The Governing Equations of Fluid Dynamics

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Integral Form of Navier-Stokes Equations

$$\frac{d}{dt}\iiint\limits_{V(t)} r \, dV + \iint\limits_{S(t)} r \, \mathbf{u}\cdot\mathbf{n} \, dS = 0$$

$$\frac{d}{dt}\iiint\limits_{V(t)} r \, \mathbf{u} \, dV + \iint\limits_{S(t)} (r \, \mathbf{u})\mathbf{u}\cdot\mathbf{n} \, dS = \iint\limits_{S(t)} \mathbf{T} \, dS$$

$$\frac{d}{dt}\iiint\limits_{V(t)} E \, dV + \iint\limits_{S(t)} r \, E \, \mathbf{u}\cdot\mathbf{n} \, dS = \iint\limits_{S(t)} \mathbf{T}\cdot\mathbf{u} \, dS - \iint\limits_{S(t)} \mathbf{q}\cdot\mathbf{n} \, dS$$

$$\mathbf{T} = -p\mathbf{n} + \overset{=}{t} \cdot \mathbf{n}$$

Most fundamental form, see Thompson’s book

Integral Form of Euler Equations

$$\frac{d}{dt}\iiint\limits_{V(t)} r\, dV + \iint\limits_{S(t)} r\,\mathbf{u}\cdot\mathbf{n}\, dS = 0$$

$$\frac{d}{dt}\iiint\limits_{V(t)} r\,\mathbf{u}\, dV + \iint\limits_{S(t)} (r\,\mathbf{u})\mathbf{u}\cdot\mathbf{n}\, dS = -\iint\limits_{S(t)} p\mathbf{n}\, dS$$

$$\frac{d}{dt}\iiint\limits_{V(t)} E\, dV + \iint\limits_{S(t)} r\, E\,\mathbf{u}\cdot\mathbf{n}\, dS = -\iint\limits_{S(t)} p\,\mathbf{u}\cdot\mathbf{n}\, dS$$

Most fundamental form

Curl, Divergence & Grad Formulae

$$\iiint_V \nabla \times \mathbf{G}\, dV = \iint_S \mathbf{G} \times \mathbf{n}\, dS$$

$$\iiint_V \nabla \cdot \mathbf{G}\, dV = \iint_S \mathbf{G} \cdot \mathbf{n}\, dS$$

$$\iiint_V \nabla H\, dV = \iint_S H\, \mathbf{n}\, dS$$

S is surface of V

Can rewrite using Divergence Theorem

$$\frac{d}{dt}\iiint_{V(t)} r\, dV + \iiint_{V(t)} \nabla\cdot(r\,\mathbf{u})\, dV = 0$$

$$\frac{d}{dt}\iiint_{V(t)} r\,\mathbf{u}\, dV + \iiint_{V(t)} \nabla\cdot((r\,\mathbf{u})\mathbf{u})\, dV = \iiint_{V(t)} \nabla p\, dV$$

$$\frac{d}{dt}\iiint_{V(t)} E\, dV + \iiint_{V(t)} \nabla(r\, E\,\mathbf{u})\, dV = -\iiint_{V(t)} \nabla\cdot(p\,\mathbf{u})\, dV$$

Most fundamental form

Can rewrite using Divergence Theorem

$$\frac{d}{dt}\iiint_{V(t)}\left[r+\nabla\cdot(r\,\mathbf{u})\right]dV \;=0$$

$$\frac{d}{dt}\iiint_{V(t)}\left[r\,\mathbf{u}+\nabla\cdot((r\,\mathbf{u})\mathbf{u})+\nabla p\right]dV \;=0$$

$$\frac{d}{dt}\iiint_{V(t)}\left[E+\nabla\cdot(r\,E\,\mathbf{u})+\nabla\cdot(p\,\mathbf{u})\right]dV \;=0$$

Most fundamental form

Reynold’s Tranport Theorem (1903)

$$\frac{d}{dt}\iiint\limits_{V(t)} F\, dV = \iiint\limits_{V(t)} \frac{\partial F}{\partial t}\, dV + \iint\limits_{S(t)} F\, \mathbf{u}\cdot\mathbf{n}\, dS$$

Differential Form of Navier-Stokes Equations (1822 & 1845)

$$\frac{\P r}{\P t} + \nabla \cdot \left(r \mathbf{V} \right) = 0$$

$$\frac{\P}{\P t}\left(r \mathbf{V} \right) + \mathbf{V}(\nabla \cdot r \mathbf{V}) + r \mathbf{V} \cdot \nabla \mathbf{V} =$$

$$-\nabla p + \nabla \cdot t_{ij} + r\, \mathbf{f}$$

$$\frac{\P E}{\P t} + \nabla \cdot E\, \mathbf{V} = r\, \mathbf{f} \cdot \mathbf{V} +$$

$$\frac{\P Q}{\P t} - \nabla \cdot \mathbf{q} - \nabla \cdot (p \mathbf{V}) + \nabla \cdot (t_{ij} \cdot \mathbf{V})$$

Conservation of Mass

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho u) + \frac{\partial}{\partial z}(\rho w) = 0$$

Conservation of Mass

$$\frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial x} + v\frac{\partial \rho}{\partial y} + w\frac{\partial \rho}{\partial z}$$

$$\rho\frac{\partial u}{\partial x} + \rho\frac{\partial v}{\partial y} + \rho\frac{\partial w}{\partial z} = 0$$

If density is constant (incompressible), then we have:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

If we define: $$\nabla \phi = \mathbf{u}$$

Then we get : $$\nabla^2 \phi = 0$$

Laplace's Equation (elliptic)

Conservation of Momentum

$$\frac{\partial r u}{\partial t} + \frac{\P}{\P x}\left[r u^{2} + p - t_{xx}\right] + \frac{\P}{\P y}\left[r uv - t_{xy}\right]$$

$$+ \frac{\P}{\P z}\left[r uw - t_{xz}\right] = r f_{x}$$

$$\frac{\partial r v}{\partial t} + \frac{\P}{\P x}\left[r uv - t_{xy}\right] + \frac{\P}{\P y}\left[r v^{2} + p - t_{yy}\right]$$

$$+ \frac{\P}{\P z}\left[r vw - t_{yz}\right] = r f_{y}$$

$$\frac{\partial r w}{\partial t} + \frac{\P}{\P x}\left[r uw - t_{xz}\right] + \frac{\P}{\P y}\left[r vw - t_{yz}\right]$$

$$+ \frac{\P}{\P z}\left[r w^{2} + p - t_{zz}\right] = r f_{z}$$

Conservation of Energy

$$
\begin{aligned}
&\frac{\partial E_t}{\partial t} - \frac{\partial Q}{\partial t} - \rho\left(f_x u + f_y u + f_z w\right) + \\
&\quad \frac{\partial}{\partial x}\left(E_t u + pu - u\tau_{xx} - u\tau_{xy} - w\tau_{xz} + q_x\right) \\
&\quad + \frac{\partial}{\partial y}\left(E_t u + pu - u\tau_{xy} - u\tau_{yy} - w\tau_{yz} + q_y\right) \\
&\quad + \frac{\partial}{\partial z}\left(E_t w + pw - u\tau_{xz} - u\tau_{yz} - w\tau_{zz} + q_z\right) = 0
\end{aligned}
$$

Constitutive Relations

Perfect Gas:

$$p = rRT$$

Sutherland:

$$m = C_1 \frac{T^{\frac{3}{2}}}{T + C_2} \qquad k = C_3 \frac{T^{\frac{3}{2}}}{T + C_4}$$

$$k = \frac{c_p m}{\mathrm{Pr}}$$

$$g = \frac{c_p}{c_v} \qquad c_v = \frac{R}{g - 1} \qquad c_p = \frac{gR}{g - 1}$$

For air at standard conditions:

$$\mathrm{R} = 286.9 \ \mathrm{J/kg/K}$$

$$g = 1.4$$

Shear Stress

$$\tau_{xx} = \frac{2}{3}\mu\left(2\frac{\partial u}{\partial x} - \frac{\partial u}{\partial y} - \frac{\partial w}{\partial z}\right)$$

$$\tau_{yy} = \frac{2}{3}\mu\left(2\frac{\partial u}{\partial y} - \frac{\partial u}{\partial x} - \frac{\partial w}{\partial z}\right)$$

$$\tau_{zz} = \frac{2}{3}\mu\left(2\frac{\partial w}{\partial z} - \frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right)$$

$$\tau_{xy} = \mu\left(\frac{\partial u}{\partial y} + \frac{\partial u}{\partial x}\right) = \tau_{yx}$$

$$\tau_{xz} = \mu\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right) = \tau_{zx}$$

$$\tau_{yz} = \mu\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial y}\right) = \tau_{zy}$$

(Assuming Stoke’s hypothesis is valid)

A more general form is:

$$t_{ij} = 2m\,(D_{ij} - \tfrac{1}{3}\, d_{ij} D_{mm}) + m_v\, d_{ij} D_{mm}$$

$$D_{ij} = \tfrac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$

$$D_{mm} = \nabla \cdot \mathbf{u}$$

$$m = \text{shear viscosity} \quad \textit{and} \quad m_v = \text{bulk viscosity}$$

Stoke's hypothesis (which he did not argue too strongly himself), was that the bulk viscosity should be set to Zero. This is only true for dilute monatomic gases. For some gases the bulk viscosity is two orders of magnitude greater than the shear viscosity. For many fluid motions, however, D_{mm} is approximately zero (incompressible flow). For acoustics and shock waves the bulk viscosity cannot be ignored.

Navier-Stokes Equations

$$\frac{\P \mathbf{U}}{\P t} + \frac{\P \mathbf{E}}{\P x} + \frac{\P \mathbf{F}}{\P y} + \frac{\P \mathbf{G}}{\P z} = 0$$

$$\mathbf{U} = \begin{bmatrix} r \\ ru \\ ru \\ rw \\ E_t \end{bmatrix}$$

$$\mathbf{E} = \begin{bmatrix} ru \\ ru^2 + p - t_{xx} \\ ruv - t_{xy} \\ ruw - t_{xz} \\ (E_t + p)u - ut_{xx} - ut_{xy} - wt_{xz} + q_x \end{bmatrix}$$

$$\mathbf{F} = \begin{bmatrix} ru \\ ruu - t_{xy} \\ ru^2 + p - t_{yy} \\ ruw - t_{yz} \\ (E_t + p)u - ut_{xy} - ut_{yy} - wt_{yz} + q_y \end{bmatrix}$$

$$\mathbf{G} = \begin{bmatrix} rw \\ ruw - t_{xz} \\ ruw - t_{yz} \\ rw^2 + p - t_{zz} \\ (E_t + p)w - ut_{xz} - ut_{yz} - wt_{zz} + q_z \end{bmatrix}$$

Incompressible Navier-Stokes Equations

$$\frac{\P u}{\P x} + \frac{\P v}{\P y} + \frac{\P w}{\P z} = 0$$

$$r \frac{Du}{Dt} + \frac{\P p}{\P x} = m \nabla^2 u$$

$$r \frac{Dv}{Dt} + \frac{\P p}{\P y} = m \nabla^2 v$$

$$r \frac{Dw}{Dt} + \frac{\P p}{\P z} = m \nabla^2 w$$

$$r \frac{De}{Dt} = \frac{\P Q}{\P t} + k \nabla^2 T + t_{ij} \frac{\P u_i}{\P x_j}$$

If there is no flow and no heat source, then

$$\rho \frac{De}{Dt} = \cancel{\frac{\partial Q}{\partial t}} + k\nabla^2 T + \tau_{ij} \cancel{\frac{\partial u_i}{\partial x_j}}$$

Then

$$\frac{\partial T}{\partial t} = \frac{k}{\rho c_v} \nabla^2 T$$

Heat equation (parabolic)

Notes

- The book in section 5.1.7 describes the non-dimensional form of the equations (and many other references do too)
- I strongly feel that the equations should be left in dimensional form.
- For theoretical work it is useful to use non-dimensional equations, but for numerical work it is better to use dimensional forms
- With 64-bit (or even 32-bit) precision, it is not necessary to artificially scale the variables so they are near 1.0
- Many, many problems in using CFD codes result from errors in interpreting the non-dimensional equations
- And once you have chemistry and flow, non-dimensional forms make no sense

3-D Euler Equations

$$\frac{\P\mathbf{U}}{\P t}+\frac{\P\mathbf{E}}{\P x}+\frac{\P\mathbf{F}}{\P y}+\frac{\P\mathbf{G}}{\P z}=0$$

$$\mathbf{U}=\begin{bmatrix} r \\ ru \\ r u \\ rw \\ E_t \end{bmatrix}$$

$$p = rRT$$

$$E = r\,(c_v T+\frac{\mathbf{v}^2}{2})$$

$$p=\frac{R}{c_v}\left(E-\frac{\mathbf{v}^2}{2}\right)$$

$$\mathbf{E}=\begin{bmatrix} ru \\ ru^2+p \\ ruv \\ ruw \\ (E_t+p)u \end{bmatrix}$$

$$\mathbf{F}=\begin{bmatrix} r u \\ ruu \\ r u^2+p \\ r uw \\ (E_t+p)u \end{bmatrix}$$

$$\mathbf{G}=\begin{bmatrix} rw \\ ruw \\ r uw \\ rw^2+p \\ (E_t+p)w \end{bmatrix}$$

Euler Equations (1755)

$$\frac{\P r}{\P t} + \frac{\P}{\P x}\left(r\,u \right) + \frac{\P}{\P y}\left(r u \right) + \frac{\P}{\P z}\left(r\,w \right) = 0$$

$$\frac{\partial\, r\,u}{\partial\, t} + \frac{\P}{\P x}\left[r\,u^{2} + p \right] + \frac{\P}{\P y}\left[r\,uv \right] + \frac{\P}{\P z}\left[r\,uw \right] = 0$$

$$\frac{D\; r\,v}{Dt} + \frac{\P}{\P x}\left[r\,uv \right] + \frac{\P}{\P y}\left[r\,v^{2} + p \right] + \frac{\P}{\P z}\left[r\,vw \right] = 0$$

$$\frac{D\; r\,w}{Dt} + \frac{\P}{\P x}\left[r\,uw \right] + \frac{\P}{\P y}\left[r\,vw \right] + \frac{\P}{\P z}\left[r\,w^{2} + p \right] = 0$$

$$\frac{\P E_t}{\P t} + \frac{\P}{\P x}\left(E_t u + pu \right) + \frac{\P}{\P y}\left(E_t u + p\,u \right) + \frac{\P}{\P z}\left(E_t w + pw \right) = 0$$

Notes on Euler Equations

- The Euler equations are not that much easier to program than the Navier-Stokes equations
- The Navier-Stokes equations also do not require that much more variables to be stored
- Then why not always solve the Navier-Stokes instead of Euler ?

Notes on Euler Equations (cont.)

- The reason people often solve Euler instead of Navier-Stokes is that the Navier-Stokes equations require roughly an order of magnitude more grid points
- In addition, we are seldom interested in laminar flows, so solving the Navier-Stokes equations without a good turbulence model does not make much sense

2-D Euler Equations

$$\frac{\P\mathbf{U}}{\P t} + \frac{\P\mathbf{E}}{\P x} + \frac{\P\mathbf{F}}{\P y} = 0$$

$$\mathbf{U} = \begin{bmatrix} r \\ ru \\ ru \\ E_t \end{bmatrix}$$

$$\mathbf{E} = \begin{bmatrix} ru \\ ru^2 + p \\ ruv \\ (E_t + p)u \end{bmatrix}$$

$$\mathbf{F} = \begin{bmatrix} ru \\ ruu \\ ru^2 + p \\ (E_t + p)u \end{bmatrix}$$

$$p = rRT$$

$$E = r\,(c_v T + \frac{\mathbf{v}^2}{2})$$

$$p = \frac{R}{c_v}\left(E - \frac{\mathbf{v}^2}{2} \right)$$

1-D Euler Equations

$$\frac{\P\mathbf{U}}{\P t} + \frac{\P\mathbf{E}}{\P x} = 0$$

$$\mathbf{U} = \begin{bmatrix} r \\ ru \\ E_t \end{bmatrix} \qquad \mathbf{E} = \begin{bmatrix} ru \\ ru^2 + p \\ \left(E_t + p\right)u \end{bmatrix}$$

$$p = rRT$$

$$E = r\,(c_v T + \frac{\mathbf{v}^2}{2})$$

$$p = \frac{R}{c_v}\left(E - \frac{\mathbf{v}^2}{2}\right)$$

1-D Euler Equations

$$\frac{\partial r}{\partial t} + \frac{\partial}{\partial x}\left(r\, u \right) = 0$$

$$\frac{\partial\, r\, u}{\partial\, t} + \frac{\partial}{\partial x}\left[r\, u^{2} + p \right] = 0$$

$$\frac{\partial E_t}{\partial t} + \frac{\partial}{\partial x}\left(E_t u + pu \right) = 0$$

LINEAR 1-D Euler Equations

Let:

$$\rho(x,t) = \rho_o + \rho'(x,t)$$

$$u(x,t) = u_o + u'(x,t)$$

$$p(x,t) = p_o + p'(x,t)$$

$$\frac{\partial \rho'}{\partial t} + \frac{\partial}{\partial x}\left(\rho_o u' + \cancel{\rho' u'}\right) = 0$$

$$\frac{\partial\left(\rho_o u' + \cancel{\rho' u'}\right)}{\partial t} + \frac{\partial}{\partial x}\left[\cancel{(\rho_o + \rho')u'^2} + p'\right] = 0$$

Then:

$$\frac{\partial \rho'}{\partial t} + \rho_o \frac{\partial u'}{\partial x} = 0$$

$$\rho_o \frac{\partial u'}{\partial t} + \frac{\partial p'}{\partial x} = 0$$

$$\frac{\partial \rho'}{\partial t} + \rho_o \frac{\partial u'}{\partial x} = 0$$

$$\rho_o \frac{\partial u'}{\partial t} + \frac{\partial p'}{\partial x} = 0$$

We can also show that: $p' = c^2 \rho'$

Then:

$$\frac{1}{c^2} \frac{\partial p'}{\partial t} + \rho_o \frac{\partial u'}{\partial x} = 0$$

$$\rho_o \frac{\partial u'}{\partial t} + \frac{\partial p'}{\partial x} = 0$$

$$\frac{\P}{\P t}\left(\frac{1}{c^{2}} \frac{\P p'}{\P t} + r_{o} \frac{\P u'}{\P x} \right) = 0$$

$$\frac{\P}{\P x}\left(r_{o} \frac{\partial u'}{\partial t} + \frac{\P p'}{\P x} \right) = 0$$

Subtracting the above two equations gives:

$$\frac{1}{c^{2}} \frac{\P^{2} p'}{\P t^{2}} - \frac{\P^{2} p'}{\P x^{2}} = 0$$

The wave equation (hyperbolic)

Quasi 1-D Euler Equations

$$\frac{\partial \mathbf{U}}{\partial t} + \frac{\partial \mathbf{E}}{\partial x} = \mathbf{S}$$

$$\mathbf{E} = \begin{bmatrix} r u A \\ (r u^2 + p) A \\ \left(E_t + p\right) u A \end{bmatrix}$$

$$\mathbf{U} = \begin{bmatrix} r A \\ r u A \\ E_t A \end{bmatrix}$$

$$\mathbf{S} = \begin{bmatrix} 0 \\ p \dfrac{dA}{dx} \\ 0 \end{bmatrix}$$

Perfect gas:

$$p = r R T$$

$$E = r \left(c_v T + \frac{\mathbf{v}^2}{2}\right)$$

$$p = \frac{R}{c_v}\left(E - \frac{\mathbf{v}^2}{2}\right)$$

Compressible Vorticity Equation

Take curl of momentum equation to get:

$$\frac{\partial \mathbf{O}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{O} = (\mathbf{O}\cdot\nabla)\mathbf{u} - \mathbf{O}(\nabla\cdot\mathbf{u}) + \nabla T \times \nabla s +$$

$$(m_v + \tfrac{4}{3}m)\left(\nabla \tfrac{1}{r} \times \nabla(\nabla\cdot\mathbf{u})\right) +$$

$$m\left(\tfrac{1}{r}\nabla^2\mathbf{O} - \nabla\tfrac{1}{r}\times(\nabla\times\mathbf{O})\right)$$

Compressible Inviscid Vorticity Equation

$$\frac{\partial}{\partial t}\left(\frac{\mathbf{O}}{r}\right)+\mathbf{u}\cdot\nabla\left(\frac{\mathbf{O}}{r}\right)=\left(\tfrac{1}{r}\,\mathbf{O}\cdot\nabla\right)\mathbf{u}+\tfrac{1}{r}\,\nabla T\times\nabla s$$

Compressible Inviscid Vorticity Equation

For grad s = 0 (viscosity and heat conduction negligible) :

$$\frac{\partial \mathbf{O}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{O} = (\mathbf{O} \cdot \nabla)\mathbf{u} - \mathbf{O}(\nabla \cdot \mathbf{u})$$

Fluid without vorticity will remain forever without vorticity

Incompressible Vorticity Equation

Take curl of momentum equation to get:

$$\frac{\partial \mathbf{O}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{O} = (\mathbf{O} \cdot \nabla)\mathbf{u} + n \nabla^2 \mathbf{O}$$

2-D Vorticity Equation:

$$\frac{\partial V}{\partial t}+u\frac{\partial V}{\partial x}+v\frac{\partial V}{\partial y}=n\left(\frac{\partial^2 V}{\partial x^2}+\frac{\partial^2 V}{\partial y^2}\right)$$

$$\frac{DV}{Dt}=n\nabla^2 V$$

$$\nabla^2 y = -V$$

Poisson Equation (elliptic)

1-D Vorticity Equation:

$$\frac{\partial V}{\partial t} + u\frac{\partial V}{\partial x} = n\frac{\partial^2 V}{\partial x^2}$$

The above is like the nonlinear, viscous Burger's equation

References

- Tannehill, Anderson, & Pletcher
- Thompson, Compressible-Fluid Dynamics