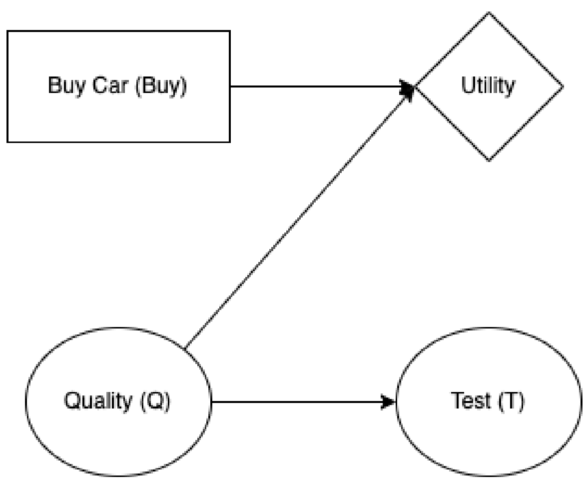
CS 6300 **HW09: VPI and HMMs** **Ryan Dalby** **Due April 19, 2022**

Please use the LaTeX template to produce your writeups. See the Homework Assignments page on the class website for details. Hand in via gradescope.

1 Decision Networks and VPI

A used car buyer can decide to carry out various tests with various costs (e.g., kick the tires, take the car to a qualified mechanic) and then, depending on the outcome of the tests, decide which car to buy. We will assume that the buyer is deciding whether to buy the car and that there is time to carry out at most one test which costs \$50 and which can help to figure out the quality of the car. A car can be in good shape (of good quality Q = +q) or in bad shape (of bad quality Q=-q), and the test might help to indicate what shape the car is in. There are only two outcomes for the test T: pass (T=pass) or fail (T=fail). The car costs \$1,500, and its market value is \$2,000 if it is in good shape; if not, \$700 in repairs will be needed to make it in good shape. The buyer's estimate is that the car has 70% chance of being in good shape.

1. Draw the decision network that represents this problem.



$U(\text{Buy}, Q)$	$+q$	$-q$
$+\text{Buy}$	$2000 - 1500 = 500$	$2000 - (1500 + 700) = -200$
$-\text{Buy}$	0	0

Q	$P(Q)$
$+q$	0.70
$-q$	0.30

2. Calculate the expected net gain from buying the car, given no test.

$$EU(+\text{Buy}|\emptyset) = P(+q|\emptyset)U(+\text{Buy}, +q) + P(-q|\emptyset)U(+\text{Buy}, -q)$$

$$EU(+\text{Buy}|\emptyset) = P(+q)U(+\text{Buy}, +q) + P(-q)U(+\text{Buy}, -q) = (0.70)(500) + (0.30)(-200) = 290$$

Also note that $\text{MEU}(\emptyset) = 290$ since $EU(-\text{Buy}|\emptyset) = 0$

3. Tests can be described by the probability that the car will pass or fail the test given that the car is in good or bad shape. We have the following information:

$$\begin{aligned} P(T = pass|Q = +q) &= 0.9 \\ P(T = pass|Q = -q) &= 0.2 \end{aligned}$$

Calculate the probability that the car will pass (or fail) its test, and then the probability that it is in good (bad) shape given each possible test outcome.

T	Q	$P(T\|Q)$
Pass	$+q$	0.9
Fail	$+q$	0.1
Pass	$-q$	0.2
Fail	$-q$	0.8

$$P(T, Q) = P(T|Q)P(Q)$$

T	Q	$P(T, Q)$
Pass	$+q$	$(0.9)(0.7) = 0.63$
Fail	$+q$	$(0.1)(0.7) = 0.07$
Pass	$-q$	$(0.2)(0.3) = 0.06$
Fail	$-q$	$(0.8)(0.3) = 0.24$

$$P(T) = \sum_Q P(T, Q)$$

T	$P(T)$
Pass	$(0.63 + 0.06) = 0.69$
Fail	$(0.07 + 0.24) = 0.31$

$$P(Q|T) = \frac{P(T, Q)}{P(T)}$$

T	Q	$P(Q\|T)$
Pass	$+q$	$\frac{0.63}{0.69} = 0.913$
Fail	$+q$	$\frac{0.07}{0.31} = 0.226$
Pass	$-q$	$\frac{0.06}{0.69} = 0.0870$
Fail	$-q$	$\frac{0.24}{0.31} = 0.774$

4. Calculate the optimal decisions given either a pass or a fail, and their expected utilities.

$$\text{MEU}(T) = \max_{\{+\text{Buy},-\text{Buy}\}} \sum_{Q} P(Q|T)U(Q|\text{Buy})$$

$$\text{MEU}(T = \text{Pass}) = \max_{\{+\text{Buy},-\text{Buy}\}} \begin{cases} P(+q|\text{Pass})U(+q| + \text{Buy}) + P(-q|\text{Pass})U(-q| + \text{Buy}) \\ P(+q|\text{Pass})U(+q| - \text{Buy}) + P(-q|\text{Pass})U(-q| - \text{Buy}) \end{cases}$$

$$\text{MEU}(T = \text{Pass}) = \max_{\{+\text{Buy},-\text{Buy}\}} \begin{cases} (0.913)(500) + (0.0870)(-200) = 439.1 \\ (0.913)(0) + (0.0870)(0) = 0 \end{cases} = 439.1$$

Optimal decision for MEU($T = \text{Pass}$) is $+$Buy (Buy)

$$\text{MEU}(T = \text{Fail}) = \max_{\{+\text{Buy},-\text{Buy}\}} \begin{cases} P(+q|\text{Fail})U(+q| + \text{Buy}) + P(-q|\text{Fail})U(-q| + \text{Buy}) \\ P(+q|\text{Fail})U(+q| - \text{Buy}) + P(-q|\text{Fail})U(-q| - \text{Buy}) \end{cases}$$

$$\text{MEU}(T = \text{Fail}) = \max_{\{+\text{Buy},-\text{Buy}\}} \begin{cases} (0.226)(500) + (0.774)(-200) = -41.8 \\ (0.226)(0) + (0.774)(0) = 0 \end{cases} = -41.8$$

Optimal decision for MEU($T = \text{Fail}$) is $-$Buy (Don't buy)

5. Calculate the value of (perfect) information of the test. Should the buyer pay for a test?

$$\text{VPI}(T) = P(\text{Pass})(\text{MEU}(\text{Pass}) - \text{MEU}(\emptyset)) + P(\text{Fail})(\text{MEU}(\text{Fail}) - \text{MEU}(\emptyset))$$
$$\text{VPI}(T) = (0.69)(439.1 - 290) + (0.31)(-41.8 - 290) = 0.021$$

No, the buyer shouldn't pay \$50 for a test since VPI(T) $= \$0.021 < \$50$

2 HMMs

You sometimes get colds, which make you sneeze. You also get allergies, which make you sneeze. Sometimes you are well, which doesn't make you sneeze (much). You decide to model the process using the following HMM, with hidden states $X \in \{well, allergy, cold\}$ and observations $E \in \{sneeze, quiet\}$:

$P(X_1)$

well	1
allergy	0
cold	0

$P(X_t \mid X_{t-1} = well)$

well	0.7
allergy	0.2
cold	0.1

$P(X_t \mid X_{t-1} = allergy)$

well	0.6
allergy	0.3
cold	0.1

$P(X_t \mid X_{t-1} = cold)$

well	0.2
allergy	0.2
cold	0.6

Transitions

$P(E_t \mid X_t = well)$

quiet	1.0
sneeze	0.0

$P(E_t \mid X_t = allergy)$

quiet	0.0
sneeze	1.0

$P(E_t \mid X_t = cold)$

quiet	0.0
sneeze	1.0

Emissions

Note that colds are "stickier" in that you tend to have them for multiple days, while allergies come and go on a quicker time scale. However, allergies are more frequent. Assume that on the first day, you are well.

Note for this problem, column vectors imply "well", "allergy", and "cold" for the unspecified state in the equation from top to bottom respectively.

1. What is the posterior distribution over your state on day 2 (X_2) if $E_1 = quiet$, $E_2 = sneeze$?

$$P(X_1|E_1 = \text{quiet}) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

This is equivalently represented as a table

X_1	$P(X_1\|E_1 = \text{quiet})$
well	1
allergy	0
cold	0

$$P(X_2|E_1 = \text{quiet}, E_2 = \text{sneeze}) = \alpha P(\text{sneeze}|X_2) \sum_{X_1} P(X_2|X_1) P(X_1|E_1 = \text{quiet})$$

$$P(X_2|E_1 = \text{quiet}, E_2 = \text{sneeze}) = \alpha \begin{bmatrix} 0 \\ 1.0 \\ 1.0 \end{bmatrix} \left(\begin{bmatrix} 0.7 \\ 0.2 \\ 0.1 \end{bmatrix} (1) + \begin{bmatrix} 0.6 \\ 0.3 \\ 0.1 \end{bmatrix} (0) + \begin{bmatrix} 0.2 \\ 0.2 \\ 0.6 \end{bmatrix} (0) \right) = \begin{bmatrix} 0 \\ 0.2 \\ 0.1 \end{bmatrix}$$

Normalizing by $\alpha = \frac{1}{2/10+1/10} = \frac{10}{3}$

$$P(X_2|E_1 = \text{quiet}, E_2 = \text{sneeze}) = \begin{bmatrix} 0 \\ 2/3 \\ 1/3 \end{bmatrix}$$

This is equivalently represented as a table

X_2	$P(X_2\|E_1 = \text{quiet}, E_2 = \text{sneeze})$
well	0
allergy	2/3
cold	1/3

2. What is the posterior distribution over your state on day 3 (X_3) if $E_1 = quiet$, $E_2 = sneeze$, $E_3 = sneeze$?

$$P(X_3|E_1 = \text{quiet}, E_2 = \text{sneeze}, E_3 = \text{sneeze}) = \\ \alpha P(\text{sneeze}|X_3) \sum_{X_2} P(X_3|X_2) P(X_2|E_1 = \text{quiet}, E_2 = \text{sneeze})$$

$$P(X_3|E_1 = \text{quiet}, E_2 = \text{sneeze}, E_3 = \text{sneeze}) = \\ \alpha \begin{bmatrix} 0 \\ 1.0 \\ 1.0 \end{bmatrix} \left(\begin{bmatrix} 0.7 \\ 0.2 \\ 0.1 \end{bmatrix} (0) + \begin{bmatrix} 0.6 \\ 0.3 \\ 0.1 \end{bmatrix} (\frac{2}{3}) + \begin{bmatrix} 0.2 \\ 0.2 \\ 0.6 \end{bmatrix} (\frac{1}{3}) \right) = \begin{bmatrix} 0 \\ 4/15 \\ 4/15 \end{bmatrix}$$

Normalizing by $\alpha = \frac{1}{4/15+4/15} = \frac{15}{8}$

$$P(X_3|E_1 = \text{quiet}, E_2 = \text{sneeze}, E_3 = \text{sneeze}) = \begin{bmatrix} 0 \\ 1/2 \\ 1/2 \end{bmatrix}$$

This is equivalently represented as a table

X_3	$P(X_3 \mid E_1 = \text{quiet}, E_2 = \text{sneeze}, E_3 = \text{sneeze})$
well	0
allergy	$1/2$
cold	$1/2$

3. What is the Viterbi (most likely) sequence for the observation sequence *quiet, sneeze, sneeze*?

$$m_1(X_1) = P(\text{quiet} \mid X_1)P(X_1)$$

$$m_1(X_1) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

This is equivalently represented as a table

X_1	$m_1(X_1)$
well	1
allergy	0
cold	0

$$m_2(X_2) = P(\text{sneeze} \mid X_2) \max_{X_1} P(X_2 \mid X_1) m_1(X_1)$$

$$m_2(X_2) = \begin{bmatrix} 0 \\ 1.0 \\ 1.0 \end{bmatrix} \max_{\text{well,allergy,cold}} \left\{ \begin{array}{l} \begin{bmatrix} 0.7 \\ 0.2 \\ 0.1 \end{bmatrix} (1) = \begin{bmatrix} 0.7 \\ 0.2 \\ 0.1 \end{bmatrix} \\ \begin{bmatrix} 0.6 \\ 0.3 \\ 0.1 \end{bmatrix} (0) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \\ \begin{bmatrix} 0.2 \\ 0.2 \\ 0.6 \end{bmatrix} (0) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \end{array} \right. = \begin{bmatrix} 0 \\ 1.0 \\ 1.0 \end{bmatrix} \begin{bmatrix} 0.7 \\ 0.2 \\ 0.1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0.2 \\ 0.1 \end{bmatrix}$$

This is equivalently represented as a table

X_2	$m_2(X_2)$
well	0
allergy	0.2
cold	0.1

$$m_3(X_3) = P(\text{sneeze}|X_3) \max_{X_2} P(X_3|X_2) m_2(X_2)$$

$$m_3(X_3) = \begin{bmatrix} 0 \\ 1.0 \\ 1.0 \end{bmatrix} \max_{\text{well,allergy,cold}} \begin{cases} \begin{bmatrix} 0.7 \\ 0.2 \\ 0.1 \end{bmatrix} (0) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \\ \begin{bmatrix} 0.6 \\ 0.3 \\ 0.1 \end{bmatrix} (0.2) = \begin{bmatrix} 0.12 \\ 0.06 \\ 0.02 \end{bmatrix} \\ \begin{bmatrix} 0.2 \\ 0.2 \\ 0.6 \end{bmatrix} (0.1) = \begin{bmatrix} 0.02 \\ 0.02 \\ 0.06 \end{bmatrix} \end{cases} = \begin{bmatrix} 0 \\ 1.0 \\ 1.0 \end{bmatrix} \begin{bmatrix} 0.12 \\ 0.06 \\ 0.06 \end{bmatrix} = \begin{bmatrix} 0 \\ 0.06 \\ 0.06 \end{bmatrix}$$

This is equivalently represented as a table

X_3	$m_3(X_3)$
well	0
allergy	0.06
cold	0.06

There are two possible most likely sequences for this observation sequence depending on if the final element in the sequence is “allergy” or “cold” (either can be chosen since m values are the same):

If “allergy” is chosen then the most likely sequence is “well” $->$ “allergy” $->$ “allergy”.

If “cold” is chosen then the most likely sequence is “well” $->$ “cold” $->$ “cold”.