ME EN 6200 | Homework 3 | Ryan Malley | Page 1

1. a)

a: $\frac{1}{s} \rightarrow \boxed{\frac{5}{s+5}} \rightarrow C(s)$

$$C(s) = \frac{1}{s} \cdot \frac{5}{s+5} = \frac{5}{s^2+5s} = \frac{5}{s(s+5)} = \frac{K_1}{s} + \frac{K_2}{s+5}$$

$$K_1 = \left.\frac{5}{s+5}\right|_{s=0} = 1 \qquad K_2 = \left.\frac{5}{s}\right|_{s=-5} = -1$$

$$\mathcal{L}^{-1}\{C(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{s} - \frac{1}{s+5}\right\} \Rightarrow \boxed{c(t) = 1 - e^{-5t}}$$

b: $\frac{1}{s} \rightarrow \boxed{\frac{20}{s+20}} \rightarrow C(s)$

$$C(s) = \frac{1}{s} \cdot \frac{20}{s+20} = \frac{20}{s(s+20)} = \frac{K_1}{s} + \frac{K_2}{s+5}$$

$$K_1 = \left.\frac{20}{s+20}\right|_{s=0} = 1 \qquad K_2 = \left.\frac{20}{s}\right|_{s=-20} = -1$$

$$\mathcal{L}^{-1}\{C(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{s} - \frac{1}{s+20}\right\} \Rightarrow \boxed{c(t) = 1 - e^{-20t}}$$

b) a: $\boxed{\tau = 1/5 \text{ sec}}$

$K=1 \quad M=1$

$c(t_s) = .98 = 1 - e^{-5t_s}$ $\boxed{t_s = 0.78 \text{ sec}}$

$c(t_r) = .9 = 1 - e^{-5t_r}$ $\boxed{t_r = 0.46 \text{ sec}}$

b: $\boxed{\tau = 1/20 \text{ sec}}$

$c(t_s) = .98 = 1 - e^{-20t_s}$ $\boxed{t_s = 0.196 \text{ sec}}$

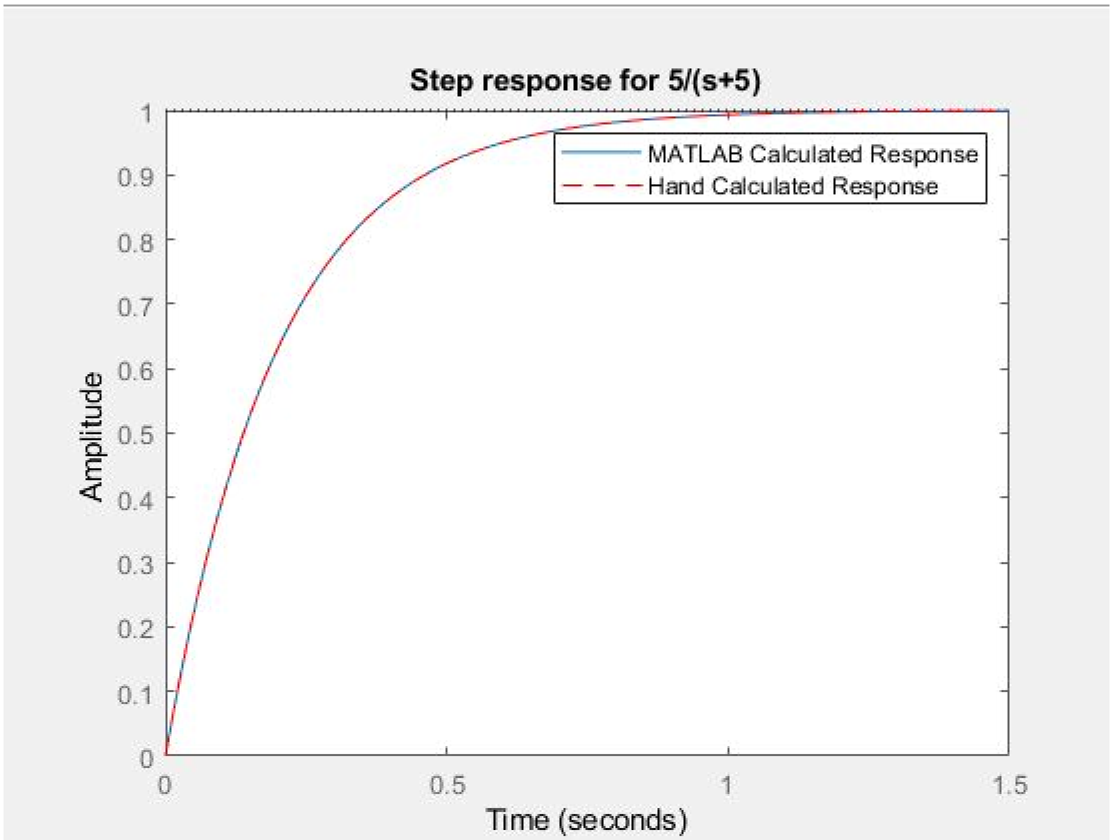
$c(t_r) = .9 = 1 - e^{-20t_r}$ $\boxed{t_r = 0.115 \text{ sec}}$

ME EN 6200
Homework 3
Ryan Dalby

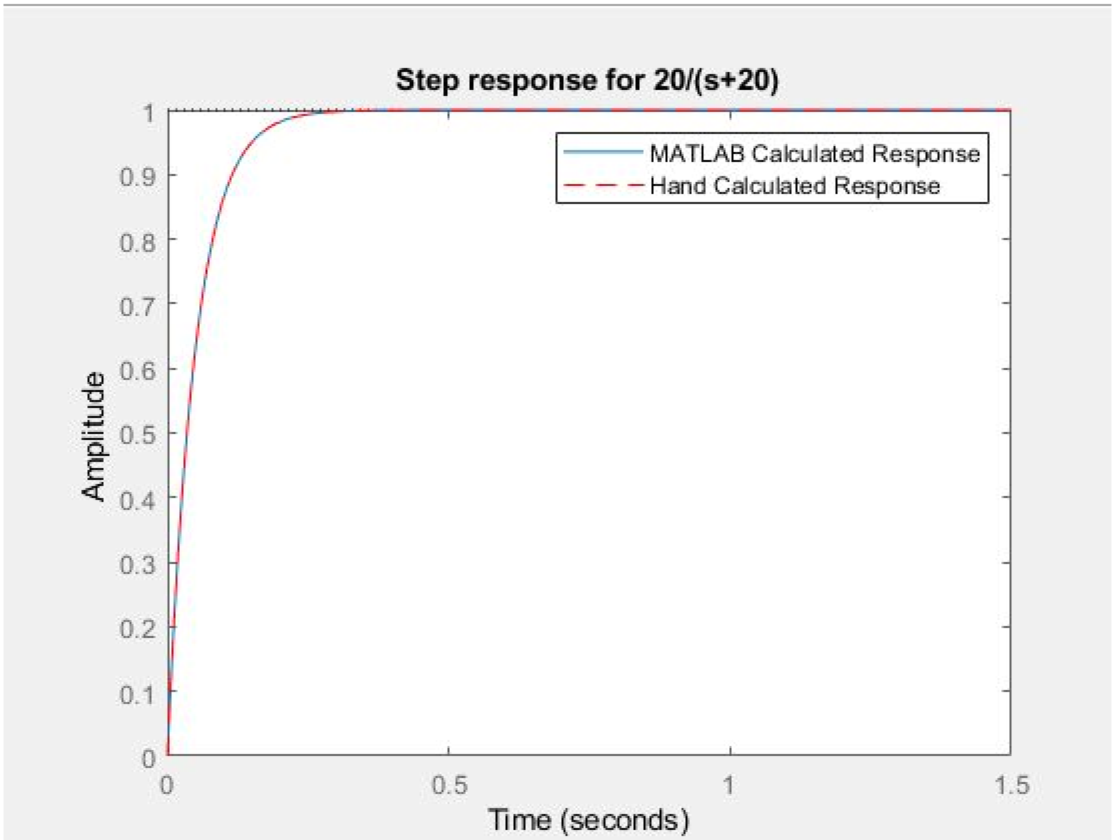
Problem 1

(c) Use Matlab's 'step' function to obtain plots of the time responses to compare your results with the equations you obtained for part (a). Submit your Matlab code for this problem (m-file or Simulink model), along with Matlab plots and label the axes!

a:



b:



Comparison:
Both transfer functions for a and b exhibit the expected unit step response. This is evidenced by the same time to rise to .9 and same time to settle to .98 for both a and b. Plotting the time solutions on top of the step response the plots are the exact same.

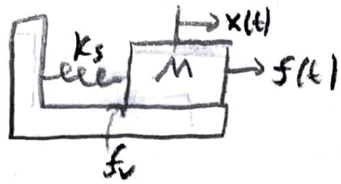
Code:

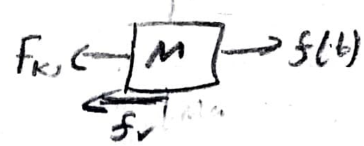
```
%% 1c
t_vals = 0:0.001:1.5;
% a
Ga = tf(5,[1,5]);
figure;
step(Ga,1.5);
title('Step response for 5/(s+5)');
hold on;
c = 1-exp(-5*t_vals);
plot(t_vals, c, 'r--');
legend('MATLAB Calculated Response', 'Hand Calculated Response');

% b
```

```
Gb = tf(20,[1,20]);
figure;
step(Gb,1.5);
title('Step response for 20/(s+20)');
hold on;
c = 1-exp(-20*t_vals);
plot(t_vals, c, 'r--');
legend('MATLAB Calculated Response', 'Hand Calculated Response');
```

2. a) $M = 1\,kg$
$K_s = 5\,[N/m]$
$f_v = 1\,[N\cdot s/m]$
$f(t) = u(t)\,[N]$





$$\Sigma F = m\ddot{x}(t) = f(t) - F_{ks} - f_v$$

$$m\ddot{x}(t) = \overset{\text{unit}}{u(t)} - k_s x(t) - f_v\dot{x}(t)$$

$$\mathcal{L}\{m\ddot{x}(t) + f_v\dot{x}(t) + k_s x(t)\} = \mathcal{L}\{\underbrace{u(t)}_{f(t)}\}$$

Assume zero initial conditions

$$mX(s)s^2 + f_v X(s)s + k_s X(s) = U(s) \leftarrow F(s)$$

$$X(s)[ms^2 + f_v s + k_s] = U(s) \qquad \frac{X(s)}{U(s)} = \frac{1}{ms^2 + f_v s + k_s}$$

$$U(s)\cdot\frac{X(s)}{U(s)} = \frac{1}{s}\left(\frac{1}{s^2 + s + 5}\right) = \frac{1}{s\left((s+1/2)^2 + 5 - 1/4\right)} = \frac{1}{s\left((s+1/2)^2 + 19/4\right)}$$

$F(s) = 1/s$; $\frac{\sqrt{19}}{2} = w$; $a = 1/2$

$$\hookrightarrow \frac{A(s+a) + Bw}{(s+a)^2 + w^2}$$

$$\frac{1}{s[(s+1/2)^2 + 19/4]} = \frac{K_1}{s} + \frac{K_2(s+1/2) + K_3\left(\frac{\sqrt{19}}{2}\right)}{(s+1/2)^2 + 19/4}$$

$A = K_2$
$B = K_3$

$$K_1 = \left.\frac{1}{(s+1/2)^2 + 19/4}\right|_{s\to 0} = 1/5 \qquad K_1 = 1/5$$

$1/4 + 19/4$

$$1 = K_1[(s+1/2)^2 + 19/4] + K_2(s+1/2)s + K_3\left(\frac{\sqrt{19}}{2}\right)s$$

$\hookrightarrow K_1 = 1/5$ Solve for K_2 and K_3

Say $s = -1/2$ $\quad 19/4\,(1/5) - K_3\frac{\sqrt{19}}{4} = 1 \quad -K_3\frac{\sqrt{19}}{4} = 1/20$

$K_3 = -1/5\sqrt{19}$

$$1 = 1/5[(s+1/2)^2 + 19/4] + \overset{-1/5}{K_2}(s+1/2)s + \frac{1}{5\sqrt{19}}\left(\frac{\sqrt{19}}{2}\right)s$$

$23/20 \quad -3/20 \quad +1/20 \quad = 20/20 = 1$ $\quad \frac{1+1}{20}$ $\quad K_2 = -1/5$

$$\frac{A(s+a) + Bw}{(s+a)^2 + w^2} = \mathcal{L}\{Ae^{-at}\cos wt + Be^{-at}\sin wt\}$$

$B = -1/5\sqrt{19}$ $\quad w = \sqrt{19}/2$
$a = 1/2$ $\quad A = -1/5$

$$\frac{K_1}{s} \quad K_1 = 1/5 \to \frac{1}{5s} = \frac{1}{5}\mathcal{L}\{1\}$$

$$\mathcal{L}^{-1}\{X(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{s(s^2+s+5)}\right\} = \frac{1}{5} - \frac{1}{5}e^{-1/2t}\cos\left(\frac{\sqrt{19}}{2}t\right) - \frac{1}{5\sqrt{19}}e^{-1/2t}\sin\left(\frac{\sqrt{19}}{2}t\right)$$

when $f(t)$ = unit step

$$\boxed{x(t) = \frac{1}{5} - \frac{1}{5}e^{-1/2t}\cos\left(\frac{\sqrt{19}}{2}t\right) - \frac{1}{5\sqrt{19}}e^{-1/2t}\sin\left(\frac{\sqrt{19}}{2}t\right)}$$

2 b) $\dfrac{X(s)}{f(s)} = \dfrac{1}{s^2 + s + 5} = k \dfrac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$

$2\zeta(\sqrt{5}) = 1$ $\qquad$ $\omega_n = \sqrt{5}$ $\qquad$ $k = 1/5$

$\zeta = \dfrac{1}{2\sqrt{5}}$

$\%OS = e^{-\left(\frac{\zeta\pi}{\sqrt{1-\zeta^2}}\right)} \times 100$

$\%OS = e^{-\left(\frac{1}{2\sqrt{5}}\pi / \sqrt{1-1/20}\right)} \times 100 = \boxed{48.6\%}$

$t_p \approx \dfrac{\pi}{\omega_n\sqrt{1-\zeta^2}} = \dfrac{\pi}{\sqrt{5}\sqrt{19/20}} = \boxed{1.44 \text{ sec}}$

$t_s \approx \dfrac{4}{\zeta\omega_n} = \dfrac{4}{\frac{1}{2\sqrt{5}} \cdot \sqrt{5}} = \boxed{8 \text{ sec}}$

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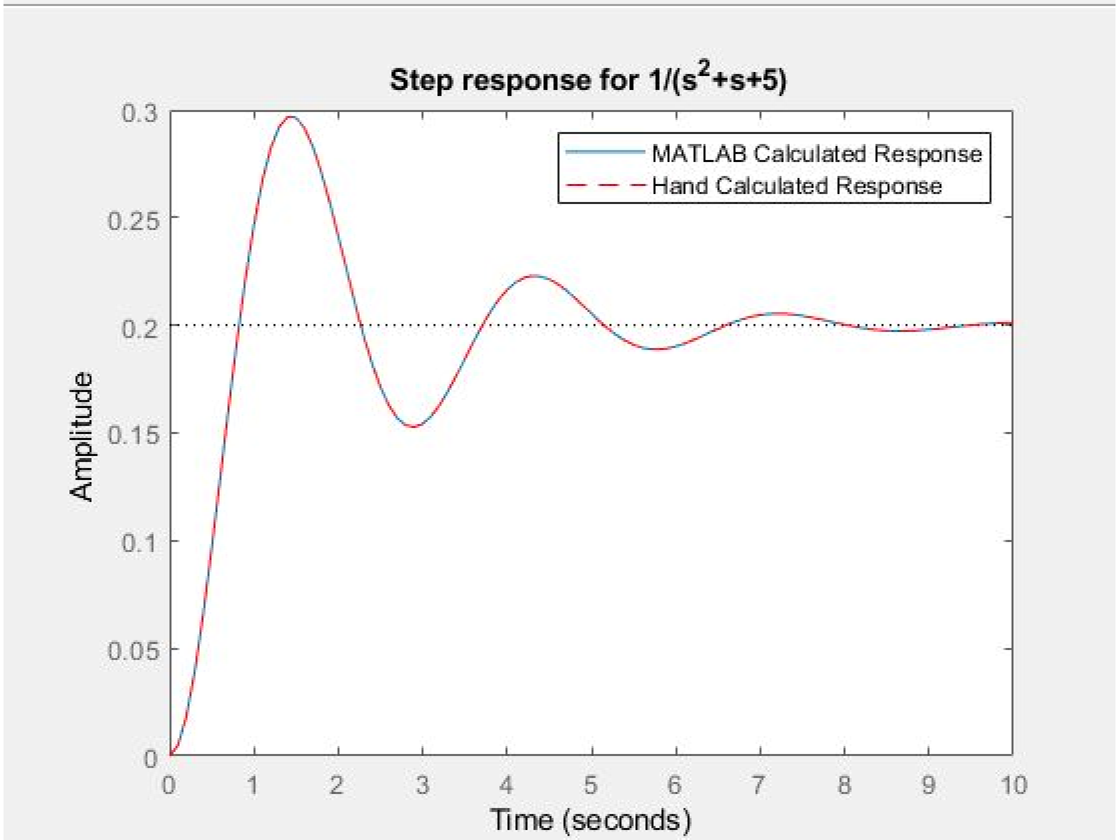
3. T(s)

Problem 2

(c) Use Matlab's 'step' function to generate a plot to compare with your analytical solutions in Part (a)
and (b). Note any differences and explain. Include your Matlab code (m-file and/or Simulink model).




```
%% 2c
t_vals = 0:0.1:10;

H = tf(1,[1,1,5]);
figure;
step(H,10);
title('Step response for 1/(s^2+s+5)');
hold on;
x = 1/5 - (1/5).*exp(-.5.*t_vals).*cos(sqrt(19).*t_vals/2) -
(1/(5.*sqrt(19))).*exp(-.5.*t_vals).*sin(sqrt(19).*t_vals/2);
plot(t_vals, x, 'r--');
legend('MATLAB Calculated Response', 'Hand Calculated Response');
```

The numerical MATLAB solution and my closed form solution for x(t) (given a step response) match up exactly as evidenced by the plots and the matching %OS, peak time, and settling time.

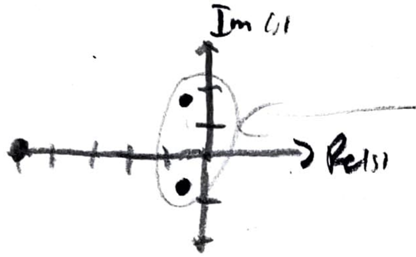
(d) Explain how damp the system is, i.e., overdamp, underdamped, etc., and justify your answer.

The system is underdamped. This is shown by the oscillatory behavior and approximately 50% overshoot. The system takes 8 seconds to settle from the oscillatory behavior that is initially shown by the system as soon as the step input begins. This long settling time where the response oscillates about a settling value also indicates that the system is underdamped.

3. $T(s) = \dfrac{14.145}{(s^2+0.842s+2.829)(s+5)}$

Poles: $s = -5$ $\qquad s = \dfrac{-0.842 \pm \sqrt{(0.842)^2 - 4(2.829)}}{2}$

$s = \dfrac{-0.842 \pm 3.2577i}{2} = -.4227 \pm 1.628i$



These poles are much closer to the Im(s) axis which implies they are closer to the boundary of the open RHP where we no longer have a decaying system thus they have a much bigger impact.

From text book if a 3rd real pole is five times farther to the left of the dominant poles then it can be neglected Pg.188

Thus $-.421 <<< -5$ (more than 5 times distance)

Thus we approximate $T(s)$ as: $T(s) = \dfrac{14.145}{s^2+0.842s+2.829}$

where $T(s) = \dfrac{K\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}$ $\qquad \omega_n = 1.682 \qquad K = 5$

$2\zeta(1.682) = 0.842 \qquad \zeta = 0.25$

$\%OS = e^{-\left(\frac{0.25\pi}{\sqrt{1-0.25^2}}\right)} \times 100 = \boxed{44.43\%}$

$t_s \approx \dfrac{4}{\zeta\omega_n} = \dfrac{4}{(.25)(1.682)} = \boxed{9.51 \text{ sec}}$

$t_r \approx \dfrac{1.8}{\omega_n} = \dfrac{1.8}{1.6819} = \boxed{1.07 \text{ sec}}$

$t_p = \dfrac{\pi}{\omega_n\sqrt{1-\zeta^2}} = \dfrac{\pi}{(1.682)\sqrt{1-0.25^2}} = \boxed{1.93 \text{ sec}}$

4. a) $t_s \approx 0.1 sec$ $t_r = 0.075$ sec

$t_s \approx 4\tau = 0.1$ $\tau \approx 0.025$ seconds $a \approx 40$

Settling value : $M\bar{K} = 2$ (1 unitstep) $\bar{K} = 2$

$$G(s) = \bar{K}\frac{1/\tau}{s + 1/\tau} = \boxed{\frac{80}{s+40}}$$

b) settling value ≈ 11 Peak value ≈ 13.5 $t_p \approx 1$ sec

%OS : $2.5/11 \cdot 100 = 22.73\%$

$$\zeta \approx \frac{-\ln(0.2273)}{\sqrt{\pi^2 + \ln^2(0.2273)}} \approx 0.43$$

$$t_p = \frac{\pi}{\omega_n\sqrt{1-(0.43)^2}} = 1$$

$\omega_n = 3.47$

$$G(s) = \bar{K}\frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

$\bar{K} = 11$ $M = 1$ unitstep

$$G(s) = 11\frac{(3.47)^2}{s^2 + 2(0.43)(3.47)s + (3.47)^2} = \boxed{\frac{132.71}{s^2 + 2.98s + 12.06}}$$

c) settling value = 1.0 Peak value : 1.4 $t_p \approx 4$ sec

%OS = $0.4/1.0 \cdot 100 = 40\%$

$$\zeta \approx \frac{-\ln(0.4)}{\sqrt{\pi^2 + \ln^2(0.4)}} \approx 0.28$$

$$t_p = \frac{\pi}{\omega_n\sqrt{1-(0.28)^2}} = 4 \text{ sec}$$

$\omega_n = 0.811$

$$G(s) = \bar{K}\frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

$\bar{K} = 1.0$ $M = 1$ unitstep

$$G(s) = \frac{(0.811)^2}{s^2 + 2(0.28)(0.811)s + (0.811)^2} = \boxed{\frac{0.67}{s^2 + 0.46s + 0.67}}$$