Hopfion Streamline Derivation

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1 Background

Hopfions are topological solitons that can be described by a normalized spin vector field $\mathbf{n}(\mathbf{r})$, and have an interesting quality. If one were to choose a direction of spin and keep it fixed, the positions of these vectors would trace a closed contour in real space. This is the preimage of the Hopf map, which defines $\mathbf{n}(\mathbf{r})$. Explicitly, the mapping is[1] $\mathbb{R}^3 \to \mathbb{S}^3 \to \mathbb{C} \to \mathbb{S}^2$. What's more is that if two preimage contours were to be traced out, corresponding to different spins, these contours would link! It turns out that this linking corresponds to a characterization of hopfions called the Hopf index, an integer which determines the stable configuration of the hopfion, i.e. the Hopf index remains invariant under continuous deformation of the spin field. This Hopf index can also be calculated via the Whitehead integral

$$Q = \int d^3r \, \mathbf{b} \cdot \mathbf{a},$$

where $\mathbf{b}$ is the hopfion's emergent magnetic field [2, Eq. 8]

$$\mathbf{b} = \mathbf{n} \cdot (\nabla \mathbf{n} \times \nabla \mathbf{n}) \tag{1}$$

and $\mathbf{a}$ is the vector potential, $\mathbf{b} = \nabla \times \mathbf{a}$.

In this paper by Azhar et al. [1], they use the idea of flux tubes to fully connect the idea of preimage linking with the Hopf index. Their calculation that relates the Whitehead integral with Gauss' linking integral is found in Appendix A of their work. Here, I verify a claim that they make in their calculation to justify Eq. (A4):

$$\frac{d\mathbf{C}_i(s)}{ds} \times \mathbf{V}_i(\mathbf{C}_i(s)) = 0, \tag{A4}$$

where $\mathbf{C}_i$ is the ith preimage contour corresponding to $\mathbf{V}_i$, which is a vector field localized at the contour, representing a field line of $\mathbf{b}$. (A4) says that the tangent vector on $\mathbf{C}_i$ must be parallel to $\mathbf{V}_i$, i.e. it is a streamline for $\mathbf{V}_i$. What I aim to show is that the preimage of $\mathbf{n}$ is a streamline for $\mathbf{b}$, specifically by showing that $\nabla \mathbf{n} \cdot \mathbf{b} = \mathbf{0}$, where $\nabla \mathbf{n}$ is the transpose Jacobian of the spin field. This is similar to the definition of equipotential contours of a scalar field $\phi(\mathbf{r})$, i.e. $\mathbf{C}(\mathbf{r})$ such that $\nabla \phi \cdot \mathbf{C}(\mathbf{r}) = 0$.

[1] Here $\mathbb{R}$ is implied to include $\{\infty\}$.

2 Calculation

To start, we recall that the vector field[2] $\mathbf{n} = n^i \, (i = 1, 2, 3)$ is normalized, so $n_i n^i = 1$ and we can write $\nabla \mathbf{n} = \partial_\alpha n^i \, (\alpha = x, y, z)$ as

$$\partial_\alpha n^i = \begin{pmatrix} \partial_x n^1 & \partial_x n^2 & \partial_x n^3 \\ \partial_y n^1 & \partial_y n^2 & \partial_y n^3 \\ \partial_z n^1 & \partial_z n^2 & \partial_z n^3 \end{pmatrix}.$$

We aim to show $\nabla \mathbf{n} \cdot \mathbf{b} = \partial_\alpha n^l b^\alpha = 0$. Since $n_i n^i = 1$, we have

$$\partial_\alpha (n_i n^i) = 0,$$

but by chain rule we also have

$$\partial_\alpha (n_i n^i) = 2 n_i \partial_\alpha n^i,$$

so we know that $n_i \partial_\alpha n^i = 0$. This tells us the column vectors of $\partial_\alpha n^i$ must be linearly dependent, so the determinant must vanish. We can write this in terms of the column vectors as

$$\varepsilon^{\alpha\beta\gamma} \partial_\alpha n^l \partial_\beta n^j \partial_\gamma n^k = 0. \tag{$*$}$$

To verify this, we can note that any distinct choices of l, j, k are equivalent to permuting the the columns of our transposed Jacobian, and since it has a determinant of 0, these permutations won't affect $(*)$. Then, if we choose any of l, j, k to be the same, we will be essentially computing the determinant of a matrix that has two identical columns, and $(*)$ holds. Next, we examine the expression for $\mathbf{b} = b^\alpha$. From (1), in index notation we have

$$b^\alpha = \varepsilon^{\alpha\beta\gamma} n^i \varepsilon_{ijk} \partial_\beta n^j \partial_\gamma n^k,$$

and calculating $\nabla \mathbf{n} \cdot \mathbf{b}$ in index notation,

$$\begin{aligned} \nabla \mathbf{n} \cdot \mathbf{b} = \partial_\alpha n^l b^\alpha &= \partial_\alpha n^l \varepsilon^{\alpha\beta\gamma} n^i \varepsilon_{ijk} \partial_\beta n^j \partial_\gamma n^k \\ &= \varepsilon_{ijk} n^i (\varepsilon^{\alpha\beta\gamma} \partial_\alpha n^l \partial_\beta n^j \partial_\gamma n^k) \\ &= 0. \end{aligned}$$

Therefore, it is true that the preimage contour is a streamline for the field lines of the emergent magnetic field of the hopfion.

References

[1] Maria Azhar et al. "3D Magnetic Textures with Mixed Topology: Unlocking the Tunable Hopf Index". In: *arXiv:2411.06929* (2024). arXiv: 2411.06929. URL: https://arxiv.org/abs/2411.06929.

[2] Wen-Tao Hou et al. "Construction of hopfion crystals". In: *Phys. Rev. Res.* 7 (4 Nov. 2025), p. L042036. DOI: 10.1103/rr4k-x9m5. URL: https://link.aps.org/doi/10.1103/rr4k-x9m5.

[2] Here I will utilize index notation with the Einstein summation convention.