Multiple Integration Note and Exercise Packet 2

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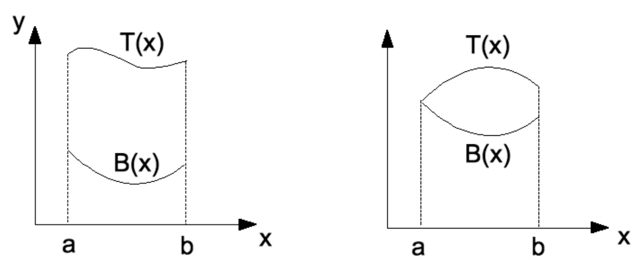
This collection of notes is intended to introduce you to the second half of Fubini's theorem. After completing this set of notes and exercises you should have a basic understanding of simple regions, and be aware of Fubini's theorem for x- simple and y-simple regions. You should be able to formulate iterated integrals over simple regions.

In a previous note we studied Fubini's theorem for rectangles. Here we develop two more advanced versions of this theorem. Each of these more advanced versions can handle domains which incorporate certain combinations of curved sides, however the application of the theorem requires that you be able to identify these **simple regions** and use them to carefully formulate the resulting iterated integrals.

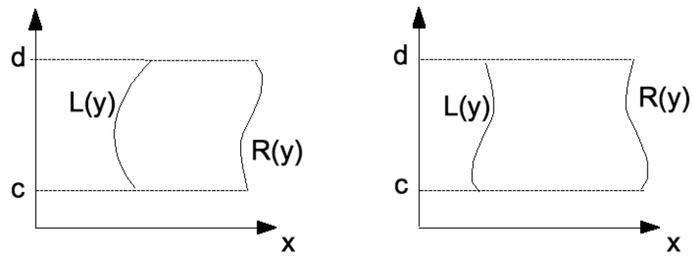
In general a **simple region** is a region which is completely enclosed by four distinct elements. These elements are various combinations of points, lines and curves. An **x-simple** region is a region which is completely enclosed by two fixed values of x which define the **left** and **right** boundaries, and two lines or curves which define the **bottom** and **top** of the region. See the figure below for two examples of **x-simple** regions.



Fubini's Theorem for x-simple regions:

$$\iint_D f(x,y)dA = \int_a^b \int_{B(x)}^{T(x)} f(x,y)dydx$$

A **y-simple** region is a region which is defined by two constant values of y which define the bottom and top of the region, and two lines or curves (which are expressed as $x = L(y)$ and $x = R(y)$) which define the left and right boundaries of the region. See the figure below for two examples of $\mathbf{y-simple}$ regions.



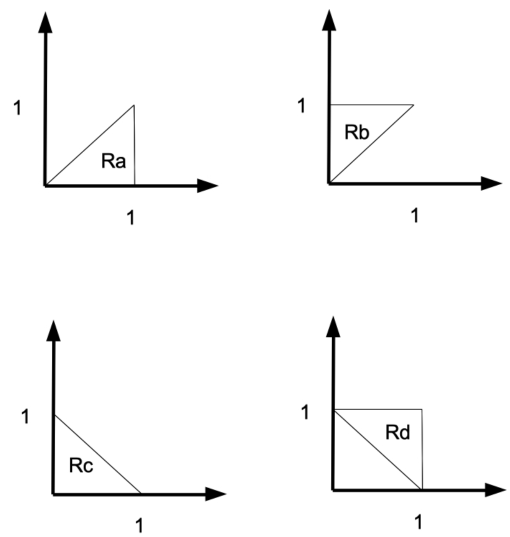
Fubini's Theorem for y-simple regions:

$$\iint_D f(x,y)dA = \int_c^d \int_{L(x)}^{R(x)} f(x,y)dxdy$$

It is possible for a region to be **both** x and y simple, neither x or y simple, or it may only be simple in one variable. You must learn to visualize regions as x and or y simple and then use that visualization to formulate and evaluate iterated integrals.

Example and Exercise Battery 1:

If the first set of examples and exercises you will be working completely with right triangular regions. These regions have two advantages: they are all both x simple and y simple, and you can compute the area of each region using your knowledge of elementary geometry. In R_a and R_b the diagonal line is defined by $y = x$, and in R_c and R_d the diagonal line is defined by $y = 1 - x$.



1. If we visualize R_a as an **x-simple** region, we choose the corresponding order of integration from the theorem. Then we decide on the limits of integration in the following way:

   (a) Find the **constant** minimum and maximum x values which bound the region. These are either vertical lines, or points. Write the corresponding left and right values as limits on the integral sign connected to x. (This should be the first one in an x-simple visualization.)

   (b) Next take your pencil (actually do this.) and shade from the bottom curve to the top curve and use the expressions defining the bottom and top curves as the limits on the inner integral sign.

   (c) Once all four limits are known, the iterated integral may be evaluated inside to outside.

$$\iint_{Ra} = \int_{.}^{.}\int_{.}^{.} dydx \to \int_{0}^{1}\int_{.}^{.} dydx \to \int_{0}^{1}\int_{0}^{x} dydx$$

2. If we visualize R_a as an **y-simple** region, we choose the corresponding order of integration from the theorem. Then we decide on the limits of integration in the following way:

   (a) Find the **constant** minimum and maximum y values which bound the region. These are either horizontal lines, or points. Write the corresponding bottom and top values as limits on the integral sign connected to y. (This should be the first one in an y-simple visualization.)

   (b) Next take your pencil (actually do this.) and shade from the **left** curve to the **right** curve and use the expressions defining the left and right as the limits on the inner integral sign.

   (c) Once all four limits are known, the iterated integral may be evaluated inside to outside.

$$\iint_{Ra} dA = \int_{.}^{.} \int_{.}^{.} dxdy \rightarrow \int_{0}^{1} \int_{.}^{.} dxdy \rightarrow \int_{0}^{1} \int_{y}^{1} dxdy$$

3. Exercise: Visualize R_b as a x simple region and formulae its area using an iterated integral. (The correct answer here should match up with the answer to (2) with x and y swapped)

4. Exercise: Visualize R_b as a y simple region and formulate its area using an iterated integral. (The corret answer here should match up with the answer to (1) with the x and y swapped.)

5. Visualize R_c as x simple, formulate the resulting iterated integral and compute its value.

6. Visualize R_c as y simple, formulate the resulting iterated integral and compute its value.

7. Visualize R_d as x simple, formulate the resulting iterated integral and compute its value.

8. Visualize R_d as y simple, formulate the resulting iterated integral and compute its value.