Line Integral Packet 2

Dr. Adam Boucher for Linearity 2

April 10, 2023

This packet of notes will cover the fundamental theorem of line integrals. We will look at the formulation of the theorem itself, establish conditions under which the theorem may be used, and cover the two simplest applications: trading curves, and finding and evaluating the potential function.

The Fundamental Theorem of Line Integrals.

The FTLI is essentially a special form of the fundamental theorem of calculus. Suppose we have a scalar function $\phi(x, y, z)$, and we assume all three of the input variables depend on a single parameter, t. If we differentiate this function using the chain rule we obtain the following structure:

$$\frac{d}{dt}\phi(x(t), y(t), z(t)) = \frac{\partial \phi}{\partial x}\frac{dx}{dt} + \frac{\partial \phi}{\partial y}\frac{dy}{dt} + \frac{\partial \phi}{\partial z}\frac{dz}{dt}$$

so it follows that if we wanted to compute the integral:

$$\int_a^b \frac{\partial \phi}{\partial x}\frac{dx}{dt} + \frac{\partial \phi}{\partial y}\frac{dy}{dt} + \frac{\partial \phi}{\partial z}\frac{dz}{dt}dt$$

it would be legitimate to apply the fundamental theorem of calculus and compute:

$$\int_a^b \frac{\partial \phi}{\partial x}\frac{dx}{dt} + \frac{\partial \phi}{\partial y}\frac{dy}{dt} + \frac{\partial \phi}{\partial z}\frac{dz}{dt}dt = \int_a^b \frac{d}{dt}\phi(x(t), y(t), z(t))dt$$
$$\int_a^b \frac{d}{dt}\phi(x(t), y(t), z(t))dt = \phi(x(b), y(b), z(b)) - \phi(x(a), y(a), z(a))$$

This is the fundamental theorem of line integrals. Most of the intellectual effort in mastering this theorem comes not from its proof, but from its application. We want to know how and when this theorem can help us with line integrals over vector fields. The technical complication is that when faced with a line integral, we know the components of the vector field, but we do not know whether they are the partials of a single potential ϕ.

Establishing the FTLI applies to a 2-d line integral

If you wish to apply the FTLI to a line integral of the form:

$$\int_C P dx + Q dy$$

Then you must first verify that:

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{dx}, \text{ or equivalently } \frac{\partial P}{\partial y} - \frac{\partial Q}{dx} = 0$$

holds in some open disk which covers, C. This provides a sufficient condition for the existence of a single ϕ which satisfies $\nabla\phi = \langle P, Q\rangle$. We can see the condition is necessary since it checks whether Clairaut's theorem is obeyed by P and Q. The sufficiency comes from an existence proof which you can find in rigorous calc textbooks. The algorithm for the construction of ϕ is pretty easy in 2-d but more challenging in 3-d.

Establishing the FTLI applies to a 3-d line integral

If you wish to apply the FTLI to a line integral of the form:

$$\int_C P dx + Q dy + R dz$$

Then you must verify that the curl of $\mathbf{F}$ is the zero vector. If you compute $\nabla \times \mathbf{F}$ you will find that each component of the curl checks a version of Clairaut's theorem which checks the equality of two of the three mixed partial derivatives.

Applying the FTLI to a 2-d integral: Finding the potential function

If you have checked the theorem applies, then you can evaluate the line integral by finding the potential function, ϕ which obeys the equation:

$$\nabla\phi = \begin{pmatrix} P \\ Q \end{pmatrix}$$

I advocate finding this ϕ through a cyclic calculation. Select **one** of the two component equations:

$$\frac{\partial\phi}{\partial x} = P(x,y), \qquad \frac{\partial\phi}{\partial y} = Q(x,y)$$

Integrate the chosen equation to partially recover ϕ, but introduce a single unknown function of the other variable to compensate for any loss from the partial derivative. Then differentiate with respect to the other variable and use the second component equation to find a first order ODE for the unknown function. Solve this ODE to fully recover the potential ϕ. Usually there will be an infinite family of ϕ's.

Abstractly following the directions above starting with the x component.

$$\begin{aligned} \int \frac{\partial\phi}{\partial x} dx &= \int P(x,y)dx \\ \phi &= \int Pdx + g(y) \\ \frac{\partial\phi}{\partial y} &= \frac{\partial}{\partial y}\left[\int Pdx\right] + \frac{\partial g}{\partial y} \\ \frac{\partial g}{\partial y} &= Q - \frac{\partial}{\partial y}\left[\int Pdx\right] \end{aligned}$$

If you have correctly established that $\langle P, Q\rangle$ is conservative and the FTLI applies, this last equation will only depend on y. (This is not obvious, and is not true when the vector field is not conservative). Then you integrate to find $g(y)$ which fully recovers the potential function ϕ. To compute the integral, you then evaluate ϕ at the end points of the curve C and take their difference. This method is particularly effective if you want to evaluate the line integral of a single conservative vector field, $\mathbf{F}$, over many different domains.

Applying the FTLI to a 2-d integral: Trading Curves.

The second variation for applying the FTLI is to exploit a corollary which is not immediately obvious.

Path Independence Corollary:

Suppose C_1 and C_2 are two different curves which start and end at the same locations. (Examples are K_4, K_5, and K_6 from ParamPacket2). Then for any conservative vector field $\mathbf{F}$ we have:

$$\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = \int_{C_2} \mathbf{F} \cdot d\mathbf{r}$$

The proof of this corollary is established by using the FTLI on both integrals, and finding that they both evaluate to $\phi(\mathbf{b}) - \phi(\mathbf{a})$. (The $\mathbf{b}$ and $\mathbf{a}$ are the starting and ending locations in space, not the starting and ending parameter values).

When we apply this version of the theorem we typically trade a curved domain for a line segment with the same starting and ending values. This avoids complicated parameterizations, and allows us to quickly evaluate line integrals of conservative vector fields.

Exercises:

(The first two problems refer back to the exercises in LineIntegralPacket1, so if you have not completed those, you should do that before attempting these problems.)

1. Verify that the vector field $\langle y, x\rangle$ is conservative by checking the condition for FTLI. This "explains" why the three different line integrals in exercise 3 had the same computed values.

2. Verify that the vector field $\langle x, x\rangle$ is **not** conservative by checking the condition for FTLI. This means that the FTLI does not apply and path independence does not hold in general. This "explains" why the three different line integrals in exercise 4 had the different computed values.

3. Suppose you were faced with $\mathbf{F} = \langle x + y, x + y\rangle$.

   (a) First verify that this is a conservative vector field.

   (b) Next consider:

   $$\int_{K_5} \mathbf{F} \cdot d\mathbf{r}$$

   First use the expanded recipe for line integrals to look at this integral as it stands. (Pretty gross right? Do NOT evaluate.)

   (c) Apply the FTLI and find the corresponding potential function for $\mathbf{F}$. (HINT: It should have three terms when you finish.) Evaluate the line integral using the potential function.

   (d) Trade K_5 for the straight line K_4 using path independence, and compute the value of that integral. You should get the exact same result as you obtained with the potential function.