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Fixed Income Models: Assessment and New Directions

1. Uses of models

2. Assessment criteria

3. Assessment

4. Open questions and new directions

   - stochastic volatility (Cox-Ingersoll-Ross)
   - volatility inputs (Heath-Jarrow-Morton)
   - multiple factors
   - pricing kernels and risk-neutral probabilities

1. Use of Models

- Valuation of new structures
  - approach 1: look at comparable assets in the market
  - approach 2: models guarantee consistency with other assets, but not necessarily good answers
    ("even a bad model can be tuned to get some prices right")
- Hedging and sensitivity
  - what's the DV01 of a swaption?
  - what's the sensitivity to a 20 bp rise in 5-7 year spot rates?
  - models can be used to do "partial derivative" calculcations, but the answer depends on the model
- Why different models for different purposes?
  - modeling = finding useful short cuts
  - you take different short cuts depending on the purpose
  - the key is to understand where short cuts hurt you

2. Assessment

- Questions for models:
  - does it reproduce current spot rates?
  - does it reproduce current term structure of volatility?
  - are sensitivities and hedge ratios reasonable?
  - does it reproduce swaption volatility matrix?
  - can volatility change randomly?
  - can it reproduce volatility smiles and skews?
  - does it allow "twists" in spot rate curve?

- Summary assessment:

	Ho-Lee	Black-Derman-Toy	Hull-White
spot rates?	yes	yes	yes
vol term str?	no	yes	no
sensitivities?	no	no?	maybe
vol matrix?	no	no	no
random vol?	no	no	no
smiles?	no	no	no
twists?	no	no	no

- All models have strengths and weaknesses

2. Assessment (continued)

- Ho and Lee
  - the innovation was to match current spot rates
  - didn't do the next step: volatilities
- Black, Derman, and Toy
  - matches volatility term structure
  - short rate or log of short rate? (not clear)
- Hull and White
  - premise: lack of mean reversion above is a mistake
  - but where does it hurt us?
    (sensitivies? volatilities on long bonds/swaps?)
  - technical trick: build mean reversion into trinomial tree
- For all the above:
  - future volatility known now
  - volatility matrix implied by models, not a flexible input
  - no volatility smiles
  - one-factor structures

3. Vasicek Revisited

- The model

$$
\begin{aligned}
-\log m_{t+1} &= \lambda^2/2 + z_t + \lambda \varepsilon_{t+1} \\
z_{t+1} &= (1-\varphi)\theta + \varphi z_t + \sigma \varepsilon_{t+1}
\end{aligned}
$$

- Pricing relation:

$$
b_t^{n+1} = E_t \left(m_{t+1} b_{t+1}^n \right)
$$

- Solution is log-linear:

$$
-\log b_t^n = A_n + B_n z_t
$$

  with

$$
\begin{aligned}
A_{n+1} &= A_n + \lambda^2/2 + B_n(1-\varphi)\theta - (\lambda + B_n \sigma)^2/2 \\
B_{n+1} &= 1 + B_n \varphi
\end{aligned}
$$

  (start with $A_0 = B_0 = 0$)

- Forward rates
  - definition is

$$
f_t^n = \log(b_t^n / b_t^{n+1})
$$

  - Vasicek solution is

$$
f_t^n = \varphi^n z_t + \text{constant}
$$

  - note: sensitivity to z declines with n if $0 < \varphi < 1$
    (think: Ho-Lee and BDT set $\varphi = 1$)

4. Stochastic Volatility

- Cox-Ingersoll-Ross model

$$\begin{aligned} -\log m_{t+1} &= (1+\lambda^2/2)z_t + \lambda z_t^{1/2}\varepsilon_{t+1} \\ z_{t+1} &= (1-\varphi)\theta + \varphi z_t + \sigma z_t^{1/2}\varepsilon_{t+1} \end{aligned}$$

Comments:

  – conditional variance varies with z: $Var_t z_{t+1} = \sigma^2 z_t$

  – square root keeps z positive

  – mean reversion

- Solution is log-linear (like Vasicek):

$$-\log b_t^n = A_n + B_n z_t$$

with

$$\begin{aligned} A_{n+1} &= A_n + B_n(1-\varphi)\theta - (\lambda + B_n\sigma)^2/2 \\ B_{n+1} &= 1 + \lambda^2/2 + B_n\varphi - (\lambda + B_n\sigma)^2/2 \end{aligned}$$

(start with $A_0 = B_0 = 0$)

- Illustrates a standard trick for allowing volatility to vary "stochastically"

5. Heath, Jarrow, and Morton

- Overview:
  - approach based on forward rates
  - allows input of volatility matrix
  - many variants — we focus on a linear one

- Pricing relation for forward rates:

$$\begin{aligned} 1 &= E_t(m_{t+1} R_{t+1}) \\ \log R_{t+1} &= r_t - \sum_{j=1}^{n} (f_{t+1}^{j-1} - f_t^j) \end{aligned}$$

  (nothing here but basic algebra)

- Behavior of forward rates (we start to get specific here):

$$f_{t+1}^{n-1} = f_t^n + \alpha_{nt} + \sigma_{nt} \varepsilon_{t+1}$$

  Comments:
  - note the volatility input σ_{nt}: varies with date t and maturity n (a matrix!)
  - linear structure common, not necessary
  - Vasicek is similar:

$$f_{t+1}^{n-1} = f_t^n + \text{constant} + \varphi^{n-1} \sigma \varepsilon_{t+1}$$

    (note the specific relation of volatility to maturity)
  - return is

$$\begin{aligned} \log R_{t+1} &= r_t - \sum_{j=1}^{n} \alpha_{jt} - \sum_{j=1}^{n} \sigma_{jt} \varepsilon_{t+1} \\ &= r_t - A_{nt} - S_{nt} \varepsilon_{t+1} \end{aligned}$$

    (A and S are partial sums)
  - other versions have multiple ε's

5. Heath, Jarrow, and Morton (continued)

- Arbitrage-free pricing
  - a pricing kernel:

$$-\log m_{t+1} \;=\; \delta_t + \lambda_t \varepsilon_{t+1}$$

  - pricing relation imposes restrictions:

$$A_{nt} - \lambda_t S_{nt} - S_{nt}^2/2 \;=\; 0$$

- Calibration
  - inputs:
    * current forward rates: $\{f_t^n\}_n$
    * volatility matrix: $\{\sigma_{nt}\}_{n,t}$
  - drift parameters $\{\alpha_{nt}\}$ chosen to satisfy arbitrage relation
  - open question: set $\lambda_t = 0$? (more shortly)

- Implementation on trees
  - a Wall Street standard
  - at each node we have complete forward rate curve
  - branches don't typically "recombine"

6. Volatility Smiles and Skews

- Approaches:
  - fit smooth curve to observed implied volatilities (but: not all smooth curves are arbitrage-free)
  - "implied binomial trees" that allow volatility to vary across states as well as dates (see Chriss's book)
  - continuous models that extend Black-Scholes logic to non-normal distributions

- Gram-Charlier smiles
  - in Vasicek, normal ε leads to Black-Scholes prices for zeros
  - Gram-Charlier expansion adds terms for skewness (γ_1) and kurtosis (γ_2) to the normal (where $\gamma_1 = \gamma_2 = 0$)
  - Wu's approximation of a volatility smile:

    $$v \;=\; \sigma \left[1 - \frac{\gamma_1}{3!} d - \frac{\gamma_2}{4!}(1 - d^2) \right]$$

    where

    $$d \;=\; \frac{\log(F/K) + v^2/2}{v}$$

  - intuition:
    * positive skewness raises value of out-of-the-money calls
    * positive kurtosis raises probability of extreme events and value of out-of-the-money puts and calls

7. Multi-Factor Models

- Problems with one-factor models
  - changes in rates of all maturities tied to a single random variable (ε)
  - shifts in spot rate curve come in only one type (parallel?)
  - correlation of spot rates across maturities restricted
  - spreads typically not variable enough

- Example: Two-factor Vasicek
  - the model:

$$\begin{aligned} -\log m_{t+1} &= \delta + \sum_i (\lambda_i^2/2 + z_{it} + \lambda_i \varepsilon_{j,t+1}) \\ z_{i,t+1} &= \varphi z_{it} + \sigma_i \varepsilon_{i,t+1} \end{aligned}$$

  - solution includes

$$f_t^n = \delta + \frac{1}{2} \sum_i \left[\lambda_i^2 - \left(\lambda_i + \frac{1-\varphi_i^n}{1-\varphi_i} \sigma_i \right)^2 \right] + \sum_i \varphi_i^n z_{it}.$$

  - standard solution: φ_1 close to one, φ_2 smaller $\Rightarrow$
    * z_1 generates almost parallel shifts, z_2 "twists"
    * long rates dominated by z_1, spreads by z_2
  - generates better behavior of spreads

8. Pricing Kernels and Risk-Neutral Probabilities

- We've taken two approaches to valuation
  - a pricing kernel (Vasicek, for example)
  - risk-neutral probabilities (binomial models)

  How are they related?

- Two-period state prices
  - q_u is value now of 1 dollar in up state next period
  - q_d is value now of 1 dollar in down state next period
  - valuation of cash flows (c_u, c_d) follows

$$p = q_u c_u + q_d c_d$$

  - one-period discount factor is $b = q_u + q_d = exp(-rh)$

- Risk-neutral probabilities
  - define $\pi_u^* = q_u/(q_u + q_d)$, $\pi_d^* = q_d/(q_u + q_d)$
  - note: (π_u^*, π_d^*) are positive and sum to one (they're probabilities!)
  - valuation follows

$$p = \exp(-rh)(\pi_u^* c_u + \pi_d^* c_d)$$

- Pricing kernel
  - define $q_u = \pi_u m_u$, $q_d = \pi_d m_d$ ("true probabilities")
  - valuation follows

$$p = \pi_u m_u c_u + \pi_d m_d c_d$$

8. Pricing Kernels and Risk-Neutral Probabilities (cont'd)

- Where's the pricing kernel in a binomial model?
  - figure it out:

$$-\log m_{t+1} = \delta^* + r_t + \lambda \varepsilon_{t+1},$$

    with

$$\begin{aligned} \delta^* &= \pi \log(\pi_u/\pi_u^*) + \pi_d \log(\pi_d/\pi_d^*) \\ 2\lambda &= \log(\pi_u/\pi_d^*) - \log(\pi_d/\pi_d^*) \end{aligned}$$

  - summary: λ is built into the difference between true and risk-neutral probabilities (and is zero when the two are equal)

- Is $\lambda = 0$ a mistake?
  - in principle yes: if kernel is wrong, valuation is wrong
  - caveat: Black-Scholes pricing doesn't depend directly on λ (so maybe it's not a bad mistake in general)
  - bottom line: who knows?

Summary

1. Models are
   - simplications of reality
   - ways to ensure consistency of pricing across assets
   - only as good s (and floors) and swaptions

2. Binomial models capture some of the elements of observed asset prices, but most versions leave some issues open:
   - stochastic volatility
   - volatility smiles (again, more work)
   - multiple factors (possible, just more work)
   - can we ignore λ?

3. Modeling remains as much art as science