2D Steady-State Conduction

Thermal Fluids and Energy Systems Lab

(ME EN 4650)

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University of Utah

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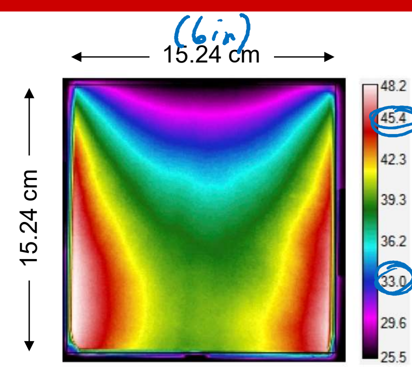
Steady-State Conduction through a Two-Dimensional Body

What we want to know:

- → temperature distribution over surface
- → heat flux across boundaries
- → validate a numerical simulation



What we can measure:

- → temperature distribution (2D temperature)
- → electrical power (digital power meter)

76,800

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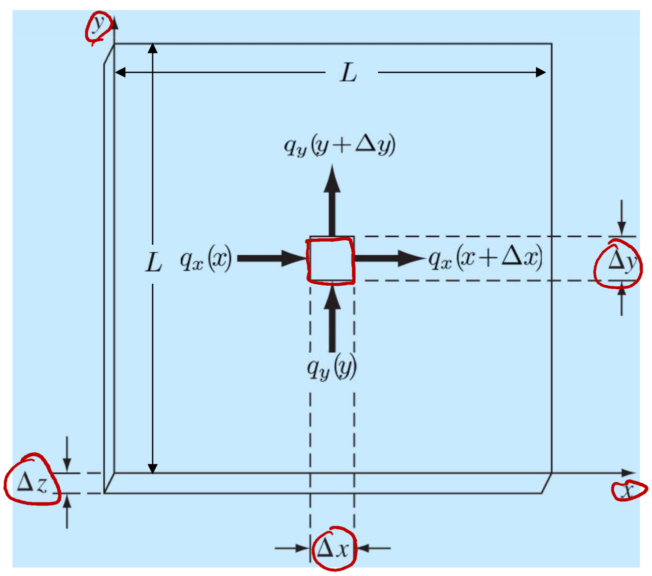
Lecture Outline

- Background (2D Conduction)
- Experimental Setup
- Measurements and Instrumentation
- Required Figures

Part 2
- Data Analysis (Finite Difference Method)

Two-Dimensional Steady State Conduction



T(x,y)

$$\dot{E}_{\text{in}} - \dot{E}_{\text{out}} = \dot{E}_{\text{st}}$$

$$-\frac{\partial q_x}{\partial x}dx - \frac{\partial q_y}{\partial y}dy - \frac{\partial q_z}{\partial z}dz = \rho\, c_p \cancel{\frac{\partial T}{\partial t}}\, dx\, dy\, dz$$

$$q_x = -k\, dy\, dz \frac{\partial T}{\partial x}$$

k: thermal conductivity

$$q_y = -k\, dx\, dz \frac{\partial T}{\partial y}$$

$$q_z = -k\, dx\, dy \frac{\partial T}{\partial z}$$

$$\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right) + \frac{\partial}{\partial y}\left(k\frac{\partial T}{\partial y}\right) + \frac{\partial}{\partial z}\left(k\cancel{\frac{\partial T}{\partial z}}\right) = 0$$

0

$$\boxed{\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0}$$

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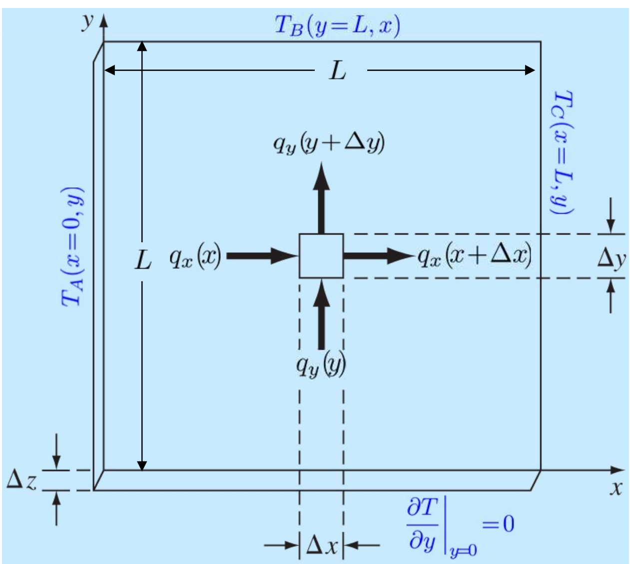
Two-Dimensional Steady State Conduction



$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0$$

2nd order PDE → 2 BCs on x, 2 BCs on y

Left: $T(x=0,y) = T_A(y)$

Top: $T(x,y=L) = T_B(x)$

Right: $T(x=L,y) = T_C(y)$

Bottom: $q''(x,y=0) = 0$

$$-k\left.\frac{\partial T}{\partial y}\right|_{y=0} = 0$$

$$\left.\frac{\partial T}{\partial y}\right|_{y=0} = 0$$

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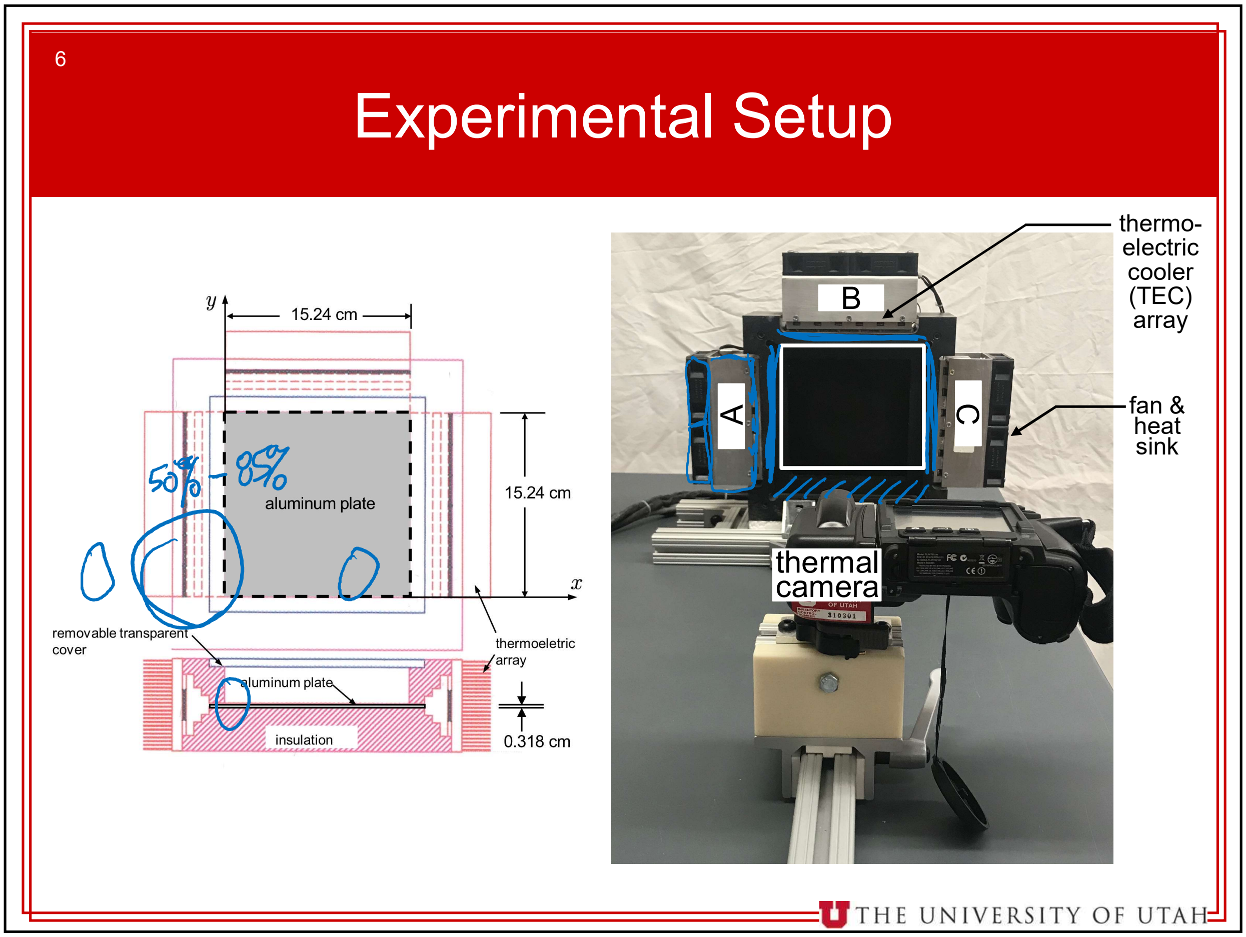
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Experimental Setup
y
15.24 cm
50% - 85%
aluminum plate
15.24 cm
x
removable transparent cover
thermoeletric array
aluminum plate
insulation
0.318 cm
thermo-electric cooler (TEC) array
B
A
C
fan & heat sink
thermal camera
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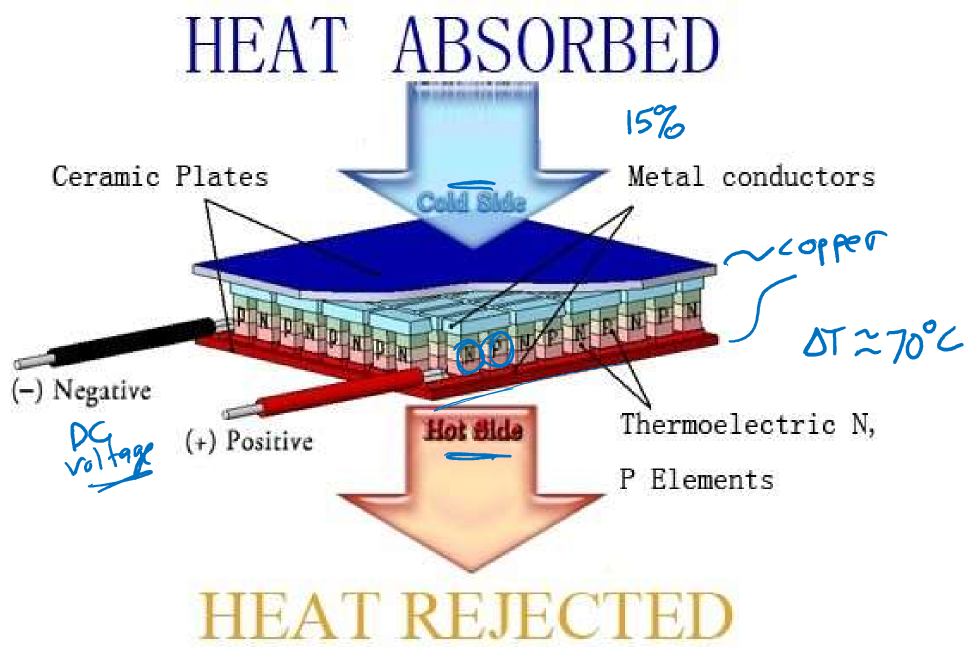
Thermoelectric Cooler (TEC)



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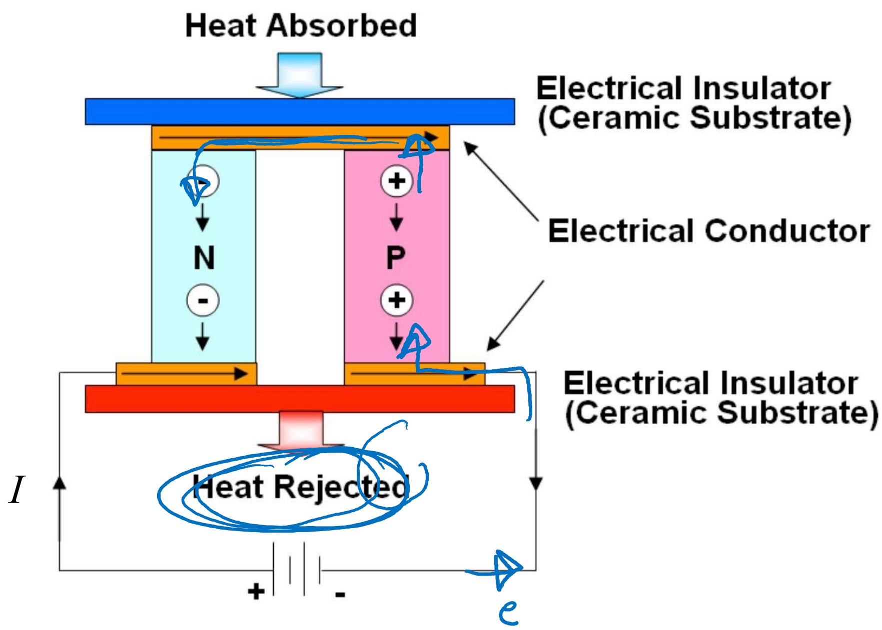
Thermoelectric Cooler (TEC)



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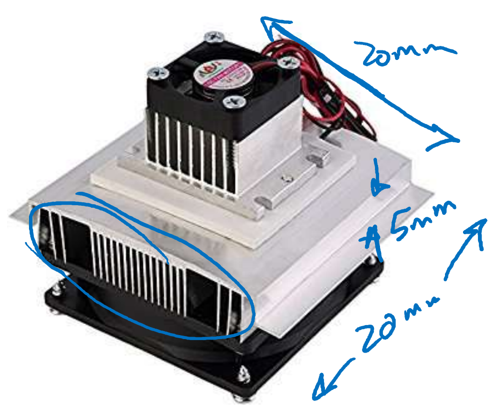
TEC with Heat Sinks



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Measurements

Quantity	Symbol	Units	Instrument
Plate temperature distribution	$T(x, y)$	°C	IR camera
Electrical power to TECs	P_E	W	power meter

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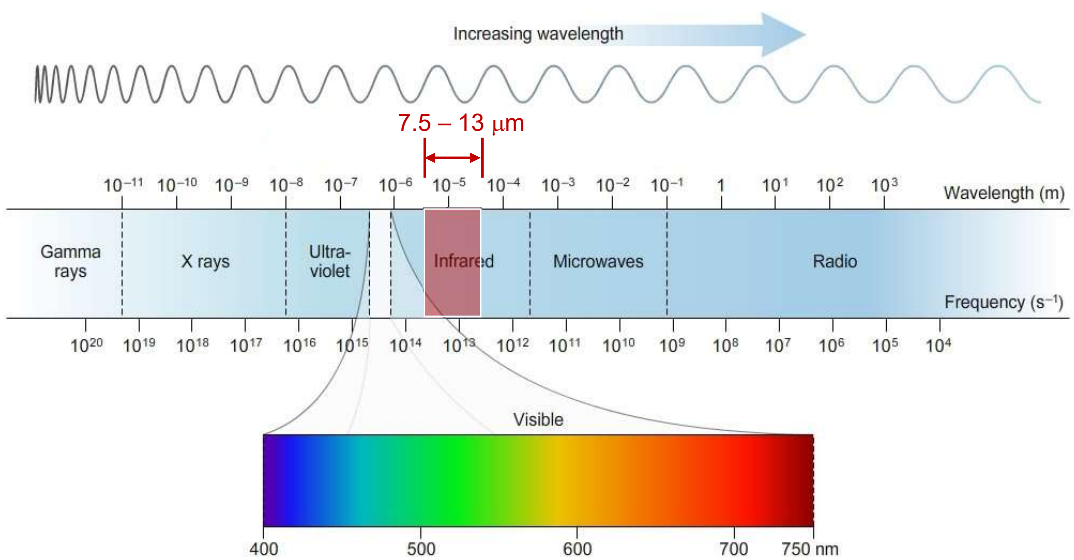
Electromagnetic Spectrum



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Thermal Imaging Camera

FLIR T420 Infrared Camera: 320 x 240, 30 Hz
(columns) (rows)

240

320

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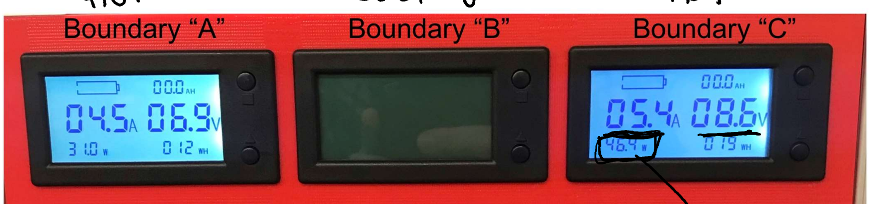
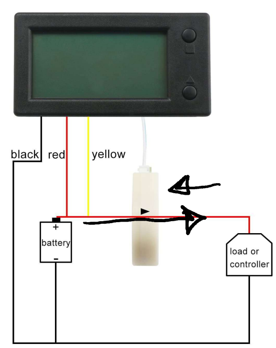
Digital Power Meter

HoT cooling Hot



*Boundary "B" was cooled in this case; while boundaries "A" and "C" were heated.

power (W)



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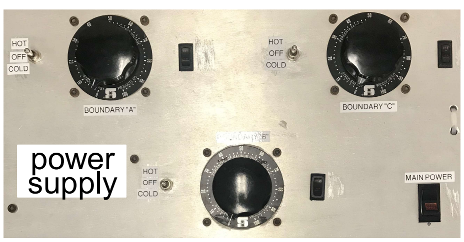
Experiments



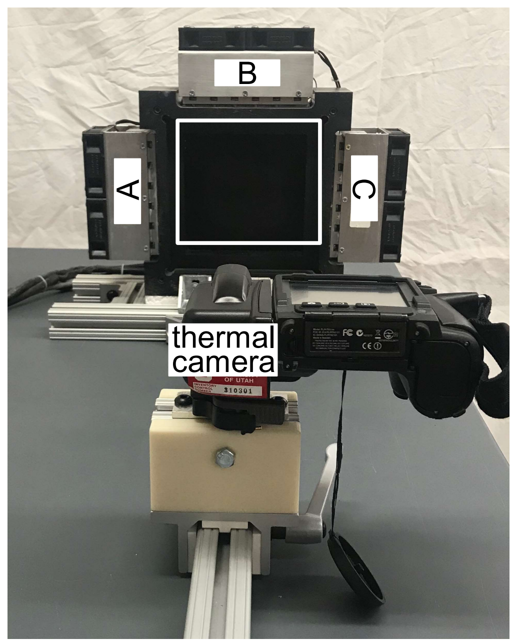


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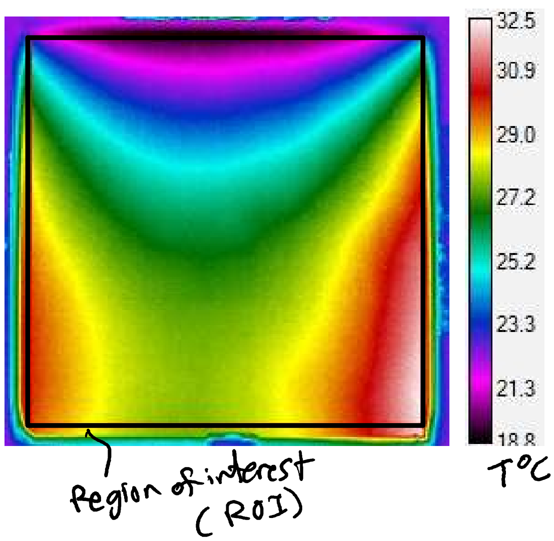
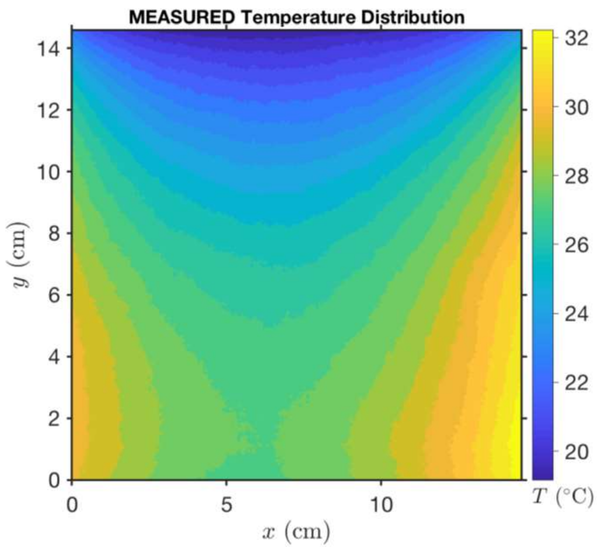
Figure 1a

Camera Image
(jpeg)



Matlab Contour



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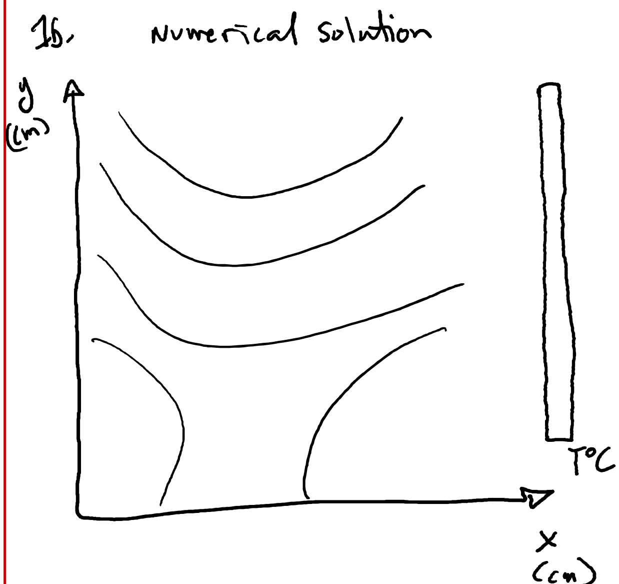
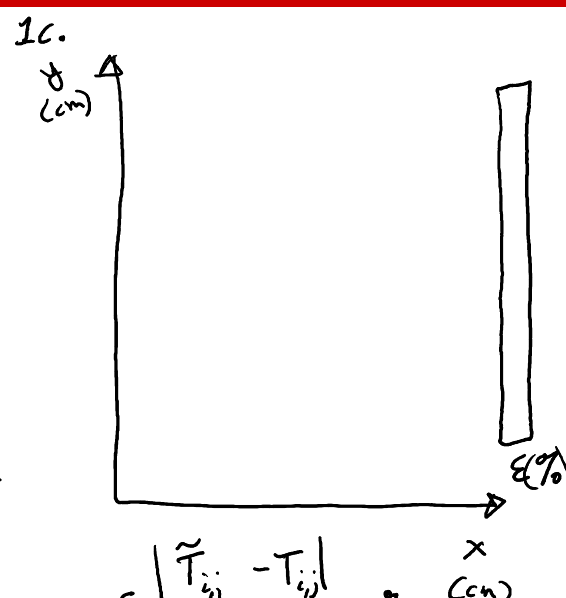
Figures1b & 1c





$$\varepsilon = \frac{\left|\tilde{T}_{i,j} - T_{i,j}\right|}{\tilde{T}_{i,j}} \cdot 100\%$$

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Figure 1d

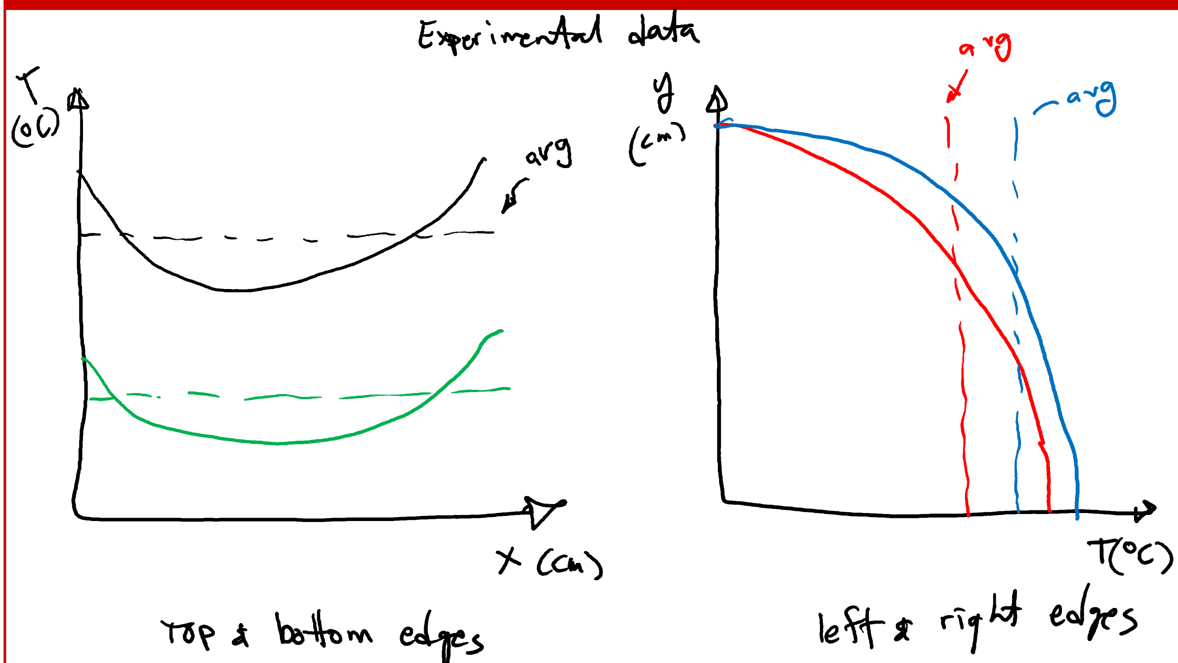

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Table 1e

Edge	q (W)	P_E (W)
Left		
Top		
Right		
Bottom		

from the data

measurements power meter (heated edges only)

2D Steady-State Conduction: Part II

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Quantities of Interest (Required Figures)

- i. JPEG image from IR camera
  ii. $T(x,y)$ color contour plot in Matlab from **measured data** one figure caption
- $T(x,y)$ color contour from **numerical model**
- $\epsilon(x,y)$ color contour
- i. T vs x (top and bottom edges) from **measured data**
  ii. T vs y (left and right edges) from **measured data** one figure caption
- q (W), P_E (W) table of values for each edge (heated only)

measured data

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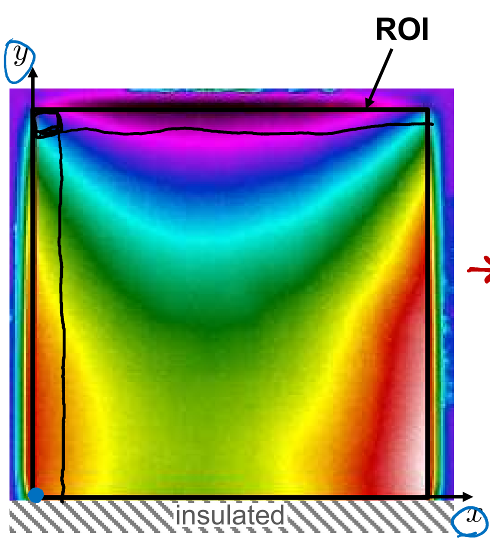
Post-Processing the Raw Data



left edge

MyData.csv

	A	B	C	D	E	F	G	H	I
1	2.27E+01	2.24E+01	2.22E+01	2.20E+01	2.20E+01	2.19E+01	2.18E+01	2.18E+01	2.16E+01
2	2.28E+01	2.25E+01	2.24E+01	2.24E+01	2.22E+01	2.22E+01	2.21E+01	2.18E+01	2.17E+01
3	2.33E+01	2.28E+01	2.28E+01	2.27E+01	2.24E+01	2.24E+01	2.22E+01	2.20E+01	2.21E+01
4	2.33E+01	2.29E+01	2.28E+01	2.27E+01	2.27E+01	2.25E+01	2.25E+01	2.22E+01	2.22E+01
5	2.35E+01	2.32E+01	2.29E+01	2.29E+01	2.29E+01	2.26E+01	2.25E+01	2.23E+01	2.23E+01
6	2.37E+01	2.35E+01	2.33E+01	2.32E+01	2.29E+01	2.28E+01	2.27E+01	2.26E+01	2.24E+01
7	2.37E+01	2.36E+01	2.36E+01	2.33E+01	2.32E+01	2.30E+01	2.29E+01	2.27E+01	2.24E+01
8	2.41E+01	2.37E+01	2.36E+01	2.33E+01	2.33E+01	2.30E+01	2.30E+01	2.28E+01	2.26E+01
9	2.41E+01	2.40E+01	2.38E+01	2.36E+01	2.35E+01	2.32E+01	2.30E+01	2.30E+01	2.29E+01
10	2.41E+01	2.41E+01	2.38E+01	2.37E+01	2.35E+01	2.34E+01	2.34E+01	2.32E+01	2.30E+01
11	2.43E+01	2.42E+01	2.40E+01	2.38E+01	2.36E+01	2.37E+01	2.33E+01	2.33E+01	2.33E+01
12	2.45E+01	2.43E+01	2.42E+01	2.39E+01	2.37E+01	2.38E+01	2.37E+01	2.36E+01	2.34E+01
13	2.45E+01	2.45E+01	2.42E+01	2.41E+01	2.41E+01	2.37E+01	2.37E+01	2.36E+01	2.35E+01
14	2.48E+01	2.45E+01	2.44E+01	2.42E+01	2.41E+01	2.41E+01	2.38E+01	2.38E+01	2.36E+01
15	2.48E+01	2.45E+01	2.45E+01	2.44E+01	2.43E+01	2.41E+01	2.41E+01	2.38E+01	2.36E+01
16	2.51E+01	2.47E+01	2.46E+01	2.44E+01	2.44E+01	2.42E+01	2.41E+01	2.40E+01	2.38E+01
17	2.51E+01	2.49E+01	2.48E+01	2.45E+01	2.44E+01	2.45E+01	2.41E+01	2.40E+01	2.40E+01
18	2.51E+01	2.50E+01	2.47E+01	2.47E+01	2.45E+01	2.45E+01	2.43E+01	2.41E+01	2.40E+01
19	2.51E+01	2.51E+01	2.49E+01	2.47E+01	2.47E+01	2.47E+01	2.43E+01	2.44E+01	2.41E+01
20	2.55E+01	2.52E+01	2.49E+01	2.48E+01	2.47E+01	2.47E+01	2.46E+01	2.43E+01	2.43E+01
21	2.56E+01	2.52E+01	2.52E+01	2.49E+01	2.48E+01	2.48E+01	2.47E+01	2.45E+01	2.43E+01
22	2.57E+01	2.53E+01	2.52E+01	2.52E+01	2.48E+01	2.48E+01	2.48E+01	2.46E+01	2.45E+01
23	2.56E+01	2.55E+01	2.52E+01	2.52E+01	2.51E+01	2.49E+01	2.47E+01	2.48E+01	2.47E+01
24	2.57E+01	2.56E+01	2.53E+01	2.52E+01	2.52E+01	2.49E+01	2.49E+01	2.47E+01	2.47E+01
25	2.57E+01	2.57E+01	2.56E+01	2.53E+01	2.52E+01	2.51E+01	2.50E+01	2.49E+01	2.49E+01

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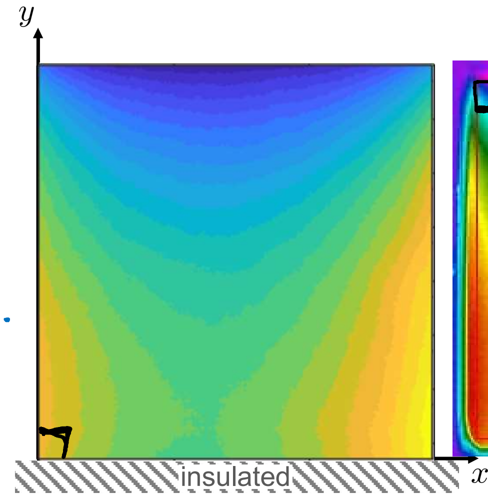
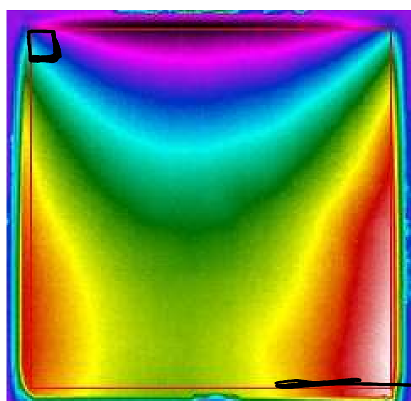
Reading the Data File into Matlab





MyData.csv

GZ	HA	HB	HC	HD
33E+01	2.34E+01	2.36E+01	2.36E+01	2.36E+01
34E+01	2.37E+01	2.40E+01	2.39E+01	2.39E+01
38E+01	2.38E+01	2.41E+01	2.41E+01	2.43E+01
39E+01	2.41E+01	2.43E+01	2.45E+01	2.46E+01
41E+01	2.43E+01	2.44E+01	2.48E+01	2.49E+01
42E+01	2.45E+01	2.46E+01	2.48E+01	2.49E+01
22E+01	3.24E+01	3.26E+01	3.26E+01	3.26E+01
23E+01	3.23E+01	3.24E+01	3.26E+01	3.26E+01
23E+01	3.23E+01	3.25E+01	3.25E+01	3.26E+01
23E+01	3.24E+01	3.23E+01	3.24E+01	3.26E+01
23E+01	3.24E+01	3.25E+01	3.25E+01	3.26E+01
23E+01	3.23E+01	3.25E+01	3.26E+01	3.26E+01
22E+01	3.23E+01	3.24E+01	3.25E+01	3.24E+01
22E+01	3.23E+01	3.23E+01	3.23E+01	3.25E+01

```
'MyData.csv');
a);
[M,N]=size(Tdata);
x=linspace(0,L,N);
y=linspace(0,H,M)';
contourf(x,y,Tdata,20); colorbar;
```

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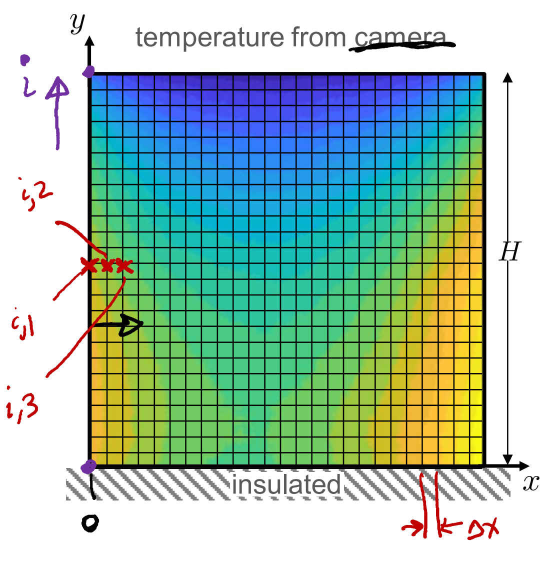
Calculating Heat Flux across Edges



left edge: $q''_L(y) = -k \left.\frac{\partial T}{\partial x}\right|_{x=0}$

thermal conductivity (aluminum) T-6061

$$q_L = t \int_{y=0}^{H} q''_L(y)\, dy$$

thickness = 1/8"

forward finite difference (at left edge):

$$q''_{L_i} = -k \left[\frac{-T_{i,3} + 4\,T_{i,2} - 3\,T_{i,1}}{2\,\Delta x} \right]$$

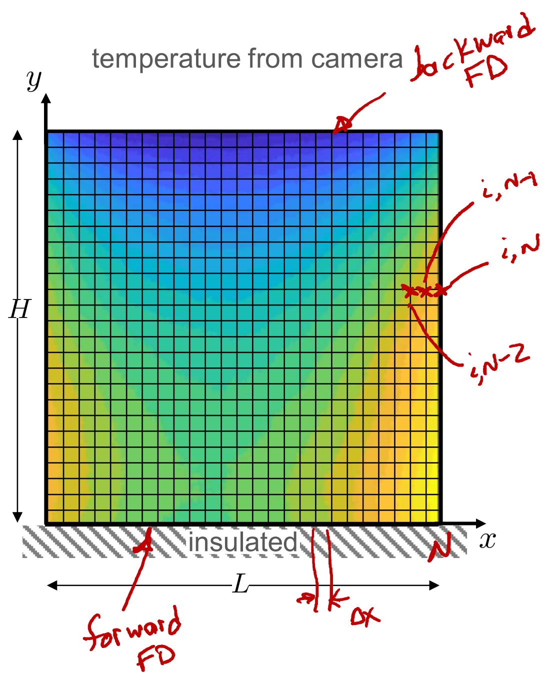
vector

trapezoidal rule (at left edge):

$$q_L = t\,\left(\frac{\Delta x}{2}\right) \left[q''_{L_1} + 2 \sum_{i=2}^{M-1} q''_{L_i} + q''_{L_M} \right]$$

in Matlab: `qL=t*trapz(y,qL_flux);`

Calculating Heat Flux across Edges



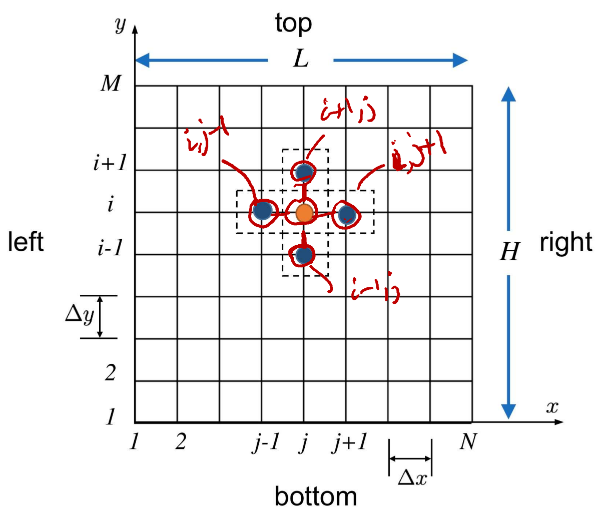
right edge: $$q''_R(y) = -k \frac{\partial T}{\partial x}\bigg|_{x=L}$$

$$q_R = t \int_{y=0}^{H} q''_R(y)\, dy$$

backward finite difference (at **right** edge):

$$q''_{R_i} = -k \left[\frac{3\,T_{i,N} - 4\,T_{i,N-1} + T_{i,N-2}}{2\,\Delta x} \right]$$

Discretization: Finite Difference Grid



$$\Delta x = \Delta y = \Delta$$

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0$$

2nd order central finite difference formulas:

$$\frac{\partial^2 T}{\partial x^2} \approx \frac{T_{i,j+1} - 2\,T_{i,j} + T_{i,j-1}}{\Delta^2}$$

$$\frac{\partial^2 T}{\partial y^2} \approx \frac{T_{i+1,j} - 2\,T_{i,j} + T_{i-1,j}}{\Delta^2}$$

Substitute into heat conduction equation:

$$T_{i,j+1} + T_{i+1,j} - 4\,T_{i,j} + T_{i,j-1} + T_{i-1,} = 0$$

Gauss-Elimination
- Direct Method (round-off error only)
- No initial guess needed

$$T_{i,j} = \tfrac{1}{4}\left(T_{i,j+1} + T_{i+1,j} + T_{i,j-1} + T_{i-1,j}\right)$$

Gauss-Seidel
- Indirect Method (convergence)
- Initial guess required

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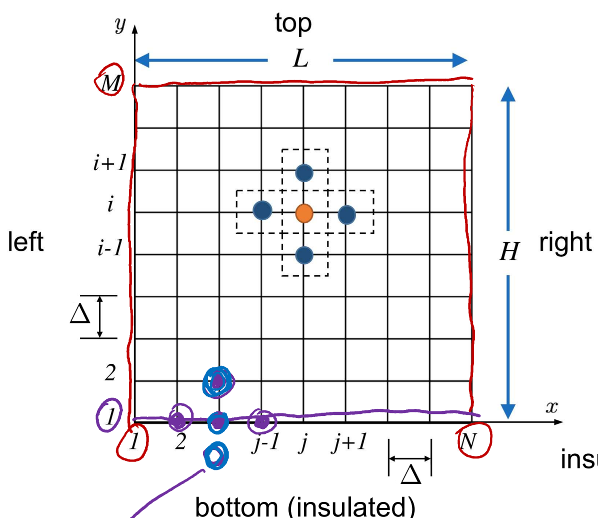
Boundary Conditions

interior: $T_{i,j} = \frac{1}{4}\left(T_{i,j+1} + T_{i+1,j} + T_{i,j-1} + T_{i-1,j}\right)$



"ghost node"

$$\boxed{\left.\frac{\partial T}{\partial y}\right|_{y=0} = 0}$$

measured

left $(j=1)$: $T_{i,1} = \tilde{T}_{i,1}$

T=zeros(M,N); T(:,1)=Tdata(:,1);

top $(i=M)$: $T_{M,j} = \tilde{T}_{M,j}$

right $(j=N)$: $T_{i,N} = \tilde{T}_{i,N}$

bottom boundary

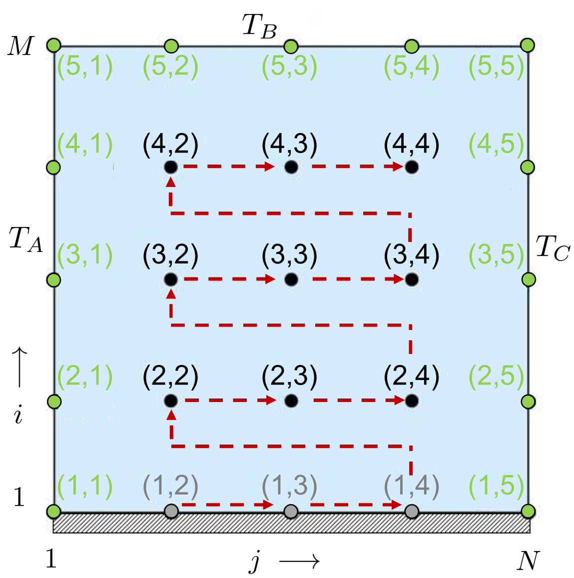
ghost node

$(i=1)$: $T_{1,j} = \frac{1}{4}\left(T_{1,j+1} + T_{2,j} + T_{1,j-1} + T_{0,j}\right)$

insulated (central difference): $\frac{T_{2,j} - T_{0,j}}{\cancel{2\Delta}} = 0$

$(i=1)$: $T_{1,j} = \frac{1}{4}\left(T_{1,j+1} + 2\,T_{2,j} + T_{1,j-1}\right)$

Gauss-Seidel Method



1. Set initial guess of $T_{i,j}$
2. Set values along left, top, and right boundaries: $T_{i,1},\ T_{5,j},\ T_{i,5}$
3. Loop until converged $\quad \epsilon \leq \underbrace{1\times 10^{-5}}_{\text{tolerance}}$

$$T_{i,j}^{\text{old}} = T_{i,j}$$

rows: for $i = 1$ to $M-1$

columns: for $j = 2$ to $N-1$

if $i = 1$:

$$T_{1,j} = \tfrac{1}{4}\left(T_{1,j-1} + 2\,T_{2,j} + T_{1,j+1}\right)$$

else:

$$T_{i,j} = \tfrac{1}{4}\left(T_{i,j-1} + T_{i,j+1} + T_{i-1,j} + T_{i+1,j}\right)$$

relaxation:

$$T_{i,j} = \lambda\, T_{i,j} + (1-\lambda)\, T_{i,j}^{\text{old}}$$

error: $\epsilon = \max\left[\dfrac{\left|T_{i,j} - T_{i,j}^{\text{old}}\right|}{T_{i,j}}\right]$

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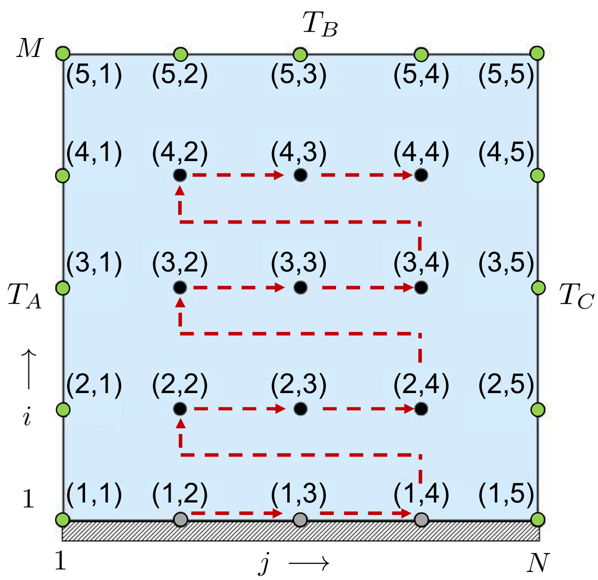
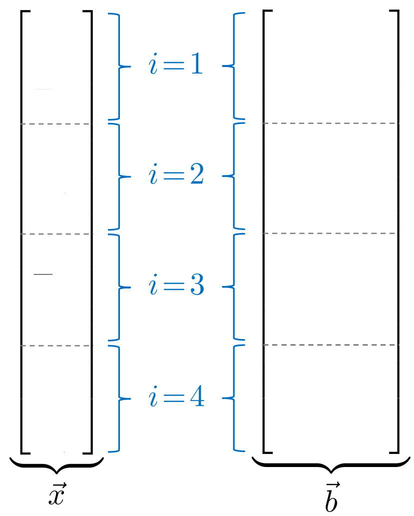
Gauss-Elimination Method

for $i=1$: $-T_{1,j+1} - 2\,T_{2,j} + 4\,T_{1,j} - T_{1,j-1} = 0$

for $2 \leq i \leq M-1$: $-T_{i,j+1} - T_{i+1,j} + 4\,T_{i,j} - T_{i,j-1} - T_{i-1,} = 0$

$\Longrightarrow \; [A]\;\vec{x} \;=\; \vec{b}$





System of Equations (3x3)

$$
\underbrace{\begin{bmatrix}
4 & -1 & 0 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 4 & -1 & 0 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 4 & 0 & 0 & -2 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 4 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & -1 & 4 & -1 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & -1 & 4 & 0 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & 4 & -1 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & -1 & 4 & -1 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & 4 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 4 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & 4 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & 4
\end{bmatrix}}_{[A]}
\underbrace{\begin{bmatrix}
T_{1,2} \\ T_{1,3} \\ T_{1,4} \\ T_{2,2} \\ T_{2,3} \\ T_{2,4} \\ T_{3,2} \\ T_{3,3} \\ T_{3,4} \\ T_{4,2} \\ T_{4,3} \\ T_{4,4}
\end{bmatrix}}_{\vec{x}}
=
\underbrace{\begin{bmatrix}
T_A \\ 0 \\ T_C \\ T_A \\ 0 \\ T_C \\ T_A \\ 0 \\ T_C \\ T_A + T_B \\ T_B \\ T_B + T_C
\end{bmatrix}}_{\vec{b}}
$$

diagonally dominant: $|a_{ii}| \geq \sum_{j \neq i} |a_{i,j}| \quad \text{for all } i$

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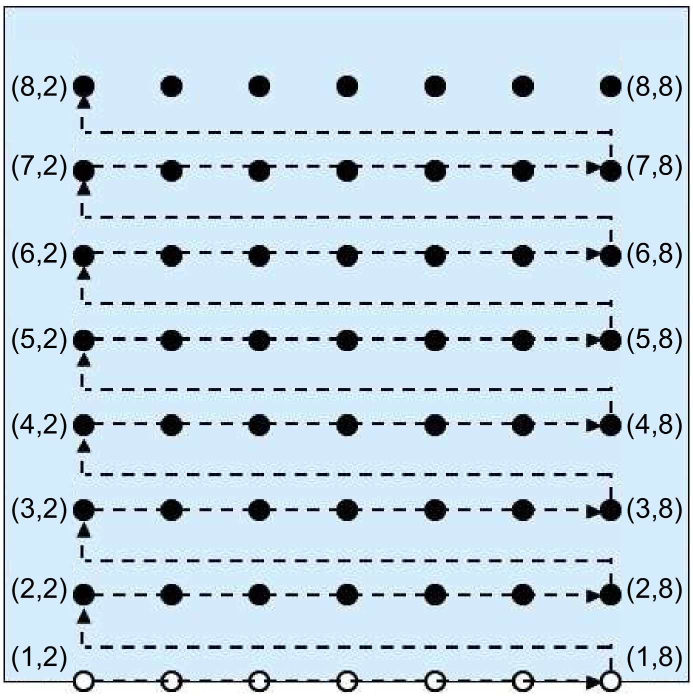
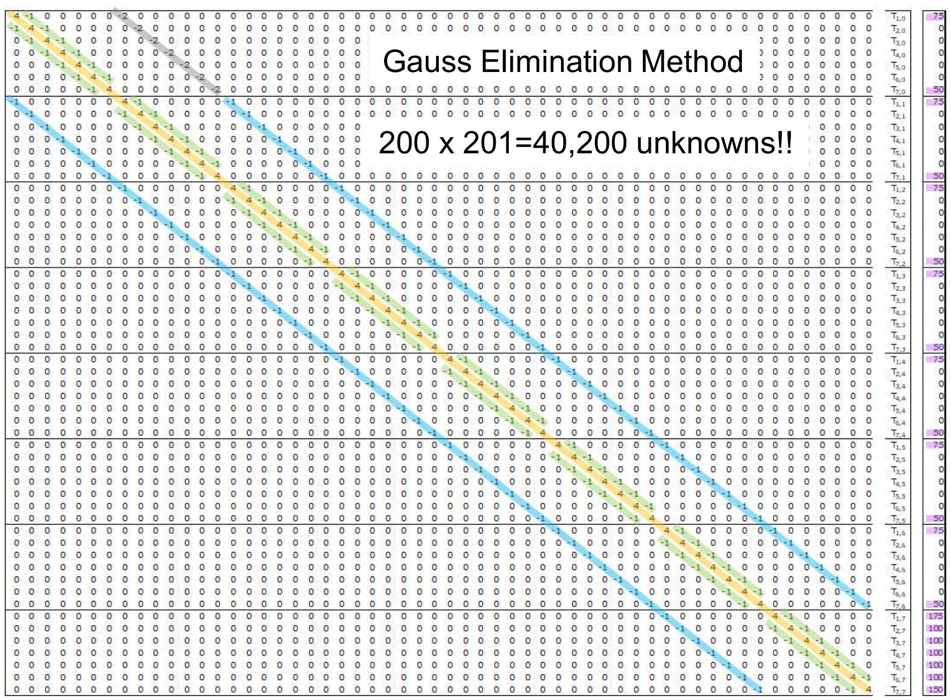
Larger System (7 x 7)

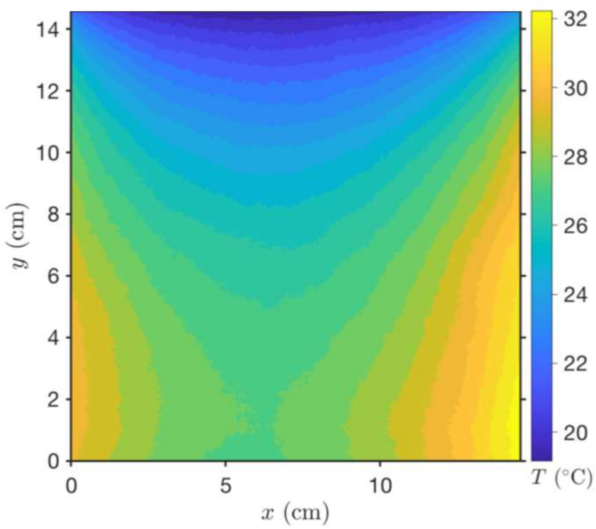


Linear System (7 x 7)

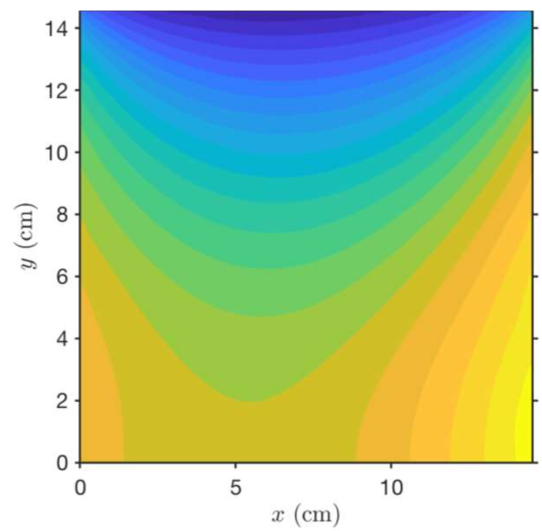


Comparison of Results

Experimental data



Numerical solution



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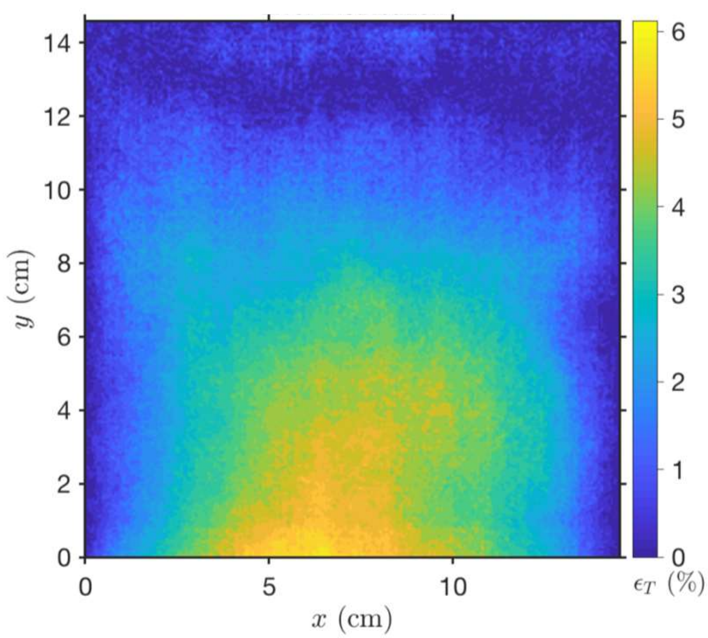
Percent Difference

$$\epsilon_{i,j} = \frac{|\tilde{T}_{i,j} - T_{i,j}|}{\tilde{T}_{i,j}} \cdot 100\%$$

measured

numerical



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Questions??

Thank you for your attention!

Let me or the TAs know if you have questions

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