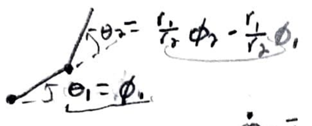
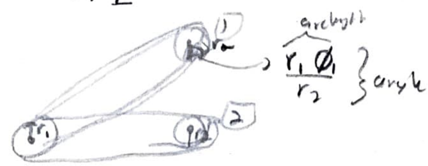
1. a) $\dot{\underline{\theta}} = J_t \dot{\underline{\phi}}_m = J_p J_g \dot{\underline{\phi}}_m = J_p \dot{\underline{\phi}}$




$\dot{\underline{\theta}} = J_p \dot{\underline{\phi}}$

We know $\dot{\theta}_1 = \dot{\phi}_1$ since there is no pulley system for motor 1

For motor 2 $\quad \dot{\theta}_2 = \frac{r_1}{r_2}\dot{\phi}_2 - \frac{r_1}{r_2}\dot{\phi}_1$, we know $\theta_2 = \frac{r_1}{r_2}\phi_2$ but we must subtract the "unwrapped" amount of $\frac{r_1}{r_2}\phi_1$

$$J_p = \begin{bmatrix} 1 & 0 \\ -\frac{r_1}{r_2} & \frac{r_1}{r_2} \end{bmatrix} \Rightarrow \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix} = J_p \begin{bmatrix} \dot{\phi}_1 \\ \dot{\phi}_2 \end{bmatrix}$$

We know $\dot{\underline{\phi}} = J_g \dot{\underline{\phi}}_m \qquad \dot{\phi}_{mi} = N_i \dot{\phi}_i$

$\dot{\phi}_1 = 1/N_1 \, \dot{\phi}_{m1}$

$\dot{\phi}_2 = 1/N_2 \, \dot{\phi}_{m2}$

$$J_g = \begin{bmatrix} 1/N_1 & 0 \\ 0 & 1/N_2 \end{bmatrix} \Rightarrow \begin{bmatrix} \dot{\phi}_1 \\ \dot{\phi}_2 \end{bmatrix} = J_g \begin{bmatrix} \dot{\phi}_{m1} \\ \dot{\phi}_{m2} \end{bmatrix}$$

$$J_t = \begin{bmatrix} 1 & 0 \\ -\frac{r_1}{r_2} & \frac{r_1}{r_2} \end{bmatrix} \begin{bmatrix} 1/N_1 & 0 \\ 0 & 1/N_2 \end{bmatrix} = \begin{bmatrix} 1/N_1 & 0 \\ -\frac{r_1}{r_2 N_1} & \frac{r_1}{r_2 N_2} \end{bmatrix}$$

$$\boxed{J_t = \begin{bmatrix} 1/N_1 & 0 \\ -\frac{r_1}{r_2 N_1} & \frac{r_1}{r_2 N_2} \end{bmatrix}}$$

b) If $r_1 = r_2$:

$$\boxed{J_t = \begin{bmatrix} 1/N_1 & 0 \\ -1/N_1 & 1/N_2 \end{bmatrix}}$$

1. c)

$$^{0}\dot{\underline{d}}_{02} = J\,\dot{\underline{\theta}} \qquad \dot{\underline{\theta}} = J_t\,\dot{\underline{\phi}}_m$$

$$^{0}\dot{\underline{d}}_{02} = J J_t \dot{\underline{\phi}}_m = J_{combined}\,\dot{\underline{\phi}}_m$$

From PS#1 :

$$J = \begin{bmatrix} -a_1 s\theta_1 - a_2 s\theta_1 c\theta_2 - a_2 c\theta_1 s\theta_2 & -a_2 c\theta_1 s\theta_2 - a_2 s\theta_1 c\theta_2 \\ a_1 c\theta_1 + a_2 c\theta_1 c\theta_2 - a_2 s\theta_1 s\theta_2 & -a_2 s\theta_1 s\theta_2 + a_2 c\theta_1 c\theta_2 \end{bmatrix}$$

trig identities

$$J J_t = \begin{bmatrix} -a_1 s\theta_1 - a_2 s(\theta_1+\theta_2) & -a_2 s(\theta_1+\theta_2) \\ a_1 c\theta_1 + a_2 c(\theta_1+\theta_2) & a_2 c(\theta_1+\theta_2) \end{bmatrix} \begin{bmatrix} 1/N_1 & 0 \\ -1/N_1 & 1/N_2 \end{bmatrix}$$

$-a_1 s\theta_1 - a_2 s(\theta_1+\theta_2) + a_2 s(\theta_1+\theta_2)$

$a_1 c\theta_1 + a_2 c(\theta_1+\theta_2) - a_2 c(\theta_1+\theta_2)$

$$\boxed{J_{combined} = \begin{bmatrix} (-a_1 s\theta_1)/N_1 & [-a_2 s(\theta_1+\theta_2)]/N_2 \\ a_1 c\theta_1/N_1 & [a_2 c(\theta_1+\theta_2)]/N_2 \end{bmatrix}}$$

where : $^{0}\dot{\underline{d}}_{02} = J_{combined}\,\dot{\underline{\phi}}_m$

d) Using principle of virtual work :

$$\tau_{m1}\dot{\phi}_{m1} + \tau_{m2}\dot{\phi}_{m2} = \tau_{joint1}\dot{\theta}_1 + \tau_{joint2}\dot{\theta}_2 \Rightarrow \begin{bmatrix}\tau_{m1}\\ \tau_{m2}\end{bmatrix}^T \begin{bmatrix}\dot{\phi}_{m1}\\ \dot{\phi}_{m2}\end{bmatrix} = \begin{bmatrix}\tau_{joint1}\\ \tau_{joint2}\end{bmatrix}^T J_t \begin{bmatrix}\dot{\phi}_{m1}\\ \dot{\phi}_{m2}\end{bmatrix}$$

$$\begin{bmatrix}\tau_{m1}\\ \tau_{m2}\end{bmatrix}^T = \begin{bmatrix}\tau_{joint1}\\ \tau_{joint2}\end{bmatrix}^T J_t \Rightarrow \begin{bmatrix}\tau_{m1}\\ \tau_{m2}\end{bmatrix} = J_t^T \begin{bmatrix}\tau_{joint1}\\ \tau_{joint2}\end{bmatrix}$$

From PS1 : $\begin{bmatrix}\tau_{joint1}\\ \tau_{joint2}\end{bmatrix} = J^T \begin{bmatrix}F_x\\ F_y\end{bmatrix}$

$$\boxed{\begin{bmatrix}\tau_{m1}\\ \tau_{m2}\end{bmatrix} = J_t^T J^T \begin{bmatrix}F_x\\ F_y\end{bmatrix}} \Rightarrow \boxed{\begin{bmatrix}\tau_{m1}\\ \tau_{m2}\end{bmatrix} = J_{combined}^T \begin{bmatrix}F_x\\ F_y\end{bmatrix}}$$

known

$(AB)^T = A^T B^T$

2. a) From PS2:

$$\begin{bmatrix}\tau_{joint1}\\ \tau_{joint2}\end{bmatrix} = \begin{bmatrix}H_{11} & H_{12}\\ H_{12} & H_{22}\end{bmatrix}\begin{bmatrix}\ddot{\theta}_1\\ \ddot{\theta}_2\end{bmatrix} + \begin{bmatrix}0 & -2h & -h\\ h & 0 & 0\end{bmatrix}\begin{bmatrix}\dot{\theta}_1^2\\ \dot{\theta}_1\dot{\theta}_2\\ \dot{\theta}_2^2\end{bmatrix} + \begin{bmatrix}G_1\\ G_2\end{bmatrix}$$

$$\underline{\tau_{joint}} = H\ddot{\underline{\theta}} + V(\dot{\underline{\theta}}, \dot{\underline{\theta}}) + G(\underline{\theta})$$

$H_{11} = I_1 + I_2 + m_2 a_1^2 + 2a_1 r_{12} m_2 c\theta_2$

$H_{12} = H_{21} = I_2 + a_1 r_{12} m_2 c\theta_2$

$H_{22} = I_2$

$h = a_1 r_{12} m_2 s\theta_2$

$G_1 = (r_{01} m_1 + a_1 m_2) g c\theta_1 + r_{12} m_2 g c(\theta_1 + \theta_2)$

$G_2 = r_{12} m_2 g c(\theta_1 + \theta_2)$

$\theta \to \phi_m \quad \to \quad \dot{\underline{\theta}} = J_t \dot{\underline{\phi}}_m$ also because $J_t$ constant: $\underline{\theta} = J_t \underline{\phi}_m \quad \ddot{\underline{\theta}} = J_t \ddot{\underline{\phi}}_m$

$\underline{\tau_m} = J_t^T \underline{\tau_{joint}}$ and $\dot{\underline{\theta}} = J_t \dot{\underline{\phi}}_m$, $\underline{\theta} = J_t \underline{\phi}_m$, $\ddot{\underline{\theta}} = J_t \ddot{\underline{\phi}}_m$ so:

$$\underline{\tau_m} = J_t^T H(J_t\underline{\phi}_m) J_t \ddot{\underline{\phi}}_m + J_t^T V(J_t\underline{\phi}_m, J_t\dot{\underline{\phi}}_m) + J_t^T G(J_t \underline{\phi}_m) + \begin{bmatrix}J_1 & 0\\ 0 & J_2\end{bmatrix}\ddot{\underline{\phi}}_m + F(\dot{\underline{\phi}}_m)$$

$$\underline{\tau_m} = H(\underline{\phi}_m)\ddot{\underline{\phi}}_m + V(\underline{\phi}_m, \dot{\underline{\phi}}_m) + G(\underline{\phi}_m) + \underbrace{\begin{bmatrix}J_1 & 0\\ 0 & J_2\end{bmatrix}}_{2\times 1}\ddot{\underline{\phi}}_m + F(\dot{\underline{\phi}}_m)$$

This is the shaft dynamics already in motor space

Note: $\underline{\theta} = J_t \underline{\phi}_m = \begin{bmatrix}\theta_1\\ \theta_2\end{bmatrix} = \begin{bmatrix}1/N_1 & 0\\ -1/N_1 & 1/N_2\end{bmatrix}\begin{bmatrix}\phi_{m1}\\ \phi_{m2}\end{bmatrix} \Rightarrow$

$\theta_1 = \phi_{m1}/N_1$

$\theta_2 = -\phi_{m1}/N_1 + \phi_{m2}/N_2$

$\dot{\theta}_1 = \dot{\phi}_{m1}/N_1$

$\dot{\theta}_2 = -\dot{\phi}_{m1}/N_1 + \dot{\phi}_{m2}/N_2$

we will make all these substitutions

$$H(\underline{\phi}_m) = \begin{bmatrix}1/N_1 & -1/N_1\\ 0 & 1/N_2\end{bmatrix}\begin{bmatrix}H_{11} & H_{12}\\ H_{12} & H_{22}\end{bmatrix}\begin{bmatrix}1/N_1 & 0\\ -1/N_1 & 1/N_2\end{bmatrix} = \begin{bmatrix}1/N_1 & -1/N_1\\ 0 & 1/N_1\end{bmatrix}\begin{bmatrix}\frac{(H_{11}-H_{12})}{N_1} & H_{12}/N_2\\ \frac{(H_{12}-H_{22})}{N_1} & H_{22}/N_2\end{bmatrix}$$

$$H(\underline{\phi}_m) = \begin{bmatrix}\frac{(H_{11}+H_{22}-2H_{12})}{N_1^2} & \frac{(H_{12}-H_{22})}{N_1 N_2}\\ \frac{(H_{12}-H_{22})}{N_1^2} & \frac{H_{22}}{N_1 N_2}\end{bmatrix}$$

$H_{11} - H_{12} - (H_{12} - H_{22})$

2. a) $$\ddot{\phi}_m = \begin{bmatrix} \ddot{\phi}_{m1}/N_1 \\ -\ddot{\phi}_{m1}/N_1 + \ddot{\phi}_{m2}/N_2 \end{bmatrix}$$

$$V(\phi_m, \dot{\phi}_m) = \begin{bmatrix} 1/N_1 & -1/N_1 \\ 0 & 1/N_2 \end{bmatrix} \begin{bmatrix} 0 & -2h & -h \\ h & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{\theta}_1^2 \\ \dot{\theta}_1\dot{\theta}_2 \\ \dot{\theta}_2^2 \end{bmatrix}$$

will sub later

$$V(\phi_m, \dot{\phi}_m) = \begin{bmatrix} -h/N_1 & -2h/N_1 & -h/N_1 \\ h/N_2 & 0 & 0 \end{bmatrix} \begin{bmatrix} \; \end{bmatrix}$$

$$G = \begin{bmatrix} 1/N_1 & -1/N_1 \\ 0 & 1/N_2 \end{bmatrix} \begin{bmatrix} G_1 \\ G_2 \end{bmatrix} = \begin{bmatrix} \frac{(G_1 - G_2)}{N_1} \\ \frac{G_2}{N_2} \end{bmatrix}$$

Now will combine with motor shaft dynamics to get final matrices
↳ Did not have to be transformed

$$H'(\phi) = \begin{bmatrix} \frac{(H_{11} + H_{22} - 2H_{12})}{N_1^2} + J_1 & \frac{(H_{12} - H_{22})}{N_1 N_2} \\ \frac{(H_{12} - H_{22})}{N_1^2} & \frac{(H_{22})}{N_1 N_2} + J_2 \end{bmatrix} = \begin{bmatrix} I_1 + 2I_2 + m_2 a_1^2 & I_2 + a_1 r_{12} m_2 c \theta_2 \ldots \\ & \end{bmatrix}$$

$$H'(\phi_m) = \begin{bmatrix} \frac{I_1 + m_2 a_1^2}{N_1^2} + J_1 & \frac{a_1 r_{12} m_2 c(-\frac{\phi_{m1}}{N_1} + \frac{\phi_{m2}}{N_2})}{N_1 N_2} \\ \frac{a_1 r_{12} m_2 c(-\frac{\phi_{m1}}{N_1} + \frac{\phi_{m2}}{N_2})}{N_1^2} & \frac{I_2}{N_1 N_2} + J_2 \end{bmatrix} \qquad \ddot{\phi}_m = \begin{bmatrix} \frac{\ddot{\phi}_{m1}}{N_1} \\ -\frac{\ddot{\phi}_{m1}}{N_1} + \frac{\ddot{\phi}_{m2}}{N_2} \end{bmatrix}$$

↳ Could isolate $\ddot{\phi}_m$s

say: $H'(\phi_m) = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \Rightarrow \begin{matrix} a\frac{\ddot{\phi}_{m1}}{N_1} - b\frac{\ddot{\phi}_{m1}}{N_1} + b\frac{\ddot{\phi}_{m2}}{N_2} \\ c\frac{\ddot{\phi}_{m1}}{N_1} - d\frac{\ddot{\phi}_{m1}}{N_1} + d\frac{\ddot{\phi}_{m2}}{N_2} \end{matrix} \Rightarrow$ Thus: $H'(\phi_m) = \begin{bmatrix} \frac{a-b}{N_1} & \frac{b}{N_2} \\ \frac{c-d}{N_1} & \frac{d}{N_2} \end{bmatrix} \quad \ddot{\phi}_m = \begin{bmatrix} \ddot{\phi}_{m1} \\ \ddot{\phi}_{m2} \end{bmatrix}$

2.a)

$$V(\underline{\phi}_m, \dot{\underline{\phi}}_m) = \begin{bmatrix} \frac{-a_1 r_{12} m_2 s\left(-\frac{\phi_{m1}}{N_1}+\frac{\phi_{m2}}{N_2}\right)}{N_1} & \frac{-2 a_1 r_{12} m_2 s\left(-\frac{\phi_{m1}}{N_1}+\frac{\phi_{m2}}{N_2}\right)}{N_1} & \frac{-a_1 r_{12} m_2 s\left(-\frac{\phi_{m1}}{N_1}+\frac{\phi_{m2}}{N_2}\right)}{N_1} \\ \frac{a_1 r_{12} m_2 s\left(-\frac{\phi_{m1}}{N_1}+\frac{\phi_{m2}}{N_2}\right)}{N_2} & 0 & 0 \end{bmatrix} \begin{bmatrix} \left(\dot{\phi}_{m1}/N_1\right)^2 \\ \left(\dot{\phi}_{m1}/N_1\right)\left(-\frac{\dot{\phi}_{m1}}{N_1}+\frac{\dot{\phi}_{m2}}{N_2}\right) \\ \left(-\frac{\dot{\phi}_{m1}}{N_1}+\frac{\dot{\phi}_{m2}}{N_2}\right)^2 \end{bmatrix}$$

$$G(\underline{\phi}_m) = \begin{bmatrix} \frac{(r_{01} m_1 + a_1 m_2) g c\left(\frac{\phi_{m1}}{N_1}\right)}{N_1} \\ \frac{r_{12} m_2 g c\left(\frac{\phi_{m2}}{N_2}\right)}{N_2} \end{bmatrix}$$

$\theta_1 + \theta_2 = \cancel{\frac{\phi_{m1}}{N_1}} - \frac{\phi_{m1}}{N}, \frac{\phi_{m2}}{N_2}$

$$F(\dot{\underline{\phi}}_m) = \begin{bmatrix} f(\dot{\phi}_{m1}) \\ f(\dot{\phi}_{m2}) \end{bmatrix}$$

$f$ could be:
$f_i = b_i \dot{\theta}_i + c_i \,\mathrm{sgn}(\dot{\theta})$
viscous friction, coulomb friction

Thus: $\underline{\tau}_m = H'(\underline{\phi}_m)\ddot{\underline{\phi}}_m + V(\underline{\phi}_m, \dot{\underline{\phi}}_m) + G(\underline{\phi}_m) + F(\dot{\underline{\phi}}_m)$

If we want to isolate $\dot{\phi}$ terms for $V(\underline{\phi}_m, \dot{\underline{\phi}}_m)$:

Say $V(\underline{\phi}_m, \dot{\underline{\phi}}_m) = \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix} \begin{bmatrix} \dot{\phi}_{m1}^2/N_1^2 \\ -\dot{\phi}_{m1}^2/N_1^2 + \frac{\dot{\phi}_{m1}\dot{\phi}_{m2}}{N_1 N_2} \\ \dot{\phi}_{m1}^2/N_1^2 + \dot{\phi}_{m2}^2/N_2^2 - 2\frac{\dot{\phi}_{m1}\dot{\phi}_{m2}}{N_1 N_2} \end{bmatrix}$

Thus:

$$V(\underline{\phi}_m, \dot{\underline{\phi}}_m) = \begin{bmatrix} \frac{a-b+c}{N_1^2} & \frac{b-2c}{N_1 N_2} & \frac{c}{N_2^2} \\ \frac{d-e+f}{N_1^2} & \frac{e-2f}{N_1 N_2} & \frac{f}{N_2^2} \end{bmatrix} \begin{bmatrix} \dot{\phi}_{m1}^2 \\ \dot{\phi}_{m1}\dot{\phi}_{m2} \\ \dot{\phi}_{m2}^2 \end{bmatrix}$$

2. b)

reflect motor friction and inertia to joint space then we can combine in joint space

Arm Dynamics w/o motor dynamics in joint space $\rightarrow \underline{\tau}_{joint} = H\ddot{\underline{\theta}} + V(\underline{\theta}, \dot{\underline{\theta}}) + G(\underline{\theta})$ joint space

motor friction (2) and motor inertia (1) in motor space $\rightarrow \begin{bmatrix} J_1 & 0 \\ 0 & J_2 \end{bmatrix} \ddot{\underline{\phi}}_m + F(\dot{\underline{\phi}}_m)$

Reflect into joint space: $\phi_m \rightarrow \theta$

$$J_t^{-T} \begin{bmatrix} J_1 & 0 \\ 0 & J_2 \end{bmatrix} J_t^{-1} \ddot{\underline{\theta}} + J_t^{-T} F(J_t^{-1} \dot{\underline{\theta}})$$

$$J_t^{-1} = \begin{bmatrix} N_1 & 0 \\ N_2/N_1 & N_2/N_1 \end{bmatrix} \qquad J_t^{-T} = \begin{bmatrix} N_1 & N_2/N_1 \\ 0 & N_2/N_1 \end{bmatrix}$$

(1)
$$\begin{bmatrix} N_1 & N_2/N_1 \\ 0 & N_2/N_1 \end{bmatrix} \begin{bmatrix} J_1 & 0 \\ 0 & J_2 \end{bmatrix} \begin{bmatrix} N_1 & 0 \\ N_2/N_1 & N_2/N_1 \end{bmatrix} \ddot{\underline{\theta}} = \begin{bmatrix} N_1 & N_2/N_1 \\ 0 & N_2/N_1 \end{bmatrix} \begin{bmatrix} N_1 J_1 & 0 \\ \frac{N_2}{N_1} J_2 & \frac{N_2}{N_1} J_2 \end{bmatrix}$$

$$= \begin{bmatrix} N_1^2 J_1 + \frac{N_2^2}{N_1^2} J_2 & \frac{N_2^2}{N_1^2} J_2 \\ \frac{N_2^2}{N_1^2} J_2 & \frac{N_2^2}{N_1^2} J_2 \end{bmatrix} \ddot{\underline{\theta}}$$

(2)
$$J_t^{-T} F(J_t^{-1} \dot{\underline{\theta}}_m) = \begin{bmatrix} N_1 & N_2/N_1 \\ 0 & N_2/N_1 \end{bmatrix} \begin{bmatrix} f(\dot{\phi}_{m1}) \\ f(\dot{\phi}_{m2}) \end{bmatrix}$$

where $\dot{\underline{\phi}}_m = J_t^{-1} \dot{\underline{\theta}}$

$$\dot{\underline{\phi}}_m = \begin{bmatrix} N_1 & 0 \\ N_2/N_1 & N_2/N_1 \end{bmatrix} \dot{\underline{\theta}}$$

$$= \begin{bmatrix} N_1 f(N_1 \dot{\theta}_1) + N_2/N_1 \, f\left(\frac{N_2}{N_1}\dot{\theta}_1 + \frac{N_2}{N_1}\dot{\theta}_2\right) \\ N_2/N_1 \, f\left(\frac{N_2}{N_1}\dot{\theta}_1 + \frac{N_2}{N_1}\dot{\theta}_2\right) \end{bmatrix}$$

$\dot{\phi}_{m1} = N_1 \dot{\theta}_1$
$\dot{\phi}_{m2} = N_2/N_1 \, \dot{\theta}_1 + N_2/N_1 \, \dot{\theta}_2$

Now combining motor inertia terms into $H(\underline{\theta})$ to make $H'(\underline{\theta})$

$$\boxed{H'(\theta) = \begin{bmatrix} I_1 + I_2 + m_2 a_1^2 + 2 a_1 r_{12} m_2 C\theta_2 + N_1^2 J_1 + \frac{N_2^2}{N_1^2} J_2 & I_2 + a_1 r_{12} m_2 C\theta_2 + \frac{N_2^2}{N_1^2} J_2 \\ I_2 + a_1 r_{12} m_2 C\theta_2 + \frac{N_2^2}{N_1^2} J_2 & I_2 + \frac{N_2^2}{N_1^2} J_2 \end{bmatrix}}$$

$$\boxed{\ddot{\underline{\theta}} = \begin{bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{bmatrix}}$$

2\. b)

$$\boxed{V(\underline{\theta},\dot{\underline{\theta}}) = \begin{bmatrix} -2(a_1 r_{12} m_2 c\theta_2) & -(a_1 r_{12} m_2 c\theta_2) \\ -(a_1 r_{12} m_2 c\theta_2) & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{\theta}_1^2 \\ \dot{\theta}_1 \dot{\theta}_2 \\ \dot{\theta}_2^2 \end{bmatrix}}$$

$$\boxed{G(\underline{\theta}) = \begin{bmatrix} (r_{01} m_1 + a_1 m_2) g c\theta_1 + r_{12} m_2 g c(\theta_1+\theta_2) \\ r_{12} m_2 g c(\theta_1+\theta_2) \end{bmatrix}}$$

$$\boxed{F(\dot{\underline{\theta}}) = \begin{bmatrix} N_1 f(N_1 \dot{\theta}_1) + N_2/N_1\, f\left(\frac{N_2}{N_1}\dot{\theta}_1 + \frac{N_2}{N_1}\dot{\theta}_2\right) \\ N_2/N_1\, f\left(\frac{N_2}{N_1}\dot{\theta}_1 + \frac{N_2}{N_1}\dot{\theta}_2\right) \end{bmatrix}}$$

Thus $\Rightarrow$ $\underline{\tau}_{joint} = H'(\underline{\theta})\ddot{\underline{\theta}} + V(\underline{\theta},\dot{\underline{\theta}}) + G(\underline{\theta}) + F(\dot{\underline{\theta}})$

c) $\underline{\tau}_m = H'(\underline{\phi}_m)\ddot{\underline{\phi}}_m + V(\underline{\phi}_m, \dot{\underline{\phi}}_m) + G(\underline{\phi}_m) + F(\dot{\underline{\phi}}_m)$

Note $\underline{\phi} = J_g \underline{\phi}_m$

$\underline{\phi}_m = J_g^{-1}\underline{\phi}$

$$\underline{\tau}_e = J_g^{-T} H'(J_g^{-1}\underline{\phi}) J_g^{-1}\ddot{\underline{\phi}} + J_g^{-T} V(J_g^{-1}\underline{\phi}, J_g^{-1}\dot{\underline{\phi}}) + J_g^{-T} G(J_g^{-1}\underline{\phi}) + J_g^{-T} F(J_g^{-1}\dot{\underline{\phi}})$$

Note: $J_g = \begin{bmatrix} 1/N_1 & 0 \\ 0 & 1/N_2 \end{bmatrix}$ $J_g^{-1} = \begin{bmatrix} N_1 & 0 \\ 0 & N_2 \end{bmatrix}$ $J_g^{-T} = \begin{bmatrix} N_1 & 0 \\ 0 & N_2 \end{bmatrix}$

Thus we can multiply using properties of a diagonal matrix

$J_g^{-1}\underline{\phi} = \underline{\phi}_m$

$\phi_{m1} = N_1 \phi_1$

$\phi_{m2} = N_2 \phi_2$

Times $N_1$ on top row

Times $N_2$ on bottom row $\rightarrow$

$$\boxed{H'(\underline{\phi}) = \begin{bmatrix} \dfrac{I_1 + m_2 a_1^2}{N_1} + J_1 N_1 & \dfrac{a_1 r_{12} m_2 c(-\phi_1+\phi_2)}{N_2} \\ \dfrac{N_2 a_1 r_{12} m_2 c(-\phi_1+\phi_2)}{N_1^2} & \dfrac{I_2}{N_1} + N_2 J_2 \end{bmatrix}}$$

$$\boxed{\ddot{\underline{\phi}} = \begin{bmatrix} \ddot{\phi}_1 \\ -\ddot{\phi}_1 + \ddot{\phi}_2 \end{bmatrix}}$$

$\hookrightarrow$ Could isolate $\ddot{\phi}$'s

Say: $H'(\underline{\phi}) = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \Rightarrow$ Thus: $\boxed{H'(\underline{\phi}) = \begin{bmatrix} a-b & b \\ c-d & d \end{bmatrix}}$ $\boxed{\ddot{\underline{\phi}}_m = \begin{bmatrix} \ddot{\phi}_1 \\ \ddot{\phi}_2 \end{bmatrix}}$

2. c)

$$V(\underline{\phi}, \underline{\dot{\phi}}) = \begin{bmatrix} -a_1 r_{12} m_2 s(-\phi_1+\phi_2) & -2a_1 r_{12} m_2 s(-\phi_1+\phi_2) & -a_1 r_{12} m_2 s(-\phi_1+\phi_2) \\ a_1 r_{12} m_2 s(-\phi_1+\phi_2) & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{\phi}_1^2 \\ (\dot{\phi}_1)(-\dot{\phi}_1+\dot{\phi}_2) \\ (-\dot{\phi}_1+\dot{\phi}_2)^2 \end{bmatrix}$$

can isolate $\dot{\phi}$s by multiplying out then

$$G(\underline{\phi}) = \begin{bmatrix} (r_{01}m_1 + a_1 m_2) g\, c(\phi_1) \\ r_{12} m_2 g\, c(\phi_2) \end{bmatrix}$$

$$F(\underline{\dot{\phi}}) = \begin{bmatrix} N_1 f(N_1 \dot{\phi}_1) \\ N_2 f(N_2 \dot{\phi}_2) \end{bmatrix}$$

Thus: $\underline{\tau_e} = H'(\underline{\phi})\,\ddot{\underline{\phi}} + V(\underline{\phi}, \underline{\dot{\phi}}) + G(\underline{\phi}) + F(\underline{\dot{\phi}})$

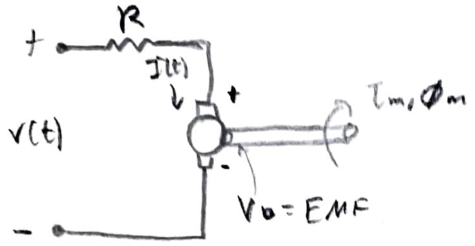
If we want to isolate $\dot{\phi}$ terms for $V(\underline{\phi}, \underline{\dot{\phi}})$:

say $V(\underline{\phi}, \underline{\dot{\phi}}) = \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix} \begin{bmatrix} \dot{\phi}_1^2 \\ -\dot{\phi}_1^2 + \dot{\phi}_1\dot{\phi}_2 \\ \dot{\phi}_1^2 + \dot{\phi}_2^2 - 2\dot{\phi}_1\dot{\phi}_2 \end{bmatrix} = \begin{bmatrix} a\dot{\phi}_1^2 - b\dot{\phi}_1^2 + b\dot{\phi}_1\dot{\phi}_2 + c\dot{\phi}_1^2 + c\dot{\phi}_2^2 - 2c\dot{\phi}_1\dot{\phi}_2 \\ d\dot{\phi}_1^2 - e\dot{\phi}_1^2 + e\dot{\phi}_1\dot{\phi}_2 + f\dot{\phi}_1^2 + f\dot{\phi}_2^2 - 2f\dot{\phi}_1\dot{\phi}_2 \end{bmatrix}$

Thus:

$$V(\underline{\phi}, \underline{\dot{\phi}}) = \begin{bmatrix} a-b+c & b-2c & c \\ d-e+f & e-2f & f \end{bmatrix} \begin{bmatrix} \dot{\phi}_1^2 \\ \dot{\phi}_1\dot{\phi}_2 \\ \dot{\phi}_2^2 \end{bmatrix}$$

3.




KVL: $V(t) - R\,I(t) - V_b = 0$

$$\tau_m = K_t\,I(t)$$
$$V_b = K_b\,\dot{\phi}_m$$

a) $\tau_m = K_t\,I(t)$

$\tau_m \to \tau_e$

$$\tau_e = J_g^{-T}\,\tau_m$$

$$\tau_e = \begin{bmatrix} N_1 & 0 \\ 0 & N_2 \end{bmatrix} \begin{bmatrix} K_t I_1(t) \\ K_t I_2(t) \end{bmatrix} = \begin{bmatrix} N_1 K_t I_1(t) \\ N_2 K_t I_2(t) \end{bmatrix}$$

Thus:
$$\begin{bmatrix} A_1 I_1(t) \\ A_2 I_2(t) \end{bmatrix} = H'(\phi)\ddot{\phi} + V(\phi,\dot{\phi}) + G(\phi) + F(\dot{\phi})$$

where $\boxed{A_1 = N_1 K_t}$ and $\boxed{A_2 = N_2 K_t}$

b) Rearrange KVL: $I(t) = \dfrac{V(t) - V_b}{R} = \dfrac{V(t) - K_b\,\dot{\phi}_m}{R}$

$$\tau_m = \frac{K_t}{R}V(t) - \frac{K_t K_b}{R}\dot{\phi}_m$$

$$\dot{\phi}_m = J_g^{-1}\dot{\phi}$$
$$\dot{\phi}_{m1} = N_1\,\dot{\phi}_1$$
$$\dot{\phi}_{m2} = N_2\,\dot{\phi}_2$$

voltage term

$$\tau_e = \begin{bmatrix} N_1 & 0 \\ 0 & N_2 \end{bmatrix}\tau_m = \begin{bmatrix} N_1\left[\frac{K_t}{R}V(t) - \frac{K_t K_b}{R}N_1\dot{\phi}_1\right] \\ N_2\left[\frac{K_t}{R}V(t) - \frac{K_t K_b}{R}N_2\dot{\phi}_2\right] \end{bmatrix}$$

friction term

Now:
$$\boxed{F'(\dot{\phi}) = \begin{bmatrix} N_1 f(N_1\dot{\phi}_1) - \dfrac{N_1^2 K_t K_b}{R}\dot{\phi}_1 \\ N_2 f(N_2\dot{\phi}_2) - \dfrac{N_2^2 K_t K_b}{R}\dot{\phi}_2 \end{bmatrix}}$$

Thus:
$$\begin{bmatrix} A_1 V_1(t) \\ A_2 V_2(t) \end{bmatrix} = H'(\phi)\ddot{\phi} + V(\phi,\dot{\phi}) + G(\phi) + F'(\dot{\phi})$$

Where: $\boxed{A_1 = \dfrac{N_1 K_t}{R}}$ $\boxed{A_2 = \dfrac{N_2 K_t}{R}}$