Please use the LaTeX template to produce your writeups. See the Homework Assignments page on the class website for details. Hand in via gradescope.

1 Decision Networks and VPI

A used car buyer can decide to carry out various tests with various costs (e.g., kick the tires, take the car to a qualified mechanic) and then, depending on the outcome of the tests, decide which car to buy. We will assume that the buyer is deciding whether to buy the car and that there is time to carry out at most one test which costs $50 and which can help to figure out the quality of the car. A car can be in good shape (of good quality Q = +q) or in bad shape (of bad quality Q=-q), and the test might help to indicate what shape the car is in. There are only two outcomes for the test T: pass (T=pass) or fail (T=fail). The car costs $1,500, and its market value is $2,000 if it is in good shape; if not, $700 in repairs will be needed to make it in good shape. The buyer's estimate is that the car has 70% chance of being in good shape.

1. Draw the decision network that represents this problem.
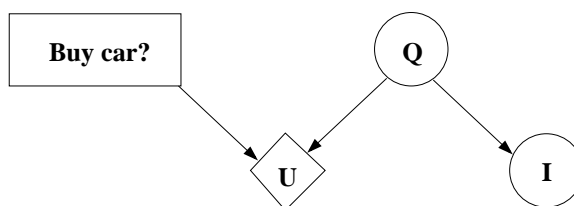
2. Calculate the expected net gain from buying the car, given no test.

$$
\begin{aligned}
EU(buy) &= P(Q = +q) * U(+q, buy) + P(Q = -q) * U(-q, buy) \\
&= 0.7 * 500 + 0.3 * -200 = 290
\end{aligned}
$$

3. Tests can be described by the probability that the car will pass or fail the test given that the car is in good or bad shape. We have the following information:

$$
\begin{aligned}
P(T = pass|Q = +q) &= 0.9 \\
P(T = pass|Q = -q) &= 0.2
\end{aligned}
$$

Calculate the probability that the car will pass (or fail) its test, and then the probability that it is in good (bad) shape given each possible test outcome.

$$P(T = pass) = \sum_q P(T = pass, Q = q)$$

$$= P(T = pass|Q = +q)P(Q = +q) + P(T = pass|Q = -q)P(Q = -q)$$

$$= 0.69$$

$$P(T = fail) = 0.31$$

$$P(Q = +q|T = pass) = \frac{P(T = pass|Q = +q)P(Q = +q)}{P(T = pass)}$$

$$= \frac{0.9 * 0.7}{0.69} = \frac{21}{23}$$

$$P(Q = +q|T = fail) = \frac{P(T = fail|Q = +q)P(Q = +q)}{P(T = fail)}$$

$$= \frac{0.1 * 0.7}{0.31} = \frac{7}{31}$$

4. Calculate the optimal decisions given either a pass or a fail, and their expected utilities.

$$EU(buy|T = pass) = P(Q = +q|T = pass) * U(+q, buy) + P(Q = -q|T = pass) * U(-q, buy)$$

$$\approx 0.91 * 500 + 0.78 * -200 = 437$$

$$EU(buy|T = fail) = P(Q = +q|T = fail) * U(+q, buy) + P(Q = -q|T = fail) * U(-q, buy)$$

$$\approx 0.22 * 500 + 0.78 * -200 = -46$$

$$EU(\neg buy|T = pass) = 0$$

$$EU(\neg buy|T = fail) = 0$$

Therefore $MEU(T = pass) = 437$ (with buy) and $MEU(T = fail) = 0$ (using $\neg buy$).

5. Calculate the value of (perfect) information of the test. Should the buyer pay for a test?

$$VPI(T) = \sum_t P(T = t)MEU(T = t) - MEU(\phi)$$

$$= 0.69 * 437 + 0.31 * 0 - 290 \approx 11.53$$

You shouldn't pay for it, since the cost is $50.

2 HMMs

You sometimes get colds, which make you sneeze. You also get allergies, which make you sneeze. Sometimes you are well, which doesn't make you sneeze (much). You decide to model the process using the following HMM, with hidden states $X \in \{well, allergy, cold\}$ and observations $E \in \{sneeze, quiet\}$:

$P(X_1)$

well	1
allergy	0
cold	0

$P(X_t \mid X_{t-1} = well)$

well	0.7
allergy	0.2
cold	0.1

$P(X_t \mid X_{t-1} = allergy)$

well	0.6
allergy	0.3
cold	0.1

$P(X_t \mid X_{t-1} = cold)$

well	0.2
allergy	0.2
cold	0.6

Transitions

$P(E_t \mid X_t = well)$

quiet	1.0
sneeze	0.0

$P(E_t \mid X_t = allergy)$

quiet	0.0
sneeze	1.0

$P(E_t \mid X_t = cold)$

quiet	0.0
sneeze	1.0

Emissions

Note that colds are "stickier" in that you tend to have them for multiple days, while allergies come and go on a quicker time scale. However, allergies are more frequent. Assume that on the first day, you are well.

1. What is the posterior distribution over your state on day 2 (X_2) if $E_1 = quiet$, $E_2 = sneeze$?

From the Bayes' net,

$$
\begin{aligned}
P(x_2 | e_{1:2}) &= \alpha P(x_2, e_{1:2}) \\
&= \alpha \sum_{x_1} P(x_1, x_2, e_{1:2}) \\
&= \alpha \sum_{x_1} P(x_1) P(e_1 | x_1) P(x_2 | x_1) P(e_2 | x_2)
\end{aligned}
$$

3

Since $P(X_1 = well) = 1$ and the others are zero, the sum simplifies to one term.

$$P(x_2|e_{1;2}) \quad = \quad \alpha P(E_1 = quiet|X_1 = well)P(x_2|X_1 = well)P(E_2 = sneeze|x_2)$$

From the tables,

X_2	$P(X_2, e_{1:2})$	$P(X_2|e_{1:2})$
well	0	0
allergy	0.2	2/3
cold	0.1	1/3

2. What is the posterior distribution over your state on day 3 (X_3) if $E_1 = quiet$, $E_2 = sneeze$, $E_3 = sneeze$?

From the forward algorithm,

$$P(x_3|e_{1:3}) \quad = \quad \alpha P(e_3|x_3) \sum_{x_2} P(x_3|x_2)P(x_2, e_{1:2})$$

$$= \quad \alpha P(E_3 = sneeze|x_3) \sum_{x_2} P(x_3|x_2)P(x_2, e_{1:2})$$

From the tables and the previous result,

X_3	$P(X_3, e_{1:3})$	$P(X_3|e_{1:3})$
well	0	0
allergy	$0 + 0.3 * 0.2 + 0.2 * 0.1 = 0.08$	0.5
cold	$0 + 0.1 * 0.2 + 0.6 * 0.1 = 0.08$	0.5

3. What is the Viterbi (most likely) sequence for the observation sequence *quiet, sneeze, sneeze*.

$$P(X_1)$$

well	1
allergy	0
cold	0

$$P(X_t \mid X_{t-1} = well)$$

well	0.7
allergy	0.2
cold	0.1

$$P(X_t \mid X_{t-1} = allergy)$$

well	0.6
allergy	0.3
cold	0.1

$$P(X_t \mid X_{t-1} = cold)$$

well	0.2
allergy	0.2
cold	0.6

Transitions

$$P(E_t \mid X_t = well)$$

quiet	1.0
sneeze	0.0

$$P(E_t \mid X_t = allergy)$$

quiet	0.0
sneeze	1.0

$$P(E_t \mid X_t = cold)$$

quiet	0.0
sneeze	1.0

Emissions

The Viterbi equations are:

$$m_1[x_1] = P(e_1|x_1)P(x_1)$$

$$m_t[x_t] = P(e_t|x_t) \max_{x_{t-1}} P(x_t|x_{t-1})m_{t-1}[x_{t-1}]$$

On day 1:

$$m_1[well] = P(quiet|well)P(well) = 1 * 1 = 1$$

$$m_1[allergy] = P(quiet|allergy)P(allergy) = 0 * 0 = 0$$

$$m_1[cold] = P(quiet|cold)P(cold) = 0 * 0 = 0$$

On day 2:

$$
\begin{aligned}
m_2[well] &= P(sneeze|well)\max \begin{cases} P(well|well)m_1[well] = 0.7*1 = 0.7 \\ P(well|allergy)m_1[allergy] = 0.6*0 = 0 \\ P(well|cold)m_1[cold] = 0.2*0 = 0 \end{cases} \\
&= 0*0.7 = 0 \\
m_2[allergy] &= P(sneeze|allergy)\max \begin{cases} P(allergy|well)m_1[well] = 0.2*1 = 0.2 \\ P(allergy|allergy)m_1[allergy] = 0.3*0 = 0 \\ P(allergy|cold)m_1[cold] = 0.2*0 = 0 \end{cases} \\
&= 1*0.2 = 0.2 \\
m_2[cold] &= P(sneeze|cold)\max \begin{cases} P(cold|well)m_1[well] = 0.1*1 = 0.1 \\ P(cold|allergy)m_1[allergy] = 0.1*0 = 0 \\ P(cold|cold)m_1[cold] = 0.6*0 = 0 \end{cases} \\
&= 1*0.1 = 0.1
\end{aligned}
$$

On day 3:

$$
\begin{aligned}
m_3[well] &= P(sneeze|well)\max \begin{cases} P(well|well)m_2[well] = 0.7*0 = 0 \\ P(well|allergy)m_2[allergy] = 0.6*0.2 = 0.12 \\ P(well|cold)m_2[cold] = 0.2*0.1 = 0.02 \end{cases} \\
&= 0*0.12 = 0 \\
m_3[allergy] &= P(sneeze|allergy)\max \begin{cases} P(allergy|well)m_2[well] = 0.2*0 = 0 \\ P(allergy|allergy)m_2[allergy] = 0.3*0.2 = 0.06 \\ P(allergy|cold)m_2[cold] = 0.2*0.1 = 0.02 \end{cases} \\
&= 1*0.06 = 0.06 \\
m_3[cold] &= P(sneeze|cold)\max \begin{cases} P(cold|well)m_2[well] = 0.1*0 = 0 \\ P(cold|allergy)m_2[allergy] = 0.1*0.2 = 0.02 \\ P(cold|cold)m_2[cold] = 0.6*0.1 = 0.06 \end{cases} \\
&= 1*0.06 = 0.06
\end{aligned}
$$

There are two equally likely explanations: (well, allergy, allergy) and (well, cold, cold).