Multiple Integration Note and Exercise Packet 8

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This packet develops the form of Fubini's theorems for triple integrals and attempts to demonstrate how to visualize and translate between three dimensional shapes and triple integrals representing their volumes. To help simplify the ideas and facilitate your understanding I will limit my attention to surfaces and shapes you have seen in geometry and can visualize easily. Many of these shapes have circular and spherical symmetry, so we will typically want to adopt either cylindrical or spherical coordinate systems to do actual computations of these integrals. This packet, however will focus on cartesian coordinates and generally avoid evaluating the integrals explicitly.

Remark: I have some doubts about how well this particular material translates into self-study form, the ideas are very visual and when I teach these in the classroom there is a combination of board drawings, pantomime and wild gesticulation to try to help the ideas meld together. Do your best to understand, my expectations about skills and results will be somewhat diminished.

Fubini's theorems in 3-d

When transitioning from 2d to 3d iterated integrals we get an increase in both complexity and versatility. The complexity increases because the whole picture of the domain 'lives' (or is embedded) in 3 dimensional space. The added versatility comes from the plethora of different orders of integration which might be used to formulate iterated integrals. There are a total of 6 orders of integration possible in standard cartesian coordinates. Rather than create a theorem and picture for each possible order of integration we will state a general form for Fubini's theorem and demonstrate a few visualization ideas which break the iterated integrals down into smaller parts.

Let the cartesian components (x, y, z) be mapped bijectively into $\{v_1, v_2, v_3\}$. Then if a domain V is v_3 simple there are corresponding limits of integration which balance:

$$\iiint_V dV = \int_a^b \int_{L(v_3)}^{U(v_3)} \int_{P(v_2,v_3)}^{Q(v_2,v_3)} dv_1 dv_2 dv_3$$

Note very carefully the dependence of the limits of integration. The two innermost limits are surfaces in v_1 which can depend on the outer two variables. The two middle limits are curves which can depend on the outermost variables. The outer limits must be constants if the iterated integral represents a fixed volume.

The theorem is unchanged if you include an integrand $f(v_1, v_2, v_3)$; just place the f inside all integrals on both sides of the equation.

The mathematical skills we want to explore involving these constructions involve formulating volumes using triple integrals, as well as understanding the volumes represented by given triple integrals. The evaluation of triple integrals involves nothing new, you simple evaluate the integrals one after the other, but now you have three integrals to work through rather than two.

Example-Exercise Battery: We will begin by visualizing given iterated integrals. The visualization process is more concrete than formulation since choices about order of integration and limits of integration are already made.

Our examples will mostly be traditional geometric shapes, we will start from the simple and move to the complex. In each example the visualization will be broken down in two ways: **Inner to Outer** and **Outer to Inner** (**Remark:** This distinction is my own, I haven't seen any texts which offer any really useful procedures for this process, most published materials at this point computer generated three dimensional plots and a library of simple functions to avoid explaining the process itself.)

Inner to Outer Visualization:
Rationale: The inner to outer visualization attempts to understand the triple iterated integral by starting from potentially the most complicated part and connecting that with a double integral exactly as we have handled before.

- Take the innermost two limits of integration as visualize them as surfaces. If we imagine v_1 as the vertical axis of the coordinate system these surfaces will define the bottom and the top sides of our three dimensional domain.
- Next, take the outer two integrals and visualize them as a v_3 simple region in 2-d. The outer two integrals define the "footprint" of the volume defined by the triple integral.
- The full region is then visualized as the space inside the footprint between the floor and ceiling defined by the innermost limits.

- If you try to formulate an integral from inner to outer, you write the top and bottom surfaces you want to use as functions of the outer-two variables, then you imagine these two surfaces collapsing together and you formulate an ordinary 2-d region for the outline of their collapsed footprint. (Finding this footprint precisely often requires finding intersections of surfaces.)

Outer to Inner Visualization:
Rationale: The outer to inner visualization attempts to understand the integral by organizing the volume into slices, then carefully investigating how those slices change throughout the object. Conceptually you build a three dimensional object out of cards, but the cards may vary in shape and size.

- Take the outer two limits of integration. These should be constants, they describe the bottom and top of the deck. The object is then visualized in flat slices perpendicular to the outermost variable's axis.
- Think of the outermost variable as a fixed parameter and investigate the inner four limits of integration. These inner limits of integration define the shape of each 'card' but since they can depend on the outer variable these cards may change throughout the object.
- If you try to formulate an integral using this visualization, you must pick the outermost integration variable and imagine the object as held or pinned between two parallel planes. These define the outermost limits. Next you imagine cutting through the object at a single fixed v_3 (whichever the outermost variable was. This cut is parallel to the holding planes.) when you make this cut, you formulate the double integral which has the correct limits of integration for the shape at the cut.

1.

$$\int_0^1 \int_0^1 \int_0^1 dxdydz$$

This iterated integral defines a cube. All side-lengths are one. One corner of the cube is at the origin, and the opposite corner is at the point $(1, 1, 1)$.

Inner to Outer Visualization:

(a) The innermost limits are visualized as the **planes** $x = 0$ and $x = 1$.

(b) We imagine flattening these two planes together and we are left with a square footprint defined by the $dydz$ integrals.

Outer to Inner Visualization:

(a) The outermost limits are visualized as holding planes $z = 0$ and $z = 1$.

(b) We imagine the cube as a bunch of slices stacked up along the z axis. Since each slice is square with sidelengths of 1, all the cards have the same size.

Exercise: Repeat these visualizations with another cube (you can change the side lengths and/or the position of the cube), but use $dzdydx$ as your order of integration.

Exercise: Change a single one of the innermost limits of integration to represent a new volume with a 'slanted side' (Many possible solutions here.)

2.

$$\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_{-\sqrt{1-x^2-y^2}}^{\sqrt{1-x^2-y^2}} dzdydx$$

This iterated integral defines the volume of a unit-sphere in cartesian coordinates.
Inner to Outer Visualization:

(a) The innermost limits are visualized as the **surfaces**:

$$z = \sqrt{1-x^2-y^2} \to x^2+y^2+z^2 = 1$$
$$z = -\sqrt{1-x^2-y^2} \to x^2+y^2+z^2 = 1$$

Both of these limits are the functional representations of the top and bottom of the unit sphere respectively.

(b) We imagine flattening these two curves together and we are left with a circular footprint. Since we flatten the surfaces vertically with no stretching, the resulting footprint is the unit circle.

Outer to Inner Visualization:

(a) The outermost limits are visualized as holding planes $x = -1$ and $x = 1$.

(b) We imagine the volume as a stack of cards aligned with the x axis. If we define $R = \sqrt{1-x^2}$ then the inner two integrals have the form:

$$\int_{-R}^{R}\int_{-\sqrt{R^2-y^2}}^{\sqrt{R^2-y^2}} dzdy$$

which is the structure of a circle of radius R. This means our integral described a deck of circular cards with radius which varies like $R = \sqrt{1-x^2}$. Since this is itself a circle, the result is a sphere.

Exercise: The sphere is very symmetric. Find at least two ways to change the given iterated integral so that it represents the volume of a hemisphere. (You don't need to capture the same hemisphere each time.) Visualize each of your constructions and make sure it is correct.

Exercise: Formulate the volume of a sphere of radius 2 rather than 1. Try changing the order of integration.

3.

$$\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_{\sqrt{x^2+y^2}}^{1} dzdydx$$

This iterated integral defines the volume of a right, circular cone with height 1 and radius 1.

(a) Perform an inner to outer visualization. Determine whether the cone is 'tip down' or 'tip up'.

(b) Perform an outer to inner visualization. Give a rough geometric description of the slices in this order of integration.

4. **Exercise:** Draw a sketch of a circular cylinder which has a height of 4 and a radius of 1. Center your cylinder in a cartesian coordinate system and align the circular axis with one of your axes. Construct an iterated integral representing the volume of your cylinder. You may use either visualization strategy, but generally it will be easier if you make the central axis either the innermost integral or the outermost integral.

(**Note:** These problems and exercises will be further utilized in the next packet where we explore the cylindrical and spherical coordinate systems.)