PHYS 615 HW4

$F = \begin{pmatrix} -y \\ x \end{pmatrix} = -y\,dx + x\,dy$

is F conservative? NO

$\vec{\nabla} \times \vec{F} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -y & x & 0 \end{vmatrix} = \hat{x}\left|\frac{\partial x}{\partial z}\right| + \hat{y}\left|\frac{\partial y}{\partial z}\right| + \hat{z}\left|\frac{\partial x}{\partial x} + \frac{\partial y}{\partial y}\right|$, 0, 0, $0 \neq \boxed{2}$

1.a) $\int_P^0 -y\,dx + x\,dy + \int_0^Q -y\,dx + x\,dy = \int_1^0 -0 \cdot dx + x \cdot 0 + \int_0^1 -y \cdot 0 + 0 \cdot dy = \boxed{0}$ a)

$dy, y = 0$

b) var: x, $y = -x+1$, $dy = -dx$

$\Rightarrow \int_1^0 -(1-x) - x\,dx = \int_1^0 -1\,dx = -x\Big|_1^0 = 0 - (-1) = \boxed{1}$ b)

c) parameterize: $x = \cos\theta$, $y = \sin\theta$, $dx = -\sin\theta\,d\theta$, $\theta: (0, \pi/2)$, $dy = \cos\theta\,d\theta$

$\int_0^{\pi/2} (-\sin\theta \cdot -\sin\theta\,d\theta) + \cos^2\theta\,d\theta = \int_0^{\pi/2} 1\,d\theta = \boxed{\frac{\pi}{2}}$ c)

2. Gradient describes a partial derivative constructed using whatever directional derivatives your coordinate system of choice defines. It represents a vector valued derivative used to describe how a function changes with direction. A line integral for a force acting on an object as it moves describes the work the force does on the object. Curl between $\vec{\nabla}$ and $\vec{F}$, notated as $\vec{\nabla} \times \vec{F}$, can help differentiate a conservative force from a non-conservative force. If $\vec{\nabla} \times \vec{F} = 0$, then F is conservative as long as it depends on position. We learned that $\vec{F} = -\vec{\nabla} U$, so the negative directional derivative of potential energy is equal to the force, which cannot be defined if the force is non-conservative. Curl also quantifies the circulation of a field.

3. Gradient: $f(x,y) = x^2 - xy$, $\vec{\nabla} f = \begin{bmatrix} 2x - y \\ -x \end{bmatrix}$ $f(x,y) = -x^4 + 4(x^2 - y^2) - 3$, $\vec{\nabla} f = \begin{bmatrix} -4x^3 + 8x \\ -8y \end{bmatrix}$

$f(x,y,z) = x - xy + z^2$, $\vec{\nabla} f = \begin{bmatrix} 1 - y \\ -x \\ 2z \end{bmatrix}$

No, fluid at $(0,0)$ has no flow

Divergence and Curl: $\vec{v}(x,y,z) = \begin{bmatrix} x^3 + y^2 + z \\ ze^x \\ xyz - 9xz \end{bmatrix}$ eval at $(0,1,2)$

$\vec{\nabla} \times \vec{v} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ v_x & v_y & v_z \end{vmatrix} \to \hat{x}(xz - e^x) + \hat{y}(1 - (yz - 9z)) + \hat{z}(ze^x - 2y)$

$\vec{\nabla} \times \vec{v}(0,1,2) = \begin{bmatrix} -1 \\ 17 \\ 0 \end{bmatrix} \to \|\vec{\omega}\| = \frac{1}{2}\sqrt{x^2 + y^2 + z^2} = \frac{1}{2}\sqrt{18}$ ≈ 8.5 rad/s

parallel to xz plane

4. $\vec{\nabla}\times\vec{F}$, $\vec{F} = \frac{\gamma}{r^2}\hat{r}$ spherical: $\vec{\nabla} = \frac{\partial F}{\partial r} + \frac{1}{r}\frac{\partial F}{\partial \theta} + \frac{1}{r\sin\phi}\frac{\partial F}{\partial \phi}$

$$\vec{\nabla}\times\frac{\gamma}{r^2}\hat{r} = \begin{vmatrix} \frac{\hat{r}}{r^2\sin\phi} & \frac{\hat{\theta}}{r\sin\theta} & \frac{\hat{\phi}}{r} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi} \\ \frac{\gamma}{r^2} & r\cdot 0 & r\sin\phi\cdot 0 \end{vmatrix} = \begin{cases} \hat{r}: \frac{1}{r^2\sin\phi}\cdot 0 = 0 ✓ \\ \hat{\theta}: \frac{1}{r\sin\phi}\cdot\frac{\partial}{\partial\theta}\left[\frac{\gamma}{r^2}\right] = 0 ✓ \\ \hat{\phi}: \frac{1}{r}\frac{\partial}{\partial\theta}\cdot\left[\frac{\gamma}{r^2}\right] = 0 ✓ \end{cases}$$

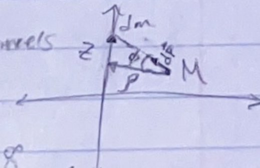
5. $\vec{\nabla}\times\vec{F}$

Cartesian:
$$\begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ u_x & u_y & u_z \end{vmatrix} = \hat{x}\left(\cancel{\frac{\partial^2 u}{\partial y\partial z}} - \cancel{\frac{\partial^2 u}{\partial z\partial y}}\right)^{0} + \hat{y}\left(\cancel{\frac{\partial u}{\partial z\partial x}} - \cancel{\frac{\partial u}{\partial x\partial z}}\right)^{0} + \hat{z}\left(\cancel{\frac{\partial u}{\partial x\partial y}} - \cancel{\frac{\partial u}{\partial y\partial x}}\right)^{0} = \boxed{0}$$

Cylindrical:
$$\begin{vmatrix} \frac{\hat{r}}{r} & \hat{\theta} & \frac{\hat{z}}{r} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial z} \\ u_r & r u_\theta & u_z \end{vmatrix} = \left(\cancel{\frac{\partial u}{r\partial z\partial\theta}} - \cancel{\frac{\partial u}{\partial z\partial\theta}}\right)\hat{r} + \left(\cancel{\frac{\partial u}{\partial z\partial r}} - \cancel{\frac{\partial u}{\partial z\partial r}}\right)^{0}\hat{\theta} + \left(\cancel{\frac{\partial u}{\partial r\partial z}} - \cancel{\frac{\partial u}{r\partial\theta\partial r}}\right) = \boxed{0}$$

Spherical:
$$\begin{vmatrix} \frac{\hat{r}}{r^2\sin\phi} & \frac{\hat{\theta}}{r\sin\phi} & \frac{\hat{\phi}}{r} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \phi} \\ u_r & r u_\theta & r\sin\phi\, u_\phi \end{vmatrix} = \left(\cancel{\frac{\partial u}{r\partial\phi\partial\theta}} - \cancel{\frac{\partial u}{r\sin\phi\,\partial\phi\partial\theta}}\right)\hat{r} + \left(\cancel{\frac{\partial u}{r\sin\phi\,\partial r\partial\phi}} - \cancel{\frac{\partial u}{\partial r\partial\phi}}\right)\hat{\theta} + \left(\cancel{\frac{\partial u}{\partial\theta\partial r}} - \cancel{\frac{\partial u}{r\partial r\partial\theta}}\right)\hat{\phi} = \boxed{0}$$

6. a) mass on rod is modelled as infinitesimal, and since this is radially symmetric, θ cancels



with model shown right of $\hat{z}$: $d\vec{F} = \frac{GM\,dm}{\rho^2+z^2}\left(-\cos\phi\,\hat{\rho} + \sin\phi\,\hat{z}\right) = GM\,dm\left(\frac{z\hat{z}}{(\rho^2+z^2)^{3/2}} - \frac{\rho\hat{\rho}}{(\rho^2+z^2)^{3/2}}\right)$

$\mu = \frac{dm}{dz}$, $dm = \mu dz$, $\cos\phi = \frac{\rho}{\sqrt{\rho^2+z^2}}$, $\sin\phi = \frac{z}{\sqrt{\rho^2+z^2}}$

$u = z^2+\rho^2$, $du = 2z\,dz$

$$d\vec{F} = GM\mu\left(\frac{z\hat{z}}{(\rho^2+z^2)^{3/2}} - \frac{\rho\hat{\rho}}{(\rho^2+z^2)^{3/2}}\right)dz \Rightarrow \vec{F} = GM\mu\int_{-\infty}^{\infty}\frac{z\cdot\hat{z}}{(\rho^2+z^2)^{3/2}}dz - GM\mu\int_{-\infty}^{\infty}\frac{\rho\cdot\hat{\rho}}{(\rho^2+z^2)^{3/2}}dz$$

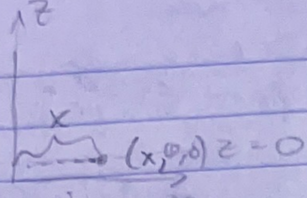
$$\Rightarrow GM\mu\left(-\frac{\hat{z}}{\sqrt{\rho^2+z^2}}\Bigg|_{-\infty}^{\infty} - \frac{2z\hat{\rho}}{\rho\sqrt{\rho^2+z^2}}\Bigg|_{0}^{\infty}\right) = -GM\mu\left[0 + \frac{2}{\rho}\hat{\rho}\right] = \boxed{\frac{-2GM\mu}{\rho}\hat{\rho}}$$ a)

$\cos\phi$ is an even function

$$\lim_{M\to\infty}\frac{2z}{\rho\cdot z\sqrt{1+\frac{\rho^2}{z^2}}}\Bigg|_0^M = \frac{2}{\rho} - 0$$

$\frac{1}{\sin} \rightarrow \frac{\sin\cdot 0 - 1\cdot\cos}{\sin^2} = \cot\csc$

6.b) $\vec{F}_g^{\,cart} \Rightarrow F_{yg}^{cart}, F_{zg}^{cart} = 0$



$$F_g^{\,x} = -\frac{GM\mu}{x}\hat{x}$$

$$\vec{\nabla}\times\vec{F}_g^{\,x} = \left(\cancel{\frac{\partial F_z}{\partial y}} - \cancel{\frac{\partial F_y}{\partial z}}\right)\hat{x} + \left(\cancel{\frac{\partial F_x}{\partial z}} - \cancel{\frac{\partial F_z}{\partial x}}\right)\hat{y} + \left(\cancel{\frac{\partial F_y}{\partial x}} - \cancel{\frac{\partial F_x}{\partial y}}\right)\hat{z} = \boxed{0} \text{ b)}$$

$$\text{c) } \nabla\times F = \left(\cancel{\frac{1}{\rho}\frac{\partial F_z}{\partial\phi}} - \cancel{\frac{\partial F_\phi}{\partial z}}\right)\hat{\rho} + \left(\cancel{\frac{\partial F_\rho}{\partial z}} - \cancel{\frac{\partial F_z}{\partial \rho}}\right)\hat{\phi} + \frac{1}{\rho}\left(\cancel{\frac{\partial(\rho F_\phi)}{\partial\rho}} - \cancel{\frac{\partial F_\rho}{\partial\phi}}\right)\hat{z} = \boxed{0} \text{ c)}$$

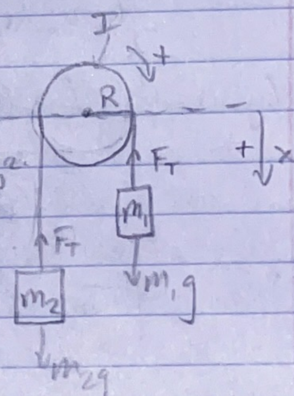
$F_\phi, F_z = 0$

$F = F(\rho)$

$$\text{d) } \vec{F} = -\frac{GM\mu}{\rho}\hat{\rho} \rightarrow U = -\int_R^\rho \frac{-GM\mu}{\rho'}d\rho'\hat{\rho} = GM\mu\ln(\rho')\Big|_R^\rho = \boxed{GM\mu\ln\left(\frac{\rho}{R}\right)}$$

arbitrary point $\rightarrow R$

7. see jupyternotebook HTML submission



$$\text{8.a) } \sum E = \frac{1}{2}I\omega^2 - g(m_1 - m_2)x + \frac{1}{2}m_1 v^2 + \frac{1}{2}m_2 v^2$$

$$\omega = \frac{v}{R} \Rightarrow \boxed{\frac{1}{2}\left(m_1 + m_2 + \frac{I}{R^2}\right)(\dot{x})^2 - gx(m_1 - m_2)} \text{ a)}$$

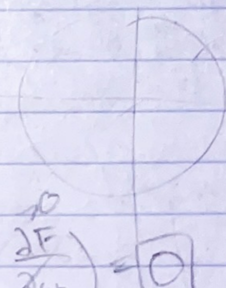
$$\text{b) } \frac{d}{dx}[E] = \boxed{\left(m_1 + m_2 + \frac{I}{R^2}\right)\ddot{x} - g(m_1 - m_2)} \text{ b)}$$

$F = ma \rightarrow \sum = m_1 g + \cancel{F_T} - \cancel{F_T} - m_2 g$

$\tau = I\alpha \rightarrow \sum = I\alpha + R^2(m_1 + m_2)\alpha \Rightarrow \alpha = \frac{a}{R} = \frac{\dot{v}}{R} = \frac{\ddot{x}}{R} \Rightarrow \ddot{x}\left(\frac{I}{R^2} + m_1 + m_2\right)$

$\Rightarrow F + \tau = \left(m_1 + m_2 + \frac{I}{R^2}\right)\ddot{x} - g(m_1 - m_2)$ ✓

9. $F = f(r)\hat{r}$, $x = r\cos\theta\sin\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\phi$



a) $\vec{\nabla}\times\vec{F}(\vec{r})\hat{r}$

$$= \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial F}{\partial x} & \frac{\partial F}{\partial y} & \frac{\partial F}{\partial z} \end{vmatrix} = \hat{x}\left(\cancel{\frac{\partial F}{\partial y\partial z}} - \cancel{\frac{\partial F}{\partial z\partial y}}\right) + \hat{y}\left(\cancel{\frac{\partial F}{\partial x\partial z}} - \cancel{\frac{\partial F}{\partial x\partial z}}\right) + \hat{z}\left(\cancel{\frac{\partial F}{\partial x\partial y}} - \cancel{\frac{\partial F}{\partial y x}}\right) = \boxed{0}$$

by spherical symmetry; $x, y, z \rightarrow 0$

9.b) $\vec{\nabla} \times \vec{F} = \hat{r} \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}\left[\sin\theta \cdot F_\phi\right] - \frac{\partial}{\partial\phi} F_\theta\right] + \hat{\theta}\left[\frac{1}{r\sin\theta} \cdot \frac{\partial}{\partial\phi} F_r - \frac{1}{r}\frac{\partial}{\partial r}(r F_\phi)\right]$

$\hat{\phi} \frac{1}{r}\left[\frac{\partial}{\partial r}(r F_\theta) - \frac{\partial}{\partial\theta} F_r\right]$

$F = F(r) \to$ no $\hat{\theta}$ or $\hat{\phi}$

$$\Rightarrow \frac{1}{r\sin\theta}\left[\cos\theta \cancel{F_\phi}^{0} - 0\right] + \left[0 - \cancel{\frac{F_\phi}{r}}^{0}\right] + \frac{1}{r}\left[\cancel{F_\theta}^{0} - \cancel{\frac{\partial}{\partial\theta} F_r}^{0}\right] = \boxed{0}$$

10. $\vec{P}_i = m_1 \vec{v}_1 \quad \vec{P}_f = (m_1+m_2)\vec{v}_2 \quad v_2 = \frac{m_1 \vec{v}_1}{(m_1+m_2)}, \; v_2^2 = \frac{m_1^2 v_1^2}{(m_1+m_2)^2}$

$\Delta T = T_f - T_i \Rightarrow \frac{1}{2}(m_1+m_2) v_2^2 - \frac{1}{2} m_1 v_1^2$

$\Rightarrow \frac{1}{2}\left(\frac{m_1^2}{m_1+m_2} v_1^2 - m_1 v_1^2\right) \to \frac{1}{2} m_1 v_1^2 \left(\frac{m_1}{m_1+m_2} - 1\right)$

$\Rightarrow \left(\frac{m_2}{m_1+m_2}\right) T_i$ if $m_2 >> m_1$ ↳ T_i is lost

if $m_1 >> m_2$ ↳ Close to zero energy is lost