Multiple Integration Note and Exercise Packet 1

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This collection of notes is intended to introduce you to double integrals, and double iterated integrals. After completing the first set of notes and exercises you should be able to evaluate double integrals with rectangular domains, and you should be aware of Fubini's theorem for rectangular regions.

1 Double integrals

Double integrals may be rigorously defined using a Riemann sum construction similar to the Riemann sums studied in elementary calculus. An **area** integral in cartesian coordinates is typically given shorthand notation:

$$\iint_R dA \equiv \lim_{\|P_j\|\to 0} \sum_{i\in P_j} \Delta A_i$$

In this definition P_j is a sequence of partitions or subdivisions of the domain $R \subset \mathbb{R}^2$. The ΔA_i is the area of a square or rectangle within the partition. In order for the sum of rectangles to converge to the geometric area of R, the magnitude of the individual rectangles must shrink as the partitions are refined.

If one wishes to compute the volume underneath a continuous function defining the surface $z = f(x, y)$ one can insert the function into the definition:

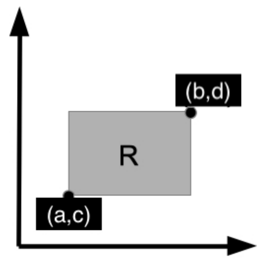
$$\iint_R f(x,y)dA \equiv \lim_{\|P_j\|\to 0} \sum_{i\in P_j} f(x_i^*, y_i^*)\Delta A_i$$

The representative (x_i^*, y_i^*) may be chosen in many ways. This construction is similar to the choices available when constructing Riemann sums in single variable calculus. (e.g. midpoints, left-points, right points etc.) however the complexity hidden in the sequence of partitions and the incremental area ΔA_i makes the definition difficult to use directly. Instead we focus on situations where multiple integrals can be evaluated one at a time. This simpler construction is called an **iterated integral**.

The relationship between double integrals and iterated integrals is explained through **Fubini's theorem**. Since the proof itself establishes that the results from the double integral definition and the iterated integral definition coincide, we will omit this proof and state the results of the theorem in two stages. We focus on the application of this theorem for evaluating integrals.

2 Fubini's Theorem on Rectangles

Suppose that one wishes to compute the area of a rectangle which is aligned with the x and y axes of the regular Cartesian plane. To establish Fubini's theorem for this situation we need only assert the value of the double integral and show that may be written in terms of two separate single variable integrals. See the figure below:



Take a moment to determine the area of this rectangle (using the product of base and height). This value is then shown to be equal to two different expressions using the following algebraic calculations:

$$\iint_R dA = (b-a)(c-d) = x|_a^b \cdot y|_c^d = \int_a^b dx \cdot \int_c^d dy$$

Fubini's theorem for rectangles is obtained when you bring one integral inside the other (since both integrals are independent) and we generalize to double integrals weighted by a function $f(x,y)$. (The generalization is not obvious.)

Fubini's Theorem for Rectangles:

$$\iint_R f(x,y)dA = \int_a^b \int_c^d f(x,y)dydx = \int_c^d \int_a^b f(x,y)dxdy$$

This result is only guaranteed when R is a finite rectangular region with horizontal and vertical edges and $f(x,y)$ is a piecewise continuous function.

This theorem is used for evaluating double integrals on rectangular domains. In this situation you are free to choose the **order of integration** as you see fit. The integral signs and the differentials are grouped inside to outside so in both iterated expressions above the a and b limits should be associated with the dx integral operation.

Examples:

1. Let R be a square with unit side lengths and its bottom left corner at the origin. The area of this square expressed as an iterated integral would be:

$$\iint_R dA = \int_0^1 \int_0^1 dydx$$

when evaluating the integral, one works inside to outside, treating only the current variable as 'active' for integration purposes.

$$\int_0^1 \int_0^1 dydx = \int_0^1 y|_0^1 dx = \int_0^1 1dx = x|_0^1 = 1$$

2. Let D be a rectangle with width of 4 and height of 2 with its center placed at the origin. Its area can be computed as follows:

$$\iint_D dA = \int_{-1}^1 \int_{-2}^2 dxdy = \int_{-1}^1 x|_{-2}^2 dy = \int_{-1}^1 4dy = 4y|_{-1}^1 = 8$$

3. Using the R from above we can calculate the volume underneath the plane $z = x + y$ on this square domain:

$$\iint_R x + ydA = \int_0^1 \int_0^1 x + ydydx = \int_0^1 xy + \frac{y^2}{2}|_{y=0}^{y=1} dx = \int_0^1 x + \frac{1}{2}dx = \frac{x^2}{2} + \frac{x}{2}|_{x=0}^{x=1} = 1$$

4. Using the D from above we can compute the volume under $z = y\cos(x)$.

$$\iint_D y\cos(x)dA = \int_{-1}^1 \int_{-2}^2 y\cos(x)dxdy = \int_{-1}^1 \int_{-1}^1 y\sin(2) - y\sin(-2)dy$$
$$= \int_{-1}^1 2y\sin(2)dy = y^2\sin(2)|_{-1}^1 = 0$$

(Finally this last example gives us a situation where changing the order of integration is helpful, by showing us the final answer in fewer steps. Watch what happens if we begin with the y integral first.)

$$\iint_D y\cos(x)dA = \int_{-2}^2 \int_{-1}^1 y\cos(x)dydx = \int_{-2}^2 \frac{y^2}{2}\cos(x)|_{y=-1}^{y=1} dx = \int_{-2}^2 0dx = 0$$

Exercises:

1. Let P be a rectangle with one side 6 units long, and the other side 3 units long. Place a corner of this rectangle at the origin and use an iterated integral to compute its area.

2. Let Q be a square with both sides π units long. Center this square on the origin and compute its area using an iterated integral.

3. Let $f(x, y) = x^2 + y^2$ compute the volume under this function using P as the domain.

4. Let $f(x, y) = \cos(x)e^y$, Compute the volume under this function using Q as the domain.