Estimating the Zero Coupon Yield Curve[1]

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The term structure of interest rates – the relationship between interest rates in the economy and the term to maturity – forms the basis for the valuation of all fixed income instruments. Modeled as a series of cashflows due at different points of time in the future, the underlying price of a fixed income security can be calculated as the net present value of the stream of cashflows. Each cashflow, in such a formulation, has to be discounted using the interest rate for the associated term to maturity. Arriving at the appropriate set of interest rates - the 'term structure' or the 'zero coupon yield curve' (ZCYC) – for the Indian debt market, is the objective of the present exercise.

Empirical analysis of the term structure of interest rates has a long history in developed countries like the US and UK [see McCulloch (1971, 1975), Carleton & Cooper (1976), Robichek & Neibuhr (1970), Schaefer (1981)]; work on term structure estimation in India, by comparison, is of more recent origin [Nag & Ghose (2000), Thomas & Saple (2000), Subramanian (2000)]. The primary reason for the latter can be traced to the dormant debt market in India prior to the phased implementation of financial sector reforms since 1991.

Among the major reforms that have been initiated in the Government securities market, the most significant, for the purpose of the present study, is the gradual shift to market-related rates of interest on Government borrowings. This was a great step forward from the pre-reform period when the return differential between different Government securities was set by statute [see Krishnan (1989) for details]. A paradigm shift in this context has been the virtual elimination of the automatic deficit financing route via the phasing out of ad-hoc T-Bills from 1997-98, and the replacement in its place of a system of Ways and Means Advances (WMA). The Government has thus had to access the market for major part of its borrowings since 1997-98. For the Reserve Bank of India (RBI), this has meant, in its role as debt manager for the Government, a gradual move towards a market-aligned yield curve to enable success of the Government's borrowing program. An estimate of the sovereign yield curve would, in such a context, provide an indication of the prices (coupons) that can be reasonably expected at primary auctions.

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[1] This is an abridged and simplified version of the paper titled 'Term Structure of Interest Rates in India: Issues in Estimation and Pricing' by Darbha, Dutta Roy & Pawaskar (2000). The document in its present form has been prepared for the express purpose of helping the reader interpret the daily estimates of the term structure provided by NSE.

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A second policy move of major significance has been the gradual reduction in statutory pre-emptions via SLR (Statutory Liquidity Ratio) prescriptions. Compulsory SLR holdings had earlier ensured a captive market for Government securities. As against which, for most part of 1997-98 for instance, banks had invested in Government securities in excess of the prescribed 25 per cent for lack of more profitable alternatives, reflecting conscious portfolio choice in favour of these instruments. An analysis of the term structure would, in this scenario, enable more efficient portfolio decisions.

Once an estimate of the term structure based on default-free government securities is obtained, it can be used to value government securities that do not trade on a given day, to construct a Government bond index, or to provide default-free valuations for corporate bonds. It can be used to price all non-sovereign fixed income instruments after adding an appropriate credit spread. Estimates of the ZCYC at regular intervals over a period of time provides us with a time-series of the interest rate structure in the economy, which can be used to analyse the extent of impact of monetary policy. Time series of ZCYC also form an input for Value at Risk (VaR) systems for fixed income systems and portfolios.

Considering the usefulness of a model of the term structure and the sparse empirical literature on the subject in the Indian context, the present exercise attempts to provide a framework for the estimation of the daily term structure taking into account important institutional details related to the Indian debt market. Section II provides a brief description of the theoretical framework underlying the econometric estimation of the term structure of interest rates. Section III provides the empirical specification of the model estimated[3]. An account of the data and related estimation issues is presented in Section IV. Results of our exercise are presented in Section V.

II. Theoretical Framework

Yield is commonly defined as the return on capital invested in fixed income earning securities. The yield on any instrument has two distinct aspects: (i) a regular income in the form of interest income (coupon payments) and (ii) changes in the market value of fixed interest bearing securities.

Some of the major factors that lead to yield differentials among fixed interest bearing securities are (i) maturity period, (ii) coupon rates, (iii) tax rates (iv) marketability and (v) risk factor. Government securities, considered the safest form of investment, also carry with them some element of risk in terms of 'purchasing power risk' and 'interest rate risk'. Purchasing power risk arises due to inflationary

[3] Readers interested in the details of the econometric methodology may please refer the full paper.

tendencies in the economy and leads to a consideration of the real rate of return[4]. Interest rate risk arises on account of fluctuating prices of securities which, in turn, requires a re-adjustment of portfolio.

A yield curve depicts the relationship between yield and maturity of a set of 'similar' securities, as on a given date. Derivation of the 'true' term structure requires a sample of bonds that are identical in every respect except in term to maturity. While Government securities, in practice, differ by coupon rates, nonetheless, these come closest to being identical[5], hence most empirical studies have concentrated on this segment of the securities market.

Valuation of a Bond

The valuation of a bond by an investor can be expressed in terms of three alternative present value formulations.

(i) Spot Interest Rates

Suppose that the *spot rates of interest* (r_t) for every future period are known, then the *present value* of an m-period bond making a series of coupon payments C every period and with redemption value R is:

Equation 1

$$PV = \frac{C}{(1+r_1)} + \frac{C}{(1+r_2)^2} + \ldots\ldots + \frac{C+R}{(1+r_m)^m}$$

The spot interest rate r_t is the interest rate applicable on a cash payment due in t periods. The set of spot rates is the *term structure of interest rates*. The term structure, also referred to as the zero coupon yield curve (ZCYC) is therefore a function that defines a relationship between maturity periods and associated spot interest rates.

The present value relation / bond price equation (1) can also be written in terms of discount factors [$\delta(t) = 1/(1+r_t)^t$] as follows:

Equation 2

$$PV = d(t_1)C + d(t_2)C + \ldots\ldots\ldots\ldots d(t_m)(C+R)$$
$$= C\sum_{j=1}^{m} d(t_i) + d(t_m)R$$

[4] Capital indexed bonds are intended to insure investors against this type of risk. The Government of India has till date issued only one bond of this type – the 6% capex bonds issued in December 1997.

[5] As compared to, say, corporate bonds which also differ in terms of creditworthiness of the issuer or default risk.

The discount function describes the present value of 1 unit payable at any time in the future. It follows that if an instrument exists that provides a single, unit cashflow t years into the future, its price should correspond to the value of the discount function at that point.

(ii) Yield to maturity

The *yield to maturity* (YTM) is the bond's internal rate of return – it is the single interest rate at which the price of a bond is equal to the present value of the stream of cashflows. It is derived from the bond price equation, constraining all cashflows to be discounted at a single rate. The *yield relationship* is expressed as:

Equation 3

$$PV = \frac{C}{(1+i)} + \frac{C}{(1+i)^2} + \ldots\ldots.. + \frac{C+R}{(1+i)^m}$$

(iii) Forward interest rates

The present value relation can also be explained in terms of *forward interest rates.* While the spot rate is the single rate of return applicable over all discrete periods from the present until the end of period j, and is defined as the average of the one-period rates applicable to periods 1,2….j, the forward interest rates describe the marginal return over period j. If γ_j represents the interest rate applicable from the end of period j-1 to end of period j, then

$1/\delta(1) = (1+r_1) = (1+\gamma_1)$

$1/\delta(2) = (1+r_2)^2 = (1+\gamma_1)(1+\gamma_2)$

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$1/\delta(j) = (1+r_j)^j = (1+\gamma_1)(1+\gamma_2)\ldots..(1+\gamma_j)$

In terms of *forward rates*, the present value relationship can be formulated as:

Equation 4

$$PV = \frac{C}{1+\gamma_1} + \frac{C}{(1+\gamma_1)(1+\gamma_2)} + \ldots\ldots. + \frac{C+R}{\prod_{1}^{m}(1+\gamma_j)}$$

The present value relation (1) forms the basis for the specification of the bond pricing equation for empirical estimation of the term structure. With equation (1) specified as the 'model' price, and taking observed prices as data, estimation involves arriving at an entire set of spot rate / maturity pairs from which the appropriate rates for the discounting of any stream of cashflows can be chosen.

Accrued Interest

There are in practice a number of factors that complicate the specification of equation (1). In the formulation above, the first coupon payment is assumed to be due in exactly one period's time. In reality, while coupon payments are only made at regular intervals during the year, bonds can be traded on any working day. Whenever a bond changes hands on a day that is not a coupon payment date, the valuation of the bond will reflect the proximity of the next coupon payment date. This is effected by the payment of *accrued interest*[6] to compensate the seller for the period since the last coupon payment date during which the seller has held the bond but for which no coupon payment is made. The *total* or *dirty* price paid by the buyer can thus be decomposed into two components: accrued interest and the *clean* or *quoted* price. A similar adjustment needs to be made to the discounting of coupon payments since, when a bond is traded on a day that is not a coupon payment date, cashflows are no longer an exact number of coupon periods into the future. The time to the earliest coupon payment date is then computed as the proportion of a coupon payment period represented by the time from the trade settlement date to the next coupon payment date[7]. The bond price equation should accordingly be modified to account for accrued interest:

Equation 5

$$p + ai = \frac{C/v}{(1 + r(t_1)/v)^{vt_1}} + \frac{C/v}{(1 + r(t_2)/v)^{vt_2}} + ...\frac{C/v + R}{(1 + r(t_m)/v)^{vt_m}}$$

where:

p = clean price of the bond

ai = accrued interest

C = annual coupon payment

R = redemption payment

t_m = maturity of bond (in years)

$r(t_j)$ = spot rate applicable to payment j due at time t_j

v = frequency of coupon payments

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[6] Computed as ai = t_0C, where C is the (annual) coupon on the bond and t_0 the proportion of a period passed since the last coupon payment was made.

[7] The market conventions to be followed for the computation of accrued interest and time to first coupon payment/maturity vary from one country to another. The commonly used ones are actual/365 and the 30/360 conventions. In the former, the actual number of days between two required dates is used for the computation, with a year comprising 365 days. In the latter, each month comprises 30 days, and a year comprises 360 days.

Note that, when we specify an estimable form of the model in terms of equation (5), the entire set of interest rates is unknown. Specification of a relation between maturity and interest rates therefore forms the first step in the estimation of the ZCYC.

III. Empirical Specification and Econometric Methodology

The Nelson-Siegel formulation [Nelson & Siegel (1987)] that we adopt in the present exercise specifies a parsimonious representation of the forward rate function given by

Equation 6

$$f(m,b) = b_0 + b_1 * \exp(-m/t) + b_2[(m/t) * \exp(-m/t)]$$

where 'm' denotes maturity and b=[β_0, β_1, β_2 and τ] are parameters to be estimated.

The forward rate function can be integrated to obtain the relevant spot rate function, the term structure:

Equation 7

$$r(m,b) = b_0 + (b_1 + b_2) * [1 - \exp(-m/t)]/(m/t) - b_2 * \exp(-m/t)$$

In the spot rate function, the limiting value of *r(m,b)* as maturity gets large is β_0 which therefore depicts the long term component (which is a non-zero constant). The limiting value as maturity tends to zero is $\beta_0 + \beta_1$, which therefore gives the implied short-term rate of interest.

With the above specification of the spot rate function, the PV relation can now be specified using the discount function[8] given by

Equation 8

$$d(m,b) = \exp\left(-\frac{r(m,b) * m}{100}\right)$$

The present value arrived at is the estimated/model price (**p_est**) for each bond. It is common to observe secondary market prices (**pmkt**) that deviate from this value. For the purpose of empirical estimation of the unknown parameters in equation 7, we postulate that the observed market price of a bond deviates from its underlying valuation by an error term e_i, which gives us the estimable relation:

Equation 9

$$pmkt_i = p\_est_i + e_i$$

This equation is estimated by minimising the sum of squared price errors. The steps followed in the estimation procedure are as follows:

i. A vector of starting parameters (β_0, β_1, β_2 and τ) is selected,
ii. The discount factor function is determined using these starting parameters,
iii. This is used to determine the present value of the bond cash flows and thereby to determine a vector of *starting* 'model' bond prices,
iv. Numerical optimisation procedures are used to estimate a set of parameters (under a given set of constraints *viz*. non-negativity of long run and short run interest rates) that minimise the sum of squared price errors,
v. The estimated set of parameters are used to determine the spot rate function (Equation 7) and therefrom the 'model' prices (this is the first set of results we compute for each day),
vi. These 'model' prices are used to compute associated 'model' YTMs for each bond (this is the second set of results).

IV. Data and Estimation Issues

As on August 18, 2000, there were 110 Government of India (GoI) dated securities outstanding, with maturity dates ranging from September 29, 2000 (11.4% 2000) to April 22, 2020 (10.7% 2020). Daily data on secondary market trades in these securities are available from two sources – trades reported on NSE-WDM and those reported on the Subsidiary General Ledger account of the Reserve Bank of India (RBI-SGL). NSE-WDM provides information on trades that are conducted as negotiated deals and subsequently reported to the exchange. The RBI-SGL data consist of trades that are reported to the RBI, information on which is disseminated only on the day of settlement.

The present exercise uses data from the NSE-WDM, which constitutes about 70 per cent of the secondary market volume. On a given trade date, the data for each individual trade include information on traded price, traded volume, settlement date, days to settlement, issue date and maturity date. This information, together with details of cashflows[9] for each bond, are used in the empirical estimation. Government dated securities in India make semi-annual coupon payments. Market conventions require computation of accrued interest on a 30/360 basis for instruments with residual maturity exceeding a year and on actual/365 basis otherwise (this includes Treasury Bills), and these are adhered to in the

[8] This is the continuous form of the discount function specified in Equation 2.
[9] This includes coupon rate and interest payment dates.

computation of coupon accrual and term to maturity of cashflows. Accrued interest, time to coupon payments and term to maturity are calculated with reference to the settlement date.

Most of the existing empirical literature on term structure estimation uses volume-weighted average (VWA) prices (for multiple trades in the same security) as indicative of ‘the’ price for a particular security on a particular day. While this method would provide reasonably accurate prices in deep, liquid and transparent markets, the loss of information on this count would be substantial in the Indian case. First, while the number of securities that are actively traded on a given day vary in the range of 13-25, it is not uncommon to observe 45-60 trades in securities considered attractive at that particular point of time. The data, in such situations, reveal significant dispersion in prices across trades in the same bond. Trades in the 12.5% 2004 security on August 9, 2000 provide a case in point. There were 67 trades in this security on this day, with prices ranging from Rs.108.9669 to Rs.109.5264. It is important to re-iterate at this point that the information relates to negotiated deals, where the scope for price discovery is limited, and this contributes to the observed dispersion in prices for different trades in the same security. Empirical estimation is complicated by the fact that the observed ‘traded prices’ are not time-stamped, which makes it difficult to control for the impact of intra-day ‘news’ in the data. However, there are important institutional details related to trade and settlement in the Indian bond market that could, to a large extent, explain such intra-security price variations. To highlight these aspects, the present exercise deviates from existing empirical literature and uses prices for each individual trade for each bond traded on a given day.

There are various factors to which intra-security variation in prices can be attributed. Within the T+5 system in vogue, trades negotiated on a given day can have settlement dates varying from current date to 5 days hence[10]. There are two mechanisms through which this can exert an impact on the price. First, for trades that do not settle on the trade date, the futures price (price for a deal that is negotiated on the trade date but settles T days in the future) differs from the spot price (price negotiated for a trade that settles on the trade date itself) by the net cost of carry. From the point of view of the seller, the opportunity cost involved in settling a deal T days into the future is approximated by the foregone return in the call money market (we proxy the call rate by the short-term rate ($\beta_0 + \beta_1$) implied by the estimated term structure), while the return is given by the coupon that accrues for these days. If the net cost of carry is positive (negative), the negotiated futures price will be higher (lower) than the spot price.

The second factor that is expected to be built into the contract is expectations about the likely directionality of interest rates, when the term structure is expected to undergo a significant change by the

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[10] While it is not uncommon to have 20-30 per cent of trades settling the next day, settlement days beyond 1 day are infrequent.

time the deal is settled. To discount the cashflows for observations that do not settle on the current day, therefore, the appropriate spot rates to be used are those that are expected to prevail on the settlement date. These are derived from the estimated term structure using the following relation

$$r_{t1}^{t2} = \left(\frac{(1+r_0^{t2})^{t2}}{(1+r_0^{t1})^{t1}} \right)^{1/(t2-t1)} - 1$$

where r_0^{t2} denotes the spot rate for maturity date t2 as on date 0, r_0^{t1} the spot rate for maturity date t1 as on date 0 and r_{t1}^{t2} denotes the spot rate for maturity date t2 expected to prevail at date t1.

V. Results

Estimate of the Term Structure

The estimated parameters for September 25, 2000 [$\beta_0 = 15.1613$, $\beta_1 = -4.8679$, $\beta_2 = -5.2543$ and $\tau = 4.6369$] have been used to plot the accompanying graph (refer Chart). The estimated parameters imply risk-free short-term and long-term rates of 10.2934 and 15.1613 respectively. Note that the long-term rate is essentially the rate of interest associated with an infinite maturity. For more realistic long-term rates (say 15 years), the rate of interest as implied by the estimated ZCYC is 12.37 per cent (refer sheet ‘calc’ in the accompanying Excel file).

Plots of the estimated term structure for any particular day can be obtained by following the procedure below:

i. Create a series of maturity values; for instance 1 to 25 years, with step lengths of (say) 0.5 years (refer column 1 in sheet named ‘calc’),
ii. For each maturity, use the estimated parameters for the required day (refer sheet named ‘para_sep’) to derive corresponding spot rates using equation 7 (refer column 2 in sheet named ‘calc’),
iii. With maturity values on the X-axis, plot the spot rates against the maturity values,
iv. Spot rate associated with any desired maturity (eg. 8.2 years) can be similarly derived by substituting the estimated parameters and m=8.2 in equation 7.

Estimated Bond Prices and Yields

‘Model’ and ‘market’ prices, along with ‘accrued interest’ and ‘errors’, have been provided in the accompanying Excel file (refer sheet named ‘model’). To re-capitulate, ‘dirty price’ enters the equation specification (refer equation 5) as ‘**pmkt**’ and the discounted value of the stream of cashflows provides the model price (**p_est**). We have therefore provided the two series inclusive of accrued interest. Deducting accrued interest from both these series, one can arrive at ‘trade prices’. ‘Error’ is the difference between ‘market’ and ‘model’ price.

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