Multiple Integration Note and Exercise Packet 6

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This set of notes presents and develops the general change of variables formula and theorem for double integrals. To anchor your understanding I will first present the change of variables theorem in general, then I will demonstrate how the change of variables theorem is used to construct the double polar integral. This serves two purposes: First you get an example of a non-trivial change of variables. Second, since we derived the form of the double polar integral in packet 5, we can feel confident that the example is correct when our answer agrees with our previous construction. Once we have performed this change of variables, we will dial back the complexity and perform some simpler coordinate changes using affine and linear combinations of the original cartesian variables.

Suppose you want to perform a change of variables from (x, y) to a new set of variables (u, v). Suppose further that these variables are connected through the following equations:

$$u = s(x, y), \qquad v = t(x, y)$$

(Note: For conversion from $(x, y) \to (u, v)$ we actually want to know the old variables in terms of the new variables, rather than the formulae given above. Both sets of conversion formulae may be helpful if you can find them.)

In order for the change of variables to be defined for a particular domain Ω we require that both $s(\Omega)$ and $t(\Omega)$ must be 1 to 1. This requirement is very important, without it the theorem doesn't apply and the change of variables cannot proceed.

The algebraic formulation for the generic change of variables in a double integral is as follows:

$$\iint_{\Omega} f(x, y) dA = \iint_{\hat{\Omega}} \hat{f}(u, v) \left| J(u, v) \right| d\hat{A}$$

Here all the complexity is hidden in the notation so let me explain each part, then we will develop a procedure for finding these parts for simple problems.

- Ω is the original integral domain. $\hat{\Omega}$ is the new integral domain, it is the **image** of all the points from the (x, y) plane as they 'come out' in the new (u, v) plane.
- If $f(x, y)$ is the original integrand, then $\hat{f}$ is the image of the function values as they come out in the (u, v) plane. In general converting f into $\hat{f}$ requires either that f is really simple, or that one can find the conversion formulae such that:

  $$x = p(u, v), \qquad y = q(u, v)$$

  with these formulae $\hat{f} = f(p(u, v), q(u, v))$.

- The new component in the converted integral $|J(u, v)|$ is the absolute value of the **Jacobian** determinant. If you have the p, and q above then this factor is computed as:

  $$|J(u, v)| = abs\left(\begin{vmatrix} \frac{\partial p}{\partial u} & \frac{\partial p}{\partial v} \\ \frac{\partial q}{\partial u} & \frac{\partial q}{\partial v} \end{vmatrix}\right)$$

  Note: here we apply an absolute value **after** we compute the determinant. One of the consequences of the the requirement that the conversion be well defined is that this Jacobian must be non-zero everywhere in the interior of the domain. This ensures the absolute value is applied once everywhere to ensure this factor is positive. This absolute value is required and cannot be neglected

  The Jacobian factor is the stretching or compressing factor which converts area in the original coordinate system into area in the new coordinate system. It is a generalization of the $\frac{du}{dx}$ which facilitates u-substitution in single variable calculus.

Example 1:

As stated in the pre-amble we begin by performing a change of variables from cartesian into polar coordinates.

We know the conversion formulae from $(x, y) \to (r, \theta)$ are given by:

$$r = \sqrt{x^2 + y^2}, \qquad \theta = \arctan\left(\frac{y}{x}\right)$$

The formulas which we use to construct the polar integral using the general change of variables are given by:

$$x = r\cos(\theta), \qquad y = r\sin(\theta)$$

We perform a change of variables on a double integral in 3 main steps. (1) Compute the Jacobian factor, (2) convert the integrand, (3) Find the new limits of integration. The example will handle these in general first, then delve into more specific examples for numerical clarity.

1. To compute the Jacobian we use the x and y formulae in terms of the new variables.

$$x = r\cos(\theta), \quad \frac{\partial x}{\partial r} = \cos(\theta), \quad \frac{\partial x}{\partial \theta} = -r\sin(\theta).$$
$$y = r\sin(\theta), \quad \frac{\partial y}{\partial r} = \sin(\theta), \quad \frac{\partial y}{\partial \theta} = r\cos(\theta).$$

   substituting these partial derivatives into the determinant yields:

$$|J(r,\theta)| = \left|r\cos^2(\theta) - (-r\sin^2(\theta))\right| = |r| = r$$

   (This yields exactly the same conversion factor we found when building the polar integral using sector areas.) This Jacobian is well defined right-up to $r = 0$. $r = 0$ can be a problem point in polar coordinates since $(0, \theta)$ coincide for all θ.

2. Since we know both sets of conversion formulae we can convert any smooth $f(x, y)$ into its polar form by using $f(r\cos(\theta), r\sin(\theta))$. (Shortcuts like replacing $x^2 + y^2$ with r^2 directly are reasonable.)

3. Last we must perform domain conversion. With polar coordinates I recommend that you sketch the domain in cartesian, then imagine the polar coordinate grid on top, and try to formulate the region as θ simple or r simple. With other coordinate systems the procedure is different. I will try to illustrate both thought processes with the numerical examples below.

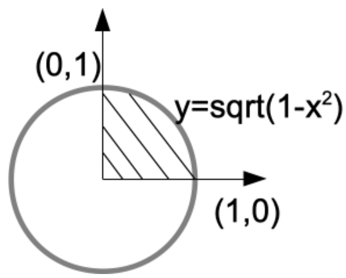
Example Battery 1
(Note the domains of the first two examples coincide with the domains from the exercises for the previous packet, so you can use these to check your answers.

1.

$$\int_0^1 \int_0^{\sqrt{1-y^2}} dxdy$$

   In this example the integrand is just 1, so the double integral represents a geometric area. To convert to polar we need to include the Jacobian factor, and we need to convert the domain.

(a) To convert as a polar area, we graph the region to obtain a sketch like the following:



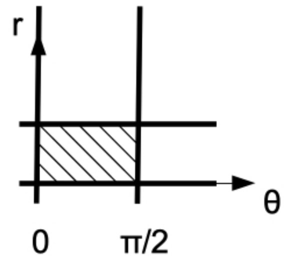
Then we re-interpret the region as a θ-simple region. We use the radial lines $\theta = 0$, and $\theta = \frac{\pi}{2}$ to make the edges of our region, and we use $r = 0$ and $r = 1$ as the inside and outside circles. The resulting integral becomes:

$$\int_0^{\frac{\pi}{2}} \int_0^1 r dr d\theta$$

(b) If we instead convert the limits of integration using the new variables directly, we instead draw a whole new picture. To find (u, v) limits after a change of variables, one first converts all equations defining the original limits of integration into equations in the new variables. The four limits of integration and their corresponding polar representations are:

$$\begin{aligned} x = 0, &\quad r\cos(\theta) = 0 \\ x = 1, &\quad r\cos(\theta) = 1 \\ y = 0, &\quad r\sin(\theta) = 0 \\ y = \sqrt{1 - x^2} \to x^2 + y^2 = 1 \to r^2 = 1 \to r &= 1 \end{aligned}$$

Normally, one now graphs these functions individually and tries to formulate an iterated integral in the new variables. This is often a bit 'fiddly' with polar coordinates, so I'll walk you through the reasoning. First, our original region only has three curves as sides, so one of these equations only matches up with a single point on the boundary of the region. That turns out to be $x = 1$, so we don't need to try to graph $r\cos(\theta) = 1$. The other three curves are satisfied by graphing $\theta = \frac{\pi}{2}$, $\theta = 0$ and $r = 1$. The final boundary point is located at $r = 0$ where both the θ curves intersect. Thus we draw the region as:



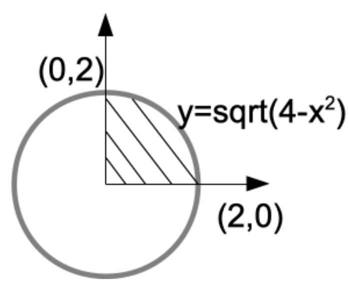
We can formulate this region as either θ simple or r simple and we obtain the same limits as when we overlaid the polar grid on top of the cartesian grid. This re-drawing procedure is how you handle more general variable changes, you try to draw the new region as if the new variables are normal horizontal and vertical lines, you generally do **not** try to see what the new variable coordinate system "looks like" on top of the regular cartesian grid.

2.

$$\int_0^2 \int_0^{\sqrt{4-y^2}} x^2 + y^2 dx dy$$

In our second example we have re-used an old exercise domain, but we have included a "nice" integrand for this conversion. We proceed as before, but now we must include the converted integrand along with the Jacobian factor.

(a) To convert as a polar area, we graph the region to obtain a sketch like the following:



Then we re-interpret the region as a θ-simple region. We use the radial lines $\theta = 0$, and $\theta = \frac{\pi}{2}$ to make the edges of our region, and we use $r = 0$ and $r = 1$ as the inside and outside circles. The resulting integral becomes:

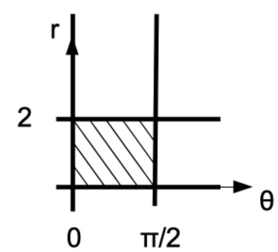
$$\int_0^{\frac{\pi}{2}} \int_0^2 r^3 dr d\theta$$

The integrand converts to r^2, while the Jacobian gives us one extra power of r.

(b) If we instead convert the limits of integration using the new variables directly, we instead draw a whole new picture. To find (u, v) limits after a change of variables, one first converts all equations defining the original limits of integration into equations in the new variables. The four limits of integration and their corresponding polar representations are:

$$\begin{aligned} x = 0, &\quad r\cos(\theta) = 0 \\ x = 2, &\quad r\cos(\theta) = 2 \\ y = 0, &\quad r\sin(\theta) = 0 \\ y = \sqrt{4 - x^2} \to x^2 + y^2 = 4 \to r^2 = 4 \to r = 2 \end{aligned}$$

Normally, one now graphs these functions individually and tries to formulate an iterated integral in the new variables. This is often a bit 'fiddly' with polar coordinates, so I'll walk you through the reasoning. First, our original region only has three curves as sides, so one of these equations only matches up with a single point on the boundary of the region. That turns out to be $x = 1$, so we don't need to try to graph $r\cos(\theta) = 1$. The other three curves are satisfied by graphing $\theta = \frac{\pi}{2}$, $\theta = 0$ and $r = 1$. The final boundary point is located at $r = 0$ where both the θ curves intersect. Thus we draw the region as:



We can formulate this region as either θ simple or r simple and we obtain the same limits as when we overlaid the polar grid on top of the cartesian grid. This re-drawing procedure is how you handle more general variable changes, you try to draw the new region as if the new variables are normal horizontal and vertical lines, you generally do **not** try to see what the new variable coordinate system "looks like" on top of the regular cartesian grid.

3.

$$\int_{-1}^{1}\int_{-\sqrt{1-y^2}}^{0} y - x dx dy$$

In our third example we are building a different polar region and including an integrand which doesn't act particularly nice. In this example we will only formulate using the polar grid, overlaid on the cartesian grid. Each piece is given below, you should try to create the polar formulation yourself and check your intuition against the answers given below.

(a) The Jacobian is the standard polar Jacobian.

(b) Our integrand converts to $r\sin(\theta) - r\cos(\theta)$.

(c) The given region should define the left half of a unit circle.

The correct polar formulation is given by:

$$\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}}\int_{0}^{1} r^2\sin(\theta) - r^2\cos(\theta) dr d\theta$$

Exercise Battery: (1 and 2 are 'standard examples based on the previous examples and notes. 3 and 4 are are important content for standardized tests etc.)

1. Reformulate the following integral in polar coordinates:

$$\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} 2-2\sqrt{x^2+y^2}dydx.$$

   (If your reformulation is correct, it should evaluate to the volume of a right circular cone with height 2 and radius 1.)

2. Reformulate the following integral into polar coordinates:

$$\int_{0}^{1}\int_{-x}^{x} dydx.$$

   (This problem is pretty easy in cartesian, but it gives a simple situation where one of the polar boundaries (and thus polar limits) is a function rather than a constant. If your reformulation is correct it should evaluate to the exact area of the domain (which is an isosceles triangle on its side in cartesian coordinates.)

3. **CLASSICAL EXERCISE:** (This problem is absolutely obligatory and forms a hidden link between π and the normal distribution in probability and statistics. I usually do this as a class demonstration showing how sometimes changing coordinates can make intractable or impossible formulations tractable. This example has the advantage of actually being useful since it allows us to normalize the normal distribution from probability.)

   Consider the following:

$$\int_{\infty}^{\infty} \int_{-\infty}^{\infty} e^{-x^2-y^2} dydx$$

   (a) Convert this improper integral into polar coordinates. (Note, you only need enough θ to cover each point once.)

   (b) Carefully evaluate the integral using $drd\theta$ order of integration. You may need a $u-sub$ or you may be able to work out the integral without it. The final result should be π.

4. **Challenge Problem:** Use careful algebraic manipulation of the above result (messing about with properties of exponents, integral position, and variable names) to show that:

$$\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}$$