Multiple Integration Note and Exercise Packet 3

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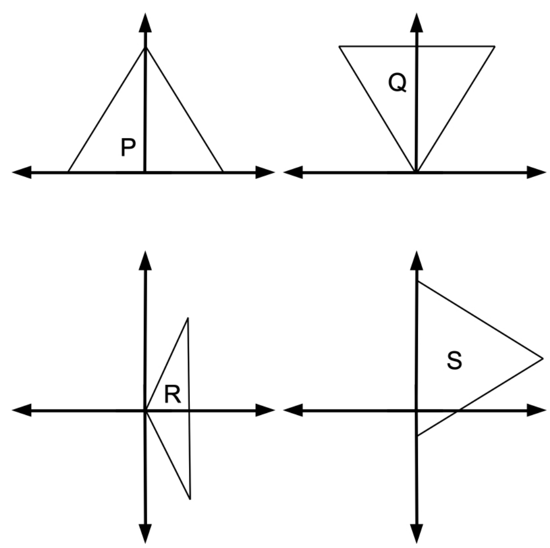
This collection of notes is intended to deepen your mastery of Fubini's Theorem to apply to regions which are not simple. We will also outline a procedure for switching the order of integration. No new mathematical results are required so we dive directly into examples and exercises.

Example Exercise Battery 2:

Look at the sketches below. These are examples of triangular regions which are simple in one variable, but **not** simple in the other. We will use these to practice visualizing and formulating iterated area and volume integrals. Examples and Exercises begin on the next page



1. **Example:** Suppose that the lines defining the edges of P are given by the following three curves (all written as $y = f(x)$ for reference.

$$y = x + 2, \quad y = -x + 2, \quad y = 0$$

P is an example of a region which is y-simple, but **not** x-simple.

(a) To set up an iterated integral using the y-simple order of integration we apply Fubini's theorem and carefully identify the **bottom**, **top** y-values. Next we identify the **left** curve and **right** curve expressed as $x = g(y)$.

$$\iint_P dA = \int_{.}^{.} \int_{.}^{.} dxdy \to \int_0^2 \int_{.}^{.} dxdy \to \int_0^2 \int_{y-2}^{2-y} dxdy.$$

(b) If we wished to use the x-simple order of integration we must break the region into pieces which are x-simple by themselves (**this is new**) Here the difficulty is that when using x-simple visualization the **top** curve changes. We split the iterated integral into two distinct pieces, one for each section where the top curve is defined by one equation.

$$\iint_P dA = \iint_{P_{left}} dA + \iint_{P_{right}} dA = \int_{-2}^{0} \int_0^{x+2} dydx + \int_0^2 \int_0^{-x+2} dydx$$

Exercise: Practice your iterated integral calculations by verifying that the integral in (a) and the set of two integrals in (b) evaluate to the same value. (And both values should agree with the area of a triangle with vertices at $(-2, 0)$, $(2, 0)$ and $(0, 2)$).

2. **Exercise:** Suppose the boundary lines for Q are given by the following three curves:

$$y = -x, \quad y = x, \quad y = 3$$

This is another example of a region which is y-simple but not x-simple.

(a) Setup the area as an iterated integral using the y-simple order of integration and compute its value.

(b) Setup the area as an iterated integral using the x-simple order of integration (this will require you to break the integral up into two pieces) and compute its value.

(c) Verify the areas you obtained for (a) and (b) are the same and that they agree with the area of a triangle whose vertices lie at $(-3,3)$, $(0,0)$, $(3,3)$.

3. **Example:** Suppose the lines defining the edges of R are given by:

$$y = 3x, \quad y = -3x, \quad x = 1$$

R is an example of a region which is x simple but not y simple, similar to the example above we formulae the integral using both orders of integration.

(a) To set up an iterated integral using the x-simple order of integration we apply Fubini's theorem and carefully identify the **left**, and **right** x-values. Next we identify the **bottom** curve and **top** curve expressed as $y = f(x)$.

$$\iint_P dA = \int_.^. \int_.^. dydx \rightarrow \int_0^1 \int_.^. dydx \rightarrow \int_0^1 \int_{-3x}^{3x} dydx.$$

(b) If we wished to use the y-simple order of integration we must break the region into pieces which are y-simple by themselves (**this is new**) Here the difficulty is that when using x-simple visualization the **left** curve changes. We split the iterated integral into two distinct pieces, one for each section where the top curve is defined by one equation.

$$\iint_P dA = \iint_{P_{bottom}} dA + \iint_{P_{top}} dA = \int_{-2}^0 \int_{-\frac{y}{3}}^1 dxdy + \int_0^2 \int_{\frac{y}{3}}^1 dxdy$$

4. **Exercise:** Suppose the boundary lines for S are given by the following three curves:

$$y = x - 1, \quad y = 5 - x, \quad x = 0$$

This is another example of a region which is x-simple but not y-simple. (Hint: You will need to know the coordinates of the intersection of the two diagonal lines to determine some limits of integration, find it early.)

(a) Setup the area as an iterated integral using the x-simple order of integration and compute its value.

(b) Setup the area as an iterated integral using the y-simple order of integration (this will require you to break the integral up into two pieces) and compute its value.

(c) Verify the areas you obtained for (a) and (b) are the same and that they agree with the area of a triangle whose vertices lie at $(-3, 3)$, $(0, 0)$, $(3, 3)$.

5. **Further Practice:** As long as a triangle has one side which is vertical or one side which is horizontal it will be a simple region in one variable, create a triangle of your own and formulate the area using both orders of integration, make sure your iterated integrals come out the same as the area of the triangle you design.