Parameterization Packet 1

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This packet attempts to review curve and surface parameterization specifically for use in multivariable integral operations such as line, surface and surface flux integral operations. The parameterizations presented here will be used in the vast majority of both exercises and examples in the packets that follow. I recommend you look over both sections, but only focus on parameterizing curves on the first-read through. Return to the parameterizing surfaces section once you start working on surface and surface flux integrals.

The process of parameterization in general has two stages: First one must construct or identify a **vector valued function** which traverses the curve or covers the surface of interest. Sometimes this object is natural and simple to find algorithmically (such as with single line segments), other times the object must be cobbled together from component parts (such as piecewise linear curves.). Second one must select the domain for the particular parameterized curve which perfectly captures the portion of the object on which integration is performed.

The component functions of the parameterization play an active role in the formulae we use to evaluate line and surface integrals, and the exact domain determines the limits of integration.

Parameterizing Curves:

We begin by outlining how to parameterize 'curves.' We give 2 generic procedures and include 2 standard examples. The exercises will give you a bit more flexibility and intuition.

The procedures for parameterizing curves don't change in a material way when changing the embedding space from 2 dimensions to 3 dimensions, but the number of component functions and thus flexibility does increase.

- Line segment parameterization
  Suppose you wish to create a straight line segment which starts at the point $\mathbf{a}$ and travels to the point $\mathbf{b}$. This can be achieved in many ways, but we will present a general method which can be quickly implemented and which always results in a very simple domain.

  We construct our parameterization by utilizing a linear combination of two 'switch' functions. Each function is a straight line, and the resulting linear combination inherits this property.

  $$\mathbf{r}(t) = \mathbf{a} \cdot (1 - t) + \mathbf{b} \cdot (t)$$

  (**Remark:** On the left hand side we are using function argument notation, so that should be read as "r of t." On the left hand side we have the points $\mathbf{a}$ and $\mathbf{b}$ scalar multiplied by the scalar functions $1 - t$ and t respectively.)

  This recipe has the advantage that the domain which traverses the exact portion of the segment from $\mathbf{a}$ to $\mathbf{b}$ is always $(0, 1)$.

  Exercise 1:
  Suppose you want a line segment to start at $(-1, 0)$ and end at $(1, 2)$. Use this recipe to construct a parameterization of that specific domain. Write out the component functions $x(t)$ and $y(t)$ explicitly. Verify that both component functions are indeed straight lines. Verify that at $t = 0$ and $t = 1$ the line passes through the desired starting and ending points.

  Exercise 2:
  Suppose you want a line segment in three dimensional space to pass through the points $(1, 0, 0)$ and the point $(1, 2, 3)$. Use exactly the same recipe, explicitly write

out the component functions, verify they are all straight lines and that they pass through the desired points when $t = 0$ and $t = 1$.

- Function parameterization.
  If you have a curve which traverses a portion of a known function in the xy plane like $y = f(x)$, then you can use the independent variable as the parameter to generate a simple parameterization. If the curve is a function of x you may use:

  $$\mathbf{r}(t) = \begin{pmatrix} t \\ f(t) \end{pmatrix}$$

  Example: Suppose I want to parameterize the bottom bend of the standard parabola. Since the parabola obeys the functional rule $y = x^2$, we can use this recipe to parameterize as:

  $$\mathbf{r}(t) = \begin{pmatrix} t \\ t^2 \end{pmatrix}$$

  If our parameterization is supposed to capture the bottom bend, we will likely see a domain like $t \in (-1, 1)$ or $t \in (-\alpha, \alpha)$, so that we capture both sides of the bend. If you want to capture different sections of the parabola, you use different intervals for the parameter.

  Exercise 3: Consider the function $y = 2x$. Use the function algorithm to create a parameterization of portion of this line between $(-1, -2)$ and $(1, 2)$. Determine the domain for your parameterization which covers the desired segment.

  Use the line segment algorithm to create a **different** parameterization. Determine the domain for this new parameterization which covers the desired segment.

Exercise 4: Suppose you want to parameterize the unit square with bottom left corner at the origin. Break it down into 4 line segments, and parameterize each one using the line segment algorithm. Write these down and hold on to them. We will use them in future problems.

Exercise 5: Suppose you want to create a parameterization of the upper half of the unit circle. Manipulate the equation for the unit circle to obtain a functional relationship between y and x and use the function algorithm to parameterize this relationship. Write this parameterization down, and hold it in reserve.

- Standard Circle Parameterization.
  A circle of radius R, centered at the point (h, k) may be parameterized with the following recipe:

$$\mathbf{r}(t) = \begin{pmatrix} h + R\cos(t) \\ k + R\sin(t) \end{pmatrix}$$

Exercise 6: Determine the standard parameterization for the unit circle centered at the origin. Determine which direction this curve is traversed. Determine a correct interval which will traverse the unit circle once.

Exercise 7: Make a simple adjustment to this parameterization which will traverse the curve in the opposite direction. (This is most easily changed directly at the parameter, but can also be achieved by a combination of shifting and switching trigonometric functions.

Example Exercise: In many integral formulae the component functions appear along side both their derivatives and their corresponding arclength.

Consider the functional parameterization for the upper unit circle. Compute both:

$$\frac{d\mathbf{r}}{dt}, \text{ and } \left\| \frac{d\mathbf{r}}{dt} \right\|$$

for that parameterization.

Compare those functions with the corresponding derivatives and arclength for the standard parameterization. In recipes where the arclength appears, the functional parameterization is typically intractable, while using the trig parameterization gives reasonable hand computations.

Parameterizing Surfaces:

We will limit our work with surfaces to two different situations. First surfaces which may be written as functions of x and y, and second surfaces which are parts of a sphere of radius R.

- Functional Parameterizations of surfaces

  If a surface is captured by the scalar equation $z = f(x, y)$, then a functional parameterization of the form:

$$\mathbf{S}(u, v) = \begin{pmatrix} u \\ v \\ f(u, v) \end{pmatrix}$$

may be used. One selects the domain in the u, v parameters to match up with the desired 'footprint' of the surface on the x y plane.

When using this parameterization it is helpful to have the following results for quick reference:

$$\frac{d\mathbf{S}}{du} = \begin{pmatrix} 1 \\ 0 \\ \frac{\partial f}{\partial u} \end{pmatrix}, \qquad \frac{d\mathbf{S}}{dv} = \begin{pmatrix} 0 \\ 1 \\ \frac{\partial f}{\partial v} \end{pmatrix},$$

(Remember that typically $u = x$ and $v = y$)
Exercise: Compute the left hand side explicitly and keep the result for reference:

$$\left\| \frac{d\mathbf{S}}{du} \times \frac{d\mathbf{S}}{dv} \right\| = \sqrt{\left(\frac{\partial f}{\partial u}\right)^2 + \left(\frac{\partial f}{\partial v}\right)^2 + 1}$$

Example:

Suppose you want to parameterize the slanted plane defined by the equation:

$$3x + 2y + z = 1$$

using the parameterization recipe we could use:

$$\mathbf{S}(u, v) = \begin{pmatrix} u \\ v \\ 1 - 3u - 2v \end{pmatrix}$$

Verify that the quantity:

$$\left\| \frac{d\mathbf{S}}{du} \times \frac{d\mathbf{S}}{dv} \right\| = \sqrt{14}$$

- Standard parameterization of a sphere of radius R. If you want to parameterize a sphere of radius R centered on the origin you may use:

  $$\mathbf{S}(u,v) = \begin{pmatrix} R\cos(u)\sin(v) \\ R\sin(u)\sin(v) \\ R\cos(v) \end{pmatrix}$$

  Note: These component functions are simply the cartesian spherical components with $\rho = R$, the parameters u chosen as the compass angle, and v chosen as the polar angle.

  When using this parameterization, we find the length of the cross product of parameter partials becomes the familiar Jacobian factor for spherical coordinates:

  $$\left\| \frac{d\mathbf{S}}{du} \times \frac{d\mathbf{S}}{dv} \right\| = R^2\cos(v).$$