Problem Set #3: Drive Train and Motor Dynamics

1. In this problem, we consider the drive train kinematics of the planar 2-DOF Quanser robot from PS#1 and 2. Figure 1 shows a diagram of the robot, where θ_i are joint angles, and ϕ_{mi} are motor angles. Note that the drive train for this robot includes a gear train on both joints, as well as a belt/pulley system on joint 2. In this case it is also helpful to define an intermediate set of angles, ϕ_i, which are located between the gear and pulley systems, and happen to coincide with the encoder placement. N_i are the gear ratios, such that $\dot{\phi}_{mi} = N_i \dot{\phi}_i$, and r_i are the radii of the pulleys.

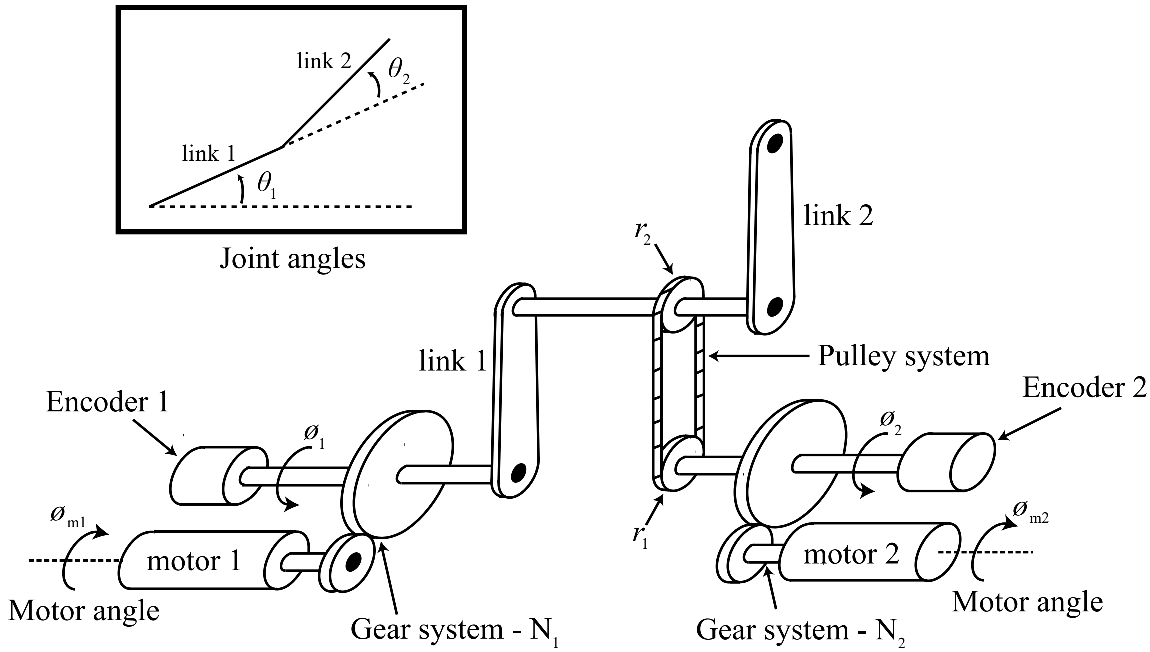

Figure 1. Schematic for the Planar 2-DOF Quanser Robot.

a. Derive the transmission Jacobian, $\mathbf{J_t}$, for this robot, including both gear and pulley effects. Note that since motor #2 is located on the base (rather than on link 1), the belt/pulley system has a coupling effect on the transmission Jacobian as we discussed in lecture. **Hint:** it may be helpful to think of the transmission Jacobian as a compounding of two matrices: a "gear" Jacobian, $\mathbf{J_g}$, and a "pulley" Jacobian $\mathbf{J_p}$.

b. Show that when the pulley radii are equal (as they are on the Quanser robots), the transmission Jacobian from part (*a*) reduces to:

$$J_t = \begin{bmatrix} \frac{1}{N_1} & 0 \\ -\frac{1}{N_1} & \frac{1}{N_2} \end{bmatrix}$$

c. Recall that the manipulator Jacobian, **J**, (from PS#1) relates the end-effector velocities $\dot{\boldsymbol{d}}_{\mathbf{02}}$ to the joint velocities $\dot{\boldsymbol{\Theta}}$:

$$^{0}\dot{\boldsymbol{d}}_{02} = \begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \mathbf{J} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix} = \mathbf{J}\dot{\boldsymbol{\Theta}}$$

Using both the manipulator Jacobian and the transmission Jacobian, find a compound Jacobian that relates the motor velocities $\dot{\boldsymbol{\Phi}}_{\boldsymbol{m}} = \begin{bmatrix} \dot{\phi}_{m1} \\ \dot{\phi}_{m2} \end{bmatrix}$ to the end-effector velocities $\dot{\boldsymbol{d}}_{\mathbf{02}}$.

d. Recalling the duality between kinematics and statics, write a matrix equation that relates the motor torques $\boldsymbol{\tau}_{\mathbf{m}} = \begin{bmatrix} \tau_{m1} \\ \tau_{m2} \end{bmatrix}$ to the end-effector forces $\mathbf{F} = \begin{bmatrix} F_x \\ F_y \end{bmatrix}$.

2. We now consider the effect of the drive train on the dynamics of the planar 2-DOF Quanser robot. From PS#2, the 2-DOF robot inverse dynamic solution has the form:

$$\boldsymbol{\tau}_{arm} = \begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix} \begin{bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{bmatrix} + \begin{bmatrix} 0 & -2h & -h \\ h & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{\theta}_1^2 \\ \dot{\theta}_1\dot{\theta}_2 \\ \dot{\theta}_2^2 \end{bmatrix} + \begin{bmatrix} G_1 \\ G_2 \end{bmatrix}$$

which can be abbreviated as:

$$\boldsymbol{\tau}_{\boldsymbol{arm}} = \boldsymbol{H}(\boldsymbol{\Theta})\ddot{\boldsymbol{\Theta}} + \mathbf{V}(\boldsymbol{\Theta}, \dot{\boldsymbol{\Theta}}) + \boldsymbol{G}(\boldsymbol{\Theta})$$

In addition to the arm dynamics in the above equation, we should now account for dynamic torques arising from the motor shaft, including frictional torques characterized by viscous damping b_i and coulomb friction c_i, as well as inertial torques due to motor shaft inertias J_i:

$$\boldsymbol{\tau}_{\boldsymbol{f}} = \begin{bmatrix} b_1\dot{\phi}_{m1} + c_1\mathrm{sgn}(\dot{\phi}_{m1}) \\ b_2\dot{\phi}_{m2} + c_2\mathrm{sgn}(\dot{\phi}_{m2}) \end{bmatrix} = \boldsymbol{F}(\dot{\boldsymbol{\Phi}}_{\boldsymbol{m}})$$

$$\boldsymbol{\tau}_{\boldsymbol{J}} = \begin{bmatrix} J_1 & 0 \\ 0 & J_2 \end{bmatrix} \ddot{\boldsymbol{\Phi}}_{\boldsymbol{m}}$$

Note that these torques are expressed in motor space, while the arm dynamics are expressed in joint space. In order to combine them, we either need to reflect the motor shaft torques into joint space $(\boldsymbol{\Theta})$ or reflect the arm dynamics into motor space $(\boldsymbol{\Phi}_{\boldsymbol{m}})$. Don't forget that when reflecting dynamics across the transmission, you need to use the transmission Jacobian twice: once to reflect the kinematics (angles/velocities/accelerations), and once to reflect the torques.

The solutions to parts (a), (b), and (c) involve large matrices. In each case, it is sufficient to solve for the matrices **H'**, **V**, **G**, and **F** individually (showing the elements in each matrix once). When writing the entire dynamic equation, just use abbreviated matrix form.

a. First we will combine the dynamics in motor space. Use the transmission Jacobian from Problem 1(b) to reflect the arm dynamics into motor space, and then combine with the motor shaft dynamics to form a new closed-form matrix equation in motor space:

$$\boldsymbol{\tau_m} = \boldsymbol{H}'(\boldsymbol{\Phi_m})\ddot{\boldsymbol{\Phi}}_{\mathbf{m}} + \mathbf{V}(\boldsymbol{\Phi_m}, \dot{\boldsymbol{\Phi}}_{\mathbf{m}}) + \boldsymbol{G}(\boldsymbol{\Phi_m}) + \boldsymbol{F}(\dot{\boldsymbol{\Phi}}_{\mathbf{m}})$$

b. Alternatively, we can combine the dynamics in joint space. Use the transmission Jacobian from Problem 1(b) to reflect the motor friction and inertia into joint space, and then combine with the arm dynamics to form the analogous closed-form matrix equation in joint space:

$$\boldsymbol{\tau_{joint}} = \boldsymbol{H}'(\boldsymbol{\Theta})\ddot{\boldsymbol{\Theta}} + \mathbf{V}(\boldsymbol{\Theta}, \dot{\boldsymbol{\Theta}}) + \boldsymbol{G}(\boldsymbol{\Theta}) + \boldsymbol{F}(\dot{\boldsymbol{\Theta}})$$

c. Finally, we should consider that on the Quanser robots, the encoders (position sensors) are actually located on the output side of the gears, such that they are measuring ϕ_i rather than ϕ_{mi}. Use the gear Jacobian $\mathbf{J_g}$ to put your equation from part (a) in terms of $\boldsymbol{\Phi}$ instead of $\boldsymbol{\Phi_m}$ (i.e. reflect the dynamics to the encoder shaft):

$$\boldsymbol{\tau_e} = \boldsymbol{H}'(\boldsymbol{\Phi})\ddot{\boldsymbol{\Phi}} + \mathbf{V}(\boldsymbol{\Phi}, \dot{\boldsymbol{\Phi}}) + \boldsymbol{G}(\boldsymbol{\Phi}) + \boldsymbol{F}(\dot{\boldsymbol{\Phi}})$$

3. Finally, we consider the electromechanical dynamics of the motors on the 2-DOF Quanser robot. Figure 2 shows an electromechanical model of a typical DC motor, where *V(t)* is the armature voltage, *R* is the armature resistance, *i(t)* is the armature current, V_b is the back EMF, τ_m is the motor torque and $\dot{\phi}_m$ is the angular velocity. We will neglect the armature inductance.

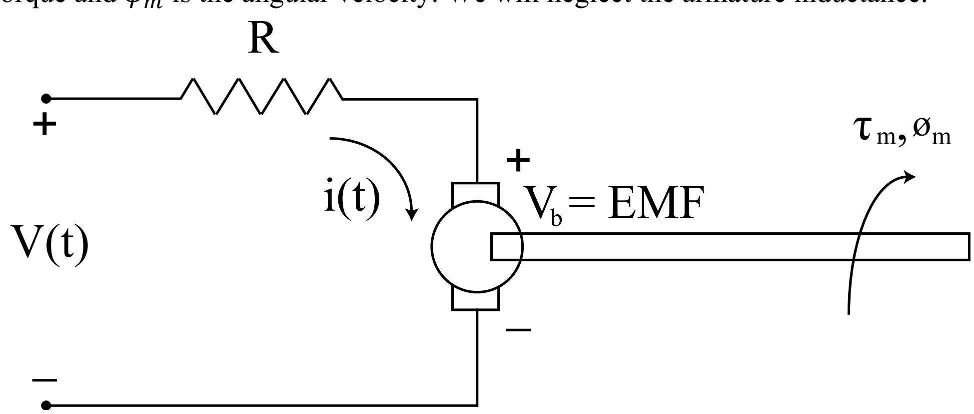

Figure 2. Electromechanical Model of a DC Motor. This model neglects motor inductance.

Since the 2-DOF system has two motors, use subscripts 1 and 2 for their respective gear ratios, *N*, resistances, *R*, motor constants, k_t, voltage sources, *V(t)*, and currents, *i(t)*. Depending on what kind of amplifier is used, either *V(t)* or *i(t)* may be the prescribed input to the motor (i.e., voltage control or current control).

a. **Current Control:** What is the relationship between motor current *i(t)* and motor torque τ_m? Substitute this formula into your dynamic equation from Problem 2(c) to get an equation of the form:

$$\begin{bmatrix} A_1 i_1(t) \\ A_2 i_2(t) \end{bmatrix} = \boldsymbol{H}'(\boldsymbol{\Phi})\ddot{\boldsymbol{\Phi}} + \mathbf{V}(\boldsymbol{\Phi}, \dot{\boldsymbol{\Phi}}) + \boldsymbol{G}(\boldsymbol{\Phi}) + \boldsymbol{F}(\dot{\boldsymbol{\Phi}})$$

What are the constants A_1 and A_2 in this case?

b. **Voltage Control:** Using the model in Figure 2, derive the relationship between voltage *V(t)* and torque τ_m. Substitute this formula into your dynamic equation from Problem 2(c) to get an equation of the form:

$$\begin{bmatrix} A_1 V_1(t) \\ A_2 V_2(t) \end{bmatrix} = \boldsymbol{H}'(\boldsymbol{\Phi})\ddot{\boldsymbol{\Phi}} + \mathbf{V}(\boldsymbol{\Phi}, \dot{\boldsymbol{\Phi}}) + \boldsymbol{G}(\boldsymbol{\Phi}) + \boldsymbol{F}'(\dot{\boldsymbol{\Phi}})$$

What are the constants A_1 and A_2 in this case? Note that the frictional torque $\boldsymbol{F}'(\dot{\boldsymbol{\Phi}})$ should now contain an extra term due to the back EMF.