Line Integral Packet 1

Dr. Adam Boucher for Linearity 2

April 18, 2020

This set of notes covers line integrals and integrals with respect to arclength. We will present each kind of integral with the same format.

- **Shorthand abstract notation.** This section will show you the most common abbreviated notations for the kind of integral under consideration. These are forms of notation you will find in textbooks and problem sets. The notation usually requires some degree of interpretation/specialization before calculations can commence.
- **Expanded Recipe**
  This section will show you how to expand the shorthand notation and convert it into either a single parametric integral which may be computed (if sufficiently simple) or approximated. We will continue to use abstract notation and function argument notation to clearly indicate how the different components (the integrand, the components of the path, the derivatives of the components of the path etc.) are assembled.
- **Example Formulations** This section will use the curves in ParamPacket 2 to apply the expanded recipe to example short hand problems.
- **Exercises** Several shorthand problems will be provided along with clues or patterns in the resulting values. If you obtain numerical agreement with the clues and patterns you can be confident that you are interpreting the recipe correctly. (Note: typos are possible, ask questions in the discussion if there are discrepancies.)

Line Integrals of scalar functions

- **Shorthand notation for integrals with respect to arclength**

$$\int_C f(\mathbf{x})ds$$

  1. The C indicates that the domain of integration is a curve (usually in $\mathbb{R}^2$ or $\mathbb{R}^3$). This curve is generally described with text or with a given parameterization. In the expanded recipe we assume that $\mathbf{r}(t)$ is a vector valued function which traverses C exactly once as the parameter $t \in (a, b)$. we use $\mathbf{x}$ to denote a point

in the embedding space and we use $x(t)$, $y(t)$ (and $z(t)$) to stand in for the component functions of $\mathbf{r}$.

2. The f is the integrand, this may be written as f, or an explicitly given formula involving the component variables x, y (and z)).
3. The ds is the only shorthand indication that you are performing an integral with respect to arclength. While s can of course be used as a variable in an ordinary integral, it is used to indicate integrals whose domain is measured by geometric length traveled along a curve in space. You distinguish between the two situations by the limits of integration on the integral symbol. If you have a lower and upper limit ds is probably ordinary integration, while an abstract limit of C indicates an integral with respect to arclength. (These are also called line integrals.)

- **Expanded Recipe for integrals with respect to arclength**

$$\int_C f(\mathbf{x})ds = \int_a^b f(x(t), y(t))\frac{ds}{dt}dt = \int_a^b f(x(t), y(t))\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dx}{dt}\right)^2}dt$$

(This is the expanded recipe when $\mathbf{r}$ is in the plane, when $\mathbf{r}$ describes a space curve in $\mathbb{R}^3$ the recipe is generalized to include $z(t)$ and $\frac{dz}{dt}$.)

- **Example Formulations:**
(Refer back to ParamPacket2 to see the parameterizations of each of the curves used as domains.)

1.

$$\int_{K_2} x + yds = \int_0^{2\pi} [\cos(t) + \sin(t)]\sqrt{(-\sin(t))^2 + (\cos(t))^2}dt = \int_0^{2\pi} \cos(t) + \sin(t)dt$$

2.

$$\int_{K_3} x^2 + y^2 ds = \int_{-1}^1 \left[t^2 + t^4\right]\sqrt{1 + 4t^2}dt$$

- **Exercises:**
Use the expanded recipe to show:

$$\int_{K_4} x - y^2 ds = \frac{2\sqrt{2}}{3}$$

Line integrals over vector fields

This kind of integral has multiple shorthand notations which emphasize different things; different disciplines and individuals may display preferences for one or the other, but either is acceptable.

- **Shorthand notations for line integrals of vector fields** (The short-hand notations are given for vector fields in $\mathbb{R}^3$, for vector fields in the plane, just remove the last component.

  $$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C P dx + Q dy + R dz$$

  The first notation is the vector form. It emphasizes the vector field in the integrand as a single thing. The second notation is the differential component form, this breaks the vector field into its components. In this notation the ordering of dx, dy, dz is important. Each of those differentials should be attached to the x y and z component functions of $\mathbf{F}$ respectively.

- **Expanded recipe for line integrals over vector fields.** The recipe for line integrals is easier than one might expect due to a serendipitous cancellation of $\frac{ds}{dt}$. I will show this in an abstract step below starting from the vector shorthand.

  $$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \mathbf{F} \cdot \frac{d\mathbf{r}}{ds} ds$$

  In the step above I have re-written the shorthand vector notation into an integral with respect to arclength. The term $\frac{d\mathbf{r}}{ds}$ stands for the unit tangent vector for the path $\mathbf{r}$, and the dot product with $\mathbf{F}$ is interpreted as a vector projection. The value of the integrand is the length of the component of $\mathbf{F}$ lying along the path. The unit tangent vector for a t-parameterization may be computed as:

  $$\frac{d\mathbf{r}}{ds} = \frac{1}{\frac{ds}{dt}} \frac{d\mathbf{r}}{dt}$$

  Using this form and the recipe we used earlier for arclength integrals we arrive at the expanded recipe we use in practice.

  $$\int_C \mathbf{F} \cdot \frac{d\mathbf{r}}{ds} ds = \int_a^b \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} \frac{1}{\frac{ds}{dt}} \frac{ds}{dt} dt = \int_a^b \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} dt$$

  if we now expand this recipe out by computing the dot product we can see the origin of the differential component notation:

  $$\int_a^b \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} dt = \int_a^b P \frac{dx}{dt} + Q \frac{dy}{dt} + R \frac{dz}{dt} dt$$

- **Example Formulations** (These examples use the planar curves from ParamPacket2, Since these example curves are in $\mathbb{R}^2$ we only use P and Q as components.)
  1. Let $\mathbf{F}$ be given by the vector field $\langle x^2, xy \rangle$.
  $$\int_{K_2} \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} \cos^2(t)(-\sin(t)) + \cos(t)\sin(t)\cos(t)dt = 0$$
  2. Let F be given by the vector field $\langle e^x, e^y \rangle$.
  $$\int_{K_4} Pdx + Qdy = \int_{-1}^{1} e^t(1) + e^t(1)dt$$

- **Exercises:**
  1. Pick two points in space for **a** and **b** to generate a simple line segment. Compute the line integral of the vector field $\mathbf{F} = \langle x^2, y^2 \rangle$ over your particular choice of line segment.
  2. Let $\mathbf{F} = \langle y, x \rangle$ (Note the x component is y and the y component is x.)

     Compute all three of:
  $$\int_{K_4} \mathbf{F} \cdot d\mathbf{r}, \qquad \int_{K_5} \mathbf{F} \cdot d\mathbf{r}, \qquad \int_{K_6} \mathbf{F} \cdot d\mathbf{r},$$
     You should find that all three of these line integrals have the same numerical value. This will be used in the next packet.
  3. Let $\mathbf{G} = \langle x, x \rangle$ Compute all three of:
  $$\int_{K_4} \mathbf{G} \cdot d\mathbf{r}, \qquad \int_{K_5} \mathbf{G} \cdot d\mathbf{r}, \qquad \int_{K_6} \mathbf{G} \cdot d\mathbf{r},$$
     You should find that these do not all have the same numerical values. This will be used in the next packet.
  4. Let C be a line segment connecting $(0, 0, 0)$ with the point $(1, 2, 3)$. Use a 3-d version of the line segment parameterization for K_1 to create a parameterization of this segment.

     Let $\mathbf{V}$ be the vector field defined by $\langle \frac{1}{\sqrt{14}}, \frac{x}{\sqrt{14}}, \frac{xy}{2\sqrt{14}} \rangle$. Use your parameterization to compute:
  $$\int_C \mathbf{V} \cdot d\mathbf{r}$$
     You should find the final numerical result is: 3.