Multiple Integration Note and Exercise Packet 9

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March 31, 2021

This packet of notes will define the cylindrical and spherical coordinate systems and their most common iterated integral formulations. In this note we will systematically transform from Cartesian to Cylindrical to Spherical components. By making the transition in stages rather than two different transformations helps you to understand the meaning and origin of each component formula. We will limit our problem solving in these coordinate systems to integrals of cones, cylinders and portions of spheres, this will make computations relatively easy and allow me to focus your attention on understanding the coordinate systems and their connections to the mathematical theory we have already developed.

Cartesian to Cylindrical Coordinates:

The standard transition to cylindrical coordinates is achieved by substituting polar coordinates for the first two cartesian components and leaving z unchanged. The "polar + z" idea is great because it allows us to leverage our existing knowledge of double integrals in polar coordinates to intuit what should happen with three dimensional integrals.

Intuition argument:

Suppose we have a triple integral which can be written as an iterated integral:

$$\iiint dV = \int_a^b \iint_R dxdydz$$

If R is a simple polar region, then we could apply a 2-d change of variables into polar coordinates on just the inner two integral operations. This would give us:

$$\int_a^b \iint_R dxdydz = \int_a^b \left(\iint_R dxdy\right) dz = \int_a^b \left(\int \int_{\hat{R}} rdrd\theta\right) dz = \iiint rdrd\theta dz$$

We would hope that this idea might carry over to other regions and orders of integration as well.

General conversion:

If one considers converting a more general triple integral from cartesian to cylindrical coordinates, then one must apply a three dimensional version of the change of variables theorem. This has the same structure and limitations as the 2-d version, but the Jacobian factor is more complex. The formulaic portion of the general change of variables theorem in 3-d is as follows:

$$\iiint_{V(x,y,z)} f dV = \iiint_{\hat{V}} \hat{f} \left| J(u,v,w) \right| d\hat{V}$$

where $|J(u,v,w)|$ is defined by:

$$J(u,v,w) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$$

When the old variables are written as functions of the new variables. In the case of cylindrical coordinates the conversion giving old in terms of new is:

$$x = r\cos(\theta), \quad y = r\sin(\theta), \quad z = w$$

And the corresponding Jacobian factor is computed as:

$$J(r,\theta,w) = \begin{vmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = r$$

which agrees with our restricted intuition based argument. (You can leave z unchanged as "z", but using a w prevents us from overloading and getting confused about z being new or old). By using the general 3-d change of variables theorem, we gain the ability to use whatever order of integration is convenient for problem solving.

Cylindrical to Spherical Coordinates:

(**Remark on Notation:** The notation for spherical coordinates has some variation between different disciplines. Two of the components define angles and these are most often chosen as the greek letters θ and ϕ. The meanings of these letters are **swapped** in most physics work. You can generally tell which angle corresponds to the **polar** angle by whichever angle symbol appears in the Jacobian factor. In my work θ will retain its meaning as the azimuthal angle which defines compass direction in the xy plane. This helps us carry our understanding of polar into the spherical coordinate system unchanged.)

The transition from Cylindrical to spherical coordinates is achieved by replacing the components r and z with two new components, ρ and ϕ. These two new components are found by imagining a point in three dimensional space defined with cylindrical coordinates.

Exercise:

Draw or imagine a point in space above a sketch of the polar coordinate grid. Use this sketch/visualization to work through the next steps.

- If you were to connect this point to the origin you would in general create a right triangle whose hypotenuse connects the origin to the point, and whose other sides are the r and z components of this triangle.
- We obtain standard spherical coordinates by replacing the legs of this right triangle with the length of the hypotenuse, and second component called the "polar" angle which is measured from the positive z-axis. The equations which define these spherical coordinates from the cylindrical coordinates are:

  $$\rho = \sqrt{r^2 + z^2}, \qquad \phi = \arctan\left(\frac{r}{z}\right)$$

  (The formula for ϕ is uncovered by a congruency relation to connect the polar angle to the "upper" angle in the triangle above the polar grid and then using $\tan(\phi) = opp/adj$.

If we write down the old coordinates in terms of new coordinates we obtain:

$$r = \rho\sin(\phi), \qquad z = \rho\cos(\phi)$$

(Again, these formulas can be 'understood' in terms of the right triangle whose legs are r and z and whose hypotenuse is the ρ length vector connect the point (r, θ, z) with the origin.)

Exercise: Take the previous formulae for r and z and substitute them into the **right hand sides** of the conversion formulae below:

$$x = r\cos(\theta), \quad y = r\sin(\theta), \quad z = z$$

The resulting formulae define the conversion from cartesian coordinates to spherical coordinates when ρ is the distance to the origin, θ is the azimuthal angle, and ϕ is the polar angle. If you reference formulae using a text, electronic reference or paper the meanings of θ and ϕ may be switched.

In order to convert a cartesian triple integral into a spherical triple integral one must compute the Jacobian.

Exercise: Verify that the Jacobian factor for the spherical coordinates defined above is:

$$J(\rho, \theta, \phi) = \rho^2 \sin(\phi)$$

(This is pretty tedious by hand, so feel free to use computational aids to facilitate differentiation/simplification/plotting.)

If you have completed all of the previous exercises correctly, you should have all the pieces to understand the conversion from a cartesian triple integral to a spherical triple integral. It is possible to get conversions directly by drawing 3-d sketches of sections of solid angle and making long drawn out trigonometric arguments, but I have found this three step transition to be the clearest way to a complete understanding of each component part. You can use cylindrical and spherical components any time a problem has circular or spherical symmetry. Cylindrical coordinates tend to be more applicable and simpler to apply in practice unless the object or domain is truly spherical.

Exercise Battery: Below are iterated integrals for generic shapes written in cartesian coordinates (other orientations and formulations are possible, try to handle these.), convert these to cylindrical or spherical coordinates as appropriate and verify that when you evaluate the correctly converted iterated integral you recover the correct volume formula from geometry.

Cylinder:

$$\int_{-R}^{R} \int_{-\sqrt{R^2-x^2}}^{\sqrt{R^2-x^2}} \int_{0}^{H} dzdydx$$

Cone:

$$\int_{0}^{H} \int_{-\frac{Rz}{H}}^{\frac{Rz}{H}} \int_{-\sqrt{\left(\frac{Rz}{H}\right)^2-x^2}}^{\sqrt{\left(\frac{Rz}{H}\right)^2-x^2}} dydxdz$$

Sphere radius R:

$$\int_{-R}^{R}\int_{-\sqrt{R^2-x^2}}^{\sqrt{R^2-x^2}}\int_{-\sqrt{R^2-x^2-y^2}}^{\sqrt{R^2-x^2-y^2}} dzdydx$$

(**Warning** the polar angle only needs to go through a range of $[0, \pi]$ to capture the whole sphere. The azimuthal (compass) angle still covers $[0, 2\pi]$.)

Hemisphere radius 1 version 1: (Formulate but do not evaluate)

$$\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_{0}^{\sqrt{1-x^2-y^2}} dzdydx$$

Hemisphere radius 1 version 2: (Formulate but do not evaluate)

$$\int_{0}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_{-\sqrt{1-x^2-y^2}}^{\sqrt{1-x^2-y^2}} dzdydx$$