Study Guide: Final Exam

The final exam will include HWs #8-10, and the following learning objectives from Chapters 4, 6 and 7.

CH 4: Laplace Transform (HW #8)

- Frequency Response of an LTIC System
  - Sketch the Bode plots of an LTIC system (4.9)
    - Lathi: 4.9-4, 9-5; *HW #8, Quiz #8*

CH 6: Fourier Series (HW #8)

- Exponential Fourier Series
  - Calculate the Fourier coefficients and construct the exponential Fourier series of a periodic signal (6.3).
    - Lathi: 6.3-1; *HW #9 (graded problem)*
- Fourier Spectra
  - Plot the amplitude and phase spectra of a periodic signal (6.3).
    - Lathi: 6.3-1, 3-7 (a); *HW #9 (graded problem), Quiz #9*
  - Determine the bandwidth of a signal (6.3).
- LTIC system response to periodic inputs
  - Determine the response of an LTIC system to a periodic input from the transfer function (6.4).
    - Lathi: 6.4-5; *HW #9 (graded problem)*
  - Apply Parseval's Theorem to calculate the power of the periodic input/output signals (6.3).
    - *HW #9 (graded problem)*

CH 7: Fourier Transform (HW #10)

- Fourier Transform
  - Determine the Fourier transform of a signal using the properties of Fourier transforms and the Fourier transform table (7.2-3)
    - Lathi: 7.3-1, 3-4, 3-5; *HW #10 (graded problem)*
- Signal energy
  - Apply Parseval's Theorem to calculate the energy of an energy signals (7.6).
    - Lathi: 7.6-2, 6-3, 6-9; *HW #10 (graded problem)*

Bode Plots

Pole = Bottom, zero = Top

Gain

need normalized form Ex: $H(s)=\frac{s(s+10)}{s^2+50s+2500}=\frac{s\left(\frac{s}{10}+1\right)}{\frac{s^2}{2500}+\frac{s}{50}+1}\frac{10}{2500}$ Gain (dB) = $20\log\left(\frac{10}{2500}\right)$

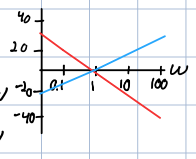
$2\zeta\omega_n$

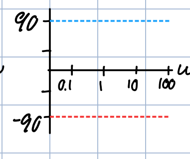
Plots are $\log_{10}$ scale

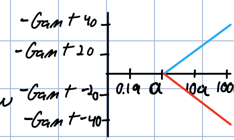
Find zeroes & poles → Zeroes: 0, −10 Poles: $-25\pm25\sqrt{3}i$

Pole at origin: amplitude decay at $-20\frac{dB}{dec}$, cross 0dB at w=1, Constant Phase −90° ∀w

Zero at origin: amplitude grows at $20\frac{dB}{dec}$, cross 0dB at w=1, Constant Phase 90° ∀w

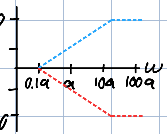






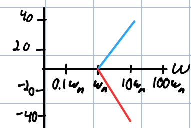
1^{st} Order Pole, w=−a, a=|zero|: amp. = 0dB for w<<a, decays at $-20\frac{dB}{dec}$ for w>>a, phase = 0° for $w<\frac{a}{10}$, = −45° for w=a, −90° for w>10a

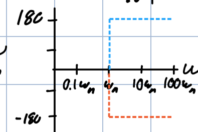
1^{st} order zero: amp. = 0dB for w<<a, grows at $20\frac{dB}{dec}$ for w>>a, phase = 0° for $w<\frac{a}{10}$, = 45° for w=a, 90° for w>10a



2^{nd} Order Complex Pole: amp = 0 for $w<<w_n$, decays @ $-40\frac{dB}{dec}$ for $w>>w_n$, phase = 0 for $w<<w_n$, −180° for $w>>w_n$

2^{nd} Order Complex zero: amp = 0 for $w<<w_n$, grows @ $40\frac{dB}{dec}$ for $w>>w_n$, phase = 0 for $w<<w_n$, 180° for $w>>w_n$





Fourier Series

T_0 = period $w_0=\frac{2\pi}{T_0}$

$x(t)=\sum_{n=-\infty}^{\infty}D_ne^{jn\omega_0t}\longleftrightarrow y(t)=\sum_{n=-\infty}^{\infty}H(jn\omega_0)D_ne^{jn\omega_0t}$ $D_n=\frac{1}{T_0}\int_{T_0}x(t)e^{-jn\omega_0t}dt$

Bandwidth

If x(t) real, then $|D_n|=|D_{-n}|$ & $\angle D_n=-\angle D_{-n}$

Parseval Theorem – Applies to periodic signal w/ Fourier transform. $P=\sum_{n=-\infty}^{\infty}|D_n|^2=|D_0|^2+2|D_1|^2+2|D_2|^2+\ldots+2|D_n|^2$

a) $T_0=\pi$ $w_0=\frac{2\pi}{T_0}=2$ Find Fourier Series

$D_n=\frac{1}{T_0}\int x(t)e^{-jnw_0t}dt=\frac{1}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}x(t)e^{-jn2t}dt=\frac{1}{\pi}\left[\int_{-\frac{\pi}{2}}^{0}e^{-jn2t}dt+\int_0^{\frac{\pi}{2}}e^{-t}e^{-jn2t}dt\right]$

$=\frac{1}{\pi}\left[\frac{e^{-jn2t}}{-jn2}\Big|_{-\frac{\pi}{2}}^{0}+\frac{e^{-t(1+jn2)}}{-(1+jn2)}\Big|_0^{\frac{\pi}{2}}\right]=\frac{1}{\pi}\left[\left(\frac{1}{-jn2}-\frac{e^{jn\pi}}{-jn2}\right)+\left(\frac{e^{-\frac{\pi}{2}(1+jn2)}}{-(1+jn2)}-\frac{1}{-(1+jn2)}\right)\right]$

u = −jn2t, du = −jn2; u = −t(1+jn2), du = −(1+jn2)

$=\frac{1}{\pi}\left[\frac{1-e^{jn\pi}}{-jn2}+\frac{e^{-\frac{\pi}{2}(1+jn2)}-1}{-(1+jn2)}\right]$

$\frac{1}{-jn2}du=1$

$D_0=\int_{-\frac{\pi}{2}}^{0}1\,dt+\int_0^{\frac{\pi}{2}}e^{-t}dt:+\frac{\pi}{2}-e^{-\frac{\pi}{2}}+1\approx0.75214$

$D_1=\frac{1}{\pi}\left[\frac{1-e^{j\pi}}{-j2}+\frac{e^{-\frac{\pi}{2}(1+j2)}-1}{-(1+j2)}\right]\approx0.077+.16j$

$D_2=\frac{1}{\pi}\left[\frac{1-e^{j2\pi}}{-j2\cdot2}+\frac{e^{-\frac{\pi}{2}(1+j2\cdot2)}-1}{-(1+j2\cdot2)}\right]\approx.015-.059j$

$D_3=\frac{1}{\pi}\left[\frac{1-e^{j3\pi}}{-j3\cdot2}+\frac{e^{-\frac{\pi}{2}(1+j3\cdot2)}-1}{-(1+j3\cdot2)}\right]\approx.01+.044j$

$D_4=\frac{1}{\pi}\left[\frac{1-e^{j4\pi}}{-jn2}+\frac{e^{-\frac{\pi}{2}(1+jn2)}-1}{-(1+jn2)}\right]\approx.0039-.031j$ ← used calculator for these

$D_0=0.75$
$D_1=.077+.16j$
$D_2=0.015-.059j$
$D_3=.01+0.044j$
$D_4=.0039-.031j$

Parseval's

D) $P_{x_4}=|D_0|^2+2|D_1|^2+2|D_2|^2+2|D_3|^2+2|D_4|^2$

$=.56+.063+.0072+.004+.0002=0.6362$

$D_0|=\sqrt{.75^2}=.75$
$D_1|=\sqrt{.077^2+.16^2}=.177$
$D_2|=\sqrt{.015^2+.059^2}=.06$
$D_3|=\sqrt{.01^2+.044^2}=.04$
$D_4|=\sqrt{.0039^2+.031^2}=.0321$

$P_y=|\tilde{D}_0|^2+2|\tilde{D}_1|^2+2|\tilde{D}_2|^2+|\tilde{D}_3|^2+|D_4|^2$

$P_y=.57+.033+.0014+.0004+.0002=0.5885$

$P_{x_4}-P_y=.0477$

$x_4(t)=D_0+D_1e^{-j2t}+D_2e^{-j4t}+D_3e^{-j6t}+D_4e^{-j8t}+D_1^*e^{j2t}+D_2^*e^{j4t}+D_3^*e^{j6t}+D_4^*e^{j8t}$

c) $y(t)=\sum_{n=-\infty}^{\infty}H(jw)D_ne^{jnw_0t}=\sum_{n=-4}^{4}H(j\omega)D_ne^{j2nt}$ w = nw0 Solve $\tilde{D}_n$'s in matlab .0027

$H(j\omega)D_n=\tilde{D}_n$ $\tilde{D}_0=.75$, $\tilde{D}_1=.12+.044j$, $\tilde{D}_2=.021-.018j$, $\tilde{D}_3=.014+.0013j$, $\tilde{D}_4=.003$

$y(t)=\tilde{D}_0+\tilde{D}_1e^{-j2t}+\tilde{D}_2e^{-j4t}+\tilde{D}_3e^{-j6t}+\tilde{D}_4e^{-j8t}+\tilde{D}_1^*e^{j2t}+\tilde{D}_2^*e^{j4t}+\tilde{D}_3^*e^{j6t}+\tilde{D}_4^*e^{j8t}$ ← Series as input to H(s)

a) Find $x(t)=\mathcal{F}^{-1}[X(\omega)]$

eq. 20 Table 7.1

$X(\omega)=\Delta\left(\frac{\omega}{2W}\right)=\Delta(\omega)$ $W=\frac{1}{2}$

$X(t)=\mathcal{F}^{-1}[X(\omega)]=\frac{\frac{1}{2}}{2\pi}sinc^2\left(\frac{\frac{1}{2}t}{2}\right)$

$=\frac{1}{4\pi}sinc^2\left(\frac{t}{4}\right)$

b) Find $y(t)=\mathcal{F}^{-1}[X(\omega)]$. Note that $Y(\omega)$ is the combination of 2 triangle functions (in red)

$Y(\omega)=X\left(\frac{1}{4}(\omega+1)\right)+X\left(\frac{1}{4}(\omega-1)\right)$

$y(t)=\mathcal{F}^{-1}[Y(\omega)]=\mathcal{F}^{-1}\left[X\left(\frac{1}{4}(\omega+1)\right)\right]+\mathcal{F}^{-1}\left[X\left(\frac{1}{4}(\omega-1)\right)\right]$

$=4\mathcal{F}^{-1}\left[\frac{1}{4}X\left(\frac{1}{4}(\omega+1)\right)\right]+4\mathcal{F}^{-1}\left[\frac{1}{4}X\left(\frac{1}{4}(\omega-1)\right)\right]$

$=4x(4t)e^{jt}+4x(4t)e^{-jt}=4\left(\frac{1}{4\pi}sinc^2\left(\frac{t}{4}\right)e^{jt}+\frac{1}{4\pi}sinc^2\left(\frac{t}{4}\right)e^{-jt}\right)$

W = 2, Eq. 20 $=\frac{1}{\pi}sinc^2\left(\frac{t}{4}\right)\left(e^{jt}+e^{-jt}\right)$

c) Use Parseval's Theorem to calculate the energy of $y(t)$.

$E=\frac{1}{\pi}\int_0^{\infty}|Y(\omega)|^2d\omega=\frac{1}{\pi}\left[\int_0^1d\omega+\int_1^3\left(-\frac{1}{2}\omega+1.5\right)^2d\omega\right]=\frac{1}{\pi}+\frac{-2u^3}{3}\Big|_{1.5}^{3=\omega}=\frac{1}{\pi}+\frac{2}{3\pi}=\frac{5}{3\pi}$

$u=-\frac{1}{2}\omega+1.5$

$-2du=1$

d) Determine the bandwidth W of $y(t)$ such that 95% of the energy is contained within the frequencies $\omega\in[-W,W]$

$\frac{1}{\pi}+\frac{-2u^3}{3\pi}\Big|_1^W=\frac{1}{\pi}+\frac{-2(-.5W+1.5)^3}{3\pi}+\frac{2}{3\pi}=\frac{W^3-9W^2+27W-7}{12\pi}$

$.95E_{y(t)}=\frac{1}{\pi}\int_0^W|Y(\omega)|^2d\omega=.95\left(\frac{5}{3\pi}\right)$ .503991

$=\frac{1}{\pi}\left[\int_0^1d\omega+\int_1^W\left|(-.5\omega+1.5)\right|^2d\omega\right]=\frac{1}{\pi}$

$.95\left(\frac{5}{3\pi}\right)=\frac{W^3-9W^2+27W-7}{12\pi}$ → $19.00001=W^3-9W^2+27W-7$

$W\approx2$

Partial Fraction Decomp

Distinct Linear Factors $=\frac{x}{x^2+x}=\frac{x}{x(x+1)}=\frac{A}{x}+\frac{B}{x+1}$

Product of linear factors $=\frac{x}{x^2(x+1)^2(x+2)}=\frac{A}{x}+\frac{B}{x^2}+\frac{C}{x+1}+\frac{D}{(x+1)^2}+\frac{E}{x+2}$

Irreducible Factors $=\frac{x}{(ax^2+bx+c)^2}=\frac{Ax+B}{(ax^2+bx+c)}+\frac{Cx+D}{(ax^2+bx+c)^2}$

Fourier Transform

Applies to aperiodic signals, $T_0\to\infty$, $w_0=\frac{2\pi}{T_0}$, series → integral

Inverse Fourier Transform: $x(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}X(\omega)e^{jt\omega}d\omega=F^{-1}[X(\omega)]$

Fourier Transform: $X(\omega)=\int_{-\infty}^{\infty}x(t)e^{-j\omega t}dt=F[x(t)]$

Frequency Response

$y(t)=A|H(j\omega)|\cos(\omega t+\angle H(j\omega)+\theta)$

$|H(j\omega)|=\frac{\sqrt{num^2}}{\sqrt{den^2}}$ $\angle H(j\omega)=\tan^{-1}\left(\frac{Im\ Num}{Re\ Num}\right)-\tan^{-1}\left(\frac{Im\ Den}{Re\ Den}\right)$

$\sin(\omega t)=\cos(\omega t-90°)$

Linearity: $ax_1(t)+bx_2(t)\longleftrightarrow aX_1(\omega)+bX_2(\omega)$, Duality: $X(t)\leftrightarrow2\pi x(-\omega)$, Conjugate Symmetry: $X(-\omega)=X^*(\omega)$ $x_1(t)x_2(t)\leftrightarrow\frac{1}{2\pi}X_1(\omega)*X_2(\omega)$

Scaling: $x(at)=\frac{1}{|a|}X\left(\frac{\omega}{a}\right)$, Time Shift: $x(t-t_0)\leftrightarrow X(\omega)e^{-j\omega t_0}$, Freq. Shift: $x(t)e^{j\omega_0t}\leftrightarrow X(\omega-\omega_0)$, Convolution: $\to x_1(t)*x_2(t)\leftrightarrow X_1(\omega)X_2(\omega)$

Signal Energy: $E_x=\frac{1}{\pi}\int_0^{\infty}|x(\omega)|^2d\omega$

Parseval Theorem: energy of x(t) results from energy contributed by all spectral component in $X(\omega)$; $|X(\omega)|^2$ is energy spectral density

Transfer Functions

impulse input

$Q(D)y(t)=P(D)x(t)\leftrightarrow H(s)=\frac{P(s)}{Q(s)}$, $y(t)=h(t)*x(t)\overset{\mathcal{L}}{\leftrightarrow}Y(s)=H(s)x(s)$

Ex) $(D^2+5D+6)y(t)=(D+1)x(t)$, $x(t)=e^{-4t}u(t)$ roots of numerator = zeroes

$(s^2+5s+6)Y(s)=(s+1)X(s)$, $H(s)=\frac{Y(s)}{x(s)}=\frac{s+1}{(s^2+5s+6)}$ roots of denom. = poles

Poles of H(s) determine BIBO stability. Stable if all poles Re<0

Determines internal stability if P(s) & Q(s) don't have common root

error response: $e(t)=Y(t)-X(t)=Y(\infty)-X(\infty)=\lim_{t\to\infty}(Y(t)-X(t))$

BIBO Stability

1. Take Z-S Response
2. Integrate $\int_{-\infty}^{\infty}$ If it exists, $\int_{-\infty}^{\infty}$
3. Finite # = Stable Else unstable

1. Find characteristic equation
2. Find roots = to 0
   - Distinct: $e^{\lambda_1t},e^{\lambda_2t}$
   - Repeated: $e^{\lambda t},te^{\lambda t}$
   - Complex: $e^{(\alpha+j\beta)t},e^{(\alpha-j\beta)t}$; $Ce^{\alpha t}\cos(\beta t+\theta)$
3. Solve y(0) conditions for C's

$\frac{-b\pm\sqrt{b^2-4ac}}{2a}$

ax^2+bx-c roots must add to b mult to c

Int. Stable = all Ch. mode converge to 0
Int. unstable = a Ch. mode diverges from 0
Int. marginally = not stable but don't grow over time
Only stable if all λ have −Re part
Only unstable if at least 1 λ has −Re
Only marginally stable if λ=0 or Re = 0 & is not repeated

If internally stable, BIBO stable
BIBO Stable could be anything