Line Integral Packet 3

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April 21, 2020

This packet is concerned with Green's Theorem. Here we will state Green's theorem first for line integrals, then we will follow this statement with a presentation of 2-d flux integrals and the flux integral version. Proving Green's theorem for doubly simple domains is a standard class activity in Linearity 2, I have also posted the corresponding in-class exercise and cheat-sheet. This proof is covered in my notes as well, but struggling through it yourself can de-mystify this theorem.

Green's Theorem for Line Integrals.

Green's theorem equates two different quantities. Both quantities are related to the components of a single vector field, however one of the quantities accumulates around the boundary of the domain, while the second quantity accumulates through the interior of the domain. The theorem **only** applies to line integrals with simple closed curve domains.

$$\int_{\delta R} \mathbf{F} \cdot d\mathbf{r} = \int_{\delta R} P dx + Q dy = \iint_R \frac{\partial Q}{dx} - \frac{\partial P}{dy} dA$$

In the notation above the curve defining the domain of the line integral, δR is the **boundary** of the domain in the double integral. This boundary is assumed to be traversed counterclockwise for the equality to hold.

When we use this theorem it is typically applied to avoid parameterizing multiple curved edges at the cost of evaluating a double integral. Certain vector fields (Conservative and/or Incompressible) are particularly vulnerable to simplification through Green's theorem.

The thought process for using **Green's Theorem** is as follows:

- When evaluating:

$$\int_C Pdx + Qdy$$

- Determine whether C is a simple, closed curve (i.e. It travels around a single region once, starting and ending at the same point.)
- If C is a simple closed curve, is the region that it bounds either x-simple or y-simple in the sense that we used for constructing iterated integrals?
- If the answer to both of those questions is yes, then the line integral is a good candidate for applying Green's Theorem.
- Compute the partials of the components of the vector field, select a convenient order of integration for R, and evaluate the corresponding iterated integral. (You may wish to use polar coordinates if R is partially circular.

(**Note:** Normal application of Green's theorem requires the curve be closed, but it is possible to use it in other circumstances by using domain additivity and "closing the curve" an example of this is provided in my notes.)

Example:

$$\int_C xydx + (x^2 + y^2)dy, \text{ where } C \text{ traverses the unit circle once.}$$

Here we have a simple closed curve so we can use Green's Theorem. Rather than parameterizing the unit circle and computing the line integral, we trade this for an iterated integral and evaluate that. Since the domain is circular, I will use polar coordinates for the evaluation, however I will include the cartesian integral stage for clarity.

$$P = xy, \qquad Q = x^2 + y^2$$
$$\frac{\partial P}{\partial y} = x, \quad \frac{\partial Q}{\partial x} = 2x$$
$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = x$$

Using this as the integrand we obtain the following integral in cartesian coordinates:

$$\int_{-1}^{1} \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} xdxdy.$$

Converting to polar coordinates for easier evaluation can now be done. (The extra r is from the Jacobian factor.)

$$\int_0^{2\pi} \int_0^1 r^2 \cos(\theta) dr d\theta = \int_0^{2\pi} \frac{1}{3} \cos(\theta) d\theta = 0$$

Exercise: Switch the components of the vector field here, apply Green's theorem and verify you obtain exactly the same value. (This does not generalize, but it's a simple exercise with a known outcome.)

Exercise: Suppose $\mathbf{F}$ is a two dimensional, conservative vector field (i.e. satisfies the conditions of the FTLI). Suppose you want to compute:

$$\oint_C \mathbf{F} \cdot d\mathbf{r}$$

(The circle indicates C is closed.) Demonstrate what happens when you apply Green's theorem and combine that with the assumption of conservation.

Flux Integrals in 2-D

While the line integral of a vector field measures the component of the vector field **along** the path, a flux integral measures and accumulates the component of the field **across** or perpendicular to the path. The parity between these two constructions is best viewed through a vector shorthand notation.

$$\text{Line Integral of vector field with tangental component } \int_C \mathbf{F}\cdot\mathbf{T}ds$$

$$\text{Flux Integral of vector field with normal component } \int_C \mathbf{F}\cdot\mathbf{N}ds$$

The $\mathbf{T}$ stands for the unit tangent vector, while the $\mathbf{N}$ stands for the unit normal vector. The standard construction of the unit normal vector relative to $\mathbf{r}(t)$ is given by:

$$\mathbf{N} = \frac{1}{\frac{ds}{dt}}\begin{pmatrix} \frac{dy}{dt} \\ -\frac{dx}{dt} \end{pmatrix}$$

(The best way to quickly give you an explanation of this structure is to remind you that perpendicular lines have negative reciprocal slopes. Since we want to capture the direction which is perpendicular to the tangent vector we can achieve that by swapping and negating one of the components of the tangent vector, the length is unchanged so we normalize by dividing by the speed, $\frac{ds}{dt}$. The choice for the location of the $-$ sign produces the "outward" flux because the chosen normal vector will point away from the interior of the region all the way around.

The recipe and shorthand component notation for a flux integral then becomes:

$$\int_C \mathbf{F}\cdot\mathbf{N}ds = \int_a^b \mathbf{F}\cdot\begin{pmatrix} \frac{dy}{dt} \\ -\frac{dx}{dt} \end{pmatrix} dt = \int_C Pdy - Qdx$$

(The use of this notation is rather subtle. You must pay close attention to the dx and dy placement, and you must be aware of the $-$ when identifying P and Q **Green's Theorem for Flux Integrals.**

The Green's Theorem for flux integrals has the same basic structure and application as the version for line integrals: it equates one integral around the boundary of a region to a different integral accumulated throughout the interior of the same region, but the pieces are somewhat jumbled about. It is **VERY EASY TO MAKE MISTAKES** with the different versions of the theorem so take very careful notes, and refer back to trusted references (calculus text books etc.)

The notational version of Green's theorem for flux integrals is below:

$$\oint_{\delta R} Pdy - Qdx = \iint_R \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y}dA$$

Vector Operator Versions of Green's Theorem: If I abuse notation slightly to map the integrands of the Green's theorem integrals onto the divergence and curl operators, then these two versions of Green's theorem can provide a stepping stone to understanding the Stokes and Divergence theorems respectively.

$$\text{Line Integral } \textbf{CURL VERSION}\text{: } \oint_{\delta R} Pdx + Qdy = \iint_R \nabla \times \mathbf{F} dA$$
$$\text{Flux Integral } \textbf{DIV VERSION}\text{: } \oint_{\delta R} Pdy - Qdx = \iint_R \nabla \cdot \mathbf{F} dA$$

(Since $\mathbf{F} = \langle P, Q \rangle$ is only 2-d, $\nabla \times \mathbf{F}$ is actually the $\hat{k}$ component of $\nabla \times \langle P, Q, 0 \rangle$.)