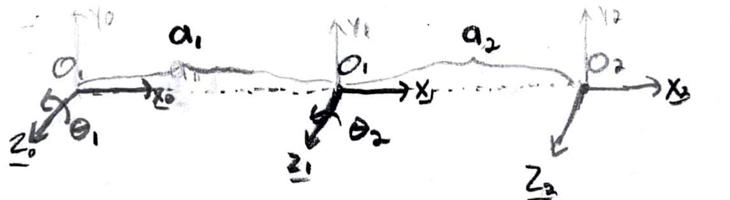
* Notation
→ $\underline{a}$ denotes vector a
→ Matrices are generally capitalized
→ Scalars are lowercase generally

1.

i	a_i	d_i	α_i	θ_i
1	a_1	0	0	*
2	a_2	0	0	*

* indicates variable value



choice of d arbitrary so choose 0 (parallel z axes)

2.

$$ {}^0T_2 = \begin{bmatrix} {}^0R_2 & {}^0\underline{d}_{02} \\ 0 & 1 \end{bmatrix} $$

$$ {}^0R_2 = {}^0R_1\,{}^1R_2 = \begin{bmatrix} c\theta_1 & -s\theta_1 & 0 \\ s\theta_1 & c\theta_1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c\theta_2 & -s\theta_2 & 0 \\ s\theta_2 & c\theta_2 & 0 \\ 0 & 0 & 1 \end{bmatrix} $$

$$ \boxed{ {}^0R_2 = \begin{bmatrix} c\theta_1 c\theta_2 - s\theta_1 s\theta_2 & -c\theta_1 s\theta_2 - s\theta_1 c\theta_2 & 0 \\ s\theta_1 c\theta_2 + c\theta_1 s\theta_2 & -s\theta_1 s\theta_2 + c\theta_1 c\theta_2 & 0 \\ 0 & 0 & 1 \end{bmatrix} } $$

$$ {}^0\underline{d}_{02} = {}^0\underline{d}_{01} + {}^0\underline{d}_{12} = {}^0\underline{d}_{01} + {}^0R_1\,{}^1\underline{d}_{12} = \begin{bmatrix} a_1 c\theta_1 \\ a_1 s\theta_1 \\ 0 \end{bmatrix} + \begin{bmatrix} c\theta_1 & -s\theta_1 & 0 \\ s\theta_1 & c\theta_1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a_2 c\theta_2 \\ a_2 s\theta_2 \\ 0 \end{bmatrix} $$

$$ \begin{bmatrix} a_2 c\theta_1 c\theta_2 - a_2 s\theta_1 s\theta_2 \\ a_2 s\theta_1 c\theta_2 \quad a_2 c\theta_1 s\theta_2 \\ 0 \end{bmatrix} $$

$$ \boxed{ {}^0\underline{d}_{02} = \begin{bmatrix} a_1 c\theta_1 + a_2 c\theta_1 c\theta_2 - a_2 s\theta_1 s\theta_2 \\ a_1 s\theta_1 + a_2 s\theta_1 c\theta_2 + a_2 c\theta_1 s\theta_2 \\ 0 \end{bmatrix} } $$

$$ \boxed{ {}^0T_2 = \begin{bmatrix} {}^0R_2 & {}^0\underline{d}_{02} \\ 0 & 1 \end{bmatrix} } $$

3. $${}^{0}\underline{d}_{02} = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a_1 C\theta_1 + a_2 C\theta_1 C\theta_2 - a_2 S\theta_1 S\theta_2 \\ a_1 S\theta_1 + a_2 S\theta_1 C\theta_2 + a_2 C\theta_1 S\theta_2 \end{bmatrix}$$

$${}^{0}\dot{\underline{d}}_{02} = \frac{d}{dt}\left({}^{0}\underline{d}_{02}\right) = \frac{\partial\, {}^{0}\underline{d}_{02}}{\partial \theta_1}\dot{\theta}_1 + \frac{\partial\, {}^{0}\underline{d}_{02}}{\partial \theta_2}\dot{\theta}_2$$

$${}^{0}\dot{\underline{d}}_{02} = J\dot{\underline{\Theta}} = \begin{bmatrix} \frac{\partial(a_1 C\theta_1 + a_2 C\theta_1 C\theta_2 - a_2 S\theta_1 S\theta_2)}{\partial \theta_1} & \frac{\partial(a_1 C\theta_1 + a_2 C\theta_1 C\theta_2 - a_2 S\theta_1 S\theta_2)}{\partial \theta_2} \\ \frac{\partial(a_1 S\theta_1 + a_2 S\theta_1 C\theta_2 + a_2 C\theta_1 S\theta_2)}{\partial \theta_1} & \frac{\partial(a_1 S\theta_1 + a_2 S\theta_1 C\theta_2 + a_2 C\theta_1 S\theta_2)}{\partial \theta_2} \end{bmatrix} \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix}$$

$${}^{0}\underline{d}_{02} = \quad J = \begin{bmatrix} -a_1 S\theta_1 + a_2 S\theta_1 C\theta_2 - a_2 C\theta_1 S\theta_2 & -a_2 C\theta_1 S\theta_2 - a_2 S\theta_1 C\theta_2 \\ a_1 C\theta_1 + a_2 C\theta_1 C\theta_2 + a_2 S\theta_1 S\theta_2 & -a_2 S\theta_1 S\theta_2 + a_2 C\theta_1 C\theta_2 \end{bmatrix}$$

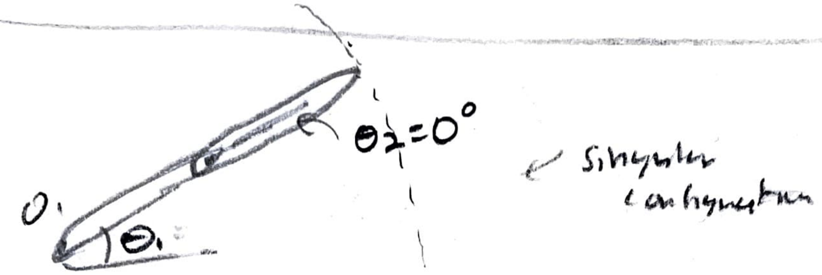
$$\det(J) = a_1 a_2 S\theta_1^2 S\theta_2 - \cancel{a_1 a_2 S\theta_1 C\theta_1 C\theta_2} + \cancel{a_2^2 S\theta_1^2 S\theta_2 C\theta_2}$$
$$- \cancel{a_2^2 S\theta_1 C\theta_1 C\theta_2^2} + \cancel{a_2^2 S\theta_1 C\theta_1 S\theta_2^2} - \cancel{a_2^2 C\theta_1^2 S\theta_2 C\theta_2}$$
$$- \left[-a_1 a_2 C\theta_1^2 S\theta_2 - \cancel{a_1 a_2 S\theta_1 C\theta_1 C\theta_2} - \cancel{a_2^2 C\theta_1^2 S\theta_2 C\theta_2}\right.$$
$$\left. - \cancel{a_2^2 S\theta_1 C\theta_1 C\theta_2^2} + \cancel{a_2^2 S\theta_1 C\theta_1 S\theta_2^2} + \cancel{a_2^2 S\theta_1^2 S\theta_2 C\theta_2}\right] = 0$$

$$a_1 a_2 S\theta_1^2 S\theta_2 + a_1 a_2 C\theta_1^2 S\theta_2 = 0$$
$$a_1 a_2 S\theta_2 (S\theta_1^2 + C\theta_1^2) = 0$$
$$a_1 a_2 S\theta_2 = 0 \qquad S\theta_2 = 0 \Rightarrow \boxed{\theta_2 = 0 + \pi k}$$

integer constant

$\det(J) = 0$

3. when $s\theta_2 = 0$ there is a singularity or when $\theta_2 = \pi k$
when k = integer



The only possible velocities in this configuration are essentially confined to a line (dashed line which has arclength $a_1 + a_2$ from O_2)

This is essentially a 1 D operating velocity space (rather than 2D planar)

4. $$\begin{bmatrix} \dot{x} \\ \dot{y} \\ w_z \end{bmatrix} = J_v \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix} = \begin{bmatrix} {}^0\underline{z}_0 \times {}^0\underline{d}_{02} & {}^0\underline{z}_1 \times {}^0\underline{d}_{12} \\ {}^0\underline{z}_0 & \underbrace{{}^0\underline{z}_1}_{={}^0\underline{z}_0} \end{bmatrix}$$

${}^0\underline{d}_{02} = a_1 {}^0\underline{x}_1 + a_2 {}^0\underline{x}_2$ ${}^0\underline{d}_{12} = a_2 {}^0\underline{x}_2$

${}^0\underline{x}_1 = c\theta_1 {}^0\underline{x}_0 + s\theta_1 {}^0\underline{y}_0$

${}^0\underline{y}_1 = c\theta_1 {}^0\underline{y}_0 - s\theta_1 {}^0\underline{x}_0$

${}^0\underline{y}_2 = c\theta_2 {}^0\underline{y}_1 - s\theta_2 {}^0\underline{x}_1$

$\underbrace{{}^0\underline{z}_0}_{={}^0\underline{z}_1={}^0\underline{z}_2} \times (a_1 {}^0\underline{x}_1 + a_2 {}^0\underline{x}_2) = a_1 {}^0\underline{y}_1 + a_2 {}^0\underline{y}_2$

${}^0\underline{z}_0 \times {}^0\underline{d}_{02} = a_1(c\theta_1 {}^0\underline{y}_0 - s\theta_1 {}^0\underline{x}_0) + a_2(c\theta_2(c\theta_1 {}^0\underline{y}_0 - s\theta_1 {}^0\underline{x}_0) - s\theta_2(c\theta_1 {}^0\underline{x}_0 + s\theta_1 {}^0\underline{y}_0))$

$${}^0\underline{z}_0 \times {}^0\underline{d}_{02} = \begin{bmatrix} -a_1 s\theta_1 - a_2 s\theta_1 c\theta_2 - a_2 c\theta_1 s\theta_2 \\ a_1 c\theta_1 + a_2 c\theta_1 c\theta_2 + a_2 s\theta_1 s\theta_2 \\ 0 \end{bmatrix}$$

$\underbrace{{}^0\underline{z}_0}_{={}^0\underline{z}_1={}^0\underline{z}_2} \times {}^0\underline{d}_{12} = a_2 {}^0\underline{y}_2 = a_2(c\theta_2(c\theta_1 {}^0\underline{y}_0 - s\theta_1 {}^0\underline{x}_0) - s\theta_2(c\theta_1 {}^0\underline{x}_0 + s\theta_1 {}^0\underline{y}_0))$

$${}^0\underline{z}_0 \times {}^0\underline{d}_{12} = \begin{bmatrix} -a_2 s\theta_1 c\theta_2 - a_2 c\theta_1 s\theta_2 \\ a_2 c\theta_1 c\theta_2 + a_2 s\theta_1 s\theta_2 \end{bmatrix}$$

4.

$$J_v = \begin{bmatrix} -a_1 s\theta_1 - a_2 s\theta_1 c\theta_2 - a_2 c\theta_1 s\theta_2 & -a_2 c\theta_1 s\theta_2 - a_2 s\theta_1 c\theta_2 \\ a_1 c\theta_1 + a_2 c\theta_1 c\theta_2 - a_2 s\theta_1 s\theta_2 & -a_2 s\theta_1 s\theta_2 + a_2 c\theta_1 c\theta_2 \\ 1 & 1 \end{bmatrix}$$

The first two rows of J_v are equivalent to J.
I would use the manipulator Jacobian J since it is a square matrix (2×2) and thus the inverse would be easier to compute and its singularities would be easier to find (as done in 3) thus it can be known when $\det(J)=0$ and J is non-invertible. J_v is 3×2 and would be more difficult to work with.
Physically this manipulator is on a plane and we have two rotary joints thus we cannot specify a w_z without another degree of freedom (joint 3)

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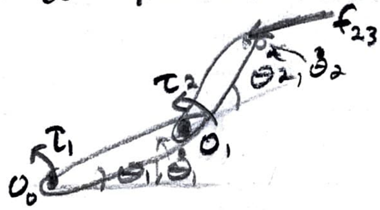
5. Using the principle of virtual work (using power, time derivative of work)

virtual Power out by motors = virtual power at end effector



3 is what is providing a force on O_2

$$f_{23} = \begin{bmatrix} F_x \\ F_y \end{bmatrix}$$

$$\tau_1\dot\theta_1 + \tau_2\dot\theta_2 = \underline{f}_{23}^T \, \underline{\dot d}_{02} \Rightarrow \underline{\dot d}_{02} = J\underline{\dot\theta}$$

$$\underbrace{\begin{bmatrix}\tau_1\\ \tau_2\end{bmatrix}^T}_{1\times2} \underbrace{\begin{bmatrix}\dot\theta_1\\ \dot\theta_2\end{bmatrix}}_{2\times1} = \underbrace{\begin{bmatrix}F_x\\ F_y\end{bmatrix}^T}_{1\times2} \underbrace{J}_{2\times2} \underbrace{\begin{bmatrix}\dot\theta_1\\ \dot\theta_2\end{bmatrix}}_{2\times1}$$

manipulator Jacobian (2x2)

Since we essentially have a dot product can eliminate $\underline{\dot\theta}$

$$\begin{bmatrix}\tau_1\\ \tau_2\end{bmatrix}^T = \begin{bmatrix}F_x\\ F_y\end{bmatrix}^T J$$

↳ By transpose property

$$\boxed{\begin{bmatrix}\tau_1\\ \tau_2\end{bmatrix} = J^T \begin{bmatrix}F_x\\ F_y\end{bmatrix}}$$

$$\begin{bmatrix} a & c \\ b & d \end{bmatrix} \begin{bmatrix}F_x\\ F_y\end{bmatrix}$$

$$\left(\begin{matrix} aF_x & cF_y \\ bF_x & dF_y \end{matrix}\right)^T$$

$$\begin{bmatrix}F_x\\ F_y\end{bmatrix}^T \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$aF_x \; cF_y$
$bF_x \; dF_y$

5. If robot at singularity $det(J) \equiv 0$ is there an $\begin{bmatrix} F_x \\ F_y \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ s.t. $\begin{bmatrix} \tau_1 \\ \tau_2 \end{bmatrix} = \underline{0}$

↳ There exists a duality between unachievable end effector velocities and when the forces are borne entirely with the robotic structure (no joint torques), this is the physical description. The mathematical description is the space of unachievable end effector velocities exists if J is singular, aka $det(J) = 0$, this space is equivalent to the Null space of the Jacobian transpose, $N(J^T)$

↳ From 3, $det(J) = 0$ when $\theta_2 = 0$

$$J_v^T = \begin{bmatrix} -a_1 s\theta_1 - a_2 s\theta_1 c\theta_2 - a_2 c\theta_1 s\theta_2 & a_1 c\theta_1 + a_2 c\theta_1 c\theta_2 - a_2 s\theta_1 s\theta_2 \\ -a_2 c\theta_1 s\theta_2 - a_2 s\theta_1 c\theta_2 & -a_2 s\theta_1 s\theta_2 + a_2 c\theta_1 c\theta_2 \end{bmatrix}$$

↳ $\theta_2 = 0$

$$J^T = \begin{bmatrix} \overbrace{-a_1 s\theta_1 - a_2 s\theta_1}^{-s\theta_1(a_1+a_2)} & \overbrace{a_1 c\theta_1 + a_2 c\theta_1}^{c\theta_1(a_1+a_2)} \\ -a_2 s\theta_1 & a_2 c\theta_1 \end{bmatrix}$$

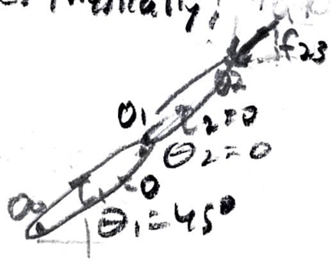
$$\begin{bmatrix} \underline{0} \end{bmatrix} = J^T \begin{bmatrix} F_x \\ F_y \end{bmatrix}$$

$$-F_x s\theta_1(a_1+a_2) + F_y c\theta_1(a_1+a_2) = 0$$
$$-F_x s\theta_1 a_2 + F_y c\theta_1 a_2 = 0$$

$$\frac{F_x s\theta_1 a_2}{c\theta_1 a_2}$$

$$\boxed{\text{All } F_y, F_x \text{ s.t. } \frac{F_y}{F_x} = \tan(\theta_1)}$$

↳ Physically, if $\theta_1 = 45°$ $\quad \frac{F_y}{F_x} = 1$ when $F_y = F_x$



This is a $[F_x, F_y] \neq [0, 0]$ s.t. $\underline{\tau} = \underline{0}$

6. τ_2 as well as $\begin{bmatrix} F_x \\ F_y \end{bmatrix}$ relate $\begin{bmatrix} \tau_1 \\ \tau_2 \end{bmatrix}$ to $\begin{bmatrix} F_x \\ F_y \\ \tau_2 \end{bmatrix}$

Using same setup as 5, principle of virtual work (power)

$$f_{23} = \begin{bmatrix} F_x \\ F_y \\ 0 \end{bmatrix} \qquad n_{23} = \begin{bmatrix} 0 \\ 0 \\ \tau_2 \end{bmatrix}$$

$$\tau_1 \dot{\theta}_1 + \tau_2 \dot{\theta}_2 = f_{23}^T \dot{\underline{d}}_{o2} + \underline{n}_{23}^T w_{o2} = \begin{bmatrix} F_x \\ F_y \\ 0 \end{bmatrix}^T \begin{bmatrix} \dot{d}_{o2} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \tau_2 \end{bmatrix}^T w_{o2}$$

like having two dot products can rearrange into a wrench and have same result

$$\begin{bmatrix} \tau_1 \\ \tau_2 \end{bmatrix}^T \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix} = \begin{bmatrix} F_x \\ F_y \\ \tau_{z2} \end{bmatrix}^T \begin{bmatrix} \dot{d}_{o2} \\ w_{o2} \end{bmatrix}$$

$J_v \dot{\underline{\theta}}$

$$\begin{bmatrix} \tau_1 \\ \tau_2 \end{bmatrix}^T \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix} = \begin{bmatrix} F_x \\ F_y \\ \tau_{z2} \end{bmatrix}^T J_v \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix}$$

velocity shown

have essentially 2 dot products so can cancel $\dot{\underline{\theta}}$

$$\begin{bmatrix} \tau_1 \\ \tau_2 \end{bmatrix}^T = \begin{bmatrix} F_x \\ F_y \\ \tau_{z2} \end{bmatrix}^T J_v$$

Transpose property

$$\boxed{\begin{bmatrix} \tau_1 \\ \tau_2 \end{bmatrix} = J_v^T \begin{bmatrix} F_x \\ F_y \\ \tau_2 \end{bmatrix}} \rightarrow [2\times1] = [2\times3][3\times1]$$

J_v^T cannot be inverted since it will be a 2×3 matrix (recall J_v was a 3×2) (Mathematically)

over determined

Physically given $\begin{bmatrix} \tau_1 \\ \tau_2 \end{bmatrix}$ there are many $\begin{bmatrix} F_x \\ F_y \\ \tau_2 \end{bmatrix}$ which will be in static equilibrium with $\begin{bmatrix} \tau_1 \\ \tau_2 \end{bmatrix}$. We have an underdetermined system for this problem thus we have many solutions, we could use a pseudo-inverse to find <u>a</u> solution that minimizes a metric.