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Short communication

A bias-compensated MUSIC for small number of samples

François Vincent[a,*], Frédéric Pascal[b], Olivier Besson[a]

[a] ISAE-Supaéro, Department Electronics Optronics Signal, 10 Avenue Edouard Belin, 31055 Toulouse, France
[b] L2S, CentraleSupélec, 91192 Gif-sur-Yvette, France

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ABSTRACT

The multiple signal classification (MUSIC) method is known to be asymptotically efficient, yet with a small number of snapshots its performance degrades due to bias in MUSIC localization function. In this communication, starting from G-MUSIC which improves over MUSIC in low sample support, a high signal to noise ratio approximation of the G-MUSIC localization function is derived. This approximation results in closed-form expressions of the weights applied to each eigenvector of the sample covariance matrix. A new method which consists in minimizing this simplified G-MUSIC localization function is thus introduced, and referred to as sG-MUSIC. Interestingly enough, this sG-MUSIC criterion can be interpreted as a bias correction of the conventional MUSIC localization function. Numerical simulations indicate that sG-MUSIC incur only a marginal loss in terms of mean square error of the direction of arrival estimates, as compared to G-MUSIC, and performs better than MUSIC.

1. Introduction and problem statement

Estimating the directions of arrival (DoA) of multiple sources impinging on an array of M sensors is a primordial task in most sonar or radar systems [1]. A reference approach to tackle this problem is by the maximum likelihood estimator (MLE) [2–5], whose performance is at best matched asymptotically, but is usually most accurate in the so-called threshold area where most estimators begin to depart from the Cramér-Rao bound (CRB). The MLE entails a global search for the maximum of a K-dimensional likelihood function, where K stands for the number of sources and can thus be prohibitive from a computational point of view. In the eighties, the paradigm of subspace-based methods was introduced, relying heavily on the low-rank structure of the noise-free covariance matrix. Exploiting the partitioning of the space as a subspace containing the signals of interest and its orthogonal complement, the K-dimensional problem was reduced to a one-dimensional problem where either K maxima, K eigenvalues or K roots of a polynomial were to be searched, see e.g. MUSIC [6,7], ESPRIT [8] or MODE [9] respectively.

MUSIC [6,7], which is one of the first subspace-based technique introduced and is applicable to any array geometry, has been extensively studied. The MUSIC DoA estimates are obtained as the K deepest minima of the localization function $\hat{L}_{\text{MUSIC}}(\theta)$ defined hereafter. In the large sample case, it was demonstrated that it is asymptotically unbiased and efficient [10–12], i.e. it achieves the CRB either as the number of snapshots T or the signal to noise ratio (SNR) grow large. Nonetheless, its performance in finite sample degrades. This is detrimental in practical situations where dynamically changing environments require carrying out DoA estimation with a possibly small number of snapshots. In [13], Kaveh and Barabell provided a detailed study of MUSIC localization function

$$\hat{L}_{\text{MUSIC}}(\theta) = \mathbf{a}^H(\theta)\,\hat{\mathbf{U}}_n\hat{\mathbf{U}}_n^H\,\mathbf{a}(\theta)$$

where $\mathbf{a}(\theta)$ stands for the array steering vector and $\hat{\mathbf{U}}_n = \begin{bmatrix} \hat{\mathbf{u}}_1 & \cdots\hat{\mathbf{u}}_{M-K} \end{bmatrix}$ where $\hat{\mathbf{u}}_m$ are the eigenvectors of the sample covariance matrix with the convention that the corresponding eigenvalues $\hat{\lambda}_m$ are sorted in ascending order. They proved that, when evaluated at a true DoA θ_k, $\hat{L}_{\text{MUSIC}}(\theta)$ has a finite sample bias, which is generally larger than the corresponding standard deviation, and is thus the main factor for the loss of resolution and accuracy. In [14], rigorous expressions for the finite sample bias of MUSIC DoA estimates were derived. In fact, resorting to random matrix theory (RMT), i.e. considering the asymptotic regime where M, $T \to \infty$ with $M/T \to c$ (denoted as RMT-regime), it was proven in [15] that the localization function of MUSIC is not consistent. As a corollary, it was demonstrated that MUSIC cannot consistently resolve sources within the main beam width. In order to cope with this problem, the G-MUSIC method was introduced which provides a consistent estimate of $\mathbf{a}^H(\theta)\,\mathbf{U}_n\mathbf{U}_n^H\,\mathbf{a}(\theta)$ in the RMT sense. G-MUSIC estimates the noise projection matrix as $\hat{\mathbf{P}}_{\text{G-MUSIC}} = \sum_{m=1}^{M} w_m\hat{\mathbf{u}}_m\hat{\mathbf{u}}_m^H$ where w_m are weights defined hereafter. The difference with the MUSIC

* Corresponding author.
E-mail addresses: francois.vincent@isae-supaero.fr, Francois.Vincent@isae.fr (F. Vincent), frederic.pascal@centralesupelec.f (F. Pascal), olivier.besson@isae-supaero.fr (O. Besson).

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projector $\hat{\mathbf{P}}_{\text{MUSIC}} = \sum_{m=1}^{M-K} \hat{\mathbf{u}}_m \hat{\mathbf{u}}_m^H$ is twofold: MUSIC uses only "noise" eigenvectors while $\hat{\mathbf{P}}_{\text{G-MUSIC}}$ makes use of *all* eigenvectors, and MUSIC does not attribute a different weighting to the eigenvectors. G-MUSIC was shown to improve over MUSIC and, although it relies on an asymptotic assumption, G-MUSIC proved to be effective in small sample support [15,16].

This said, the weights of G-MUSIC are not easy to obtain: computing them requires finding the roots of a Mth degree polynomial or finding the eigenvalues of a $M \times M$ matrix, see below for details. Additionally, it is difficult to have a simple and intuitive interpretation of these weights. In this communication, we start from G-MUSIC which performs well for small T, and try to simplify calculation of its weights and to obtain more insightful expressions. Our approach is based on a high SNR approximation of the G-MUSIC weights and results in a simple, closed-form expression. Interestingly enough, the so-approximated weights can be interpreted as a correction of the bias in MUSIC localization function. The new scheme is thus simpler than G-MUSIC without sacrificing accuracy, as will be shown in the numerical simulations.

2. Derivation of sG-MUSIC

In this section, we derive an approximated and simplified expression of G-MUSIC projection estimate

$$\hat{\mathbf{P}}_{\text{G-MUSIC}} = \sum_{m=1}^{M} w_m \hat{\mathbf{u}}_m \hat{\mathbf{u}}_m^H \tag{1}$$

and relate the so-obtained estimate to a bias compensation of MUSIC.

2.1. Background and approach

The weights w_m of G-MUSIC are given by [15]

$$w_m = \begin{cases} 1 + \sum_{k>M-K} \left(\frac{\hat{\lambda}_k}{\hat{\lambda}_m - \hat{\lambda}_k} - \frac{\hat{\mu}_k}{\hat{\lambda}_m - \hat{\mu}_k} \right) & m \leq M-K \\ -\sum_{k \leq M-K} \left(\frac{\hat{\lambda}_k}{\hat{\lambda}_m - \hat{\lambda}_k} - \frac{\hat{\mu}_k}{\hat{\lambda}_m - \hat{\mu}_k} \right) & m > M-K \end{cases} \tag{2}$$

where $\hat{\lambda}_k$ are the eigenvalues of the sample covariance matrix and $\hat{\mu}_k$, $k = 1, \ldots, M$ denote the roots of

$$f(\mu) = \sum_{m=1}^{M} \frac{\hat{\lambda}_m}{\hat{\lambda}_m - \mu} = \frac{M}{c} = T \tag{3}$$

sorted in ascending order. Note that, when $c < 1$, we have the interlacing property that $\hat{\lambda}_{m-1} < \hat{\mu}_m < \hat{\lambda}_m$ [17]. It follows that, at high signal to noise ratio where there is a clear separation between signal and noise eigenvalues, the last K values $\hat{\mu}_m$ will be well above the cluster of the $M-K$ smallest $\hat{\mu}_m$, which should lie around the white noise power (WNP), and the latter is assumed to be small. Moreover, observe from (2) that the $M-K$ smallest $\hat{\mu}_m$ will impact the weights of the signal eigenvectors while the weights of the noise eigenvectors depend on the K largest $\hat{\mu}_m$ only.

Our approximation relies on finding the roots of (3) by considering the two clusters of solutions independently. Rewriting the function in (3) as $f(\mu) = \sum_{m=1}^{M} f_m(\mu)$, where $f_m(\mu) = \frac{\hat{\lambda}_m}{\hat{\lambda}_m - \mu}$ one can thus make the following partitioning

$$f(\mu) = \sum_{m=1}^{M-K} f_m(\mu) + \sum_{m=M-K+1}^{M} f_m(\mu) = f_n(\mu) + f_s(\mu).$$

First, we use the fact that, when searching for the $M-K$ smallest values of μ, $f_s(\mu)$ is approximately constant, which leads to an approximation of $\hat{\mu}_m$ for $m \leq M-K$ and hence of the signal eigenvectors weights. As for the w_m, $m \leq M-K$, we will provide a high SNR approximation of them directly.

2.2. Approximating the signal eigenvectors weights

Proposition 1. *At high signal to noise ratio, the weights w_m of Eq. (2) applied to the signal eigenvectors can be approximated as*

$$w_m \approx -\hat{\lambda}_m^{-1} (T-K)^{-1} \left(\sum_{k=1}^{M-K} \hat{\lambda}_k \right) \quad m > M-K. \tag{4}$$

Proof. First note w_m for $m > M-K$ is related to the $M-K$ smallest solutions of (3). The latter will be typically of the same magnitude as the WNP (due to the interlacing property $\hat{\lambda}_{m-1} < \hat{\mu}_m < \hat{\lambda}_m$) and hence negligible compared to $\hat{\lambda}_{M-K+1}, \cdots, \hat{\lambda}_M$. Hence, they belong to some interval $\mathcal{I}_n$ where $\hat{\lambda}_m / (\hat{\lambda}_m - \mu) \approx 1$ for $m > M-K$ which results in $f_s(\mu) \approx K$ when $\mu \in \mathcal{I}_n$. Consequently, the $M-K$ smallest values of μ are obtained by solving

$$\begin{aligned} f_n(\mu) + K = T &\Leftrightarrow 1 - \frac{1}{T-K} \sum_{m=1}^{M-K} \frac{\hat{\lambda}_m}{\hat{\lambda}_m - \mu} = 0 \\ &\Leftrightarrow 1 - \frac{1}{T-K} \sqrt{\hat{\boldsymbol{\lambda}}_n}^T \left(\hat{\boldsymbol{\Lambda}}_n - \mu \mathbf{I} \right)^{-1} \sqrt{\hat{\boldsymbol{\lambda}}_n} = 0 \\ &\Leftrightarrow \det \left(\hat{\boldsymbol{\Lambda}}_n - \frac{1}{T-K} \sqrt{\hat{\boldsymbol{\lambda}}_n} \sqrt{\hat{\boldsymbol{\lambda}}_n}^T - \mu \mathbf{I} \right) = 0 \end{aligned} \tag{5}$$

where $\hat{\boldsymbol{\lambda}}_n = \begin{bmatrix} \hat{\lambda}_1 & \cdots & \hat{\lambda}_{M-K} \end{bmatrix}^T$, $\hat{\boldsymbol{\Lambda}}_n = \text{diag}(\hat{\boldsymbol{\lambda}}_n)$ and where the last equivalence is obtained by multiplying by $\det\left(\hat{\boldsymbol{\Lambda}}_n - \mu \mathbf{I}\right)$. It follows that $\hat{\mu}_m$ for $m = 1, \cdots, M-K$ are approximately the eigenvalues of $\hat{\boldsymbol{\Lambda}}_n - (T-K)^{-1} \sqrt{\hat{\boldsymbol{\lambda}}_n} \sqrt{\hat{\boldsymbol{\lambda}}_n}^T$.

Let us accordingly consider an approximation of the weights w_m, $m > M-K$. Let us introduce the notation $\Delta_k = \hat{\lambda}_k - \hat{\mu}_k$. Note that, at high SNR, we have $\hat{\lambda}_k \hat{\lambda}_m^{-1} \ll 1$ for $k = 1, \cdots, M-K$ and, since $\hat{\mu}_k < \hat{\lambda}_k$, it follows that $\hat{\mu}_k \hat{\lambda}_m^{-1} \ll 1$. It then ensues that, for $m > M-K$

$$\begin{aligned} w_m &= -\sum_{k=1}^{M-K} \left(\frac{\hat{\lambda}_k}{\hat{\lambda}_m - \hat{\lambda}_k} - \frac{\hat{\mu}_k}{\hat{\lambda}_m - \hat{\mu}_k} \right) \\ &= -\sum_{k=1}^{M-K} \frac{\hat{\lambda}_m \Delta_k}{\hat{\lambda}_m^2 (1 - \hat{\lambda}_k \hat{\lambda}_m^{-1})(1 - \hat{\mu}_k \hat{\lambda}_m^{-1})} \\ &\approx -\hat{\lambda}_m^{-1} \sum_{k=1}^{M-K} \Delta_k = -\hat{\lambda}_m^{-1} \left[\sum_{k=1}^{M-K} \hat{\lambda}_k - \sum_{k=1}^{M-K} \hat{\mu}_k \right] \\ &\approx -\hat{\lambda}_m^{-1} \left[\sum_{k=1}^{M-K} \hat{\lambda}_k - \text{Tr}\{ \hat{\boldsymbol{\Lambda}}_n - (T-K)^{-1} \sqrt{\hat{\boldsymbol{\lambda}}_n} \sqrt{\hat{\boldsymbol{\lambda}}_n}^T \} \right] \\ &= -\hat{\lambda}_m^{-1} (T-K)^{-1} \left(\sum_{k=1}^{M-K} \hat{\lambda}_k \right). \quad \square \end{aligned} \tag{6}$$

2.3. Approximating the noise eigenvectors weights

Proposition 2. *At high signal to noise ratio, the weights applied to the noise eigenvectors can be approximated as*

$$w_m \approx 1 \quad m \leq M-K. \tag{7}$$

Proof. Let us write, for $m \leq M-K$

$$w_m = 1 + \sum_{k=M-K+1}^{M} \left(\frac{\hat{\lambda}_k}{\hat{\lambda}_m - \hat{\lambda}_k} - \frac{\hat{\mu}_k}{\hat{\lambda}_m - \hat{\mu}_k} \right)$$

$$= 1 + \sum_{k=M-K+1}^{M} \frac{\hat{\lambda}_m(\hat{\lambda}_k - \hat{\mu}_k)}{(\hat{\lambda}_m - \hat{\lambda}_k)(\hat{\lambda}_m - \hat{\mu}_k)}$$

$$= 1 + \sum_{k=M-K+1}^{M} \frac{(\hat{\lambda}_m \hat{\lambda}_k^{-1})(\hat{\lambda}_k \hat{\mu}_k^{-1} - 1)}{(1 - \hat{\lambda}_m \hat{\lambda}_k^{-1})(1 - \hat{\lambda}_m \hat{\mu}_k^{-1})}. \tag{8}$$

Under the high SNR assumption, we have that $\hat{\lambda}_m \hat{\lambda}_k^{-1} \ll 1$ for $m \leq M-K$ and $k > M-K$. Let us now show that $\hat{\lambda}_m \hat{\mu}_k^{-1} \ll 1$ for $m \leq M-K$ and $k > M-K$. As $\hat{\mu}_k \in \left]\hat{\lambda}_{k-1}, \hat{\lambda}_k\right[$, it follows directly from the high SNR assumption that, for $k > M-K+1$ and $m \leq M-K$, $\hat{\lambda}_m \hat{\mu}_k^{-1} < \hat{\lambda}_m \hat{\lambda}_{k-1}^{-1} \ll 1$. It remains to examine the special case of $k_{\min} = M-K+1$, that is of the smallest signal eigenvalue since, in this case, $\hat{\mu}_{k_{\min}}$ lies between the largest noise eigenvalue and the smallest signal eigenvalue. We now prove that $\hat{\mu}_{k_{\min}}$ is close to $\hat{\lambda}_{k_{\min}}$. For $k \geq k_{\min}$

$$\frac{\hat{\lambda}_k}{\hat{\lambda}_k - \hat{\mu}_{k_{\min}}} \leq \frac{\hat{\lambda}_{k_{\min}}}{\hat{\lambda}_{k_{\min}} - \hat{\mu}_{k_{\min}}} \tag{9}$$

which implies that

$$\sum_{k=M-K+1}^{M} \frac{\hat{\lambda}_k}{\hat{\lambda}_k - \hat{\mu}_{k_{\min}}} \leq K \frac{\hat{\lambda}_{k_{\min}}}{\hat{\lambda}_{k_{\min}} - \hat{\mu}_{k_{\min}}}$$

$$\Leftrightarrow \sum_{k=1}^{M} \frac{\hat{\lambda}_k}{\hat{\lambda}_k - \hat{\mu}_{k_{\min}}} \leq K \frac{\hat{\lambda}_{k_{\min}}}{\hat{\lambda}_{k_{\min}} - \hat{\mu}_{k_{\min}}} + \sum_{k=1}^{M-K} \frac{\hat{\lambda}_k}{\hat{\lambda}_k - \hat{\mu}_{k_{\min}}}$$

$$\Leftrightarrow T \leq K \frac{\hat{\lambda}_{k_{\min}}}{\hat{\lambda}_{k_{\min}} - \hat{\mu}_{k_{\min}}} + \sum_{k=1}^{M-K} \frac{\hat{\lambda}_k}{\hat{\lambda}_k - \hat{\mu}_{k_{\min}}}$$

$$\Leftrightarrow K \frac{\hat{\lambda}_{k_{\min}}}{\hat{\lambda}_{k_{\min}} - \hat{\mu}_{k_{\min}}} \geq T - \sum_{k=1}^{M-K} \frac{\hat{\lambda}_k}{\hat{\lambda}_k - \hat{\mu}_{k_{\min}}}. \tag{10}$$

The right-hand side of last equation being strictly greater than T, it follows that

$$\hat{\mu}_{k_{\min}} > \hat{\lambda}_{k_{\min}}\left(1 - \frac{K}{T}\right). \tag{11}$$

The previous equation shows that $\hat{\mu}_{k_{\min}}$ is rather close to the upper bound of the interval $\left]\hat{\lambda}_{k_{\min}-1}, \hat{\lambda}_{k_{\min}}\right[$ to which it belongs. Similar derivations as for (10) can show that, for any $k > M-K$, $\hat{\mu}_k > \hat{\lambda}_k\left(1 - \frac{M-k+1}{T}\right)$, and hence as the eigenvalues increase, $\hat{\mu}_k$ comes closer to $\hat{\lambda}_k$. Furthermore, (11) implies that $\hat{\lambda}_m \hat{\mu}_{k_{\min}}^{-1} < \hat{\lambda}_m \hat{\lambda}_{k_{\min}}^{-1}\left(1 - \frac{K}{T}\right)^{-1} \ll 1$. Coming back to (8) it follows that, at high SNR, $w_m \approx 1$ for $m \leq M-K$. □

2.4. sG-MUSIC and its relation to MUSIC bias compensation

Combining (4) and (7), it follows that $\hat{\mathbf{P}}_{\text{G-MUSIC}}$ can be approximated by

$$\hat{\mathbf{P}}_{\text{sG-MUSIC}} = \hat{\mathbf{U}}_n \hat{\mathbf{U}}_n^H - \frac{\left(\sum_{k=1}^{M-K} \hat{\lambda}_k\right)}{T-K} \hat{\mathbf{U}}_s \hat{\mathbf{\Lambda}}_s^{-1} \hat{\mathbf{U}}_s^H$$

$$= \hat{\mathbf{P}}_{\text{MUSIC}} - \frac{\left(\sum_{k=1}^{M-K} \hat{\lambda}_k\right)}{T-K} \hat{\mathbf{U}}_s \hat{\mathbf{\Lambda}}_s^{-1} \hat{\mathbf{U}}_s^H \tag{12}$$

where $\hat{\mathbf{U}}_s = [\hat{\mathbf{u}}_{M-K+1} \cdots \hat{\mathbf{u}}_M]$ and $\hat{\mathbf{\Lambda}}_s = \text{diag}(\hat{\boldsymbol{\lambda}}_s)$ with $\hat{\boldsymbol{\lambda}}_s = \left[\hat{\lambda}_{M-K+1} \quad \cdots \quad \hat{\lambda}_M\right]^T$. The projector in (12) provides an approximation of $\hat{\mathbf{P}}_{\text{G-MUSIC}}$ which relies only on the eigenvalues and eigenvectors and thus avoids the need to solve (3). One can observe that the noise eigenvectors are attributed a common weight equal to one as in MUSIC, while the signal eigenvectors are weighted by $\hat{\lambda}_m^{-1}(T-K)^{-1}\left(\sum_{k=1}^{M-K} \hat{\lambda}_k\right)$, which tends to zero as T increases and/or the signal to noise ratio increases, which seems logical.

Interestingly enough, $\hat{\mathbf{P}}_{\text{sG-MUSIC}}$ can be viewed as a correction of the bias of MUSIC localization function. More precisely, we will show that the corrective term in the second line of (12) can be interpreted as a compensation of MUSIC bias due to finite sample support. As shown in [13], see also [1, Chapter 9], one has

$$\mathbb{E}\left\{\hat{\mathbf{P}}_{\text{MUSIC}}\right\} - \mathbf{U}_n \mathbf{U}_n^H = -\sum_{i=M-K+1}^{M} \sum_{j=1, j\neq i}^{M} \frac{\lambda_i \lambda_j}{T(\lambda_i - \lambda_j)^2}\left[\mathbf{u}_j \mathbf{u}_j^H - \mathbf{u}_i \mathbf{u}_i^H\right]. \tag{13}$$

We can rewrite the previous equation as

$$\begin{aligned}\mathbb{E}\left\{\hat{\mathbf{P}}_{\text{MUSIC}}\right\} - \mathbf{U}_n \mathbf{U}_n^H &= -\sum_{i=M-K+1}^{M} \sum_{j=1}^{M-K} \frac{\lambda_i \lambda_j}{T(\lambda_i - \lambda_j)^2}\left[\mathbf{u}_j \mathbf{u}_j^H - \mathbf{u}_i \mathbf{u}_i^H\right] \\ &= -\sum_{i=M-K+1}^{M} \sum_{j=1}^{M-K} \frac{\lambda_i \sigma^2}{T(\lambda_i - \sigma^2)^2}\left[\mathbf{u}_j \mathbf{u}_j^H - \mathbf{u}_i \mathbf{u}_i^H\right] \\ &= \left[-\sum_{i=M-K+1}^{M} \frac{\lambda_i \sigma^2}{T(\lambda_i - \sigma^2)^2}\right]\mathbf{U}_n \mathbf{U}_n^H + \frac{(M-K)\sigma^2}{T} \sum_{i=M-K+1}^{M} \frac{\lambda_i \mathbf{u}_i \mathbf{u}_i^H}{(\lambda_i - \sigma^2)^2}\end{aligned} \tag{14}$$

where σ^2 stands for the WNP. Therefore, when evaluated at a true DoA θ_k, the average value of MUSIC localization function is given by

$$\begin{aligned}\mathbb{E}\left\{\hat{L}_{\text{MUSIC}}(\theta_k)\right\} &= \frac{(M-K)\sigma^2}{T} \sum_{i=M-K+1}^{M} \frac{\lambda_i |\mathbf{a}^H(\theta_k)\mathbf{u}_i|^2}{(\lambda_i - \sigma^2)^2} \\ &\approx \frac{(M-K)\sigma^2}{T} \sum_{i=M-K+1}^{M} \lambda_i^{-1} |\mathbf{a}^H(\theta_k)\mathbf{u}_i|^2 \\ &= \mathbf{a}^H(\theta_k)\left[\frac{(M-K)\sigma^2}{T} \mathbf{U}_s \mathbf{\Lambda}_s^{-1} \mathbf{U}_s^H\right]\mathbf{a}(\theta_k).\end{aligned} \tag{15}$$

Comparing (12) to (15), one can interpret the correction $\hat{\mathbf{P}}_{\text{sG-MUSIC}} - \hat{\mathbf{P}}_{\text{MUSIC}}$ as an $O(T^{-1})$ estimate of the bias of MUSIC. In other words, $\hat{\mathbf{P}}_{\text{sG-MUSIC}}$ can be viewed as a modification of MUSIC by attempting to remove bias. It is very interesting to note that the theory which led to G-MUSIC is completely different from the theory from which (13) originates. With this respect, the new $\hat{\mathbf{P}}_{\text{sG-MUSIC}}$ enables to sort of establish a bridge between the two approaches. It can either be viewed as an approximation and simplification of G-MUSIC and/or a correction of MUSIC.

3. Numerical simulations

In this section, we compare the mean-square error (MSE) of the DoA estimates obtained by MUSIC, G-MUSIC and sG-MUSIC. We consider the same scenario as in [15] i.e., a uniform linear array of $M = 20$ sensors, spaced a half-wavelength apart. Two equi-powered sources are assumed to be present in the field of view of the array, with power P and DoA 35° and 37°. The measurements are corrupted by white Gaussian noise with power σ^2 and we define the signal to noise ratio as $\text{SNR} = \frac{P}{\sigma^2}$. We consider as figure of merit the mean square error defined as $\text{MSE} = \sum_{p=1}^{P} \mathbb{E}\left\{(\hat{\theta}_p - \theta_p)^2\right\}$. 5000 Monte-Carlo simulations are run to estimate MSE. In Figs. 1–3 we plot MSE as a function of SNR for various values of T, namely $T = 15$, $T = 25$ and $T = 75$. As can be observed, sG-MUSIC performs nearly as well as G-MUSIC in the

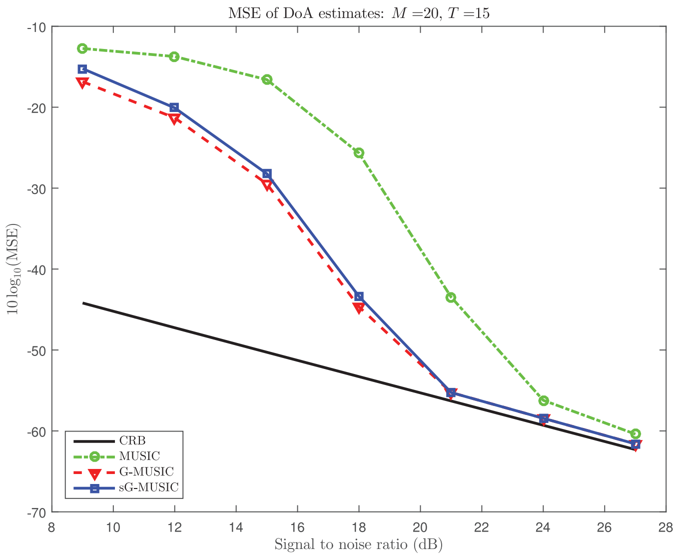

Fig. 1. MSE of MUSIC, G-MUSIC and sG-MUSIC DoA estimates versus SNR. $T = 15$.

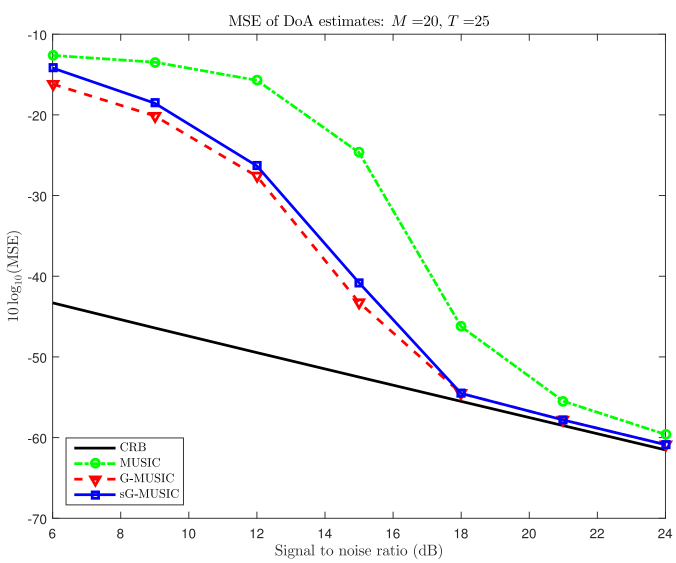

Fig. 2. MSE of MUSIC, G-MUSIC and sG-MUSIC DoA estimates versus SNR. $T = 25$.

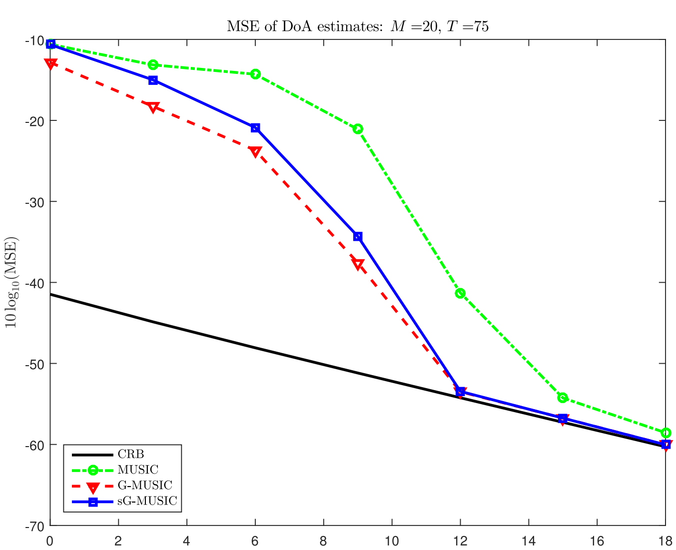

Fig. 3. MSE of MUSIC, G-MUSIC and sG-MUSIC DoA estimates versus SNR. $T = 75$.

threshold area (especially for small T) and much better than MUSIC. As SNR increases the difference between the three methods vanishes. Therefore, sG-MUSIC offers a very good compromise: it is somewhat simpler than G-MUSIC and undergoes a marginal loss in the threshold area, at least when a limited number of samples is available. On the other hand, sG-MUSIC has a complexity similar to that of MUSIC but provides more accurate estimates.

4. Conclusions

In this communication, starting from the G-MUSIC localization function, we have presented an approximation that can be viewed as a bias compensated version of MUSIC. Indeed, the new method corresponds to a modification of MUSIC localization function which somehow removes the bias in the latter. Moreover, the weights to be applied to the eigenvectors of the sample covariance matrix are obtained in closed-form, similarly to MUSIC, but do not require to find the M roots of a non-linear equation as in G-MUSIC. Numerical simulations indicate that the new scheme performs nearly as well as G-MUSIC, especially in low sample support, and better than MUSIC.

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