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Biquaternion cumulant-MUSIC for DOA estimation of noncircular signals ☆

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ABSTRACT

Direction-of-arrival (DOA) estimation for noncircular sources is addressed within the hypercomplex framework utilizing fourth-order (FO) cumulants and a MUSIC-like estimator is proposed. Simulation results show the better performance of the proposed method compared to its complex counterpart in terms of both accuracy and robustness to model errors due to the stronger orthogonality in the biquaternion domain.

1. Introduction

Direction-of-arrival (DOA) estimation for noncircular signals, which are omnipresent in radar and communication systems such as binary phase shift keying (BPSK) and amplitude modulation (AM) signals, has been raising increasing attention. Various algorithms based on second-order statistics, as well as high-order statistics (HOS), have been presented for DOA estimation. In [1], a MUSIC-like algorithm was proposed for noncircular signals (named NC-MUSIC) with increased processing capacity and then the spectrum searching was replaced for polynomial rooting in [2]. A detailed study of MUSIC-like algorithms for noncircular signals can be found in [3]. For centro-symmetric arrays, the unitary ESPRIT algorithm was meliorated for noncircular signals in [4]. An extended $2q$-MUSIC algorithm [5] for noncircular signals ($q=1$) based on second-order statistics and non-Gaussian noncircular signals ($q>1$) based on HOS, named NC-$2q$-MUSIC, was proposed in [6] and was shown to possess a further extended aperture and better performance compared with original algorithm.

In the array signal processing literature, signals are generally processed in complex numbers. The extended data structure acquired either physically (e.g., recorded by vector-sensors) or from signal processing (e.g., the case of receiving noncircular signals) is generally set out to be a prolonged vector. Recently, several algorithms for DOA estimation using vector-sensors were proposed in the hypercomplex framework using quaternions, biquaternions and quad-quaternions [7–10]. Instead of laying the data recorded by different components of a vector-sensor array on sequential places to create a "long vector", they relate them to different imaginary units to generate a hypercomplex vector. Due to the stronger orthogonality in the hypercomplex domain compared to the complex one, better performance is shown in both accuracy and robustness. Motivated by this fact, we herein propose a MUSIC-like algorithm in the hypercomplex framework using fourth-order (FO) cumulants. Though there are $2^4=16$ FO cumulant matrices, three of them are sufficient to achieve further aperture extension. To utilize the three matrices symmetrically, biquaternions are used.

The rest of the paper is organized as follows. In Section 2, we introduce the mathematical preliminaries on biquaternion algebra. We formulate the problem in Section 3 and propose our algorithm in Section 4. Some numerical examples to illustrate the performances of the proposed

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algorithm are given in Section 5 and we conclude the paper in Section 6.

2. Mathematical preliminaries

W.R. Hamilton's biquaternions [11,12] form an eight-dimensional space and can be viewed as "complexified quaternions" or "quartered complex numbers", namely, a biquaternion $b \in \mathbb{H}_{\mathbb{C}}$ possesses eight parts (one real and seven imaginary parts) and can be expressed by

$$\begin{aligned} b &= q_0 + \mathtt{I} q_1 \\ &= c_0 + \mathtt{i} c_1 + \mathtt{j} c_2 + \mathtt{k} c_3 \\ &= b_{00} + \mathtt{i} b_{10} + \mathtt{j} b_{20} + \mathtt{k} b_{30} + \mathtt{I} b_{01} + \mathtt{i}\mathtt{I} b_{11} + \mathtt{j}\mathtt{I} b_{21} + \mathtt{k}\mathtt{I} b_{31} \end{aligned} \tag{1}$$

where $\{q_n\}_{n=0}^{1}$, $\{c_n\}_{n=0}^{3}$, and $\{b_{nm}\}_{n=0,m=0}^{n=3,m=1}$ are quaternions, complex numbers with imaginary unit of $\mathtt{I}$, and real numbers, respectively. In this paper, we denote the sets of biquaternions, quaternions, complex numbers with imaginary unit of $\dot{u}$, and real numbers as $\mathbb{H}_{\mathbb{C}}$, $\mathbb{H}$, $\mathbb{C}_{\dot{u}}$, and $\mathbb{R}$, respectively, where $\dot{u} = \mathtt{i},\mathtt{j},\mathtt{k},\mathtt{I}$, which are subject to the following constraints:

$$\begin{aligned} &\mathtt{i}^2 = \mathtt{j}^2 = \mathtt{k}^2 = \mathtt{I}^2 = -1 \\ &\mathtt{ij} = -\mathtt{ji} = \mathtt{k}, \mathtt{jk} = -\mathtt{kj} = \mathtt{i}, \mathtt{ki} = -\mathtt{ik} = \mathtt{j} \\ &\mathtt{iI} = \mathtt{Ii}, \mathtt{jI} = \mathtt{Ij}, \mathtt{kI} = \mathtt{Ik} \end{aligned} \tag{2}$$

In addition, we use superscripts T, H, and $*$ to denote transpose, Hermitian transpose, and conjugation of a complex matrix. We then present a brief list of properties of biquaternions involved in formulating the algorithm.

(1) A biquaternion matrix $\boldsymbol{B} \in \mathbb{H}_{\mathbb{C}}^{M\times N}$ is defined as an $M \times N$ matrix with biquaternion entries, i.e.,

$$\begin{aligned} \boldsymbol{B} = \boldsymbol{B}_{00} + \mathtt{i}\boldsymbol{B}_{10} + \mathtt{j}\boldsymbol{B}_{20} + \mathtt{k}\boldsymbol{B}_{30} + \mathtt{I}\boldsymbol{B}_{01} \\ + \mathtt{i}\mathtt{I}\boldsymbol{B}_{11} + \mathtt{j}\mathtt{I}\boldsymbol{B}_{21} + \mathtt{k}\mathtt{I}\boldsymbol{B}_{31} \end{aligned} \tag{3}$$

where $\{\boldsymbol{B}_{nm}\}_{n=0,m=0}^{n=3,m=1} \in \mathbb{R}^{M\times N}$.

(2) The total conjugate of biquaternion b and Hermitian transpose of biquaternion matrix $\boldsymbol{B}$ are respectively given by

$$\begin{aligned} b^{\diamond} = b_{00} - \mathtt{i} b_{10} - \mathtt{j} b_{20} - \mathtt{k} b_{30} - \mathtt{I} b_{01} \\ + \mathtt{i}\mathtt{I} b_{11} + \mathtt{j}\mathtt{I} b_{21} + \mathtt{k}\mathtt{I} b_{31} \end{aligned} \tag{4}$$

$$\begin{aligned} \boldsymbol{B}^{\dagger} = \boldsymbol{B}_{00}^{\mathrm{T}} - \mathtt{i}\boldsymbol{B}_{10}^{\mathrm{T}} - \mathtt{j}\boldsymbol{B}_{20}^{\mathrm{T}} - \mathtt{k}\boldsymbol{B}_{30}^{\mathrm{T}} - \mathtt{I}\boldsymbol{B}_{01}^{\mathrm{T}} + \mathtt{i}\mathtt{I}\boldsymbol{B}_{11}^{\mathrm{T}} \\ + \mathtt{j}\mathtt{I}\boldsymbol{B}_{21}^{\mathrm{T}} + \mathtt{k}\mathtt{I}\boldsymbol{B}_{31}^{\mathrm{T}} \end{aligned} \tag{5}$$

For two biquaternions $a,b \in \mathbb{H}_{\mathbb{C}}$ and two biquaternion matrices $\boldsymbol{A} \in \mathbb{H}_{\mathbb{C}}^{M\times N}, \boldsymbol{B} \in \mathbb{H}_{\mathbb{C}}^{N\times K}$, there holds [8]

$$(ab)^{\diamond} = b^{\diamond} a^{\diamond} \tag{6}$$

$$(\boldsymbol{AB})^{\dagger} = \boldsymbol{B}^{\dagger}\boldsymbol{A}^{\dagger} \tag{7}$$

(3) A square biquaternion matrix $\boldsymbol{B} \in \mathbb{H}_{\mathbb{C}}^{N\times N}$ is Hermitian if [12]

$$\boldsymbol{B} = \boldsymbol{B}^{\dagger} \tag{8}$$

(4) The norm of a biquaternion vector $\boldsymbol{b} \in \mathbb{H}_{\mathbb{C}}^{N\times 1}$ is defined as

$$\|\boldsymbol{b}\| = \sqrt{\mathcal{R}(\boldsymbol{b}^{\dagger}\boldsymbol{b})} \tag{9}$$

where operator $\mathcal{R}(\cdot)$ denotes the real part of a biquaternion and operator $\sqrt{\cdot}$ denotes the square-root operation [8].

(5) Two biquaternion vectors $\boldsymbol{a},\boldsymbol{b} \in \mathbb{H}_{\mathbb{C}}^{N\times 1}$ are orthogonal if [8]

$$\boldsymbol{a}^{\dagger}\boldsymbol{b} = 0 \tag{10}$$

(6) The eigenvalue decomposition (EVD) of a Hermitian biquaternion matrix $\boldsymbol{B} \in \mathbb{H}_{\mathbb{C}}^{N\times N}$ is given by

$$\boldsymbol{B} = \sum_{n=1}^{2N} \lambda_n \boldsymbol{u}_n \boldsymbol{u}_n^{\dagger} \tag{11}$$

where $\{\lambda_n\}_{n=1}^{2N} \in \mathbb{R}$ are real eigenvalues and $\{\boldsymbol{u}_n\}_{n=1}^{2N} \in \mathbb{H}_{\mathbb{C}}^{N\times 1}$ are unit-norm orthogonal eigenvectors [8].

(7) For a complex matrix $\boldsymbol{C} = \boldsymbol{C}_0 + \mathtt{i}\boldsymbol{C}_1$, where $\boldsymbol{C} \in \mathbb{C}_{\mathtt{i}}^{M\times N}$, $\boldsymbol{C}_0,\boldsymbol{C}_1 \in \mathbb{R}^{M\times N}$, there holds

$$\mathtt{j}\mathtt{I}\boldsymbol{C} = \mathtt{j}\mathtt{I}(\boldsymbol{C}_0 + \mathtt{i}\boldsymbol{C}_1) = (\boldsymbol{C}_0 - \mathtt{i}\boldsymbol{C}_1)\mathtt{j}\mathtt{I} = \boldsymbol{C}^{*}\mathtt{j}\mathtt{I} \tag{12}$$

3. Problem formulation

3.1. On second-order noncircularity

Second-order (SO) noncircular signals possess nonzero conjugate SO moments in addition to the SO moments. A particular kind of SO noncircular signals, called rectilinear signals, e.g., binary phase shift keying (BPSK) and amplitude modulation (AM) signals, are widely studied in the array signal processing literature. The SO moment and conjugate SO moment of a rectilinear signal $s(t) \in \mathbb{C}_{\mathtt{i}}$ are given by

$$E\{|s(t)|^2\} = \sigma^2 \tag{13}$$

$$E\{s^2(t)\} = \sigma^2 e^{\mathtt{i}\varpi} \tag{14}$$

respectively, where $\varpi \in [0,2\pi)$ is referred to as the noncircular phase and $\sigma^2 > 0$ denotes the power of $s(t)$.

We assume that $s(t)$ is also non-Gaussian, and it possesses a nonzero kurtosis which is defined as

$$\kappa = \mathrm{cum}\{s(t),s^*(t),s(t),s^*(t)\} = -2\sigma^4 \tag{15}$$

where operator "cum" denotes the FO cumulant and is defined as

$$\begin{aligned} &\mathrm{cum}\{s_1(t),s_2(t),s_3(t),s_4(t)\} \\ &\quad = E\{s_1(t)s_2(t)s_3(t)s_4(t)\} - E\{s_1(t)s_3(t)\}E\{s_2(t)s_4(t)\} \\ &\qquad - E\{s_1(t)s_4(t)\}E\{s_2(t)s_3(t)\} - E\{s_1(t)s_2(t)\}E\{s_3(t)s_4(t)\} \end{aligned} \tag{16}$$

For simplicity, we denote $\mathrm{cum}\{\boldsymbol{x},\boldsymbol{y},\boldsymbol{z},\boldsymbol{p}\}$ (where $\boldsymbol{x},\boldsymbol{y},\boldsymbol{z},\boldsymbol{p} \in \mathbb{C}_{\mathtt{i}}^{N\times 1}$) as a $N^2 \times N^2$ matrix of which the $((k_1-1)N+k_2, (k_3-1)N+k_4)$-th entry is $\mathrm{cum}\{x_{k_1},y_{k_2},z_{k_3},p_{k_4}\}$, where $\{k_n\}_{n=1}^{4} = 1,2,\ldots,N$.

3.2. Array model for noncircular sources

We assume an N-element scalar-sensor array illuminated by M statistically independent noncircular sources $\{s_m(t)\}_{m=1}^{M} \in \mathbb{C}_{\mathtt{i}}$ from $\{\theta_m\}_{m=1}^{M} \in [-\pi/2,\pi/2)$, steered by

$\{\boldsymbol{a}_m\}_{m=1}^{M} \in \mathbb{C}_{\mathrm{i}}^{N\times 1}$, in the presence of Gaussian noise $\boldsymbol{n}(t) \in \mathbb{C}_{\mathrm{i}}^{N\times 1}$. The output vector of the array $\boldsymbol{x}(t) \in \mathbb{C}_{\mathrm{i}}^{N\times 1}$ is then given by

$$\boldsymbol{x}(t) = \sum_{m=1}^{M} \boldsymbol{a}_m s_m(t) + \boldsymbol{n}(t) = \boldsymbol{A}\boldsymbol{s}(t) + \boldsymbol{n}(t) \tag{17}$$

where $\boldsymbol{A} = [\boldsymbol{a}_1, \boldsymbol{a}_2, \ldots, \boldsymbol{a}_M]$ and $\boldsymbol{s}(t) = [s_1(t), s_2(t), \ldots, s_M(t)]^{\mathrm{T}}$.

The array's three FO cumulant matrices are defined as [6]

$$C_{\mathrm{xx1}} = \mathrm{cum}\{\boldsymbol{x}(t), \boldsymbol{x}^*(t), \boldsymbol{x}^*(t), \boldsymbol{x}(t)\} = \tilde{A}_1 \tilde{C}_{ss1} \tilde{A}_1^{\mathrm{H}} \in \mathbb{C}_{\mathrm{i}}^{N^2\times N^2} \tag{18}$$

$$C_{\mathrm{xx2}} = \mathrm{cum}\{\boldsymbol{x}(t), \boldsymbol{x}(t), \boldsymbol{x}^*(t), \boldsymbol{x}(t)\} = \tilde{A}_2 \tilde{C}_{ss2} \tilde{A}_1^{\mathrm{H}} \in \mathbb{C}_{\mathrm{i}}^{N^2\times N^2} \tag{19}$$

$$C_{\mathrm{xx3}} = \mathrm{cum}\{\boldsymbol{x}^*(t), \boldsymbol{x}^*(t), \boldsymbol{x}(t), \boldsymbol{x}(t)\} = \tilde{A}_2^* \tilde{C}_{ss3} \tilde{A}_2^{\mathrm{T}} \in \mathbb{C}_{\mathrm{i}}^{N^2\times N^2} \tag{20}$$

respectively, where

$$C_{\mathrm{xx1}}^{\mathrm{H}} = C_{\mathrm{xx1}}, C_{\mathrm{xx3}}^{\mathrm{H}} = C_{\mathrm{xx3}} \tag{21}$$

$$\tilde{A}_1 = \boldsymbol{A} \otimes \boldsymbol{A}^* \in \mathbb{C}_{\mathrm{i}}^{N^2\times M^2}, \tilde{A}_2 = \boldsymbol{A} \otimes \boldsymbol{A} \in \mathbb{C}_{\mathrm{i}}^{N^2\times M^2} \tag{22}$$

$$(\tilde{C}_{\mathrm{ss1}})_{(m_1-1)M+m_2,(m_3-1)M+m_4} = \mathrm{cum}\{s_{m_1}(t), s_{m_2}^*(t), s_{m_3}^*(t), s_{m_4}(t)\} \tag{23}$$

$$(\tilde{C}_{\mathrm{ss2}})_{(m_1-1)M+m_2,(m_3-1)M+m_4} = \mathrm{cum}\{s_{m_1}(t), s_{m_2}(t), s_{m_3}^*(t), s_{m_4}(t)\} \tag{24}$$

$$(\tilde{C}_{\mathrm{ss3}})_{(m_1-1)M+m_2,(m_3-1)M+m_4} = \mathrm{cum}\{s_{m_1}^*(t), s_{m_2}^*(t), s_{m_3}(t), s_{m_4}(t)\} \tag{25}$$

$$\{\tilde{C}_{ssn}\}_{n=1}^{3} \in \mathbb{C}_{\mathrm{i}}^{M^2\times M^2}, \{m_n\}_{n=1}^{4} = 1, 2, \ldots, M \tag{26}$$

and "$\otimes$" denotes the Kronecker product. Since the sources are independent, we can rewrite the three FO cumulant matrices as

$$C_{\mathrm{xx1}} = A_1 \boldsymbol{\Gamma}_1 A_1^{\mathrm{H}} = \sum_{m=1}^{M} \kappa_m \boldsymbol{c}_{1m} \boldsymbol{c}_{1m}^{\mathrm{H}} \tag{27}$$

$$C_{\mathrm{xx2}} = A_2 \boldsymbol{\Gamma}_2 A_1^{\mathrm{H}} = \sum_{m=1}^{M} \kappa_m e^{\mathrm{i}\varpi_m} \boldsymbol{c}_{2m} \boldsymbol{c}_{1m}^{\mathrm{H}} \tag{28}$$

$$C_{\mathrm{xx3}} = A_2^* \boldsymbol{\Gamma}_1 A_2^{\mathrm{T}} = \sum_{m=1}^{M} \kappa_m \boldsymbol{c}_{2m}^* \boldsymbol{c}_{2m}^{\mathrm{T}} \tag{29}$$

respectively, where

$$A_1 = \boldsymbol{A} \circ \boldsymbol{A}^* \in \mathbb{C}_{\mathrm{i}}^{N^2\times M}, A_2 = \boldsymbol{A} \circ \boldsymbol{A} \in \mathbb{C}_{\mathrm{i}}^{N^2\times M} \tag{30}$$

$$\boldsymbol{c}_{1m} = \boldsymbol{a}_m \otimes \boldsymbol{a}_m^* \in \mathbb{C}_{\mathrm{i}}^{N^2\times 1}, \boldsymbol{c}_{2m} = \boldsymbol{a}_m \otimes \boldsymbol{a}_m \in \mathbb{C}_{\mathrm{i}}^{N^2\times 1}, m = 1, 2, \ldots, M \tag{31}$$

$$\boldsymbol{\Gamma}_1 = \mathrm{diag}(\kappa_1, \kappa_2, \ldots, \kappa_M) \in \mathbb{R}^{M\times M} \tag{32}$$

$$\boldsymbol{\Gamma}_2 = \mathrm{diag}(\kappa_1 e^{\mathrm{i}\varpi_1}, \kappa_2 e^{\mathrm{i}\varpi_2}, \ldots, \kappa_M e^{\mathrm{i}\varpi_M}) \in \mathbb{C}_{\mathrm{i}}^{M\times M} \tag{33}$$

and "$\circ$" denotes the Khatri–Rao product.

4. Proposed DOA estimation algorithm

4.1. Biquaternion cumulant MUSIC

In the NC-4-MUSIC ($l=1$) approach, an extended cumulant matrix is formed as

$$\boldsymbol{E} = \begin{bmatrix} C_{\mathrm{xx1}} & C_{\mathrm{xx2}}^{\mathrm{T}} \\ C_{\mathrm{xx2}}^* & C_{\mathrm{xx3}} \end{bmatrix} \tag{34}$$

and the steering vector of $s_m(t)$ is given by

$$\boldsymbol{e}_m = \begin{bmatrix} \boldsymbol{c}_{1m} \\ e^{-\mathrm{i}\varpi_m} \boldsymbol{c}_{2m}^* \end{bmatrix} \in \mathbb{C}_{\mathrm{i}}^{2N^2\times 1} \tag{35}$$

We consider an alternate construction of $\boldsymbol{e}_m$ in the hypercomplex framework as follows:

$$\boldsymbol{b}_m = \boldsymbol{c}_{1m} + e^{-\mathrm{i}\varpi_m} \boldsymbol{c}_{2m}^* \mathrm{j}\,\mathrm{I} \in \mathbb{H}_{\mathbb{C}}^{N^2\times 1} \tag{36}$$

As a consequence, the pseudo cumulant matrix is formed as

$$\begin{aligned} B &= \sum_{m=1}^{M} \kappa_m \boldsymbol{b}_m \boldsymbol{b}_m^{\dagger} \\ &= \sum_{m=1}^{M} \kappa_m (\boldsymbol{c}_{1m} + e^{-\mathrm{i}\varpi_m} \boldsymbol{c}_{2m}^* \mathrm{j}\,\mathrm{I})(\boldsymbol{c}_{1m}^{\mathrm{H}} + \mathrm{j}\,\mathrm{I} e^{\mathrm{i}\varpi_m} \boldsymbol{c}_{2m}^{\mathrm{T}}) \\ &= \sum_{m=1}^{M} \kappa_m (\boldsymbol{c}_{1m} \boldsymbol{c}_{1m}^{\mathrm{H}} + e^{-\mathrm{i}\varpi_m} \boldsymbol{c}_{1m} \boldsymbol{c}_{2m}^{\mathrm{H}} \mathrm{j}\,\mathrm{I} \\ &\quad + \mathrm{j}\,\mathrm{I} e^{\mathrm{i}\varpi_m} \boldsymbol{c}_{2m} \boldsymbol{c}_{1m}^{\mathrm{H}} + \boldsymbol{c}_{2m}^* \boldsymbol{c}_{2m}^{\mathrm{T}}) \\ &= C_{\mathrm{xx1}} + C_{\mathrm{xx2}}^{\mathrm{H}} \mathrm{j}\,\mathrm{I} + \mathrm{j}\,\mathrm{I} C_{\mathrm{xx2}} + C_{\mathrm{xx3}} \end{aligned} \tag{37}$$

It should be noted that $\boldsymbol{E}$ has $2N^2$ eigenvalues, hence in order to create a hypercomplex matrix that has the same number of eigenvalues, biquaternions are chosen (N^2 in the case of quaternions).

According to the theory of signal and noise subspaces, $\boldsymbol{b}_m$ is orthogonal to the noise subspace, denoted U_{n}, out of B, where the EVD of B is given by

$$\begin{aligned} B &= \sum_{n=1}^{2N^2} \lambda_n \boldsymbol{u}_n \boldsymbol{u}_n^{\dagger} \\ &= [\,U_{\mathrm{s}} \quad U_{\mathrm{n}}\,] \begin{bmatrix} \boldsymbol{\Sigma}_{\mathrm{s}} & \boldsymbol{O} \\ \boldsymbol{O} & \boldsymbol{O} \end{bmatrix} \begin{bmatrix} U_{\mathrm{s}}^{\dagger} \\ U_{\mathrm{n}}^{\dagger} \end{bmatrix} \end{aligned} \tag{38}$$

and a MUSIC-like estimator can be established as follows:

$$(\hat{\theta}, \hat{\varpi}) = \arg\min_{\theta,\varpi} J(\theta, \varpi) = \arg\min_{\theta,\varpi} \| U_{\mathrm{n}}^{\dagger} \boldsymbol{b}(\theta, \varpi) \|^2 \tag{39}$$

where

$$\boldsymbol{b}(\theta, \varpi) = \boldsymbol{c}_1(\theta) + e^{-\mathrm{i}\varpi} \boldsymbol{c}_2^*(\theta) \mathrm{j}\,\mathrm{I} \tag{40}$$

From the construction of B, we know that

$$U_{\mathrm{n}} = U_{\mathrm{n0}} + \mathrm{j}\,\mathrm{I} U_{\mathrm{n1}} \tag{41}$$

where both U_{n0} and U_{n1} are complex-valued matrices. Hence,

$$U_{\mathrm{n}} U_{\mathrm{n}}^{\dagger} = V_{\mathrm{n0}} + \mathrm{j}\,\mathrm{I} V_{\mathrm{n1}} \tag{42}$$

where

$$\begin{aligned} V_{\mathrm{n0}} &= U_{\mathrm{n0}} U_{\mathrm{n0}}^{\mathrm{H}} + U_{\mathrm{n1}}^* U_{\mathrm{n1}}^{\mathrm{T}}, V_{\mathrm{n0}}^{\mathrm{H}} = V_{\mathrm{n0}} \\ V_{\mathrm{n1}} &= U_{\mathrm{n1}} U_{\mathrm{n0}}^{\mathrm{H}} + U_{\mathrm{n0}}^* U_{\mathrm{n1}}^{\mathrm{T}}, V_{\mathrm{n1}}^{\mathrm{T}} = V_{\mathrm{n1}} \end{aligned} \tag{43}$$

We then rewrite the cost function as

$$\begin{aligned}J(\theta,\varpi) &= \mathcal{R}[\boldsymbol{b}^{\dagger}(\theta,\varpi)U_{\mathrm{n}}U_{\mathrm{n}}^{\dagger}\boldsymbol{b}(\theta,\varpi)]\\ &= \mathcal{R}\{[\boldsymbol{c}_1^{\mathrm{H}}(\theta)+\mathtt{j}\,\mathtt{I}e^{\mathrm{i}\varpi}\boldsymbol{c}_2^{\mathrm{T}}(\theta)](V_{\mathrm{n}0}+\mathtt{j}\,\mathtt{I}V_{\mathrm{n}1})[\boldsymbol{c}_1(\theta)\\ &\quad+e^{-\mathrm{i}\varpi}\boldsymbol{c}_2^{*}(\theta)\mathtt{j}\,\mathtt{I}]\}\\ &= \boldsymbol{c}_1^{\mathrm{H}}(\theta)V_{\mathrm{n}0}\boldsymbol{c}_1(\theta)+2\mathcal{R}[e^{\mathrm{i}\varpi}\boldsymbol{c}_1^{\mathrm{H}}(\theta)V_{\mathrm{n}1}^{*}\boldsymbol{c}_2(\theta)]\\ &\quad+\boldsymbol{c}_2^{\mathrm{H}}(\theta)V_{\mathrm{n}0}^{*}\boldsymbol{c}_2(\theta)\end{aligned} \tag{44}$$

Thus

$$\forall\theta,\min_{\varpi}J(\theta,\varpi)=\boldsymbol{c}_1^{\mathrm{H}}(\theta)V_{\mathrm{n}0}\boldsymbol{c}_1(\theta)-2\left|\boldsymbol{c}_1^{\mathrm{H}}(\theta)V_{\mathrm{n}1}^{*}\boldsymbol{c}_2(\theta)\right|+\boldsymbol{c}_2^{\mathrm{H}}(\theta)V_{\mathrm{n}0}^{*}\boldsymbol{c}_2(\theta) \tag{45}$$

The spatial spectrum is then given by

$$J(\theta)=\boldsymbol{c}_1^{\mathrm{H}}(\theta)V_{\mathrm{n}0}\boldsymbol{c}_1(\theta)-2\left|\boldsymbol{c}_1^{\mathrm{H}}(\theta)V_{\mathrm{n}1}^{*}\boldsymbol{c}_2(\theta)\right|+\boldsymbol{c}_2^{\mathrm{H}}(\theta)V_{\mathrm{n}0}^{*}\boldsymbol{c}_2(\theta) \tag{46}$$

and the noncircular phase is then estimated by

$$e^{-\mathrm{i}\hat{\varpi}}=-\frac{\boldsymbol{c}_1^{\mathrm{H}}(\hat{\theta})V_{\mathrm{n}1}^{*}\boldsymbol{c}_2(\hat{\theta})}{\left|\boldsymbol{c}_1^{\mathrm{H}}(\hat{\theta})V_{\mathrm{n}1}^{*}\boldsymbol{c}_2(\hat{\theta})\right|} \tag{47}$$

We name the DOA estimator formulated above as Biquaternion Noncircular Cumulant MUSIC (BNC-MUSIC). For the NC-4-MUSIC ($l=1$) approach, the expressions of spatial spectrum $F(\theta)$ and the noncircular phase are similar to BNC-MUSIC, and are respectively given by [6]

$$\begin{aligned}F(\theta)&=\boldsymbol{c}_1^{\mathrm{H}}(\theta)\boldsymbol{U}_{\mathrm{n}1}\boldsymbol{U}_{\mathrm{n}1}^{\mathrm{H}}\boldsymbol{c}_1(\theta)-2\left|\boldsymbol{c}_1^{\mathrm{H}}(\theta)\boldsymbol{U}_{\mathrm{n}1}\boldsymbol{U}_{\mathrm{n}2}^{\mathrm{H}}\boldsymbol{c}_2^{*}(\theta)\right|\\ &\quad+\boldsymbol{c}_2^{\mathrm{T}}(\theta)\boldsymbol{U}_{\mathrm{n}2}\boldsymbol{U}_{\mathrm{n}2}^{\mathrm{H}}\boldsymbol{c}_2^{*}(\theta)\end{aligned} \tag{48}$$

$$e^{\mathrm{i}\hat{\varpi}}=-\frac{\boldsymbol{c}_1^{\mathrm{H}}(\hat{\theta})\boldsymbol{U}_{\mathrm{n}1}\boldsymbol{U}_{\mathrm{n}2}^{\mathrm{H}}\boldsymbol{c}_2^{*}(\hat{\theta})}{\left|\boldsymbol{c}_1^{\mathrm{H}}(\hat{\theta})\boldsymbol{U}_{\mathrm{n}1}\boldsymbol{U}_{\mathrm{n}2}^{\mathrm{H}}\boldsymbol{c}_2^{*}(\hat{\theta})\right|} \tag{49}$$

where $\boldsymbol{U}_{\mathrm{n}1}$ and $\boldsymbol{U}_{\mathrm{n}2}$ are the two equal-dimensional submatrices of the noise subspace $\boldsymbol{U}_{\mathrm{n}}$ of $\boldsymbol{E}$, i.e., $\boldsymbol{U}_{\mathrm{n}}=[\boldsymbol{U}_{\mathrm{n}1}^{\mathrm{T}},\boldsymbol{U}_{\mathrm{n}2}^{\mathrm{T}}]^{\mathrm{T}}$.

4.2. Performance analysis compared with NC-4-MUSIC

Two main reasons are expected to lead to performance improvements compared with NC-4-MUSIC ($l=1$), namely

(1) As is shown in [8], the orthogonality between two biquaternion vectors is generally stronger than its complex counterpart, subspace-based estimators using $\boldsymbol{b}_m$ is expected to have a better performance both in estimation accuracy and robustness to model errors. For example, we assume two $N^2\times 1$ complex vectors $\boldsymbol{x},\boldsymbol{y}$. If $\boldsymbol{e}_m$ is orthogonal to $\boldsymbol{z}=[\boldsymbol{x}^{\mathrm{T}},\boldsymbol{y}^{\mathrm{T}}]^{\mathrm{T}}$, it leads to

$$\boldsymbol{e}_m^{\mathrm{H}}\boldsymbol{z}=\boldsymbol{c}_{1m}^{\mathrm{H}}\boldsymbol{x}+e^{\mathrm{i}\varpi_m}\boldsymbol{c}_{2m}^{\mathrm{T}}\boldsymbol{y}=0 \tag{50}$$

However, if $\boldsymbol{b}_m$ is orthogonal to the hypercomplex counterpart of $\boldsymbol{z}$, namely $\boldsymbol{z}'=\boldsymbol{x}+\boldsymbol{y}\mathtt{j}\,\mathtt{I}$, it leads to

$$\boldsymbol{b}_m^{\dagger}\boldsymbol{z}'=\boldsymbol{c}_{1m}^{\mathrm{H}}\boldsymbol{x}+e^{-\mathrm{i}\varpi_m}\boldsymbol{c}_{2m}^{\mathrm{H}}\boldsymbol{y}^{*}+\mathtt{j}\,\mathtt{I}(e^{\mathrm{i}\varpi_m}\boldsymbol{c}_{2m}^{\mathrm{T}}\boldsymbol{x}+\boldsymbol{c}_{1m}^{\mathrm{T}}\boldsymbol{y}^{*})=0 \tag{51}$$

which means that

$$\boldsymbol{c}_{1m}^{\mathrm{H}}\boldsymbol{x}+e^{-\mathrm{i}\varpi_m}\boldsymbol{c}_{2m}^{\mathrm{H}}\boldsymbol{y}^{*}=0,\quad e^{\mathrm{i}\varpi_m}\boldsymbol{c}_{2m}^{\mathrm{T}}\boldsymbol{x}+\boldsymbol{c}_{1m}^{\mathrm{T}}\boldsymbol{y}^{*}=0 \tag{52}$$

The extra constraint leads to the "stronger" orthogonality in the biquaternion domain compared with the complex one.

(2) In the construction of the pseudo cumulant matrix B, the cumulant matrices are actually smoothed, i.e., (a) between $C_{\mathrm{xx}1}$ and $C_{\mathrm{xx}3}$; (b) between $C_{\mathrm{xx}2}$ and $C_{\mathrm{xx}2}^{\mathrm{T}}$.

4.3. Resolution capacity

The NC-2q-MUSIC has an estimation capacity of $(q+l)(N-1)$ for a uniform linear array (ULA) composed of N elements [5,6,13,14]. As $\boldsymbol{b}_m$ shares the same aperture extension ability with $\boldsymbol{e}_m$, BNC-MUSIC is capable of estimating $(2+1)(N-1)=3(N-1)$ noncircular sources for an ULA, which will be later demonstrated via numerical examples.

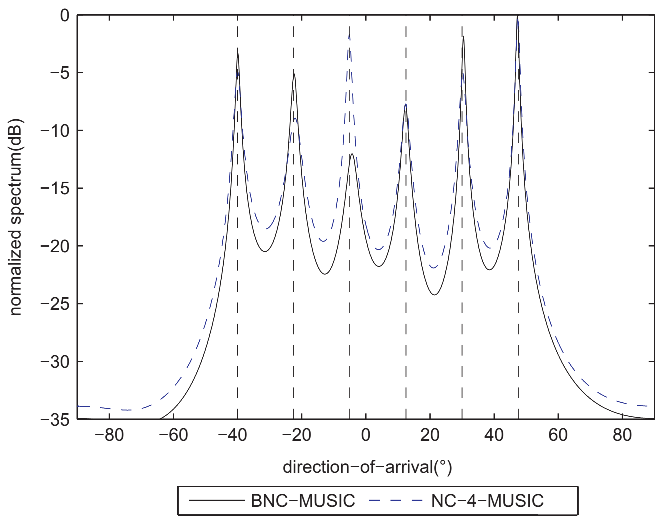

Fig. 1. Normalized spectrum of BNC-MUSIC and NC-4-MUSIC.

4.4. Computational complexity compared with NC-4-MUSIC

(1) The complexity to compute the three cumulant matrices is about $27N^4K$ complex multiplications, where K denotes the snapshot number [6].
(2) The EVD of the $N^2 \times N^2$ Hermitian biquaternion matrix B, can be deduced to the EVD of the $2N^2 \times 2N^2$ quaternionic adjoint matrix $\gamma(B)$ of B, then to the EVD of the $4N^2 \times 4N^2$ complex adjoint matrix $\chi[\gamma(B)]$ of $\gamma(B)$. It leads to that the complexity of the EVD of B is about $O(64N^6)$ complex multiplications, compared with $O(8N^6)$ in the case of NC-4-MUSIC [6].
(3) The searching process shares the same complexity with NC-4-MUSIC.

Table 1
Actual element responses.

No.	Magnitude	Phase
1	1	0°
2	0.98	1°
3	1.01	2°

4.5. Algorithmic difficulties related to biquaternions

Since the EVD of B can be transformed into the EVD of its complex adjoint matrix $\chi[\gamma(B)]$ and the searching process is done in the complex domain, all the hypercomplex computations involved in BNC-MUSIC can be converted into complex computations. Hence, even though biquaternions form a non-multiplicatively commutative, non-normed division algebra, there would not be algorithmic difficulties for BNC-MUSIC.

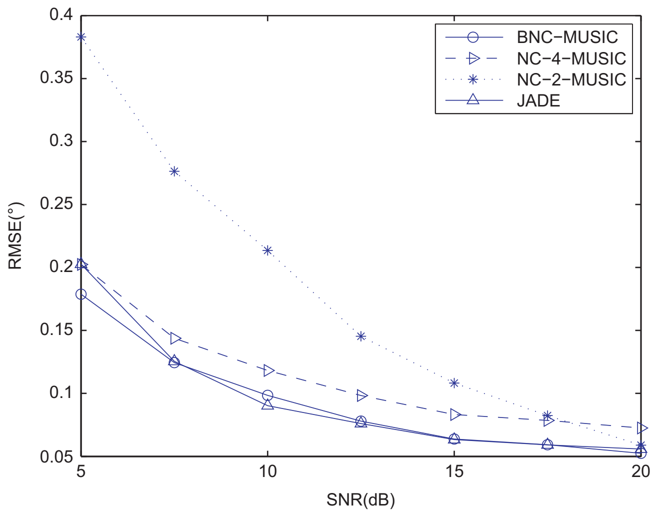

Fig. 2. RMSEs versus SNR of BNC-MUSIC, NC-4-MUSIC, NC-2-MUSIC, and JADE.

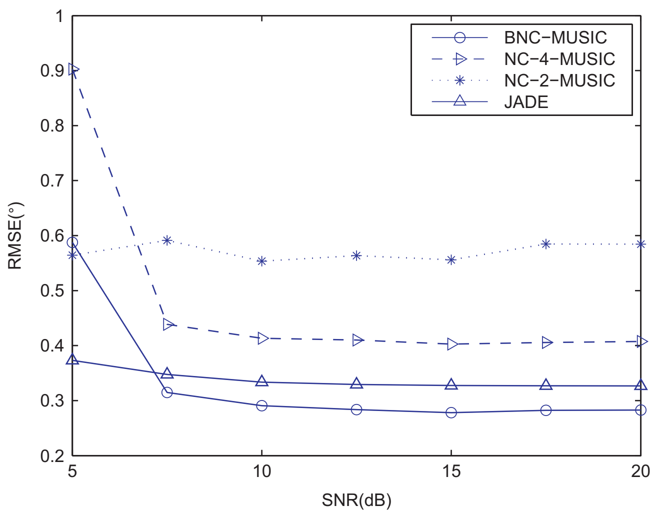

Fig. 3. RMSEs versus SNR of BNC-MUSIC, NC-4-MUSIC, NC-2-MUSIC, and JADE in the presence of channel mismatch.

4.6. Discussions on diagonalization

In this subsection, we present some remarks on the diagonalization issue:

(1) The biquaternion diagonalization of the pseudo-cumulant matrix B is given by $B=\sum_{n=1}^{R} \boldsymbol{v}_n \mu_n \boldsymbol{v}_n^{\dagger}$, where $R \le N^2$, $\{\mu_n\}_{n=1}^{R} \in \mathbb{H}_{\mathbb{C}}$, and $\{\boldsymbol{v}_n\}_{n=1}^{R} \in \mathbb{H}_{\mathbb{C}}^{N^2\times 1}$. However, the diagonal elements $\{\mu_n\}_{n=1}^{R}$ contain no information of incident directions, which is different and hence not suitable for the biquaternion matrix diagonalization (BMD) approach proposed in [15].

(2) Since $C_{xx1}=A_1\boldsymbol{\Gamma}_1A_1^{\mathrm{H}}$, $C_{xx3}=A_2^{*}\boldsymbol{\Gamma}_1A_2^{\mathrm{T}}$, $\boldsymbol{\Gamma}_1$ is a real diagonal matrix, and A_1 cannot be expressed by a linear permutation of A_2, it is generally impossible to jointly diagonalize C_{xx1} and C_{xx3}.

5. Simulation results

In this section, we provide some numerical examples to illustrate the performance of the proposed DOA estimator. We assume an ULA of 3 scalar-sensors spaced half wavelength apart, receiving equal-power statistically independent BPSK signals with phases uniformly distributed in $(0,2\pi)$, in the presence of complex-valued additive white Gaussian noise.

5.1. Processing capacity

The estimation capacity of BNC-MUSIC is $3(N-1)=6$. Assume the array is illuminated by 6 signals from -40°, -22.5°, -5°, 12.5°, 30°, 47.5°, respectively. The SNR is 20 dB and the snapshot number is 10^4. Fig. 1 depicts the normalized spatial spectrum $1/J(\theta)$ of BNC-MUSIC algorithm compared with NC-4-MUSIC and demonstrates the capacity.

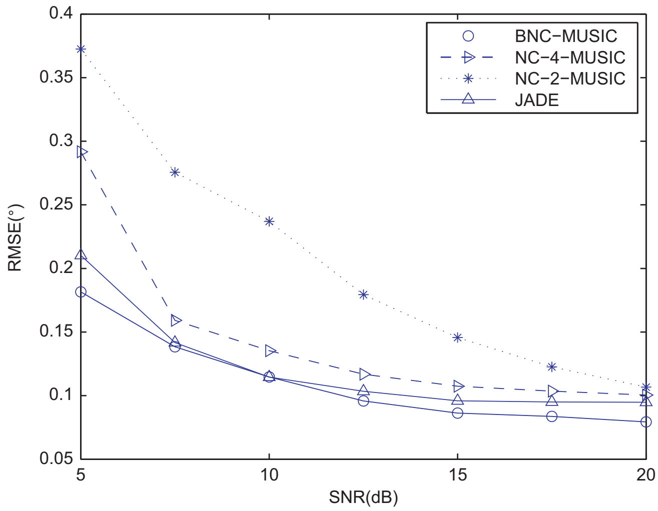

Fig. 4. RMSEs versus SNR of BNC-MUSIC, NC-4-MUSIC, NC-2-MUSIC, and JADE in the presence of element position errors.

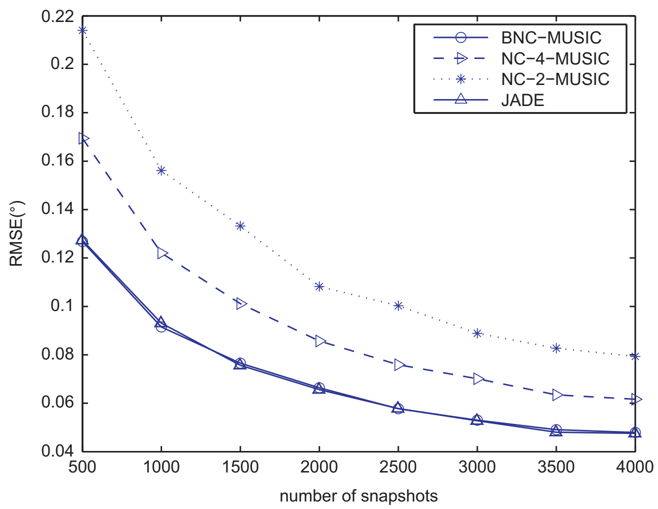

Fig. 5. RMSEs versus snapshot number of BNC-MUSIC, NC-4-MUSIC, NC-2-MUSIC, and JADE.

5.2. Estimation accuracy

Assume the array is illuminated by 2 signals from 0° and 7°, respectively. All the results are averaged via 1000 Monte-Carlo simulation runs. The root-mean-square error (RMSE) curves versus signal-to-noise ratio (SNR) of BNC-MUSIC, NC-4-MUSIC ($l=1$), NC-2-MUSIC, and Joint Approximate Diagonalization of Eigen-matrices (JADE) [16] in the cases of perfect array calibration, channel mismatch (as shown in Table 1), and element position errors (3%) for 2000 snapshots are shown in Figs. 2, 3 and 4, respectively. The RMSE curves versus snapshot number of the four estimators in the same three aforementioned scenarios for SNR=15 dB are depicted in Figs. 5, 6 and 7, respectively.

(1) We can see from Figs. 2–7 that generally the three FO statistics-based algorithms, namely BNC-MUSIC, NC-4-MUSIC and JADE, have better estimation precision compared with the SO statistics-based NC-2-MUSIC algorithm, mainly due to the larger aperture and the suppression of Gaussian noise.
(2) The BNC-MUSIC has better estimation precision compared with NC-4-MUSIC, as is explained in Section 4.2, which can be observed in Figs. 2–7.
(3) It is observed in Figs. 2 and 5 that BNC-MUSIC and JADE have almost the same estimation precision in ideal conditions. Though JADE is a blind approach that can blindly estimate the array manifold, the extraction of incident directions from the manifold might be affected by model errors. MUSIC-like algorithms find incident directions based on the orthogonality between the signal and noise subspaces and might be less sensitive to model errors. Thus BNC-MUSIC could have better performance compared with JADE in the presence of model errors, which can be seen in Figs. 3–4 and Figs. 6–7.

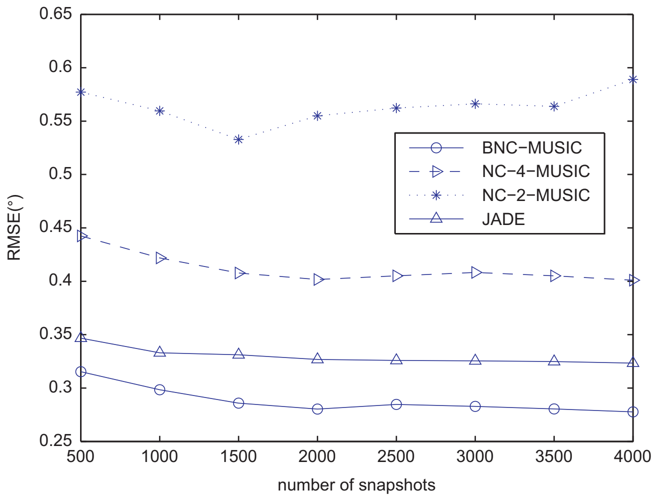

Fig. 6. RMSEs versus snapshot number of BNC-MUSIC, NC-4-MUSIC, NC-2-MUSIC, and JADE in the presence of channel mismatch.

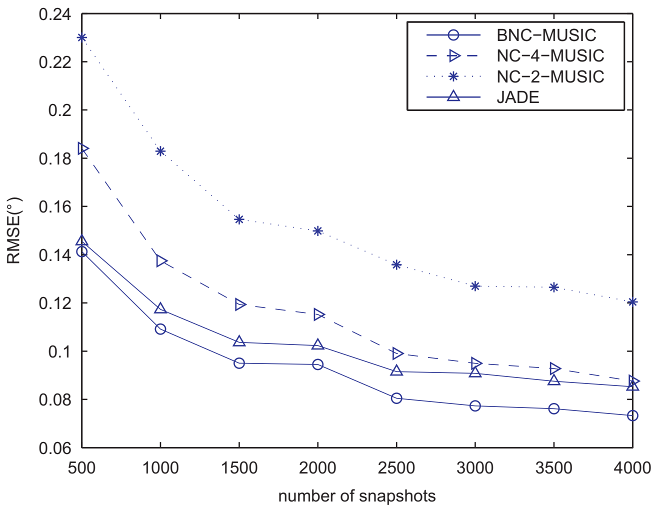

Fig. 7. RMSEs versus snapshot number of BNC-MUSIC, NC-4-MUSIC, NC-2-MUSIC, and JADE in the presence of element position errors.

(4) The presence of channel mismatch seems to have more serious influence compared with element position errors, since the curves in Figs. 3 and 6 converge at relatively small SNR values and numbers of snapshots, respectively.

Conclusively, BNC-MUSIC has a better estimation performance compared with NC-2q-MUSIC ($q=1$, 2), and is more robust to modeling errors compared with JADE in terms of direction finding.

6. Conclusion

We have proposed a biquaternion-based cumulant MUSIC (BNC-MUSIC) for noncircular signals. Both the theoretical formulation and computer simulations show that the BNC-MUSIC estimator has better performance than its complex counterpart in terms of RMSE while sharing the same estimation capacity. To conclude the paper, we note that the proposed algorithm can be easily extended to higher even orders, e.g., sixth-order.

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