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Options 1: The Term Structure of Volatility

1. Approaches to valuation (review and extension)

2. The Black-Scholes formula (history and meaning)

3. Option sensitivities

4. Term structure of volatility

5. Extended Vasicek

6. Options on zeros

7. Options on coupon bonds

8. Options on eurocurrency futures

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10. Summary and final thoughts

1. The Importance of Black-Scholes

- Elegant solution to option price (once you get used to it)
- A purely academic innovation
- Now the industry standard in most markets
- Played critical role in growth of option markets
- Together with related work by Samuelson and Merton, triggered an explosion of work that laid the basis for modern finance in universities and on Wall Street
- New approach to valuation: replication, a precursor to the kinds of arbitrage-free models we are using
- Constant interest rate: fortunately not essential or applications to fixed income would have died out rapidly

2. Two Approaches to Valuation

- Problem: value random future cash flows c_{t+n}
- Recursive valuation (one period at a time):
  - In period $t+n-1$, value is
    $$p_{t+n-1} = E_{t+n-1}\left(m_{t+n}c_{t+n}\right)$$
  - In period $t+n-2$, value is
    $$p_{t+n-2} = E_{t+n-2}\left(m_{t+n-1}p_{t+n-1}\right)$$
  - Etc.
  - In period t, value is
    $$p_t = E_t\left(m_{t+1}p_{t+1}\right)$$
- "All-at-once" valuation:
  - Apply multiperiod "stochastic discount factor" $M_{t,t+n}$:
    $$p_t = E_t\left(M_{t,t+n}c_{t+n}\right)$$
- Same answer if $M_{t,t+n} = m_{t+1}m_{t+2}\cdots m_{t+n}$
  - Two-period zero: recursive approach
    $$\begin{aligned} p_t &= E_t\left(m_{t+1}p_{t+1}\right) \\ &= E_t\left(m_{t+1}E_{t+1}\left[m_{t+2}\times 1\right]\right) \\ &= E_t\left[m_{t+1}m_{t+2}\times 1\right]. \end{aligned}$$
  - Two-period zero: "all-at-once" approach
    $$p_t = E_t\left(M_{t,t+2}\times 1\right).$$

3. Option Basics

- A European *call option* is the right to buy an asset at a fixed price K at a fixed date n periods in the future
  - S is the price of the *underlying*
  - K is the *strike price*
  - n is the *maturity* of the option
  - a call generates cash flow at $t+n$ of

    $$c_{t+n} = (S_{t+n} - K)^+$$

    where x^+ means $\max(x, 0)$ (the positive part of x)

- The price C of a call satisfies

  $$C_t^n = E_t \left[M_{t,t+n} (S_{t+n} - K)^+ \right].$$

- A *put option* is a comparable right to sell an asset at price K:

  $$c_{t+n} = (K - S_{t+n})^+$$

- A *forward contract* arranged at date t specifies the purchase at $t+n$ of an asset for price F_t^n (the forward price, fixed at t)

- Prices of puts (P) and calls (C) are related by *put-call parity*:

  $$C_t^n - P_t^n = b_t^n (F_t^n - K)$$

  (This allows us to concentrate on calls, knowing we can convert any answer we get to puts.)

4. The Black-Scholes Formula

- Assume: $(\log M_{t,t+n}, \log S_{t+n})$ is bivariate normal
  (we say $(M_{t,t+n}, S_{t+n})$ is log-normal)

- Then (Black-Scholes formula):

$$C_t^n = b_t^n F_t^n \Phi(d) - b_t^n K \Phi(d - w)$$

$$\begin{aligned} b_t^n &= n\text{-period discount factor} \\ F_t^n &= \text{forward price of the underlying} \\ \Phi &= \text{cumulative normal distribution function} \\ w^2 &= Var_t\left(\log S_{t+n}\right) \\ d &= \frac{\log(F_t^n/K) + w^2/2}{w} \end{aligned}$$

- Comments:
  - a little different from the original, but more useful for our purposes
  - industry standard in most markets
  - similar formulas were derived by Merton (dividend-paying stocks), Black (futures), and Biger-Hull and Garman-Kohlhagen (currencies)
  - one of the best parts: the only component of the formula that's not observed in cash markets is volatility
    (we can turn this around: given call price, find volatility)
  - describes (implicitly) how changes in (F, W, b) affect C

4. The Black-Scholes Formula (continued)

- Sample inputs (streamlined notation):
  - $F = 95.00$
  - $K = 97.50$ (slightly out of the money)
  - maturity $= 1$ year
  - $b = \exp(-.0500) = 0.9512$
  - $w = 17.30\%$

- Intermediate calculations:
  - $d = -0.0636$
  - $\Phi(d) = 0.4746$
  - $d - w = -0.2366$
  - $\Phi(d - w) = 0.4065$

- Call price:

$$\begin{aligned} C &= b\,[F\Phi(d) - K\Phi(d-w)] \\ &= 0.9512\,[(95)(0.4746) - (97.5)(0.4065)] \\ &= 5.193 \end{aligned}$$

- Put price for same strike (use put-call parity):

$$\begin{aligned} P &= C - b(F - K) \\ &= 5.193 - (0.9512)(95 - 97.5) \\ &= 7.571 \end{aligned}$$

  (we most often use this in reverse: convert a put to a call and use the Black-Scholes call formula.)

5. Black-Scholes: How We Got There

Step 1: Integral formulas

- Setting: let $(x_1, x_2) = (\log X_1, \log X_2)$ be bivariate normal with mean μ and variance Σ:

$$\mu = \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix}, \quad \Sigma = \begin{bmatrix} \sigma_1^2 & \sigma_{12} \\ \sigma_{12} & \sigma_2^2 \end{bmatrix}$$

  [Think of $(x_1, x_2) = (\log M_{t,t+n}, \log S_{t+n})$.]

- Formula 1:

$$\begin{aligned} E(X_1) &= (2\pi\sigma_1)^{-1} \int_{-\infty}^{\infty} e^{x_1} e^{-(x_1-\mu_1)^2/2\sigma_1^2} dx_1 \\ &= \exp(\mu_1 + \sigma_1^2/2) \end{aligned}$$

- Formula 2:

$$\begin{aligned} E\left(X_1 | X_2 \geq K\right) &= (2\pi)^{-1} |\Sigma|^{-1/2} \int_{-\infty}^{\infty} \int_{\log K}^{\infty} e^{x_1} e^{\left[-(x-\mu)^\top \Sigma^{-1}(x-\mu)/2\right]} dx_2 dx_1 \\ &= E(X_1)\Phi(d), \end{aligned}$$

  with

$$d = \frac{\mu_2 - \log K + \sigma_{12}}{\sigma_2}$$

- Formula 3:

$$\begin{aligned} E\left(X_1 X_2 | X_2 \geq K\right) &= (2\pi)^{-1} |\Sigma|^{-1/2} \int_{-\infty}^{\infty} \int_{\log K}^{\infty} e^{x_1+x_2} e^{\left[-(x-\mu)^\top \Sigma^{-1}(x-\mu)/2\right]} dx_2 dx_1 \\ &= E(X_1 X_2)\Phi(d), \end{aligned}$$

  with

$$d = \frac{\mu_2 - \log K + \sigma_{12} + \sigma_2^2}{\sigma_2}$$

5. Black-Scholes: How We Got There (continued)

Step 2: Black-Scholes

- Call prices satisfy

$$C_t^n = E_t\left(M_{t,t+n} S_{t+n} | S_{t+n} \geq K\right) - K E_t\left(M_{t,t+n} | S_{t+n} \geq K\right)$$

- A forward contract arranged at date t specifies the purchase at $t+n$ of an asset for a price F_t^n set at t

  – Forward price F_t^n satisfies

$$0 = E_t\left[M_{t,t+n}\left(S_{t+n} - F_t^n\right)\right]$$

  (The usual pricing relation with current payment of zero.)

  – Solution includes (use Formula 1):

$$\begin{aligned} b_t^n F_t^n &= E_t\left(M_{t,t+n} S_{t+n}\right) = \exp(\mu_1 + \mu_2 + [\sigma_1^2 + \sigma_2^2 + 2\sigma_{12}]/2) \\ b_t^n &= E_t\left(M_{t,t+n}\right) = \exp(\mu_1 + \sigma_1^2/2) \\ F_t^n &= E_t\left(M_{t,t+n} S_{t+n}\right) / E_t\left(M_{t,t+n}\right) = \exp(\mu_2 + \sigma_{12} + \sigma_2^2/2). \end{aligned}$$

- Term 1 (use Formula 3):

$$E_t\left(M_{t,t+n} S_{t+n} | S_{t+n} \geq K\right) = b_t^n F_t^n \Phi(d)$$

  with

$$d = \frac{\mu_2 - \log K + \sigma_{12} + \sigma_2^2}{\sigma_2} = \frac{\log(F_t^n/K) + \sigma_2^2/2}{\sigma_2}.$$

- Term 2 (use Formula 2):

$$E_t\left(M_{t,t+n} | S_{t+n} \geq K\right) K = b_t^n K \Phi(d'),$$

  with

$$d' = \frac{\mu_2 - \log K + \sigma_{12}}{\sigma_2} = \frac{\log(F_t^n/K) - \sigma_2^2/2}{\sigma_2} = d - \sigma_2$$

5. Black-Scholes: How We Got There (continued)

- Comments
  - Clever substitution:
    - volatility w is the only thing not observable directly from other markets
    - the features of M matter only via their effect on b and F
  - Role of log-normality:
    - assumption is that log of underlying — whatever that might be – is normal
    - prices of many assets exhibit more frequent large changes than this implies (kurtosis, skewness)
  - Is model appropriate?
    - it's a good starting point
    - and the industry standard in most markets
    - a useful benchmark even when not strictly appropriate mathematically

6. Implied Volatility

- Annual units
  - it's standard practice to measure maturity in years
  - if periods are h years, then maturity is $N = hn$
  - volatility (v below) in annual units is:

$$N v^2 = w^2 \quad \Rightarrow \quad v = w / N^{1/2}$$

  - v is often multiplied by 100 and reported as a percentage

- Black-Scholes formula becomes (and this will be our standard):

$$C_t^N = b_t^N F_t^N \Phi(d) - b_t^N K \Phi(d - N^{1/2} v)$$

$$d = \frac{\log(F_t^N / K) + N v^2 / 2}{N^{1/2} v}$$

- Implied volatility: given a call price, find the value of v that satisfies the formula
  - since the price is increasing in v, we can generally find a solution numerically
  - limits: the minimum call price is $b_t^N (F_t^N - K)^+$ ($v = 0$), the maximum is $b_t^N F_t^N$ (for $v = \infty$)
  - example (continued): if call price is 6.00, $v = 19.54\%$

6. Implied Volatility (continued)

- Numerical methods benefit from good first guess
- If solution fails, check limits
- Brenner-Subrahmanyam approximations for options at-the-money forward ($F = K$):

$$
\begin{aligned}
N^{1/2}v &= \frac{(2\pi)^{1/2}C}{bF} \\
C &= \frac{N^{1/2}v\ bF}{(2\pi)^{1/2}}
\end{aligned}
$$

- Example (continued)
  - Let $K = 95$ (at-the-money forward)
  - With $v = 17.30\%$ (same as before), call price is $C = 6.229$
  - Brenner-Subrahmanyam approximation:

$$
\begin{aligned}
v &= \frac{(2\pi)^{1/2}6.229}{(0.9512)(95)} \\
&= 17.28\%!
\end{aligned}
$$

  - Comment: this is more accuracy than is justified by the underlying data, but it illustrates the quality of the approximation.

7. Sensitivities

- Effect of small changes in F:

$$\begin{aligned} \text{“delta”} &= \frac{\partial C}{\partial F} \\ &= b\Phi(d) \end{aligned}$$

(in-the-money options more sensitive than out-of-the-money)

- Effect of small changes in v:

$$\begin{aligned} \text{“vega”} &= \frac{\partial C}{\partial v} \\ &= bF\Phi'(d)N^{1/2} \end{aligned}$$

(more sensitive at $d = 0$ — roughly at the money)

- Effect of small changes in r (use $b = e^{-rN}$):

$$\begin{aligned} \text{“rho”} &= \frac{\partial C}{\partial r} \\ &= -NC \end{aligned}$$

- Comments:
  - standard approach to quantifying risk in options
  - based on Black-Scholes: other models imply different sensitivities
  - volatility critical: dealers tend to be short options (customers buy puts and calls), so they are typically “short volatility” even if they are “delta hedged”
  - fixed income poses basic problem with the approach: hard to disentangle (say) *delta* and *rho*, since both reflect changes in interest rates
    $\Rightarrow$ need to tie all of the components to underlying variables — a z, so to speak

8. Volatility Term Structures

- In the theoretical environment that led to Black-Scholes, volatility v is the same for options of different maturities and strikes

- In fixed income markets, volatility v varies across several dimensions:
  - maturity of the option (N)
  - "tenor" of underlying asset
  - strike price (K)

- Volatility term structures for March 19, 1999
  - interest rate caps:

Maturity (yrs)	Volatility (%)
1	12.50
2	15.50
3	16.55
5	17.25
7	17.15
10	16.55

  - swaption volatilities (%):

	Tenor				
Maturity	1 yrs	2 yrs	5 yrs	10 yrs	20 yrs
1 mo	10.00	11.75	13.88	13.88	11.30
6 mos	13.00	14.00	14.75	14.60	11.80
1 yr	15.65	15.55	15.20	14.90	11.90
5 yrs	16.40	15.85	15.10	14.20	10.55

9. Extended Vasicek Model

- Model generates Black-Scholes option prices for zeros

- Extended Vasicek model (translated from Hull and White):

  – model consists of

$$
\begin{aligned}
-\log m_{t+1} &= \lambda^2/2 + z_t + \lambda\varepsilon_{t+1} \\
z_{t+1} &= (1-\varphi)\theta_t + \varphi z_t + \sigma\varepsilon_{t+1}
\end{aligned}
$$

  – bond prices satisfy

$$
\begin{aligned}
\log b_t^n &= -A_{nt} - B_n z_t \\
B_n &= 1 + \varphi + \varphi^2 + \cdots + \varphi^{n-1}
\end{aligned}
$$

  – details:

    * fits structure of Black-Scholes formula: $(\log M_{t,t+n}, \log b^\tau_{t+n})$ are bivariate normal for all maturities n and tenors τ
    * "extended" refers to the "t" subscripts on θ and A_n (allows us to reproduce current spot rates exactly)

- Forward prices:

  – a forward contract locks in a price F on an $(n+\tau)$-period zero in n periods

  – price now (b^n times F) must equal current price:

$$
b^n F = b^{n+\tau}
$$

  – for completeness, we should probably write F as $F^{n,\tau}$

- Key issue: what are the volatilities?

10. Volatility Term Structures in Vasicek

- How does volatility vary (in theory!) with maturity and tenor?

- Volatility of bond prices:

$$\begin{aligned}\log b^{\tau}_{t+n} &= -A_{\tau} - B_{\tau} z_{t+n} \\ Var_t\left(\log b^{\tau}_{t+n}\right) &= (B_{\tau})^2\, Var_t(z_{t+n})\end{aligned}$$

- Effect of maturity:

$$\begin{aligned}Var_t(z_{t+n}) &= \sigma^2\left(1 + \varphi^2 + \varphi^4 + \cdots + \varphi^{2(n-1)}\right) \\ &= \sigma^2\left(\frac{1-\varphi^{2n}}{1-\varphi^2}\right)\end{aligned}$$

  If $0 < \varphi < 1$, this grows more slowly than n

- Effect of tenor: how B_{τ} varies with τ

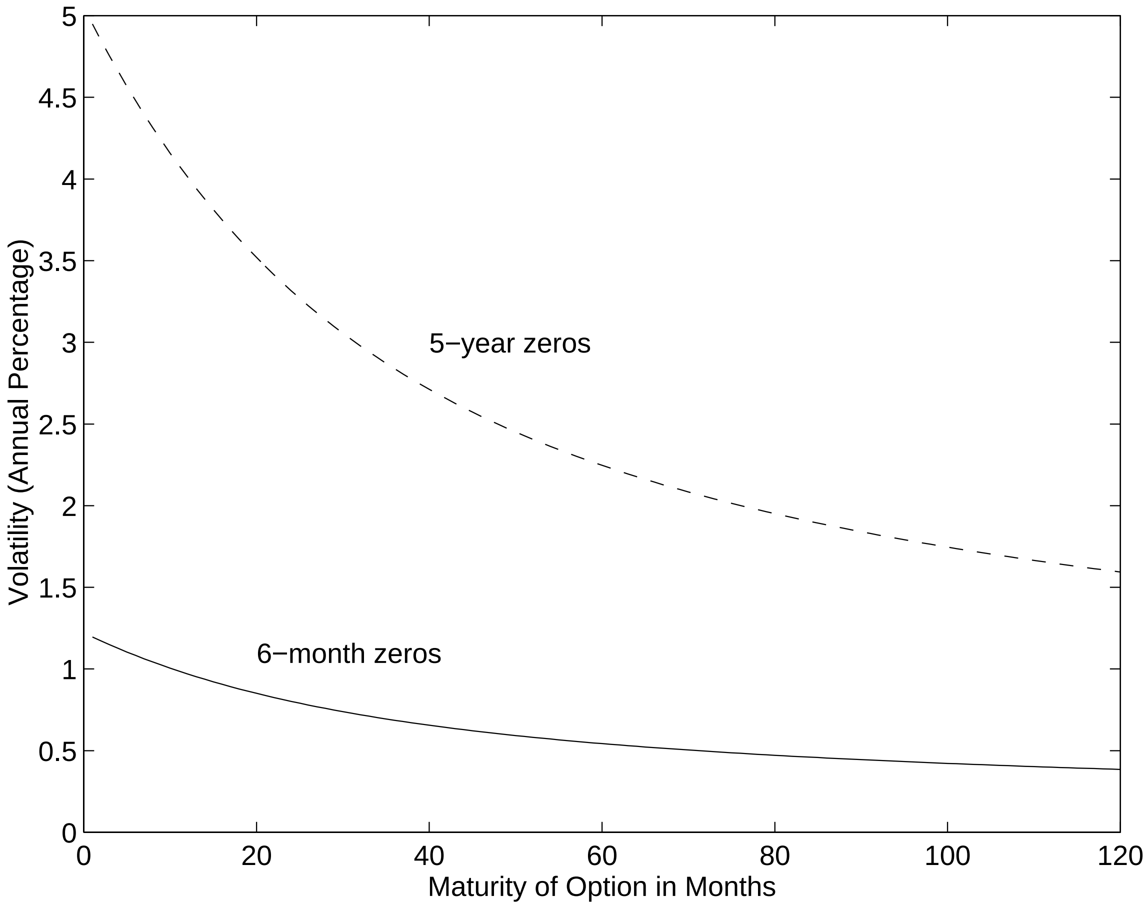
- Annualized per-period volatility is

$$v_{n,\tau} = (B_{\tau})^2 \frac{Var_t(z_{t+n})}{hn} = (B_{\tau})^2 \left(\frac{\sigma^2}{hn}\right)\left(\frac{1-\varphi^{2n}}{1-\varphi^2}\right)$$

- Comments:
  - note that volatility has two dimensions: it's a matrix, like the swaption volatility matrix we'll see shortly
  - effects of mean reversion apparent through
    - maturity: v declines with n if $0 < \varphi < 1$ (constant if $\varphi = 1$)
    - tenor: v increases with τ, but the rate of increase is greater if φ is closer to one
  - model required — not optional — for B_{τ}, $Var_t(z_{t+n})$

10. Volatility Term Structures in Vasicek (continued)

- Volatility term structures for 6-month and 5-year zeros



- Less mean reversion (φ closer to one) . . .
  - reduces rate of decline with maturity
  - increases volatility of long zeros

11. Options on Coupon Bonds

- A coupon bond has known cash flows on fixed dates
  - label set of dates by $J = \{n_1, n_2, n_3, \ldots\}$
    eg, with a monthly time interval and semiannual payments, a 5-year bond would have payments at
$$J \;=\; \{6, 12, 18, 24, 30, 36, 42, 48, 54, 60\}$$
  - label cash flows by c_j for each $j \in J$

- Bond price is
$$p_t \;=\; \sum_{j \in J} c_j b_t^{n_j}$$

- If prices of zeros are log-normal, price of a coupon bond is not (log of a sum not equal to sum of logs)
$$\Rightarrow \quad \text{Black-Scholes not strictly appropriate}$$

- Nevertheless: treat Black-Scholes as
  - a convenient reporting tool (eg, its implied volatility)
  - an approximation to the exact formula

12. Options on Eurodollar Futures

- Money market futures:
  - similar contracts on eurodollars (CME), short sterling (LIFFE), euros (LIFFE), euroyen (TIFFE), etc
  - maturities out 10-15 years
  - price F related to "yield" Y by $F = 100 - Y$ (settles at Y = 3-m LIBOR)

- Call options on these contracts are
  - claims to future cash flow

$$(F - K)^+ \times \frac{90}{360}$$

  - equivalent to put options on the yield

$$\begin{aligned}([100 - Y] - K)^+ \times \frac{90}{360} &= ([100 - K] - Y)^+ \times \frac{90}{360} \\ &= (K_Y - Y)^+ \times \frac{90}{360}\end{aligned}$$

- Similarly: put options are equivalent to calls on the yield

$$\begin{aligned}(K - F)^+ \times \frac{90}{360} &= ([100 - K] - [100 - Y])^+ \times \frac{90}{360} \\ &= (Y - K_Y)^+ \times \frac{90}{360}\end{aligned}$$

12. Options on Eurodollar Futures (continued)

- Option quotes on Mar 16 for options on the June contract:
  - $F = 94.955$
  - $N = 0.25$ (3 months)
  - $b = 0.9874$ (roughly 5% for 3 months)
  - call price (mid of bad/ask) for $K = 95.00$: $C = 0.0425$
  - put price (mid of bad/ask) for $K = 95.00$: $P = 0.0925$

- Yield-based implied volatilities (streamlined notation):

$$C = b(100 - F)\Phi(d) - b(100 - K)\Phi(d - N^{1/2}v)$$

$$d = \frac{\log[(100 - F)/(100 - K)] + Nv^2/2}{N^{1/2}v}$$

  (This is more work than finding volatility for the futures directly, but connects futures to related OTC markets for rates.)

  - put is equivalent to a call on the yield:
    * volatility (solution to formula) is $v = 0.0687$
  - call is equivalent to a put on the yield:
    * put-call parity gives comparable call price as

$$\begin{aligned} C_Y &= P_Y + b[(100 - F) - (100 - K)] \\ &= 0.0425 + (.9874)(0.045) \\ &= 0.0869 \end{aligned}$$

    * volatility is $v = 0.0629$

- Comment: do this slowly and carefully!

12. Options on Eurodollar Futures (continued)

- Current term structure of yield volatilities:

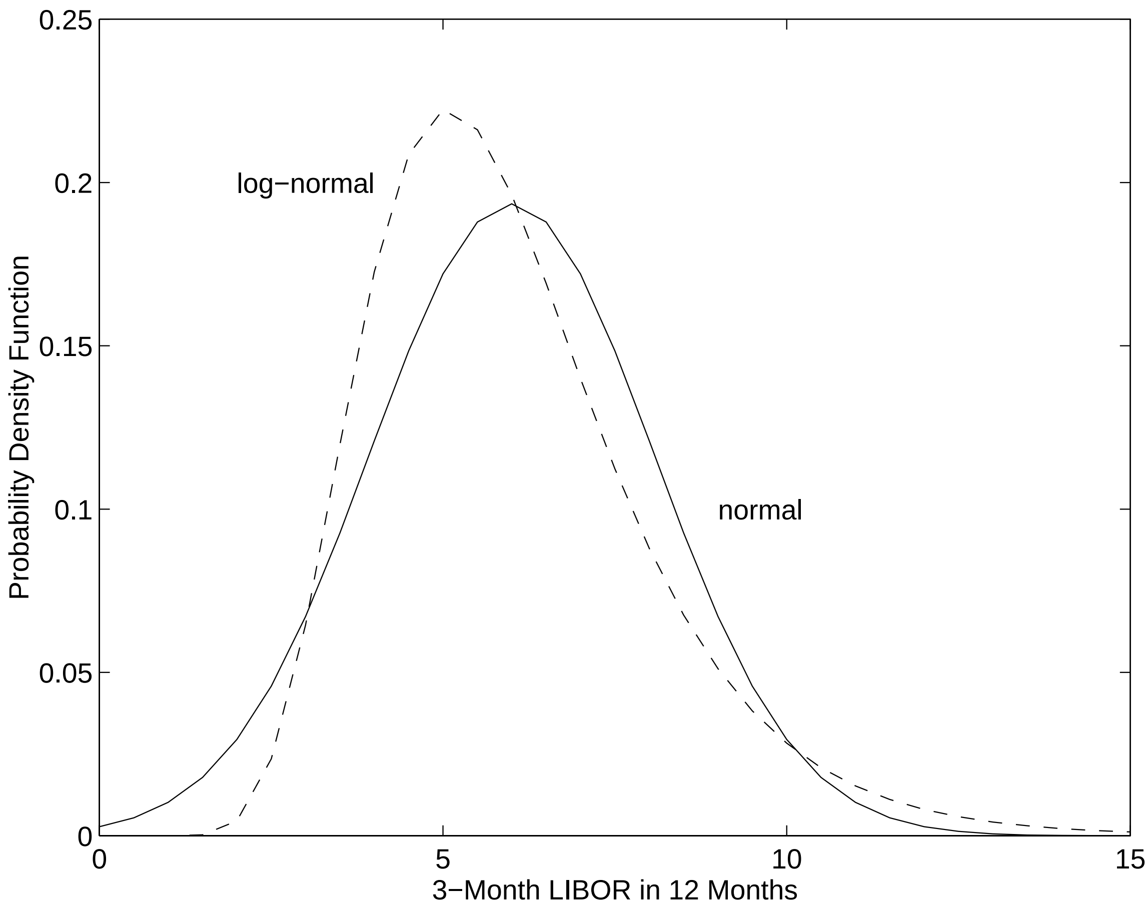
Contract	Maturity	Futures Price	Strike	Put Price	Volatility
Jun 99	0.25	94.955	95.00	0.0925	0.0687
Sep 99	0.50	94.860	94.75	0.1075	0.1087
Dec 99	0.75	94.550	94.50	0.2500	0.1502
Mar 00	1.00	94.640	94.75	0.3650	0.1541
Jun 00	1.25	94.550	94.50	0.3500	0.1628
Sep 00	1.50	94.475	94.50	0.5700	0.1766

  Strikes are closest to at-the-money. Discount factors are $b^N = e^{-Nr}$, with $r = 0.05$.

- Comments:
  - contracts are for fixed tenor but different maturities
  - good source of yield volatilities (standard model input)
  - suggests that markets are more complex than Vasicek (standard pattern shows declines in volatility at long maturities, like Vasicek, but increases at short end)

13. On Normals and Log-Normals

- We have made two different assumptions about interest rates
  - rates are normal: built into Vasicek (logs of prices are normal, interest rates are (approx) normal)
  - rates are log-normal: our analysis of eurodollar futures is based on log-normal yields; caps and swaptions are generally treated the same way



- This apparently technical issue can have a large effect on option prices, especially for options far out of the money

Summary

1. The Black-Scholes formula is both a theoretical benchmark for option pricing and an industry standard for quoting prices.

2. Black-Scholes can be viewed as a theoretical framework for thinking about options (a model), a convention for quoting prices (a formula), or both.

3. In fixed income markets, volatility varies across both the maturity of the option and the "tenor" of the underlying instrument.

4. In Vasicek, the relation between volatility across maturities and tenors is controlled largely by the mean reversion parameter.

5. Market prices suggest two issues that deserve further attention:
   - the difference between normal and log-normal interest rates
   - the shape of the volatility term structure