Analytic Solution Techniques and Applications of Ordinary and Partial Differential Equations

Adam Boucher

Contents

This collection of notes is intended to survey the elementary techniques and theory of ordinary and partial differential equations. The notes are divided into three parts, and each part has different mathematical prerequisites. The first part provides an overview of initial value problems for ordinary differential equations. The second part provides an overview of the theory and solution techniques for boundary value problems, including a survey of Sturm-Liouville theory fundamentals of distribution theory and a short exposition of green's function techniques for solving ordinary differential equations. The example problems are drawn primarily from mathematical physics, and we include a short interlude explaining the derivation of the Euler-Lagrange equation. In the third section we introduce the canonical equations of mathematical physics. Our derivations use the models studied in the second part as the foundation for the derivations of the equations so these equations are built naturally from our previous study, rather than pulled out of the air. We introduce some of the fundamental techniques for solving partial differential equations including separation of variables, and basic fundamental solution techniques for the Cauchy problem for the heat and wave equations. Emphasis is placed on obtaining a well grounded understanding of the qualitative behavior of solutions to these problems, and we spend time developing interesting case studies which connect back to the elementary theory of ordinary differential equations.

The level of the material in the first section is roughly equivalent to the level of material presented in an undergraduate ordinary differential equations text, and the student is expected to be familiar with the techniques and theorems of single variable calculus including elementary integration and differentiation as well as a basic understanding of the properties of Taylor series. When discussing linear ordinary differential equations we draw freely from the theory of linear algebra. Students without a background in linear algebra should be prepared to revisit many of the techniques and example problems involving linear algebra once they understand the relevant background in order to gain a deeper understanding of why the various techniques and 'tricks' are justified.

The level of the material in the second section is more advanced, and while we aim to give a self contained presentation of the material, a student with a good understanding of the first section may not possess the mathematical maturity necessary to fully understand the second section. A well prepared student for this material should have a good background in calculus, linear algebra, and a full understanding of the proofs will require some understanding of mathematical analysis.

In some ways the material presented in the second section is more difficult than the material in the third section. Many students will be able to use separation of variables and eigenfunction expansion techniques from the third section to actually solve problems without having an understanding of why the methods work. The only additional background material necessary to handle the third section would be a good grounding in multivariable calculus including integral vector calculus.

In order to streamline the presentation and make the text more functional each part is divided into two sections: theory and practice. The theory sections are designed to outline the theoretical structure of the material including the results which are either essential for understanding or justifying the techniques that we actually use. The theory sections contain abstract derivations as well as proofs (of varying levels of rigor) and their readability is strongly dependent upon your background and level of mathematical maturity. First time students should read through the relevant theory sections (proofs can probably be omitted upon a first read) and should try to understand the results before turning to the practice section. In the practice section we present both numerical and abstract examples with copious annotations and explanation. Whenever a technique has different flavors or variations, we try to present at least one example of each of the possible cases that might arise in practice. When learning the material try to remember that the theory section is designed to help you understand the 'why' of the material, while the practice section is designed to help you with the more practical 'how' of the material.

CHAPTER 0

Preliminaries

In this chapter we cover the essential fundamentals that should be addressed before one even looks at a differential equation. We define differential equations, we explain what it means to 'solve' a differential equation and we explain why differential equations provide important and **useful** insight into the physical world around us. In addition to these somewhat humble beginnings we will outline some very important notational considerations that you **should** have gained from calculus, but probably didn't.

1. What is a differential equation?

A differential equation is an equation involving a function and one or more of that function's derivatives. Differential equations are used to describe dynamic balances. That is balances between the rates of change of functions and other factors. Physically speaking we often use differential equations as models to describe all of the factors which combine to influence and change quantities of interest. The most familiar example is probably Newton's second law. This particular law can be thought of as a definition of force as well as a framework for defining interesting differential equations. We may recall the law as 'Force is equal to mass times acceleration', but if we recall the definition of acceleration as the time derivative of velocity, then we can just as easily write this law as a differential equation:

$$m\frac{dv}{dt} = F$$

We then use this equation as a framework to translate different physical situations into mathematical statements. The absolutely mind blowing result is that if we are clever enough to correctly describe the forces in a particular situation, then we can often use this differential equation to make predictions abut how those physical situations will evolve and change. Put more crudely, differential equations can be used to predict the future.

Now, before you run off and buy a $TI-\infty$ differential equation solving crystal ball to uncover your future loves, or dump your life savings into the stock market expecting to be able to predict the future of everything, I should probably qualify that statement. Complex physical systems are exactly that: complex. As a result they are notoriously difficult to model exactly, and even in cases where we possess models which we feel accurately capture the important features of the physical system, our ability to write down the equations

far outstrips our ability to actually 'solve' the equations and to find the solutions which contain those most important bits of information such as: What will next weeks lottery numbers be, or where is that asteroid going to hit the Earth? However all is not lost, because we can make predictions about simple model problems, and in many situations these model problems give us valuable insight which allows us to actually **understand** the world around us, and this greater level of understanding allows us to nudge the physical world in directions that we feel might give us happiness, or improve our lot in life. (From a more pessimistic perspective, I should also note that mathematics makes it possible to make bigger and more accurate explosions, so greater understanding also gives us more opportunity for destructive capability).

To give you a more concrete feel for what it means to solve a differential equation we will use a **very** simple example from elementary physics, but we will explore the example with probably more mathematical sophistication than you have previously used. We begin with a very simple physical situation.

Remark: If you are at all familiar with high school or college kinematics you probably know the answer to this particular problem and will find this argument rather long-winded. You should pay attention to the structure of the mathematical problem and the terminology I use, because these are features which will be salient when we generalize to more interesting and difficult problems.

Suppose you throw a ball vertically into the air. How high will the ball go? This is a very simple question, but one which might be a little difficult to measure experimentally. We will use this question as an elementary model from which we can begin to explore the modeling process, and see how and why one goes about 'solving' differential equations. As we proceed we will introduce many of the technical terms which arise when modeling physical problems.

In order to model the situation with differential equations, we need to know how to translate the physical situation into mathematical notation. Usually one begins by defining variables, and connecting those variables using known or assumed physical laws. Once the variables and laws have been translated into mathematical notation we attempt to solve the problem mathematically. In this particular problem, we will translate the problem as a whole into mathematical notation and then solve it exactly. In more difficult problems, the interaction between the mathematical model and the physical situation must be treated like a conversation rather than a closed package.

The physical principle that we will use to formulate the mathematical version of this problem is Newton's second law. As stated earlier, Newton's second law balances the change in velocity of an object with the total of all forces exerted on the object. In order to specify the differential equation we simply need to decide on the forces which we feel are important when we throw an object into the air. We will begin with the simplest model available, which is to assume that gravity is the only force exerted on the ball, and that the gravitational force is constant. We will assume the gravitational force is proportional to the mass of the object, thus allowing us to write down the following ordinary differential

Remark: When hearing these assumptions, many students tend to react with a hubris which is totally undeserved. Relax and absorb everything you can out of the simple version of this problem because we **will** relax these assumptions, and you'll find its a very short road from chest bumping overconfidence to brown trousers and a more humble disposition.

equation:

$$m\frac{dv}{dt} = -mg$$

in this equation we have used m to denote the mass of the ball, $v = v(t)$ represents the velocity of the ball (note that this is a function of time), and we use g to denote a constant which describes the local gravitational acceleration on a unit mass. We have selected a coordinate system where the positive y-axis points 'upwards.' As such we have included a negative sign to indicate that the direction of the graviataional force is directed towards the ground. Once the differential equation is written down, we are free to manipulate it algebraically. In this case, we might divide out the mass parameter completely:

$$\frac{dv}{dt} = -g$$

Next we generally attempt to 'solve' this equation. **Solving a differential equation means finding the function or functions which when substituted into the equation actually make the equation true.** In this case the equation tells us that the first derivative of the velocity (which is the unknown function here) is equal to a constant value. This is a particularly simple differential equation because it describes one of the derivatives of v explicitly. In order to recover v we simply integrate the derivative away.

$$\int \frac{dv}{dt}dt = \int -gdt$$

On the left hand side we appeal to the fundamental theorem of calculus and note that the integral of a perfect derivative with respect to the same differentiation variable, must return the original function. On the right hand side we integrate under the assumption that g is constant with respect to the integration variable.

$$\int dv = -g\int dt$$
$$v = -gt + C$$

At this stage we have 'solved' this differential equation and we have obtained a formula for the velocity of the ball...unfortunately there are two problems. First because this formula includes a constant of integration it does not define a unique velocity function. In order to select a single velocity function we need to include a supplementary condition which tells us about the value of the velocity at some point in time. Such side conditions are called **initial conditions** and they help us to distinguish between different possible instances of physical systems which have the same governing dynamics. In this case we can specify a unique velocity function by selecting the initial velocity of the ball as it first leaves our hand.

$$v(0) = v_0$$

Remark: Note carefully that we are performing the **same** operation on both sides of this equation. The goal of this operation is to recover $v(t)$. A differential equation is a genuine equation, and if you are going to perform any operation on an equation you must perform the same operation to both sides.

Remark: Since we performed an indefinite integral, we add a constant of integration. One constant of integration is sufficient, so you need not add constants to both sides of the equation.

We use this condition to define the correct value of C. To apply this initial condition we substitute $t = 0$ into our general solution (the solution including all arbitrary constants of integration) and we set the general solution equal to the desired initial value.

$$\begin{aligned} v(0) &= -g(0) + C \\ v_0 &= -g(0) + C \\ v_0 &= C \end{aligned}$$

Once we have applied all initial conditions to specify the constants of integration, the result is called a particular solution. In the example problem the particular solution for the velocity is given by:

$$v(t) = -gt + v_0$$

The second problem with this solution, is that it doesn't actually answer our original question, which was 'How high will the ball go?' To answer this question we must pass to yet another differential equation. We know that velocity is the time derivative of position. So we link the velocity function to the position of the ball using the following differential equation:

$$\frac{dy}{dt} = v$$

Solving this equation using our formula for v and the same integration process as before we find:

$$y = -g\frac{t^2}{2} + v_0 t + C_2$$

where C_2 is another constant of integration. If we apply the initial condition $y(0) = 0$, then we find $C_2 = 0$. This gives us an explicit formula describing the height of the object relative to its initial height, and one which we can exploit to find an answer to our original equation. In order to answer our question we must determine the maximum height achieved by the ball, so we must solve a simple optimization problem (reconnecting to your study of calculus again!) in order to extract the correct information from the height function. Briefly we have:

$$\begin{aligned} \max_t y(t) \to \frac{dy}{dt} &= v(t_{max}) = 0. \\ 0 &= -gt_{max} + v_0, \qquad t_{max} = \frac{v_0}{g} \\ y(t_{max}) &= -\frac{g}{2}\left(\frac{v_0}{g}\right)^2 + v_0\frac{v_0}{g} \\ y(t_{max}) &= \frac{v_0^2}{2g} \end{aligned}$$

So we see the maximum height obtained by the ball varies with the square of the initial velocity and is inversely proportional to the strength of the gravitational acceleration. By

changing our assumptions about the form and importance of the forces in this problem we can obtain different differential equations and try to follow the same basic procedure to obtain the same kind of information in more varied and interesting circumstances.

From the perspective of applications, differential equation are extremely practical. Suppose we are interested in predicting some quantity of interest, $q(t)$. If we ask the very general question, 'What will q be tomorrow?' we are faced with an ambiguous and ill posed problem. The question is so blunt, that there is very little that we can use as a foothold to analyze. By asking the more specific question 'What factors or physical processes affect the value of $q(t)$, we may be able to use our experience and intuition to write down a credible model describing all of the influences on $q(t)$. By treating the sum of all these influences like forces (and ensuring they are at least dimensionally consistent) we can try to solve the resulting differential equation (which may be very difficult) to obtain a quantitative model which can make predictions about the value of $q(t)$ in the future. In this way differential equations give us a way to pass from influences in the past and present to predictions about the future. The quality of our predictions will vary greatly, and will depend upon a combination of the accuracy with which we can account for the true influencing factors, our ability to accurately measure the current or past values of $q(t)$, and our ability to solve the resulting differential equations.

To give you a practical application of the application of differential equations to the real world, consider weather prediction. Here is a big physical system in which we all have at least a passing interest, and one which can be modeled quite effectively using differential equations in many different ways. If you ask the question 'What is the weather going to be tomorrow?' then getting an answer to that question directly is a difficult proposition. If however you start to model the factors which influence the weather in a particular region: current and past temperature, current and past pressure, wind speed and precipitation, etc. And you link these factors together into differential equations modeling the motion of fluids under reasonable circumstances, then you start to be able to make some fairly accurate predictions for some time into the future. Obviously, these predictions are not perfect and cannot predict the exact weather and cloud cover in a pointwise fashion, however these predictions have excellent agreement on say, countywide scales for short to medium term projections 1-3 days. This is pretty impressive given the relatively small density of weather measurements made on the whole globe.

Part 1

Initial Value Problems for Ordinary Differential Equations: Theory

CHAPTER 1

First order equations

We begin our study of differential equations by laying out the primary goal: Given a differential equation involving an unknown function what manipulations can we perform to identify the unknown function. From a purely mechanical perspective, we can manipulate the differential equation as an algebraic equation so we can add or subtract terms from both sides, multiply or divide the equation by non-zero terms, but such manipulations are generally insufficient to obtain a closed form representation of the unknown function purely in terms of the independent variable.

For example suppose we are faced with the differential equation:

$$\frac{dy}{dt} + 2ty = 0$$

it is legitimate to 'solve' this equation algebraically for y to obtain:

$$y = -\frac{\frac{dy}{dt}}{2t}.$$

While this is a valid manipulation, and a true statement whenever the original differential equation is true, the formula depends upon $\frac{dy}{dt}$ and thus, we haven't really uncovered the identity of the unknown function, we have only re-written the formula in a new way. In order to fully 'solve' this equation in the sense of differential equations it is necessary to uncover a formula for y which make the equation a true statement for all t of interest. As such, in addition to algebraic manipulation, we must be prepared to integrate the equation in order to recover the function. It is legitimate to integrate both sides of an equation provided you integrate with respect to the variable on both sides, and you use the same interval of integration (when using definite integrals). Because a differential equation usually an unknown function of the independent variable, this integration process generally requires a more delicate touch than the procedure most students master in elementary calculus. As such, we begin by reviewing the fundamental theorem of calculus in a rather more technical light, and we proceed to show how this theorem is immediately useful for solving simple explicit differential equations.

Remark: In this particular example the general solution would be given by $y = Ce^{-t^2}$. You can verify that this function 'solves' the differential equation by carefully differentiating the function with respect to t and substituting the function and its derivative into the equation.

1. Method of Separation

The version of the fundamental theorem of calculus which is immediately useful to us is given by the following identity:

$$\int f'(t)dt = f(t) + C$$

which states that the integral of the derivative of **any** differentiable function is equal to that function modulo a constant. This tells us that differentiation and integration work as inverse operations. The beauty of this identity is its generality. We could plug **any** differentiable function into this statement and the statement would remain true. Re-writing this statement using Leibniz notation we find a way that we can remove derivatives, and legitimately solve differential equations

$$\int \frac{d}{dt}\Big[\ \Big]dt = \Big[\ \Big] + C$$

The only thing required for this idea to work is that we must be able to manipulate the equation into a form where **all** the dependence on the unknown function falls within the brackets. If we have any y terms outside the brackets we will not be able to integrate the equation successfully.

we return to the equation presented at the beginning of this section to illustrate how this idea might be applied:

$$\frac{dy}{dt} + 2ty = 0$$

If we attempt to integrate this equation with respect to t we obtain:

$$\int \frac{dy}{dt} + 2tydt = \int 0dt$$
$$\int \frac{dy}{dt}dt + \int 2tydt = \int 0dt$$

Remark: You might be tempted to think that the integral here is simply t^2y. However this is fundamentally **wrong**. Since y represents an unknown function of t, its influence in the integral cannot be ignored, and it cannot be treated as a constant. This is huge. Until you understand this point, you are not ready to proceed.

Now we can integrate the right-hand side with no difficulty, and we can apply the fundamental theorem of calculus to help us with the first term on the left hand side, however we cannot honestly integrate the second term on the left, since y depends on t in an as yet, unknown way. Technically we obtain the following:

$$y + C_1 + \int 2tydt = C_2$$
$$y = \hat{C} - \int 2tydt$$

While this is still valid, and indeed correct, it doesn't get us any closer to obtaining a explicit formula for $y(t)$. Returning to the original equation, we perform one algebraic

manipulation, then we repeat the same integration process.

$$\frac{dy}{dt} + 2ty = 0$$
$$\frac{1}{y}\frac{dy}{dt} + 2t = 0$$

Written in this way, all of the explicit y dependence is restricted to a single term in the equation. If we try to integrate the equation now, we find a very different outcome awaits us:

$$\int \frac{1}{y}\frac{dy}{dt} + 2tdt = \int 0dt$$
$$\int \frac{1}{y}\frac{dy}{dt}dt + \int 2tdt = \int 0dt$$

Here we can easily integrate the right hand side and the second term on the left hand side using the power rule for integration (note that there is no y term to spoil the integral this time.

$$\int \frac{1}{y}\frac{dy}{dt}dt + t^2 + C_1 = C_2$$

More importantly, if we are careful we can see that by recalling the technique of u substitution, the remaining integral may be simplified and integrated as follows:

$$\int \frac{1}{y}\frac{dy}{dt}dt = \int \frac{1}{y}dy = \ln|y| + C.$$

Substituting this into our equation and collecting the constants of integration we obtain:

$$\ln|y| = \hat{C} - t^2$$

at this stage we have legitimately solved the 'differential' part of the of the equation, and it remains to be seen if we can solve this implicit equation exactly for y. By exponentiating both sides we obtain:

$$|y| = e^{\hat{C}-t^2}$$

Using properties of exponents we have:

$$e^{\hat{C}-t^2} = e^{\hat{C}}e^{-t^2} = k_+e^{-t^2} \text{ for some positive } k_+$$

Finally, since the absolute value function either acts as an identity operator or as a negative sign we can remove this function by selecting a formula which is capable of capturing all possible solutions.

$$y = ke^{-t^2} \text{ for some } k \text{ either positive, zero or negative.}$$

This methodology can be applied to any explicit or separable equation that one might encounter. The actual technique itself is extremely brief, so the primary difficulties one encounters are in the identification of separability and the actual technical difficulty of

Remark: Because the exponential function is positive definite we either have $y(t)$ is positive for all t, zero, or negative for all t. In the first case we have $|y| = y$ so $y = k_+e^{-t^2}$ is the solution. In the second case we have $|y| = 0$, so $y = 0$ is a solution, and in the third case we can use $y = -k_+e^{-t^2}$ as a formula for the solution.

performing the required integration.

Separable equations allow one to 'separate' the dependent and independent variables and then integrate each side independently.

Let us assume we have a differential equation which may be manipulated, or 'blocked out' into the following form:

$$\frac{dy}{dx} = F(y, x)$$

This form restrict us to first order equations, but we are making no assumptions about the actual form of $F(y, x)$

The idea behind separation is that we need to be able to think of $F(y, x)$ as being composed of two disjoint pieces, one which is a function of y alone, and the other which is a function of x alone.
Mathematically:

$$F(y, x) = M(y) \cdot N(x)$$

If we can identify F with the component functions M and N, then we say the equation is separable.
Next we divide by $M(y)$, thus separating all y terms and x terms.

$$\frac{1}{M(y)}\frac{dy}{dx} = N(x)$$

next we try to integrate both sides with respect to x and (hopefully) we can solve the resulting equation for y.

$$\int \frac{1}{M(y)}\frac{dy}{dx}dx = \int N(x)dx$$
$$\int \frac{1}{M(y)}dy = \int N(x)dx$$

This method is illustrated on several example equations given in the practice section.

2. Method of Integrating Factors

The method of integrating factors is a procedure for analytically solving first order, linear ordinary differential equations with variable coefficients. I will proceed first by outlining the idea behind the method then we will derive the method from first principles, and derive a representation formula for the solution. Several examples of this method are given in the practice section.

Just like the method of separation, the method of integrating factors is a technique which converts a differential equation into a new form which allows both sides to be integrated exactly. In this case we attempt to re-write the equation in such a way that a

perfect derivative appears, however this time we use the product rule rather than the chain rule to obtain perfect derivative structure.

Technically the method proceeds by introducing a 'helper function' called the integrating factor. This function is selected specially in order to create an instance of product rule structure on one side of the equation. Once this perfect derivative structure is created, the equation may be integrated immediately using the fundamental theorem to obtain a formula for the unknown function.

Consider an equation with the following form:

$$\frac{dy}{dt} + p(t)y = q(t),$$
$$y' + py = q, \text{ (in more compact notation)}$$

Here p, and q are not necessarily constants but may be functions of t. We note this problem cannot be solved using direct integration for arbitrary p and q because these functions interfere with our ability to separate the equation. This means a new technique must be devised.

Remark: If you need to do some algebra to get your equation into this form then do it! The arguments that follow assume that you start with something that has **exactly** this form.

Put this equation out of your mind for the moment and recall calculus 1 and the product rule. In compact notation the product rule might be written as:

$$\frac{d}{dt}\left[u(t) \cdot v(t)\right] = uv' + u'v$$

The goal of the method of integrating factors is to manipulate our differential equation so that the left hand side mimics the structure of the product rule. Then, when we integrate the left hand is a perfect derivative and the fundamental theorem may be applied directly.

Our current left hand side has the following form:

$$y' + py$$

we need it to have the form:

$$uv' + u'v$$

So let us multiply our whole differential equation by some unknown function, μ. This is our integrating factor and it will give us some freedom to adjust the left hand side into the form we desire.

$$\mu y' + \mu p y = \mu q \tag{1}$$

Now our left hand is almost perfect, let us identify v and y. It follows that we must also identify $u = \mu$. Now if we can choose μ so that $\mu' = \mu p$ then our left hand side will have exactly the same form as the product rule. So we solve this simpler differential equation:

$$\mu' = \mu p$$

This equation we can solve by the method of separation:

$$\frac{d\mu}{dt} = \mu p$$
$$\frac{\frac{d\mu}{dt}}{\mu} = p$$
$$\int \frac{d\mu}{\mu} = \int p dt$$

Remark: Without a particular p to work on, we can't really finish solving this equation, but by leaving the integral in the problem we can obtain a handy formula which we can specialize whenever p is known.

Integrating the left hand side, and we can find a formula for μ that will work for any nice p.

$$\ln|\mu| = \int p dt$$
$$\mu = e^{\int p dt}$$

With this formula in hand we return to the differential equation. By using any μ given by the formula above we are guaranteed to obtain product rule structure on the left hand side of our equation.

$$\mu y' + \mu' y = \mu q$$

Remark: This step is often difficult for students to justify, and many students try to invent an algebraic justification for the simplification. This is generally wrong-headed. The method of integrating factors makes this happen **by design**, so you don't need to justify this step, you need to understand that this step is the whole purpose of the method of integrating factors. If you wish to check your work up to this step, you should apply the product rule carefully to $\frac{d}{dt}[\mu y]$ if you obtain the same grouping of terms that appear in the previous step you can be confident that you have found the correct integrating factor.

Taking advantage of this structure, we simplify and integrate both sides:

$$\frac{d}{dt}[\mu y] = \mu q$$
$$\int \frac{d}{dt}[\mu y] dt = \int \mu q dt$$
$$\mu y = \int \mu q dt$$

Again, until we actually plug in a form for q we cannot evaluate the integral on the right hand side. However we solve the above equation for y and we will obtain a formula for y which, provided we can evaluate the integrals for μ and μg will give us the general solution to this differential equation.

$$y(t) = \frac{\int \mu g dt}{\mu}$$

In practice the application of this method is much quicker than the derivation, and is usually performed in four main steps.

(1) Convert the equation into the standard form used in the derivation. Identify $p(t)$ and $g(t)$, being careful to attach the appropriate sign to $p(t)$.
(2) Integrate $p(t)$ to find an appropriate integrating factor. (If possible you should simplify the formula for the integrating factor as much as possible. When computing the integrating factor it is always permissible to select the constant of integration as zero.)

(3) Mulitply the differential equation by the integrating factor and integrate both sides, you can pass immediately to the simplified equation:

$$\mu y = \int \mu g(t) dt$$

(4) Complete the integration process, apply any initial conditions and solve for y.

Examples of this technique in action are given in the practice section.

3. Modeling with First Order Equations

Now that we know how to solve a few different kinds of first order equations (namely separable equations and linear equations) we proceed to explore some different situations which can be effectively modeled with these kinds of equations. As you read the examples that follow you should try to maintain some intellectual distance from the problems themselves. Our concern is primarily to gain more experience in the application and solution of differential equations which are simple enough to be tractable by hand. As such the models that we first consider will be simplistic and cartoonish, but still have the ability to make qualitatively correct predictions.

The example problems in this section have been at least partially fictionalized to ease the computational burden. In the practice section we consider several more realistic and experimentally grounded example problems. In each problem we will begin with a paragraph which briefly describes the relevant background information and context for the problem, this paragraph is followed by a lengthy exposition which describes how one must extract the formal mathematical model from the problem description, and then proceeds to show how the problem may be solved.

Before we begin the following 'meta-considerations' may be helpful in solving these problems:

- Always keep in mind that the problems you see on tests, and homework have been at least partially 'cooked' in order to make the problems tractable, thus you should always try to keep a positive outlook with these problems. The problems **are** solvable, and when you do the problems correctly the calculations probably won't take more than a page.

- These notes are intended to help you learn about differential equations, so each problem will involve at least one differential equation.

- We know that in order to extract actual predictions out of a differential equation we also need initial condions to help specify the solution. Keep a lookout for linguistic clues, and adjectives such as 'initially' or 'originally' which can help to

pick initial conditions.

- Keep a running dialogue with yourself when you assess your work. While the models we use will be simple, they also give reasonable results so if you get an answer which has patently unphysical behavior, then you should question it.

3.1. Proportionality with Radioactive Decay. In our first example we explore the concept of proportionality as it relates to mathematical modeling. Proportionality is an important and powerful idea because it has a unique mathematical interpretation. This interpretation simplifies the transition from the physical situation of interest to the formalized mathematical model.

A particular radioactive element, Unobtainium-479, is observed to decay at a rate which is proportional to the amount present. Given that at some point $t = 0$ there are U_o grams of Unobtainum in a particular location, Create a model which will tell you how much Unobtainium remains at any subsequent time.

Remark: In the exposition that follows I'll be leaving gaps between some of the paragraphs. These gaps are a subtle clue that **you** should answer the questions yourself. This is not a story-book: in order to learn this material you must become a participant, not just an observer.

In order to solve this problem we must have an agenda which addresses all of the different features of the problem that we must address. The first order of business is to come to terms with what it actually means to solve this problem. We need to find a constructive answer to the question: What is this question asking for?

Once you have determined the objective of the question, you must create a plan of attack which will help you to reach toward the desired quantity. In this problem we must be able to predict the amount of unobtainium at any time after $t = 0$. To be able to model this situation effectively we must **define** notation which will help us to describe the situation qualitatively. When you analyze systems with multiple parts, you can't just call everything 'x' and expect to have anyone follow you. Take some care and try to use notation which is clear and suggestive enough so that people will be able to guess the intended meanings of the variables.

Let's define U as the amount of unobtainium present at time, t. Thus $U = U(t)$, so U is a function of t. Usually the simplest way to create a first order differential equation for a quantity of interest is to define the derivative using appropriate notation consistent with our choice of notation above, and then set that derivative equal to the sum of all the different influencing factors. Given this approach what is the appropriate notation for the rate of change (or the derivative) of the amount of unobtainium present?

Now to obtain an expression for this derivative we return to the problem statement. If we look at the first sentence:

...Unobtainium-479, is observed to decay at a rate which is proportional to the amount present.

we are told something about how unobtainium changes... *it decays* ...and we are told something about **how** it decays...*at a rate proportional to the amount present.*
Since this is all we are told in the entire problem statement about the dynamics of unobtainium, we assume that it tells us everything important and we build our model using only this information.

We know that we want to model how U changes, so we start with the following template:

$$\frac{dU}{dt} = ?$$

Looking at the main clause, "decays at a rate proportional to the amount present." we start to formalize this sentence with the goal of translating it into a form which we can drop into our template above.

We have assigned the symbol U to measure the amount of unobtainium present. Substituting this symbol into the main clause yields: " decays at a rate proportional to U."

Since we are only concerned with one changing quantity in this particular problem, the decay rate **must** refer to the derivative. Further translating our main clause gives: "$\frac{dU}{dt}$ decays in proportion to U."

Now this statement basically begs to be turned into an equation. Proportionality is probably a topic which you discussed in high school, but its doubtful that you ever used it to help derive differential equations. Briefly we review the concept of proportionality.
The statement that "A is proportional to B" is equivalent to the following equation:

$$\frac{A}{B} = k$$
$$k \in \Re$$

Applying that to our current problem we get:

$$\frac{\frac{dU}{dt}}{U} = k$$

Solve for $\frac{dU}{dt}$ we obtain something that fits into our template:

$$\frac{dU}{dt} = kU$$

We are also told that U "decays." When something decays, it gets smaller or wears away, so we can either think of k as a negative constant or add a negative sign explicity. I

Remark: Here we see a general heuristic for determining the signs of terms in a first order differential equation. If a particular term increases the quantity, then that term should be positive, whereas terms which tend to decrease a quantity should be negative.

choose the latter.

$$\frac{dU}{dt} = -kU$$

We now have a differential equation which we can solve by the method of separation.

$$\int \frac{dU}{U} = \int -k dt$$

$$U(t) = Ce^{-kt}$$

The second sentence in our problem statement tells us the value of $U(0)$ so this acts as the initial condition, $U(0) = U_o$ and this allows us to pick the constant $C = U_o$.

$$U(t) = U_o e^{-kt}$$

At this stage we have obtained a model which tells us how U changes in time. with this solution we can answer such questions as:

What will happen to the unobtainium after a long time?
Since $e^{-kt} \to 0$ for $k > 0$ and $t \to \infty$ we can say that after a long time the unobtainium will decay away to nothing.

Remark: You might be wondering about 'k.' In this problem no specific information is given which allows us to pick k. Check the practice section for several examples that involve k.

3.2. Proportionality with Heat Transfer. As a second example which utilizes proportionality as a basis for constructing a differential equation we consider the simplest viable model for heat transfer between an object and its environment. The model is based on 'Newton's Law of Cooling,' which is a postulate about the rate of heat transfer between an object and its environment. The model makes the simplifying assumption that the transfer of heat energy between the object of interest and the surrounding environment makes no significant contribution to the overall environment temperature.

Newton's Law of cooling states: *The rate of change of temperature of an object is proportional to the temperature difference between the object and the environment.*

Using our mathematical translation of proportionality we propose the following differential equation model to capture Newton's law of cooling. Let Q be the temperature of the object, and let E be the temperature of the environment.

$$\frac{dQ}{dt} = k(E - Q)$$

Remark: We have written the temperature difference as $E - Q$, so that when k is a positive constant temperature flows in the correct 'direction' from warmer to cooler.

In order to illustrate how this model can help us to predict the temperature changes in objects of interest we consider the following physical situation:

- At exactly $6am$ my coffee pot heats newly brewed coffee to a temperature of 180^oF.
- I keep my home at a cozy 70^oF.
- I have found that if I let the coffee sit in my home until $7am$ it is too cold to drink at a temperature of 81^oF.
- If the ideal drinking temperature for coffee is 114^oF, when exactly should I drink it?

Remark: Here for clarity we list the information rather than encoding it in paragraph form

If we assume Newton's law of cooling is an appropriate model for this situation, then we can proceed. First we convert this information to a mathematically usable form.

We are told the temperature of the coffee at various times. For simplicity I set $t = 0$ to be 6am and measure my times relative to then. With this assumption the measurements for the coffee temperature become the following conditions:

$$Q(0) = 180, \quad Q(1) = 81$$

(t is measured in hours.)

We are also told the temperature of the apartment is kept at 70^o. Assuming this temperature is constant, this sets the value of the parameter $E = 70$. Substituting into the equation we have:

$$\frac{dQ}{dt} = k(70 - Q)$$

Remark: Alternately you might use integrating factors to help solve this problem.

Now we solve this problem using the method of separation.

$$\frac{1}{Q-70}\frac{dQ}{dt} = -k$$

By separating the equation in this way (with a negative sign attached to k), the integration is slightly easier. If you are careful to respect the chain rule, you can separate this equation in other ways.

$$\int \frac{1}{Q-70}\frac{dQ}{dt}dt = \int -kdt$$
$$\int \frac{1}{Q-70}dQ = -kt + C$$
$$\ln|Q-70| = -kt + C$$

Solving for Q we obtain:

$$Q = \hat{C}e^{-kt} + 70$$

(For completeness, $\hat{C} = \pm e^C$)

Now we apply the information on the coffee temperatures. These conditions will allow us to select values for $\hat{C}$ and k.

$$Q(0) = 180, \quad 180 = \hat{C} + 70, \quad \hat{C} = 110$$
$$Q(1) = 110, \quad 81 = 110e^{-k} + 70, \quad k = \ln(10)$$

These values specify the solution exactly, and we can solve the problem to find the ideal drinking time:

$$S = 110e^{-\ln(10)t} + 70$$
$$S = 110(10)^{-t} + 70$$
$$44 = 110(10)^{-t}$$
$$\frac{2}{5} = 10^{-t}$$
$$t = -\log_{10}(\frac{2}{5}) = -\log_{10}(0.4) = 0.3979$$

Converting this time to minutes, we find that the coffee will be at the ideal temperature just before 6 : 24.

3.3. Input, Output and Dimensional Analysis with Mixing. In the next example we consider a tongue-in-cheek modeling problem which demonstrates how one can create differential equations which model average chemical concentrations in 'well-mixed' environments simply by enumerating all of the different inputs and outputs of the relevant chemicals. Monitoring an average concentration rather than a pointwise concentration is useful in real world applications whenever the diffusion time scale is small relative to the transport time scale, this is essentially the 'well-mixed' hypothesis, and the small scale dynamics of the flow aren't immediately important (such as when monitoring chemical concentrations in lakes, rivers and public water supplies.)

This example also gives us a wonderful opportunity to explore the principle of dimensional consistency which is a valuable tool for constructing and evaluating models of physical phenomena.

(The numbers are made up, but the concepts are real.)

Severe insect infestations are often handled by fumigating the affected area with toxic gases. Almost anything which 'kills bugs dead' probably isn't the healthiest gas to be around while practicing deep breathing exercises. Thus fumigators are faced with an interesting problem. They need to spray gases at a high enough concentration to become lethal to the insects, but they also need to make sure that the concentration drops below safe levels before they allow people to return to the area being fumigated.

Let's assume that the fumigators are using a gas which becomes deadly to termites at a concentration of $1\frac{g}{m^3}$. Let's further assume that the pure gas is stored at a concentration of $10\frac{g}{m^3}$. and that the gas can be pumped into the 'death zone' at a rate of $2m^3/min$. Let's also assume that the room containing the gas is sealed off, but the well mixed air-gas mixture is able to escape the room at the same rate of $2m^3/min$.

(1) If we fumigate a cubical room with 10m long walls, how long after you turn on the gas will it take for the gas to become effective?

(2) Once the gas has reached a concentration which is deadly to the insects you turn off the gas and use industrial fans to pump clean air into the room at a rate of 100 cubic meters per minute. If the gas is harmless to humans when it falls below a concentration of $0.1g/m^3$ how long must you wait before the room is safe to enter? (You may assume that the concentration of gas in the room reached $1\frac{g}{m^3}$ before you turned off the knockout gas)

In order to solve the first problem we are going to need to know how much gas is present in the room at any time.

Let G represent the amount of gas (measured in grams) in the room at any time. In order to find $G(t)$, we must construct a differential equation capturing the change in G.

$$\frac{dG}{dt} = ?$$

Remark: We **choose** to use the mass of gas in the room as our dependent variable. Modeling mass is a more robust approach than trying to monitor concentration since we can use the same approach in situations where the overall volume is changing.

In order to correctly capture this change we will enforce dimensional consistency on the equation, and experiment with the given information to create a physically reasonable expression for $\frac{dG}{dt}$.

The principle of dimensional consistency states that all of the terms in an equation which is physically correct must have the same units. Mathematically, dimensional consistency and dimensional analysis have a more sophisticated interpretation. Any time one is permitted to choose a unit of measure (such as a length unit, time unit, mass unit etc.) this choice of unit of measure should **not** play a material role in the accuracy of predictions from a physically correct mathematical model. For example, we require that our mathematical models provide accurate predictions regardless of whether we choose to work in imperial units or metric units. Since both sets of units are 'made up' one set of units shouldn't be preferred over the other. This reasonable condition has a rather large mathematical requirement attached to it. By trying to enforce this consistency between different choices of units, we impose a scaling symmetry on each physical model for each unit (or dimension) we are permitted to select. These symmetries require that whenever we have a physical equation, changing units does not change the truth of the equation.

In order to apply the principle of dimensional consistency to this problem we first look at the physical dimensions of $\frac{dG}{dt}$. Once we have determined the dimensions of this term, we will require all other terms to have exactly the same dimensions. G is measured in grams, and t is measured in minutes, so in order for a term to have the same units as $\frac{dG}{dt}$ it should be measured in units of $\frac{g}{min}$. By looking over the given information we see that none of the given rates or concentrations have exactly those units.

Dimensional consistency requires that we try to to combine the numbers in our problem in ways that make both physical and dimensional sense.

First let's think about what is coming into the room...we are told we have gas coming in at a concentration of "$10\frac{g}{m^3}$, at a rate of $2\frac{m^3}{min}$." Dimensional consistency forbids us from adding these terms since they have different units, so lets try multiplying them:

Remark: Note that units and dimensions are treated algebraically as if they were variables.

$$10\frac{g}{m^3} \times 2\frac{m^3}{min} = 20m\frac{gm^3}{m^3 min} = 20\frac{g}{min}$$

This operation returns a number which is dimensionally consistent with (has the same units as) $\frac{dG}{dt}$. Think about this operation: does it make physical sense to multiply this rate and concentration to determine how much of the gas is entering the room?

The concentration tells us how much stuff is in a certain volume, and the rate tells us how

much volume moves into the room per unit time, so multiplying them tells us how much stuff moves into the room per unit time.

We make this term positive in our equation, since it is adding gas to the room:

$$\frac{dG}{dt} = 20\frac{g}{min} + ?$$

Next we consider the filter system which is removing air from the room. We are told that this filter removes the well mixed air at the same rate $2\frac{m^3}{min}$, however we are not told the concentration of the air leaving the room. Naturally the stuff which leaves the room must depend on the stuff that is actually in the room, but trying to pin that down is a little subtle.

Because we are adding gas into the room, the concentration of gas in the room will be changing. In order to find the concentration of gas in the room we again use dimensional consistency to help us select the correct term. A concentration should have units of $\frac{g}{m^3}$ so we take the total mass of gas in the room (G g) and divide it by the volume of the room ($V = 10 \times 10 \times 10m^3 = 1000m^3$). Multiplying by the outgoing rate yields:

$$\frac{Gg}{Vm^3} 2\frac{m^3}{min} = \frac{2G}{1000}\frac{g}{min} = \frac{G}{500}\frac{g}{min}$$

This term has the appropriate units, but it represents the gas leaving the room, so we subtract the term in our differential equation:

$$\frac{dG}{dt} = 20 - \frac{G}{500}$$

We have accounted for all of the gas entering or leaving the room, and now we only need to solve the differential equation we have created. This equation is amenable to either direct integration or the method integrating factors, it general solution will be given by the following:

$$G(t) = (G(0) - 10,000)e^{-\frac{t}{500}} + 10,000$$

Since there was no knockout gas in the room before $t = 0$ we can set $G(0) = 0$ and we obtain the following:

$$G(t) = 10,000(1 - e^{-\frac{t}{500}})$$

Remember that we are interested in finding out when the concentration within the room reaches an effective level, so we want to find the time t_1 when $\frac{G}{V} = 1\frac{g}{m^3}$.

$$\begin{aligned} C(t) &= \frac{G}{V} \\ C(t) &= 10(1 - e^{-\frac{t}{500}}) \\ 1 &= 10(1 - e^{-\frac{t}{500}}) \\ \ln\frac{9}{10} &= -\frac{t}{500} \\ t &= -500\ln(0.9) \\ t &\approx 52.7 minutes \end{aligned}$$

Thus you will need to wait nearly an hour before the gas becomes effective...I hope you brought a book.

In order to answer the second question we apply the same kind of analysis as before, but we must be careful to adjust our equations to reflect the new structure of the system. We begin again with:

$$\frac{dG}{dt} = ?$$

Since you have turned off the knockout gas, there is no gas entering the system. All change in the gas will be due to gas leaving the system and we are told that the air filtration system is now pumping the air out of the room at a rate of $100\frac{m^3}{min}$, to find the actual amount of gas leaving the system, apply the same reasoning as before. Take the amount of gas in the room G divide by the total volume of air in the room V then multiply by the rate at which the air is being removed. All this gives the following differential equation:

$$\begin{aligned} \frac{dG}{dt} &= -\frac{G}{1000}(100) \\ \frac{dG}{dt} &= -\frac{G}{10} \end{aligned}$$

This is an equation we can easily solve, and if we apply the given assumption that $C(0) = 1\frac{g}{m^3}$ then $G(0) = 1000g$, and we obtain the following formula for $G(t)$.

$$G(t) = 1000e^{-\frac{t}{10}}$$

Now if we want to find when the concentration falls to safe levels, we simply solve for t with a sufficiently small G.

$$\begin{aligned} 100 &= 1000e^{-\frac{t}{10}} \\ 10\ln(10) &= t \\ t &\approx 23 minutes \end{aligned}$$

3.4. Input-Output: Mixing with variable volume. In this section we present a second example of a mixing problem, but this time we illustrate how to handle a variable volume of fluid. This example illustrates why modeling mass rather than concentration is a more robust approach.

A certain $100gal$ hydrodynamics tank is initially filled with 1 gallon of fresh water. This tank has two inlet valves and one output valve. One input value is connected to a reservoir of fresh water, while the other valve is connected to a reservoir of saline at a concentration of $3\frac{g}{gal}$. If the inlet valves each allow fluid to pass at a rate of $2\frac{gal}{min}$, and the output valve allows well mixed fluid to leave the tank at a rate of $3\frac{gal}{min}$, Determine the mass of salt in the tank when it first fills up.

To solve this problem we need to keep track of the volume of fluid, V and the amount of salt, S.

Looking at the volume first we find an equation for $\frac{dV}{dt}$. Enforcing dimensional consistency we require that each term in this equation have the same units as $\frac{dV}{dt}$. This turns out to be very easy. We add the volume of fluid coming into the tank and subtract the volume of fluid leaving the tank.

$$\frac{dV}{dt} = Input_1 + Input_2 - Output_1$$
$$\frac{dV}{dt} = 2 + 2 - 3 = 1$$

Solving this equation we find a function representing the volume of fluid in the tank at any time.

$$V = t + V(0)$$

Initially the tank has $1gallon$ of fresh water, so we apply the initial condition, $V(0) = 1$. We obtain the exact volume in the tank as a function of time:

$$V = t + 1$$

Next we find an equation representing the amount of salt in the tank at any time. By following the same line of argument as the previous problem and enforcing dimensional consistency on each term we obtain the following:

$$\begin{array}{rccccc} \frac{dS}{dt} = & Input_1 & + & Input_2 & - & Output_1 \\ \frac{dS}{dt} = & 2 \times 0 & + & 2 \times 3 & - & 3 \times \frac{S}{V} \end{array}$$

The first input contributes nothing since fresh water doesn't contain salt. Since the volume is changing, we use the known formula for V in this equation. Simplifying, we obtain the

following linear differential equation:

$$\frac{dS}{dt} = 6 - 3\frac{S}{t+1}$$

We solve this problem using the method of integrating factors. Re-writing in the canonical form:

$$\frac{dS}{dt} + 3\frac{S}{t+1} = 6$$

We multiply by the integrating factor, $(t+1)^3$.

$$\begin{aligned}
(t+1)^3 S' + 3(t+1)^2 S &= 6(t+1)^3 \\
\frac{d}{dt}\left[(t+1)^3 S\right] &= 6(t+1)^3 \\
\int \frac{d}{dt}\left[(t+1)^3 S\right] dt &= \int 6(t+1)^3 dt \\
(t+1)^3 S &= \frac{3}{2}(t+1)^4 + C \\
S &= \frac{3}{2}(t+1) + \frac{C}{(t+1)^3}
\end{aligned}$$

Since the tank initially had fresh water, we apply the initial condition $S(0) = 0$.

$$S = \frac{3}{2}(t+1) - \frac{3}{2(t+1)^3}$$

This model will be valid until the tank overflows. The volume of the tank is 100 gallons, so the overflow time is $t = 99$. Substituting this into the solution we obtain the amount of salt in the tank at overflow.

$$\begin{aligned}
S &= \frac{3}{2}(100) - \frac{3}{2(100)^3} \\
S &= 150 - 1.5 \times 10^{-6} g
\end{aligned}$$

Looking at this answer, we see that lots of the salt has passed through the tank. If we just look at the amount of salt put into the tank we get $2\frac{gal}{min} \times 3\frac{g}{gal} \times 99min = 594g$, whereas the amount of salt that actually stayed in the tank was around $\frac{1}{4}$ of that total. Hopefully this will help to demonstrate how these models can defy our intuition.

4. Specialized Analytic Solution Techniques

In this section we illustrate several alternate techniques for solving first order equations which have different structures which may be exploited. These methods are more specialized than the method of separation and the method of integrating factors, but may become useful in certain circumstances. Awareness of these methods is important, but relative to the methods discussed previously these techniques are more specialized and lower priority.

4.1. Exact equations. Exact equations are a class of first order differential equations which arise from a perfect derivative structure important in multivariable calculus. Let $z(x, y)$ be a differentiable function of two variables. When one studies multivariable functions, partial derivatives provide the natural extension of ordinary derivatives, and permit one to compute partial rates of change when all other arguments are fixed. We define partial derivatives similar to ordinary derivatives:

$$\frac{\partial z}{\partial x} = z_x \equiv \lim_{h\to 0} \frac{z(x+h, y) - z(x, y)}{h}, \qquad \frac{\partial z}{\partial y} = z_y \equiv \lim_{h\to 0} \frac{z(x, y+h) - z(x, y)}{h}$$

In certain situations expressions involving partial derivatives are equal to ordinary derivatives, one of these situations in particular gives rise to exact differential equations:

Suppose that $z(x, y)$ represents the elevation of a mountain range. Suppose further that $y = y(x)$ is a particular is a curve representing a trail through this mountain range. A hiker might naturally be interested in knowing how steep the trail is at different locations. This steepness is naturally calculated by computing the rate of change of height of the trail, with respect to the distance traveled. Symbolically we would like to find:

$$\frac{dz}{dx}$$

Since z is a surface and depends on both x and y, actually computing this derivative requires that we use partial derivatives to help organize and subdivide the differentiation process. Using the chain rule and assuming that the trail follows the parametric curve $(x, y(x))$, we have:

$$\frac{dz}{dx} = \frac{\partial z}{\partial x} + \frac{\partial z}{\partial y}\frac{dy}{dx}$$

This formula gives us a way of computing $\frac{dz}{dx}$, without needing to plug in a formula for the trail $y(x)$ explicitly. From the perspective of differential equations this gives us a new kind of structure to exploit in solving first order equations.

In order to see how this structure can be exploited in order to solve first order equations we begin by blocking out the basic process:

Suppose that we start with a generic first order equation:

$$\frac{dy}{dx} = F(x, y)$$

we must mentally block out a numerator and a denominator with the following structure:

$$\frac{dy}{dx} = \frac{-M(x, y)}{N(x, y)}$$

Once we have chosen the M and N in this form we can rearrange as:

$$M(x, y) + N(x, y)\frac{dy}{dx} = 0$$

In this form we have roughly the same 'shape' as the desired structure, but we **need** to make sure that the $M(x,y)$ and $N(x,y)$ that we have selected are actually connected by being the correct partial derivatives of a single function $z(x,y)$. Lining up our differential equation with the exact form we have:

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0$$
$$\frac{\partial z(x,y)}{\partial x} + \frac{\partial z(x,y)}{\partial y}\frac{dy}{dx} = 0$$

So in order for us to proceed we **need** to have:

$$M = \frac{\partial z}{\partial x}, \quad N = \frac{\partial z}{\partial y}$$

Before we do too many calculations, we can check for exact structure now by using Clairaut's theorem from multivariable calculus. This theorem states that when a function is sufficiently smooth, partial derivatives commute. Algebraically, the condition reads:

$$\frac{\partial}{\partial y}\frac{\partial z}{\partial x} = \frac{\partial}{\partial x}\frac{\partial z}{\partial y}$$

Applying this to the known functions M and N we find the following compatibility condition:

$$\frac{\partial}{\partial y}M = \frac{\partial}{\partial x}N$$

If this condition is false, you cannot go further. If this condition is true, then you can proceed to uncover the function $z(x,y)$.

To find $z(x,y)$ we need to use both of the equations from above:

$$M = \frac{\partial z}{\partial x}, \quad N = \frac{\partial z}{\partial y}$$

In order to recover z take either equation and integrate with respect to the relevant variable. The fundamental theorem of calculus applies in either case, but since you are removing a partial derivative, the constant of integration may depend upon y in the case of an x integral, or x in the case of a y integral.

Supposing we start with the equation for M we would obtain the following:

$$M(x,y) = \frac{\partial z}{\partial x}$$
$$\int M dx = \int \frac{\partial z}{\partial x} dx$$
$$\int M dx + h(y) = z(x,y)$$

In order to fully recover z we need to know the correct form of $h(y)$. We use the other equation for N to uncover this:

$$\begin{aligned} N(x,y) &= \frac{\partial z}{\partial y} \\ N(x,y) &= \frac{\partial}{\partial y}\left[\int M dx + h(y)\right] \\ N(x,y) &= \frac{\partial}{\partial y}\int M dx + \frac{dh}{dy} \end{aligned}$$

Remark: If the equation has exact structure the differential equation you obtain for $\frac{dh}{dy}$ should be independent of x. If any x terms remain, you have made a mistake

This yields a first order differential equation for $\frac{dh}{dy}$. By solving this equation you possess a formula which fully recovers z.

Returning to our differential equation we have:

$$\begin{aligned} M(x,y) + N(x,y)\frac{dy}{dx} &= 0 \\ \frac{\partial z(x,y)}{\partial x} + \frac{\partial z(x,y)}{\partial y}\frac{dy}{dx} &= 0 \\ \frac{dz}{dx} &= 0 \\ \int \frac{dz}{dx}dx &= \int 0 dx \\ z(x,y) &= C \end{aligned}$$

Since this process uncovers a formula for $z(x,y)$, the differential equation is completely solved.

4.2. Homogeneous equations. A first order homogeneous equation is one which can be expressed solely in terms of the ratio $\frac{y}{x}$. This symmetry in the equation can be exploited by making a special change of variables. We suppose that we have a differential equation written in the following block form:

$$\frac{dy}{dx} = f\left(\frac{y}{x}\right)$$

Our assumption states that the equation depends on only a single configuration of the dependent and independent variables, to exploit this structure we use exactly that configuration as a new dependent variable:

$$v = \frac{y}{x}$$

In order to fully change variables using this new variable, we need to do two things, first we must replace every instance of $\frac{y}{x}$ with the new variable, and second we must replace $\frac{dy}{dx}$. The first task is trivial since $f\left(\frac{y}{x}\right) = f(v)$. This second task is not difficult, but is a little

more subtle. We find:

$$y = vx$$

so differentiating both sides with respect to x and using the product rule we have:

$$\frac{dy}{dx} = \frac{d}{dx}\left[vx\right] = x\frac{dv}{dx} + v$$

Substituting these into our differential equation yields a new equation which is separable:

$$x\frac{dv}{dx} + v = f(v)$$
$$\frac{1}{f(v) - v}\frac{dv}{dx} = \frac{1}{x}$$

We proceed by integrating both sides, this may be more or less difficult depending upon the form of $f(v)$. After obtaining the general solution (implicitly) we can change back to our original variables and apply any initial conditions.

CHAPTER 2

Constant Coefficient Linear Equations of arbitrary order

As we pass from first order equations to higher order equations, the number of useful analytic solution techniques diminishes rapidly. In general, one has a plethora of techniques which are effective for linear equations, and very specialized techniques which work for specific non-linear equations. As we work with constant coefficient equations we will develop several different techniques for solving the same class of problem. There will be considerable overlap in the applicability of these techniques, but different techniques generalize differently so it is important to note the strengths and weaknesses of each technique.

1. Higher Order Homogeneous Equations

We begin with the study of a homogeneous, n-th order linear, constant coefficient ordinary differential equation. Such an equation may be written in the form:

$$\alpha_n \frac{d^n y}{dt^n} + \alpha_{n-1} \frac{d^{n-1} y}{dt^{n-1}} + \cdots + \alpha_1 \frac{dy}{dt} + \alpha_0 y = 0$$

this equation expresses a balance between some linear combination of the function $y(t)$ and its derivatives. Specifically some combination of y and its derivatives sum to **exactly** zero at each instant of time. If we use some intuition from calculus one, we can actually get quite close to writing down a general solution for this type of equation. Let us take the first order version of this problem, and use it to inspire a guess for the more general n-th order equation. If we look at the homogeneous first order constant coefficient linear ordinary differential equation, we must solve:

$$\alpha_1 \frac{dy}{dt} + \alpha_0 y = 0$$

Using techniques from the previous chapter we know the general solution to this equation may be written as:

$$y = Ce^{-\frac{\alpha_0}{\alpha_1} t}$$

Here we see the **exponential** function has the right kind of behavior. An exponential function keeps its functional character under the operation of differentiation, so if we assume our solution has an exponential character, the time-dependent portion of the solution will drop out of the n-th order homogeneous equation leaving us with an interesting condition which must be satisfied by the coefficient in the argument of the exponential.

Explicitly, we assume that $y = Ce^{\lambda t}$ and we substitute this form into the n-th order equation:

$$\begin{aligned}
\alpha_n \frac{d^n y}{dt^n} + \alpha_{n-1} \frac{d^{n-1} y}{dt^{n-1}} + \cdots + \alpha_1 \frac{dy}{dt} + \alpha_0 y &= 0 \\
\alpha_n \frac{d^n}{dt^n} Ce^{\lambda t} + \alpha_{n-1} \frac{d^{n-1}}{dt^{n-1}} Ce^{\lambda t} + \cdots + \alpha_1 \frac{d}{dt} Ce^{\lambda t} + \alpha_0 Ce^{\lambda t} &= 0 \\
\alpha_n \lambda^n Ce^{\lambda t} + \alpha_{n-1} \lambda^{n-1} Ce^{\lambda t} + \cdots + \alpha_1 \lambda Ce^{\lambda t} + \alpha_0 Ce^{\lambda t} &= 0
\end{aligned}$$

Now by dividing out the common factor of $Ce^{\lambda t}$ we are left with a polynomial constraint on the possible values of λ.

$$\alpha_n \lambda^n + \alpha_{n-1} \lambda^{n-1} + \ldots \alpha_1 \lambda + \alpha_0 = 0$$

This constraint is called the characteristic polynomial, or characteristic equation of the ODE. The roots of this characteristic equation yield the viable λ's which may be used in the exponential functions. If this polynomial possesses a full set of distinct roots, then we have sufficient solutions to solve any initial value problem involving this homogeneous equation. In the case of repeated or complex roots we must work a little harder to ensure that we can construct a satisfactory solution for arbitrary initial conditions.

We fully develop the theory for second order equations with the understanding that since any polynomial can be fully factored over the complex numbers, these ideas generalize directly to constant coefficient equations of arbitrary order.

1.1. Repeated Roots in the Characteristic equation. A repeated root in the characteristic equation means that there are too few elementary exponential solutions to construct a fundamental solution set for arbitrary initial conditions. Thus we must seek additional solutions with a character which is more complicated than a simple exponential. We do this via the technique of reduction of order. Ultimately we will discover these additional solutions have a very simple and predictable form, however rather than stating the final result, we derive them using a more general technique: Reduction of Order. Reduction of Order is a bootstrapping technique whereby one uses partial information about the solutions of an ordinary differential equation to derive simpler (lower order) equations which may be solved to uncover new solutions. The technique is valuable in its own right and is effective on any linear equation.

Reduction of order works by assuming a solution may be constructed out of a product of a known solution and a 'helper' function. One then substitutes this product solution into the differential equation and works out the details. Since the assumed solution is a product of a known homogeneous solution and this helper function by applying the product rule repeatedly the sum of all terms involving the undifferentiated function cancel exactly. This cancellation leaves a new lower order equation for the first derivative of the helper function

which can be solved and then integrated to find the exact value of the helper function. If one has multiple repeated roots, this procedure can be applied recursively until sufficient solutions are found to form a basis for the space of solutions of the homogeneous equation.

We illustrate the technique on an arbitrary linear equation of second order.

$$p(t)y'' + q(t)y' + r(t)y = 0.$$

Assume the coefficient functions p, q, r are all given continuous functions. Further suppose that we have found one solution to this differential equation given by the function $y_1(t)$. as indicated above we assume that we can find another solution of the following form:

$$y_2 = u(t)y_1(t)$$

This guess is very general, since $u(t)$ is unspecified, we haven't really imposed any restrictions by making such a guess. The motivation for this guess is that since y_1 is a solution to the differential equation, this might yield a simpler problem for the unknown function $u(t)$.

After making this guess we differentiate it and substitute it into our differential equation.

$$\begin{aligned} y_2' &= u'y_1 + uy_1' \\ y_2'' &= u''y_1 + 2u'y_1' + uy_1'' \end{aligned}$$

plugging this into the differential equation we obtain:

$$\begin{aligned} p(t)y_2'' + q(t)y_2' + r(t)y_2 &= 0 \\ p(t)\left[u''y_1 + 2u'y_1' + uy_1''\right] + q(t)\left[u'y_1 + uy_1'\right] + r(t)uy_1 &= 0 \end{aligned}$$

which looks quite complex. Next we rearrange these terms seeking cancellation. Let's arrange the terms by the derivatives of u that appear:

$$p(t)y_1u'' + \left[2py_1' + q(t)y_1\right]u' + \left[p(t)y_1'' + q(t)y_1' + r(t)y_1\right]u = 0$$

The remarkable observation here is that since y_1 is a solution to the original differential equation, the whole coefficient of u is equal to zero. This yields:

$$p(t)y_1u'' + \left[2py_1' + q(t)y_1\right]u' = 0$$

This problem might still be formidable, but it's intrinsically easier than the full equation, since it is a **first order** equation in u'. The fact that we only have to solve a first order differential equation means that either the method of separation or the method of integrating factors can be used to help assist us with this problem. Using separation we

can even write down a formula for the function $u(t)$. If neither coefficient is zero we find:

$$\begin{aligned}
\frac{u''}{u'} &= -\frac{[2py_1' + q(t)y_1]}{y_1 p(t)} \\
\int \frac{u''}{u'} dt &= \int -\frac{[2py_1' + q(t)y_1]}{y_1 p(t)} dt \\
\ln(u') &= \int -\frac{[2py_1' + q(t)y_1]}{y_1 p(t)} dt \\
u &= \int e^{\int -\frac{[2py_1' + q(t)y_1]}{y_1 p(t)} dt} dt
\end{aligned}$$

Which looks pretty terrible, but can actually be pretty easy to handle in practice. We apply the method to several explicit examples in the practice section.

1.2. Complex Roots in the Characteristic equation. When the characteristic equation yields complex roots, but we expect only real valued solutions to be of physical relevance we need to connect the complex exponential functions to real valued function which we can use. The argument which takes us from complex solutions to real valued solutions is a long but beautiful argument which draws from calculus and the theory of polynomial equations by way of two equations known as the Euler identities which we will derive shortly. We will work with 'the' standard canonical example, but keep in mind that the structure that we see here generalizes in a dependable way to other problems.

Take the following second order, constant coefficient ordinary differential equation:

$$y'' + y = 0, \qquad y(0) = \alpha, \qquad y'(0) = \beta$$

By substituting our exponential guess into this equation we arrive at the characteristic polynomial:

$$\lambda^2 + 1 = 0$$

this equation has two, purely imaginary complex conjugate roots. $\lambda = \pm i$. Polynomial equations with real coefficients always have complex roots appear in complex conjugate pairs, this becomes important for cancellation later in the argument.

Since we have these complex roots, we substitute them back into our exponential guess and try to make sense of the resulting function. From these roots we have the following representation of our solution as a complex function:

$$y = K_1 e^{it} + K_2 e^{-it}$$

where K_1 and K_2 are possibly complex coefficients. In order to extract a real valued solution here, we first use the power series representation of the exponential function to split each of these into real and imaginary parts. This representation gives rise to the Euler

identities.

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \dots.$$

Using this representation and the property that $i^2 = -1$ we find:

$$e^{it} = 1 + it - \frac{t^2}{2} - i\frac{t^3}{3!} + \dots.$$

re-ordering the terms of the series to group the real and imaginary parts together yields something amazing:

$$e^{it} = \left(1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!} - \dots\right) + i\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots\right).$$

The two series that result are **exactly** the power series representations of the Cosine and Sine respectively. We thus arrive at the celebrated Euler identity:

$$e^{it} = \cos(t) + i\sin(t).$$

Similarly, we have:

$$e^{-it} = \cos(t) - i\sin(t).$$

By using these identities in our general complex solution we can extract real-valued functions by choosing K_1 and K_2 in clever ways. By choosing $K_1 = \frac{1}{2}$, and $K_2 = \frac{1}{2}$ we find:

$$y_1 = \cos(t),$$

and by selecting the less-intuitive $K_1 = \frac{i}{2}$, $K_2 = -\frac{i}{2}$ we find:

$$y_2 = \sin(t)$$

Since the equation is linear, these particular combinations are both genuine solutions to the original equation, and we can use these as a general solution rather than the complex solutions we derived from the exponential guess.

Rather than going through this argument for every problem we encounter it is helpful to generalize the results for arbitrary complex roots.

If the second order, linear, constant coefficient, homogeneous, ordinary differential equation has a characteristic polynomial with complex roots $\lambda = \mu \pm i\sigma$, then one can adopt the following anzatz for a real valued general solution.

$$y = C_1 e^{\mu t}\cos(\sigma t) + C_2 e^{\mu t}\sin(\sigma t)$$

2. Higher Order Non-homogeneous equations

As we pass from homogeneous equations to non-homogeneous equations it is helpful to adopt an operator theoretic perspective. Given a constant coefficient differential equation (of second order for example) we will organize the equation as:

$$ay'' + by' + cy = f(t).$$

we then think of the whole left-hand side an a operator which transforms the function $y(t)$. So we might define:

$$\mathcal{L}(y) = ay'' + by' + cy$$

This perspective allows us to approach the solution of linear differential equation using the same conceptual foundations as the solutions of matrix-vector equations. Here we will see that the solvability properties of the non-homogeneous equation:

$$\mathcal{L}(y) = f(t),$$

is intrinsically linked with the solutions of the homogeneous equations. The analogy with matrix-vector equations is more pronounced with boundary value problems, however it is helpful to introduce the perspective now. When we take the operator theoretic perspective, the process of solving a differential equation is exactly analogous to inverting a matrix. Similarly, the homogeneous solutions of a differential equation form the null-space of the differential operator.

Just like in linear algebra, differential operators which possess non-trivial homogeneous solutions possess multiple solutions, and side conditions such as initial or boundary conditions must be used to specify a unique solution of interest. Explicitly suppose that one has found a function $\phi(t)$ which satisfies the non-homogeneous linear, differential equation:

$$\mathcal{L}(y) = f(t),$$

Then, since $\mathcal{L}(y)$ is linear the sum of $\phi(t)$ and any homogeneous solutions will still be a solution to the non-homogeneous equation.

When faced with a non-homogeneous equation there are a variety of different techniques which can be used to solve the problem. The first technique we will discuss is called the Method of Undetermined Coefficients. The technique is effective when $f(t)$ is a sum or product of elementary polynomial, exponential and trigonometric functions, but does not generalize to more complex equations. The technique works by making an educated guess about the form of the solution, then substituting the guess and selecting the correct coefficients to satisfy the desired equation.

2.1. Method of Undetermined Coefficients. The method of undetermined coefficients is best mastered through practice and experimentation, thus we give a brief overview of the procedure from a theoretical prespective, and then we recommend the student turn to the practice section for a copious selection of examples which outline what can go right and what can go wrong with the method.

Faced with a non-homogeneous, linear, constant coefficient equation of the form:

$$\mathcal{L}(y) = f(t)$$

one tries to find the solution y by making a guess based upon the functional form of $f(t)$. If one can guess the correct prototype for a given $f(t)$, then acting on this prototype using the operator $\mathcal{L}$ yields an output function of the same **kind**, then one varies the amount of the guess to obtain the exact solution. This generally involves solving a system of linear equations to balance the output of the prototype function and the desired output, $f(t)$.

Below we give a short list of the naive prototype functions for elementary $f(t)$. In this table lowercase letters, a, b, c denote known constant parameters, while upper case parameters A, B, C denote the undetermined coefficients.

$f(t)$	—	Prototype
e^{at}	—	Ae^{at}
$\cos(at)$	—	$A\cos(at) + B\sin(at)$
$\sin(at)$	—	$A\cos(at) + B\sin(at)$
$at^n + bt^{n-1} + \cdots + mt + n$	—	$At^n + Bt^{n-1} \cdots + Mt + N$

The prototypes for sums and products of these functions are obtained by adding and/or multiplying the respective protoypes and including an undetermined coefficient for each of the resulting functional terms. For example:

$f(t)$	—	Prototype
$e^{at}\sin(bt)$	—	$Ae^{at}\cos(bt) + Be^{at}\sin(bt)$
t^2e^{at}	—	$At^2e^{at} + Bte^{at} + Ce^{at}$

These prototypes will be sufficient to solve the simplest non-homogeneous constant coefficient equations, however and interesting situation arises when one or more of the prototype functions belongs to the null space of the differential operator. Here the prototype must be modified to find a guess which works. Fortunately the modification is simple, and universal, however the resulting calculations have a tendency to be tedious, so it is important to make all necessary modifications to the prototype before beginning any actual computation.

When **any** term belonging to the prototype for a particular forcing function $f(t)$ is also a solution to the equation $\mathcal{L}(y) = 0$, multiply the entire prototype for that function by t. After multiplying by t, re-evaluate the modified prototype, if any of the modified prototype functions are still solutions of the homogeneous equation then multiply by t again. Repeat the process until none of the prototype functions are homogeneous solutions. Because of the structure of the homogeneous solutions to ordinary differential equations, you will never need to repeat this process more times than the degree of the equation. The reason and justification for this modification process is given in the later examples in the practice section.

2.2. How do I pick the right guess? The method of undetermined coefficients can be a very difficult method for students to master, because in order for the method to work,

one must come up with the correct guess for input. Many textbooks include a table of appropriate guesses and this is a valid approach to mastering the method, however more knowledge can be gained by understanding **why** those are the correct guesses.

Let's consider the thought process which is necessary to come up with a correct guess.

When you consider the differential equation $\mathcal{L}(y) = F$, your goal is to find y. To make an appropriate guess for y you need to use the given information, F, along with an understanding of the operator $\mathcal{L}$.
To make a guess which will be adequate on the first try, you will need to include any terms which might pop up.

Let's consider the first order equation:

$$y' + y = F$$

If you know the function F, then your guess should reflect the fact that F is made up of adding y and it's first derivative. Your guess should include a function that looks just like F, **and** a function which has F as a derivative. Let's try a real simple example:

$$y' + y = e^t$$

Here the output is the exponential function. This should be an easy example because the exponential function has a very simple relationship to it's derivatives: They are all identical. Knowing that the exponential function stays the same when we differentiate it, means that we might be able to get away with just guessing Ae^t. This is in fact the case. If we make this guess, differentiate it and plug it in we find:

$$Ae^t + Ae^t = e^t$$

If we pick $A = \frac{1}{2}$ we are done.

Sometimes, though we need to expand our guess because the function isn't quite as simple. (This is where it can get a little tricky). Consider:

$$y' + y = t^2$$

Here we might try the same tact as before just guessing At^2 and hoping that it works. (It doesn't). With this simple guess we have some leftover functions which don't balance:

$$A2t + At^2 = t^2$$

This gives us conflicting information, since to balance the t^2 we want to pick $A = 1$, but to get rid of the $2t$ term, we want to pick $A = 0$. Since we can't have it both ways, this guess must be wrong. So we try a more complicated guess: $At^2 + Bt$, plugging this in we find:

$$A2t + B + At^2 + Bt = t^2$$

We did a little better here, but we still can't balance all of the terms, a factor of B is left over. We now try an even more complicated guess: $At^2 + Bt + C$. This guess will work

because there are exactly the right types of functions to balance everything that appears.

It's important to note that, we're not just making the guess more complicated by adding random functions, we're only adding those functions which remain unbalanced. This is the key to using the method of undetermined coefficients efficiently, you want to make a guess composed of all and only those function types which could arise when trying to make F come out of the operator $\mathcal{L}$.

3. Solving Non-homogeneous Initial Value Problems

In this section we provide the most efficient procedure for handling non-homogeneous initial value problems. No matter what technique you apply for solving a non-homogeneous equation, the order of steps chosen in this procedure will save you from unnecessary and incorrect computations. The technique involves combining our knowledge of homogeneous and non-homogenous equations and using an appropriate combination of functions to satisfy the initial conditions.

First we outline the procedure in a stepwise fashion. We follow these steps in the practice section for solving problems with method of undetermined coefficients and with the variation of parameters technique explained in the next section.

Consider the non-homogeneous initial value problem:

$$ay'' + by' + cy = F(t), \quad y(t_o) = \alpha, \quad y'(t_o) = \beta$$

with $a, b, c, \alpha, \beta, t_o$ all given numbers and $F(t)$ a given function.

(1) Solve the homogeneous equation and find the general solution. $y_h = C_1y_1 + C_2y_2$.

(2) Find a solution to the non-homogeneous equation. If you are using method of undetermined coefficients, check whether $F(t)$ is one of the homogeneous solutions and modify your guess appropriately.

(3) Create a linear combination of the non-homogeneous solution and an arbitrary linear combination of the homogeneous solutions. Apply the initial conditions to the combined solution and solve the resulting linear system for the coefficients of the homogeneous solution terms.

Remark: Finding the general homogeneous solution **first** saves time since you won't need to waster time with incorrect guesses.

Remark: You **must** apply the initial conditions to the **combined** solution. If you do not include the non-homogeneous solution the resulting answer will not satisfy the correct initial conditions.

4. Variation of Parameters

The method of undetermined coefficients is useful for constant coefficient equations whenever the forcing function is an elementary function with predictable derivatives. The method does not generalize well, and fails completely for arbitrary variable coefficient equations. Variation of parameters is a more powerful method for solving non-homogeneous

linear equations. Conceptually the method is similar to reduction of order, in that we bootstrap a non-homogeneous solution by modifying the homogeneous solutions. The method is far more elegant than one might expect, but the elegance only appears after considerable calculation.

We will derive the method explicitly for a general abstract problem, however to the more practically minded student, the method may be applied directly by using the variation of parameters described at the end of the section.

We consider the following general example problem:

$$y'' + p(t)y' + q(t)y = F(t)$$

We assume that the homogeneous equation is fully solved and we know a fundamental solution set given by:

Remark: explicitly y_1 and y_2 satisfy: $y_i'' + py_i' + qy_i = 0$, exactly. We use this to great effect later in the calculation

$$y_h = C_1 y_1 + C_2 y_2$$

With these assumptions in place we make the guess that we can solve the non-homogeneous equation using a 'functional' linear combination of the homogeneous solutions:

$$\text{Guess: } Y = u_1(t)y_1(t) + u_2(t)y_2(t)$$

If we can find two functions u_1 and u_2 which make this combination solve the differential equation, then we are done.

We note at the outset that since we have two functions to pick, we can impose **two** conditions on these functions. One condition will be that this solution satisfies the differential equation. The other condition is chosen to ease the computational difficulty.

We proceed by taking derivatives of this assumed solution and plugging them into the differential equation:

$$Y' = u_1'y_1 + u_1y_1' + u_2'y_2 + u_2y_2'$$

We impose our first condition here, rather than deal with this whole first derivative we will pick u_1 and u_2 so that they satisfy the following equation:

$$u_1'y_1 + u_2'y_2 = 0$$

This is a first order differential equation involving u_1' and u_2' and known functions. By imposing this condition several things happen. First our first derivative becomes considerably simpler:

$$Y' = u_1y_1' + u_2y_2'$$

Correspondingly our second derivative also becomes much simpler:

$$Y'' = u_1'y_1' + u_1y_1'' + u_2'y_2' + u_2y_2''$$

When we plug these equations back into our differential equation we obtain the following:

$$\left[u_1'y_1' + u_1y_1'' + u_2'y_2' + u_2y_2'\right] + p\left[u_1y_1' + u_2y_2'\right] + q\left[u_1(t)y_1(t) + u_2(t)y_2(t)\right] = F(t)$$

Next, we group the terms in the equations by the u_i terms since these are the only unknown functions.

$$u_1'y_1' + u_2'y_2' + u_1\left[y_1'' + py_1' + qy_1\right] + u_2\left[y_2'' + py_2' + qy_2\right] = F(t)$$

Here we see that the coefficients of the u_1 term and the u_2 term are exact copies of the original differential operator applied to the homogeneous solutions. We see that each of these coefficients is **exactly** zero (See the remark at the beginning of the derivation.). This leaves the following equation:

$$u_1'y_1' + u_2'y_2' = F(t)$$

Combining this with the previous condition we have two **algebraic** equations for u_1' and u_2'.

$$\begin{aligned} u_1'y_1' + u_2'y_2' &= F(t) \\ u_1'y_1 + u_2'y_2 &= 0 \end{aligned}$$

We can solve these equations explicitly to obtain:

$$\begin{aligned} u_1' &= -\frac{y_2F(t)}{y_1y_2' - y_2y_1'} = \frac{y_2F(t)}{W(y_1,y_2)} \\ u_2' &= \frac{y_1F(t)}{y_1y_2' - y_2y_1'} = \frac{y_2F(t)}{W(y_1,y_2)} \end{aligned}$$

By integrating both of these equations we can obtain the desired solutions:

$$\begin{aligned} u_1 &= \int_{t_0}^{t} -\frac{y_2F(t)}{W(y_1,y_2)}dt, \\ u_2 &= \int_{t_0}^{t} \frac{y_1F(t)}{W(y_1,y_2)}dt, \end{aligned}$$

Finally we recall that our solution was given by the following expression:

$$\begin{aligned} Y &= u_1(t)y_1(t) + u_2(t)y_2(t) \\ Y &= -y_1\int_{t_0}^{t} \frac{y_2F(t)}{W(y_1,y_2)}dt + y_2\int_{t_0}^{t} \frac{y_1F(t)}{W(y_1,y_2)}dt \end{aligned}$$

If we had an initial value problem we could then add an appropriate multiple of the homogeneous solution to satisfy the initial conditions.

Now that we are in possession of the formulae for variation of parameters we need not re-compute these formulae for every problem we face.

5. Transform Techniques: Laplace Transforms

In this section we develop the basic theory of the Laplace transform as an alternative technique for solving constant coefficient ordinary differential equations. The method has the advantage of versatility in that homogeneous and non-homogeneous equations are handled identically, but has the disadvantage of being very detail intensive; without an extensive reference table of Laplace transforms, one must be able to perform complex-contour integration in order to invert the transforms that arise in arbitrary problems. In our discussion we will present a rough overview of the fundamental theory and explore some of the more advanced properties which pave the way for understanding advanced techniques for solving boundary value problems.

The Laplace transform is effectively a change of variables which converts differential equations into algebraic equations. Before demonstrating the method in action we prove some of the most fundamental properties:

The Laplace transform of a piece-wise continuous function $f(t)$ is given by the following formula:

$$\mathcal{L}(f(t)) = F(s) = \int_0^\infty e^{-st} f(t)dt$$

Note that this transformation contains an improper integral and may fail to exist if f doesn't satisfy certain properties.

If f is piecewise continuous and f obeys the bound $f(t) \leq Ke^{at}$ for any $t \geq M$ for positive constants K, a, M then the Laplace transform of f exists for any $s > a$.

Looking at these conditions in some detail we can verify that the Laplace transform exists.

$$\mathcal{L}(f) = \int_0^M e^{-st} f(t)dt + \int_M^\infty e^{-st} f(t)dt$$

The first integral exists since it is a proper integral defined over a finite interval. The second integral exists because of the assumed bound on $f(t)$

$$\int_M^\infty e^{-st} f(t)dt \leq \int_M^\infty e^{-st} K e^{at} dt = K \int_M^\infty e^{(a-s)t} dt$$

Remark: In assuming that the limit goes to zero, I have introduced an implicit assumption that the transform parameter $s > 0$, such assumptions are very common when evaluating Laplace transforms by direct integration. In practice as long as the transform exists for some s, the technique is valid.

Now that we have defined the Laplace transform we can easily use it to find the transform of some known functions:
Let $f(t) = 1$ then we find:

$$\mathcal{L}(1) = \int_0^\infty e^{-st} dt = \lim_{t\to\infty} -\frac{1}{s} e^{-st} + \frac{1}{s} = \frac{1}{s}$$

Let $f(t) = \cos(\omega t)$ with $\omega \in \mathbb{R}$, then:

$$\mathcal{L}\left(\cos(\omega t)\right) = \int_0^\infty e^{-st}\cos(\omega t)dt$$

By applying integration by parts to this integral twice we are able to find its value:

$$\begin{aligned}
\int_0^\infty e^{-st}\cos(\omega t)dt &= \frac{1}{\omega}e^{-st}\sin(\omega t)|_0^\infty + \frac{s}{w}\int_0^\infty e^{-st}\sin(\omega t)dt \\
\int_0^\infty e^{-st}\cos(\omega t)dt &= \frac{1}{\omega}e^{-st}\sin(\omega t)|_0^\infty + \frac{s}{w}\left[-\frac{1}{\omega}e^{-st}\cos(\omega t)|_0^\infty - \frac{s}{\omega}\int_0^\infty e^{-st}\cos(\omega t)dt\right] \\
(1+\frac{s^2}{\omega^2})\int_0^\infty e^{-st}\cos(\omega t)dt &= \frac{1}{\omega}e^{-st}\sin(\omega t)|_0^\infty + \frac{s}{w}\left[-\frac{1}{\omega}e^{-st}\cos(\omega t)|_0^\infty\right]
\end{aligned}$$

Evaluating each of the terms on the right hand side reveals:

$$(1+\frac{s^2}{\omega^2})\int_0^\infty e^{-st}\cos(\omega t)dt = 0 + \frac{s}{\omega}\left[\frac{1}{\omega}\right]$$

Rearranging, we find the value of this improper integral becomes:

$$\begin{aligned}
\int_0^\infty e^{-st}\cos(\omega t)dt &= \frac{s}{\omega^2+s^2} \\
\mathcal{L}(\cos(\omega t)) &= \frac{s}{\omega^2+s^2}
\end{aligned}$$

As is clear from the previous example, the computation of Laplace transforms is not necessarily easy. To increase our speed we include a brief table of Laplace transforms which can be used to avoid calculating these transforms for elementary functions:

$f(t) = \mathcal{L}^{-1}(F(s))$	$F(s) = \mathcal{L}(f(t))$
1	$\frac{1}{s}$
e^{at}	$\frac{1}{s-a}$
te^{at}	$\frac{1}{(s-a)^2}$
$t^n, \quad n \in \mathbb{N}$	$\frac{n!}{s^{n+1}}$
$\sin(at)$	$\frac{a}{s^2+a^2}$
$\cos(at)$	$\frac{s}{s^2+a^2}$
$u_c(t)$	$e^{-cs}\frac{1}{s}$
$u_c(t)f(t-c)$	$e^{-cs}F(s)$
$e^{ct}f(t)$	$F(s-c)$
$\delta(t-c)$	e^{-cs}

5.1. Basic Properties of the Laplace Transform. Since the Laplace Transform is performed through integration and integration is a linear operation, the Laplace transform is also linear:

$$\begin{aligned}\mathcal{L}[\alpha f(t) + \beta g(t)] &= \int_0^\infty e^{-st}(\alpha f(t) + \beta g(t))\,dt \\ &= \alpha \int_0^\infty e^{-st} f(t)dt + \beta \int_0^\infty e^{-st} g(t))dt \\ &= \alpha\mathcal{L}(f) + \beta\mathcal{L}(g)\end{aligned}$$

The real defining feature of the Laplace transform is the relationship between $\mathcal{L}(f)$ and $\mathcal{L}(f')$, we will see that the Laplace transforms of derivatives take a particularly simple form which reduces linear differential equations to algebraic equations.

Let us assume that f is a continuous, differentiable function, and that the derivative f' is at least piecewise continuous. Let us also assume that f obeys the bound $f(t) \leq Ke^{at}$ for each $t \geq M$, then we can compute the Laplace transform of f' totally in terms of F and initial conditions on f.

$$\mathcal{L}(f') = \int_0^\infty e^{-st} f' dt$$

Integrating by parts we find:

$$\int_0^\infty e^{-st} f' dt = f(t)e^{-st}|_{t=0}^{t\to\infty} + s\int_0^\infty e^{-st} f dt$$

$$\int_0^\infty e^{-st} f' dt = -f(0) + s\mathcal{L}(f)$$

This relationship can be iterated to compute the laplace transforms of higher derivatives. If we assume f obeys the exponential bound and that f has $n-1$ continuous derivatives then we can compute the Laplace transform of $f^{(n)}$.

$$\mathcal{L}(f'') = s\mathcal{L}f' - f'(0)$$

$$\mathcal{L}(f'') = s\left[s\mathcal{L}(f) - f(0)\right] - f'(0)$$

See the practice section for the application of the Laplace transform to initial value problems.

5.2. Laplace transforms and discontinuous forces. One advantage of the Laplace transform technique is the ability to handle discontinuous forces in an elegant and efficient manner. In this section we define the heaviside distribution and prove the basic properties of this distribution under Laplace transforms. These properties are already recorded in our table of Laplace transforms, but the proofs and explanation are contained here.

Since Laplace transforms are defined for piecewise continuous functions, we can hope that we can make sense of the Laplace transforms of some simple discontinuous functions by defining them in an appropriate way.

We will define piecewise continuous functions by using a combination of Heaviside step functions. By developing a concise notation for these functions, we will be able to construct arbitrary piecewise continuous functions and write these functions in a simple manner.

We will denote the heaviside step function with the following notation:

$$u_c(t) = \begin{cases} 0 & t < c \\ 1 & t \geq c \end{cases}$$

This function mimics a light switch which is 'off' before $t = c$, and 'on' afterward.

By combining various heaviside functions we can create arbitrary piecewise continuous functions.

$$u_1(t) - u_2(t)$$

Here we have one function which turns on at $t = 1$ we subtract a different function which turns on at $t = 2$. The net effect from these two functions is a square wave which is positive on $1 \leq t \leq 2$ and zero elsewhere.

Taking the Laplace transform of an individual step function is very easy:

$$\mathcal{L}(u_c(t)) = \int_0^\infty e^{-st} u_c(t) dt = \int_c^\infty e^{-st}(t) dt = \frac{e^{-sc}}{s}$$

In addition we can also take the Laplace transform of a function which is started along with our heaviside function:

$$\mathcal{L}(u_c f(t-c)) = \int_0^\infty e^{-st} u_c(t) f(t-c) dt = \int_c^\infty e^{-st} f(t-c) dt = e^{-sc} F(s)$$

And similarly if we try and invert the Laplace transform of:

$$\mathcal{L}^{-1}(e^{-cs} F(s)) = u_c(t) f(t-c)$$

In a strict sense, such discontinuities should cause some alarm with our mathematical intuition. The heaviside function $u_c(t)$ doesn't have a smooth derivative at the point $t = c$. Including such functions in our forcing functions, may cause similar discontinuities in our solution or it's derivatives. Generally speaking if our forcing function is piecewise continuous we can expect the solution to an nth order linear equation to have $n - 1$ continuous derivatives, thus the discontinuities in our forcing function will give rise to corresponding discontinuities in our highest derivative.

5.3. Intermediate Properties of the Laplace Transform: Convolution. In this section we derive the convolution property of the Laplace transform which gives us a way to invert products of Laplace transforms. This essentially algebraic result allows us to find integral representation formulae for the solutions to non-homogeneous initial value problems which are more compact than the variation of parameters formulae.

Let $F(s)$ and $G(s)$ be the Laplace transforms of the functions $f(t)$ and $g(t)$. If we try to invert the Laplace transform of their product, we can write the result in a very elegant way.

$$\begin{aligned} F(s)G(s) &= \int_0^\infty e^{-s\xi} f(\xi) d\xi \int_0^\infty e^{-s\tau} g(\tau) d\tau \\ F(s)G(s) &= \int_0^\infty g(\tau) \int_0^\infty e^{-s(\xi+\tau)} f(\xi) d\xi d\tau \\ F(s)G(s) &= \int_0^\infty g(\tau) \int_\tau^\infty e^{-s(t)} f(t-\tau) dt d\tau \end{aligned}$$

If we interchange the integrals then we obtain:

$$F(s)G(s) = \int_0^\infty e^{-st} \int_0^t g(\tau) f(t-\tau) d\tau dt$$

This is the Laplace transform of the convolution of the functions $f(t)$ and $g(t)$.

In order to apply this result to the solution of initial value problems we consider the following model problem:

$$ay'' + by' + cy = f(t), \quad y(0) = 0, \quad y'(0) = 0$$

If we opt to solve this problem via the Laplace transform we will obtain:

$$as^2Y + bsY + cY = F(s)$$

$$Y = \frac{1}{as^2 + bs + c}F(s)$$

This will occur regardless of the choice of constants a, b, c and regardless of the choice of forcing function. By using our convolution formula for the product $G(s)F(s)$ we can invert this transform immediately as an integral.

Let's assume $\mathcal{L}^{-1}(G(s)) = g(t)$, then the solution of this problem will be given by the following convolution:

$$y(t) = \int_0^t g(t-\tau)f(\tau)d\tau$$

The function $G(s)$ is called the transfer function for the given differential operator and it depends only on the coefficients of the operator. The transfer function effectively translates the given forces into the natural motions of the system at hand. The function $g(t-\tau)$ is called the impulse response of the system because it gives the response of a unit impulse at the time τ. The function $g(t-\tau)$ solves the following initial value problem:

$$ag'' + bg' + cg = 0, \quad g(\tau) = 0, g'(\tau) = 1$$

Remark: We will see that the transfer function also satisfies a distributional problem involving an initial impulse.

5.4. Laplace Transforms and Distributions. In addition to handling piecewise defined forcing functions in an elegant manner, the Laplace transform can also solve impulsive forcing problems, where the applied force is not a function in the classical sense. In an impulsive forcing problem we can imagine a situation where the physical situation changes very rapidly, so rapidly in fact that the forcing is only non-zero for an instant. This mathematical model is an idealization that can be very useful in physical contexts. For example one might conduct ballistics experiments to determine the amount of energy stored in a bullet fired by a high powered rifle. One such experiment is conducted by firing the bullet into a very large, heavy block of material such as a tree trunk, or slab of ballistic gel, while said block is suspended from the ceiling as a pendulum. By measuring the mass of the tree trunk and its resultant displacement after being hit by the bullet, one can estimate the amount of energy stored in the bullet. In situations like this, one could reasonably neglect the actual impact and burrowing of the bullet into the tree trunk and treat the bullet impact as an impulse which introduced momentum into the combined tree/bullet pendulum.

To handle impulsive forces we introduce a very powerful analytic tool called the Dirac delta distribution. This distribution allows us to model ideal impulses and handle them in a very elegant way using the Laplace transform technique. Before we begin it is important to stress that the Dirac delta distribution is not a function in the classical sense, but it may be approximated by classical functions.

We will take the following as our definition of the Dirac delta distribution:

$$\delta(t-c) = \lim_{\tau \to 0} d_\tau(t-c)$$

where $d_\tau(t-c)$ are piecewise constant function defined below:

$$d_\tau(t-c) = \begin{cases} \frac{1}{2\tau} & |t-c| \leq \tau \\ 0 & |t-c| > \tau \end{cases}$$

Remark: Proper use of this definition requires careful timing of the limit operation. If one takes the limit of this sequence of functions in isolation, the limit does not exist in the classical sense, we find that for each $t_o \neq c$, $\delta(t_o - c) = 0$, but $\delta(0)$ is undefined. However, if one tries to integrate this sequence of functions first then something sensible generally comes out. The rule of thumb is that if you are not sure how the Dirac delta distribution behaves, you can replace it with d_τ simplify and work through any operations which are present and then take the limit as $\tau \to 0$. The result of this computation will be identical to the result involving the δ distribution.

Remark: Sometimes we might write an equation involving the δ distribution and treat it algebraically as if it were a regular function. Such formal computations only make sense if at some point we integrate the result. Without an integration operation, the δ distribution doesn't make classical sense and cannot be used as a traditional function.

Now let's look at some of the amazing formal properties of the delta distribution. First, the integral of the delta distribution is identically one if the region of integration includes c, and zero otherwise.

$$\int_{-\infty}^{\infty} \delta(t-c)dt = \lim_{\tau \to 0} \int_{-\infty}^{\infty} d_\tau(t-c)dt$$

Since d_τ is zero most of the time, we can adjust the region of integration to only capture the place where it is non-zero.

$$\int_{-\infty}^{\infty} \delta(t-c)dt = \lim_{\tau \to 0} \int_{c-\tau}^{c+\tau} d_\tau(t-c)dt = \lim_{\tau \to 0} \int_{c-\tau}^{c+\tau} \frac{1}{2\tau} dt$$

$$\int_{-\infty}^{\infty} \delta(t-c)dt = \frac{t}{2\tau}\Big|_{c-\tau}^{c+\tau}$$

$$\int_{-\infty}^{\infty} \delta(t-c)dt = \frac{c+\tau}{2\tau} - \frac{c-\tau}{2\tau} = 1$$

If the region of integration doesn't include c, then if we make τ sufficiently small the interval $(c-\tau, c+\tau)$ which lies in the region of integration becomes vanishingly small. The result is zero in that case.

The second amazing property of the delta distribution is it's ability to pick out a single value of a function. (This is the chief property we will exploit to help us with differential equations.) Suppose we consider integrating the delta distribution against a continuous function, $f(t)$

$$\int_{-\infty}^{\infty} \delta(t-c)f(t)dt = \lim_{\tau\to 0}\int_{-\infty}^{\infty} d_\tau(t-c)f(t)dt$$

Again we use the properties of d_τ to determine exactly what the delta distribution should do:

$$\int_{-\infty}^{\infty} \delta(t-c)f(t)dt = \lim_{\tau\to 0}\int_{c-\tau}^{c+\tau} \frac{f(t)}{2\tau}dt$$

Suppose $f(t)$ has an antiderivative given by $F(t)$, then we could complete the integration to obtain:

$$\int_{-\infty}^{\infty} \delta(t-c)f(t)dt = \lim_{\tau\to 0}\frac{F(c+\tau)-F(c-\tau)}{2\tau}$$

The result is exactly the centered difference definition of the derivative of $F(c)$, which by construction is our original function, f evaluated at $t=c$. This gives us the following elegant result:

$$\int_{-\infty}^{\infty} \delta(t-c)f(t)dt = f(c)$$

This property is absolutely stunning. Here is a strange distribution which acts like a magnifying glass that selects a **single** function value while ignoring all other values of the function.

At this point we know enough to deduce how the delta function would behave in an initial value problem if we use the method of Laplace transforms.

6. Application: Mechanical Vibrations

6.1. Linear Vibrations. In this section we develop the theory of linear vibrations. This application of constant coefficient ordinary differential equations is central to both classical physics and mechanical and electrical engineering. The concepts we discuss here will be re-visited and used in other applications, so you should try to master and internalize the ideas here.

We consider the one dimensional motion of a mass which is affixed to a linear spring. The governing equation for linear vibrations often appears as a leading order approximation

for the dynamics of physical systems, we will see it when we study the two body problem, and again when we analyze the motion of the simple pendulum. Our goal in studying this equation will be to understand the motion of this mass, and to be able to predict the position of the mass at any later time when the position and velocity are known at an initial time.

In order to derive the differential equation for linear vibrations, we apply Newton's second law. We set the acceleration proportional to the total force applied, with the mass as the constant of proportionality.

$$m\frac{d^2y}{dt^2} = \sum F$$

(We have replaced the acceleration with the second time derivative of the displacement, y in anticipation of the desire to predict the displacement of the body at any point in time.)

If we imagine the body hanging vertically from a linear spring, then two forces will act to influence the position of the body. The gravitational force and the restoring force of the spring:

$$\sum F = F_g + F_s$$

We will assume that the gravitational force is a constant and the restoring force is proportional to the displacement, but opposite in direction. The constant of proportionality for the restoring force represents the 'strength' of the spring. (This is simply assuming the spring obeys Hooke's Law)

$$\sum F = -mg - ky$$

Replacing these two forces in Newton's second law we are left with a second order, linear ordinary differential equation which describes the motion of this body.

$$m\frac{d^2y}{dt^2} = -mg - ky$$

In writing the equation in this way we have defined the displacement y as the displacement from the rest position of the spring, but it turns out that by a different choice of variables we can work with a simpler, but equivalent model equation. Rather than measuring our displacement from the equilibrium position of the spring, we could measure the displacement from the rest position of the hanging body.

To accomplish this change of variables to $\hat{y}$ we only need to shift our measurement coordinate by a constant:

$$\hat{y} = y + \frac{mg}{k} \text{ or } y = \hat{y} - \frac{mg}{k}$$

If we substitute this change of variables we find:

$$m\frac{d^2}{dt^2}\left[\hat{y} - \frac{mg}{k}\right] = -mg - k\left(\hat{y} - \frac{mg}{k}\right)$$

Simplifying and using the fact that m, g and k are assumed constant we obtain:

$$m\frac{d^2\hat{y}}{dt^2} = -k\hat{y}$$

From this point on, we will work with this reduced model, understanding that our displacement is measured from the rest position of the body, not from the equilibrium position of the spring. Will will also drop the 'hat' for brevity.

We know from our previous study of second order linear ODE's that in order to have a unique solution this differential equation should be coupled to two initial conditions which physically represent the position and velocity of the body at some point in time.

Grouping all unknown terms on one side of the equation we arrive at:

$$my'' + ky = 0, \quad y(0) = \alpha, \quad y'(0) = \beta,$$

as the governing model for a hanging mass undergoing free vibrations due to a linear spring.

Solving this model in general we can assume an exponential solution and arrive at the purely complex roots:

$$mr^2 + k = 0, \quad r = \pm\sqrt{\frac{k}{m}}i$$

This leads us to a general solution of sines and cosines:

$$y_{gen} = C_1 \cos\left(\sqrt{\frac{k}{m}}t\right) + C_2 \sin\left(\sqrt{\frac{k}{m}}t\right)$$

Which leads us to the observation that masses on springs have a tendency to bounce up and down. With this general solution we can further see how changing the physical situation will impact the behavior of the model. The parameters k and m both have an influence on the frequency of oscillation of the spring mass system.

At this stage we should look at our physical experience and try to be a little critical of our model. If we dangle a body on a spring and give it a good solid yank, then the body will definitely bounce up and down for a while, but the size of each bounce will decrease with time. This model predicts that the body bounces up and down over and over for all time. The difference between our physical experience and our model's predictions should cause us some degree of unease. If our model doesn't behave like the physical world, then we must be missing some critical piece of information

6.2. Damped Linear Oscillations. In order to capture the physical behavior of gradually decaying oscillations we introduce another force to our problem. We introduce a damping force whose effects are intended to slow or weaken the oscillations of the body on the spring. (**Remark:** The actual form of the damping force that we use is chosen for mathematical reasons, not based upon physical considerations.)

For damped oscillations we consider the influence of two forces on the motion of the body on the spring:

$$\sum F = F_s + F_d,$$

we continue to use Hooke's Law to define our restoring force, and we adopt a linear drag law to define our damping force. Qualitatively we want the damping force to slow down the motion of the body, but the damping force should disappear when the object is stationary. The simplest damping force we can use that will have this qualitative behavior will be the following:

$$F_d = -\gamma \frac{dy}{dt}$$

Here the parameter, $\gamma > 0$, is a damping coefficient which regulates the 'strength' of the damping force. The velocity is used so that the damping force is slows the body down when it is in motion, but disappears when the object is motionless. (**Remark**: Negative choices of γ are possible, but generally non-physical these types of models are used to display certain types of self-excitation behaviors in mechanical systems, but usually aren't of more general use. Qualitatively negative choices of γ behave like 'flubber' the fictional material used in a series of old disney movies.)

This form of the damping force yields the general model for Damped Linear Vibrations:

$$my'' + \gamma y' + ky = 0, \qquad y(0) = \alpha, y'(0) = \beta$$

Solving this model in general yields a variety of different behaviors depending upon the relative sizes of the different parameters. If we assume an exponential solution to this model then the characteristic polynomial will have roots given by the quadratic formula:

$$r = \frac{-\gamma \pm \sqrt{\gamma^2 - 4mk}}{2m}$$

Let's ask the question, how does the behavior of the solution depend on the roots of this polynomial? We know the roots can be real or imaginary, and that those cases give us different functions for the general solution. The roots of this quadratic will switch between being imaginary and real when the discriminant $\gamma^2 - 4mk$ passes through zero.

If the damping is small (i.e. the discriminant is negative: $\gamma^2 - 4mk < 0$), then the roots will have complex components and the solution will have the general form:

$$y_{gen} = C_1 e^{\lambda t}\cos(\mu t) + C_1 e^{\lambda t}\sin(\mu t)$$
$$\text{with } \lambda = \frac{-\gamma}{2m}, \text{ and } \mu = \frac{\sqrt{\gamma^2 - 4mk}}{2m}.$$

This means that the solution will have an oscillatory character, but the oscillations will decrease in amplitude over time. This situation is called **under-damped**

If the damping is large (i.e. the discriminant is positive: $\gamma^2 - 4mk > 0$), then the roots will be real and distinct and the solution will have an exponentially decaying character:

$$y_{gen} = C_1 e^{r_1 t} + C_2 e^{r_2 t}$$

with both r_1 and r_2 negative. This situation is called over-damped.

When the discriminant is exactly equal to zero the situation is called "**critically - damped**." Qualitatively the behavior is basically the same as the over-damped case, the repeated root gives rise to a different general solution, but the solutions still decay exponentially. (**Remark:** The choice of terminology here refers to the critical point at which the behavior changes, it does **not** mean that the damping is at a 'critical' level. If all parameters except damping are held constant, an over-damped system has more damping than a critically damped system.)

6.3. Forced Linear Oscillations. Now that we have looked at the governing models for both free and damped linear oscillations, we can consider the more vibration problem:

$$my'' + \gamma y' + ky = F(t)$$

In this problem we have added an external force, $F(t)$ which influences the motion of the body on the spring. (Ig γ is set equal to zero, the model is reduced to free vibrations.)

We now ask ourselves how the motion of the body on the spring can be influenced by imparting and additional external force to the body. We will focus on the analogy of a person on a swing (to leading order a person on a swing is much like a pendulum, which is much like a linear spring for small displacements).

When we try to play on a swing set as children we learn to kick our feet in time with the motion of the swing, and we soon discover that the swing behaves differently when we kick our feet at different times. If we kick our feet in time with the motion of the swing we are able to build up momentum and increase the amplitude of our swinging. If we struggle and kick our feet wildly, however, the swing usually doesn't get any higher, and it might even slow down. We can see the same type of behavior with Free vibration if we study what happens with periodic forcing functions.

CHAPTER 3

Variable Coefficient Linear Equations of arbitrary order

As we pass from constant coefficent equations to variable coefficient equations, the subject becomes much more difficult. This difficulty comes not from the underlying mathematics, but because the solutions do not generally possess compact or familiar representations. Even in situations where the coefficients are polynomial, solutions can seldom be written in terms of finite linear combinations of elementary functions. Thus to deal with such equations we must be willing to seek solutions in terms of more complicated representations. The classical method for dealing with variable coefficient equations analytically is to employ power series representations. These representations are convenient computationally as they allow many problems to be broken down into an ordered search for the coefficients of the power series. By appealing to a power series representation one transforms the solution into a series of known structure, and it reduces the problem of finding an unknown function to a problem of finding a sequence of coefficients.

The first theoretical questions which arises from such an approach is the question of validity. If we are to assume a solution possesses a power series representation, what conditions would ensure that such a representation actually exists. If we can establish these conditions, then we will know that such an approach is valid and any resulting series can be trusted to converge to an actual solution to the problem.

When dealing with linear equations, the question of the existence of a power series representation is usually split into two distinct cases, the case of **ordinary** points, and the case of **singular** points. We will define these situations and briefly outline sufficient conditions under which power series representations will be guaranteed to exist.

We consider a homogeneous, second order linear equation

$$P(t)y'' + Q(t)y' + R(t)y = 0$$

If the coefficients P, Q and R are analytic at t_0 (i.e. each function possesses a power series representation with at least a positive radius of convergence) and $P(t_0) \neq 0$, then the equation may be formally solved for the highest order term:

$$\frac{d^2y}{dt^2} = -\frac{Q}{P}\frac{dy}{dt} - \frac{R}{P}y$$

RemarkYou can generalize these results by isolating the highest order term in higher order equations

Since $P(t_0) \neq 0$ one can construct power series representations for $\frac{Q}{P}$ and $\frac{R}{P}$ which will have radii of convergence no smaller than the smallest of P, Q, R, and by employing repeated differentiation this equation and two initial conditions will provide a unique value for each $\frac{d^n y}{dt^n}|_{t_0}$. This series must have a radius of convergence no smaller than the smallest of P, Q, R. Points, t_0 when $P(t_0) \neq 0$ are called **ordinary** points. At ordinary points, we see behavior similar to what we would expect, the solution is as nice as the coefficients in the equation.

Points where $P(t_0) = 0$ are termed singular points. Here the differential equation itself changes order, and may give rise to unbounded solutions or other unpleasantness. When things work out, the singular point is called **regular** while when things give rise to problems (unbounded solutions and the like) the points are termed **irregular**.

Next we give an abstract outline of calculations necessary to determine the coefficients of a power series representation about a regular point.

To simplify the computations somewhat, we assume that we have a second order equation of the form:

$$y'' + q(t)y' + r(t)y = 0$$

and we introduce power series representations of the coefficient functions (In practice these are often simply polynomials so this outline will be more computationally intense than most of the problems your would be likely to face in coursework).

$$q(t) = \sum_{n=0}^{\infty} q_n (t - t_0)^n, \qquad r(t) = \sum_{n=0}^{\infty} r_n (t - t_0)^n$$

With these pieces in hand we assume that the solution possesses a power series representation about t_0 and we formally differentiate term by term (assuming that we stay within the radius of convergence of the series). We then substitute into the equation, and create an equation for the coefficient of each power of $(t - t_0)$.

$$y'' = \sum_{n=2}^{\infty} n(n-1)a_n (t - t_0)^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}(t - t_0)^n$$

$$y' = \sum_{n=1}^{\infty} n a_n (t - t_0)^{n-1} = \sum_{n=0}^{\infty} (n+1)a_{n+1}(t - t_0)^n$$

Above we have shifted the index of summation to simplify the process of grouping by powers.

$$\sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}(t - t_0)^n + \left[\sum_{n=0}^{\infty} q_n (t - t_0)^n\right]\left[\sum_{n=0}^{\infty} (n+1)a_{n+1}(t - t_0)^n\right] + \\ + \left[\sum_{n=0}^{\infty} r_n (t - t_0)^n\right]\left[\sum_{n=0}^{\infty} a_n (t - t_0)^n\right] = 0$$

From this we can now extract the coefficient equations starting with $n = 0$. In general we expect initial conditions will determine a_0 and a_1, so the equations are used to define a_n with $n \geq 2$. One can construct two linearly independent solutions to the problem by extracting the dependence of the coefficients on a_0 and a_1 separately. We proceed by formally multiplying out the power series and grouping by powers as explained earlier.

$$\begin{aligned} n &= 0 \\ 2a_2 + q_0a_1 + r_0a_0 &= 0 \\ n &= 1 \\ 3 \cdot 2 \cdot a_3 + 2q_0a_2 + +q_1a_1 + r_0a_1 + r_1a_0 &= 0 \\ n &= 2 \\ 4 \cdot 3 \cdot a_4 + 3q_0 \cdot a_3 + 2q_1a_2 + q_2a_1 + r_0a_2 + r_1a_1 + r_2a_0 &= 0 \end{aligned}$$

The patterns continue to grow as the sequence of coefficients expands. In **very** simple situations one may be able to deduce a closed form formula for the coefficients of the solution, however this is not to be expected in general. Taking these first few equations, we solve for the coefficents and extract the first few terms of each linearly independent solution:

Remark: I would recommend employing a computer algebra system to facilitate calculations such as these calculating anything beyond a_3 accurately by hand would not be an efficient use of time.

$$\begin{aligned} a_2 &= \frac{-q_0a_1}{2} - \frac{r_0a_0}{2} \\ a_3 &= -\frac{1}{3}q_0\left(-\frac{1}{2}q_0a_1 - \frac{1}{2}r_0a_0\right) - \frac{1}{6}q_1a_1 - \frac{1}{6}r_0a_1 - \frac{1}{6}r_1a_0 \\ a_4 &= -\frac{1}{12}r_0\left(-\frac{1}{2}q_0a_1 - \frac{1}{2}r_0a_0\right) - \frac{1}{4}q_0\left(-\frac{1}{3}q_0\left(-\frac{1}{2}q_0a_1 - \frac{1}{2}r_0a_0\right) - \frac{1}{6}q_1a_1 - \frac{1}{6}r_0a_1 - \frac{1}{6}r_1a_0\right) \\ &\quad - \frac{1}{6}q_1(-\frac{1}{2} - q_0a_1 - \frac{1}{2}r_0a_0) - \frac{1}{12}q_2a_1 - \frac{1}{12}r_1a_1 - \frac{1}{12}r_2a_0 \end{aligned}$$

Finally to isolate the first few terms of each linearly independent solution we group each of the coefficients by dependence on either a_0 or a_1. This gives us the following generic result:

$$\begin{aligned} a_2 &= -\frac{r_0a_0}{2} - \frac{q_0a_1}{2}, \qquad a_3 = \frac{q_0r_0 - r_1}{6}a_0 + \frac{q_0^2 - q_0 - r_0}{6}a_1 \\ a_4 &= \left(\frac{r_0^2 + 2q_1r_0 + q_0r_1 - q_0^2r_0 - 2r_2}{24}\right)a_0 + \left(\frac{2q_0r_0 - q_0^3 + 4q_1q_0 - 2r_1 - 2q_2}{24}\right)a_1 \end{aligned}$$

Thus we extract the first few terms of each solution as:

$$\begin{aligned} y_a &= a_0\left[1 - \frac{r_0}{2}(t-t_0)^2 + \frac{q_0r_0 - r_1}{6}(t-t_0)^3 + \left(\frac{r_0^2 + 2q_1r_0 + q_0r_1 - q_0^2r_0 - 2r_2}{24}\right)(t-t_0)^4 + \dots\right] \\ y_b &= a_1\left[(t-t_0) - \frac{q_0}{2}(t-t_0)^2 - \frac{q_0^2 - q_0 - r_0}{6}(t-t_0)^3 + \left(\frac{2q_0r_0 - q_0^3 + 4q_1q_0 - 2r_1 - 2q_2}{24}\right)(t-t_0)^4 + \dots\right] \end{aligned}$$

We note that upon substitution directly into the original differential equation these solutions leave a remainder of $O(t-t_0)^3)$, as we should expect since the coefficients were chosen to balance the constant, linear and quadratic terms within the equation.

1. Euler Equations

In order to develop the theory of regular singular points one develops a full set of solutions for Euler equations, then generalizes the technique used to solve these equations to more complex equations. An Euler equation is a linear, homogeneous equation where the coefficients are monomials with integer powers which match the order of differentiation. The model second order Euler equation is given by:

$$x^2y'' + axy' + by = 0.$$

This type of equation possesses a regular singular point at $x = 0$. This point is termed regular since it is always possible to describe 2 linearly independent solutions which are defined in a neighborhood with the singular point at the boundary. We note that is not generally possible to satisfy arbitrary initial conditions if they are imposed at the singular point, however we will be able to satisfy initial conditions away from the singularity. Because differentiation of a power of x reduces the power by exactly one, the form of the equation suggests that a monomial guess might be productive. Upon the substitution $y = Ax^r$, and imposing the restriction $x > 0$ we arrive at a polynomial equation which is termed the indicial equation.

$$Ax^r\left[r(r-1) + ar + b\right] = 0$$
$$r^2 + (a-1)r + b = 0$$

This equation plays essentially the same role as the characteristic equation for constant coefficient linear problems.

Remark: Since roots of this equation are potentially negative, we can see the appearance of functions which are unbounded as $x \to 0^+$.

We consider a small menagerie of example problems to explore the possible solutions:
Example 1:

$$x^2y'' - 2xy' + 2y$$

This equation has the indicial equation:

$$r^2 - 3r + 2 = 0$$

Here we see 2 distinct roots, which means our guess has provided 2 distinct possible solutions. We take the general solution to be:

$$y = \frac{C_1}{x} + \frac{C_2}{x^2}$$

Example 2:

$$x^2y'' + 5xy' + 4y = 0$$

Here the indicial equation has repeated roots, so a procedure like reduction of order must be employed to obtain a second solution:

$$r^2 + 4r + 4 = 0$$

Upon substitution the reduction of order guess $y_2 = u(x) \cdot y_1(x)$, yields the following differential equation:

$$\begin{aligned} x^2\left[u''y_1 + 2u'y_1' + uy_1''\right] + 5x\left[u'y_1 + uy_1'\right] + 4uy_1 &= 0 \\ x^2\left[u''y_1 + 2u'y_1'\right] + 5x\left[u'y_1\right] &= 0 \\ x^2\left[u''\frac{1}{x^2} + 2u'\frac{-2}{x^3}\right] + 5x\left[u'\frac{1}{x^2}\right] &= 0 \\ u'' + \frac{1}{x}u' &= 0 \end{aligned}$$

We can solve this equation easily by the use of an integrating factor and we obtain:

$$\begin{aligned} \frac{d}{dx}\left[xu'\right] &= 0 \\ xu' &= C \\ u &= \int \frac{C}{x}dx \\ u &= C\ln(x) + C_2 \end{aligned}$$

Example 3: Finally we provide an example where the indicial equation possesses complex roots:

$$x^2y'' + 5xy' + 5y = 0$$

Here we obtain the indicial equation:

$$r^2 + 4r + 5 = 0$$

which possesses the complex conjugate roots: $r = -2 \pm i$. From here we define $x^{i\mu} = e^{i}\mu\ln(x)$ and by taking appropriate linear combinations of our solutions we can obtain the real valued solution set:

$$y = C_1x^{-2}\cos(\ln(x)) + C_2x^{-2}\sin(\ln(x))$$

CHAPTER 4

Systems of Linear Equations

While a single ordinary differential equation can model the evolution of a single continuously changing quantity, a system of differential equations allows one to monitor multiple quantities whose value and evolution may be linked. In order to deal with systems of differential equations, direct integration is usually not feasible, so one must either find a technique for decoupling the equations of the system or approximate the solutions using a suitable numerical technique. In this chapter we will deal only with constant coefficient systems of linear equations, but we will be drawing from many of the techniques and ideas presented in previous chapters.

1. Conversion to First Order Systems, and Matrix Notation

In order to level the playing field for our analysis, we will show how any high order differential equation may be converted into an equivalent system of first order equations. Once this reformulation is complete it will be sufficient to deal only with systems of first order equations since this conversion can be performed on any collection of ordinary differential equations without a loss of generality.

We assume that we start with a single nth order ordinary differential equation.

$$\frac{d^n y}{dt^n} = f\left(\frac{d^{n-1}y}{dt^{n-1}}, \ldots, \frac{d^2 y}{dt^2}, \frac{dy}{dt}, y\right)$$

We introduce new variables for every lower order derivatives of y, so:

$$a_0 = y, \qquad a_1 = \frac{dy}{dt}, \ldots, a_{n-1} = \frac{d^{n-1}y}{dt^{n-1}}$$

We then replace each instance of the lower order derivatives with the new variables in the original equation and the definitions we just created. We end up with a system of n ordinary differential equations for the functions $a_0, \ldots a_{n-1}$.

$$\begin{aligned}
\frac{da_0}{dt} &= a_1 \\
\frac{da_1}{dt} &= a_2 \\
\frac{da_2}{dt} &= a_3 \\
&\vdots \\
\frac{da_{n-2}}{dt} &= a_{n-3} \\
\frac{d^n a_{n-1}}{dt^n} &= f\left(a_{n-1}, a_{n-2} \ldots, a_2, a_1, a_0\right)
\end{aligned}$$

as an explicit example of this process consider the following third order ordinary differential equation:

$$\frac{d^3y}{dt^3} = \frac{d^2y}{dt^2} + 4y$$

Since this is a third order equation, we introduce variables for the unknown and its first and second derivatives:

$$a_0 \equiv y, \qquad a_1 = \frac{dy}{dt} \qquad a_2 = \frac{d^2}{dt^2}$$

Taking these definitions and substituting into the original equation:

$$\frac{da_2}{dt} = a_2 + 4a_0$$

Putting all of the differential equations together we get a system of three first order ordinary differential equations:

$$\begin{aligned}
\frac{da_0}{dt} &= a_1 \\
\frac{da_1}{dt} &= a_2 \\
\frac{da_2}{dt} &= a_2 + 4a_0
\end{aligned}$$

When dealing with systems of differential equations, especially linear ones, it is often efficient to adopt matrix notation:

$$\frac{d}{dt}\begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 4 & 0 & 1 \end{pmatrix}\begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix}$$

The entries in the matrix are the coefficients for the unknowns functions. One selects a vector for the unknown functions and uses that vector structure whenever possible. This

makes it possible to use short hand notation to represent the equation:

$$\frac{d}{dt}\mathbf{a} = M\mathbf{a}$$

2. Solution Techniques for Constant Coefficient Systems

If we are faced with a constant coefficient homogeneous system of first order equations:

$$\frac{d}{dt}\mathbf{x} = A\mathbf{x},$$

we can approach this system using the same educated guesses that were helpful for solving individual constant coefficient equations. By adopting an exponential guess and substituting into this equation we can convert this system of equations into a pure linear algebra problem (assuming A is a constant valued matrix).

$$\mathbf{x} = \begin{pmatrix} \xi_1 e^{\lambda t} \\ \xi_2 e^{\lambda t} \\ \vdots \\ \xi_n e^{\lambda t} \end{pmatrix} = \begin{pmatrix} \xi_1 \\ \xi_2 \\ \vdots \\ \xi_n \end{pmatrix} e^{\lambda t}$$

Plugging this anzatz into out model equation we obtain:

$$\begin{aligned} \frac{d}{dt}\left[\boldsymbol{\xi} e^{\lambda t}\right] &= A\boldsymbol{\xi} e^{\lambda t}, \\ \lambda \boldsymbol{\xi} e^{\lambda t} &= A\boldsymbol{\xi} e^{\lambda t} \\ \lambda \boldsymbol{\xi} &= A\boldsymbol{\xi} \end{aligned}$$

By dividing out the appearance of the exponential function we end up with an equation which should be familiar to anyone who has studied linear algebra. This is the eigenvalue problem. Essentially, when A is a constant matrix, the exponential guess allows us to remove the time dependence from the system of differential equations, leaving only an algebraic system. The resulting algebra problem is more difficult than just solving an algebraic linear system, but it is a well known problem and the theory surrounding the eigenvalue problem will provide us with a great deal of inspiration for techniques to handle systems of linear ODEs.

Before proceeding any further we include a 'brief' overview of the solution to the eigenvalue problem. The eigenvalue problem is covered in any decent linear algebra text, so we only include the absolute essentials.

Given a square, constant valued matrix, A the eigenvalue problem consists of finding all scalars, λ termed the eigenvalues, and all corresponding vectors $\boldsymbol{\xi}$ which satisfy the following equation:

$$A\boldsymbol{\xi} = \lambda\boldsymbol{\xi}.$$

In an operator theoretic sense we are seeking to discover any situations where the transformation induced by A has a particularly simple effect on the input vector $\boldsymbol{\xi}$. The eigenvectors are the inputs which retain their direction after being acted on by A.

We solve this problem in two stages: First we find the eigenvalues, and second we find the corresponding eigenvectors. The eigenvalues are found by manipulating the governing system and appealing to some important results in theoretical linear algebra.

We bring all terms to the same side of the equation and attempt to 'factor out' the eigenvector.

$$A\boldsymbol{\xi} - \lambda\boldsymbol{\xi} = \mathbf{0}.$$

Since subtraction between matrices and scalars is not well defined we must introduce a matrix placeholder in order to draw out the common factor of $\boldsymbol{\xi}$ in the equation. The identity matrix plays the role of this placeholder.

$$A\boldsymbol{\xi} - \lambda I\boldsymbol{\xi} = \mathbf{0},$$
$$(A - \lambda I)\,\boldsymbol{\xi} = \mathbf{0}.$$

In this form we seek vectors which lie in the null space of the matrix $A - \lambda I$. Solvability theory for linear algebra tells us that this problem only has interesting (non-zero) solutions when $A - \lambda I$ is singular. This condition is equivalent to the computational condition:

$$det\,(A - \lambda I) = 0$$

where det is the determinant function. This result is helpful because this determinant equation contains only the unknown scalar, λ and in practice is a polynomial equation which can be solved numerically or algebraically in simple cases.

Once this polynomial equation is solved, each real root will give rise to at least one eigenvector which can be found by using traditional gaussian elimination and a judicious selection of degrees of freedom. To find the eigenvectors take each each eigenvalue and solve the linear system :

$$(A - \lambda_i I)\,\boldsymbol{\xi} = \mathbf{0}.$$

This concludes our outline for solving the eigenvalue problem. Refer to a linear algebra text to see a more detailed overview on special situations such as repeated or complex eigenvalues.

If you are able to fully solve the eigenvalue for the matrix A, then exploiting the linear structure of the system you can construct a general solution by substituting all eigenvalue eigenvector pairs into the solution anzatz and forming a linear combination.

$$\mathbf{x}_{gen} = C_1\boldsymbol{\xi}_1 e^{\lambda_1 t} + C_2\boldsymbol{\xi}_2 e^{\lambda_2 t} + \cdots + C_n\boldsymbol{\xi}_n e^{\lambda_n t}$$

If you are given an initial condition of the form $\mathbf{x}(t_0) = \mathbf{b}$ simply substitute t_0 into the general solution and solve the resulting linear system for the coefficients $C_1 \dots C_n$.

In the practice section we include several examples of different cases that can arise when solving the eigenvalue problem, these include repeated eigenvalues with a defective matrix, and complex eigenvalues.

3. Non-Homogeneous systems and Diagonalization

If one is faced with a non-homogeneous system of constant coefficient ordinary differential equations, then one needs more than just an exponential guess to solve the problem. In this section we explore how to use matrix diagonalization techniques to solve systems of first order ordinary differential equations.

Our model problem is given by:

$$\frac{d}{dt}\mathbf{x}(t) = A\mathbf{x}(t) + \mathbf{f}(t)$$

where A is a constant valued matrix, and $\mathbf{f}(t)$ is a given, smooth vector valued-function of t. If we can solve the eigenvalue problem for A, then we can convert this problem to an equivalent problem involving de-coupled first order ordinary diffrential equations. We include a brief outline of the diagonalization procedure.

Assume the $n \times n$matrix A possesses n linearly independent eigenvectors (a sufficient condition for this would be that A has n distinct eigenvalues). Each of these eigenvectors satisfies an equation of the form:

$$A\mathbf{e}_i = \lambda_i \mathbf{e}_i$$

Form a matrix E using these eigenvectors as it's columns. Now if we perform the matrix multiplication AE, the result has a great deal of structure:

$$AE = \begin{pmatrix} \vdots & \vdots & \vdots & \dots \\ \lambda_1\mathbf{e}_1 & \lambda_2\mathbf{e}_2 & \lambda_3\mathbf{e}_3 & \dots \\ \vdots & \vdots & \vdots & \dots \end{pmatrix}$$

Thus the result is E with it's columns scaled by the respective eigenvalues. By manipulating this matrix further we can extract the matrix E.

$$AE = \begin{pmatrix} \vdots & \vdots & \vdots & \dots \\ \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 & \dots \\ \vdots & \vdots & \vdots & \dots \end{pmatrix} \begin{pmatrix} \lambda_1 & 0 & 0 & \dots \\ 0 & \lambda_2 & 0 & \dots \\ 0 & 0 & \lambda_3 & \dots \end{pmatrix} = E\Lambda$$

By itself this is a bit of a curiosity, however it leads to the slightly more devious construction $E^{-1}AE$. combined with the above argument we find that if these three matrices are

grouped together, the result is a diagonal matrix with the eigenvalues along the diagonal.

This structure can be used to de-couple systems of ordinary differential equations. Once the decoupling procedure is performed 'any' technique for solving scalar equations may be applied to the newly decoupled equations. One then reconstructs the solutions to the original problem by taking an appropriate linear combination of the decoupled pieces. We demonstrate the technique on an abstract system of constant coefficient equations.

$$\frac{d}{dt}\mathbf{x}(t) = A\mathbf{x}(t) + \mathbf{f}(t)$$

To diagonalize this system of equations we must construct the matrix product $E^{-1}AE$ within the equation. We can do this by first left-multiplying both sides of the equation by E^{-1} and then inserting an identity matrix between A and $\mathbf{x}$. This is the matrix analog of multiplying and dividing by the same number to make a desired constant factor appear.

$$\begin{aligned}
E^{-1}\frac{d}{dt}\mathbf{x}(t) &= E^{-1}A\mathbf{x}(t) + E^{-1}\mathbf{f}(t),\\
E^{-1}\frac{d}{dt}\mathbf{x} &= E^{-1}AI\mathbf{x} + E^{-1}\mathbf{f},\\
E^{-1}\frac{d}{dt}\mathbf{x} &= E^{-1}A\left(EE^{-1}\right)\mathbf{x} + E^{-1}\mathbf{f},\\
\frac{d}{dt}E^{-1}\mathbf{x} &= \left(E^{-1}AE\right)E^{-1}\mathbf{x} + E^{-1}\mathbf{f}.
\end{aligned}$$

When we group the terms in this way, two helpful things happen. First $E^{-1}AE$ reduces to the diagonal matrix or eigenvalues. Second every appearance of $\mathbf{x}$ is preceded by the constant matrix E^{-1}, this suggests introducing a new variable $\mathbf{y} = E^{-1}\mathbf{x}$, giving the now decoupled system of equations:

$$\frac{d}{dt}\mathbf{y} = \Lambda\mathbf{y} + E^{-1}\mathbf{f}$$

Since each row in this system has the form:

$$\frac{dy_i}{dt} = \lambda y_i + g_i$$

you can pick from any of the viable techniques for solving first order linear constant coefficient equations including method of integrating factors, or the method of Laplace transforms.

4. Phase Plane Analysis for Linear Equations

In this section we will explore some qualitative aspects of systems of first order equations, our study here will be relatively abstract, but in the sections that follow we will show how linear systems can often be used to approximate and help understand and predict the behavior of more complex non-linear systems. Our examples and procedures will be specially geared towards systems of 2 first order equations since the behavior of solutions

to these systems can be viewed parametrically in the phase plane. The concepts we will discuss: equilibria and their stability, and linearization for approximating behaviors near equilibria generalize, but the phase portraits do not.

When you have a system of two ordinary differential equations, the phase plane is the two dimensional coordinate system generated by using the two different components of the solution as the coordinate axes. A curve in this plane is parametrically given by the independent variable (usually time in our applications) $(x_1(t), x_2(t))$. We draw a phase portrait for a system of ordinary differential equations by sketching a family of different solutions which capture the different behaviors that system is capable of exhibiting.

We begin by describing and plotting several representative phase portraits for homogeneous linear systems. Our examples will be extensive, but not completely exhaustive. We include procedures for constructing these phase portraits by hand, but in most cases a computer algebra system will be able to create them fast and with more precision than hand sketches.

Given a homogeneous system of two constant coefficient ordinary differential equations a reasonable phase portrait can be constructed using the eigenvalues and eigenvectors of the coefficient matrix. Different root configurations give rise to different behaviors and thus, different phase portraits.

We first concern ourselves with the case of distinct real roots, so we assume that the matrix A in the system:

$$\mathbf{x}' = A\mathbf{x}$$

Possesses two distinct, real eigenvalues. In this situation the general solution may be represented as some linear combination:

$$\mathbf{x} = C_1\boldsymbol{\xi}_1 e^{\lambda_1 t} + C_2\boldsymbol{\xi}_2 e^{\lambda_2 t}$$

If we wish to draw a representative solution set we must first decide how these functions will appear in the phase plane. Consider a single function of the form:

$$\boldsymbol{\eta} e^{\lambda t}.$$

This is a constant vector, multiplied by a scalar function of t. Since the vector is constant, the time dependence only affects the length of the vector and since the function is exponential, the length either grows without bound as $t \to \infty$ or shrinks to zero in the same limit. In the phase plane a function consisting of exactly one of these solutions (e.g. a solution where either $C_1 = 0$ or $C_2 = 0$) will appear as a straight line in the phase plane.

Thus the first step in constructing a phase portrait in the case of real roots is to draw straight lines through the origin in the directions of the eigenvectors. Once these straight lines are drawn we indicate with small arrows the direction of travel along these lines as

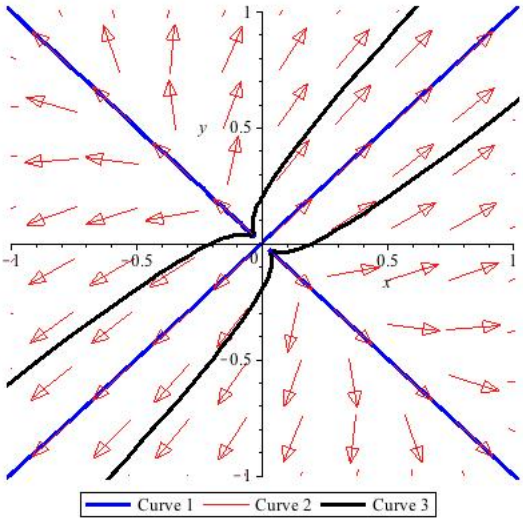


FIGURE 1. Two distinct positive roots

$t \to \infty$ (So outward going arrows for positive λ and inward going arrows for negative λ.)

Solutions constructed from both components of the general solution will be curves if the λ's are not equal, this follows from the fact that each point on the solution curve is a linear combination of two constant vectors, but the lengths of those vectors are changing at different rates. When drawing phase portrait qualitatively by hand, the most important feature to capture is the asymptotic shape of the curved solutions. When the roots are different in magnitude but share the same sign, the curves in each region between the straight line solution asymptotically approach the closest eigen-direction of the dominant (larger in magnitude) eigenvalue as the curve leaves the origin, and they asymptotically approach the sub-dominant eigen-direction as the curve nears the origin. Below are the computer generated phase portraits of the three simplest root configurations.

In these figures, the straight line eigensolutions are drawn in blue, the underlying slope field is drawn in red arrows, and intermediate solutions are drawn in black. The direction of the arrows indicates the direction along the solution as $t \to \infty$ In the case of complex eigenvalues, the periodic components of the solution (i.e. the cosines and sines which appear as a results of Euler's identity) induce spiraling in the phase plane. These spirals can be captured quite easily using the eigenvalues and the coefficient matrix itself.

To draw a qualitatively accurate phase portrait of a system with complex eigenvalues, one has to determine whether solutions are periodic, closed curves; stabilizing spirals which tend towards the origin, or unstable spirals which head away from the origin. ONe must also determine the direction of spiraling.

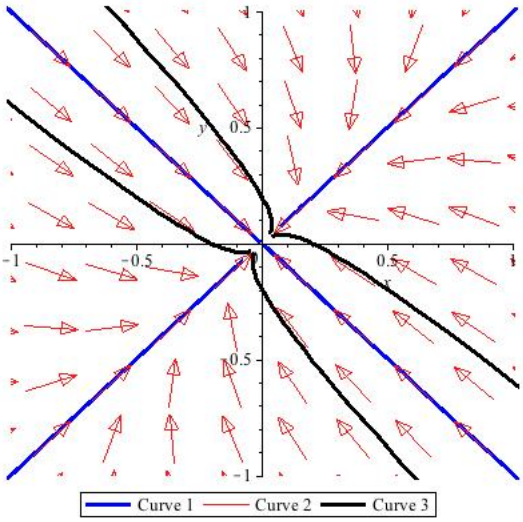


FIGURE 2. Two distinct negative roots

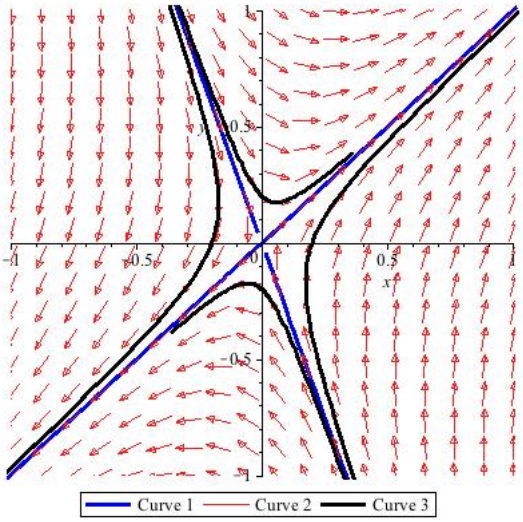


FIGURE 3. One positive, one negative root

If the eigenvalues of the coefficient matrix are given by $\lambda = \mu \pm \sigma i$, then the stability of the spirals is determined by μ. Positive μ give rise to spirals which grow and head away from the origin. Negative μ give rise to spirals that shrink and spin towards the origin, and $\mu = 0$ gives rise to periodic solutions which may look like circles or ellipses in the phase

plane.

To determine the direction of rotation of the spirals, one can effectively construct the slope at a few points, then fit the spirals to the direction indicated by those slopes. By multiplying the points $(1, 0)$ and $(0, 1)$ by the coefficient matrix one can find the slope of the solution at each of those points, by drawing a short arrow with the slopes at each point the direction of spiraling will be unambiguous.

CHAPTER 5

Regular Perturbation Theory for initial value problems

In this chapter we briefly outline how parametric series expansions may be used to develop approximations for problems which are somehow 'close' to linear problems. Our discussion in this chapter will be limited to variations on the one dimensional projectile problem. (i.e. dropping an object)

As a first example of a regular perturbation expansion we consider a problem which can be solved analytically. This will allow us to conduct a critical examination of our approximations by comparing them to the exact solution.

We consider the problem of predicting the velocity of an object dropped in a uniform gravitational field with the inclusion of a 'small' drag force. Here we will be focusing primarily on the mechanics of this approximation technique, to understand the theoretical technicalities present in its practical use we point the reader to the classic Introduction to Perturbation Methods by Mark Holmes. (Springer)

Example 1: Velocity of a Falling object with small drag.

The exact problem we will analyze is given by:

$$m\frac{dv}{dt} = -mg + \alpha v, \qquad v(0) = 0.$$

here m denotes the mass, g the local gravitational acceleration, and α an experimentally fitted drag coefficient indicating the 'strength' of the drag force.

In order to simplify the mathematical analysis we select a set of units customized to the problem and systematically divide these units out in a process called 'non-dimensionalization.'

We non-dimensionalize with the following scales:

$$t^* = \frac{t}{\tau}, L^* = \frac{L}{\tau^2 g}, \qquad v^* = \frac{v}{\tau g}.$$

Upon substitution, simplification and dropping the non-dimensional 'starring' we obtain the following dimensionless problem:

$$\frac{dv}{dt} = -1 + \epsilon v, \qquad v(0) = 0$$

where $\epsilon = \frac{\alpha\tau}{m}$, and we make the claim that $\epsilon << 1$ is a 'small' parameter. Once the problem is written in this form, we are ready to present and analyze the regular perturbation theory

Remark: This process allows us to make rigorous, scale invariant comparisons for different forces in the problem and help to create an objective measure of 'smallness' which will be exploited by the regular perturbation technique.

anzatz. A regular perturbation expansion is conducted in essentially three steps. One assumes the solution to a problem depends smoothly on a small problem parameter, by virtue of this assumption we suppose that this smooth dependence implies the existence of a series expansion which is valid for small values of the parameter. Here we would propose:

$$v = v_0(t) + \epsilon v_1(t) + \epsilon^2 v_2(t) + \dots$$

Once the series expansion is proposed, we substitute into the original problem, and split the problem into a **sequence** of problems which are to be solved successively to find the coefficient functions in the series expansion. For the problem at hand, we substitute our expansion into the differential equation and the initial condition:

$$\begin{aligned} \frac{d}{dt}\left[v_0 + \epsilon v_1 + \epsilon^2 v_2 + \dots\right] &= -1 + \epsilon\left(v_0 + \epsilon v_1 + \epsilon^2 v_2 \dots\right) \\ v_0(0) + \epsilon v_1(0) + \epsilon^2 v_2(0) &= 0 \end{aligned} \tag{2}$$

We then split the full equation into a sequence of distinct equations by making the following argument (informally at this point). If our series expansion is valid as $\epsilon \to 0$, and each of the coefficient functions is uniformly bounded (which would be true if these functions are continuous and we restrict t to a compact interval) then each of the different orders of epsilon must balance independently because as $\epsilon \to 0$ the different powers undergo an arbitrarily large scale separation.

For this problem we see the following equations at order 1, ϵ and ϵ^2.

$$\begin{aligned} O(1) \quad & v_0' = -1, \quad v_0(0) = 0, \\ O(\epsilon) \quad & v_1' = v_0, \quad v_1(0) = 0, \\ O(\epsilon^2) \quad & v_2' = v_1, \quad v_2(0) = 0, \end{aligned}$$

Note that these equations are all linear, and that they may be solved sequentially. Only v_0 is required to solve the equation for v_1, and only v_1 is required to solve the equation for v_2. The solutions to these problems are given by:

$$v_0 = -t, \quad v_1 = \frac{-t^2}{2}, \quad v_2 = \frac{-t^3}{3!}$$

Placing these functions back into the assumed series expansion we obtain:

$$v(t) \approx -t + \epsilon\frac{-t^2}{2} + \epsilon^2\frac{-t^3}{3!}$$

Because the original problem is both linear and constant coefficient, we can compare the solution here to the exact solution. Without proof note that the exact solution is:

$$v = \frac{1}{\epsilon} - \frac{1}{\epsilon}e^{\epsilon t},$$

The correspondence of the approximation to the solution becomes apparent if we use the Taylor series representation of the exponential function expanded around zero.

$$\frac{1}{\epsilon} - \frac{1}{\epsilon}\left[1 + \epsilon t + \frac{(\epsilon t)^2}{2} + \frac{(\epsilon t)^3}{3!} + \frac{(\epsilon t)^4}{4!} + \dots\right]$$

From here, we see the approximations are building a term by term construction of the Taylor expansion for the exact solution. To give some idea of the power of the technique we next solve a more complex variation of the same problem.

Example 1: Velocity of a Falling object with small quadratic drag. Unlike the previous problem in this example we solve a non-linear problem which is still analytically solvable, but to which regular perturbation theory provides an easier sequence of problems to solve. Using a similar scaling as the previous example and suitable altered definitions for α and ϵ we consider the following dimensionless problem:

$$\frac{dv}{dt} = -1 + \epsilon^2 v^2, \qquad v(0) = 0$$

We again assume the solution has a power series expansion in ϵ and we substitute a truncated version of that expansion into both the differential equation and the initial condition.

$$v(t) = v_0(t) + \epsilon v_1(t) + \epsilon^2 v_2(t) + \dots$$

we then substitute this series expansion into the differential equation and organize the result by requiring that each the coefficient for each term in the resulting series be identically zero.

Remark: The use of ϵ^2 is intended to implicitly demonstrate what happens when you assume a series expansion which has unnecessary terms.

$$v_0' + \epsilon v_1' + \epsilon^2 v_2' = -1 + \epsilon^2 \left(v_0 + \epsilon v_1 + \epsilon^2 v_2 + \dots\right)^2$$
$$v_0(0) = 0, \qquad v_1(0) = 0, \qquad v_2(0) = 0$$

The terms independent of ϵ give us the no-drag problem for velocity:

$$v_0' = -1, \qquad v_0(0) = 0,$$

we can solve this problem easily by integrating and using the initial condition to determine the constant of integration.

$$v_0(t) = -t$$

Next we turn to the terms which are $O(\epsilon)$ to find the following differential equation:

$$v_1' = 0, \qquad v_1(0) = 0, ,$$

This problem has the zero solution, so we proceed to the $O(\epsilon^2)$ problem.

$$v_2' = v_0^2$$

Since we already know v_0 we can solve this problem by integrating:

$$v_2' = t^2$$
$$v_2 = \frac{t^3}{3}$$

This gives us the approximate solution:

$$v \approx v_0 + \epsilon v_1 + \epsilon^2 v_2$$
$$v \approx -\frac{1}{2}t + 0 + \epsilon^2 \frac{t^3}{3}.$$

This is a good example for you because this approximation is not perfect, it has many drawbacks, fortunately we can solve this problem exactly and we can investigate how the approximation behaves. To enrich the comparision we will compute the approximation a little bit further.

At $O(\epsilon^3)$ we have:

$$v_0' + \epsilon v_1' + \epsilon^2 v_2' + \epsilon^3 v_3' + \epsilon^4 v_4' + \ldots = -g + \epsilon^2 \left(v_0 + \epsilon v_1 + \epsilon^2 v_2 + \ldots\right)^2$$
$$v_i(0) = 0,$$
$$\epsilon^3 v_3' = \epsilon^2(\epsilon v_0 v_1)$$

Since $v_1 \equiv 0$, we find $v_3 = 0$ as well.

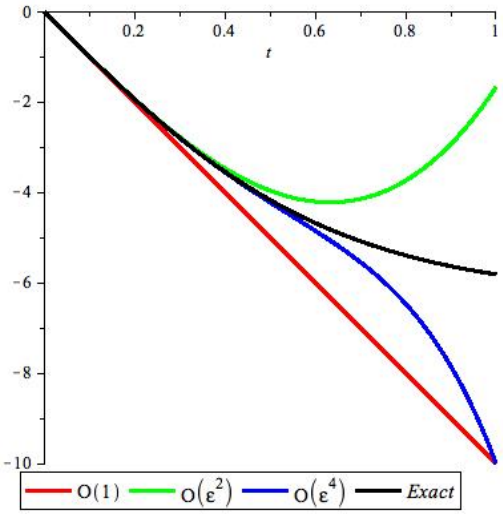
At $O(\epsilon^4)$ we have:

$$v_4' = 2v_0 v_2 + v_1^2$$
$$v_4' = 2(-t)(\frac{t^3}{3})$$
$$v_4 = -\frac{2}{15}t^5$$

This gives us the following approximation:

$$v \approx -\frac{g}{2}t + 0 + \epsilon^2 \frac{t^3}{3} - \epsilon^4 \frac{2}{15}t^5.$$

Comparing these approximations with the exact solution we have:



From this graph we see behavior similar to the previous example, but some salient points become important. First the approximation gets better as we include more terms, second the quality of the approximation is not uniform, the approximation is only good while the 'smallness' of epsilon dominates the growth of the polynomial terms in the coefficient functions.

Part 2

Initial Value Problems for Ordinary Differential Equations: Practice

CHAPTER 1

First Order Equations

When learning to solve differential equations, being able to distinguish between different classes and types of equation is probably the most critical factor. One must be able to accurately identify different kinds of mathematical structure in order to determine which techniques may be applicable. Before we present any worked examples, we present a short list of the kinds of structure one should look for when solving first order equations. Note that a particular equation may fall into several different categories, so more than one technique may be applicable. In order to be prepared for any eventuality you should learn to identify each of the structures given below, and learn to correctly apply each of the solution techniques

- **Explicit**

$$\frac{dy}{dt} = f(t)$$

  When one is faced with an explicit equation, direct integration is the simplest and most direct route to a solution.

- **Autonomous:**

$$\frac{dy}{dt} = f(y)$$

  An autonomous equation does not depend upon the independent variable (t in this case) explicitly, as such equations of this form are always separable, however other methods may also apply.

- **Separable:**

$$\frac{dy}{dt} = M(y)N(t)$$

  A separable equation is a more general structure than autonomy, but requires that one is able to algebraically isolate the dependence on the independent variable, t. If this structure is identified, then the method of separation may be applied.

- **Linear** Linear equations may appear in different forms, but they can all be algebraically manipulated into the canonical form:
$$\frac{dy}{dt} + p(t)y = g(t)$$
First order linear equations may be solved via the method of integrating factors.

- **Exact** An equation written in the form:
$$M(x,y)\frac{dy}{dx} + N(x,y) = 0$$
is exact whenever the functions M and N satisfy the additional relationship:
$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}.$$
Exact equations may be solved by recovering the potential function $\phi(x,y)$ which satisfies the differential equation $\nabla\phi = \langle M, N\rangle$.

1. Method of Separation

The method of separation is a formalization of the direct integration procedure applied to first order equations, the method requires that the equation be **separable**. This is defined in P 1, C 1, S 2.

Example #1

Consider the following equation:

$$\frac{dy}{dx} = xy$$

This is about the simplest separable equation you could imagine (in fact it's linear, you should use the method of integrating factors and make sure both methods yield the same general solution.)
In order to use the method of separation we need to identify $F = xy$ with component functions $M(y)$, and $N(x)$.

$$M(y) = y \qquad N(x) = x$$

These are clearly the only choices we have available. We now divide both sides of our equation by the appropriate $M(y)$, integrate and solve the resulting equation.

$$\begin{aligned} \frac{1}{y}\frac{dy}{dx} &= x \\ \int \frac{dy}{y} &= \int x dx \\ \ln|y| &= \frac{x^2}{2} + c \\ y &= ke^{\frac{x^2}{2}} \end{aligned}$$

(With an appropriately defined k from initial conditions.)

Example #2

Moving on to a more complicated example, consider the following:

$$\frac{dy}{dx} = \frac{y}{(x^2+1)(y+1)}$$

Thinking of the right hand side in terms of component functions:

$$\frac{dy}{dx} = \frac{y}{y+1}\frac{1}{x^2+1}$$

Now we divide to separate the y's and x's and proceed as before:

$$\frac{y+1}{y}\frac{dy}{dx} = \frac{1}{x^2+1}$$

$$\int \frac{y+1}{y}dy = \int \frac{1}{x^2+1}dx$$

We can split the integral on the left hand side using linearity, then each term is a perfect derivative. On the right-hand side we need to recognize that we have a perfect derivative of the $\arctan(x)$.

$$\int 1 + \frac{1}{y}dy = \arctan(x) + c$$

$$y + \ln|y| = \arctan(x) + c$$

At this stage we have solved the differential equation and we have obtained an equation which relates the x values to the y values, however, in this case the equation is *implicit*. We cannot solve the equation to isolate y. While this may be somewhat frustrating, it's a fact. In order to obtain information from an implicit equation such as this, one might use a computer algebra system to plot x as a function of y and back out desired information about the solution graphically.

2. Method of Integrating Factors

The method of integrating factors is applicable for first order, linear ordinary differential equation. The equation may be either constant or variable coefficient. Here we apply the method on several simple example problems.

Example 1
Consider the following initial value problem:

$$y' + ty = t, \qquad y(0) = 2$$

This equation is a first order, linear equation with variable coefficients (the t's), so we employ the method of integrating factors. First, we need to find the appropriate integrating factor. We recall from the derivation of this method that the integrating factor, μ is given by the following formula:

$$\mu = e^{\int p dt}$$

In our case, $p = t$ so we substitute and integrate to find μ.

$$\mu = e^{\int t dt} = e^{\frac{t^2}{2}}$$

We multiply the whole equation by this integrating factor, simplify and integrate:

$$\mu y' + \mu t y = \mu t$$
$$e^{\frac{t^2}{2}} y' + t e^{\frac{t^2}{2}} y = t e^{\frac{t^2}{2}}$$
$$\frac{d}{dt}\left[e^{\frac{t^2}{2}} y\right] = t e^{\frac{t^2}{2}}$$
$$\int \frac{d}{dt}\left[e^{\frac{t^2}{2}} y\right] dt = \int t e^{\frac{t^2}{2}} dt$$

On the left hand side the integral and derivative cancel eachother out, while the right hand side we actually have to integrate (you can use a u substitution or simply realize its the perfect derivative of μ).

$$e^{\frac{t^2}{2}} y = e^{\frac{t^2}{2}} + c$$
$$y(t) = \frac{e^{\frac{t^2}{2}} + c}{e^{\frac{t^2}{2}}}$$

Substitute our initial condition to obtain the final solution:

$$y(0) = 2 = \frac{1 + c}{1} \qquad c = 1$$

Thus our final specific solution is given by:

$$y(t) = \frac{e^{\frac{t^2}{2}} + 1}{e^{\frac{t^2}{2}}} = 1 + e^{\frac{-t^2}{2}}$$

Example #2

Consider the following initial value problem:

$$\frac{dy}{dt} + \frac{y}{t} = e^{-t}, \qquad y(1) = -2e^{-1}$$

We proceed as before, first we find the appropriate integrating factor:

$$\mu = e^{\int \frac{1}{t} dt} = e^{\ln(t)} = t$$

We multiply by the integrating factor, simplify and integrate.

$$\begin{aligned} t\frac{dy}{dt} + y &= te^{-t} \\ \frac{d}{dt}[ty] &= te^{-t} \\ \int \frac{d}{dt}[ty]dt &= \int te^{-t}dt \\ ty &= \int te^{-t}dt \end{aligned}$$

In order to integrate the right hand side, we must use the technique of integration by parts. (work omitted)

$$ty = -te^{-t} - e^{-t} + c$$

Solving for y, rearranging and substituting our initial condition yields the final specific solution.

$$y = -e^{-t} - \frac{e^{-t} + c}{t}$$

Substituting our initial condition yields $c = 0$, giving the final specific solution.

$$y = -e^{-t} - \frac{e^{-t}}{t}$$

Example #3

Occasionally you might need to manipulate the equation slightly in order to successfully apply the method of integrating factors.
Consider:

$$t^2y' - 2t^3y = t^3 \quad y(1) = 1$$

If we try to apply the method of integrating factors to this equation directly we will get everything wrong because we derived the method of integrating factors with an equation which did not have a coefficient function on the y' term (check equation (1)). Thus we must divide by the coeffcient of y' to get the equation into the proper form.

$$y' - 2ty = t$$

Now we can proceed as before.

$$\begin{aligned}
\mu &= e^{\int -2tdt} = e^{-t^2} \\
e^{-t^2}y' - 2te^{-t^2}y &= te^{-t^2} \\
\frac{d}{dt}[e^{-t^2}y] &= te^{-t^2} \\
\int \frac{d}{dt}[e^{-t^2}y]dt &= \int te^{-t^2}dt \\
e^{-t^2}y &= \int te^{-t^2}dt
\end{aligned}$$

The right hand side can be integrated by utilizing a simple u-substitution (work omitted)

$$e^{-t^2}y = -\frac{1}{2}e^{-t^2} + c$$

Solving for y and substituting the initial condition yields the following:

$$\begin{aligned}
y &= ce^{t^2} - \frac{1}{2} \\
y &= \frac{3}{2e}e^{t^2} - \frac{1}{2}
\end{aligned}$$

3. Modeling with First Order Equations

Below are more examples and exercises exploring different concepts and ideas which can be modeled with first order equations. We leave most of the work in this section as an exercise, but we try to provide extensive directions and explanations for any concepts or terms not explained explicitly in the theory section.

3.1. Proportionality with Bacterial Growth. This first problem gives a simple example of the application of proportional reasoning applied to simple population modeling. In these problems **A certain petri dish is filled with a population of bacteria which are observed to reproduce through mitotic division at a rate proportional to the number of bacteria present. The population size is observed to double in 4 hours. Assuming that initially a single bacterium is present and there is not predation or shortage of nutrients, predict the number of bacteria which will be present after a whole day has passed**

Remark: We leave the first part of the problem as an exercise, but include a discussion of how to use the 'doubling time' information to determine the rate constant.

The general solution to this problem is given by:

$$B(t) = B_0 e^{rt}$$

In order to determine the rate constant, r, we appeal to the information that the population doubles in four hours. If we have a population of B at time t, then we have a population of $2B$ at $t+4$ whenever t is measured in hours. Writing these conditions algebraically we have:

$$B = B_0 e^{rt}, \quad 2B = B_0 e^{r(t+4)}$$

Remark: The result here is indicative of a more general pattern. If you are given a doubling time, τ the rate constant is $\frac{\ln(2)}{\tau}$. If you are given a tripling time,θ the rate constant is $\frac{\ln(3)}{\theta}$. Alternately, if you are given a 'half-life,' τ then the rate constant is $-\frac{\ln(2)}{\tau}$.

Dividing one equation by the other and solving for r we obtain:

$$\begin{aligned} \frac{2B}{B} &= \frac{B_0 e^{r(t+4)}}{B_0 e^{rt}} \\ 2 &= \frac{e^{r(t+4)}}{e^{rt}} \\ 2 &= e^{4r} \\ r &= \frac{\ln(2)}{4} \end{aligned}$$

3.2. Proportionality with Radiocarbon dating. This problem explores the use of radioactive trace elements to find the age of certain archeological sites. The decay of radioactive elements is generally assumed to happen at a rate proportional to the amount of radioactive material present. This steady decay means that by knowing the constant rate of decay, and having some estimate about the amount of radioactive material present in a particular location, we can make predictions about the time elapsed since that collection of radioactive material was deposited. This idea can be applied to certain organic materials in certain circumstances to help estimate the age. Specifically Carbon-14 is a radioactive isotope of carbon which is produced in the atmosphere and continually ingested in trace quantities by living things. Carbon-14 decays very, very slowly, with a half-life of approximately 5730 years (i.e. use the remark from the previous problem to help quickly deduce the appropriate rate constant when time is measured in years).

In 1991, hikers in the Tyrolean Alps of Europe made a remarkable discovery. They found an almost perfectly preserved body of a prehistoric man, whom scientists named Otzi. The discovery was made possible because recent warming of the atmosphere had caused glaciers in the region to retreat, exposing objects that had been buried under the ice for millennia. Otzis fate was matched by a variety of well- preserved plant and animal species that were found close by. As discoveries of such quality are rare, the event was a genuine treasure trove for scientists. They reasoned that Otzi and the other organisms must have been trapped by a sudden snowfall and virtually flash frozen. This singular event was followed immediately by an extended cold period that preserved the specimens until the present glacial retreat. Samples collected from the site had an average of 53% remaining carbon-14. Solve an appropriate model describing the amount of carbon-14 in Otzi's body at any time and use the information about the half-life of carbon-14 to find an estimate for the time of death (i.e. yesterday or 50,000 years ago). (Hint: If you work through the problem carefully, you will find that you do not need to know the amount of Carbon-14 originally present since the data is given in terms relative to the original amount.)

This problem is taken from the webpage of Nelson Eby at Umass Lowell:
`http://faculty.uml.edu/nelson_eby/Forensic%20Geology/Exercises/Carbon-14%20Dating.pdf`

3.3. Input, Output and Dimensional Analysis: Torricelli's Principle: Part 1. In this problem we explore some further applications of dimensional analysis and dimensional consistency. This problem attempts to help bridge the gap between theoretical and experimental modeling. You will begin by studying a simple situation and we theoretically deriving a differential equation which describes this situation. After solving this differential equation, we explore the problem of parameter selection for our model, and demonstrate how we can run a simple experiment and use the experimental data to find approximations of the model parameters.

Consider a cylindrical tank with circular cross section. If the tank is filled with fluid and a small hole is opened on the side of the tank, then the fluid escapes through the hole and the height of fluid in the tank changes over time. In this set of exercises we will derive and solve a mathematical model which will determine the height of fluid above the hole at any time.

(1) We begin by deriving an equation for the rate of change of fluid volume within the tank. If we assume that the change in volume depends only on the water escaping through the hole in the side of the tank, then we might propose a model of the form:

$$\frac{dV}{dt} = f(v, a).$$

where v is the velocity of water escaping from the hole, and a is the area of the hole. At first glance $f(v, a)$ is an unknown function, but dimensional consistency is a very powerful condition in this case. Since we can choose a unit for the length dimension, and we can choose a unit for the time dimension, this equation must possess two different scaling symmetries, furthermore since f is assumed to be a function of only v and a, we find that the **only** a power law of the form:

Remark: Here k is an unknown numerical constant without units, we will approximate k later using experimental data.

$$f(v, a) = kv^{\alpha}a^{\beta},$$

could possibly have the correct dimensions. **Choose the powers α and β so that the units of $f(v, a)$ are the same as the units of $\frac{dV}{dt}$.**

(2) Torricelli's principle claims that the horizontal velocity of water escaping from a large tank is given by:

$$v = \sqrt{2gh}$$

where g is the local gravitational acceleration, and h is the height of the water above the center of the hole. Since $h = h(t)$ varies with time as the tank empties, we deduce that the rate of change of volume will vary with time. **Use the formula for the volume of a circular cylinder to find a relationship between V and h. Use this relationship to obtain a single differential equation which describes $\frac{dh}{dt}$**

(3) Solve the resulting differential equation for an explicit formula for h as a function of t and all of the model parameters.

3.4. Parameter selection: Torricelli's Principle: Part 2. In this problem we will use physical measurements and experimental data to find approximations for the model parameters present in the simple model from part 1. It is possible to complete this part of the problem in two different ways.

(1) Independent Experimentation:
(If you choose to complete this problem independently, you must find a circular cylinder (tupperware works well) which you will need to ruin by drilling or poking a small hole near the base.)
   (a) In order to fully specify the model, we must specify all of the model parameters. Carefully measure all geometric and environmental model parameters that you can for your experimental cylinder. (The area of the hole, the cross sectional area of the tank, the local gravitational acceleration).

   (b) In order to estimate the value of k you must run a series of experiments. To obtain decent measurements, the easiest thing to do is to temporarily plug the hole, fill the cylinder with water to a fixed height above the hole (which will become your initial height $h(0)$) , and create a movie which captures the water level of the cylinder as it empties on a digital camera, or smartphone. If you place a measuring stick or grid along side the tank, you can then cross reference the water level with the time-codes on the movie and obtain a discrete data set of $(t, h(t))$ values. (Note: This process is somewhat messy, and is best conducted outside)

   (c) Once you have collected the data, take the model equation for $h(t)$ and solve it algebraically for k. Substitute all of your model parameters along with the initial condition specified by your experiment to determine the formula for k. Using this formula each data point you recorded from your experiment $(t, h(t))$ provides a different estimate for the model parameter k. Since the data points will specify different values of k, we want to find the best approximation which uses all of the data. In this simple case with only a single model parameter, the arithmetic average of all of the predictions gives us the best approximation of k. Find the value of k.

(2) Extract data from a sample experiment. (This option allows you to collect experimental data from a movie file without actually getting your hands dirty)
   (a) In order to fully specify the model, we must specify all of the model parameters. The geometric measurements of the experimental cylinder used in the movie file *Torricelli.mov* are provided below. Note that length measurements are given in terms of two different units, you should convert everything to a single length unit before you proceed with the calculations (I recommend using 'lines' since these are the easiest measurements to extract from the movie.

Apparatus Specifications:

$d_{hole} = .45(lines), \quad d_{cylinder} = 14(lines), \quad g \approx 1200(lines)/s^2$

The (lines) unit of measure is the distance between the white lines on the cylinder presented in the movie. If you wish to work in metric units, $1(line) = \frac{9}{11}cm$.

(b) In the movie file, I have recorded a cylinder of constant cross section in the process of leaking from a small hole. I have taken the liberty of measuring the diameter of the cylinder and the diameter of the hole (which has a roughly circular cross section) and I have placed a regular grid on the cylinder so you can extract a collection of h(t) values, along with the corresponding t values. (You might want to use excel or another spreadsheet to help handle the data.)

Use the 'time elapsed' in the movie to collect a data set of h vs. t values for this particular cylinder. (10-15 data points should suffice, you don't need to watch the full 4 minutes of draining glory).

(c) Once you have collected the data, use the given parameters along with the initial condition specified by the movie to construct a set of equations (one for each data point) which can be solved for k. Since the different equations will specify different values of k we want to find the best approximation which uses all of the data. It turns out that in this case we can use the arithmetic average of all of the predictions to create our approximation of k. Find the value of k.

(**Remark:** A physical interpretation of k. Our model assumes the simplified picture that the water flows at a uniform velocity at each point coming out of the hole, and the stream of water has exactly the same cross section as the area of the whole. More realistically, there are two important factors which we have ignored, first the stream of water will not be uniform, but will narrow significantly as the stream leaves the cylinder, this is sometimes called the 'vena contracta' and it is different for different fluids. Second there are small but noticable friction effects which come from the edges of the whole slowing down the theoretical speed of the jet. Thus, the value of k is like an effective volume fraction. The parameter$k$ would be identically one if the stream of water left at exactly the velocity predicted by Torricelli's Principle, and had exactly the same cross sectional area as the hole. The fitted value of k should be somewhat less than one.)

4. Specialized Analytic Solution Techniques

4.1. Exact Equations. Example 1:

We begin with a simple differential equation which can be solved by exploiting exact structure.

$$\frac{dy}{dx} = \frac{y^2 - x^2}{y - 2xy}$$

In order to get this into a form convenient to study exact structure, we multiply by the given denominator and bring all terms to the same side of the equation:

$$x^2 - y^2 + (y - 2xy)\frac{dy}{dx} = 0$$

Here we follow the theoretical formulation and identify $M(x, y) = x^2 - y^2$ and $N(x, y) = y - 2xy$. Next we check to see whether the equation has the correct form to be exact. We compute:

$$\frac{\partial M}{\partial y} = -2y, \qquad \frac{\partial N}{\partial x} = -2y$$

since these two partial derivatives are equal, we know the equation has exact structure. Thus to solve the problem, we only need to seek a function $z(x, y)$ which satisfies **both**:

$$M = \frac{\partial z}{\partial x}, \qquad N = \frac{\partial z}{\partial y}$$

Imposing the first condition we have:

$$x^2 - y^2 = \frac{\partial z}{\partial x}$$

Integrating both sides with respect to x, we partially recover z.

$$\int x^2 - y^2 dx = \int \frac{\partial z}{\partial x} dx$$

$$\frac{x^3}{3} - xy^2 + h(y) = z$$

To uncover the form of $h(y)$, we take a partial derivative with respect to y and use the connection between z and N.

$$\frac{\partial}{\partial y}\left[\frac{x^3}{3} - xy^2 + h(y)\right] = \frac{\partial z}{\partial y}$$

$$\frac{\partial}{\partial y}\left[\frac{x^3}{3} - xy^2 + h(y)\right] = N$$

$$-2xy + \frac{dh}{dy} = y - 2xy$$

Simplifying this equation we find:

Remark: Note this equation is **independent** of x as it should be.

$$\frac{dh}{dy} = y$$

Solving this equation we find:

$$h(y) = \frac{y^2}{2}$$

We now fully recover z as:

$$z(x, y) = \frac{x^3}{3} - 2xy^2 + \frac{y^2}{2}$$

Since our equation has exact structure, we now write the general solution implicitly using the following argument:

$$\begin{aligned} x^2 - y^2 + (y - 2xy)\frac{dy}{dx} &= 0 \\ \frac{\partial z}{\partial x} + \frac{\partial z}{\partial y}\frac{dy}{dx} &= 0 \\ \frac{dz}{dx} &= 0 \\ z(x, y) &= C \\ \frac{x^3}{3} - 2xy^2 + \frac{y^2}{2} &= C \end{aligned}$$

And for this simple equation we can actually recover an explicit formula for y.

$$\begin{aligned} y^2 &= \frac{C - \frac{x^3}{3}}{\frac{1}{2} - 2x} \\ y &= \pm\sqrt{\frac{C - \frac{x^3}{3}}{\frac{1}{2} - 2x}} \\ y &= \pm\sqrt{\frac{C_2 - 2x^3}{3 - 12x}} \end{aligned}$$

where the correct branch of the square root would be selected by applying an appropriate initial condition.

Example 2: As a second example of exact equations, we present an equation which is **not** exact, but can be made exact by using an idea similar to the method of integrating factors.

$$\frac{dy}{dx} = \frac{-2xy}{1-2y^2}$$

Seeking exact structure we write this in the simplest form for exact structure:

$$2xy + (1-2y^2)\frac{dy}{dx} = 0$$

Next we identify M and N, and check the compatibility condition for exact structure:

$$\frac{\partial M}{\partial y} = 2x, \qquad \frac{\partial N}{\partial x} = 0$$

Since these two derivatives are **not** equal, the equation is **not** exact as written. We cannot proceed without manipulating the equation into a form which actually has exact structure. Without encumbering you with details we multiply the whole equation by the function $e^{x^2-y^2}$ and check again for exact structure:

$$2xye^{x^2-y^2} + (1-2y^2)e^{x^2-y^2}\frac{dy}{dx} = 0$$

We now check this new equation for exact structure:

$$\frac{\partial M}{\partial y} = 2xe^{x^2-y^2} - 4xy^2e^{x^2-y^2}, \qquad \frac{\partial N}{\partial x} = (1-2y^2)2xe^{x^2-y^2}$$

we find these quantities are now equal, so the equation is exact when written in this form. We proceed to find the equation using the same procedure as the previous example.

$$2xye^{x^2-y^2} = \frac{\partial z}{\partial x}$$
$$ye^{x^2-y^2} + h(y) = z$$

Differentiating with respect to y, and substituting for N we obtain:

$$e^{x^2-y^2} - 2y^2e^{x^2-y^2} + \frac{dh}{dy} = e^{x^2-y^2} - 2y^2e^{x^2-y^2}$$
$$\frac{dh}{dy} = 0$$

We find $h(y) = \text{const.}$. Solving the original equation we find:

$$ye^{x^2-y^2} = C$$

4.2. Homogeneous Equations. Consider the equation:

$$\frac{dy}{dx} = \frac{y}{y-x}$$

If we divide the top and the bottom of the right hand side expression by x, the value will remain unchanged.

$$\frac{dy}{dx} = \frac{\frac{y}{x}}{2\frac{y}{x}-1}$$

In this form we only see the configuration $\frac{y}{x}$. This implies the equation is homogeneous. To take advantage of this symmetry we change dependent variables from y to $v = \frac{y}{x}$.

In order to completely change variables we will need to replace $\frac{dy}{dx}$ with something depending only on v, x and $\frac{dv}{dx}$. Differentiating carefully we obtain:

$$\frac{dy}{dx} = x\frac{dv}{dx} + v$$

Changing variables in the differential equation yields:

$$x\frac{dv}{dx} + v = \frac{v}{2v-1}$$

The resulting equation is separable (we still need to do a little work to get it separated, but a separable equation is better than a non-separable one.)

$$\begin{aligned}
x\frac{dv}{dx} &= -v + \frac{3v}{2v-1} \\
x\frac{dv}{dx} &= -\frac{v(2v-1)}{2v-1} + \frac{v}{2v-1} \\
x\frac{dv}{dx} &= \frac{-2v^2+2v}{2v-1} \\
x\frac{dv}{dx} &= -2\frac{v^2-v}{2v-1}
\end{aligned}$$

Now we can solve this equation by separation:

$$\begin{aligned}
x\frac{dv}{dx} &= -2\frac{v^2-v}{2v-1} \\
\frac{2v-1}{v^2-v}\frac{dv}{dx} &= \frac{-2}{x} \\
\int \frac{2v-1}{v^2-v}\frac{dv}{dx}dx &= \int \frac{-2}{x}dx \\
\int \frac{1}{u}du &= -2\ln(x) + C \\
\ln(v^2-v) &= -2\ln(x) + C
\end{aligned}$$

Substituting $v = \frac{y}{x}$ we can find the solution to the original problem:

$$\ln(\frac{y^2}{x^2} - \frac{y}{x}) = -2\ln(x) + C$$
$$\frac{y^2}{x^2} - \frac{y}{x} = e^{-2\ln(x)+C}$$
$$\frac{y^2}{x^2} - \frac{y}{x} - k\frac{1}{x^2} =$$
$$y = x^2 \frac{\frac{1}{x} \pm \sqrt{\frac{1}{x^2} - \frac{4k}{x^4}}}{2}$$
$$y = \frac{x \pm \sqrt{x^2 - 4k}}{2}$$

CHAPTER 2

Higher Order Equations

1. Reduction of Order

We perform reduction of order on the following homogeneous second order equation to demonstrate the full technique.

$$y'' + 4y' + 4y = 0$$

If we try to solve this method via assuming an exponential solution we only find a single solution: $y_1 = e^{-2t}$. We assume this problem has another solution $y_2 = ue^{-2t}$ and we plug into the equation. (I'll plug into the equation abstractly and cancel terms, since this is faster that slogging through all the details with e^{-2t}.

$$\begin{aligned} y_2'' + 4y_2' + 4y_2 &= 0 \\ e^{-2t}u'' + 2u'y_1' + uy_1'' + 4\left[u'y_1 + uy_1'\right] + 4uy_1 &= 0 \\ e^{-2t}u'' + \left[2y_1' + 4y_1\right]u' + \left[y_1'' + 4y_1' + 4y_1\right]u &= 0 \end{aligned}$$

We know the coefficient of u is zero since we're using a solution to this equation as our choice of y_1. Plugging in the gory details for this particular example we can proceed:

$$\begin{aligned} e^{-2t}u'' + \left[2y_1' + 4y_1\right]u' &= 0 \\ e^{-2t}u'' + \left[2(-2e^{-2t}) + 4e^{-2t}\right]u' &= 0 \\ e^{-2t}u'' &= 0 \\ u'' &= 0 \end{aligned}$$

Solving this very simple equation by integrating twice we obtain:

$$u = C_1 t + C_2$$

Thus our second solution is:

$$y_2 = \left[C_1 t + C_2\right] e^{-2t}$$

Note that this solution contains the old solution y_1 as a subset of possible solutions, but that we also get a new solution with an extra factor of t:

$$y_2 = C_1 t e^{-2t} + C_2 e^{-2t}$$

2. Method of Undetermined coefficients

2.1. Naive Guesses. Example 1:

Let $\mathcal{L}(y)$ be the following constant coefficient differential operator.

$$\mathcal{L}(y) \equiv y'' - 2y' - 3y$$

We use the method of undetermined coefficients to find a function satisfying the non-homogeneous equation:

$$\mathcal{L}(y) = e^t, \text{ explicitly we solve:} \qquad y'' - 2y' - 3y = e^t.$$

Using the non-homogeneous term e^t, we guess the prototype of this function and substitute it into the differential equation. Since e^t is invariant under differentiation, it is its own prototype. Thus we make the guess:

$$Y = Ae^t, \text{ with } A \in \mathbb{R} \text{ an undetermined coefficient.}$$

Substituting this into our operator yields:

$$\mathcal{L}(Y) = (Ae^t)'' - 2(Ae^t)' - 3(Ae^t) = Ae^t - 2Ae^t - 3Ae^t = -4Ae^t.$$

Next we set this output equal to the desired function and set the coefficients of each function appearing equal to one another. This yields a system of algebraic equations which we solve for the values of the undetermined coefficients. If we are able to solve this algebraic system exactly, then the method of undetermined coefficients was a success, but if we are unable to solve the resulting system then undetermined coefficients was a failure with the chosen guess.

Here we have:

$$\begin{aligned} -4Ae^t &= e^t \\ -4A &= 1 \\ A &= \frac{-1}{4} \end{aligned}$$

We obtain an exact value for A, and thus find the function $y = \frac{1}{4}e^t$ is a solution to the equation $\mathcal{L}(y) = e^t$.

Example 2:

Using the same differential operator, we now solve the non-homogeneous equation:

$$\mathcal{L}(y) = 3e^{2t}$$

Since we have a different right hand side, we select a new guess using the prototype for this function. Guessing and computing derivatives carefully we obtain:

$$\begin{aligned} \text{Guess: } Y &= Ae^{2t} \\ Y' &= 2Ae^{2t} \\ Y'' &= 4Ae^{2t} \end{aligned}$$

Substituting into our differential operator we obtain:

$$\begin{aligned} \mathcal{L}(Y) = Y'' - 2Y' - 3Y &= 3e^{2t} \\ 4Ae^{2t} - 2(2Ae^{2t}) - 3Ae^{2t} &= -3Ae^{2t} \end{aligned}$$

Next we set this output equal to the desired output and solve for the undetermined coefficient, A.

$$\begin{aligned} -3Ae^{t} &= 3e^{2t} \\ -3A = 3, \quad A &= -1 \end{aligned}$$

We find the particular solution $-e^{2t}$ solves this non-homogeneous equation.

Example 3: (Incorrect guesses)
In this example we illustrate what happens when one makes an incorrect guess. We leave $\mathcal{L}$ the same, but we again change the right-hand side:

We solve the non-homogeneous equation:

$$\mathcal{L}(y) = 65\cos(2t)$$

Since the right-hand side is a cosine function we might try the (incorrect) guess:

$$Y = A\cos(2t)$$

On computing derivatives and substituting into the differential operator we obtain:

$$\mathcal{L}(Y) = -4A\cos(2t) + 4A\sin(2t) - 3A\cos(2t)$$

We now set this equal to the right-hand side of the non-homogeneous equation and **attempt**, set coefficients of each function equal to one another and attempt to solve the resulting system of linear equations. Here we require:

$$\begin{aligned} -4A\cos(2t) + 4A\sin(2t) - 3A\cos(2t) &= 65\cos(2t) \\ -7A\cos(2t) + 4A\sin(2t) &= 65\cos(2t) \end{aligned}$$

Since we see two different functions appearing in this equation we require that the coefficients for each function balance independently, so:

$$\begin{aligned} -7A &= 65 \text{ for the cosine terms to balance,} \\ 4A &= 0 \text{ for the sine terms to balance.} \end{aligned}$$

This system of equations is inconsistent, so the method of undetermined coefficients fails with this guess. This failure explains why the prototype guess for both cosine and sine functions includes **both** functions. We can get the method of undetermined coefficients to succeed for this problem if we make the correct guess:

$$\begin{aligned} \text{Guess: } Y &= A\cos(2t) + B\sin(2t) \\ Y' &= -2A\sin(2t) + 2B\cos(2t) \\ Y'' &= -4A\cos(2t) - 4B\sin(2t) \end{aligned}$$

Substituting these derivatives into the differential operator and setting the result equal to the desired right hand side we obtain the following equation:

$$\begin{array}{cccc} Y'' & -2\left[\quad Y' \quad\right] & -3\left[\quad Y \quad\right] & = 65\cos(2t) \\ [-4A\cos(2t) - 4B\sin(2t)] & -2\left[-2A\sin(2t) + 2B\cos(2t)\right] & -3\left[A\cos(2t) + B\sin(2t)\right] & = 65\cos(2t) \end{array}$$

Balancing the coefficients for each functional form yields the following system of equations:with the same functional form together (Using $\mathbf{C}$ for $65\cos(2t)$, and $\mathbf{S}$ for $\sin(2t)$)

$$\begin{aligned} -7A - 4B &= 65 \\ 4A - 7B &= 0 \end{aligned}$$

Using elementary algebra to solve this system yields the unique solution $A = -7$ and $B = -4$.

We find that the function $-7\cos(2t) - 4\sin(2t)$ satisfies the given non-homogeneous differential equation.

Example 4: In this example we select a new differential operator. Define:

$$\mathcal{L} = y'' - 2y$$

We solve the non-homogeneous equation:

$$\mathcal{L}(y) = 2t^2 + 4t.$$

Here the right hand side is a polynomial, so the correct prototype is a generic polynomial of the same degree with undetermined coefficients.

Remark: Note that our guess contains a constant term, even when the right hand side does not. Using the recommended prototypes will save you a great deal of frustration and incorrect guesses.

$$\begin{aligned} \text{Guess: } Y &= At^2 + Bt + C \\ Y' &= 2At + B \\ Y'' &= 2A \end{aligned}$$

Substituting

$$\begin{aligned} Y'' - 2Y &= 2t^2 + 4t \\ 2A - 2(At^2 + Bt + C) &= 2t^2 + 4t \end{aligned}$$

Grouping the coefficients of the different functional groupings we obtain three equations:
t^2

$$-2A = 2$$

t

$$-2B = 4$$

1

$$2A - 2C = 0$$

Solving these equations we find $A = -1$, $B = -2$ and $C = -1$. We conclude that $Y = -t^2 - 2t - 1$ solves the equation $\mathcal{L}(Y) = 2t^2 + 4t$.

Note: The naive guesses we are making here will work as long as the forcing function on the right hand side is not a homogeneous solution to the differential equation. When this occurs, our guesses must be modified in an appropriate way.

3. Non-Homogeneous Initial value problems

Example 1:

Consider:

$$y'' + y' - 6y = 2e^t, \quad y(0) = 1, y'(0) = 0$$

We begin by calculating the general homogeneous solution:

$$y_h'' + y_h' - 6y_h = 0$$

the characteristic polynomial and roots for this equation are given by:

$$r^2 + r - 6 = 0 \quad (r-2)(r+3) = 0, \quad r = 2, -3$$

The general homogeneous solution will then be given by:

$$y_h = C_1 e^{2t} + C_2 e^{-3t}$$

Next we proceed to solve the non-homogeneous equation:

$$Y'' + Y' - 6Y = 2e^t.$$

Since e^t is different from both of our homogeneous solutions we can proceed exactly as before:

$$\begin{aligned} \text{Guess: } Y &= Ae^t \\ Y' &= Ae^t \\ Y'' &= Ae^t \end{aligned}$$

$$\begin{aligned} Ae^t + Ae^t - 6Ae^t &= 2e^t \\ -4A = 2, \quad A &= -\frac{1}{2} \end{aligned}$$

Now we combine the homogeneous solution and non-homogeneous solution and apply the initial conditions:

$$\begin{aligned} y &= y_h + Y \\ y &= C_1 e^{2t} + C_2 e^{-3t} - \frac{1}{2} e^t \\ 1 &= C_1 + C_2 - \frac{1}{2} \\ 0 &= 2C_1 - 3C_2 - \frac{1}{2} \end{aligned}$$

Solving these equations we find:

$$\begin{aligned} C_1 &= \frac{3}{2} - C_2 \\ 0 &= 3 - 5C_2 - \frac{1}{2} \\ C_2 &= \frac{1}{2}, \quad C_1 = 1 \end{aligned}$$

We conclude that the solution to this initial value problem is given by:

$$y = e^{2t} + \frac{1}{2}e^{-3t} - \frac{1}{2}e^{t}$$

Let us now consider the same operator and initial conditions, but change the right-hand side to be one of the homogeneous solutions.

Example:

$$y'' + y' - 6y = e^{2t}, \quad y(0) = 1, y'(0) = 0$$

We have already solved for the homogeneous solution to this problem and we have discovered that it is given by:

$$y_h = C_1 e^{2t} + C_2 e^{-3t}$$

Since our forcing function $F(t) = e^{2t}$ is one of our homogeneous solutions, the naive guess of Ae^{2t} will not work, we must reinforce our guess with another function in order to get an output of e^{2t}. In short to make the correct guess for a forcing function which is a homogeneous solution we multiply by factors of the independent variable, t until it is no longer a homogeneous solution.

Here we make the modified guess:

$$\begin{aligned} \text{Guess: } Y &= Ate^{2t} \\ Y' &= Ae^{2t} + 2Ate^{2t} \\ Y'' &= 4Ae^{2t} + 4Ate^{2t} \end{aligned}$$

$$4Ae^{2t} + 4Ate^{2t} + Ae^{2t} + 2Ate^{2t} - 6Ate^{2t} = e^{2t}$$

Here we have two function groupings, te^{2t} and e^{2t}. Since e^{2t} is a homogeneous solution, the equation for one of these groupings will automatically be satisfied:
e^{2t}

$$4A + A = 1$$

te^{2t}

$$4A + 2A - 6A = 0$$

The second equation is automatically satisfied and we can take $A = \frac{1}{5}$.

Now we continue and apply the initial conditions to our combined homogeneous and non-homogeneous solution:

$$y = C_1 e^{2t} + C_2 e^{-3t} + \frac{1}{5} t e^{2t}$$

We get the following linear equations to solve for C_1 and C_2.

$$\begin{aligned} 1 &= C_1 + C_2 + \frac{1}{5} \\ 0 &= 2C_1 - 3C_2 + \frac{1}{5} \\ C_1 &= \frac{4}{5} - C_2 \\ 0 &= \frac{8}{5} - 5C_2 + \frac{1}{5} \\ C_2 = \frac{9}{25}, &\quad C_1 = \frac{11}{25} \end{aligned}$$

The final solution becomes:

$$y = \frac{11}{25} e^{2t} + \frac{9}{25} e^{-3t} + \frac{1}{5} t e^{2t}$$

Example:
Let's consider the initial value problem:

$$y'' + 9y = \sin(\omega t) \quad y(0) = 0, \quad y'(0) = 1$$

Where ω is an integer that we haven't picked just yet.

Solving the homogeneous equation here we find:

$$r^2 + 9 = 0, \quad r = \pm 3i$$

Pure imaginary roots yields a solution of sines and cosines:

$$y_h = \sin(3t) + \cos(3t)$$

Now if $\omega \neq 3$ we can use our regular guess to solve the particular problem at hand:

$$\begin{aligned} \text{Guess: } Y &= A\cos(\omega t) + B\sin(\omega t) \\ Y' &= -\omega A \sin(\omega t) + \omega B \cos(\omega t) \\ Y'' &= -\omega^2 A \cos(\omega t) - \omega^2 B \sin(\omega t) \end{aligned}$$

$$\begin{aligned} -\omega^2 A\cos(\omega t) - \omega^2 B\sin(\omega t) + 9A\cos(\omega t) + B\sin(\omega t) &= \sin(\omega t) \\ (9-\omega^2)A\cos(\omega t) + (9-\omega^2)B\sin(\omega t) &= \sin(\omega t) \\ A = 0, \quad B &= \frac{1}{9-\omega^2} \end{aligned}$$

So when $\omega \neq 3$ we get a reasonable solution. Substituting the initial conditions we find:

$$y = \frac{(9-\omega^2)-\omega}{3(9-\omega^2)}\sin(3t) + \frac{1}{(9-\omega^2)}\sin(\omega t)$$

When $\omega = 3$ our guess for the non-homogeneous equation has to change.

Let's solve:

$$y'' + 9y = \sin(3t) \quad y(0) = 0, \quad y'(0) = 1$$

We know the homogeneous solution is given by:

$$y_h = C_1\cos(3t) + C_2\sin(3t)$$

This means we need a better guess than our naive guess. We try the trick of multiplying by t and see what happens:

$$\begin{aligned}
\text{Guess: } Y &= At\cos(3t) + Bt\sin(3t) \\
Y' &= -3At\sin(3t) + A\cos(3t) + 3Bt\cos(3t) + B\sin(3t) \\
Y'' &= -9At\cos(\omega t) - 2(3)A\sin(3t) - 9Bt\sin(\omega t) + 2(3)B\cos(3t)
\end{aligned}$$

$$\begin{aligned}
-9At\cos(\omega t) - 2(3)A\sin(3t) - 9Bt\sin(\omega t) + 2(3)B\cos(3t) + 9\left[At\cos(3t) + Bt\sin(3t)\right] &= \sin(3t) \\
-6A\sin(3t) + 6B\cos(3t) &= \sin(3t) \\
A = -\frac{1}{6}, B &= 0
\end{aligned}$$

Note that A was the coefficient of the cosine, not the sine, so our forcing solution becomes:

$$Y = -\frac{1}{6}t\cos(3t)$$

Now we can apply the initial conditions:

$$\begin{aligned}
0 &= C_1 \\
1 &= 3C_2 - \frac{1}{6} \\
C_2 &= \frac{7}{18}
\end{aligned}$$

We obtain the final solution:

$$y = \frac{7}{18}\sin(3t) - \frac{1}{6}t\cos(3t)$$

4. Variation of Parameters

Example 1:

Use variation of parameters to find a particular solution to:

$$y'' - 5y' + 6y = 3e^t, \quad y(0) = 0, \quad y'(0) = 0$$

First we solve the homogeneous problem to find suitable y_1 and y_2. Assume an exponential solution then we obtain:

$$\begin{aligned} y'' - 5y' + 6y &= 0 \\ r^2 - 5r + 6 &= 0 \\ (r-2)(r-3) &= 0, \quad r = 2, 3 \end{aligned}$$

$$y_1 = Ce^{2t}, \quad y_2 = Ce^{3t}$$

we compute the Wronskian of these two solutions for two reasons, first we need to make sure the solutions are linearly indepedent, and second we need it for the Variation of Parameters formulae:

$$W(y_1, y_2) = \begin{vmatrix} e^{2t} & e^{3t} \\ 2e^{2t} & 3e^{3t} \end{vmatrix} = 3e^{5t} - 2e^{5t} = e^{5t}$$

Since the Wronskian is non-zero we proceed:

$$\begin{aligned} Y &= u_1 y_1 + u_2 y_2 \\ Y &= -e^{2t} \int_0^t \frac{e^{3s} 3e^s}{e^{5s}} ds + e^{3t} \int_0^t \frac{e^{2s} 3e^s}{e^{5s}} ds \\ Y &= -e^{2t} \int_0^t 3e^{-t} dt + e^{3t} \int_0^t 3e^{-2t} dt \\ Y &= -e^{2t} \left[-3e^{-t} + 3e^0 \right] + e^{3t} \left[-\frac{3}{2} e^{-2t} + \frac{3}{2} e^0 \right] \end{aligned}$$

Multiplying these terms out we get the exact solution:

$$\begin{aligned} Y &= 3e^t - 3e^{2t} - \frac{3}{2} e^t + \frac{3}{2} e^{3t} \\ Y &= \frac{3}{2} e^t - 3e^{2t} + \frac{3}{2} e^{3t} \end{aligned}$$

You can check this result with method of undetermined coefficients.

Example 2:

Now we consider another problem that we could not solve by using the method of undetermined coefficients:

$$y'' + y = \frac{1}{\sin(t)\cos(t)}, \quad y(\frac{\pi}{4}) = 0, \quad y'(\frac{\pi}{4}) = 0$$

Solving the homogeneous problem we would find the following homogeneous solutions:

$$y_1 = C_1 \cos(t), \quad y_2 = C_2 \sin(t)$$

Computing the Wronskian we obtain:

$$W(y_1, y_2) = \begin{vmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{vmatrix} = 1$$

Using variation of parameters we find:

$$Y = u_1 y_1 + u_2 y_2$$
$$Y = -\cos(t)\int_{\frac{\pi}{4}}^{t} \frac{\sin(s)}{\sin(s)\cos(s)} ds + \sin(t)\int_{\frac{\pi}{4}}^{t} \frac{\cos(s)}{\sin(s)\cos(s)} ds$$
$$Y = -\cos(t)\int_{\frac{\pi}{4}}^{t} \sec(s) ds + \sin(t)\int_{\frac{\pi}{4}}^{t} \csc(s) ds$$
$$Y = -\cos(t)\left[\ln|\sec(t)+\tan(t)| - \ln|\sqrt{2}+1|\right] + \sin(t)\left[-\ln|\csc(t)+\cot(t)| + \ln|\sqrt{2}+1|\right]$$

Example 3: We consider a final example, again one which could not be solved using method of undetermined coefficients:

$$2t^2 y'' - 2ty' = 2t^{\frac{5}{2}}, \quad y(1) = 0, \quad y'(1) = 0$$

Two homogeneous solutions to this problem are given by:

$$y_1 = 1 \quad y_2 = t^2$$

Taking the Wronskian of these two solutions we find the following:

$$W(y_1, y_2) = \begin{vmatrix} 1 & t^2 \\ 0 & 2t \end{vmatrix} = 2t$$

This Wronskian is non-zero as long as $t > 0$. We proceed using variation of parameters:

First we must get the equation into standard form:

$$y'' - \frac{1}{t}y' = \sqrt{t}$$

$$Y = u_1 y_1 + u_2 y_2$$
$$Y = -\int_1^t \frac{s^2\sqrt{s}}{2s} ds + t^2 \int_1^t \frac{\sqrt{s}}{2s} ds$$
$$Y = -\int_1^t \frac{1}{2} s^{\frac{3}{2}} ds + t^2 \int_1^t \frac{1}{2} s^{-\frac{1}{2}} ds$$
$$Y = -\left[\frac{1}{5}t^{\frac{5}{2}} - \frac{1}{5}\right] + t^2\left[\sqrt{t} - 1\right]$$

5. Laplace Transforms

5.1. Applying the Laplace Transform to IVPs. Since the Laplace transform is a linear operator, and we can write the transforms of the derivatives of an arbitrary function in terms of the regular transform we can actually use this operation to solve linear differential equations:

Example 1: Consider the problem:

$$y'' - 4y' + 3y = 0, \quad y(0) = 2, \quad y'(0) = 0$$

Let us take the laplace transform of this problem and see what happens. we will use $Y(s)$ to denote the transform of $y(t)$.

$$\begin{aligned}\mathcal{L}(y'') - 4\mathcal{L}(y') + 3\mathcal{L}(y) &= 0\\ s\left[s\mathcal{L}(y) - y(0)\right] - y'(0) - 4\left[s\mathcal{L}(y) - y(0)\right] + 3\mathcal{L}(y) &= 0\\ s\left[sY - 2\right] - 4\left[sY - 2\right] + 3Y &= 0\\ s^2Y - 2s - 4sY + 8 + 3Y &= 0\end{aligned}$$

Now after taking the transform we solve the resulting algebraic equation for the transform, $Y(s)$.

$$Y = \frac{2s-8}{s^2-4s+3}$$

Now if we can determine how to transform this back into the t variable we will have solved our initial value problem. Here we can apply a partial fraction decomposition to try and simplify this transform:

$$\begin{aligned}\frac{2s-8}{s^2-4s+3} = \frac{2s-8}{(s-3)(s-1)} &= \frac{A}{s-3} + \frac{B}{s-1}\\ 2s-8 &= A(s-1) + B(s-3)\end{aligned}$$

We can either plug in conspicuous values of s here or solve the resulting system of two equations and two unknowns.
Plugging $s = 1$ and $s = 3$ into this equation yields the values of B and A respectively:

$$-6 = -2B, \quad B = 3; \qquad -2 = 2A, \quad A = -1$$

This gives us the simplified transform:

$$Y(s) = -1\frac{1}{s-3} + 3\frac{1}{s-1}$$

In practice one would now consult a table of Inverse Laplace Transforms to determine the function $y(t)$. Since we can determine the form of the general solution to this problem by our guess and check method we might deduce that the inverse transforms of these functions must be exponentials, and we would be correct:

$$\begin{aligned}\mathcal{L}^{-1}(F(s)) &= f(t)\\ \mathcal{L}^{-1}\left(\frac{1}{s-a}\right) &= e^{at}\end{aligned}$$

So:

$$y(t) = \mathcal{L}^{-1}(Y(s)) = -1e^{3t} + 3e^{t}$$

The primary application of the Laplace transform is in solving non-homogeneous linear problems with constant coefficients. With a large table of Laplace transforms at hand, one

can solve a huge variety of linear problems.

Example 2: Let us now consider the problem:

$$y'' + y = \sin(t) \quad y(0) = 1; \quad y'(0) = 0$$

This is a problem that we could handle with method of undetermined coefficients, but which is very tractable with Laplace transforms as well.

Taking a Laplace transform of the whole differential equation we obtain:

$$\begin{aligned}
\mathcal{L}(y'') + \mathcal{L}(y) &= \mathcal{L}(\sin(t)) \\
s[sY - y(0)] - y'(0) + Y &= \frac{1}{1+s^2} \\
s[sY - 1] + Y &= \frac{1}{1+s^2} \\
(1+s^2)Y - s &= \frac{1}{1+s^2}
\end{aligned}$$

Solving for the transformed solution, Y, we get:

$$Y(s) = \frac{1}{(1+s^2)^2} + \frac{s}{1+s^2}$$

Looking in a table of Laplace transforms we find that each term here is the transform of a different function:

$$\begin{aligned}
y(t) &= \mathcal{L}^{-1}\left(\frac{1}{(1+s^2)^2}\right) + \mathcal{L}^{-1}\left(\frac{s}{1+s^2}\right) \\
y(t) &= \mathcal{L}^{-1}\left(\frac{1}{(1+s^2)^2}\right) + \cos(t) \\
y(t) &= \frac{1}{2}\left(\sin(t) - t\cos(t)\right) + \cos(t)
\end{aligned}$$

Example 3:
In this example we consider a problem with discontinuous forcing function:

$$y'' + 5y' + 6y = u_1(t)e^{-(t-1)} + u_2 e^{-(t-2)}, \quad y(0) = 1, y'(0) = 1$$

Applying the Laplace transform to this problem yields:

$$s[sY - 1] - 1 + 5(sY - 1) + 6Y = e^{-s}\frac{1}{s+1} + e^{-2s}\frac{1}{s+1}$$

Solving for Y we find:

$$Y = \frac{s}{s^2+5s+6} + \frac{6}{s^2+5s+6} + e^{-s}\frac{1}{(s+1)(s^2+5s+6)} + e^{-2s}\frac{1}{(s+1)(s^2+5s+6)}$$

To Complete the problem we must invert the transform. Using a partial fractions decomposition is our best bet. (We can ignore the e^{-cs} terms, because we know that these will

turn into heaviside functions when we invert the transform.)

$$Y = \frac{s+6}{s^2+5s+6} + e^{-s}\frac{1}{(s+1)(s^2+5s+6)} + e^{-2s}\frac{1}{(s+1)(s^2+5s+6)}$$

We deal with the first term first:

$$\frac{s+6}{s^2+5s+6} = \frac{s+6}{(s+2)(s+3)} = \frac{A}{s+2} + \frac{B}{s+3}$$
$$s+6 = A(s+3) + B(s+2)$$

Plugging in sneaky values of s will give us convenient equations:
$s = -2$

$$4 = A$$

$s = -3$

$$3 = -B, \quad B = -3$$

This completes this partial fraction decomposition:

$$\frac{s+6}{s^2+5s+6} = \frac{4}{s+2} - \frac{3}{s+3}$$

Now we perform a partial fraction decomposition of the second and third term:

$$e^{-s}\frac{1}{(s+1)(s^2+5s+6)} - e^{-2s}\frac{1}{(s+1)(s^2+5s+6)} = (e^{-s}-e^{-2s})\frac{1}{(s+1)(s+2)(s+3)}$$
$$= (e^{-s}-e^{-2s})\left[\frac{A}{s+1} + \frac{B}{s+2} + \frac{C}{s+3}\right]$$

$1 = A(s+2)(s+3) + B(s+1)(s+3) + C(s+1)(s+2)$

Again plugging in sneaky values of s will give us simplified equations:

$s = -1$

$$1 = A(1)(2), \quad A = \frac{1}{2}$$

$s = -2$

$$1 = B(-1)(1), \quad B = -1$$

$s = -3$

$$1 = C(-2)(-1), \quad C = \frac{1}{2}$$

This completes this round of partial fraction decomposition:

$$(e^{-s}-e^{-2s})\frac{1}{(s+1)(s+2)(s+3)} = (e^{-s}+e^{-2s})\left[\frac{1}{2}\frac{1}{s+1} - \frac{B}{s+2} + \frac{1}{2}\frac{C}{s+3}\right]$$

Combining all of these individual terms we are able to invert the Laplace transform for each one:

$$Y(s) = \frac{4}{s+2} - \frac{3}{s+3} + (e^{-s} + e^{-2s}) \left[\frac{1}{2} \frac{1}{s+1} - \frac{1}{s+2} + \frac{1}{2} \frac{C}{s+3} \right]$$

$$y(t) = 4e^{-2t} - 3e^{-3t} + u_1(t) \left[\frac{1}{2} e^{-(t-1)} - e^{-2(t-1)} + \frac{1}{2} e^{-3(t-1)} \right]$$

$$+ u_2(t) \left[\frac{1}{2} e^{-(t-2)} + e^{-2(t-2)} + \frac{1}{2} e^{-3(t-2)} \right]$$

5.2. Laplace Transforms of distributional equations. Suppose we were faced with the problem:

$$y'' + y = \delta(t-\pi), \qquad y(0) = 0, y'(0) = 0$$

Here we have a spring-mass system, which is initially at rest, but which is given an impulsive jolt of energy at $t = \pi$. We can solve this problem using Laplace transforms. (I'll include an extra step, where I use the definition of the Laplace transform, this allows us to see where the delta distribution is integrated, and will help you to understand the Laplace transform of the δ distribution.

$$\mathcal{L}(y'') + \mathcal{L}(y) = \mathcal{L}(\delta(t-\pi)),$$

$$s^2 Y - sy(0) - y'(0) + Y = \int_0^\infty e^{-st} \delta(t-\pi) dt,$$

Using the special integration property of the δ distribution (and the initial conditions) we obtain:

$$s^2 Y + Y = e^{-s\pi},$$

$$Y = \frac{e^{-s\pi}}{s^2+1}$$

Now we can take the inverse Laplace transform to uncover the solution:

$$y(t) = u_\pi(t) \left[\sin(t-\pi) \right]$$

We find that the answer contains a heaviside function that switches 'on' exactly when the impulse hits.

CHAPTER 3

Variable Coefficient Equations

CHAPTER 4

Systems of Linear Equations

Example 1:

In our first example we will look at a system of two coupled first order equations. We will first solve the problem using an exponential guess, then we will consider a non-homogeneous version of the problem that we will solve using diagonalization. In both cases the eigenvalue problem for the matrix of coefficients plays a central role.

Consider the initial value problem:

$$\frac{dx_1}{dt} = 4x_1 + 3x_2$$
$$\frac{dx_2}{dt} = x_1 + 2x_2$$

along with the initial conditions:

$$x_1(0) = 1, \quad x_2(0) = 1.$$

We first write this system in matrix notation:

$$\mathbf{x}' = \begin{pmatrix} 4 & 3 \\ 1 & 2 \end{pmatrix} \mathbf{x}, \quad \mathbf{x}(0) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Next we assume the solution has an exponential character and we substitute that guess into the differential equations.

$$\mathbf{x} = \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} e^{\lambda t}$$

on substitution and simplification the system reduces to the eigenvalue problem for the parameter λ and the vector $\boldsymbol{\xi}$.

$$\lambda \boldsymbol{\xi} = \begin{pmatrix} 4 & 3 \\ 1 & 2 \end{pmatrix} \boldsymbol{\xi}.$$

We briefly solve the eigenvalue problem, again seek out a decent linear algebra text for a more detailed outline of the solution of the eigenvalue problem. The characteristic

polynomial which defines the eigenvalues is given by:

$$\begin{aligned} det\begin{pmatrix} 4-\lambda & 3 \\ 1 & 2-\lambda \end{pmatrix} &= 0 \\ (4-\lambda)(2-\lambda) - 3 &= 0 \\ \lambda^2 - 6\lambda + 5 &= 0 \end{aligned}$$

Solving this quadratic leads to two distinct eigenvalues: $\lambda = 5, 1$.
For $\lambda = 5$ the eigenvector solves the linear system:

$$\begin{aligned} \begin{pmatrix} 4-5 & 3 \\ 1 & 2-5 \end{pmatrix} \boldsymbol{\xi} &= 0 \\ \begin{pmatrix} -1 & 3 \\ 1 & -3 \end{pmatrix} \boldsymbol{\xi} &= 0 \end{aligned}$$

For $\lambda = 1$ the eigenvector solves the linear system:

$$\begin{aligned} \begin{pmatrix} 4-1 & 3 \\ 1 & 2-1 \end{pmatrix} \boldsymbol{\xi} &= 0 \\ \begin{pmatrix} 3 & 3 \\ 1 & 1 \end{pmatrix} \boldsymbol{\xi} &= 0 \end{aligned}$$

Thus we can take the eigenpairs given by:

$$\lambda = 5, \boldsymbol{\xi} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}; \quad \lambda = 1, \boldsymbol{\xi} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

Substituting these eigenvectors into our exponential guess, we construct the general solution using an arbitrary linear combination:

$$\mathbf{x} = C_1 \begin{pmatrix} 3 \\ 1 \end{pmatrix} e^{5t} + C_2 \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{t}$$

to find the particular solution satisfying the desired initial condition, we substitute the initial condition into the general solution and solve for C_1 and C_2.

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \end{pmatrix}$$

Solving this system yields:

$$C_1 = \frac{1}{2}, \quad C_2 = \frac{1}{2}$$

Example 1b

Next we consider a non-homogeneous version of the same problem. Since we have already solved the eigenvalue problem for this coefficient matrix, we can immediately begin work on the diagonalization procedure and demonstrate how the problem transforms.

$$\frac{dx_1}{dt} = 4x_1 + 3x_2 + t$$
$$\frac{dx_2}{dt} = x_1 + 2x_2 + e^{-2t}$$

in vector notation we would write this as:

$$\mathbf{x}' = \begin{pmatrix} 4 & 3 \\ 1 & 2 \end{pmatrix} \mathbf{x} + \begin{pmatrix} t \\ e^{-2t} \end{pmatrix}$$

We use the eigenvalues and eigenvectors for this coefficient matrix to convert this to a decoupled system of 2 first order equations.

We construct the matrix E from the eigenvectors, and we must also compute the inverse matrix since it will be used to convert the forcing functions into the new formulation.

$$E = \begin{pmatrix} 3 & -1 \\ 1 & 1 \end{pmatrix}, \quad E^{-1} = \frac{1}{4} \begin{pmatrix} 1 & 1 \\ -1 & 3 \end{pmatrix}$$

Next we diagonalize this system by forming the matrix product $E^{-1}AE$ described in the theory section. We left multiply both sides of our equation by E^{-1} and insert $I = EE^{-1}$ between the coefficient matrix and the vector of unknowns. Then by defining the new variables:

$$\mathbf{y} = E^{-1}\mathbf{x}$$

we arrive at the diagonal system:

$$\mathbf{y}' = \begin{pmatrix} 5 & 0 \\ 0 & 1 \end{pmatrix} \mathbf{y} + E^{-1} \begin{pmatrix} t \\ e^{-2t} \end{pmatrix}$$

Which in scalar notation yields the following decoupled ordinary differential equations:

$$\frac{dy_1}{dt} = 5y_1 + \frac{1}{4}t + \frac{1}{4}e^{-2t}$$
$$\frac{dy_2}{dt} = y_2 - \frac{1}{4}t + \frac{3}{4}e^{-2t}$$

Since these equations are decoupled we are free to use any applicable technique for solving scalar first order, linear equations.

Once these equations are solved one can take the general solution found here and convert it back to the original variables using E.

$$\mathbf{x} = E\mathbf{y}$$
$$x_1 = 3y_1 - y_2,$$
$$x_2 = y_1 + y_2$$

Example 2:

In certain situations the exponential guess may lead to complex eigenvalues and complex eigenvectors, this problem can be resolved by utilizing the Euler identities in a similar fashion to the scalar equation case.

Consider the following system of ordinary differential equations given in vector notation:

$$\mathbf{x}' = \begin{pmatrix} -2 & -1 \\ 1 & -2 \end{pmatrix} \mathbf{x}$$

Assuming an exponential character for the solution, substitution again brings us to the eigenvalue problem, but here the eigenvalues are given by:

$$\begin{aligned} det \begin{pmatrix} -2-\lambda & -1 \\ 1 & -2-\lambda \end{pmatrix} &= 0 \\ \lambda^2 + 4\lambda + 5 &= 0 \\ \lambda &= -2 \pm i \end{aligned}$$

Just like in the scalar equation case we use the Euler identities in combination with the complex conjugate structure of the roots (and eigenvectors) to extract purely real valued solutions to this problem.

The eigenvectors are found as in the real valued case, but one or more components will involve complex numbers. The complex conjugate structure of the roots carries over to the eigenvectors, so it is only necessary to find one eigenvector explicitly.

$$\begin{aligned} \begin{pmatrix} -2-(-2+i) & -1 \\ 1 & -2-(-2+i) \end{pmatrix} \boldsymbol{\xi} &= \mathbf{0} \\ \begin{pmatrix} -i & -1 \\ 1 & -i \end{pmatrix} \boldsymbol{\xi} &= \mathbf{0} \end{aligned}$$

These two rows are complex multiples of one another, so this matrix has a non-trivial null space spanned by the single eigenvector:

$$\boldsymbol{\xi} = \begin{pmatrix} 1 \\ i \end{pmatrix}$$

The second vector is given by the complex conjugate, we obtain this by simply replacing each appearance of i with$-i$.

$$\boldsymbol{\xi} = \begin{pmatrix} 1 \\ -i \end{pmatrix}$$

Now we can construct real valued solutions using the same type of linear combinations that we used for extracting real valued solutions from the scalar equation, however there is a simple technique which allows us to construct a general solution by manipulating a single complex solution.

Take a single solution to this system:

$$\begin{pmatrix} 1 \\ i \end{pmatrix} e^{(-2+i)t}$$

Use the Euler identities to replace the complex part of the exponential with cosines and sines.

$$\begin{pmatrix}1\\ i\end{pmatrix} e^{-2t}\left(\cos(t)+i\sin(t)\right)$$

Multiply the trigonometric functions into the eigenvector and distribute all multiplication.

$$\begin{pmatrix}\cos(t)+i\sin(t)\\ i\cos(t)-\sin(t)\end{pmatrix} e^{-2t}$$

Split this solution into it's real and imaginary parts.

$$\begin{pmatrix}\cos(t)\\ -\sin(t)\end{pmatrix} e^{-2t}+i\begin{pmatrix}\sin(t)\\ \cos(t)\end{pmatrix} e^{-2t}$$

and finally a suitable general solution is obtained by creating a linear combination of the real and imaginary parts as two linearly independent solutions.

$$\mathbf{x}=C_1\begin{pmatrix}\cos(t)\\ -\sin(t)\end{pmatrix} e^{-2t}+C_2\begin{pmatrix}\sin(t)\\ \cos(t)\end{pmatrix} e^{-2t}$$

the last step in this calculation is **not algebraic** The two expressions are not equal, but the final leap is justified by exploiting linear combinations of the two complex conjugate solutions we obtained from the exponential guess.

Diagonalization for constant coefficient systems

When one deals with systems of linear equations, often techniques from linear algebra can be used to reformulate the system in a simpler way. In the following examples we will look at how diagonalization using the eigen-basis can be used to convert coupled systems of first order equations into decoupled equations which are vulnerable to other techniques we know. This example is not intended to be a first introduction to the eigenvalue problem, so if you have not seen this concept in a linear algebra class, you may wish to skip this technique.

Consider the following constant coefficient system of equations:

$$\begin{aligned}\frac{dx}{dt}&=-2x-y\\ \frac{dy}{dt}&=x-2y\end{aligned}$$

Looking at this system in vector notation, we will diagonalize using a basis of the eigenvectors of the coefficient matrix.

$$\mathbf{x}'=\begin{pmatrix}-2&-1\\ -1&-2\end{pmatrix}\mathbf{x}$$

Through direct computation, one can check that this coefficient matrix has two distinct eigenpairs:

$$\lambda_1 = -1, \quad \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \qquad \lambda_2 = -3, \quad \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

In order to diagonalize we perform the following abstract calculation. (The goal is to create the correct matrix configuration to diagonalize the system, we then adopt a change of variables which allows us to solve the diagonal system in a natural way.

Let E be the matrix of eigenvectors for our coefficient matrix, and let E^{-1} be its inverse. Then:

$$E^{-1}\mathbf{x}' = E^{-1}A\mathbf{x}$$

to complete the diagonalization we need the product $E^{-1}AE$, so we insert an identity matrix between A and $\mathbf{x}$ and split it appropriately.

$$E^{-1}\mathbf{x}' = E^{-1}A\mathcal{I}\mathbf{x}$$
$$E^{-1}\mathbf{x}' = E^{-1}AEE^{-1}\mathbf{x}$$

now we adopt the new variable $\mathbf{y} = E^{-1}\mathbf{x}$ and solve:

$$\mathbf{y}' = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \mathbf{y}$$

for this example we have:

$$y_1' = -y_1,$$
$$y_2' = -3y_2$$

You can solve these equations independently using any technique you wish. The general solutions here become:

$$y_1 = C_1e^{-t}, \qquad y_2 = C_2e^{-3t}$$

we recover the solutions in the original variables by reversing our change of variables:

$$\mathbf{x} = E\mathbf{y}$$

we then apply initial conditions as necessary.

In a situation where the coupled equations are non-homogeneous, but still constant coefficient we can apply the same procedure, however the forcing functions change due to the change of variables.

You can compute for yourself, that given a non-homogeneous problem:

$$\mathbf{x}' = A\mathbf{x} + \mathbf{f}$$

diagonalizing the coefficient matrix A will yield the following decoupled system of equations:

$$\mathbf{y}' = \Lambda\mathbf{y} + E^{-1}\mathbf{f}$$

where $\mathbf{y}$ is defined exactly like the previous example, and $E^{-1}\mathbf{f}$ are the adjusted forces.

Part 3

Boundary Value Problems for Ordinary Differential Equations: Theory

Part 4

Boundary Value Problems for Ordinary Differential Equations: Practice

Part 5

Introduction to Partial Differential Equations: Theory

Part 6

Introduction to Partial Differential Equations: Practice