Project:Deadline May 14

1. Solve the advection equation $u_t - u_x = 0$ for $x \in (0,1)$ and $T = 2$ with FTCS, BTCS and Leapfrog schemes. The initial data is given by

$$u(0,x) = e^{\frac{-(x-0.2)^2}{0.01}},$$

use the periodic boundary condition. Discuss about the stability limits for these schemes? Hand in plots of the numerical solution together with the exact solution.

2. Program the central finite difference method for

$$\begin{cases} (\beta(x)u_x)_x - \gamma(x)u = f(x), & x \in (0,1) \\ u(0) = 1 \quad au(1) + bu'(1) = c; \end{cases}$$

using a uniform grid and the central finite difference scheme

$$\frac{\beta_{i+\frac{1}{2}}\frac{U_{i+1}-U_i}{h} - \beta_{i-\frac{1}{2}}\frac{U_i - U_{i_1}}{h}}{h} - \gamma_i U_i = f_i.$$

- Test your code for the case where

$$\beta(x) = 1 + x^2, \quad \gamma(x) = x, a = 2, b = -3,$$

and the other functions or parameters are determined from the exact solution
$$u(x) = e^{-x}(x-1)^2.$$

- Plot the computed solution and the exact solution, and the error for a particular grid $n = 100$.

- (Optional: This part has extra credit) Do the grid refinement analysis, to determine the order of accuracy of the global solution.
- Can your code handle the case when $a = 0$ or $b = 0$?
- If the central finite difference scheme is used for the equivalent differential equation

$$\beta u^{''} + \beta' u^{'} - \gamma(x)u = f(x).$$

what are the advantages or disadvantages?

3. (This part is for NAEDP students) Consider the one-phase obstacle problem on domain $\Omega = (-2, 2)$. Let the obstacle is given by

$$\phi(x) = \begin{cases} 0 & -2 \le x \le -\frac{7}{4}, \\ -(x+\frac{7}{4})(x+\frac{1}{2}) & -\frac{7}{4} \le x \le -\frac{1}{2}, \\ 0 & -\frac{1}{2} \le x \le 0, \\ -2x^2+3x & 0 \le x \le \frac{3}{2} \\ 0 & \frac{3}{2} \le x \le 2. \end{cases} \tag{1}$$

The boundary value $g = 0$.

- Solve the one phase problem

$$\min(-u_{xx} - f, u - \phi) = 0,$$

for $f = 0, 10$ and the number ok unknowns $n = 50, 100, 500, 1000$. Check the rate of convergence.

- Implement Semi-Smooth Newton Method.
- (This part is for student have Numerical Analysis of Partial Differential Equations) Use Two level Multi-grid method.

4. Implement and compare the Crank–Nicolson and the ADI method using Matlab for the heat equation:

$$\begin{cases} u_t = u_{xx} + u_{yy} + f(x, y, t), 0 < x < 1, 0 < y < 1 \quad 0 < t \le 1, \\ u(x, y, 0) = u_0(x, y), \end{cases}$$

For initial value and Dirichlet boundary conditions and f choose

$$u = (1 + t)(x^2 + y^2),$$

to test and debug your code. Write a short report about the two methods. Your discussion should include the grid refinement analysis (with a fixed final time, say $T = 1$, error and solution plots, comparison of cpu time, and any conclusions you can draw from your results.

5. (This part is for MCF students) Consider a call option with the following parameters:

$$K = 100, \quad r = 0.05, \quad \sigma = 0.2, \quad \text{and } T = 1.$$

- Calculate the values for European call option for $S = 80, 85, \cdots, 135, 140$. You can use the formulas given at the Page 40 of the file numeric_pde.pdf. (Note that the strike price or exercise price is denoted by E).
- Implement explicit and Crank-Nicholson schemes to approximate the values of option. For explicit scheme use $dt = 0.0001$ $ds = 1$ and for Crank-Nicholson use $dt = 0.01$ and $ds = 0.5$. Compare approximated values with values from previous step and obtain the accuracy.
- Explain the difference between American option and European option.
- Use Finite Difference scheme to approximate American call option.