ME 2450 Design Project

Name: ______________________

Due: April 25, 2019

Objective

A computer program that determines parameters that minimize the time required for a compressed air train, shown in Figure 1, to traverse a track of specified length will be determined using numerical methods learned in class. To receive credit for the project

- ☐ (6 points) write a program that finds the optimal design parameters; and
- ☐ (4 points) write a memo, based on the memo template found on Canvas, describing the design process and final (optimal) design parameters is to be turned in via Canvas by the time and date indicated on Canvas. Relevant code shall be attached electronically to the memo. Be sure to refer to attached code files by name in the memo, otherwise they will not be considered.

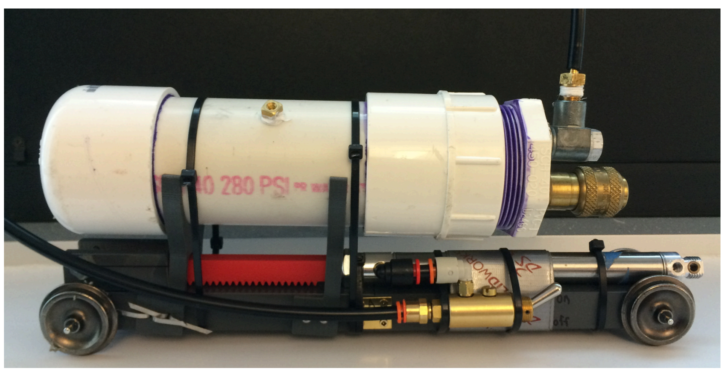

Figure 1: Photograph of a fully built and functioning compressed air train.

Train Design Competition

The project involves the design of a model train to compete in a race as described below. Note, you will NOT be building anything in this class, but rather specifying the physical dimensions of the train and locomotion system in order to be competitive in the race.

Race Track

The race track consists of a 10m section of straight track with a single tunnel that starts 6m after the start line, as shown in Figure 2. There is a 1.5m length of "set-up" track before the start line, and a 2.5m length of "run-out" track after the finish line. The tunnel has a minimum internal width of 0.2m, a length of 1m, and a maximum internal height of 0.23m above the track. The track used in the competition is standard model G-scale railroad track (rails spaced 45mm apart). The train should use steel wheels (radius of 20mm) and axles.

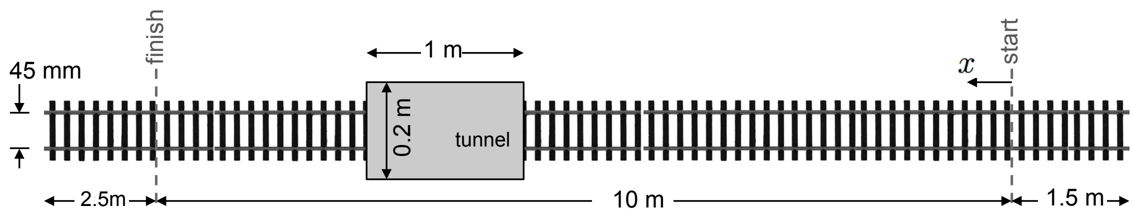

Figure 2: Layout of the race track. The distance traveled by the train is denoted by $x(t)$, with $x(0) = 0$ located at the start line.

Goal of the Race

The goal of the competition is to design a train that completes the race from start to finish in the least amount of time without damaging the tunnel and without running off the end of the track.

Power Source

The train is to be powered by compressed air stored on-board in a pressurized tank having a cylindrical geometry. The tank will be filled prior to the start of the race using a manual bicycle pump with an inline dial-type pressure gage. According to the rules of the competition, the initial tank pressure must not exceed 30psig.

Locomotion System

In order to simplify the modeling equations, we will restrict the locomotion system to one consisting of a pneumatic piston connected to a rack and pinion gear that drives the rear axle of the train. Furthermore, we will restrict the actuation of the piston to a single stroke.

The locomotion system will provide an "initial push" to accelerate the train up to its maximum velocity; after which time, the train simply coasts (decelerates) to the finish line. In this manner, the propulsion system is only active during a portion of the race, as illustrated in Figure 3. We will denote the acceleration stage from $0 \geq x \leq L_a$ and the deceleration stage from $L_a < x \leq L$, where L is the total length traveled. Here, $x(t)$ represents the distance traveled by the model train as a function of time, where $x = 0$ represents the starting line.

We will see in the next section that the propulsive force acting on the pneumatic piston decreases with x (i.e., the driving force is non-constant over the acceleration stage). Contrast this case to that of a gravitational force (as one would experience rolling down a hill), which is constant and simply proportional to the weight of the object.

Physical Model

Train Dynamic Model

In order to obtain a design solution for a winning train, we need to be able to faithfully simulate the dynamics of the train. We start with a free-body diagram of the train (Figure 4), from which Newton's

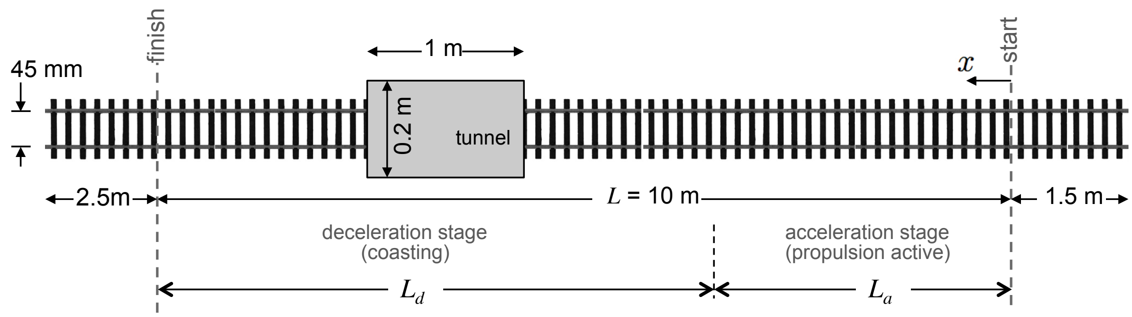

Figure 3: Schematic highlighting the locomotion strategy whereby the train is initially accelerated over a distance $0 \leq x \leq L_a$, and then allowed to coast (decelerate) to the finish line over the distance $L_a < x \leq L$.

2nd law can be written. The forces acting on the train include the rolling friction, F_r, the aerodynamic drag force, F_d, and the traction force, F_t. Note, the traction force depends on the type of locomotion system used and is only applicable during the *acceleration stage*. The engineering theory needed to derive a mathematical model of the system is presented below.

For the case of a train moving down a flat track, Newton's 2nd law in the x-direction parallel to the track is

$$\text{acceleration:} \qquad m\frac{d^2x}{dt^2} = F_t - F_d - F_r, \tag{1}$$

$$\text{deceleration:} \qquad m\frac{d^2x}{dt^2} = \quad - F_d - F_r. \tag{2}$$

where m denotes the total mass of the train and $a(= d^2x/dt^2)$ denotes the acceleration.

The aerodynamic drag force can be written in terms of a drag coefficient as

$$F_d = \frac{1}{2}C_d \rho A V^2,$$

where ρ denotes the density of the air, A is the frontal area of the train, and V is the velocity of the train ($V = dx/dt$). The drag coefficient C_d depends on the shape of the object and the surface roughness. The model train to be used in the race will consist of a pressurized cylindrical tank on wheels. A good estimate for the drag coefficient of a circular cylinder oriented axially to the flow is $C_d \approx 0.8$. The

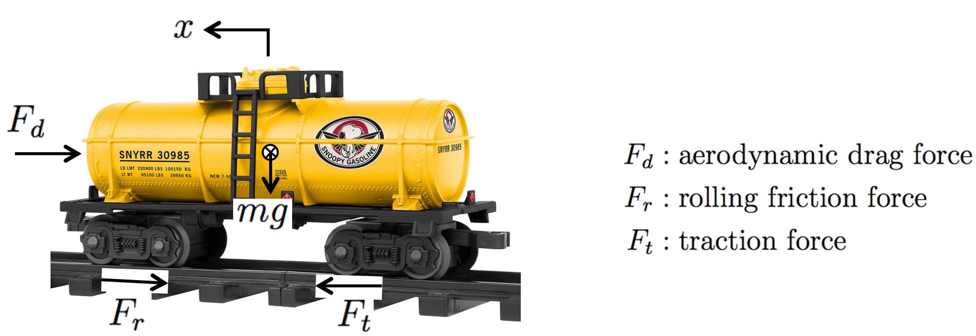

Figure 4: Free body diagram of the forces acting on the train during the race.

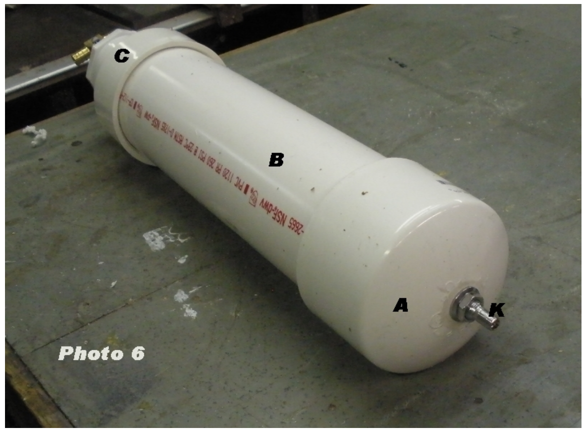

Figure 5: Example of a pressurized tank using PVC pipe with end caps.

rolling friction force between the wheel of the train and the rail of the train track is parameterized by the rolling resistance coefficient C_r according to the expression

$$F_r = C_r mg.$$

One can empirically (using experiments) determine a value for the rolling resistance. The value should be $C_r \approx 0.03$. We will assume that C_d and C_r are constant, which is a very good assumption under steady state conditions (i.e., the train is traveling at a constant speed). During the race, the model train never reaches a steady state condition; therefore, some error is expected by assuming a constant C_d. Substituting the expressions for F_d and F_r into (1) and (2) yields

$$\text{acceleration:} \qquad m\frac{d^2x}{dt^2} = F_t - \frac{1}{2}C_d\rho A\left(\frac{dx}{dt}\right)^2 - C_r mg, \tag{3}$$

$$\text{deceleration:} \qquad m\frac{d^2x}{dt^2} = \quad - \frac{1}{2}C_d\rho A\left(\frac{dx}{dt}\right)^2 - C_r mg. \tag{4}$$

Traction Force

The traction force F_t is the frictional force that the track rail exerts on the rotating train wheel. The traction force is applied at the point of contact between the wheel and the rail (see Figure 6). The

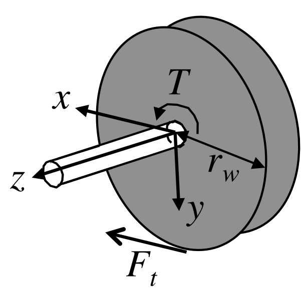

Figure 6: Schematic of the train wheel illustrating the traction force, F_t, between the wheel and rail (not shown).

traction force can be related to the applied torque on the wheel through conservation of angular momentum, which states that the sum of the torque is equal to the moment of inertia of the wheel (I) times the angular acceleration (α) of the wheel. Applying conservation of momentum about the axis of the wheel gives

$$T - r_w F_t = I\alpha,$$

where r_w = 20 mm is the radius of the train wheel. Approximating the train wheel as a solid disc, we can write the moment of inertia of the wheel as $I = \frac{1}{2} m_w r_w^2$, where m_w denotes the mass of the wheel. Note, since there are two wheels per axle, we need to multiply I by two. Therefore, the traction force is

$$F_t = \frac{T}{r_w} - m_w r_w \alpha. \tag{5}$$

Note, wheel slip will occur if the traction force is greater than the static friction force,

$$\text{wheel-slip criterion:} \quad F_t > \mu_s \frac{m}{2} g, \tag{6}$$

where μ_s is the coefficient of static friction between the wheel and the rail. The factor of $\frac{1}{2}$ is used because we assume the propulsion force only drives one of the axles, and that the total mass of the train is distributed equally between the front and rear axles. The value of μ_s depends on the type of materials in contact. Your code will need to check for wheel-slip criterion in (6) and give an appropriate error if it is violated.

Applied Torque

The applied torque driving the wheel, T in (5), is produced by the locomotion system, which in this case includes a rack and pinion gear as shown in Figure 7. Therefore, the applied torque is equal to the driving force of the rack, F_p, multiplied by the radius of the pinion gear, r_g, i.e.,

$$T = r_g \cdot F_p. \tag{7}$$

The rack is to be connected to a pneumatic piston as shown in Figure 8. A photograph of an actual pneumatic piston that can be purchased off-the-shelf is given in Figure 9.

The force F_p driving the rack and pinion is equal to the *net pressure* acting on the piston times the area of the piston head A_p,

$$F_p = P_{\text{net}} \cdot A_p. \tag{8}$$

The net pressure is equal to the pressure inside the piston chamber, P, minus the pressure acting on the outside of the piston which is assumed to be atmospheric pressure, i.e., $P_{\text{net}} = P - P_{\text{atm}}$. Therefore, we can write

$$F_p = (P - P_{\text{atm}}) A_p. \tag{9}$$

Note, the pressure inside the piston chamber is decreasing with time as the piston expands.

Piston Pressure

The pressure inside the piston chamber is governed by the ideal gas law,

$$PV = nRT, \tag{10}$$

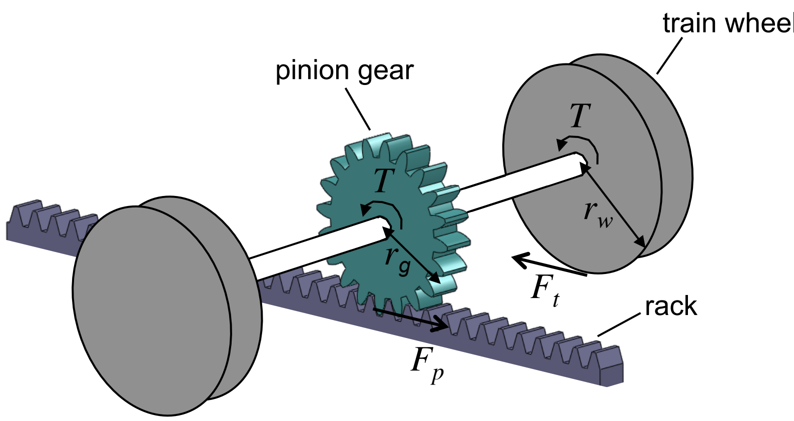

Figure 7: Schematic of the train axle showing one possible configuration of the rack/pinion.

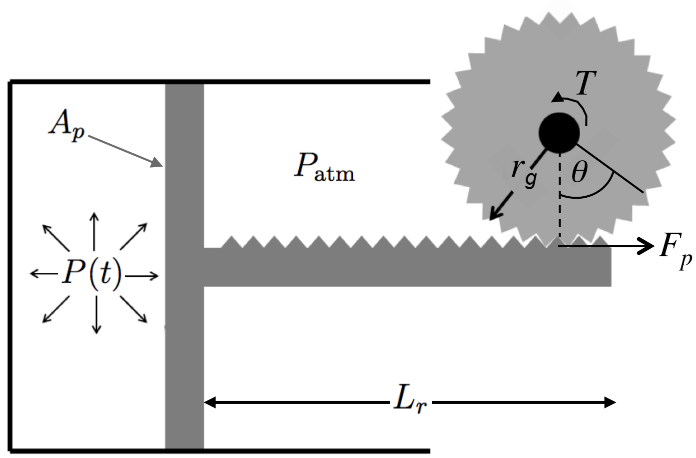

Figure 8: Schematic of the piston pushing against the rack thereby causing an applied torque T on the pinion gear.

where V is the volume of the gas in the tank, n is the amount of gas in moles, R is the universal gas constant, and T is the absolute temperature of the gas. Note, P in (10) must be expressed in terms of an absolute pressure and NOT a gauge pressure. Since no gas is exhausted to the surroundings during locomotion, the mass of gas, and hence n, remains constant. If we assume that the expansion of the piston occurs isothermally, i.e., temperature remains constant, then the left hand side of (10) must be constant and equal to its initial value. This means (10) can be rewritten as

$$PV = P_0 V_0, \tag{11}$$

where P_0 and V_0 denote the initial tank pressure and volume, respectively. At a given time t, the total volume of the gas is equal to the initial tank volume plus the volume of the piston chamber, $V = V_0 + \Delta V$. Solving (11) for P gives

$$P = \frac{P_0 V_0}{V_0 + \Delta V}. \tag{12}$$

As the piston expands, the volume increases by an amount $\Delta V = A_p \ell$ as shown in Figure 10. Since the piston is connected to the rack and pinion gear, ℓ is determined by the angular rotation of the pinion

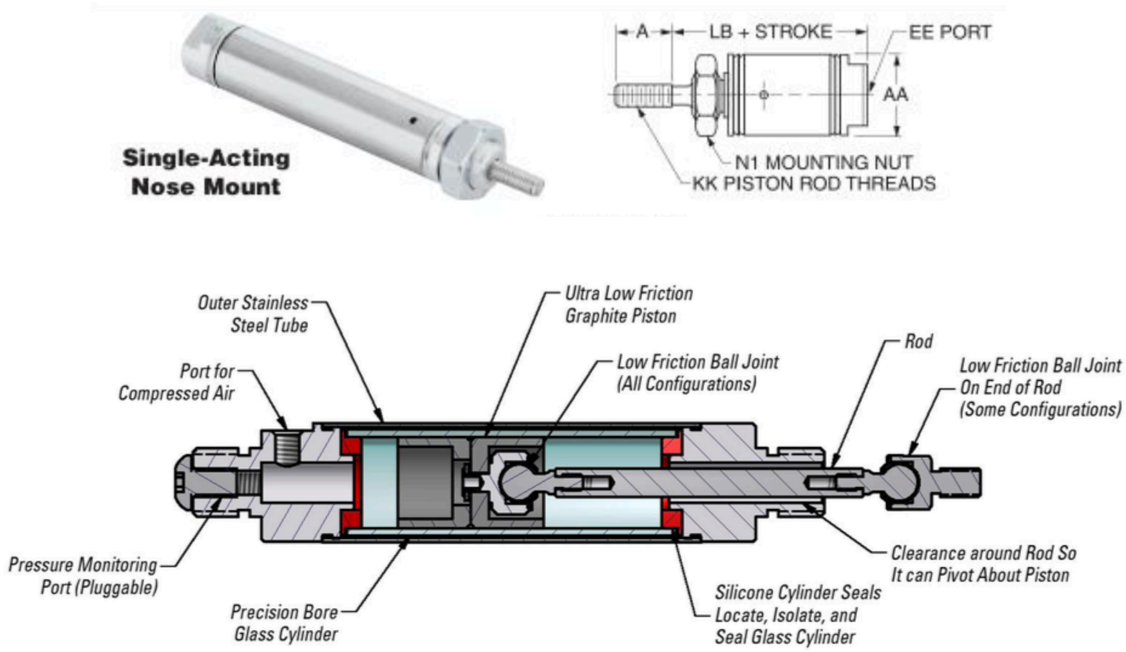

Figure 9: Example of a pneumatic piston that can be purchased off-the-shelf.

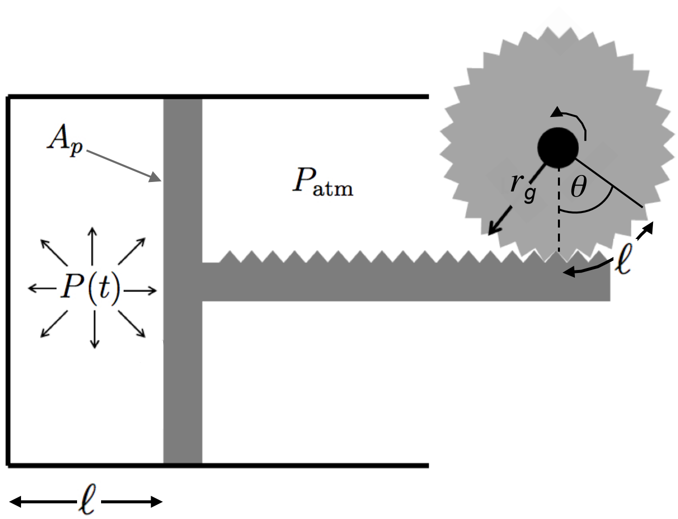

Figure 10: Schematic of the piston head displaced by a distance ℓ. The displaced volume is equal to $A_p\ell$.

gear, $\ell = r_g\theta$, as illustrated in Figure 10. Therefore, (12) becomes

$$P = \frac{P_0 V_0}{V_0 + A_p r_g \theta}. \tag{13}$$

Final System of ODEs

Finally, we can substitute (13), (9), and (7) into (5) to obtain an expression for the traction force in terms of wheel rotation θ,

$$F_t = \frac{r_g A_p}{r_w}\left[\frac{P_0 V_0}{V_0 + A_p r_g \theta} - P_{\text{atm}}\right] - m_w r_w \alpha. \tag{14}$$

We now recognize that, since the pinion gear and train wheel are connected to the same axle, the wheel/gear rotation is determined by the distance traveled,

$$x = r_w \theta. \tag{15}$$

This means we can write both the angular rotation of the wheel, θ, and the angular acceleration of the wheel, α, in terms of x as

$$\theta = \frac{x}{r_w} \qquad \text{and} \qquad \alpha = \frac{d^2\theta}{dt^2} = \frac{1}{r_w}\frac{d^2 x}{dt^2}. \tag{16}$$

Substituting (16) into (14) yields

$$F_t = \frac{r_g A_p}{r_w}\left[\frac{P_0 V_0}{V_0 + A_p \frac{r_g}{r_w} x} - P_{\text{atm}}\right] - m_w \frac{d^2 x}{dt^2}. \tag{17}$$

Note, the traction force is inversely proportional to x.

Finally, we can substitute (17) back into the original equations of motion

$$\text{acceleration:} \quad m\frac{d^2x}{dt^2} = \frac{r_g A_p}{r_w}\left[\frac{P_0 V_0}{V_0 + A_p \frac{r_g}{r_w} x} - P_{\text{atm}}\right] - m_w \frac{d^2 x}{dt^2} - \frac{1}{2} C_d \rho A \left(\frac{dx}{dt}\right)^2 - C_r m g, \tag{18}$$

$$\text{deceleration:} \quad m\frac{d^2x}{dt^2} = - \frac{1}{2} C_d \rho A \left(\frac{dx}{dt}\right)^2 - C_r m g. \tag{19}$$

Dividing through by m and rearranging the top equation to get the derivative term on the left hand side produces the final set of governing equations

$$\text{Acceleration:} \quad \frac{d^2x}{dt^2} = \frac{1}{m + m_w}\left[A_p \frac{r_g}{r_w}\left(\frac{P_0 V_0}{V_0 + A_p (r_g/r_w) x} - P_{\text{atm}}\right) - \frac{1}{2} C_d \rho A \left(\frac{dx}{dt}\right)^2 - C_r m g\right], \tag{20}$$

$$\text{Deceleration:} \quad \frac{d^2x}{dt^2} = \frac{1}{m}\left[-\frac{1}{2} C_d \rho A \left(\frac{dx}{dt}\right)^2 - C_r m g\right]. \tag{21}$$

You will solve these ODEs in one of your lab sessions using the 4th order Runge-Kutta method.

Note, your code will need to test when the acceleration stage ends and the deceleration stage begins. This is set by the length of the rack L_r as shown in Figure 8. Accordingly, the propulsion system will continue to apply a traction force as long as $r_g \theta < L$, or substituting (16) for θ, gives the following conditions

$$\text{acceleration stage:} \quad x \le L_r \frac{r_w}{r_g} \tag{22}$$

$$\text{deceleration stage:} \quad x > L_r \frac{r_w}{r_g} \tag{23}$$

Solution of (20) and (21) for a competitive train might look like the plot in Figure 11, which shows the distance traveled x versus time t. The train crosses the finish line at a time $t_f \approx 8.5$ s, and comes to a full stop at $x_s \approx 11.6$ m before reaching the end of the run-out track.

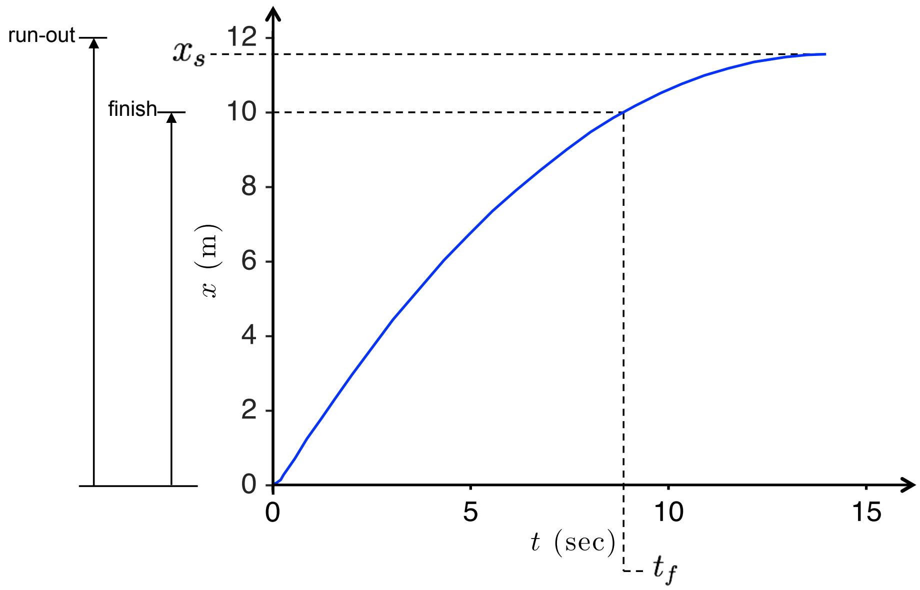

Figure 11: Behavior of a competitive train, where $x(t)$ represents the distance traveled as a function of time t.

Multidimensional Optimization Problem

Tables 1 and 2 list relevant physical parameters that appear in equations (20) and (21) and, therefore, can affect the performance of the train. The parameters in Table 1 are fixed at the values shown; whereas, the parameters in Table 2 will be optimized as part of the design solution. Recall that the value for the initial tank pressure used in (20) is the absolute pressure calculated using

$$P_0 = P_{0_{\text{gage}}} + P_{\text{atm}}.$$

You will need to determine realistic values for each of the parameters in Table 2.

The model train is assumed to be made in the shape of a cylinder, consisting of a pipe/tube with PVC end caps and can be made out of PVC, steel, stainless steel, aluminum, or any other material that is readily purchased in tube/pipe form. You should consider at least three different materials in the design process. The densities of various tube/pipe materials is given in Table 3.

Your program should include the following features

1. Define ranges for physical parameters

   Use the specified values for the *fixed* parameters in Table 1. For each of the *design* parameters in Table 2, select appropriate ranges of values. You should consult information from manufacturers in order to determine realistic values for off-the-shelf parts that need to be purchased. Your

Table 1: Relevant Physical Parameters

Parameter	**Symbol**	**Value**	**Units**
air density	ρ_a	1	kg/m^3
atmospheric pressure	P_{atm}	101325	Pa
drag coefficient	C_d	.8	1
rolling friction coefficient	C_r	0.03	1
coefficient of static friction	μ_s	0.7	1
wheel radius	r_w	20	mm
mass of wheels and axles	m_w	100	g

Table 2: Relevant Physical Parameters

Parameter	**Symbol**	**Range of Values**	**Units**
length of train	L_t		mm
outer radius of train	r_o		mm
density of train material	ρ_t		kg/m^3
initial tank gage pressure	$P_{0_{\text{gage}}}$		Pa
pinion gear radius	r_g		mm
length of piston stroke	L_r		mm
radius of piston	r_p		mm

Table 3: Density of Various Materials used for Pipes/Tubes

Material	**Density** (kg/m^3)
PVC	1400
acrylic	1200
galvanized steel	7700
stainless steel	8000
titanium	4500
copper	8940
aluminum – 6061	2700

optimization procedure should eliminate any combination of parameters that violates the design constraints (see Item 2).

For simplicity's sake, assume the following:

- The inside radius r_i of the pipe tube is proportional to the outer radius ($r_o = 1.15r_i$).
- The mass of the pneumatic piston is proportional to its volume, i.e., $M_p = \rho_p(\pi r_p^2 L_p)$, with a proportionality constant of $\rho_p = 1250$ kg/m^3.
- The total length of the piston is proportional to its stroke ($L_p = 1.5L_r$).
- The length of the rack is equal to the length of the stroke of the pneumatic piston.

Note, you need to consider where the piston will be placed. Options are underneath the train or behind the train like a caboose. If the piston is placed underneath the train, then enough clearance must be ensured, which will affect the total height of the train. If the piston is placed behind the train, then the total length of the train is the length of the tank plus the length of the piston. There are design constraints on both the total length of the train and the height of the train as discussed below.

2. Check design constraints

   Your code should eliminate any set of parameters that violates one or more of the design constraints:

   train height/width the train must be capable of passing through the tunnel. Therefore, the height cannot exceed 0.23 m; and, the width cannot exceed 0.2 m.

   train length the entire train must fit on the track at the starting line. Since the length of "set-up" track is only 1.5 m, the total length of the train plus propulsion system cannot exceed 1.5 m.

   gear radius the radius of the pinion gear must be less than the radius of the train wheel (i.e., $r_g/r_w < 1$).

   wheel slippage the wheels will slip if the maximum traction force exceeds the force due to static friction, as described by (6). The maximum traction force will be experienced at start up when the pressure inside the tank is a maximum. Using the relation for the traction force in equation (14), one can derive a condition on the maximum allowable initial tank pressure to avoid wheel slippage. If the initial tank pressure exceeds the maximum allowable pressure, it is inadmissible and an appropriate error should be raised.

3. Perform optimization

   Using the scripts provided,

   - `findmin.[m,py]`
   - `rk4.[m,py]`
   - `plot_results.[m,py]`

   set up and run an optimization problem to determine the optimal train parameters. See the projectile motion test in `test_findmin.[m,py]` for an example of setting up an optimization problem. You will have to calculate quantities such as the frontal area of the train A, the initial tank volume V_0, the total mass of the train m, etc., based on the values provided for the design parameters.

4. Plot distance traveled and train velocity versus elapsed time (optimum design solution)

   For the optimum set of parameters obtained, plot x versus t and v versus t based on your numerical simulation. You can use the provided plotting function, or write your own. Be sure to include the plots in your memo.

5. Display optimum design solution

   Once you have found the optimum set of design parameters, your code should calculate the following quantities, which are to be included in your memo in tabular format as shown below. Be sure to include the table in your memo.

Parameter	Symbol	Optimum Value	Units
length of train	L_t		m
outer radius of train	r_o		mm
height of train	H_t		m
material of train	–		
total mass of train	m		kg
train frontal area	A		m^2
initial tank gage pressure	$P_{0_{\text{gage}}}$		Pa
tank volume	V_0		m^3
pinion gear radius	r_g		mm
length of piston stroke	L_r		m
total length of piston	L_p		m
radius of piston	D_p		mm
mass of piston	M_p		kg

6. Select *realistic* components

   You may not be able to build an actual model train having the exact specifications given by your optimum solution, because materials and components will typically only be available in standard sizes. You are now tasked with selecting realistic components for your train that match as close as possible to the parameter values given by your optimum design solution. In order to do this, you will need to search the internet to identify manufacturers that make pipe/tubing, pneumatic pistons, and rack/pinion gears. Based on your research put together a parts list for your train including the size, model number, vendor, and estimated price for each item. Include the parts lists in your memo.

   If your *realistic* train has different parameter values than your optimum numerical solution, then you will need to rerun your simulation on the *realistic* train in order to verify that the *realistic* train still completes the race without running off the track. In your memo, you should state the time to cross the finish line for your *realistic* train.