Multiple Integration Note and Exercise Packet 5

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This set of notes will explore double integrals in polar coordinates. The notes will first derive the iterated integral for polar area using a geometric argument. We will then compute a few simple polar areas and volumes. After this theoretical base is established, the next packet we will present the change of variables theorem used for converting multiple integrals from one coordinate system to another. By learning about polar coordinates first, we can ensure that the results we obtained for that coordinate system match up with the general theorem.

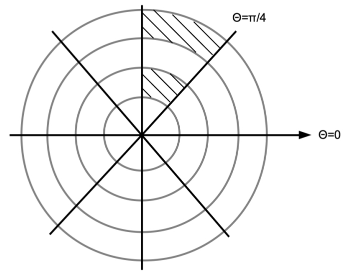
Polar Coordinates: Conversion and Peculiarities

When working with multiple coordinate systems you should be aware of conversion formulae which allow you to convert the coordinates of one system into the other. Be prepared to use both sets of formulae when switching coordinate systems.

The conventional polar coordinate system is obtained by matching $\theta = 0$ with the positive x-axis, and having θ increase while rotating in the counterclockwise direction. The formulae used to convert individual components are given below:

$$x = r\cos(\theta), \qquad y = r\sin(\theta);$$
$$r = \sqrt{x^2 + y^2}, \qquad \theta = \arctan\left(\frac{y}{x}\right).$$

When constructing an iterated polar integral one must deal with the non-uniform structure of the polar grid. Consider the following shaded sectors of the polar grid:



Suppose the circles are drawn at $r = 1, 2, 3$ etc. In this situation the changes in r and the changes in θ are held fixed for the different chunks. However, the areas are not the same. We want to compensate for this non-uniformity in our integration process so we can correctly capture polar area. Recall from geometry the area of a sector (full pie slice) can be obtained by thinking of the sector as an angular fraction of a full circle (and we all recall the area of a circle is πr^2).

$$A_s = \frac{\Delta\theta}{2\pi}(\pi r^2) = \frac{\Delta\theta^2}{r}$$

We can use this geometric formula to find the exact area of a chunk of our polar grid. Using the above picture for specificity we have:

$$A_{innerchunk} = \frac{\frac{\pi}{4}}{2}(2)^2 - \frac{\frac{\pi}{4}}{2}(2)^2(1)^2 = \frac{3\pi}{8}$$
$$A_{outerchunk} = \frac{\frac{\pi}{4}}{2}(4)^2 - \frac{\frac{\pi}{4}}{2}(2)^2(3)^2 = \frac{7\pi}{8}$$

As we already noticed by eye, these areas are different. With just these two examples in hand, it would be hard to deduce a pattern which captures the non-uniformity, however since I know where we are going I can show you the pattern here which will then give you some intuition for the general limiting argument which we use to derive the polar integral itself.

Note that the center of the inner shaded chunk occurs at $r = 1.5$, and the center of the outer shaded chunk occurs at $r = 3.5$. Both of the areas we obtained agree with the formula:

$$A_{generalchunk} = \Delta\theta\Delta r(r_{center})$$
$$A_{innerchunk} = \left(\frac{\pi}{4}\right)(1)(1.5) = \frac{3\pi}{8}$$
$$A_{outerchunk} = \left(\frac{\pi}{4}\right)(1)(3.5) = \frac{7\pi}{8}$$

Thus we expect the that the non-uniformity in polar area should depend linearly on the radial variable. If we approach this from a calculus perspective, we create a partition of a polar area into these polar chunks (two sides made from radial lines, the other two sides segments of circles centered on the origin). We then consider what happens to the areas of these chunks as the partition is refined.

The area of a single polar chunk centered at r having radial width Δr and angular width $\Delta\theta$ will have the following area:

$$A = \frac{\Delta\theta}{2}\left(r + \frac{\Delta r}{2}\right)^2 - \frac{\Delta\theta}{2}\left(r - \frac{\Delta r}{2}\right)^2$$

Exercise: Multiply this out and simplify.

You should obtain:

$$A = r\Delta\theta\Delta r$$

From here one applies a standard limiting argument (under the right hypotheses on the partition we should find that riemann sums of these chunks should converge to corresponding integrals of the limiting polar areas.

Thus an integral representing an area in polar coordinates should be computed as:

$$\iint_R dA = \int\int r dr d\theta$$

Following our notation for iterated integrals, this order of integration may be termed a θ-simple region. It occurs when the region of interest has two straight radial sides (two lines passing through the origin) and inner and outer sides which may be expressed as polar functions. The simplest examples are circles which occur at constant r. The $d\theta dr$ order of integration is less common in practice, but occurs when the domain has circular segments as inner and outer boundaries and on each side θ may written as a (different) function of r.

Example-Exercise Battery 1:
These examples will verify that the polar integral agrees with geometric formulae we know from high-school.

1. An iterated polar integral representing the area of the unit circle can be written as:

$$\int_0^{2\pi} \int_0^1 r dr d\theta$$

   Compute the value of this integral and verify its value agrees with the corresponding geometric formula for area of a circle.

2. An iterated polar integral representing the **volume** of a right circular cone with a height of 1 unit and a outer radius of 1 unit may be written as:

$$\int_0^{2\pi} \int_0^1 (1-r) r dr d\theta$$

   Compute the value of this integral and verify its value agrees with the corresponding geometric formula for volume of a cone.

3. Formulate and evaluate a polar integral representing the area of a circle of radius 9. Verify your result is correct.

4. Formulate and evaluate a polar integral representing the upper half of the unit circle. Verify your result is correct.

5. Formulate and evaluate a polar integral representing the **right half** of a circle of radius 4. Verify your result is correct.

6. Formulate and evaluate a polar integral representing the volume of a circular cone with a height of 4 units, but an outer radius of 2. Verify your result is correct.

Example-Exercise Battery 2:
These examples go a little bit deeper into the process for converting from cartesian double integrals to double polar integrals. For these problems I recommend you sketch the domain in cartesian coordinates, then re-visualize the region with the polar coordinate system on top.

1. The iterated integrals below are all portions of circles. Sketch the domains and determine the corresponding double polar integral.

   (a)

   $$\int_0^1 \int_0^{\sqrt{1-y^2}} dxdy$$

   (b)

   $$\int_0^2 \int_0^{\sqrt{4-y^2}} dxdy$$

   (c)

   $$\int_0^3 \int_{-\sqrt{9-y^2}}^{\sqrt{9-y^2}} dxdy$$

   (d)

   $$\int_0^{\sqrt{2}} \int_{-x}^{x} dydx + \int_{\sqrt{2}}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} dydx$$