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TR-MUSIC—A robust frequency estimation method in impulsive noise

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Abstract

Sinusoidal frequency estimation has been studied for many years. The MUSIC method represents a class of super-resolution methods based on subspace decomposition. However, the MUSIC method has poor performance in impulsive noise environments due to the prevalence of outliers and very large noise variance. A more robust method called trimmed correlation based-MUSIC (TR-MUSIC) method is proposed in this paper. Through a trimming operation, outliers in the samples participating in the correlation calculation are discarded, yielding a correlation sequence that is closer to the true underlying correlation. The amount of trimming is determined by the Mahalanobis distance in which robust estimates of location and scale are utilized to compensate for outlier effects. Frequency estimation results from the eigendecomposition of the trimmed correlation matrix. In the simulations, we take α-stable noise ($\alpha > 1$) as an example of impulsive noise. The proposed method is very robust and performs better than the conventional MUSIC and other robust methods. Furthermore, it can be applied to real signals as well as complex signals.

Keywords: Frequency estimation; MUSIC method; Impulsive noise; Trimmed-correlation

1. Introduction

Frequency estimation of sinusoidal signals embedded in white noise is frequently encountered in many applications such as radar, sonar and speech processing. It has been a classical problem in signal processing for many years. Among the methods proposed to address this problem, subspace-based methods such as MUSIC [1] and its variants, ESPRIT [2], and minimum norm method [3], have been extensively used due to their super-resolution capability. These methods utilize the orthogonality between the signal subspace and the noise subspace to estimate the frequencies through either eigendecomposition or singular value decomposition of the autocovariance matrix. These approaches give very accurate frequency estimates in conventional white noise case.

In some circumstances, however, noise may exhibit impulsive characteristics. For example, α-stable noise [4] has heavier tails than Gaussian noise and does not even possess finite variance. Impulsive noise occurs quite often in some practical situations, examples of which include radar clutter, underwater acoustics, and seismological measurements [5]. As

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such, many signal processing problems have been studied under an impulsive noise framework, e.g. [6–8]. The outliers appearing in the noise significantly distort the original signal, deteriorate the conventional autocorrelation estimates, and consequently degrade the performance of the aforementioned subspace-based methods [9,10]. Therefore, robust frequency estimators need to be developed to properly handle impulsive noise environments.

Several robust methods for either frequency or spectral estimation have been reported in the literature. For example, Schroeder et al. proposed a method based on L_p normed spectral estimation to find the frequencies [11]. This method changes the least-square criterion to L_p normed criterion and estimates the frequencies by finding the minimum of the modified cost function. In [12], a robust covariation-based MUSIC method is developed under the α-stable noise model. This method replaces the covariance matrix in the conventional MUSIC method by the covariation matrix, which can be constructed from the fractional lower order moments of the α-stable process. A robust method to calculate the periodogram under impulsive observations is given in [13], where Huber's minimax regression estimation [14] is employed. A method based on sign and rank correlation is proposed in [9] that estimates the autocovariance matrix by the sign and rank correlations, thereby leading to robust frequency estimation from MUSIC. In an alternative approach, a new median correlation theory from which robust frequency estimation results is presented in [15]. Among these methods, [9,12,15] rely on eigendecomposition of the autocovariance matrix or the covariation matrix and are thus reminiscence of the MUSIC method. The approach in [13] is non-parametric and thus has limited resolution. Also, [11] is an extension of the modified covariance method [16], in which the cost function is given by the L_p norm of the prediction error.

Nearly all spectral estimation methods need to calculate the autocorrelation of the data. The subspace methods, such as MUSIC, provide high resolution for closely spaced frequencies since they exploit the particular structure of the sinusoidal signal and the fact that the signal subspace is orthogonal to the noise subspace. However, the estimated autocovariance matrix becomes unreliable when impulsive noise is present [9]. In this paper, a new frequency estimation method which is based on MUSIC, but is more robust to impulsive noise, is proposed. In order to obtain a cleaned version of the data, a trimming operation is applied to the two vectors contributing to the correlation calculation. The resulting correlation estimate is referenced to as the α-*trimmed correlation*, since it is analogous to the α-trimmed mean [17]. The MUSIC spectrum is computed from the α-trimmed correlation. The method is therefore called trimmed correlation based-MUSIC (TR-MUSIC). The proposed method has several useful features:

(1) The estimated trimmed-autocorrelation matrix retains the Hermitian symmetric structure.
(2) It can be applied to complex signals as well as real signals.
(3) No specific model for impulse noise is imposed except that noise samples are i.i.d., which is necessary for the applicability of the MUSIC method, and noise has finite first-order moment[1] (mean) and is zero-mean.

Simulation studies show that the proposed method performs very well even in very impulsive noise environments where the conventional MUSIC method fails to locate the correct frequencies. A comparison with two other robust methods [9,11] demonstrates that the proposed TR-MUSIC method outperforms these methods in the sense that the TR-MUSIC has smaller mean squared error, although all three methods yield better results than the conventional MUSIC in impulsive noise.

The remainder of this paper is organized as follows. Section 2 formulates the frequency estimation problem, briefly reviews the MUSIC method, and also investigates the problems of the MUSIC method in impulsive noise. In Section 3, a robust TR-MUSIC method is proposed for both real and complex signals. Corroborating simulations are given in Section 4 to illustrate the improvement in performance achieved by the proposed method over the other two robust methods and the conventional MUSIC method.

2. Preliminaries and motivation

In this paper, the signals of interest are complex (real) sinusoids corrupted by additive noise. Such a signal is sometimes called a harmonic model. For real signals, complex exponentials form a complex conjugate pair of sinusoids, while for complex

[1]In the context of α-stable noise, this condition implies $\alpha > 1$.

signals, they occur at a single frequency. In general, a complex harmonic model can be described by

$$x(n) = \sum_{i=1}^{p} A_i \exp[\mathrm{j}(2\pi f_i n + \phi_i)] + w(n), \quad n = 0, 1, \ldots, N-1, \tag{1}$$

where p is the total number of signal sources and is assumed to be known here, N is the total length of the observed data, f_i is the frequency of interest from the ith source, ϕ_i is the phase and is assumed to be uniformly distributed in $[0, 2\pi)$, A_i is the unknown amplitude, and $w(n)$ is the zero mean additive noise which is impulsive in nature. In addition, the noise samples are assumed to be i.i.d., which is analogous to the conventional case where the noise is white. Our aim is to find the p frequencies in (1) through the MUSIC spectral estimation.

2.1. The MUSIC method

In the MUSIC method, the autocorrelation matrix of input signal $\mathbf{R}_{xx}$ is formed by the data vector $\mathbf{x}(n) = [x(n), x(n-1), \ldots, x(n-M+1)]^{\mathrm{T}}$ of sliding window length M. Applying eigendecomposition to $\mathbf{R}_{xx}$, we obtain

$$\begin{aligned} \mathbf{R}_{xx} &= \mathbf{V}\mathbf{\Lambda}\mathbf{V}^H \\ &= \sum_{i=1}^{M} \lambda_i \mathbf{v}_i \mathbf{v}_i^H \\ &= \sum_{i=1}^{p} (MP_i + \sigma_w^2)\mathbf{v}_i \mathbf{v}_i^H + \sum_{i=p+1}^{M} \sigma_w^2 \mathbf{v}_i \mathbf{v}_i^H, \end{aligned} \tag{2}$$

where $P_i = A_i^2$ is the signal power. The first p eigenvectors $\mathbf{v}_i$, $i = 1, \ldots, p$ span the signal subspace, while the rest $M - p$ eigenvectors span the noise subspace. Utilizing the orthogonal relationship between the two subspaces, the MUSIC spectrum can be calculated as

$$S_{\mathrm{MUSIC}}(f) = \frac{1}{\sum_{i=p+1}^{M} |\mathbf{s}^H(f)\mathbf{v}_i|}. \tag{3}$$

Since the vector $\mathbf{s}(f) = [1, \mathrm{e}^{-\mathrm{j}2\pi f}, \ldots, \mathrm{e}^{-\mathrm{j}2\pi f(M-1)}]^{\mathrm{T}}$ is orthogonal to $\mathbf{v}_i$, $i = p+1, \ldots, M$ at $f = f_i$, $i = 1, \ldots, p$, the summation in the denominator of (3) is zero. Consequently, the MUSIC spectrum will have peaks at these frequencies. Therefore, the frequencies can be estimated by locating the p frequencies where $S_{\mathrm{MUSIC}}(f)$ attains peaks.

2.2. Problems in impulsive noise case

MUSIC spectral estimation provides an easy and effective method to extract the frequency information from signals in additive white noise. It has high resolution and works very well even at moderately low SNR. However, when the noise exhibits impulsive characteristics, several issues of concern arise.

First, consider the impact of impulsive noise on the power spectrum. For the signal model given in (1), the autocorrelation sequence is

$$r_{xx}(m) = E\{x^*(n)x(n+m)\} = \sum_{i=1}^{p} A_i^2 \exp(\mathrm{j}2\pi f_i m) + \sigma_w^2 \delta(m). \tag{4}$$

The power spectrum $S_{xx}(f)$ is the Fourier transform of the autocorrelation sequence

$$S_{xx}(f) = \sum_{i=1}^{p} A_i^2 \delta(f - f_i) + \sigma_w^2. \tag{5}$$

As an example, for two complex sinusoids with frequencies at f_1 and f_2 embedded in conventional white noise, the theoretical power spectrum is plotted in Fig. 1(a).

The difference between conventional white noise and impulsive noise is caused by the characteristics of their distribution tails. In order to quantify how impulsive a distribution is, "tail heaviness" given by $\Pr(|X| > a)$ as $a \to \infty$ is used as a measure [18]. Even though tail heaviness reflects the extent of impulsiveness, its impact on the MUSIC method requires further investigation on the noise variance since the MUSIC method is a subspace method that relies on second-order statistics (correlation matrix).

In the case of impulsive noise, the variance of the noise is significantly increased and may even becomes infinite, for instance, in α-stable noise case. This can be shown through some analysis on the noise variance. Suppose we have two types of noise: $w(n)$ being impulsive and $v(n)$ being conventional white noise (non-heavy-tailed). Then from [18], we know that the tail heaviness of $w(n)$ is much heavier than that of $v(n)$. Therefore, $\Pr(|W| > a) \gg \Pr(|V| > a)$ as $a \to \infty$. Using Chebyshev inequality, we have

$$\Pr(|W| > a) < \frac{\sigma_w^2}{a^2}, \tag{6a}$$

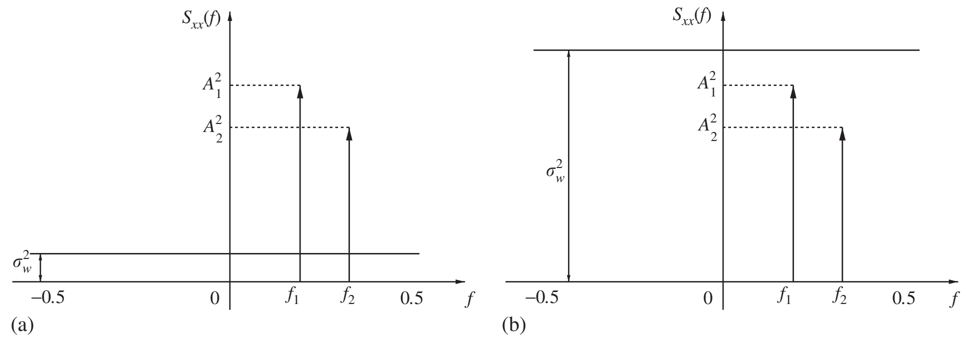

Fig. 1. Power spectrum in (a) conventional white noise and (b) impulsive noise.

$$\Pr(|V|>a)<\frac{\sigma_v^2}{a^2}. \tag{6b}$$

When $a \to \infty$, the bound is tight and the inequalities in (6a) and (6b) become equalities. Thus, the noise variances can be written as $\sigma_w^2 = a^2 \Pr(|W|> a)$ and $\sigma_v^2 = a^2 \Pr(|V|>a)$ when $a \to \infty$. Finally, the ratio of the two noise variances is

$$\frac{\sigma_w^2}{\sigma_v^2} = \lim_{a\to\infty} \frac{\Pr(|W|>a)}{\Pr(|V|>a)} \geqslant 1, \tag{7}$$

which indicates that the noise variance is significantly increased due to the heavier tail of impulsive noise.

Due to the increased noise power, the underlying signal spectrum may be overwhelmed by the noise spectrum. In practice, the peaks corresponding to the frequencies in the spectrum have finite values, and thus will be indistinguishable if the noise variance is sufficiently large. This phenomenon is illustrated by Fig. 1(b), where the two peaks are totally submerged by the noise.

The MUSIC spectrum is actually a pseudo-spectrum since it is calculated from the eigendecomposition rather than from the original definition of the power spectrum. Thus, it is not shaped as shown in Fig. 1(b). However, the MUSIC method still relies on autocorrelation estimates, which are significantly degraded by the impulsive noise. Note that in (4), if the noise is impulsive, or in the case of α-stable noise, the variance is infinite, then the autocorrelation $r_{xx}(m)$ is far less accurate than the conventional white noise case, especially at lag 0. Furthermore, the eigenstructure of the autocorrelation matrix is dramatically altered by large noise variance. It follows from (2) that the diagonal matrix containing the eigenvalues of $\mathbf{R}_{xx}$ has the following structure:

$$\mathbf{\Lambda} = \begin{bmatrix} MP_1+\sigma_w^2 & & & & & & \\ & MP_2+\sigma_w^2 & & & & \mathbf{0} & \\ & & \ddots & & & & \\ & & & MP_p+\sigma_w^2 & & & \\ & & & & \sigma_w^2 & & \\ & \mathbf{0} & & & & \ddots & \\ & & & & & & \sigma_w^2 \end{bmatrix} \tag{8}$$

The eigenvalues are normally sorted in descending order such that $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \lambda_p \geqslant \lambda_{p+1} = \lambda_{p+2} = \cdots = \lambda_M = \sigma_w^2$. However, due to the large and possibly infinite variance of the noise, the eigenvalues in the signal subspace and the eigenvalues in the noise subspace tend to be identical. In other words, the condition number of $\mathbf{R}_{xx}$ is

$$\begin{aligned} \mathrm{cond}(\mathbf{R}_{xx}) &= \frac{\lambda_{\max}}{\lambda_{\min}} \\ &= \lim_{\sigma_w^2\to\infty} \frac{MP_1+\sigma_w^2}{\sigma_w^2} \\ &= 1. \end{aligned} \tag{9}$$

This result has a detrimental effect on the conventional MUSIC method as in this case it is difficult to correctly select the $M-p$ eigenvectors in the noise subspace since they cannot be clearly identified by their corresponding eigenvalues. Consequently, separation of signal and noise subspaces becomes a difficult problem in the impulsive noise scenario. Hence, in the following section, we propose a new method to improve the applicability of the MUSIC method in the impulsive noise case.

3. Proposed method: TR-MUSIC

The conventional MUSIC method applies to the signal model in (1) quite well if the noise $w(n)$ is not impulsive. The difference between impulsive noise and non-impulsive noise in the time domain is that impulsive noise usually contains outliers whose magnitudes are significantly larger than the rest of the noise samples. The distribution tails of impulsive noise are heavier than those of non-impulsive one. For example, both double exponential (Laplacian) and α-stable distributions have heavier tails than the commonly assumed Gaussian distribution. In view of this fact, we choose to reduce the effects of those observed samples corrupted by outlying noise samples, thus obtaining a cleaned version of the observed data. One method for varying the effect that samples can have on correlation estimates is to weight the samples in the correlation estimate. However, this approach will possibly alter the inherent correlation structure among the samples, and is therefore not considered here.

From robust statistics, we know that one way to estimate the mean is using the α-trimmed mean [17]. Here, similar to the α-trimmed mean, we introduce the α-trimmed correlation, or simply trimmed correlation. In the conventional sample correlation calculation

$$\hat{r}_{xx}(m) = \frac{1}{N} \sum_{n=0}^{N-m-1} x^*(n)x(n+m), \tag{10}$$

all samples $x(0), x(1), \ldots, x(N-1)$ participate in the calculation. By eliminating those outlying samples, we may improve the correlation estimate. To elaborate on this, we divide the problem into two cases: real signals and complex signals. The method for the real signals is described first followed by later extensions to complex signals.

3.1. Real signal case

In the real signal case, the sample correlation at time lag m is given by

$$\hat{r}_{xx}(m) = \begin{cases} \dfrac{1}{N} \displaystyle\sum_{n=0}^{N-m-1} x(n)x(n+m) & \text{if } m \geqslant 0, \\ \hat{r}_{xx}(-m) & \text{if } m < 0. \end{cases} \tag{11}$$

Without loss of generality, we examine the case $m \geqslant 0$. If we denote the vector $\mathbf{x}^b = [x(0), x(1), \ldots, x(N-m-1)]^{\mathrm{T}}$ as the lagging vector and $\mathbf{x}^f = [x(m), x(m+1), \ldots, x(N-1)]^{\mathrm{T}}$ as the leading vector, the autocorrelation is simply the inner product of these two vectors, i.e., $\hat{r}_{xx} = (\mathbf{x}^b)^{\mathrm{T}} \cdot \mathbf{x}^f/N$. Note that both vectors may be contaminated by some outliers from the impulsive noise. Therefore, to obtain more accurate correlation estimates, those outlying samples are discarded. We can carry out a trimming operation on both the lagging vector $\mathbf{x}^b$ and the leading vector $\mathbf{x}^f$.

In the α-trimmed correlation, samples in the two vectors are first rank ordered. Then α percent of samples are trimmed on both ends. The remaining samples are further trimmed based on the temporal and rank order relationship. Finally, the trimmed correlation is calculated based on the remaining samples.

The parameter α determines how many samples are trimmed from the original samples and is calculated as the proportion of outlying samples

$$\alpha = \frac{n_{\mathrm{o}}}{N}, \tag{12}$$

where n_{o} is the total number of outliers. To appropriately trim the outliers, we first order the samples in the vector $\mathbf{x}^b$ and $\mathbf{x}^f$, thus obtaining two new ordered vectors

$$\mathbf{x}_{\mathrm{o}}^b = [x_{(1)}^b, x_{(2)}^b, \ldots, x_{(N-m)}^b]^{\mathrm{T}} \tag{12a}$$

$$\mathbf{x}_{\mathrm{o}}^f = [x_{(1)}^f, x_{(2)}^f, \ldots, x_{(N-m)}^f]^{\mathrm{T}}, \tag{12b}$$

where the samples are arranged in non-decreasing order, i.e., $x_{(1)}^b \leqslant x_{(2)}^b \leqslant \cdots \leqslant x_{(N-m)}^b$ and $x_{(1)}^f \leqslant x_{(2)}^f \leqslant \cdots \leqslant x_{(N-m)}^f$. After the percent of trimming α is determined, the number of samples trimmed is set as

$$t = \lfloor \alpha(N-m) \rfloor. \tag{14}$$

Then we trim the t smallest and largest samples in both $\mathbf{x}_{\mathrm{o}}^b$ and $\mathbf{x}_{\mathrm{o}}^f$ and the remaining samples are stored for further processing. Note that the trimming is bilateral since outliers can be either positive or negative for real-valued signals.

To determine the parameter α, a quantitative measure is needed to detect the outliers in the samples. One such measure is given by the *Mahalanobis distance* (MD) [19], which measures the distance of the data vector to the central mass of the whole set of data:

$$d(\mathbf{x}, \boldsymbol{\mu}_x) = (\mathbf{x} - \boldsymbol{\mu}_x)^{\mathrm{T}} \mathbf{C}^{-1} (\mathbf{X} - \boldsymbol{\mu}_x), \tag{15}$$

where $\boldsymbol{\mu}_x$ is the mean vector and $\mathbf{C}$ is the covariance matrix. In the case at hand, we have only single time series and can redefine the MD of the scalar random variable as

$$d(x(n), \mu_x) = \frac{[x(n) - \mu_x]^2}{\sigma_x^2}. \tag{16}$$

Similar to (15), μ_x is the mean and σ_x^2 is the variance.

In order to calculate the percentage of trimming α, a threshold ξ must be selected to determine the total number of outliers n_o. The selection of the threshold ξ can be based on the computation of the MD for the whole data set $\{x(n)\}_{n=0}^{N-1}$. We should note that outliers have much larger MD than normal samples, i.e., outliers are farther away from central mass of the data than normal samples. Therefore, after calculating the MD for all data samples, most outliers have MD greater than a certain value while most normal samples have MD below this value as can be seen from Fig. 2. Then this value is chosen as the threshold. Upon selection of ξ, we start from sample $x(0)$ and compare its MD with the threshold. If $d(x(0), \mu_x) > \xi$, we increase n_o by 1; otherwise, n_o remains unchanged. Continue doing this until we reach the last sample $x(N-1)$. The final value of n_o we obtain is used to calculate α by (12).

Note that in practice, μ_x and σ_x^2 in the calculation of (16) are unknown and must be estimated from the data. Sample mean and sample variance could be used if there were only very few outliers. Utilizing this approach in the presence of a significant number of outliers, however, makes the MD of outliers less than those of the normal samples. Hence, we adopt more robust estimates for the location and scale parameters. For μ_x, we replace it by the sample median

$$\hat{\mu}_x = \mathrm{med}(x(0), x(1), \ldots, x(N-1)). \tag{17}$$

For σ_x^2, we replace it by the square of the median absolute deviation (MAD) [14]

$$\hat{\sigma}_x^2 = \left(\mathrm{med}\left\{\left|x(n) - \underset{0 \leqslant n \leqslant N-1}{\mathrm{med}} x(n)\right|\right\}\right)^2. \tag{18}$$

Since the autocorrelation involves temporal relationship between samples, the temporal information of the remaining samples after trimming must be retained. In addition, when a sample $x(n)$ in $\mathbf{x}_o^b$ is trimmed, its counterpart $x(n+m)$ in $\mathbf{x}_o^f$ should also be eliminated from correlation calculation and vice versa. However, the trimming procedures

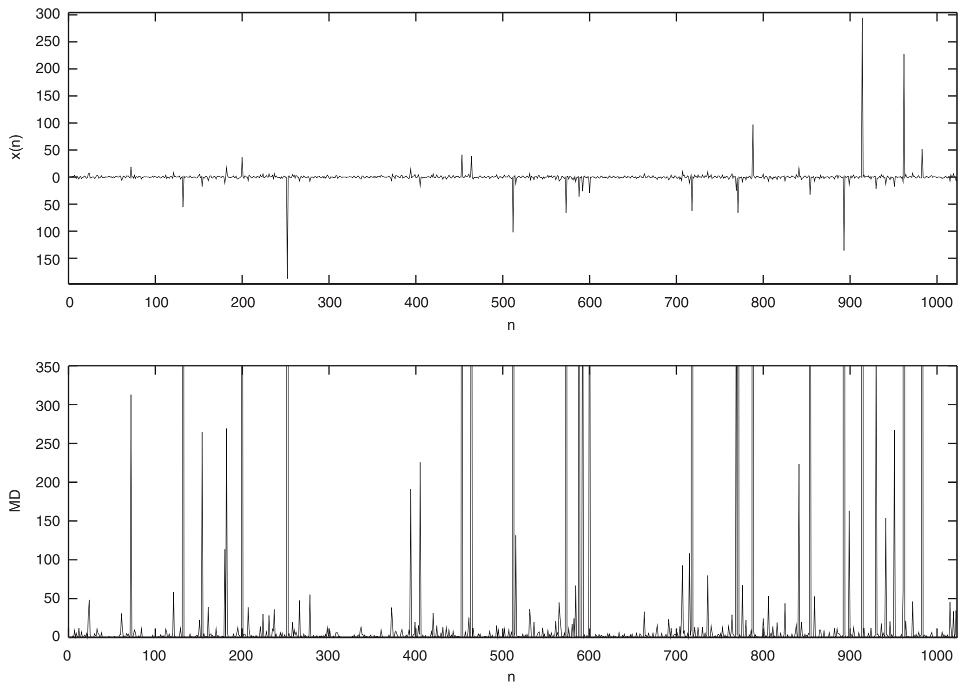

Fig. 2. Data samples and their Mahalanobis distances.

described above does not take this into account. In the following, we consider this problem by using set operations. We name such kind of procedure as *retrimming*.

Denote the set of the time indices of samples in the ordered lagging vector $\mathbf{x}_o^b$ as

$$\mathscr{I}^b = \{i_1, \ldots, i_t, i_{t+1}, \ldots, i_{N-m-t}, i_{N-m-t+1}, \ldots, i_{N-m}\}. \tag{19}$$

We also define $\mathscr{T}^b$ as the set of time indices corresponding to the samples trimmed from $\mathbf{x}_o^b$

$$\mathscr{T}^b = \{i_1, \ldots, i_t, i_{N-m-t+1}, \ldots, i_{N-m}\}. \tag{20}$$

Consequently, the set of the time indices of the surviving samples after trimming in $\mathbf{x}_o^b$ is given by

$$\begin{aligned}\mathscr{R}^b &= \mathscr{I}^b - \mathscr{T}^b \\ &= \{i_{t+1}, i_{t+2}, \ldots, i_{N-m-t}\}.\end{aligned} \tag{21}$$

Similarly, we can define the sets of the time indices for the total samples, trimmed samples, and surviving samples in $\mathbf{x}_o^f$ as

$$\begin{aligned}\mathscr{I}^f = \{&k_1, \ldots, k_t, k_{t+1}, \ldots, k_{N-m-t}, \\ &k_{N-m-t+1}, \ldots, k_{N-m}\},\end{aligned} \tag{22}$$

$$\mathscr{T}^f = \{k_1, \ldots, k_t, k_{N-m-t+1}, \ldots, k_{N-m}\}, \tag{23}$$

$$\mathscr{R}^f = \{k_{t+1}, k_{t+2}, \ldots, k_{N-m-t}\}. \tag{24}$$

Note that the time indices in the set $\mathscr{I}^f$ are those in $\mathscr{I}^b$ shifted by the autocorrelation lag m. Therefore, subtracting k_l in $\mathscr{T}^f$ by m will map the time indices in $\mathscr{T}^f$ to those in $\mathscr{T}^b$. We can thus define a new index set

$$\begin{aligned}\bar{\mathscr{T}}^f = \{&k_1 - m, \ldots, k_t - m, \\ &k_{N-m-t+1} - m, \ldots, k_{N-m} - m\},\end{aligned} \tag{25}$$

which is formed from the index set $\mathscr{T}^f$ with each element shifted by m. Since the trimming operation is performed on both $\mathbf{x}_o^b$ and $\mathbf{x}_o^f$, the complete set of trimmed samples must be taken into account. When an outlier $x(i_p)$ in $\mathbf{x}^b$ with time index $i_p \in \mathscr{T}^b$ is trimmed, its counterpart $x(i_p + m)$ in $\mathbf{x}^f$ should also trimmed since the trimmed correlation involves the sum of the product of two samples. The same rule applies if an outlier is trimmed in $\mathbf{x}^f$ (we then trim its counterpart in $\mathbf{x}^b$). The union of $\mathscr{T}^b$ and $\bar{\mathscr{T}}_f$ is the complete set of all time indices of trimmed samples after we consider this interaction between the lagging and leading vectors. Therefore, the time index set of the remaining samples that will participate in the trimmed correlation calculation is given by

$$\mathscr{R} = \mathscr{I}^b - (\mathscr{T}^b \cup \bar{\mathscr{T}}^f). \tag{26}$$

Finally, the trimmed autocorrelation is defined as

$$\hat{r}_{xx}^{\mathrm{tr}}(m) = \begin{cases} \dfrac{1}{N_{\mathscr{R}}} \displaystyle\sum_{n \in \mathscr{R}} x(n)x(n+m) & \text{if } m \geqslant 0, \\ \hat{r}_{xx}^{\mathrm{tr}}(-m) & \text{if } m < 0, \end{cases} \tag{27}$$

where $N_{\mathscr{R}} = |\mathscr{R}|$ is the cardinality of $\mathscr{R}$.

The trimmed autocorrelation retains the symmetry property since for negative lags, we only need to switch the role of the lagging vector $\mathbf{x}_o^b$ and the leading vector $\mathbf{x}_o^f$. This is a desirable property for the autocorrelation since it results in a symmetric autocorrelation matrix:

$$\hat{\mathbf{R}}_{xx}^{\mathrm{tr}} = \begin{bmatrix} \hat{r}_{xx}^{\mathrm{tr}}(0) & \hat{r}_{xx}^{\mathrm{tr}}(1) & \cdots & \hat{r}_{xx}^{\mathrm{tr}}(M-1) \\ \hat{r}_{xx}^{\mathrm{tr}}(1) & \hat{r}_{xx}^{\mathrm{tr}}(0) & \cdots & \hat{r}_{xx}^{\mathrm{tr}}(M-2) \\ \vdots & \vdots & \ddots & \vdots \\ \hat{r}_{xx}^{\mathrm{tr}}(M-1) & \hat{r}_{xx}^{\mathrm{tr}}(M-2) & \cdots & \hat{r}_{xx}^{\mathrm{tr}}(0) \end{bmatrix}. \tag{28}$$

In the α-trimmed correlation, trimming is performed based on the parameter α, which represents the percentage of outliers and is determined by computing the MD for the whole data. As an alternative approach, one may trim the leading and lagging vectors by directly computing the MD for each sample in the two vectors and discard those samples with MD greater than a preset threshold. Similar procedures can be carried out to retrim samples with temporal correspondence in the two vectors as is done in the α-trimmed correlation. However, this approach is inefficient because it needs to evaluate the MD for each samples at each correlation lag. If we need to calculate correlation for a large number of lags, the computational burden is formidable. In addition, we find through the simulations that trimming based on α-trimmed correlation yields better performance than trimming directly based on MD. Hence, we adopt α-trimmed correlation and utilize it to estimate the frequencies.

The trimmed correlation-based MUSIC spectrum is formed on the basis of eigendecomposition of $\hat{\mathbf{R}}_{xx}^{\mathrm{tr}}$, which can be readily computed as

$$S_{\mathrm{TR-MUSIC}}^{r}(f) = \frac{1}{\sum_{i=2p+1}^{M} |\mathbf{s}^H(f)\mathbf{v}_i^{\mathrm{tr}}|}, \tag{29}$$

where $\mathbf{v}_i^{\mathrm{tr}}$ is the noise subspace eigenvector of the trimmed autocorrelation matrix $\hat{\mathbf{R}}_{xx}^{\mathrm{tr}}$. Note that for real signals, the dimension of the signal subspace is $2p$ because each real sinusoid is the sum of two complex exponentials. The frequencies are estimated by locating p peaks in $S^r_{\mathrm{TR-MUSIC}}(f)$ over the normalized frequency range $[0, 1/2]$ for real signals.

3.2. Complex signal case

The proposed TR-MUSIC method can also be applied to complex signals. Following the derivations in the previous section, some modifications need to be made in order to apply this method to complex signals.

First, for the ordering of the samples as in (13), we order the samples according to their magnitudes. Also, the Mahalanobis distance in (16) is changed to reflect the magnitude ordering

$$d^c(x(n), \mu_x) = \frac{|x(n) - \mu_x^c|^2}{\sigma_x^{2\,c}}. \tag{30}$$

Again, the location μ_x^c and the variance $\sigma_x^{2\,c}$ estimates are calculated as

$$\hat{\mu}_x^c = \mathrm{medc}(x(0), x(1), \ldots, x(N-1)), \tag{31}$$

$$\hat{\sigma}_x^{2c} = \left(\mathrm{med}\left\{\left|x(n) - \underset{0\leqslant n\leqslant N-1}{\mathrm{medc}}\; x(n)\right|\right\}\right)^2, \tag{32}$$

where $\mathrm{medc}(\cdot)$ takes the sample with median magnitude for complex data.

After the amount of trimming t is determined as per (14), the t samples with the largest magnitudes are trimmed in both lagging and leading vector. Note that the trimming for complex signal is unilateral, which is different from the real signal case since only those samples with large magnitudes are considered as outliers. Finally, the trimmed autocorrelation for a complex signal is given by

$$\hat{r}_{xx}^{\mathrm{trc}}(m) = \begin{cases} \dfrac{1}{N_{\mathscr{R}^c}} \displaystyle\sum_{n\in\mathscr{R}^c} x^*(n)x(n+m) & \text{if } m \geqslant 0, \\ (\hat{r}_{xx}^{\mathrm{trc}}(-m))^* & \text{if } m < 0. \end{cases} \tag{33}$$

In (33), $\mathscr{R}^c$ is the set containing the time indices of the remaining samples after trimming t samples and retrimming based on the temporal and rank order relationship. Accordingly, the TR-MUSIC spectrum for complex signals is computed as

$$S^c_{\mathrm{TR-MUSIC}}(f) = \frac{1}{\sum_{i=p+1}^{M} |\mathbf{s}^H(f)\mathbf{v}_i^{\mathrm{trc}}|}, \tag{34}$$

where $\mathbf{v}_i^{\mathrm{trc}}$ is the noise subspace eigenvector of the trimmed autocorrelation matrix.

4. Case studies

We use simulations to demonstrate the results obtained from the TR-MUSIC method proposed in this paper for both real and complex signals. In the first example, the signal model is real. We show the performance improvement of the TR-MUSIC over the MUSIC and also compare it with the SIGN-MUSIC proposed in [9]. In the second example, we show the results of the TR-MUSIC method applied to complex signals and compare it with the L_p norm linear predictive spectral estimation method in [11].

There exists several different ways to model impulsive noise that contains outliers. Among them, the α-stable distribution is perhaps the most well suited and widely used since it obeys the Generalized Central Limit Theorem [4] and the extent of impulsiveness is determined by its distribution parameters. Therefore, we use it to model the impulsive noise in the presented simulations. In the following two examples, both spectrum figures and normalized mean squared error (NMSE) figures are given for comparison. In addition, for the real signal example, we also compare the TR-MUSIC based on α-trimmed correlation and the TR-MUSIC based on direct MD trimming. Each spectrum figure is generated by 30 overlaid independent realizations. The NMSE figure is plotted from the results of 1000 independent Monte Carlo trials. The total data length N is 1024. For the definition of signal-to-noise ratio (SNR) in α-stable noise case, we adopt the SNR definition utilized in [20], in which the geometric power of α-stable noise is used

$$\mathrm{SNR} = \frac{P_s}{P_n} = \frac{A^2}{(C_g\gamma)^{2/\alpha'}/C_g^2}, \tag{35}$$

where A^2 is the signal power, γ is the dispersion of the α-stable noise, α' is the characteristic exponent and $C_g \approx 1.7811$ is the exponential of the Euler constant. To compare the performance of the TR-MUSIC with the MUSIC and other robust methods, we quantify the performance using the normalized mean squared error

$$\mathrm{NMSE} = \frac{1}{N_t}\sum_{i=1}^{N_t} \frac{\|\hat{\mathbf{f}}_i - \mathbf{f}_0\|^2}{\|\mathbf{f}_0\|^2}. \tag{36}$$

In the above expression, N_t is the number of trials, $\|\cdot\|$ is the norm of a vector, $\hat{\mathbf{f}}_i$ is the vector of frequency estimates for the ith trail, and $\mathbf{f}_0$ is the true frequency vector.

Example 1. In this example, the TR-MUSIC method is applied to a real signal consisting of two sinusoidal signals with frequencies $f_1 = 0.2$ and $f_2 = 0.222$ embedded in α-stable noise. The parameters of the α-stable noise are $\alpha' = 1.1$, $\beta = 0$ and $\mu = 0$. The threshold ξ is chosen to be 30 since we observe that most outliers have MD greater than 30 while the normal samples have MD below this value. The trimmed correlation parameter α is calculated accordingly based on (12). The simulation results are shown in Fig. 3. To better illustrate the point that α-trimmed correlation performs better than trimming based on MD only, we also plot the curves for trimming based on MD for comparison. Figs. 3(a)–(d) show the spectrum

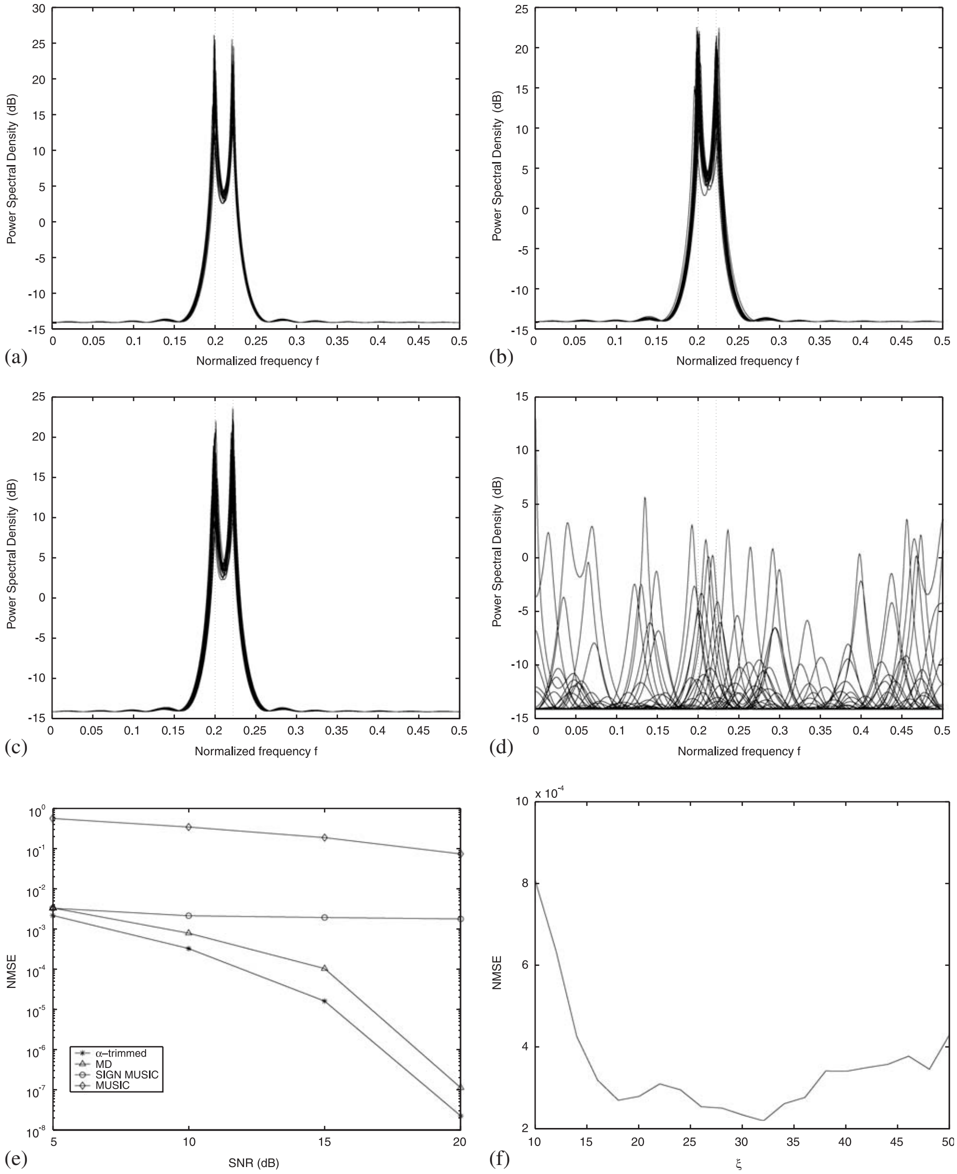

Fig. 3. Simulation results for real signal (in Figs. (a)–(d), 30 realizations are plotted and SNR = 5 dB). (a) TR-MUSIC spectrum ($\xi = 30$). (b) MD-based TR-MUSIC spectrum ($\xi = 30$). (c) SIGN-MUSIC spectrum. (d) MUSIC spectrum. (e) SNR vs. NMSE. (f) Threshold vs. NMSE.

estimated from the TR-MUSIC, MD-based TR-MUSIC, SIGN-MUSIC, and MUSIC, respectively, for SNR = 5 dB. It can be seen from these figures that both TR-MUSIC and SIGN-MUSIC methods are capable of identifying the two frequencies in the presence of impulsive noise, while the conventional MUSIC completely fails. Also, from Fig. 3(e), we can see that both TR-MUSIC methods have smaller MSE than the SIGN-MUSIC method. The α-trimmed based TR-MUSIC performs slightly better than the MD-based TR-MUSIC. Furthermore, the MSE of the TR-MUSIC decreases with increasing SNR while the MSE of the SIGN-MUSIC is nearly insensitive to the SNR. Finally, the effect of threshold ξ on the performance of the algorithm is illustrated in Fig. 3(f). We can see that the relationship between NMSE and the threshold ξ is quite complicated. In general, if the threshold is too small, we would expect NMSE increases as Fig. 3(f) shows because some samples not contaminated by outliers are erroneously trimmed. Conversely, if the threshold is too large, NMSE will also increase since some outliers are not trimmed. The guideline of choosing the threshold given in Section 3.1 is applied here and we can see from Fig. 3(f) that the chosen threshold 30 is nearly optimal.

Example 2. Although the SIGN-MUSIC is a robust method for estimating frequencies of the sinusoidal model, it cannot be used in the complex signals case. The TR-MUSIC, however, can be used for complex exponentials as well. To judge the performance of the TR-MUSIC method applied to complex signals, we compare it with the L_p norm linear predictive spectral estimator in [11]. The signal in this case consists of two complex exponentials with the same frequencies as in Example 1. The additive impulsive noise is complex α-stable noise with the same parameters as in Example 1. We select the same threshold value as in Example 1. Simulation results are given in Fig. 4. Note that in the L_p norm method, an iterative reweighted least squares (IRLS) algorithm is used with an AR model order of $L = N/3$, which is the value usually selected [11]. Again, we can see that the TR-MUSIC method performs much better than the L_p norm method and the conventional MUSIC method. The simulations show that the proposed TR-MUSIC method has

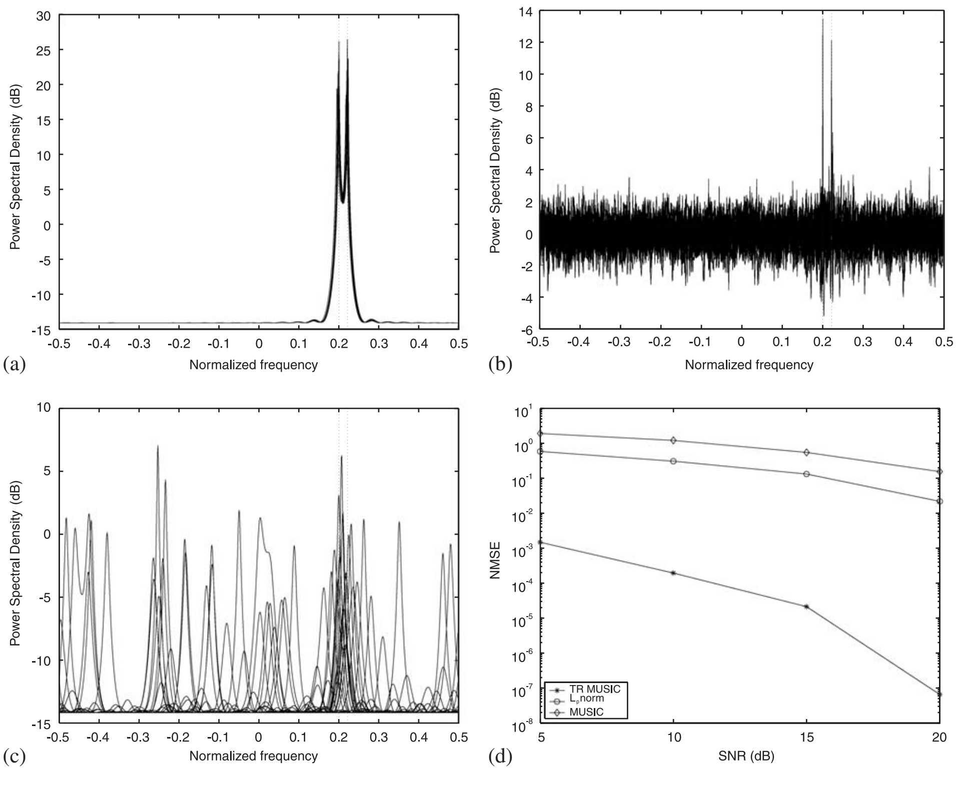

Fig. 4. Simulation results for complex signal (in Figs. (a)–(c), 30 realizations are plotted and SNR = 5 dB). (a) TR-MUSIC spectrum ($\xi = 30$). (b) L_p norm method spectrum. (c) MUSIC spectrum. (d) Normalized mean-squared error.

superior performance in both real and complex signal cases.

5. Conclusion

Frequency estimation of sinusoidal signals in impulsive noise is addressed in this paper. Conventional subspace methods, like MUSIC, are not guaranteed to find the frequencies correctly in this scenario. A new robust trimmed correlation based-MUSIC method is proposed for this purpose. In calculating the autocorrelation of the signal, trimming is performed on both lagging and leading vectors so that outliers are removed from the samples involved in the correlation estimates. Before the trimming process, we select a threshold based on a priori knowledge derived from Mahalanobis distance of data samples. The autocorrelation matrix is formed based on the trimmed correlation sequence and the frequencies are estimated from the eigendecomposition of the robust trimmed autocorrelation matrix. Simulations show that the TR-MUSIC performs better than the MUSIC and other robust methods in impulsive noise (α-stable noise as an example in this paper) environments for both real and complex signals.

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