Multiple Integration Note and Exercise Packet 7

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This packet gives several simple examples and exercises applying the change of variables theorem to iterated integrals. The basic procedure and statement of the theorem are in Multipacket6.

Example-Exercise Pairing 1:
Consider the following iterated integral:

$$\int_{-1}^{1}\int_{1-y}^{3-y} dxdy$$

The region described by these limits of integration is a parallelogram. If we look at the algebraic representations of the boundary curves, they can inspire a change of variables which can simplify the limits of integration. (**Remark:** With a starting problem as simple as this this change is probably not worthwhile in absolute terms, but this is an illustration of the theorem in a non-polar coordinate setting.)

The equations defining the boundary curves are:

$$\begin{aligned} y &= 1, & x &= 3-y \\ y &= -1, & x &= 1-y \end{aligned}$$

Re-arranging the two inner limits we can obtain:

$$x+y=3, \text{ and } x+y=1$$

The appearance of the expression $x+y$ multiple times suggests this might be a good choice for a new variable. To keep the transformation easy, we can leave the y variable unchanged in our new coordinate system. These ideas together suggest the following change of variables (new from old).

$$u = x+y, \quad v = y$$

In order to apply the change of variables formula we invert this system algebraically to get equations which describe x and y completely in terms of these new variables:

$$x = u - v, \qquad y = v$$

With these equations in hand we follow the standard procedure for performing a change of variables.

1. Compute the Jacobian factor:

$$|J(u, v)| = \begin{vmatrix} 1 & -1 \\ 0 & 1 \end{vmatrix} = 1.$$

2. Re-formulate the limits of integration in the new variables¿

$$\begin{aligned} v &= 1, & u &= 3 \\ v &= -1, & u &= 1 \end{aligned}$$

From these equations you create a new sketch all of the curves in the u v plane. If the original integral was set up in the conventional way (left to right and low to high in x and y respectively) then you select an order of integration which is suitable for the new coordinates and proceed to formulate the limits using your new sketch as you normally would. Here the region is rectangular and aligned with the new coordinates so any order of integration is reasonable.

$$\int_{-1}^{1} \int_{1}^{3} 1 du dv$$

(**Exercise 1:** Switch the equations defining u and v. Compute the new Jacobain. This is why we always apply the absolute value when determining the Jacobian factor.)

Exercise 2: Perform a similar analysis to select and execute a change of variables on the following iterated integral:

$$\int_0^1 \int_{-x}^{2-x} dydx$$

Exercise 3: Suppose you were trying to represent a physical area of 1 square foot as an iterated integral:

$$\int_0^1 \int_0^1 dxdy$$

Keep the origin the same, but invent new variables which convert the x and y scales both to inches. Perform the change of variables (don't forget the Jacobian) and show that the result is numerically equal to the original integral.

Example-Exercise Pairing 2: In this example the change of variables is selected to simplify the integrand rather than the limits of integration:

$$\int_0^1 \int_0^x \sqrt{2x-y}(x+y)^2 dydx$$

This integral is somewhat messy because the product in the integrand is coupled together. If we can change variables to uncouple the integrand the integration might be far simpler. This suggests the change of variables:

$$u = 2x - y, \quad v = x + y$$

Since the change of variables is linear in x and y and the two equations are linearly independent, we can be certain that nothing unpleasant will happen. Inverting the change of variables yields:

$$x = \frac{u+v}{3}, \quad y = \frac{2v-u}{3}$$

From here we can compute the Jacobian:

$$J(u,v) = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}$$
$$J(u,v) = \frac{2}{9} + \frac{1}{9} = \frac{1}{3}$$

Due to the linear form of the change of variables our Jacobian is a nice solid constant. Now we need to determine the region of integration in the new variables. Our boundary curves become the following equations when converted into the new variables:

$$0 = u + v, \quad 3 = u + v, \quad 2v - u = 0, \quad u + v = 2v - u$$

Choosing the order of integration $dudv$ and sketching this region in $u-v$ coordinate plane we find the integral may be written in the following way:

$$\int_0^1 \int_{\frac{v}{2}}^{2v} \sqrt{u}v^2 \frac{1}{3} dudv + \int_1^2 \int_{\frac{v}{2}}^{3-v} \sqrt{u}v^2 \frac{1}{3} dudv$$

While this integral isn't exactly simple, the easier integration should make solving the problem in stages a bit more tractable.

Exercise 4: Consider the following simpler version of the problem above:

$$\int_{-1}^1 \int_{\frac{-2-x}{2}}^{\frac{2-x}{2}} (x+2y)^2 dydx$$

Select a linear change of variables inspired by the linear combination appearing in the integrand and the limits of integration and execute the change of variables on this iterated integral.