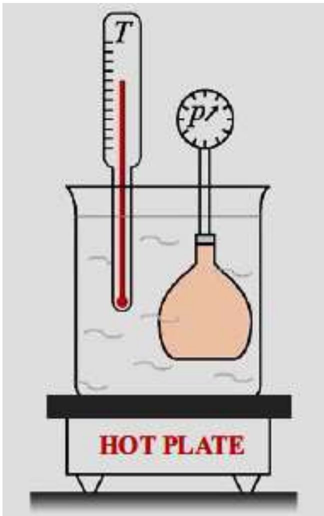
Homework #3

Due Date : Apr. 14, 2020 (23.59pm)

(Upload both your report and C program)

Problem 1 [35pts]. (Textbook Example 6-1) According to Charles's law for an ideal gas, at constant volume, a linear relationship exists between the pressure, p, and temperature, T. In the experiment shown in the figure, a fixed volume of gas in a sealed container is submerged in ice water ($T = 0°C$). The temperature of the gas is then increased in ten increments up to $T = 100°C$ by heating the water, and the pressure of the gas is measured at each temperature.



The data from the experiment is:

T	0	10	20	30	40	50	60	70	80	90	100
p	0.94	0.96	1.00	1.05	1.07	1.09	1.14	1.17	1.21	1.24	1.28

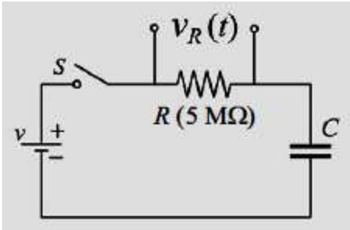
Extrapolate the data to determine the absolute zero temperature, T_0. This can be done using the following steps:

(a) Make a plot of the data (p versus T).

(b) Use linear least-squares regression to determine a linear function in the form $p = a_1 T + a_0$ that best fits the data points. First calculate the coefficients by hand using only the four data points: $0, 30, 70$, and $100\,°C$. Then write your C function that calculates the coefficients of the linear function for any number of data points and use it with all the data points to determine the coefficients.

(c) Using the Matlab, plot the linear function estimated in (b), and extend the line (extrapolate) until it crosses the horizontal (T) axis. This point is an estimate of the absolute zero temperature, T_0. Determine the value of T_0 from the function.

Problem 2 [35pts]. (Textbook Example 6-2) An experiment with an RC circuit is used for determining the capacitance of an unknown capacitor. In the circuit, shown on the right and in the following figure, a $5M\Omega$ resistor is connected in series to the unknown capacitor C and a battery.



The experiment starts by closing the switch and measuring the voltages, v_R, across the resistor every 2 seconds for 30 seconds. The data measured in the experiment is:

$t\,[sec]$	2	4	6	8	10	12	14	16	18	20	22	24	26	28	30
$v_R[V]$	9.7	8.1	6.6	5.1	4.4	3.7	2.8	2.4	2.0	1.6	1.4	1.1	0.85	0.69	0.6

Theoretically, the voltage across the resistor as a function of time is given by the exponential function:

$$v_R = v \cdot e^{-\frac{t}{RC}}$$

Determine the capacitance of the capacitor by curve fitting the exponential function to the data.

Hint

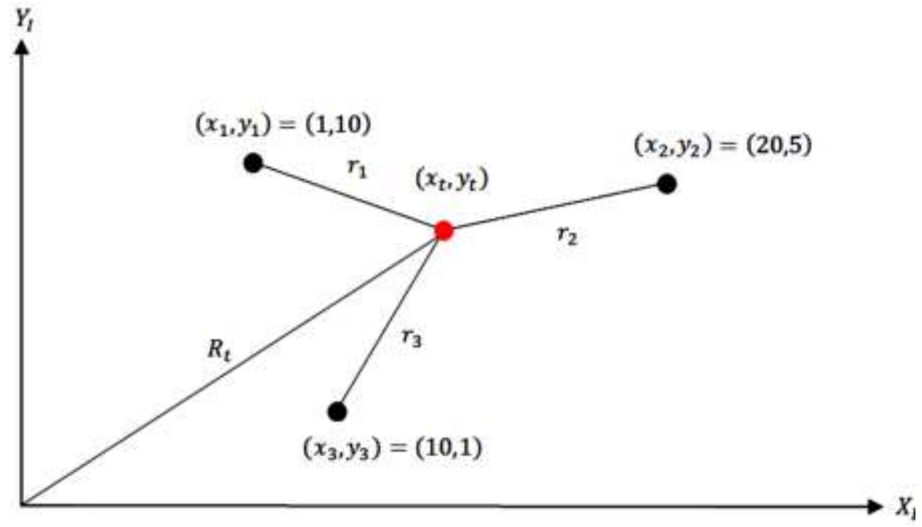
A certain curve fitting(approximation or estimation) problem associated with a nonlinear function can be converted into the linear least squares problem by considering the corresponding linear measurement equation as summarized in the following table.

Nonlinear equation	**Linear form**	**Relationship to** $Y = a_1X + a_0$	**Values for linear least-squares regression**	**Plot where data points appear to fit a straight line**
$y = bx^m$	$\ln(y) = m\ln(x) + \ln(b)$	$Y = \ln(y),\ X = \ln(x)$ $a_1 = m,\ a_0 = \ln(b)$	$\ln(x_i)$ and $\ln(y_i)$	y vs. x plot on logarithmic y and x axes. $\ln(y)$ vs. $\ln(x)$ plot on linear x and y axes.
$y = be^{mx}$	$\ln(y) = mx + \ln(b)$	$Y = \ln(y),\ X = x$ $a_1 = m,\ a_0 = \ln(b)$	x_i and $\ln(y_i)$	y vs. x plot on logarithmic y and linear x axes. $\ln(y)$ vs. x plot on linear x and y axes.
$y = b10^{mx}$	$\log(y) = mx + \log(b)$	$Y = \log(y),\ X = x$ $a_1 = m,\ a_0 = \log(b)$	x_i and $\log(y_i)$	y vs. x plot on logarithmic y and linear x axes. $\log(y)$ vs. x plot on linear x and y axes.
$y = \dfrac{1}{mx+b}$	$\dfrac{1}{y} = mx + b$	$Y = \dfrac{1}{y},\ X = x$ $a_1 = m,\ a_0 = b$	x_i and $1/y_i$	$1/y$ vs. x plot on linear x and y axes.
$y = \dfrac{mx}{b+x}$	$\dfrac{1}{y} = \dfrac{b}{m}\dfrac{1}{x} + \dfrac{1}{m}$	$Y = \dfrac{1}{y},\ X = \dfrac{1}{x}$ $a_1 = \dfrac{b}{m},\ a_0 = \dfrac{1}{m}$	$1/x_i$ and $1/y_i$	$1/y$ vs. $1/x$ plot on linear x and y axes.

Problem 3 [15pts]. Consider the target localization problem using sensor network as shown in the following figure. In the figure, the black and red circles indicate the sensor and target locations, respectively. Each sensor provides the range measurement $\tilde{r}_i$ corrupted by the measurement noise δr_i. That is, there exits the following relation between the true range r_i and measured range $\tilde{r}_i$.

$$\tilde{r}_i = r_i + \delta r_i$$

Under the assumption that the sensor position (x_i, y_i) is exactly known without uncertainty, we would like to obtain the estimate of target location $(\hat{x}_t, \hat{y}_t)$, and $\hat{R}_t^2$.



(a) Defining the unknown vector $\boldsymbol{x} = \begin{bmatrix} x_t \\ y_t \\ R_t^2 \end{bmatrix}$, derive the measurement equation of the form:

$$\boldsymbol{y}_k = H\boldsymbol{x} + \boldsymbol{v}_k$$

where $\boldsymbol{y}_k$ is the measurement vector and $\boldsymbol{v}_k$ is the measurement error vector at time instant k.

(b) Make your C program with functions:

- Data Import

  The attached data file “hw3_3b.dat” which contains the range measurements. The data structure is as follows:

k	$\tilde{r}_{1,k}$	$\tilde{r}_{2,k}$	$\tilde{r}_{3,k}$
1	6.6869	16.718	7.3259
2	6.0597	17.259	7.5072
3	5.7886	16.954	7.7358
⋮	⋮	⋮	⋮
100	6.8595	16.947	7.6694

  Open the data file and store the range measurements into the double arrays zR1[100], zR2[100] and zR3[100].

- Recursive Weighted Least Squares Estimation
  Using the imported data, recursively calculate the target location estimate $\hat{\boldsymbol{x}}_k$. Store the calculation result in the double arrays hatXt[100], hatYt[100], and hatRt2[100]. Use the following design parameters which are roughly set for implementing the recursive least squares estimator.

$$\hat{\boldsymbol{x}}_0 = \begin{bmatrix} \hat{x}_{t,0} \\ \hat{y}_{t,0} \\ \hat{R}^2_{t,0} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix},\ P_0 = \begin{bmatrix} 15^2 & 0 & 0 \\ 0 & 15^2 & 0 \\ 0 & 0 & 15^2 \end{bmatrix},\ R = \begin{bmatrix} 10^2 & 0 & 0 \\ 0 & 10^2 & 0 \\ 0 & 0 & 10^2 \end{bmatrix}$$

  c.f.
  While the above design parameters are not mathematically rigorous, they are still meaningful in practice. This is because, in many cases, the recursive least squares estimator is robust against the uncertainty in the design parameters.

- Export the Estimation Results
  Export the stored estimates hatXt[100], hatYt[100], and hatRt2[100] to the output file named "hw3_3b.out".

- Using Matlab, plot the target location estimates and attach it on your report. Confirm that your estimation results converge to true value $\boldsymbol{x} = \begin{bmatrix} x_t \\ y_t \\ R_t^2 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \\ 5^2 \end{bmatrix}$ as $k \to 100$.

Problem 4 [15pts]. Consider a DC motor has a tachometer which generates the voltage v_o proportional to the angular speed of the motor, $\omega = 2\pi f[rad/s]$. It is well-known that, issuing the sinusoidal driving voltage $v_c(t) = \sin(\omega t)$ for an ideal DC motor, the noiseless tachometer output voltage v_o can be expressed as $s(t) = A\sin(\omega t + \phi)$ where A and ϕ are often called as the gain attenuation and phase shift of $s(t)$ with respect to the driving voltage $v_c(t)$. As you already know, these parameters act an important role in modeling the motor dynamics because they represent the frequency response of the motor.

Let us consider the case that we use a computer to analyze and model the motor characteristics. To do this, the computer acquires the tachometer output voltage through its embedded A/D(analog to digital converter) with sampling frequency $f_s[Hz]$. Therefore, the acquired noiseless output voltage of the tachometer can be rewritten as

$$s_k = s\left(t = \frac{k}{f_s}\right) = A\sin\left(\frac{2\pi f}{f_s}k + \phi\right) = A\sin(\varpi k + \phi),\quad \varpi \triangleq \frac{2\pi f}{f_s}$$

In the above equation, it is assumed that the normalized angular frequency ϖ is known exactly.

(a) Unfortunately, the actual tachometer output y is inevitably contaminated by the measurement noise v. Hence, the available measurement will be

$$y_k = s_k + v_k$$

Given output sequences $y_1, y_2, \cdots y_N$, we would like to find the best estimates $\hat{A}$ and $\hat{\phi}$ in the setting of nonlinear least squares estimation. Note that the nonlinear least squares estimates $\hat{A}$ and $\hat{\phi}$ are the minimizing solution to the following cost function.

$$J_N = \sum_{k=1}^{N} (y_k - A\sin(\bar{\omega}k + \phi))^2$$

Derive the necessary condition(stationary condition) for the above minimization problem.

(b) From the above stationary condition, one can see that the nonlinear least squares estimates $\hat{A}$ and $\hat{\phi}$ satisfy the following relation.

$$\hat{A} = \frac{b_1 \cos\hat{\phi} + b_2 \sin\hat{\phi}}{a_1 + a_2 \cos\hat{\phi}\sin\hat{\phi} + a_3 \sin^2\hat{\phi}} = \frac{-b_1 \sin\hat{\phi} + b_2 \cos\hat{\phi}}{\frac{1}{2}a_2 - a_2 \sin^2\hat{\phi} + a_3 \sin\hat{\phi}\cos\hat{\phi}}$$

Express the parameters a_1, a_2, a_3, b_1 and b_2 in terms of y_k and $\bar{\omega}$.

(c) Make your own C program to find $\hat{A}$ and $\hat{\phi}$ under the assumption that $|\hat{\phi}| < \frac{\pi}{2}$. Use the data file "hw3_4c.dat" which contains the tachometer output(measurement) y_k for a motor rotating in $\bar{\omega} = 2\pi \cdot 0.176$. Your program must have following functions.

- Importing the measured tachometer output sequence $y_1, y_2, \cdots y_{4000}$
- Calculating batch form of the nonlinear least squares solution and displaying it on the screen

In order to check whether your program works properly, using Matlab, plot the measured data y_k and its estimate $\hat{y}_k = \hat{A}\sin(\bar{\omega}k + \phi)$ in the same graph.

Hint

In order to the nonlinear least squares estimation problem tractable, first define $x = \sin\phi$ and then convert the problem into the root finding one, $f(x) = 0$.