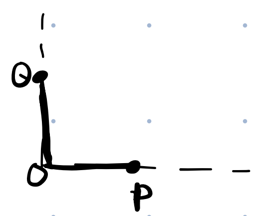
Homework 5: Energy, Work

Problem 1: (Also found in the class activity)

Line integrals: $\vec{F} = (-y, x)$

$$W = \int F \cdot dr = \int F_x\, dx + \int F_y\, dy$$

a) Path $P = (1, 0)$ to $Q = (0, 1)$



$$W = \int_P^O F_x dx + \cancel{\int_P^O F_y dy}$$

$$+ \cancel{\int_O^Q F_x dx} + \int_O^Q F_y dy$$

$$W = \int_P^O F_x dx + \int_O^Q F_y dy = -\int_P^O y\, dx + \int_O^Q x\, dy$$

$= \underline{0}$ → because $y = 0$ from $P \to O$
$x = 0$ from $O \to Q$

b) $y = 1 - x$ $\quad dy = -dx$

$$W = \int_P^Q F_x dx + \int_P^Q F_y dy = -\int_P^Q y\, dx + \int_P^Q x\, dy$$

Writing in terms of x

$$W = -\int_P^Q (1-x)\, dx - \int_P^Q x\, dx = -\int_P^Q dx$$

$Q \rightarrow 0 \quad P \rightarrow 1$ (Only limits of x)

$$W = -(-1) = \underline{\underline{1}}$$

c) Quarter of a circle, Same integral appl

$$W = -\int_P^Q y\, dx + \int_P^Q x\, dx$$

How do we define the relation of x and y for a circle? $\rightarrow$ Use polar coordinates with $\rho = 1$ (or $r = 1$

$x = \cos\phi \qquad y = \sin\phi$

$dx = -\sin\phi \, d\phi \qquad dy = \cos\phi \, d\phi$

Bundanes also change.

$Q \Rightarrow \phi = \pi/2 \qquad P \Rightarrow \phi = 0$

$$W = \int_0^{\pi/2} \cos^2\phi \, d\phi + \int_0^{\pi/2} \sin^2\phi \, d\phi$$

$$W = \int_0^{\pi/2} (\cos^2\phi + \sin^2\phi) \, d\phi = \int_0^{\pi/2} d\phi$$

$$W = \phi \Big|_0^{\pi/2} \longrightarrow \underline{\underline{W = \frac{\pi}{2}}}$$

2. Own reflection (I will read it)

3. Own reading (I will read it)

Problem 4

The coulomb force is $\bar{F} = \frac{\gamma}{r^2} \hat{r} \quad \gamma = kqQ$

The force only depends on the position ✓
Work should be independent of the path:

$$\vec{\nabla} \times \vec{F} = \begin{vmatrix} \dfrac{\hat{r}}{r^2 \sin\theta} & \dfrac{\hat{\theta}}{r\sin\theta} & \dfrac{\hat{\phi}}{r} \\ \partial/\partial r & \partial/\partial\theta & \partial/\partial\theta \\ \gamma/r^2 & 0 & 0 \end{vmatrix}$$

$$= \frac{\hat{r}}{\sin\theta}\left[0\right] + \cancel{\frac{\hat{\theta}}{r\sin\theta}\left[\frac{\partial}{\partial\theta}\left(\frac{\gamma}{r^2}\right)\right]}^{\,0}$$

$$+ \frac{\hat{\phi}}{r}\left[-\cancel{\frac{\partial}{\partial\theta}\left(\frac{\gamma}{r^2}\right)}^{\,0}\right] = 0$$

Problem 5

- Cartesian $\quad \vec{\nabla} u = \dfrac{\partial u}{\partial x}\hat{x} + \dfrac{\partial u}{\partial y}\hat{y} + \dfrac{\partial u}{\partial z}\hat{z}$

$$\begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} & \dfrac{\partial u}{\partial z} \end{vmatrix} = \left.\begin{aligned} &\left[\underline{\frac{\partial}{\partial y}\frac{\partial u}{\partial z}} - \underline{\frac{\partial}{\partial z}\frac{\partial u}{\partial y}}\right]\hat{x} \\ &+\left[\underline{\frac{\partial}{\partial z}\frac{\partial u}{\partial x}} - \underline{\frac{\partial}{\partial x}\frac{\partial u}{\partial z}}\right]\hat{y} \\ &+\left[\underline{\frac{\partial}{\partial x}\frac{\partial u}{\partial y}} - \underline{\frac{\partial}{\partial y}\frac{\partial u}{\partial x}}\right]\hat{z} \end{aligned}\right\} = 0$$

Same Derivatives in Partial derivatives

$$\frac{\partial}{\partial \xi}\frac{\partial}{\partial \eta} = \frac{\partial}{\partial \eta}\frac{\partial}{\partial \xi}$$

In Cylindrical: $\vec{\nabla} u = \frac{\partial u}{\partial \rho}\hat{\rho} + \frac{1}{\rho}\frac{\partial u}{\partial \phi}\hat{\phi} + \frac{\partial u}{\partial z}\hat{z}$

$$\vec{\nabla}\times\vec{u} = \begin{vmatrix} \frac{\hat{\rho}}{\rho} & \hat{\phi} & \hat{z}/\rho \\ \frac{\partial}{\partial \rho} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial z} \\ \frac{\partial u}{\partial \rho} & \cancel{\rho}\frac{1}{\cancel{\rho}}\frac{\partial u}{\partial \phi} & \frac{\partial u}{\partial z} \end{vmatrix} = \cancelto{0}{\left[\frac{\partial}{\partial \phi}\frac{\partial u}{\partial z} - \frac{\partial}{\partial z}\frac{\partial u}{\partial \phi}\right]}\hat{r}/\rho$$

$$+ \cancelto{0}{\left[\frac{\partial}{\partial z}\frac{\partial u}{\partial \rho} - \frac{\partial}{\partial z}\frac{\partial u}{\partial \rho}\right]}\hat{}$$

$$+ \cancelto{0}{\left[\frac{\partial}{\partial \rho}\frac{\partial u}{\partial \phi} - \frac{\partial}{\partial \phi}\frac{\partial u}{\partial \rho}\right]} z/\rho$$

In Spherical: $\vec{\nabla} u = \frac{\partial u}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial u}{\partial \theta}\hat{\theta} + \frac{1}{r\sin\theta}\frac{\partial u}{\partial \varphi}\hat{\varphi}$

$$\vec{\nabla}\times u = \begin{vmatrix} \frac{\hat{r}}{r^2\sin\theta} & \frac{\hat{\theta}}{r\sin\varphi} & \frac{\hat{\phi}}{r} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial \varphi} \\ \frac{\partial u}{\partial r} & \frac{\cancel{r}}{\cancel{r}}\frac{\partial u}{\partial \theta} & \frac{\cancel{r\sin\theta}}{\cancel{r\sin\theta}}\frac{\partial u}{\partial \varphi} \end{vmatrix}$$

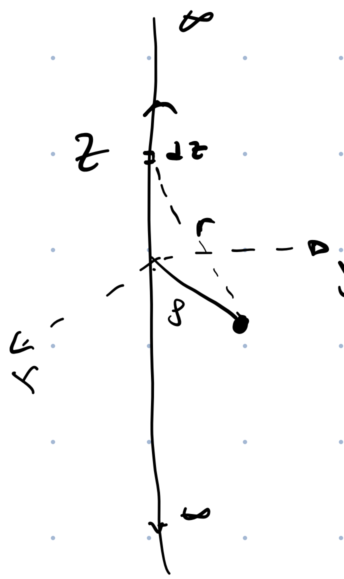
$$= \frac{\hat{r}}{r^2\sin\theta}\cancelto{0}{\left[\frac{\partial}{\partial \theta}\frac{\partial u}{\partial \varphi} - \frac{\partial}{\partial \varphi}\frac{\partial u}{\partial \theta}\right]}$$

$$+ \frac{\hat{\theta}}{r\sin\varphi}\cancelto{0}{\left[\frac{\partial}{\partial \varphi}\frac{\partial u}{\partial r} - \frac{\partial}{\partial r}\frac{\partial u}{\partial \varphi}\right]}$$

$$+ \frac{\hat{\phi}}{r}\cancelto{0}{\left[\frac{\partial}{\partial r}\frac{\partial u}{\partial \theta} - \frac{\partial}{\partial \theta}\frac{\partial u}{\partial r}\right]}$$

$$= 0$$

Problem 6



μ [mass/length]

ρ is known

(a) **Force on m from gravity**

In general $\bar{F} = -\frac{GMm}{r^2}$

where $\bar{F}$ must accont for every piece in the road dz

$dM = \mu\, dz \qquad F \Rightarrow dF \}$ for each piece.

$$dF = \frac{Gm}{r^2}\mu\, dz$$

The force must be along the $\hat{\rho}$ direction:

$$dF_\rho = -\frac{Gm\mu}{r^2}\cos\theta\, dz \qquad \text{where } \cos\theta = \frac{\rho}{r}$$

$$\rho^2 + z^2 = r^2$$

$$dF_\rho = -\frac{Gm\mu\rho}{r^3}\, dz \qquad \rho \text{ is known}$$

Suming over all the pieces $\int dF_\rho = -Gm\mu\rho \int_{-\infty}^{\infty} \frac{z}{(\rho^2+z^2)^{3/2}}$

This is an integral that we cansolve by using $z = \rho\tan\theta$, $dz = \rho\sec^2\theta\, d\theta$

$z \to \infty$ when $\theta = \pi/2$

$z \to -\infty$ when $\theta = -\frac{\pi}{2}$

$$F_\rho = -Gm\mu\rho \int_{-\pi/2}^{\pi/2} \frac{\rho\sec^2\theta\, d\theta}{(\rho^2+\rho^2\tan^2\theta)^{3/2}}$$

$$F_\rho = -\frac{Gm\mu}{\rho}\int_{-\pi/2}^{\pi/2} \frac{\sec^2\theta\, d\theta}{\underbrace{(1+\tan^2\theta)}_{\sec^2\theta}{}^{3/2}} = -\frac{Gm\mu}{\rho}\int_{-\pi/2}^{\pi/2}\frac{d\theta}{\sec\theta} = -\frac{Gm\mu}{\rho}\int_{-\pi/2}^{\pi/2}\cos\theta\, d\theta$$

$$F_\rho = -\frac{Gm\mu}{\rho}\sin\theta\Big|_{-\pi/2}^{\pi/2} = \underline{\underline{\frac{-2Gm\mu}{\rho}}}$$

So $\vec{F} = -\frac{2Gm\mu}{\rho}\hat{\rho}$

b) In cartesian coordinates

$$\rho^2 = x^2 + y^2 \qquad \hat{\rho} = \frac{\rho}{|\rho|} = \frac{x}{\sqrt{x^2+y^2}}\,\hat{x} + \frac{y}{\sqrt{x^2+y^2}}\,\hat{y}$$

$$F = -2Gm\mu\left(\frac{x}{\sqrt{x^2+y^2}}\,\hat{x} + \frac{y}{\sqrt{x^2+y^2}}\,\hat{y}\right)$$

$$\vec{\nabla}\times\vec{F} = -2Gm\mu\begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{x}{\sqrt{x^2+y^2}} & \frac{y}{\sqrt{x^2+y^2}} & 0 \end{vmatrix}$$

$$= -2\hat{x}Gm\mu\left[\cancel{-\frac{\partial}{\partial z}\frac{y}{\sqrt{x^2+y^2}}}\right] \to 0$$

$$-2\hat{y}Gm\mu\left[\cancel{\frac{\partial}{\partial z}\frac{x}{\sqrt{x^2+y^2}}}\right] \to 0$$

$$-2yGm\mu\left[\cancel{\underbrace{\frac{\partial}{\partial x}\frac{y}{\sqrt{x^2+y^2}} - \frac{\partial}{\partial y}\frac{x}{\sqrt{x^2+y^2}}}_{\text{similar } x \longleftrightarrow y}}\right] \to 0$$

c In cylindrical coordinates $F = -\frac{2GM\mu}{\rho}\hat{\rho}$

$\vec{\nabla}\times\vec{F} \to$ All the terms go to zero since $F = F(\rho)\hat{\rho}$
So all the derivatives are zero

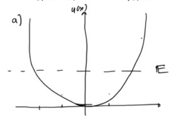
d) $$U = -\int_{\rho_0}^{\rho} F_\rho\, d\rho = -\int_{\rho_0}^{\rho} -\frac{2Gm\mu}{\rho}\,d\rho = 2Gm\mu \ln\rho\Big|_{\rho_0}^{\rho}$$

$$\boxed{U = 2Gm\mu\ln\left(\frac{\rho}{\rho_0}\right)}$$

Problem 7

$U = Kx^4 \qquad K>0$

a)



At $x=0$
kick to the right
at $t=0$

The kick moves the particle

so $\dot{x}(t=0) = V$ Goes to a maximum

where $T=0$ and U is maximum

So $E_0 = T_0$ in the max $U = U_{max}$ and by conservation of energy $T_0 = U_{max}$

(b) In general $E = \underbrace{T+U}_{\text{Any point}} = E_0 \rightarrow$ Initial energy

$T = \frac{1}{2} m \left(\frac{dx}{dt}\right)^2 \qquad U = Kx^4$

$$E_0 = \frac{1}{2} m \left(\frac{dx}{dt}\right)^2 + Kx^4$$

$$\sqrt{\frac{2}{m}}\left(E_0 - Kx^4\right)^{1/2} = \frac{dx}{dt}$$

$$\int_0^t dt = \int_{x_0}^{x_f} \frac{dx}{\sqrt{\frac{2}{m}}\left(E_0 - Kx^4\right)^{1/2}}$$

E_0 can be expressed as KA^4

$$t = \frac{1}{\sqrt{\frac{2}{m}}\sqrt{K}} \int_{x_0}^{x_f} \frac{dx}{(A^4 - x^4)^{1/2}}$$

The time to get to A
$x_0 = 0$
$x = A$

$$t = \sqrt{\frac{m}{2K}} \int_0^A \frac{dx}{(A^4 - x^4)^{1/2}}$$

c) The period is 4 times $0 \rightarrow A$ since it is to come back to the same plac

$$\tau = 4\sqrt{\frac{m}{2K}} \int_0^A \frac{dx}{\sqrt{A^4 - x^4}}$$

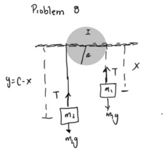
d) If $m = K = A = 1$

$$\tau = \frac{4}{\sqrt{2}} \underbrace{\int_0^1 \frac{dx}{\sqrt{1-x^4}}}_{\sim 1.311}$$

(I will see your code)

$$\underline{\underline{\tau \simeq 3.70}}$$

Problem 8



"height" for m_2
l = length of the string
$\underbrace{l - x - \pi R}_{\text{Constant} = C} = y$
$y = C - x$

a) Total Kinetic Energy $= T = T_{m_1} + T_{m_2} + T_{pulley}$

$$T = \frac{1}{2} m_1 \dot{x}^2 + \frac{1}{2} m_2 \dot{x}^2 + \frac{1}{2} I\omega^2$$

ω and $\dot{x}$ are related $\rightarrow \omega = \frac{\dot{x}}{R}$

$$T = \frac{1}{2} m_1 \dot{x}^2 + \frac{1}{2} m_2 \dot{x}^2 + \frac{1}{2R^2} I \dot{x}^2$$

$$T = \frac{1}{2}\left[m_1 + m_2 + \frac{I}{R^2}\right]\dot{x}^2$$

The total potential energy $U = U_{m_1} + U_{m_2} + \cancel{U_{pulley}}^{0}$ We set it to zero

$$U = -m_1 g x - m_2 g (C - x)$$

The total energy : $E = T + U$

$$E = \frac{1}{2}\left[m_1 + m_2 + \frac{I}{R^2}\right]\dot{x}^2 - \left[m_1 g x + m_2 g (C - x)\right]$$

Constants can be ignored

$$E = \frac{1}{2}\left[m_1 + m_2 + \frac{I}{R^2}\right]\dot{x}^2 - \left[m_1 + m_2\right] g x$$

(b) The equation of motion is $\ddot{x}(t) = \dots$

We have a first diff equation

$$\frac{1}{2}\left[m_1 + m_2 + \frac{I}{R^2}\right]\dot{x}^2 - \left[m_1 + m_2\right] g x = E$$

$$\frac{dE}{dt} = \frac{1}{\cancel{2}}\cancel{2}\left[m_1 + m_2 + \frac{I}{R}\right]\ddot{x}\cancel{\dot{x}} - \left[m_1 + m_2\right] g \cancel{\dot{x}}$$

Equation of motion

$$\underline{\left[m_1 + m_2 + \frac{I}{R}\right]\ddot{x} = \left[m_1 + m_2\right] g}$$

By Newton's law

Body m_1: Body m_2: Pulley

(1) $m_1 g - T_1 = m_1 \ddot{x}$ (2) $T_2 - m_2 g = m_2 \ddot{x}$ $I\dot{\omega} = [T_1 - T_2]R$

Combining (2)+1

$$\underbrace{T_2 - T_1}_{-\frac{I\dot{\omega}}{R} = -\frac{I}{R}\ddot{x}} + m_1 g - m_2 g = (m_1 + m_2)\ddot{x}$$

$$-\frac{I}{R}\ddot{x} + m_1 g - m_2 g = (m_1 + m_2)\ddot{x}$$

$$\underbrace{(m_1 + m_2) g = (m_1 + m_2 + I/R)\ddot{x}}_{\text{Same equation}}$$

Problem 9

$$\vec{F}(\vec{r}) = f(r)\hat{r}$$

So $\vec{F}(r) = f(r)\frac{\vec{r}}{r} = \underbrace{\frac{f(r)}{r}}_{D(r)}\left[x\hat{x} + y\hat{y} + z\hat{z}\right]$ $\leftarrow$ Other function of r

$$\vec{\nabla} \times \vec{F} = \begin{vmatrix} \hat{x} & y & z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ D(r)x & D(r)y & D(r)z \end{vmatrix}$$

$$= \hat{x}\left[\frac{\partial}{\partial y}\left[D(r)z\right] - \frac{\partial}{\partial z}\left[D(r)y\right]\right] + \hat{y}\left[\frac{\partial}{\partial z}\left[D(r)x\right] - \frac{\partial}{\partial x}\left[D(r)z\right]\right] + \hat{z}\left[\frac{\partial}{\partial x}\left[D(r)y\right] - \frac{\partial}{\partial y}\left[D(r)x\right]\right]$$

We know $\frac{\partial}{\partial y}(D(r)) = \frac{\partial D(r)}{\partial r}\frac{\partial r}{\partial y}$ Same for all variables

$$= \hat{x}\left[z\frac{\partial D(r)}{\partial r}\frac{\partial r}{\partial y} - y\frac{\partial D(r)}{\partial r}\frac{\partial r}{\partial z}\right]$$