mathbf{a}(\theta_p, \phi_p)$, the vectors $\mathbf{a}_{q,p,1}^{(v,l)}$ and $\mathbf{a}_{q,p,2}^{(v,l)}$ are defined by

$$\mathbf{a}_{q,p,1}^{(v,l)} = \mathbf{a}_p^{\otimes l} \otimes \mathbf{a}_p^{*\otimes(v-l)} \tag{7}$$

$$\mathbf{a}_{q,p,2}^{(v,l)} = \mathbf{a}_p^{\otimes(q-v+l)} \otimes \mathbf{a}_p^{*\otimes(q-l)} = \mathbf{a}_{q,p,1}^{(2q-v,q-v+l)}. \tag{8}$$

4. The rectangular $2q$-MUSIC algorithm

To develop the rectangular $2q$-MUSIC algorithms for the arrangement $\mathbf{C}_{2q,x}^{(v,l)}$, we introduce some other assumptions:

1. The number of correlated sources within each group is smaller than the number of sensors of the array: $P_g < N$, $1 \le g \le G$.
2. The matrices $\mathbf{A}_g^{\otimes l} \otimes \mathbf{A}_g^{*\otimes(v-l)}$ and $\mathbf{A}_g^{\otimes(q-v+l)} \otimes \mathbf{A}_g^{*\otimes(q-l)}$, spanning the left and right signal subspaces of the g-th group of sources in $\mathbf{C}_{2q,x}^{(v,l)}$, have a rank greater than or equal to $P_g^{\min(v,2q-v)}$, $1 \le g \le G$, i.e., the smaller signal subspace matrix has full rank.
3. The dimension of the left signal subspace is less than or equal to the smallest dimension of the rectangular cumulant matrix:

$$P(G, v, q) = \sum_{g=1}^{G} P_g^{\min(v,2q-v)} \begin{cases} < N^v & \text{if } v \le q \\ \le N^{2q-v} & \text{if } v > q \end{cases}$$

4. The matrices $\overline{\mathbf{A}}_{q,1}^{(v,l)} = [\mathbf{A}_1^{\otimes l} \otimes \mathbf{A}_1^{*\otimes(v-l)}, \ldots, \mathbf{A}_G^{\otimes l} \otimes \mathbf{A}_G^{*\otimes(v-l)}]$ and $\overline{\mathbf{A}}_{q,2}^{(v,l)} = [\mathbf{A}_1^{\otimes(q-v+l)} \otimes \mathbf{A}_1^{*\otimes(q-l)}, \ldots, \mathbf{A}_G^{\otimes(q-v+l)} \otimes \mathbf{A}_G^{*\otimes(q-l)}]$, spanning the left and right signal subspaces of $\mathbf{C}_{2q,x}^{(v,l)}$, have a rank greater than or equal to $P(G, v, q)$, i.e., the smaller signal subspace matrix has full rank.

In general, the matrix $\mathbf{C}_{2q,s_g}^{(v,l)}$ has full rank, $\min(P_g^v, P_g^{2q-v})$, i.e., P_g^{2q-v} for $v \ge q$, since the components of $\mathbf{s}_g(t)$ are statistically dependent. We then deduce from A1 to A4 that, for $q > 1$, the matrix $\mathbf{C}_{2q,x}^{(v,l)}$ has a rank equal to $P(G, q, v)$. To build a MUSIC-like algorithm from the matrix $\mathbf{C}_{2q,x}^{(v,l)}$, for $q > 1$, we first compute the singular value decomposition (SVD) of the latter:

$$\mathbf{C}_{2q,x}^{(v,l)} = [\mathbf{U}_{2q,s}^{(v,l)}\ \ \mathbf{U}_{2q,n}^{(v,l)}] \begin{bmatrix} \mathbf{\Sigma}_{2q,s}^{(v,l)} & \mathbf{0}_{2q,1}^{(v,l)} \\ \mathbf{0}_{2q,2}^{(v,l)} & \mathbf{\Sigma}_{2q,n}^{(v,l)} \end{bmatrix} \begin{bmatrix} \mathbf{V}_{2q,s}^{(v,l)\mathrm{H}} \\ \mathbf{V}_{2q,n}^{(v,l)\mathrm{H}} \end{bmatrix} \tag{9}$$

where $\mathbf{\Sigma}_{2q,s}^{(v,l)}$ is the diagonal matrix of the $P(G, q, v)$ non zero singular values of $\mathbf{C}_{2q,x}^{(v,l)}$ and the unitary matrices $\mathbf{U}_{2q,s}^{(v,l)}$ and $\mathbf{V}_{2q,s}^{(v,l)}$ contain, respectively, the left and right singular vectors associated with the non zero singular values. The diagonal matrix $\mathbf{\Sigma}_{2q,n}^{(v,l)}$ and the unitary matrices $\mathbf{U}_{2q,n}^{(v,l)}$ and $\mathbf{V}_{2q,n}^{(v,l)}$ are associated with the $(N^{2q-v} - P(G, q, v))$ zero singular values and $\mathbf{0}_{2q,1}^{(v,l)}$ and $\mathbf{0}_{2q,2}^{(v,l)}$ are null matrices. As $\mathrm{span}\{\mathbf{U}_{2q,s}^{(v,l)}\} = \mathrm{span}\{\overline{\mathbf{A}}_{q,1}^{(v,l)}\}$, we deduce that all the columns of all the matrices $\mathbf{A}_g^{\otimes l} \otimes \mathbf{A}_g^{*\otimes(v-l)}$, $1 \le g \le G$, are orthogonal to all the columns of $\mathbf{U}_{2q,n}^{(v,l)}$. Let $\mathbf{a}_{ig}$ be the steering vector of the ith source of the gth group. Then the vector $\mathbf{a}_{ig}^{\otimes l} \otimes \mathbf{a}_{ig}^{*\otimes(v-l)}$ corresponds to one column of $\mathbf{A}_g^{\otimes l} \otimes \mathbf{A}_g^{*\otimes(v-l)}$. Hence, all vectors $\{\mathbf{a}_{ig}^{\otimes l} \otimes \mathbf{a}_{ig}^{*\otimes(v-l)}, 1 \le i \le P_g, 1 \le g \le G\}$ are orthogonal to the columns of $\mathbf{U}_{2q,n}^{(v,l)}$ and are solutions of the following equation

$$[\mathbf{a}^{\otimes l} \otimes \mathbf{a}^{*\otimes(v-l)}]^{\mathrm{H}} \mathbf{U}_{2q,n}^{(v,l)} \mathbf{U}_{2q,n}^{(v,l)\mathrm{H}} [\mathbf{a}^{\otimes l} \otimes \mathbf{a}^{*\otimes(v-l)}] = 0 \tag{10}$$

which corresponds to the heart of the rectangular $2q$-MUSIC algorithm for the arrangement $\mathbf{C}_{2q,x}^{(v,l)}$, called $2q$-MUSIC(v, l). In practical situations, the matrix $\mathbf{U}_{2q,n}^{(v,l)}$ has to be estimated from the observations and the DOAs of the sources may be found by searching for the minima of the lefthand side of Eq. (10).

5. Properties and performance of rectangular $2q$-MUSIC

5.1. Performance of rectangular $2q$-MUSIC

As the best performance of $2q$-MUSIC(v, l) is obtained for statistically independent sources [14], we consider in this section statistically independent sources for which (5) reduces to (6). Under this assumption, we deduce from (6) that each source p contributes to the rectangular cumulant matrix $\mathbf{C}_{2q,x}^{(v,l)}$ through a rank one matrix $c_{2q,s_p} \mathbf{a}_{q,p,1}^{(v,l)} \mathbf{a}_{q,p,2}^{(v,l)\mathrm{H}}$. The vectors $\mathbf{a}_{q,p,1}^{(v,l)}$ and $\mathbf{a}_{q,p,2}^{(v,l)}$ correspond to the $(N^v \times 1)$ left and $(N^{2q-v} \times 1)$ right virtual steering vectors for the considered array of N sensors and are defined by (7) and (8), respectively. It has been shown in [15] that $\mathbf{a}_{q,p,1}^{(v,l)}$ and $\mathbf{a}_{q,p,2}^{(v,l)}$ can be considered as true steering vectors of the source p but for two VA of N^v and N^{2q-v} virtual sensors (VS), called hereafter left and right VA, respectively. From [15] we deduce that the positions of the left and right VS are defined by

$$\mathbf{p}_{q,1,m_1}^{(v,l)} = \sum_{j=1}^{l} \mathbf{p}_{k_j} - \sum_{u=1}^{v-l} \mathbf{p}_{k_{u+l}} \tag{11}$$

$$\mathbf{p}_{q,2,m_2}^{(v,l)} = \sum_{j=1}^{q-v+l} \mathbf{p}_{k_j} - \sum_{u=1}^{q-l} \mathbf{p}_{k_{u+q-v+l}} \tag{12}$$

where $1 \le k_j \le N$ for $1 \le j \le j_{\max}$ with $j_{\max}$=v and $j_{\max}$=$2q - v$ for the left and right VA, respectively, while $m_1 = \sum_{j=1}^{v} N^{v-j}(k_j - 1) + 1$ and $m_2 = \sum_{j=1}^{2q-v} N^{2q-v-j}(k_j - 1) + 1$. Fig. 1 shows examples of the left and right VAs associated with different arrangements of the 4th order statistics for a UCA with 3 sensors. These concepts of left and right VA extend to rectangular arrangements $\mathbf{C}_{2q,x}^{(v,l)}$ the HO VA concept introduced in [15] for square arrangements $\mathbf{C}_{2q,x}^{(q,l)}$ for which the left and right virtual steering vectors and the VAs coincide. The $2q$-MUSIC(v, l) pseudo-spectrum is computed based on the vectors $\mathbf{a}(\theta, \phi)^{\otimes l} \otimes \mathbf{a}(\theta, \phi)^{*\otimes(v-l)}$, that are associated with the left VA and which correspond, for the DOAs (θ_p, ϕ_p), p=1,...,P, to the steering vectors $\mathbf{a}_{q,p,1}^{(v,l)}$. We note that the pseudo-spectrum does not depend on the steering vectors $\mathbf{a}_{q,p,2}^{(v,l)}$ associated with the right VA. This leads to the following property:

Property 1. *The performance of* $2q$*-MUSIC*(v, l) *is directly controlled*

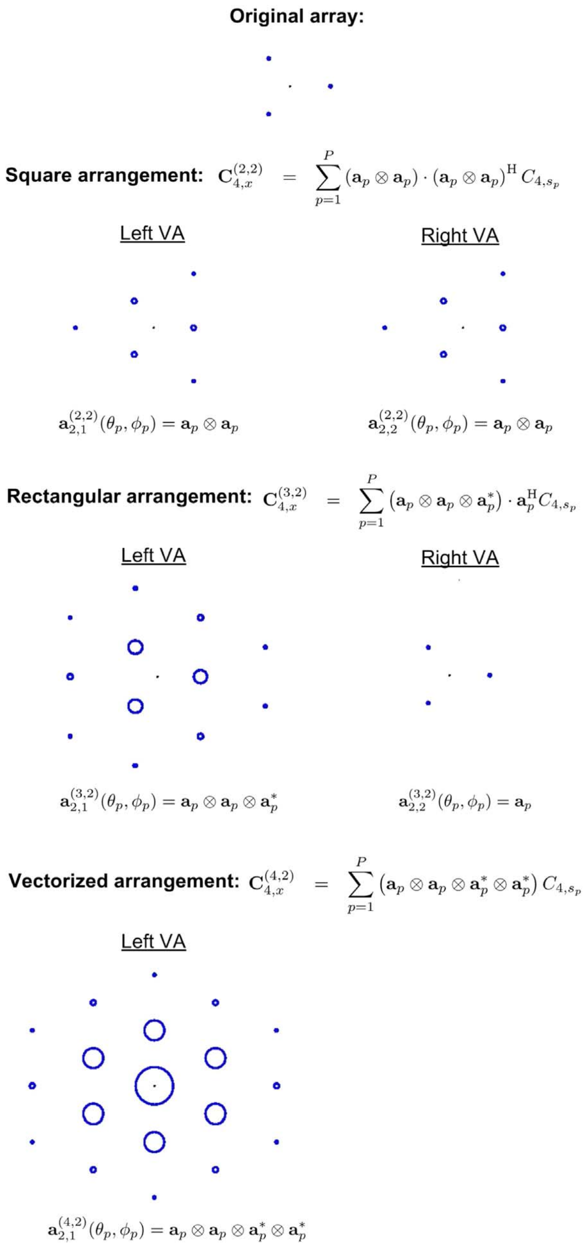

Fig. 1. Virtual arrays associated with different rectangular arrangements of the 4-th order cumulants for a UCA of 3 sensors. The size of the blue circles marking the sensor positions is proportional to the multiplicity of each virtual sensor. The small black dots mark the centers of the arrays.

by the left VA associated with $\mathbf{C}_{2q,x}^{(v,l)}$ *and increases with* v.

This result enlightens the interest of rectangular arrangements with respect to square ones and is demonstrated by the following example: consider two integers q_1 and q_2 such that $1 \le q_2 < q_1 \le 2q_2$ and choose l such that $q_1 - q_2 \le l \le q_2$. It is straightforward to verify that $\mathbf{a}_{q_2,p,1}^{(q_1,l)}$ and $\mathbf{a}_{q_1,p,1}^{(q_1,l)}$ exist and coincide. This means that the left VAs associated with $\mathbf{C}_{2q_2,x}^{(q_1,l)}$ and $\mathbf{C}_{2q_1,x}^{(q_1,l)}$ also coincide. This result proves that using a rectangular arrangement, $\mathbf{C}_{2q_2,x}^{(q_1,l)}$, of the $2q_2$th-order circular cumulants of the data allows to achieve the same performance in terms of potential resolution and robustness to modelling errors, as using a square arrangement, $\mathbf{C}_{2q_1,x}^{(q_1,l)}$, of circular cumulants of the data with an order $2q_1$ higher than $2q_2$. However, for given performance, the rectangular arrangement $\mathbf{C}_{2q_2,x}^{(q_1,l)}$ generates both a complexity reduction and a lower variance in the statistics estimate than the square arrangement $\mathbf{C}_{2q_1,x}^{(q_1,l)}$, hence an overall best global performance. This result proves that for a given statistics order $2q$, the parameter $v > q$ of the rectangular arrangement $\mathbf{C}_{2q,x}^{(v,l)}$ indicates the statistics order, $2v > 2q$, of the square arrangement, $\mathbf{C}_{2v,x}^{(v,l)}$, whose performance can be achieved.

5.2. Identifiability of rectangular 2q-MUSIC

Let us denote by $N_{2q,1}^{(v,l)}$ and $N_{2q,2}^{(v,l)}$ the number of different VS of the left and right $2q$th-order VA associated with $\mathbf{C}_{2q,x}^{(v,l)}$. In analogy with $2q$-MUSIC [14], the maximal number of sources $P_{2q,\max}^{(v,l)}$ which may be processed by $2q$-MUSIC(v, l) is upper bounded by $N_{2q,1}^{(v,l)} - 1$. Furthermore, $P_{2q,\max}^{(v,l)}$ is limited by the maximal rank $r_{2q,x}^{(v,l)}$ of $\mathbf{C}_{2q,x}^{(v,l)}$ in the absence of noise. It is obvious from (6)–(8) that $r_{2q,x}^{(v,l)} = \min(N_{2q,1}^{(v,l)}, N_{2q,2}^{(v,l)})$. This means that among the left and right VA associated with $\mathbf{C}_{2q,x}^{(v,l)}$, $r_{2q,x}^{(v,l)}$ is limited by the one having the smallest number of distinct VSs. In summary, $P_{2q,\max}^{(v,l)} = N_{2q,1}^{(v,l)} - 1$ for $N_{2q,1}^{(v,l)} \le N_{2q,2}^{(v,l)}$ and $P_{2q,\max}^{(v,l)} = N_{2q,2}^{(v,l)}$ for $N_{2q,1}^{(v,l)} > N_{2q,2}^{(v,l)}$. As, for given values of q and l, $N_{2q,1}^{(v,l)}$ and $N_{2q,2}^{(v,l)}$ are increasing and decreasing functions of v respectively, we obtain:

Property 2. *The maximal number of sources which may be identified by* $2q$-MUSIC(v, l) *is equal to* $P_{2q,\max}^{(v,l)} = \min(N_{2q,1}^{(v,l)} - 1, N_{2q,2}^{(v,l)})$ *and is maximized for* $v = q$, *i.e., for the square arrangement* $\mathbf{C}_{2q,x}^{(q,l)}$.

5.3. Performance – identifiability trade-off

As the performance of $2q$-MUSIC(v, l), for given values of q and l, increases with v (Property 1), in practice v should be chosen such that $q \le v \le 2q - 1$ to optimize the performance for a given number of sources (greater than one) to be processed. This means that $\mathbf{C}_{2q,x}^{(v,l)}$ must be either square or tall. For square arrangements, $N_{2q,1}^{(v,l)} = N_{2q,2}^{(v,l)}$ and $P_{2q,\max}^{(v,l)} = N_{2q,2}^{(v,l)} - 1$ and for tall arrangements, $P_{2q,\max}^{(v,l)} = N_{2q,2}^{(v,l)}$ (Property 2), i.e., the identifiability is controlled by the right VA. As $P_{2q,\max}^{(v,l)}$ is maximal for $v = q$ and decreases with increasing v while the performance improves, we obtain:

Property 3. *There is a trade - off between the maximal number of sources that can be processed by* $2q$*-MUSIC*(v, l) *and its performance in the context of multiple sources.*

Thus, a compromise has to be found in practice. The possibility to adjust the compromise between the number of sources to be processed by $2q$-MUSIC(v, l) and its performance for multiple sources is one of the main interests of rectangular arrangements with respect to square ones.

5.4. Optimal arrangement index l

For given values of q and $v \ge q$, the arrangement index l for the rectangular arrangement $\mathbf{C}_{2q,x}^{(v,l)}$ is such that $v - q \le l \le q$. For a sufficiently large number of snapshots K, the performance of $2q$-MUSIC(v, l) for multiple sources is directly controlled by the 3 dB-beamwidth of the left VA associated with $\mathbf{C}_{2q,x}^{(v,l)}$ (see section 5.1). However, this 3 dB-beamwidth has been shown to be independent of l [15]. In this case, the optimal index l is the one which maximizes the number of sources to be processed from $\mathbf{C}_{2q,x}^{(q,l)}$, i.e., which maximizes $N_{2q,2}^{(v,l)}$. Using the results of [15], we deduce that the optimal index l, denoted by l_{opt}, is such that $q - v + l_{\mathrm{opt}} = (2q - v)/2$ if v is even and $q - v + l_{\mathrm{opt}} = (2q - v - 1)/2$ if v is odd. This results in:

Property 4. *The optimal arrangement index, which maximizes the number of sources to be processed, is given by*

$$l_{\mathrm{opt}} = \begin{cases} v/2 & \text{if } v \text{ is even} \\ (v-1)/2 & \text{if } v \text{ is odd.} \end{cases}$$

5.5. Computer simulations

The results of the previous sections and subsections are illustrated in this subsection by computer simulations for both overdetermined and underdetermined mixtures of sources. The sources are assumed to have a zero elevation angle and to be zero-mean stationary. We only consider independent sources because the performance decrease due to source correlation is a common effect and not specific to rectangular $2q$-MUSIC.

5.5.1. Performance criteria

Two performance criteria are considered for each of the P sources. The first one is the probability of non-aberrant results, i.e., the probability that the estimated left-hand side of (10) is lower than a threshold η. In the following $\eta = 0.1$. The second one is the averaged root mean square error (RMSE), computed from the non-aberrant results. These criteria have been presented in detail in [15] and are not presented again in this paper.

5.5.2. Overdetermined mixture of sources

To illustrate the performance of $2q$-MUSIC(v, l) for overdetermined mixtures of sources, i.e., such that $P < N$, we assume that two synchronized statistically independent QPSK sources are received by a uniform circular array (UCA) composed of N=3 omnidirectional sensors. The radius r of the array is such that $r = 0.3\lambda$, where λ is the wavelength. The two sources have the same symbol duration and the same input SNR equal to 10 dB and are sampled at the symbol rate. The DOAs are equal to $\theta_1 = 90^{\circ}$ and θ_2, respectively. Fig. 2 shows the variations, as a function of the number of snapshots K, of the RMSE for the source 1, RMSE_1, and the associated probability of non-abberant

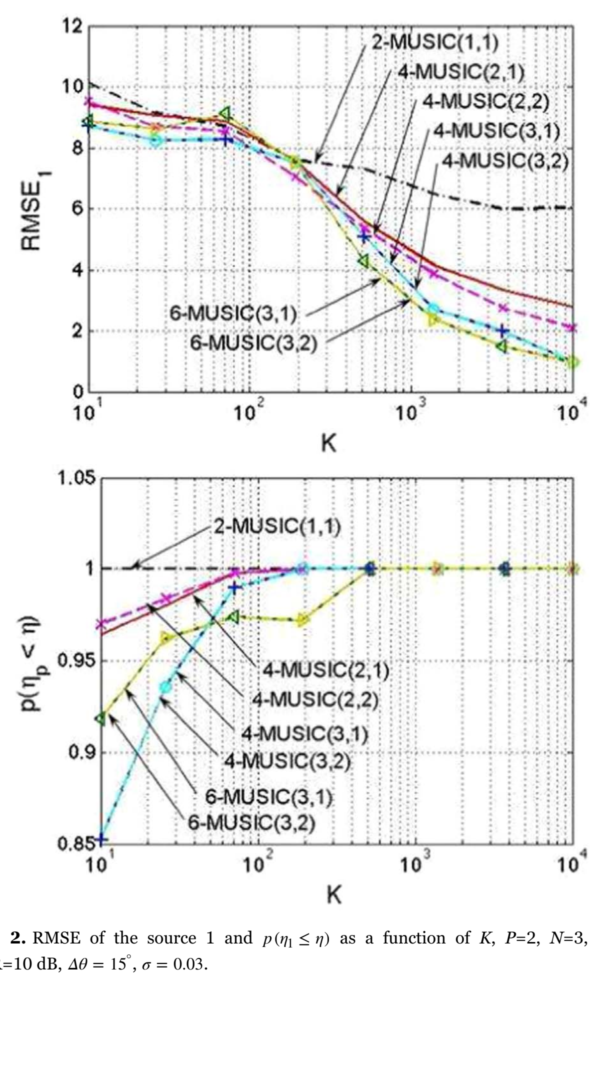

Fig. 2. RMSE of the source 1 and $p(\eta_1 \le \eta)$ as a function of K, P=2, N=3, UCA, SNR=10 dB, $\Delta\theta = 15^{\circ}$, $\sigma = 0.03$.

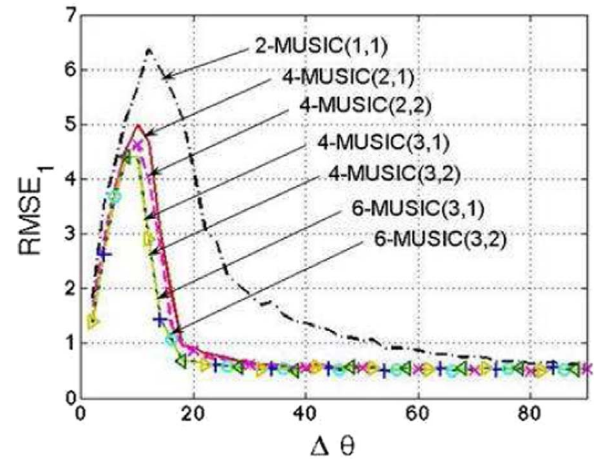

Fig. 3. RMSE of the source 1 as a function of $\Delta\theta$, P=2, N=3, UCA, SNR=10 dB, $K = \infty$, $\sigma = 0.03$.

results, $p(\eta_1 \le \eta)$, (we obtain similar results for the source 2), estimated from M=500 realizations, at the output of 2-MUSIC(1, 1), 4-MUSIC(2, 1), 4-MUSIC(2, 2), 4-MUSIC(3, 1), 4-MUSIC(3, 2), 6-MUSIC(3, 1) and 6-MUSIC(3, 2). For these figures, $\theta_2 = 105^{\circ}$ and the steering vectors $\mathbf{a}_p(1 \le p \le 2)$ are corrupted by a zero-mean circular Gaussian modelling error vector $\mathbf{e}_p$ such that $\mathbb{E}[\mathbf{e}_p\mathbf{e}_s^{\text{H}}] = \sigma^2\delta_{ps}\mathbf{I}_N$ where $\sigma = 0.03$. Note that beyond K=500 snapshots, the probability of non-aberrant results is equal to 1 for all the methods and that the RMSE_1 of the rectangular 4-MUSIC algorithms, 4-MUSIC(3, 1) and 4-MUSIC(3, 2), is smaller than that of the square 4-MUSIC algorithms, 4-MUSIC(2, 1), 4-MUSIC(2, 2). In this case the rectangular 4-MUSIC and the square 6-MUSIC algorithms achieve almost the same performance. To complete the previous results, we consider again the previous scenario with modelling errors but with now an infinite number of snapshots K and a variable value of $\theta_2 = \theta_1 + \Delta\theta$. Under these assumptions, Fig. 3 shows the variations of the RMSE_1 as a function of $\Delta\theta$ at the output of the previous methods. We note that the rectangular 4-MUSIC algorithms, 4-MUSIC(3, 1) and 4-MUSIC(3, 2), outperform the square 4-MUSIC methods, 4-MUSIC(2, 1) and 4-MUSIC(2, 2), especially for close sources and allow to obtain the performance of the square 6-MUSIC methods, 6-MUSIC(3, 1) and 6-MUSIC(3, 2), using 4th-order statistics only.

5.5.3. Underdetermined mixture of sources

To illustrate the performance of $2q$-MUSIC(v, l) for underdetermined mixtures of sources, i.e., such that $P \ge N$, we consider the same array as previously but we now assume that four synchronized statistically independent QPSK sources having the same symbol duration and sampled at the symbol rate, are received by the array. The four sources have the same input SNR equal to 10 dB and DOAs equal to $\theta_1 = 90^{\circ}$, $\theta_2 = 130^{\circ}$, $\theta_3 = 170^{\circ}$ and $\theta_4 = 50^{\circ}$. Fig. 4 shows the variations, as a function of the number of snapshots K, of the highest RMSE, $\text{RMSE}_{\max}$, and the lowest probability of non-abberant results, $p_{\min}$, over the sources, estimated from M=500 realizations, at the output of 4-MUSIC(2, 1), 4-MUSIC(2, 2), 6-MUSIC(4, 1), 6-MUSIC(4, 2), 6-MUSIC(4, 3), 6-MUSIC(3, 1) and 6-MUSIC(3, 2). No modelling errors are present at the reception. To achieve a probability of non-aberrant results which is approximately equal to 1, for 4-MUSIC(2, 1) and 4-MUSIC(2, 2), K=500 snapshots are required, whereas for 6-MUSIC(4, 1), 6-MUSIC(4, 2), 6-MUSIC(4, 3), 6-MUSIC(3, 1) and 6-MUSIC(3, 2), K=5000 snapshots are required. This can be explained by the increased variance of the HO statistics estimate (for FO statistics, an analysis of this variance can be found in [11]). Furthermore, we note that the probability of non-aberrant results decreases with increasing v. For $K > 5000$ snapshots, 6-MUSIC(4, 1), 6-MUSIC(4, 2), 6-MUSIC(4, 3), 6-MUSIC(3, 1) and 6-MUSIC(3, 2) lead to comparable perfor-

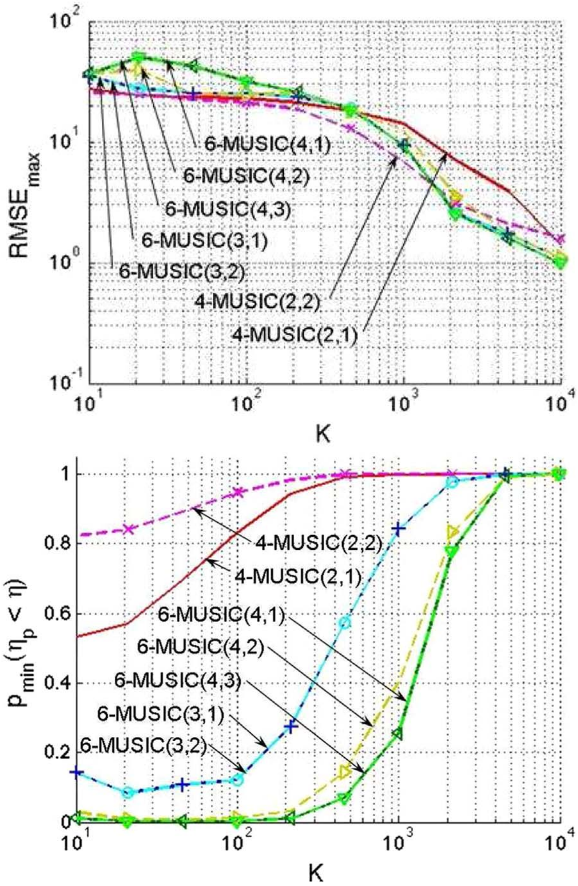

Fig. 4. Highest RMSE and lowest probability of non-aberrant results as a function of K, P=4, N=3, UCA, SNR=10 dB, $\theta_1 = 90^{\circ}$, $\theta_2 = 130^{\circ}$, $\theta_3 = 170^{\circ}$, $\theta_4 = 50^{\circ}$, $\sigma = 0$.

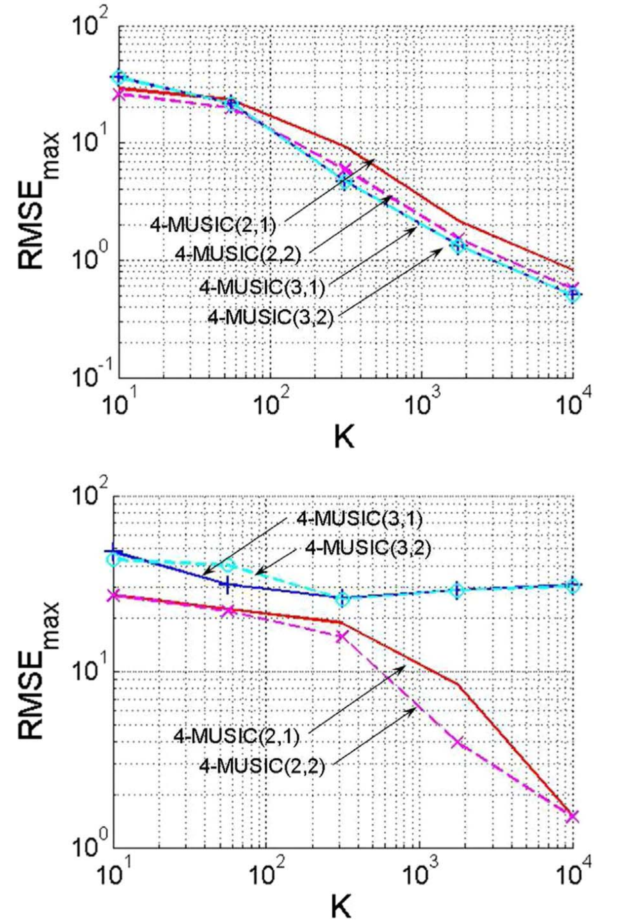

Fig. 5. Highest RMSE as a function of K, N=3, UCA, SNR=10 dB, $\sigma = 0$, $\theta_1 = 90^{\circ}$, $\theta_2 = 130^{\circ}$, $\theta_3 = 170^{\circ}$, $\theta_4 = 50^{\circ}$, for P=3 (left) and for P=4 (right).

mances in terms of the RMSE and the probability of non-aberrant results and outperform 4-MUSIC(2,1) and 4-MUSIC(2,2). This shows the interest of rectangular 6-MUSIC algorithms for underdetermined mixtures of sources.

5.5.4. Identifiability

To illustrate the identifiability results of $2q$-MUSIC(v, l), we consider two different scenarios. In both cases, we assume that the array has the same geometry as described in Section 5.5.2, that there are no modelling errors, and that all sources are statistically independent QPSK sources with identical symbol durations, and an input SNR equal to 10 dB. In the first scenario, P=3 sources with DOAs $\theta_1 = 90^{\circ}$, $\theta_2 = 130^{\circ}$, $\theta_3 = 170^{\circ}$ are received at the array whereas in the second scenario, we add a fourth source with DOA $\theta_4 = 50^{\circ}$. We evaluate the highest RMSE, $\mathrm{RMSE}_{\max}$, averaged over M=500 realizations, for different numbers of snapshots K at the output of 4-MUSIC(2,1), 4-MUSIC(2,2), 4-MUSIC(3,1), and 4-MUSIC(3,2). As Fig. 5 shows, 4-MUSIC(2,1) and 4-MUSIC(2,2) permit to find all sources in both cases since $P_{4,\max}^{(2,1)} = N_{4,1}^{(2,1)} - 1 = N^2 - N = 6$ and $P_{4,\max}^{(2,2)} = N_{4,1}^{(2,2)} - 1 = (N^2 + N)/2 - 1 = 5$ (which is greater than 3 or 4), whereas 4-MUSIC(3,1) and 4-MUSIC(3,2) only permit to identify a maximum of $P_{4,\max}^{(3,1)} = N_{4,2}^{(3,1)} = P_{4,\max}^{(3,2)} = N_{4,2}^{(3,2)} = N = 3$ sources, but cannot resolve $P = 4 > N$ sources.

6. The non-redundant rectangular $2q$-MUSIC algorithms

6.1. Non-redundant rectangular arrangements of the $2q$th-order statistics

For given values of q, v and l, the rectangular arrangement $\mathbf{C}_{2q,x}^{(v,l)}$ contains some redundancy due to the symmetry properties of cumulants. For statistically independent sources p ($1 \le p \le P$), this redundancy generates redundant left and right virtual steering vectors $\mathbf{a}_{q,p,1}^{(v,l)}$ and $\mathbf{a}_{q,p,2}^{(v,l)}$ (7), (8), i.e., vectors having some components which are the same functions of (θ_p, ϕ_p). Equivalently, this generates redundant left and right $2q$th-order VAs, which have some elements that coincide. The number of redundant VSs depends on the array geometry, the number and type of sensors, as well as indices q, v, and l and has been analyzed in detail in [14,15]. A VS with an order of multiplicity $M > 1$ (number of coinciding VSs) can be considered as weighted in amplitude by a factor equal to M and the associated VA becomes an amplitude-tapered VA [15]. The amplitude tapering increases the 3 dB-beamwidth of the VA, which generates a loss in resolution. Moreover, due to this redundancy, the size of the virtual steering vectors is artificially increased with no additional information. To decrease the complexity and to improve the resolution of $2q$-MUSIC(v, l), the amplitude tapering of the associated left and right VAs can be removed by eliminating the redundancies of the left and right virtual steering vectors. The number of elements of the associated non-redundant VAs then corresponds to the number of different VSs. This gives rise to non-redundant rectangular arrangements of the $2q$th-order cumulants, also exploitable for correlated sources.

To remove the redundancy of the rectangular arrangement $\mathbf{C}_{2q,x}^{(v,l)}$, we introduce the left and right reduction matrices, $\mathbf{Z}_{q,1}^{(v,l)}$ and $\mathbf{Z}_{q,2}^{(v,l)}$, associated, for statistically independent sources, with the left and right steering vectors, $\mathbf{a}_{q,p,1}^{(v,l)}$ and $\mathbf{a}_{q,p,2}^{(v,l)}$. These reduction matrices map all the redundant elements of the associated virtual steering vectors into only one element of a new, non-redundant virtual steering vector. The non-redundant ($N_{2q,1}^{(v,l)} \times 1$) left and ($N_{2q,2}^{(v,l)} \times 1$) right virtual steering vectors are given by

$$\mathbf{a}_{q,p,1,\mathrm{nr}}^{(v,l)} = \mathbf{Z}_{q,1}^{(v,l)}\mathbf{a}_{q,p,1}^{(v,l)} \tag{13}$$

$$\mathbf{a}_{q,p,2,\mathrm{nr}}^{(v,l)} = \mathbf{Z}_{q,2}^{(v,l)}\mathbf{a}_{q,p,2}^{(v,l)} \tag{14}$$

and are associated with the non-redundant left and right VAs having $N_{2q,1}^{(v,l)}$ and $N_{2q,2}^{(v,l)}$ VSs with multiplicity one, respectively. The reduction matrices $\mathbf{Z}_{q,1}^{(v,l)}$ and $\mathbf{Z}_{q,2}^{(v,l)}$ are $(N_{2q,1}^{(v,l)} \times N^v)$ and $(N_{2q,2}^{(v,l)} \times N^{2q-v})$ real-valued matrices of full rank $N_{2q,1}^{(v,l)}$ and $N_{2q,2}^{(v,l)}$, respectively, such that VSs with multiplicity 1 are kept with their multiplicity 1 in the non-redundant VA and such that VSs with identical positions and complex amplitude patterns are combined together to generate the same VS but with a multiplicity 1. Please note that a more detailed description of the properties of reduction matrices as well as examples of reduction matrices are provided in the supplementary material.

The $(N_{2q,1}^{(v,l)} \times N_{2q,2}^{(v,l)})$ non redundant rectangular arrangements of the $2q$th-order cumulants are obtained by applying the reduction matrices to $\mathbf{C}_{2q,x}^{(v,l)}$:

$$\mathbf{C}_{2q,x,\mathrm{nr}}^{(v,l)} = \mathbf{Z}_{q,1}^{(v,l)}\mathbf{C}_{2q,x}^{(v,l)}\mathbf{Z}_{q,2}^{(v,l)\mathrm{H}}. \tag{15}$$

Using (5), (13), (14) and (16), we finally obtain

$$\mathbf{C}_{2q,x,\mathrm{nr}}^{(v,l)} = \sum_{g=1}^{G} \mathbf{Z}_{q,1}^{(v,l)}[\mathbf{A}_g^{\otimes l} \otimes \mathbf{A}_g^{*\otimes(v-l)}]\mathbf{C}_{2q,s_g}^{(v,l)}[\mathbf{A}_g^{\otimes(q-v+l)} \otimes \mathbf{A}_g^{*\otimes(q-l)}]^{\mathrm{H}}\mathbf{Z}_{q,2}^{(v,l)\mathrm{H}} + \eta_2\mathbf{Z}_{1,1}^{(v,l)}\mathbf{V}(v,l)\mathbf{Z}_{1,2}^{(v,l)\mathrm{H}}\delta(q-1). \tag{16}$$

For statistically independent sources, using (6), (13) and (14), (16) becomes

$$\mathbf{C}_{2q,x,\mathrm{nr}}^{(v,l)} = \sum_{p=1}^{P} c_{2q,s_p}\mathbf{a}_{q,p,1,\mathrm{nr}}^{(v,l)}\mathbf{a}_{q,p,2,\mathrm{nr}}^{(v,l)\mathrm{H}} + \eta_2\mathbf{Z}_{1,1}^{(v,l)}\mathbf{V}(v,l)\mathbf{Z}_{1,2}^{(v,l)\mathrm{H}}\delta(q-1). \tag{17}$$

In the following, we consider two different strategies to remove the redundancy of the rectangular arrangement $\mathbf{C}_{2q,x}^{(v,l)}$.

6.2. Two reduction matrix strategies

6.2.1. Selection strategy

The selection strategy to remove the redundancy of $\mathbf{C}_{2q,x}^{(v,l)}$ is built assuming statistically independent sources and is then applied regardless of the correlation of the sources. In this case, for both the left and right VA associated with $\mathbf{C}_{2q,x}^{(v,l)}$, the selection strategy consists in keeping only one VS among each group of coinciding VSs. This strategy, which has been presented in [21] for specific array geometries, ULA or URA, is optimal in terms of complexity reduction, and asymptotically optimal (for a large number of snapshots) in terms of performance in the absence of modelling errors.

6.2.2. Averaging strategy

This strategy to remove the redundancy of $\mathbf{C}_{2q,x}^{(v,l)}$ is an alternative to the selection strategy, not described in [21]. It consists, for both the left and right VA associated with $\mathbf{C}_{2q,x}^{(v,l)}$, in averaging the components of a virtual steering vector associated with the same VS. In terms of performance and in the absence of modelling errors, this strategy is asymptotically optimal and is better than the selection strategy due to an averaging of the noise in the associated observation components. Moreover, this strategy generally outperforms the selection strategy in the presence of modelling errors due to an averaging of the latter in the associated observation components. From a complexity point of view, the averaging strategy is less powerful than the selection strategy since the computation of $\mathbf{C}_{2q,x,\mathrm{nr}}^{(v,l)}$ requires the computation of all the elements of $\mathbf{C}_{2q,x}^{(v,l)}$ contrary to the selection strategy. In practice, the averaging strategy is of interest for the left VA only, because removing the redundancy of the right VA may only contribute to the reduction of the computational complexity of the cumulant estimation (since it does not intervene in the calculation of the pseudo-spectrum, cf. section 6.3).

6.3. Non-redundant rectangular $2q$-MUSIC algorithms

The non-redundant rectangular $2q$-MUSIC algorithms, denoted by NR-$2q$-MUSIC(v, l), for the arrangement $\mathbf{C}_{2q,x,\mathrm{nr}}^{(v,l)}$ defined by (16) can be derived in an analogous way to the rectangular $2q$-MUSIC algorithms. In the absence of noise, $\mathbf{C}_{2q,x,\mathrm{nr}}^{(v,l)}$ and $\mathbf{C}_{2q,x}^{(v,l)}$ have the same rank equal to $P(G, q, v)$. The ranks of the signal and noise subspaces of $\mathbf{C}_{2q,x,\mathrm{nr}}^{(v,l)}$ are thus identical to those of $\mathbf{C}_{2q,x}^{(v,l)}$, but the left signal subspace of $\mathbf{C}_{2q,x,\mathrm{nr}}^{(v,l)}$, is now spanned by the reduced matrix $\overline{\mathbf{A}}_{q,1,\mathrm{nr}}^{(v,l)} = \mathbf{Z}_{q,1}^{(v,l)}\overline{\mathbf{A}}_{q,1}^{(v,l)}$. The NR-$2q$-MUSIC$(v, l)$ algorithm then aims at minimizing the pseudo-spectrum

$$\hat{\mathrm{P}}_{\mathrm{NR}-2q-\mathrm{Music}(v,l)}(\theta, \phi) = \mathbf{a}_{q,p,1,\mathrm{nr}}^{(v,l)\mathrm{H}}\hat{\mathbf{U}}_{2q,n,\mathrm{nr}}^{(v,l)}\hat{\mathbf{U}}_{2q,n,\mathrm{nr}}^{(v,l)\mathrm{H}}\mathbf{a}_{q,p,1,\mathrm{nr}}^{(v,l)} \tag{18}$$

based on the non-redundant virtual sterring vector $\mathbf{a}_{q,p,1,\mathrm{nr}}^{(v,l)}$ and the left noise subspace $\hat{\mathbf{U}}_{2q,n,\mathrm{nr}}^{(v,l)}$ estimated from the SVD of $\mathbf{C}_{2q,x,\mathrm{nr}}^{(v,l)}$.

The properties of NR-$2q$-MUSIC(v, l) are similar to those of $2q$-MUSIC(v, l). Therefore, in the following, we concentrate on their differences concerning performance and computational complexity.

6.4. Performance of non-redundant rectangular $2q$-MUSIC

As the best performance of NR-$2q$-MUSIC(v, l) is obtained for statistically independent sources [14], we consider in this section statistically independent sources for which (16) reduces to (17). Applying to (16) a similar reasoning as in section 5.1, we deduce that the performance of NR-$2q$-MUSIC(v, l), in terms of resolution for a finite observation duration and asymptotic robustness to modelling errors, is directly controlled by the 3 dB-beamwidth of the left non-redundant VA associated with $\mathbf{C}_{2q,x,\mathrm{nr}}^{(v,l)}$. To compute this 3 dB-beamwidth, let us calculate the spatial correlation coefficient of two sources, with directions $\boldsymbol{\theta}_p = (\theta_p, \phi_p)$ and $\boldsymbol{\theta}_0 = (\theta_0, \phi_0)$, respectively, for the non-redundant left VA associated with $\mathbf{C}_{2q,x,\mathrm{nr}}^{(v,l)}$. This coefficient, denoted by $\alpha_{2q,\mathrm{nr}}^{(v,l)}(\boldsymbol{\theta}_p, \boldsymbol{\theta}_0)$, is defined by the normalized inner product of the non-redundant virtual steering vectors $\mathbf{a}_{q,p,1,\mathrm{nr}}^{(v,l)} = \mathbf{Z}_{q,1}^{(v,l)}\mathbf{a}_{q,p,1}^{(v,l)} = \mathbf{Z}_{q,1}^{(v,l)}[\mathbf{a}(\theta_p, \phi_p)^{\otimes l} \otimes \mathbf{a}(\theta_p, \phi_p)^{*\otimes(v-l)}]$ and $\mathbf{a}_{q,0,1,\mathrm{nr}}^{(v,l)} = \mathbf{Z}_{q,1}^{(v,l)}\mathbf{a}_{q,0,1}^{(v,l)} = \mathbf{Z}_{q,1}^{(v,l)}[\mathbf{a}(\theta_0, \phi_0)^{\otimes l} \otimes \mathbf{a}(\theta_0, \phi_0)^{*\otimes(v-l)}]$ and can be written as

$$\alpha_{2q,\mathrm{nr}}^{(v,l)}(\boldsymbol{\theta}_p, \boldsymbol{\theta}_0) = \frac{\mathbf{a}_{q,p,1}^{(v,l)\mathrm{H}}\mathbf{Z}_{q,1}^{(v,l)\mathrm{H}}\mathbf{Z}_{q,1}^{(v,l)}\mathbf{a}_{q,0,1}^{(v,l)}}{[\mathbf{a}_{q,p,1}^{(v,l)\mathrm{H}}\mathbf{Z}_{q,1}^{(v,l)\mathrm{H}}\mathbf{Z}_{q,1}^{(v,l)}\mathbf{a}_{q,p,1}^{(v,l)}]^{1/2}[\mathbf{a}_{q,0,1}^{(v,l)\mathrm{H}}\mathbf{Z}_{q,1}^{(v,l)\mathrm{H}}\mathbf{Z}_{q,1}^{(v,l)}\mathbf{a}_{q,0,1}^{(v,l)}]^{1/2}} \tag{19}$$

The 3 dB-beamwidth of the left non-redundant VA associated with $\mathbf{C}_{2q,x,\mathrm{nr}}^{(v,l)}$ corresponds to the smallest value of $2\Delta\theta = 2(\theta_p - \theta_0)$ such that $|\alpha_{2q}^{(v,l)}(\boldsymbol{\theta}_p, \boldsymbol{\theta}_0)|^2 = 0.5$. It is difficult to simplify this expression in the general case of an arbitrary reduction matrix $\mathbf{Z}_{q,1}^{(v,l)}$, but for a given array of sensors, authorized parameters (q, v, l) and reduction matrix $\mathbf{Z}_{q,1}^{(v,l)}$, $\alpha_{2q,\mathrm{nr}}^{(v,l)}(\boldsymbol{\theta}_p, \boldsymbol{\theta}_0)$ can easily be computed for arbitrary values of $(\boldsymbol{\theta}_p, \boldsymbol{\theta}_0)$. Note that in the absence of modelling errors and noise, the selection and averaging strategies are equivalent. For the following illustrations, we assume that the elevation angle of the sources is 0 and we denote by $\alpha_{2q}^{(v,l)}(\boldsymbol{\theta}_p, \boldsymbol{\theta}_0)$, the spatial correlation coefficient of the two sources for the redundant left VA associated with $\mathbf{C}_{2q,x}^{(v,l)}$. It has been shown in [15] that, for $v = q$, $|\alpha_{2q,\mathrm{nr}}^{(v,l)}(\boldsymbol{\theta}_p, \boldsymbol{\theta}_0)|$ does not depend on l. Under these assumptions, Fig. 6 shows, for a UCA of N=5 omnidirectional sensors whose radius is equal to $r = 0.3\lambda$, for $(q, v, l) = (2, 2, 1)$ and $(q, v, l) = (2, 2, 2)$, the variations of both $|\alpha_{2q,\mathrm{nr}}^{(v,l)}(\boldsymbol{\theta}_p, \boldsymbol{\theta}_0)|^2$ (for the selection or averaging strategy) and $|\alpha_{2q}^{(v,l)}(\boldsymbol{\theta}_p, \boldsymbol{\theta}_0)|^2$ as a function of $\Delta\theta = \theta_p - \theta_0$. In order to obtain results that are independent of the angle θ_0, the quantities $|\alpha_{2q,\mathrm{nr}}^{(v,l)}(\boldsymbol{\theta}_p, \boldsymbol{\theta}_0)|^2$ and $|\alpha_{2q}^{(v,l)}(\boldsymbol{\theta}_p, \boldsymbol{\theta}_0)|^2$ are averaged over 360 values of θ_0. Note from Fig. 6 the smaller 3 dB-beamwidth of the left non-redundant VA with respect to the left redundant VA, and then the better asymptotic resolution of the $2q$th-order array processing method exploiting $\mathbf{C}_{2q,x,\mathrm{nr}}^{(v,l)}$ instead of $\mathbf{C}_{2q,x}^{(v,l)}$. Note also that this 3 dB-beamwidth depends on l contrary to that of the redundant left VA and that the best performance is obtained with l=2. The 3 dB-beamwidth thus seems to be minimized by an arrange-

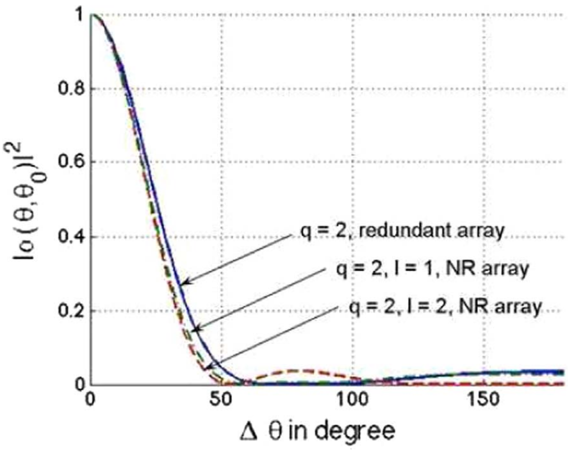

Fig. 6. $|\alpha_{2q,\mathrm{nr}}^{(v,l)}(\boldsymbol{\theta}_p, \boldsymbol{\theta}_0)|^2$ (Selection or Averaging) and $|\alpha_{2q}^{(v,l)}(\boldsymbol{\theta}_p, \boldsymbol{\theta}_0)|^2$ as a function of $\Delta\theta = \theta_p - \theta_0$, N=5, UCA, θ_0 is averaged over 360 values.

Table 1
Number of complex operations required by $2q$-MUSIC(v,l) and NR-$2q$-MUSIC(v,l) algorithms, P statistically independent sources, arrays with minimum redundancy.

	Computational complexity
$2q$-MUSIC(v,l)	$F_{2q}N^{2q} + (N^v + 1)(N^v - P)L$
S-$2q$-MUSIC(v,l)	$F_{2q}N_{2q,1}^{(v,l)}N_{2q,2}^{(v,l)} + (N_{2q,1}^{(v,l)} + 1)(N_{2q,1}^{(v,l)} - P)L$
A-$2q$-MUSIC(v,l)	$F_{2q}N^{2q} + (N_{2q,1}^{(v,l)} + 1)(N_{2q,1}^{(v,l)} - P)L$

ment which differs from the optimal arrangement (in terms of identifiability) of the redundant matrix $\mathbf{C}_{2q,x}^{(v,l)}$.

6.5. Computational complexity

The use of reduction matrices in (16) does not only improve the performance of $2q$-MUSIC(v, l), as shown in section 6.4, but also diminishes the computational complexity of both the cumulant estimation (selection strategy) and the NR-$2q$-MUSIC(v, l) search procedure. Indeed, for the selection strategy, as $\mathbf{C}_{2q,x,\mathrm{nr}}^{(v,l)}$ is obtained by discarding some rows and columns of $\mathbf{C}_{2q,x}^{(v,l)}$, it is not necessary to compute the latter. In this case, the computation of only $N_{2q,1}^{(v,l)}N_{2q,2}^{(v,l)}$ data $2q$th-order cumulants is required instead of N^{2q}. However, this property does not hold for the averaging strategy for which all the elements of $\mathbf{C}_{2q,x}^{(v,l)}$ have to be computed. On the other hand, whatever the chosen strategy, the redundancy removing generates left non-redundant virtual steering vectors, $\mathbf{a}_{q,p,1,\mathrm{nr}}^{(v,l)}$, of size $N_{2q,1}^{(v,l)}$ instead of N^v, which diminishes the number of operations to compute at each point of the NR-$2q$-MUSIC(v, l) pseudo-spectrum. Note that the number of operations required for all the other functions, such as the SVD computation of the rectangular cumulant matrix or the building of the non-redundant virtual steering vectors, can be neglected with respect to the data cumulant matrix estimation or to the NR-$2q$-MUSIC(v, l) search procedure. As the number of additions required for an operation is generally of the same order as the number of multiplications, we limit the analysis of the computational complexity to the determination of the number of multiplications. The computational complexities of the $2q$-MUSIC(v, l) and NR-$2q$-MUSIC(v, l) algorithms are synthesized in Table 1 as a function of the number of sensors N, of sources P, of snapshots K and of the number of computation points, L, of the

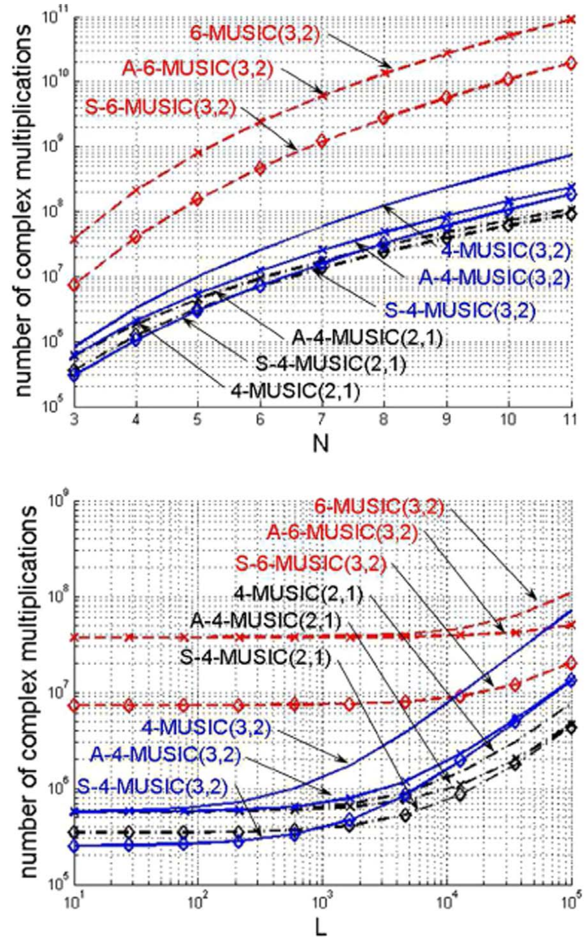

Fig. 7. Number of complex operations required by $2q$-MUSIC(v, l) and NR-$2q$-MUSIC(v, l) as a function of N for $(L, K, P) = (360, 1000, 2)$ (a) and of L for $(N, K, P) = (3, 1000, 2)$ (b).

Table 2
Number of complex operations required by $2q$-MUSIC(v,l) and NR-$2q$-MUSIC(v,l) algorithms, $(q, v, l) = (2, 2, 1), (2, 3, 2), (3, 3, 2)$, P statistically independent sources, arrays with minimum redundancy.

	$(q, v, l) = (2, 2, 1)$	$(q, v, l) = (2, 3, 2)$	$(q, v, l) = (3, 3, 2)$
$2q$-MUSIC(v,l)	$7KN^4$ $+(N^2 + 1)(N^2 - P)L$	$7KN^4$ $+(N^3 + 1)(N^3 - P)L$	$51KN^6$ $+(N^3 + 1)(N^3 - P)L$
S-$2q$-MUSIC(v,l)	$7K(N^2 - N + 1)^2$ $+(N^2 - N + 1)^2L$ $+((N^2 - N + 1)(1 - P) - P)L$	$7K(N^4 - N^3 + 2N^2)$ $+(N^3 - N^2 + 1)^2L/4$ $+((N^3 - N^2 + 1)(1 - P)/2 - P)L$	$51K(N^3 - N^2 + 2N)^2/4$ $+(N^3 - N^2 + 1)^2L/4$ $+((N^3 - N^2 + 1)(1 - P)/2 - P)L$
A-$2q$-MUSIC(v,l)	$7KN^4$ $+(N^2 - N + 1)^2L$ $+((N^2 - N + 1)(1 - P) - P)L$	$7KN^4$ $+(N^3 - N^2 + 1)^2L/4$ $+((N^3 - N^2 + 1)(1 - P)/2 - P)L$	$51KN^6$ $+(N^3 - N^2 + 1)^2L/4$ $+((N^3 - N^2 + 1)(1 - P)/2 - P)L$

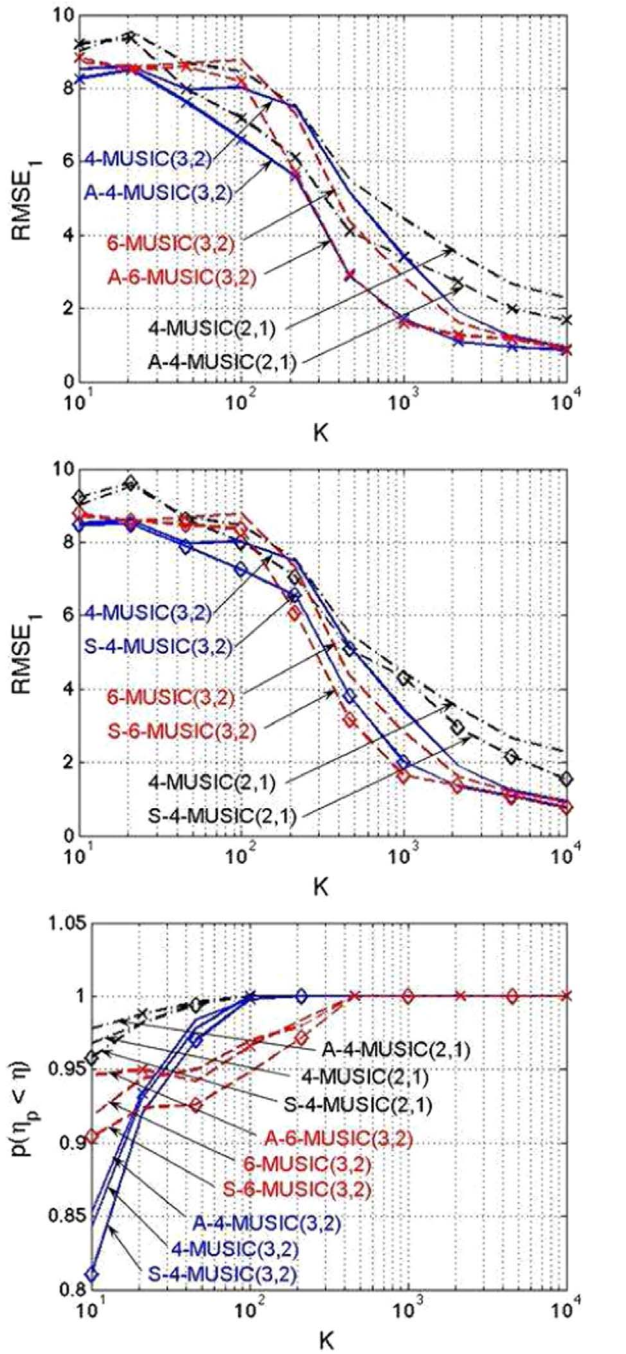

Fig. 8. RMSE of the source 1 and $p(\eta_1 \le \eta)$ as a function of K, P=2, N=3, UCA, SNR=10 dB, $\Delta\theta = 15^\circ$, $\sigma = 0.03$.

pseudo-spectrums. The results of Table 1 assume statistically independent sources and minimum redundancy arrays such as UCAs with a prime number of sensors. Moreover, in this table, S-2q-MUSIC(v, l) and A-2q-MUSIC(v, l) correspond to NR-2q-MUSIC(v, l) for the selection and the averaging strategies, respectively. The quantity $F_{2q}(K)$ denotes the computational cost associated with the estimation of one 2qth-order cumulant from K data samples. Note that the number of required operations presented in Table 1 does not take into account all the symmetries of the data cumulant matrix and these results can then be seen as worst case indications. To enlighten the values of Table 1, Table 2 presents the latter for some particular values of (q, v, l) corresponding to (2, 2, 1), (2, 3, 2) and (3, 3, 2) respectively. The variations of these values are then illustrated in Fig. 7 as a function of N for $(L, K, P) = (360, 1000, 2)$ and as a function of L for $(N, K, P) = (3, 1000, 2)$ respectively. Note that for a given value of (q, v, l), 2q-MUSIC(v, l) always exhibits the highest complexity, followed by A-2q-MUSIC(v, l) and S-2q-MUSIC(v, l). Moreover, we also note that the complexity reduction related to the removing of redundancy increases with q. For small values of L, the cost of the cumulants estimation prevails, and the complexity of A-2q-MUSIC(v, l) is approximately the same as the complexity of 2q-MUSIC(v, l) (cf. 6-MUSIC(3,2) and A-6-MUSIC(3,2) in Fig. 7 (top)). For increasing values of L, the computational complexity of A-2q-MUSIC(v, l) approaches that of S-2q-MUSIC(v, l) (cf. Fig. 7 (bottom)) because the computational cost of the pseudo-spectra, which is the same for all NR-2q-MUSIC(v, l) methods, plays an increasingly important role.

6.6. Computer simulations

To compare the performance of NR-2q-MUSIC(v, l) and 2q-MUSIC(v, l) for overdetermined mixtures of sources, we consider the scenario of section 5.5.2. Fig. 8 shows the variations, as a function of the number of snapshots K, of the RMSE for the source 1, RMSE$_1$, and the associated probability of non-aberrant results, $p(\eta_1 \le \eta)$, (we obtain similar results for the source 2), estimated from M=500 realizations, at the output of different versions of 4-MUSIC and 6-MUSIC. Note that the probability of non-aberrant results is equal to 1 beyond K=100 snapshots for 4-MUSIC(2, 1), S-4-MUSIC(2, 1), A-4-MUSIC(2, 1), 4-MUSIC(3, 2), S-4-MUSIC(3, 2), A-4-MUSIC(3, 2), and beyond K=500 snapshots for 6-MUSIC(3, 2), S-6-MUSIC(3, 2) and A-6-MUSIC(3, 2) due to a higher variance in the cumulant estimation. In terms of global performance, regardless of the values of (q, v, l), NR-2q-MUSIC(v, l) outperforms 2q-MUSIC(v, l) and A-NR-2q-MUSIC(v, l) is always the best beyond K=100 for q=2 and beyond K=500 for q=3. For large numbers of time samples, rectangular 4-MUSIC(3, 2), S-4-MUSIC(3, 2) and A-4-MUSIC(3, 2) achieve comparable performances corresponding to those of square 6-MUSIC(3, 2), S-6-MUSIC(3, 2) and A-6-MUSIC(3, 2), which are clearly better than those achieved by square 4-MUSIC(2, 1), S-4-MUSIC(2, 1) and A-4-MUSIC(2, 1).

7. Conclusion

In this paper, rectangular arrangements of the 2qth-order ($q \ge 1$) circular cumulants of the data instead of square ones (as for classical 2q-MUSIC) have been considered for high resolution direction finding from arbitrary arrays of sensors instead of specific linear arrays (as for the nested array strategy), giving rise to the rectangular 2q-MUSIC algorithms. The performance and the identifiability of rectangular 2q-MUSIC have been analyzed for statistically independent sources through the extension of the HO VA concept, initially introduced in [15] for square arrangements of the HO data statistics, to rectangular arrangements. It has been shown that rectangular arrangements allow optimizing the trade-off between performance and maximal number of sources to be estimated. In addition, they allow a complexity reduction for a given level of performance. To further improve the previous trade-off while still decreasing the complexity of rectangular 2q-MUSIC, two strategies of removing the redundancy of rectangular arrangements of the data statistics have been considered and compared, giving rise to the non-redundant rectangular 2q-MUSIC algorithms. A performance and a complexity analysis of these new algorithms have shown that the selection strategy is optimal in terms of complexity reduction while the averaging strategy is optimal in terms of performance. These results should open new areas of research in the domain of tensorial array processing methods [23–25] and should allow the development of new powerful array processing methods. One important open question concerns in particular the optimal unfolding of a tensor, i.e., the optimal combination of its rectangular modes for a given application (direction finding, blind source separation) and a given level of performance.

Appendix A. Supplementary data

Supplementary data associated with this article can be found in the online version at http://dx.doi.org/10.1016/j.sigpro.2016.10.020.

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