Multiple Integration Note and Exercise Packet 4

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This set of notes will focus on double integrals with non-trivial integrands. We will transition from formulating iterated integrals to analyzing and evaluating given iterated integrals. We will also consider integration regions which involve curved boundaries. Note that the procedures we use will have graphical steps which are essential for mastery. Your sketching does not need to be perfect, but it does need to be qualitatively correct (i.e. you want to capture where and how curves intersect (do they cross over? which curve is larger etc.).

We will begin with the simple integrand of $f(x, y) = x - y$, this is easy but it allows us to track how we treat x and y differently when performing iterated integrals.

The initial examples will be simple and flexible, they are intended to build confidence and facilitate the harder problems where not all steps are tractable.

Each initial example and exercise will perform the following steps:

1. Evaluate the given iterated integral as it stands.
2. Change the order of integration by creating a sketch of the domain and **re-visualizing** using the reversed order of integration.
3. Evaluate the iterated integral in the new order of integration and verifying the result is the same as the original.

Example-Exercise Battery 1:

1.

$$\int_0^1 \int_0^x x - y dy dx$$

We evaluate this iterated integral.

$$\int_0^1 \int_0^x x - y dy dx = \int_0^1 xy - \frac{y^2}{2}|_{y=0}^{y=x} dx = \int_0^1 x^2 - \frac{x^2}{2} dx = \frac{x^3}{6}|_0^1 = \frac{1}{6}$$

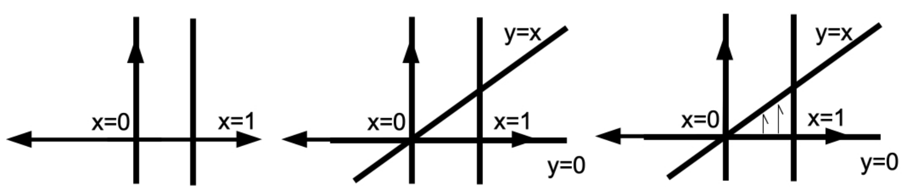
(This represents the volume between the planes $z = x - y$ and $z = 0$ bounded by the triangle R_a sketched in MultiPacket2.)

2. Next we **change the order of integration**. We do this using the limits of integration as follows:

   (a) We identify the current order of integration as $dydx$, this means the integral represents an **x-simple** region. We sketch it by identifying the limits of integration as:

   $$\int_{LEFT}^{RIGHT} \int_{BOTTOM}^{TOP} dydx$$

   To create our sketch we draw **vertical** lines at the left and right values of x. We then sketch the lines/curves for the bottom and the top curves. Finally we **shade** by dragging a pencil/pen starting at the bottom curve and stopping at the top curve. This sketching process (for this example) is shown below in three stages, You only need to make a single sketch.



   After the region is drawn, you re-visualize it as a **y-simple** region and formulate the integral using the same steps we used in MultiPacket3. For quick reference: Draw the **horizontal lines** which define the bottom and top of your shaded region. Make sure you have a unique line or curve defining the left and right sides of the region. Take your pencil and drag it from the left curve to the right curve to re-shade the region. This process should yield the limits of integration used in the last part of the example.

(b) Finally we evaluate the iterated integral in the new order of integration and verify the result is the same as 1.

$$\int_0^1 \int_y^1 x - y dx dy = \int_0^1 \frac{x^2}{2} - xy|_y^1 dy = \int_0^1 \frac{1}{2} - y + \frac{y^2}{2} dy = \frac{y}{2} - \frac{y^2}{2} + \frac{y^3}{6}|_0^1 = \frac{1}{6}$$

The result is the same as the original formulation. Note changing the order of integration **does not effect the integrand.** So the original integration under consideration is carried over to this formulation without any change.

Exercise:

Perform the same sequence of steps on the following iterated integral. (HINT: The domain of integration should line up with region R_b from MultiPacket 2 so you can use that to check your intermediate steps.)

1. Evaluate as given:

$$\int_0^1 \int_x^1 x - y dy dx$$

(Note your answer here should be negative, this makes sense since the integrand is negative on this domain.)

2. Sketch the region using the given order of integration.

3. Change the order of integration by **re-visualizing** as a **y-simple** region.

4. Evaluate the iterated integral using the new order of integration and verify the result is the same as part 1 of this exercise.

Example-Exercise Battery 2

1. $$\int_0^2 \int_0^{2-y} x - y dy dx$$

   We evaluate this iterated integral.

   $$\int_0^2 \int_0^{2-y} x - y dx dy = \int_0^2 \frac{x^2}{2} - xy|_0^{2-y} dy = \int_0^2 \frac{(2-y)^2}{2} - (2-y)y dy$$
   $$= \int_0^2 \frac{3y^2}{2} - 4y + 2dy = \frac{y^3}{2} - 2y^2 + 2y|_0^2 = 0$$

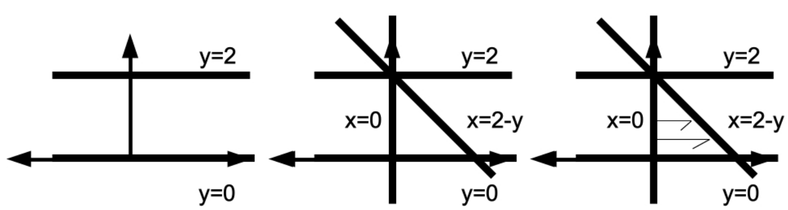
   (This represents the volume between the planes $z = x - y$ and $z = 0$ bounded by a triangle similar to R_c sketched in MultiPacket2. The resulting value is zero since the plane has an equal volume above and below $z = 0$ on the domain of integration.)

2. Next we **change the order of integration**. We do this using the limits of integration as follows:

   (a) We identify the current order of integration as $dxdy$, this means the integral is formulated as a **y-simple** region. We sketch it by identifying the limits of integration with:

   $$\int_{BOTTOM}^{TOP} \int_{LEFT}^{RIGHT} dx dy$$

   To create our sketch we draw **horizontal** lines at the bottom and top values of y. We then sketch the lines/curves for the left and the right curves. Finally we **shade** by dragging a pencil/pen starting at the left curve and stopping at the right curve. This sketching process (for this example) is shown below in three stages, You only need to make a single sketch.

   After the region is drawn, you re-visualize it as a **x-simple** region and formulate the integral using the same steps we used in MultiPacket3. For quick reference: Draw the **vertical lines** which define the bottom and top of your shaded region. Make sure you have a unique line or curve defining the left and right sides of the region. Take your pencil and drag it from the left curve to the right curve to re-shade the region. This process should yield the limits of integration used in the last part of the example.

(b) Finally we evaluate the iterated integral in the new order of integration and verify the result is the same as 1.

$$\int_0^2 \int_0^{2-x} x - y dy dx = \int_0^2 xy - \frac{y^2}{2}\Big|_0^{2-x} dx = \int_0^2 x(2-x) - \frac{(2-x)^2}{2} dx$$
$$= \int_0^2 -\frac{3x^3}{2} + 4x - 2dx = \frac{x^3}{2} + 2x^2 - 2x\Big|_0^2 = 0$$

The result is the same as the original formulation. Note changing the order of integration **does not effect the integrand.** So the original integration under consideration is carried over to this formulation without any change.

Example-Exercise Battery 3:

In the previous examples you have been given one example changing $dydx$ to $dxdy$ and one example exercise pair changing $dxdy$ to $dydx$. In this battery we demonstrate some simple examples where the order of integration impacts the actual calculus, so we use changing the order of integration to make intractable problems tractable. These problems are "cooked" to make the things work one way and not the other. Problems of this flavor (multiple integrals requiring an order change to complete) are common on standardized tests in math/engineering and physics. In practice, one would just approximate an integral numerically rather than hunting for progress by hand calculation.

In the problems below we explain the difficulty with handling the problem as is; apply a change of order and outline the steps to evaluate the resulting integral. You should practice as much of the computations as you require to become comfortable with changing the order of integration as well as handling the difficulty you see in these problems. **Example 3.1**

- $$\int_0^1 \int_y^1 e^{x^2} dxdy$$

- As is, this integral is intractable by hand, it is missing factors for a u-sub and does not have multiple parts to attempt integration by parts.
- We change the order of integration (This domain was used in the first example battery, but here we change the order back.) to obtain:

$$\int_0^1 \int_0^x e^{x^2} dydx$$

- This new order of integration allows us to complete the first integral immediately, and owing to the cooked nature of the problem, we get exactly the factor we need to complete the problem using a u-substitution.

$$\int_0^1 \int_0^x e^{x^2} dydx = \int_0^1 ye^{x^2}|_0^x dx = \int_0^1 xe^{x^2} dx$$

Complete the problem by performing a $u = x^2$ u substitution.

Example 3.2: In this example, the problem is actually tractable in both orders of integration, but 'easier' after a change.

$$\int_0^1 \int_{x^2}^1 \frac{x}{4-y^2} dydx$$

- As is the integrand requires an application of partial fraction decomposition to proceed with the first integral. The second integrals also require a bit of manipulation to compute.
- Changing the order of integration yields the following new iterated integral problem:

$$\int_0^1 \int_0^{\sqrt{y}} \frac{x}{4-y^2} dxdy$$

- The first integral is now an easy application of the power rule, which after simplification leads to a pretty simple u-substitution.

Exercises: The following problems will give you more practice changing the order of integration. Some are possible using both orders of integration, while others require a specific order of integration.

1.
$$\int_0^1 \int_x^1 \cos(y^2)dydx$$

(Proceed similarly to example 1.)

2.
$$\int_1^e \int_{\ln(y)}^1 \ln(y)dxdy$$

3.
$$\int_0^\pi \int_x^\pi \frac{\sin(y)}{y}dydx$$

4.
$$\int_0^3 \int_{\sqrt{\frac{x}{3}}}^1 e^{y^3}dydx$$