ME EN 2450 Final Project Memo

To:	Jacob Hochhalter
From:	Ryan Dalby
Date:	April 25th, 2019
Subject:	Compressed Air Train Project

Summary

In order to determine the optimal design choices to get a train down a 10.0-meter track in the least amount of time, it was necessary to describe the motion of the train through physical laws. Then using numerical methods, the motion of the train was fully described and related to all of the design parameters in a way that could be optimized by the way of an objective function. Once optimized, these values were changed to meet more realistic values that could be purchased to create the train, and then re-simulated. It was found that a train of 0.278m in length with an outer radius of 84.1375mm made out of 1400kg/m^3 PVC filled with pressured air of 87668.5Pa with a pneumatic piston attached to the back of 0.4572 m in length with a radius of 9.525mm and a 6.35mm radius pinion gear could go the 10.0m in 8.10755 seconds and stop by 10.0392m.

Numerical Methods

In order to determine the optimal train design solution for this project is was first necessary to describe the motion of the train through physical laws. Using Newton's 2nd Law for two sections of the track it was possible to model the train motion. For the first section of the track, the train is being thrust by the pneumatic piston, so here there is a thrust force, drag due to air, and rolling resistance. For the second part of the track, the train is coasting so only drag from the air and rolling resistance from the wheels act. The following equations describe the train motion:

$$\text{Acceleration:} \quad \frac{d^2x}{dt^2} = \frac{1}{m+m_w}\left[A_p\frac{r_g}{r_w}\left(\frac{P_0V_0}{V_0 + A_p(r_g/r_w)x} - P_{\text{atm}}\right) - \frac{1}{2}C_d\rho A\left(\frac{dx}{dt}\right)^2 - C_r mg\right], \tag{20}$$

$$\text{Deceleration:} \quad \frac{d^2x}{dt^2} = \frac{1}{m}\left[-\frac{1}{2}C_d\rho A\left(\frac{dx}{dt}\right)^2 - C_r mg\right]. \tag{21}$$

A description of the constant parameters and design parameters are as follows:

Constant Parameters

Parameter	Symbol	Value	Units
air density	ρ_a	1	kg/m^3
atmospheric pressure	P_{atm}	101325	Pa
drag coefficient	C_d	.8	1
rolling friction coefficient	C_r	0.03	1
coefficient of static friction	μ_s	0.7	1
wheel radius	r_w	20	mm
mass of wheels and axles	m_w	100	g

Design Parameters

Parameter	Symbol	Range of Values	Units
length of train	L_t		mm
outer radius of train	r_o		mm
density of train material	ρ_t		kg/m^3
initial tank gage pressure	$P_{0_{\text{gage}}}$		Pa
pinion gear radius	r_g		mm
length of piston stroke	L_r		mm
radius of piston	r_p		mm

When the first section of the track ends and the second section of track begins depends on the length of the rack, which is equal to the length of the stroke of the piston, and the ratio of the radius of the wheel to the radius of the pinion gear. The relations are as follows:

$$\text{acceleration stage:} \quad x \le L_r \frac{r_w}{r_g}$$

$$\text{deceleration stage:} \quad x > L_r \frac{r_w}{r_g}$$

Once the motion of the train was described it was broken into a set of first-order ODEs by stating v = dy/dt (physically meaning velocity) which resulted in two first-order differential equations in code as follows:

```
Fd = 0.5 * Cd * rhoA * At * velocity**2 #part of drag term
Fr = Cr * mTotal * g #part of rolling resistance term

#Determine derivative values of dydt and dvdt
if(position <= (Lr * (rw/rg))): #accelerating
    Ft = Ap * (rg/rw) * (((P0*V0) / (V0 + Ap*(rg/rw)*position)) - Patm)  #part of thrust term
    dydt = velocity
    dvdt = (1.0 / (mTotal + mw)) * (Ft - Fd - Fr)
else: #decelerating
    dydt = velocity
    dvdt = (1.0 / mTotal) * (-Fd - Fr)
return dydt, dvdt
```

It was also necessary to relate all parameters in our differential equation to the design parameters which we are trying to optimize:

```
P0Eq = lambda P0gage: P0gage + Patm #Pa initial tank absolute pressure
mTotalEq = lambda Lt, r0, rhoT, Lr, rp: mw + (rhoT*np.pi*Lt*(r0**2 - riEq(r0)**2)) + MpEq(rp, Lr)  #kg total mass of train
AtEq = lambda r0: np.pi * r0**2 #m^2 train frontal area
V0Eq = lambda r0, Lt: np.pi * riEq(r0)**2 * Lt #m^3 tank volume
LpEq = lambda Lr: 1.5 * Lr #m total length of pistion
MpEq = lambda rp, Lr: 1250.0 * (np.pi * rp**2 *LpEq(Lr)) #kg mass of piston
ApEq = lambda rp: np.pi * rp**2 #m^2 area of the piston
riEq = lambda r0: r0/1.15 #m inside radius of tank
```

It was then necessary to solve for position using a 4th order Runge Kutta method for a system of differential equations. This was done in a driver script which further restricted our output to positive velocity values, stopping when velocity hit 0, physically meaning the train was stopped.

The next step was to optimize the design parameters. This was done by creating an objective function which took in a given set of design parameters and gave a time value for the train to complete the task which was to be used by the optimizer to minimize the time to get to the finish. It was also necessary to impose constraints on what design parameters were considered optimal in the objective function. This was done by sending our output to a maximum time value if the given set of design parameters violated a constraint. The constraints that were imposed on the train were:

- Stop between 10.0 and 12.5 meters to not fall off the track.
- The height had to be less than 0.23 meters to fit in the tunnel.
- The width had to be less than 0.20 meters to fit in the tunnel.
- The total train length including the piston on the back had to be less than 1.5 meters to fit on to the starting line of the track.
- The pinion gear radius had to smaller than the radius of the wheels.
- The traction force of the wheels could not exceed the force of static friction which would result in wheel slippage. To do this it was assumed if slipping were to occur it would occur initially when it would be presumed that the traction force was the greatest. These relationships were as follows: (It was also presumed that the m_wd^2x/dt^2 term was 0)

wheel-slip criterion: $F_t > \mu_s \frac{m}{2} g,$

$$F_t = \frac{r_g A_p}{r_w}\left[\frac{P_0 V_0}{V_0 + A_p \frac{r_g}{r_w} x} - P_{\text{atm}}\right] - m_w \frac{d^2 x}{dt^2}.$$

To be able to perform optimization realistically it was necessary to select realistic design parameter ranges to be passed into our optimizer with the objective function, and fix parameters that would be constant for our testing. The drag coefficient C_d, rolling resistance coefficient C_r, air density ρ_a, the mass of the wheels m_w, the radius of the wheels r_w, atmospheric pressure P_{atm}, gravitational constant g, and coefficient of static friction μ_s were all constant with values as seen below.

For the design parameters, the following ranges were initially selected after determining realistic limits in the actual parts that would be achievable. This restricted our design space which allowed the optimizer to better find values that met the constraints.

Table 1- Description of Parameters:

Parameter	Ranges	Units
length of the train	0.03 to 1.25	meters
outer radius of the train	0.003 to 0.095	meters
density of train material	1200 to 9000	kg/m^3
initial tank gauge pressure (constrained to < 30psig)	20684.3 to 206843.0	pascals
pinion gear radius	0.001 to 0.020	meters
length of the piston stroke (equal to length of rack)	0.01 to 0.70	meters
radius of piston	0.003 to 0.095	meters

Table 1: Relevant Physical Parameters

Parameter	Symbol	Value	Units
air density	ρ_a	1	kg/m^3
atmospheric pressure	P_{atm}	101325	Pa
drag coefficient	C_d	.8	1
rolling friction coefficient	C_r	0.03	1
coefficient of static friction	μ_s	0.7	1
wheel radius	r_w	20	mm
mass of wheels and axles	m_w	100	g

Numerical Results

Table 2- Optimal Solution Summary:

execution time(s)	11.6897
x postion at finish(m)	12.1377
time to finish(s)	5.93188
length of train(m)	0.460132
outer radius of train(mm)	94.3567
material of train	density = 6060.27kg/m^3
initial tank gage pressure(Pa)	137036
pinion gear radius(mm)	5.85061
length of piston stroke(m)	0.478787
radius of piston(mm)	19.0498
total mass of train(kg)	20.1432

Max iterations used in driver/RK4: 1000
Step size used in driver/RK4: 0.1

Figure 1- Optimal Solution Plot:

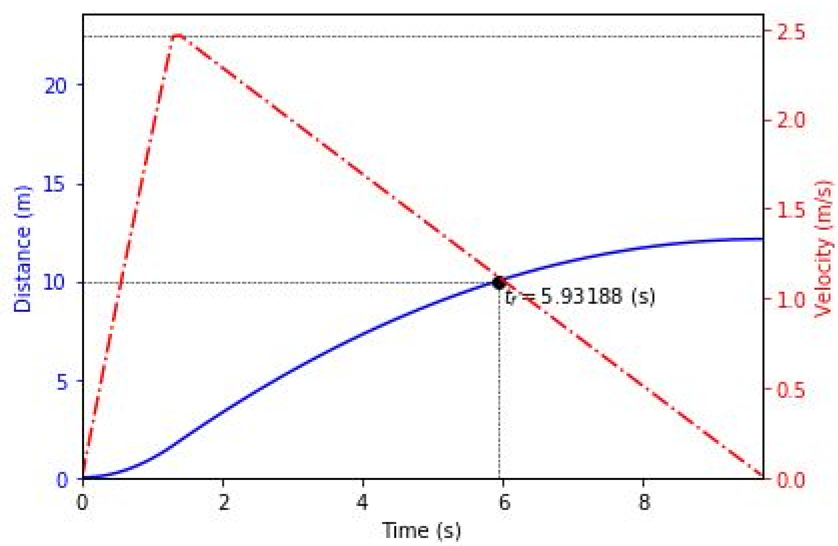


Table 3- Physical Parameters of Optimal Solution:

method name	monte carlos
time to finish(s)	5.93188
x position at finish(m)	12.1377
length of train(m)	0.460132
outer radius of train(mm)	94.3567
height of train(m)	0.228713
material of train	density = 6060.27kg/m^3
total mass of train(kg)	20.1432
train frontal area(m^2)	0.0279702
initial tank gauge pressure(Pa)	137036
tank volume(m^3)	0.00973155
pinion gear radius(mm)	5.85061
length of piston stroke(m)	0.478787
total length of piston(m)	0.71818
radius of piston(mm)	19.0498
mass of piston(kg)	1.02347
execution time(s)	11.6897

Max iterations used in driver/RK4: 1000
Step size used in driver/RK4: 0.1

By implementing the described numerical methods it was possible to determine a set of optimal design parameters. Table 2 shows the optimal values of the design parameters, the time to compute them, the time to pass the finish line and the stopping x position. With the optimal design parameters, it takes 5.93188 seconds to reach 10.0 meters on the track. The train stops before the end of the track at 12.1377 meters. Looking at the plot on Figure 1 our results seem reasonable, as we see the relatively linear thrust term dominating for the first part of the track and the relatively linear rolling resistance term dominating for the second part of the track. Looking at Table 3 we see that all of the physical parameters meet the design constraints. One important thing to note is that these optimal parameters may not be realistic to buy parts for the train, this will be addressed in the following section. It is also important to note that assumptions the relationships between some of our physical parameters and our design parameters may not hold as well as the dimensions change drastically. One example of this would be the outer radius of the train being 1.15 times as big as the inner radius. With a larger radius of the train, it is likely that parts that could be purchased keep a similar wall thickness even as the radius gets bigger, thus the total mass wouldn't be as high. Once again selecting more realistic parameters will be addressed in the following section.

Realistic Train Design

Now realistic parts were found which became the actual design parameters that would be used. It was determined that the values for the outer radius of the train, and the train material(density of the material), would be strictly set by the piping chosen. The length of the train and the initial tank pressure could be varied continuously by cutting the pipe to size and adjusting the pressure respectively(PVC was chosen as the material because on its ease of being cut to different length and variety of sizes). The radius of the pinion gear was strictly set by the chosen pinion gear and rack. The length of the piston stroke and the radius of the piston would be strictly set by the chosen pneumatic piston. Then the optimal realistic solution was found by fixing parameters that had discrete values one by one and letting the optimizer choose values for the length of the train and the initial tank pressure. The following was found:

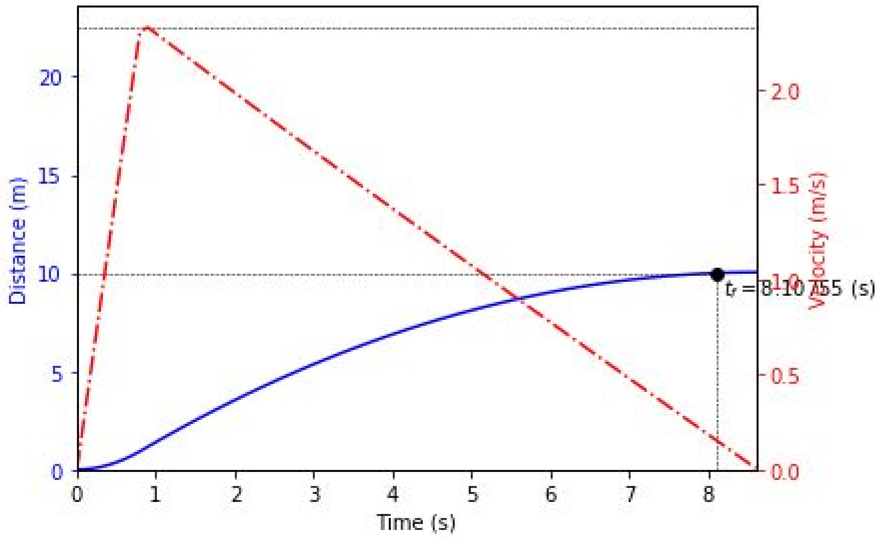
Table 4- Optimal Realistic Solution:

method name	monte carlos
time to finish(s)	8.10755
x position at finish(m)	10.0392
length of train(m)	0.277699
outer radius of train(mm)	84.1375
height of train(m)	0.208275
material of train	PVC = 1400.00kg/m^3
total mass of train(kg)	2.37135
train frontal area(m^2)	0.0222397
initial tank gauge pressure(Pa)	87668.5
tank volume(m^3)	0.0046699
pinion gear radius(mm)	6.35
length of piston stroke(m)	0.3048
total length of piston(m)	0.4572
radius of piston(mm)	9.525
mass of piston(kg)	0.162891
execution time(s)	7.48397

Max iterations used in driver/RK4: 1000

Step size used in driver/RK4: 0.1

Figure 2- Optimal Realistic Solution Plot:



Part List:

- pneumatic piston: https://www.mcmaster.com/6453k134 $152.03
  - ¾ inch bore(diameter) = 9.525 mm radius of piston
  - 12 inch stroke = .3048 m stroke
- train body: https://www.mcmaster.com/48925k25 $53.36
  - 6.625 outer diameter = 84.1375 mm outer radius of train
  - length of 5ft = 1.524 m which can be cut to 0.277699 m
  - PVC density = 1400.00kg/m^3
  - gage pressure will be set to 87668.5 Pa
- end caps of train body: https://www.mcmaster.com/4880k141 $13.98 x 2 = $27.96
- pinion gear: https://www.mcmaster.com/6325k89 $15.17
  - 0.5 inch bore(diameter) = 6.35mm
- rack: https://www.mcmaster.com/6295k11 $16.91
  - 2ft = 0.6096 m which will be cut to length of piston stroke of .3048 m
- wheels and axles: $1.68 x 4 (wheels) + $0.50 x 2 (axles)
  - 1.625 in diameter wheels approx 40 mm diameter wheels: https://www.mcmaster.com/2337t1 (Will change what wheels specifically with more details of track)
  - axles: https://www.mcmaster.com/23595t11

It can be seen from Table 4 what the optimal realistic design parameters result in an 8.10755 second time to 10.0m with the train stopping at 10.0392 meters. The plot in Figure 2 is very similar to the optimal solution plot, although slightly slower time to 10.0 meters. The part list above has all of the parts needed and is based on the realistic design parameters chosen. Both Table 4 and Figure 2 show the performance of the realistic design parameters.