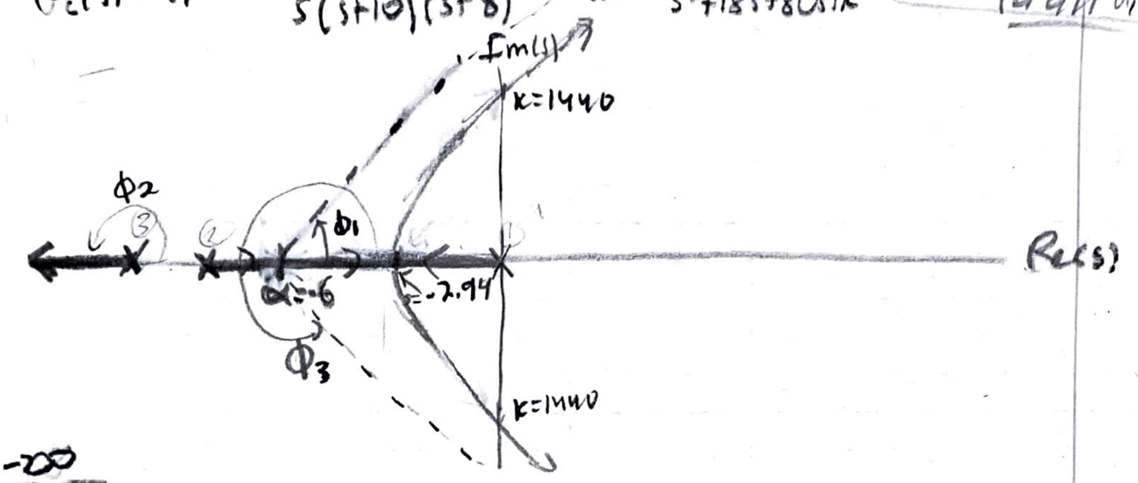
1. a) $G_c(s)G(s) = \dfrac{K}{s(s+10)(s+8)}$ CLTF: $\dfrac{K}{s^3+18s^2+80s+K}$ $0<K<\infty$ for all a, b, c, d



As $K\to\infty$

$\hat{G}(s) = \dfrac{1}{s(s+10)(s+8)} = \dfrac{1}{s^3+18s^2+80s}$ Zeros: None Poles: $s=0$ $s=-10$ $s=-8$

No zeros so $m=0$: $n-m=3$ 3 asymptotes

$\phi_1 = \dfrac{180^\circ + \cancel{360^\circ(0)}}{3} = \boxed{60^\circ}$

$\phi_2 = \dfrac{180^\circ+360^\circ(1)}{3} = \boxed{180^\circ}$

$\phi_3 = \dfrac{180^\circ+360^\circ(2)}{3} = \boxed{300^\circ}$

$\boxed{\alpha = \dfrac{-18}{3} = -6}$

Angle of Departure (Trivial)
① $\phi_{dep} = 0-180-0 = -180^\circ$
② $\phi_{dep} = -180-180-0 = 0^\circ = -360^\circ$
③ $\phi_{dep} = -360-180-0 = -540 = 180^\circ$

jw Crossover

$s^3+18s^2+80s+K$

s^3	1	80	0
s^2	18	K	0
s^1	$-K/18+80$	0	0
s^0	K	0	0

$-K/18+80=0$

jw Crossover at $K=1440$

Breakaway / Breakin Points

$G(s) = \dfrac{1}{s(s+10)(s+8)}$ (s^3+18s^2+80s) $= \dfrac{b(s)}{a(s)}$ $\dfrac{d}{ds}\left(-\dfrac{1}{G(s)}\right)=0 = \dfrac{d}{ds}\left(-\dfrac{s(s+10)(s+8)}{1}\right)$

$b=1$ $a=s(s+10)(s+8)$

$\dfrac{db}{ds}=0$ $\dfrac{da}{ds} = 3s^2+36s+80$

$-\dfrac{1}{b^2}\left(b\dfrac{da}{ds} - \cancel{a\dfrac{db}{ds}}\right)=0 = \dfrac{-\frac{da}{ds}}{b} = \dfrac{-3s^2-36s-80}{1} = 0$

roots$(-3s^2-36s-80) =$

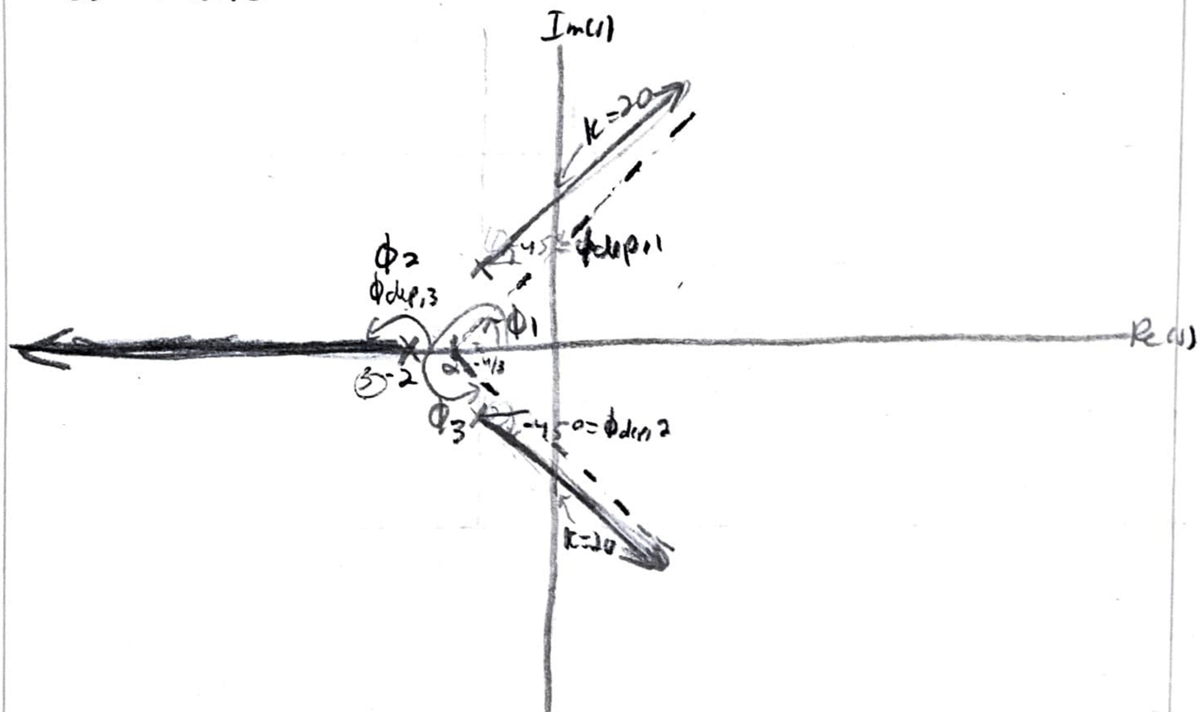
$\boxed{s=-2.9449}$ ← Breakaway point   ~~-9.0551~~ not on root locus

1. b) $G_c(s)G(s) = \dfrac{K}{(s^2+2s+2)(s+2)}$ CLTF: $\dfrac{K}{s^3+4s^2+6s+4+K}$

$\hat{G}(s) = \dfrac{1}{(s^2+2s+2)(s+2)} = \dfrac{1}{s^3+2s^2+2s+2s^2+4s+4} = \dfrac{1}{s^3+4s^2+6s+4}$

Poles: $s = \dfrac{-2 \pm \sqrt{4-8}}{2} = -1 \pm j$ $s=-1+j$ $s=-1-j$ $s=-2$

Zeroes: None



As $k \to \infty$

No zeroes so $m=0$: $n-m=3$ 3 asymptotes

$$\boxed{\begin{aligned}\phi_1 &= \frac{180°}{3} = 60° \\ \phi_2 &= \frac{540}{3} = 180° \\ \phi_3 &= \frac{900}{3} = 300°\end{aligned}}$$

$$\boxed{\alpha = \frac{(-1 + -1 + -2)}{3} = -4/3}$$

Departure Angles for poles

$$\boxed{\begin{aligned}&① \ \phi_{dep,1} = -(90°+45°)+180° = 45° \\ &② \ \phi_{dep,2} = -(270°+315°)+180° = -405 = -45° \\ &③ \ \phi_{dep,3} = -(45+315)+180 = -180°\end{aligned}}$$

$j\omega$ crossing:

s^3	1	6	0
s^2	4	4+k	0
s^1	5-k/4	0	0
s^0	4+k	0	0

$-\dfrac{(4+k-24)}{4}$ $\dfrac{20-k}{4}$ $5 - k/4 = 0$ $k=20$

$\boxed{j\omega \text{ crossing at } k=20}$

Breakaway points

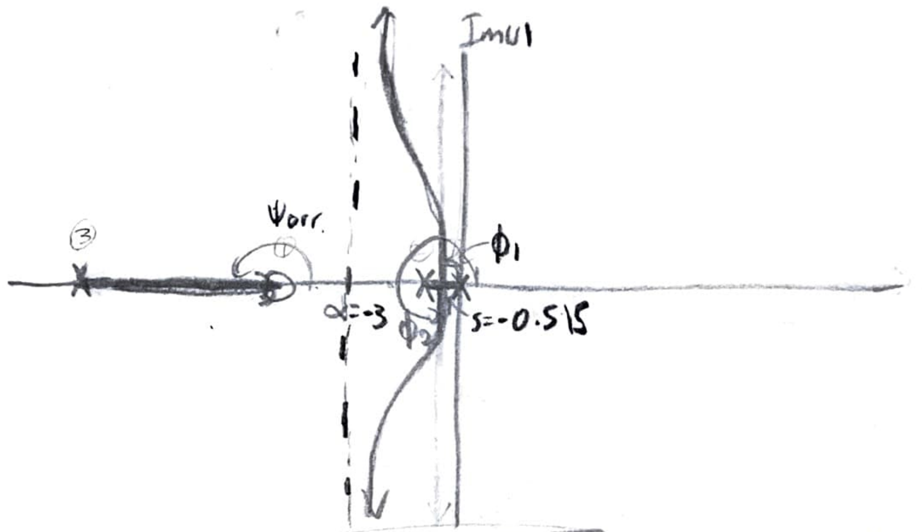
$\hat{G}(s) = \dfrac{b(s)}{a(s)} = \dfrac{1}{s^3+4s^2+6s+4}$ $b=1$ $a=s^3+4s^2+6s+4$ $\dfrac{db}{ds}=0$ $\dfrac{da}{ds} = 3s^2+8s+6$

$-\dfrac{1}{b^2}\left(b\dfrac{da}{ds} - a\dfrac{db}{ds}\right) = 0 = -\dfrac{\frac{da}{ds}}{5} = -\dfrac{da}{ds} = -3s^2-8s-6$ $s = -1.33 \pm 0.47j$

(No breakaway points (obviously))

1. c) $G_c(s)G(s) = \dfrac{K(s+5)}{s(s+1)(s+10)}$ CLTF: $\dfrac{K(s+5)}{s(s+1)(s+10)+K(s+5)} = \dfrac{K(s+5)}{s^3+11s^2+(10+K)s+5K}$

$\hat{G}(s) = \dfrac{s+5}{s(s+1)(s+10)} = \dfrac{s+5}{s^3+11s^2+10s}$ Poles: $s=0,\ s=-1,\ s=-10$
Zeros: $s=-5$



As $K\to\infty$
$3 \to n-m=2$

$\phi_1 = \dfrac{180^\circ+0}{2} = 90^\circ$
$\phi_2 = \dfrac{180^\circ+360^\circ}{2} = 270^\circ$

$\alpha = \dfrac{(0+-1+-10)-(-5)}{2} = -3$

Departure angles for poles → Trivial

① $\phi_{dep} = [0]-[0]-180 = -180^\circ = 180^\circ$
② $\phi_{dep} = [0]-[180]-180 = -360^\circ = 0^\circ$
③ $\phi_{dep} = [180]-[360]-180 = -360^\circ = 0^\circ$

Arrival Angle for Poles → Trivial

ψ_{arr} ① $= 180^\circ - 0 + 180 = 540 = 180^\circ$ → Trivial

jω crossing:

s^3	1	10+K	0
s^2	11	5K	0
s^1	$\frac{105}{11}+K$	0	
s^0	5K	0	

$\dfrac{5K-110-11K}{11} = -\left|\dfrac{-105}{11}-K\right| = \dfrac{105}{11}+K$

$\dfrac{105}{11}+K=0$

$K = -\dfrac{105}{11}$ for jω axis crossing
↳ Does not occur if $K>0$

If $K>0$
no jω crossing

Break away and break in points

$\hat{G}(s) = \dfrac{s+5}{s^3+11s^2+10s} = \dfrac{b(s)}{a(s)}$ $b = s+5$ $a = s^3+11s^2+10s$ $\dfrac{db}{ds} = 1$ $\dfrac{da}{ds} = 3s^2+22s+10$

$-\dfrac{1}{b^2}\left(b\dfrac{da}{ds} - a\dfrac{db}{ds}\right)=0$ $-\dfrac{1}{(s+5)^2}\left((s+5)(3s^2+22s+10) - (s^3+11s^2+10s)(1)\right)=0$

$3s^3+22s^2+10s+15s^2+110s+50-(s^3+11s^2+10s)$
$2s^3+26s^2+110s+50=0$
$s^3+13s^2+55s+25=0$

Break away point at $s=-0.515$ ($-6.24\pm3.084j$)

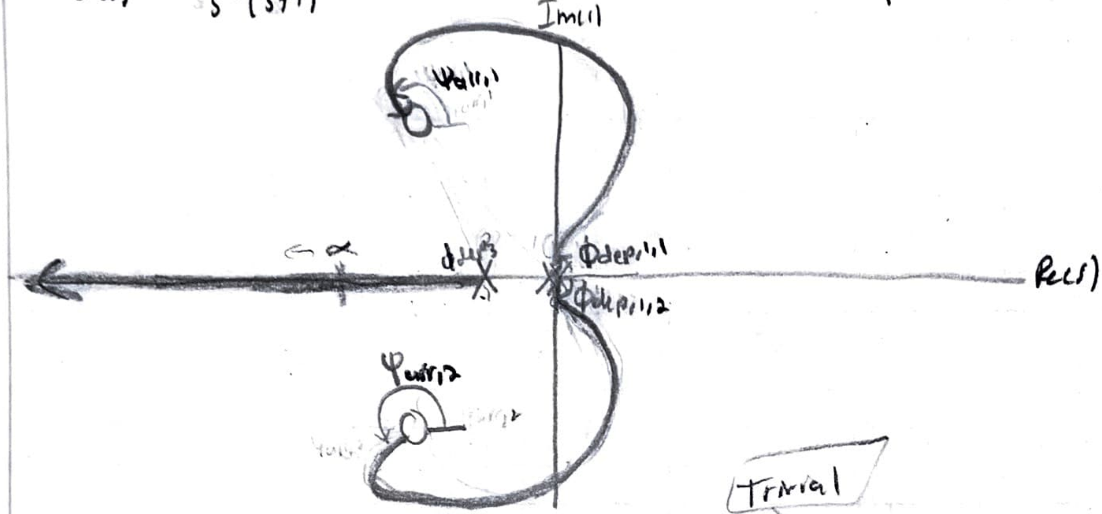
1. d) $G_c(s)G(s) = \dfrac{K(s^2+4s+8)}{s^2(s+1)}$ CLTF: $\dfrac{K(s^2+4s+8)}{s^2(s+1)+K(s^2+4s+8)}$ $s^3+s^2+Ks^2+4Ks+8K$

$\sqrt{16} = 4$

$s = \dfrac{-4 \pm \sqrt{16-32}}{2}$

$-2 \pm 2j$

$\hat{G}(s) = \dfrac{s^2+4s+8}{s^2(s+1)}$ Poles: $s=0, s=0, s=-1$ Zeros: $s=-2+2j$, $s=-2-2j$ CLTF: $\dfrac{K(s^2+4s+8)}{s^3+(1+K)s^2+4Ks+8K}$



Trivial

As $K \to \infty$

$n - m = 1$ ($n = 3$, $m = 2$)

$\phi_1 = \dfrac{180° + 0}{1} = 180°$ $\alpha = \dfrac{-1+4}{1} = -3$

Departure Angle for poles

① $l_q = 2$: $2\phi_{dep} = [315° + 45°] - 0 - 180°$ $\boxed{\phi_{dep,1,1} = 90°}$

$2\phi_{dep} = [315° + 45°] - 0 - 180 - 360$ $\boxed{\phi_{dep,1,2} = -90°}$

② $\phi_{dep} = [296.56° + 63.43°] - 360° - 180°$ $\phi_{dep,2} = -180°$

Arrival Angles

$\psi_{arr,1} = (2(135°) + 153.43°) - 90 + 180° = 513.43$

$\boxed{\psi_{arr,1} = 153.43°}$

$\psi_{arr,2} = (2(225°) + 206.57°) - 270 + 180 = 566.57$

$\boxed{\psi_{arr,2} = 206.57°}$

$j\omega$-crossing

s^3	1	$4K$	0
s^2	$1+K$	$8K$	0
s^1	C_1	0	0
s^0	$8K$		

$C_1 = \dfrac{-(8K - 4K - 4K^2)}{1+K} = \dfrac{-(4K - 4K^2)}{1+K} = \dfrac{-4K(1-K)}{1+K}$

$C_1 = 0$ $4K(1-K) = 0$

$j\omega$ crossing at $\boxed{K = 1 \quad K = 0}$

Multiple $j\omega$ crossings

Break Away and Break in points $\hat{G}(s) = \dfrac{s^2+4s+8}{s^2(s+1)} = \dfrac{b(s)}{a(s)}$ $b = s^2+4s+8$, $a = s^3+s^2$

$\dfrac{db}{ds} = 2s+4$ $\dfrac{da}{ds} = 3s^2+2s$

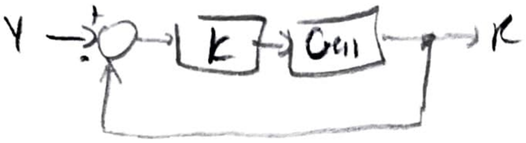
$-\dfrac{1}{b^2}\left(b\dfrac{da}{ds} - a\dfrac{db}{ds}\right) = 0$ $-\dfrac{1}{(s^2+4s+8)^2}\left((s^2+4s+8)(3s^2+2s) - (s^3+s^2)(2s+4)\right) = 0$

$\dfrac{-(s^4+8s^3+28s^2+16s)}{(s^2+4s+8)^2} = 0$

roots$(s^4+8s^3+28s^2+16s)$: $\boxed{s=0}$, -0.69874, $-3.6506 \pm 3.047j$

$\boxed{\text{Breakaway point at } s=0}$

2. a) $\hat{G}(s) = \dfrac{1}{s(s+3)(s^2+2s+2)}$ CLTF: $\dfrac{K}{s(s+3)(s^2+2s+2)+K}$



CLTF: $\dfrac{K}{s^4+5s^3+8s^2+6s+K}$

$-2 \pm \sqrt{4-8}$ / 2

$-1 \pm i$

$\hat{G}(s) = \dfrac{b(s)}{a(s)} = \dfrac{1}{s(s+3)(s^2+2s+2)}$ $b = 1$

$a = s(s+3)(s^2+2s+2) = s(s^3+5s^2+8s+6) = s^4+5s^3+8s^2+6s$

$\dfrac{db}{ds} = 0$ $\dfrac{da}{ds} = 4s^3+15s^2+16s+6$

$\dfrac{d}{ds}\left(-\dfrac{a(s)}{b(s)}\right) = -\dfrac{1}{b^2}\left(b\dfrac{da}{ds} - a\cancel{\dfrac{db}{ds}}\right) = 0 = -\dfrac{1}{b}\left(\dfrac{da}{ds}\right)$

$-(4s^3+15s^2+16s+6) = 0$ $4s^3+15s^2+16s+6 = 0$

roots$(4s^3+15s^2+16s+6)$ $\boxed{s = -2.2886}$, $-0.73 \pm 0.3486j$

b)

s^4	1	8	K	0
s^3	5	6	0	0
s^2	34/5	K	0	0
s^1	C_1	0	0	0
s^0	K			

$\dfrac{-6-40}{5}$ -34 $\dfrac{-5K + \frac{204}{5}}{\frac{34}{5}}$ $-\dfrac{25}{34}K + 6$

$\dfrac{-0-5K}{5}$

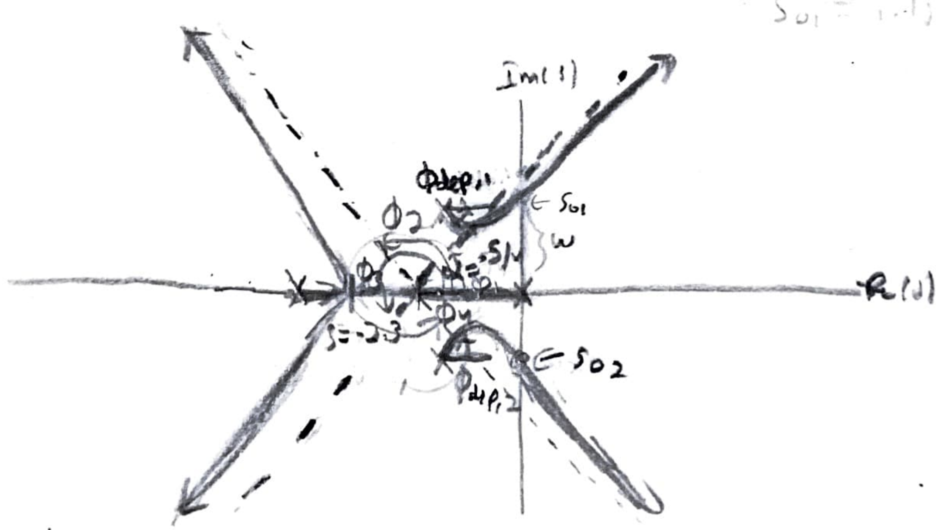
jw crossing when $C_1 = 0$ at $K = 8.16$ $C_1 = -\dfrac{25}{34}K + 6 = 0$ $\boxed{\text{jw crossing when } K = 8.16}$

Look at auxiliary polynomial → Factor of characteristic equation $A(s) = 5s^3 + 6s = s(5s^2+6) = 0$

↳ $A(s) = 0 = 5s^2 + 6$ $s = \pm\sqrt{-6/5} = \pm 1.1j$

$\boxed{s_{01} = 1.1j}$ $\boxed{s_{02} = -1.1j}$

c)



$\hat{G}(s)$ poles: $s = 0$, $s = -3$, $s = -1+i$, $s = -1-i$

Zeroes: None

2. c) As $K \to \infty$

$n-m=4$

$$\phi_1 = \frac{180+0}{4} = \boxed{45^\circ} \qquad \phi_3 = \frac{180+720}{4} = \boxed{225^\circ}$$

$$\phi_2 = \frac{180+360}{4} = \boxed{135^\circ} \qquad \phi_4 = \frac{180^\circ+1080^\circ}{4} = \boxed{315^\circ}$$

$$\boxed{\alpha = \frac{[0+-3+-1+-1]}{4} = -5/4}$$

Check departure angle of $s=-1+i$ and $s=-1-i$

$$\phi_{dep,1} = [0]-[135^\circ+90^\circ+26.56^\circ]-180^\circ = -431.56 = \boxed{-71.56^\circ}$$

$$\phi_{dep,2} = [0]-[225^\circ+270^\circ+333.43^\circ]-180^\circ = -1008.43 = -288.43 \boxed{= 71.57^\circ}$$

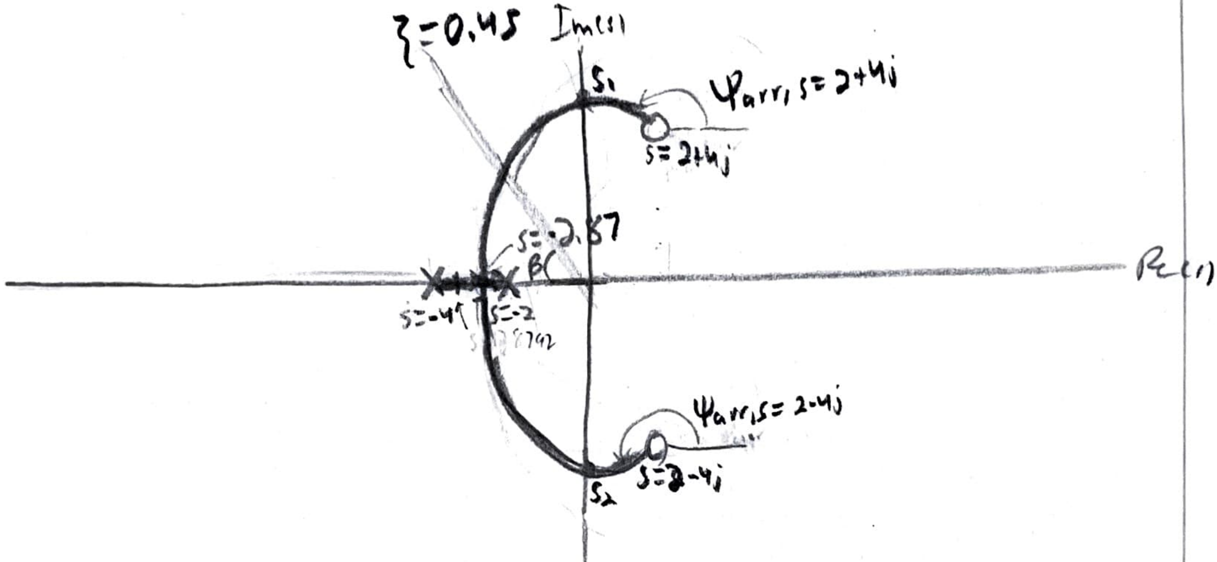
Rest are trivial

3.a) $\hat{G}(s) = \dfrac{(s^2-4s+20)}{(s+2)(s+4)} = \dfrac{(s^2-4s+20)}{s^2+6s+8}$ CLTF: $\dfrac{K(s^2-4s+20)}{(1+K)s^2+(6-4K)s+(8+20K)}$

$\dfrac{4 \pm \sqrt{16-80}}{2}$

$2 \pm 4j$

Zeroes: $s=2+4j$ $s=2-4j$
Poles: $s=-2$ $s=-4$



As $k \to \infty$

poles will approach zeroes, No asymptote, $n-m=0$

Departure Angle for poles:

$\phi_{dep, s=-2} = [225° + 135°] - [0] - 180° = 180°$

$\phi_{dep, s=-4} = [214° + 146°] - [180°] - 180° = 0°$

Arrival Angle for zeroes:

$\psi_{arr, s=2+4j} = [45° + 33.7°] - [90°] + 180° = 168.7°$

$\psi_{arr, s=2-4j} = [-45° + -33.7°] - [270°] + 180° = -168.7° = 191.3°$

$j\omega$ axis crossing

s^2	$1+k$	$8+20k$
s^1	$6-4k$	0
s^0	$8+20k$	0

$6-4k=0$ $\boxed{j\omega \text{ crossing at } k = 6/4}$

Break away and break in points $\hat{G}(s) = \dfrac{b(s)}{a(s)}$ $b=s^2-4s+20$ $a=s^2+6s+8$

$\dfrac{d}{ds}\left(-\dfrac{1}{\hat{G}(s)}\right) = \dfrac{d}{ds}\left(-\dfrac{a(s)}{b(s)}\right) = -\dfrac{1}{b^2}\left(b\dfrac{da}{ds} - a\dfrac{db}{ds}\right) = 0$ $\dfrac{da}{ds} = 2s+6$ $\dfrac{db}{ds} = 2s-4$

$\dfrac{-1}{(s^2-4s+20)^2}\left((s^2-4s+20)(2s+6) - (s^2+6s+8)(2s-4)\right) = 0$

$\dfrac{2(5s^2-12s-76)}{(s^2-4s+20)^2} = 0$

$5s^2-12s-76=0$

$s = \dfrac{12 \pm \sqrt{144+1520}}{10}$

$s = \cancel{5.2792}, -2.8792$

a) We know $\cos(\beta) = 0.45$

The angle at which the $\zeta = 0.45$ lies is $180° - \beta$

Thus at $180° - \cos^{-1}(0.45) = 116.74°$ the root locus crosses the $j\omega$ axis

Plot line from 0 to $10\angle 116.74° = -4.5 + 8.93j$ =

arbitrary length, far enough to intersect

↳ From matlab: Intersection at approximately:

$\boxed{s_0 = -1.53 + 3.03j}$ → $\boxed{s_0 = 3.39\angle 116.74°}$

$$\text{mag}\left(\frac{(s_0 - (2+4j))\,(s_0 - (2-4j))}{(s_0+2)\,(s_0+4)}\right) = \text{mag}\left(-\frac{1}{K}\right)$$

$\boxed{K = 0.4171}$

b) From $j\omega$ crossover on last page: we have $j\omega$ crossover when $K = 6/4$

The auxillary polynomial is: $A(s) = (1+K)s^2 + (8+20K)$ where $K = 6/4$

$A(s) = 10/4\, s^2 + 38$ which divides the characteristic equation so finding the poles at $K = 6/4$ on $j\omega$ axis yields $10/4\, s^2 + 38 = 0$ $s = \pm 3.9j$

$\boxed{s_1 = 3.9j}$ $\boxed{s_2 = -3.9j}$ $\boxed{\text{at } K = 6/4}$

c) Breakaway point occurs at $\boxed{s = -2.8792}$

↳ See previous page for work

d) $\boxed{0 < K < 6/4}$ System is stable

↳ see last page

ME EN 6200
Homework 7
Ryan Dalby

Problem 3

a)

```
sys = tf([1 -4 20], [1 6 8]);
start_line = 0;
end_line = -4.5 + 8.93i;
damp_line = linspace(start_line, end_line);

figure;
[r,~] = rlocus(sys);
rlocus(sys);
hold on;
plot(damp_line);

upper_line = r(2,:);

[real_inter,
imag_inter]=polyxpoly(real(upper_line),imag(upper_line),real(damp_line),ima
g(damp_line));

s0 = real_inter + imag_inter*1i;

mag_G_hat = abs(((s0-(2+4i))*(s0-(2-4i)))/((s0+2)*(s0+4)));

k = 1/mag_G_hat;

fprintf('Intersection at: %.2f + %.2fj where   k=%.4f \n', real_inter,
imag_inter, k);
```

```
Intersection at: -1.53 + 3.03j and k=0.4171
```