Bond Relative Value Models

Detecting relative value refers to the process of comparing the potential returns of alternative investments. This paper describes practical Excel/VBA implementations of two such relative value models which provide information on over-, respectively underpriced bonds. The first one is a simple model based on traditional yield to maturity analysis (explained in section 1) and the second one demonstrates the slightly more complex JP Morgan discount function model, an approach of term-structure of interest modelling (shown in section 2). Both have been tested in practical applications in bond relative value research but they have here been adapted for academic purposes. In particular, the real time data feeds have been generalized to be independent of a particular data provider. To use them in a "real world" setting, they would require links to current price and bond data as well as utility macros to maintain the bond baskets analysed.

The Excel files containing the models can be downloaded from the following website:

http://www.mngt.waikato.ac.nz/kurt/

The detailed link is currently http://www.mngt.waikato.ac.nz/kurt/frontpage/ModelMainpages/FixedIncomeModels.htm#BondRelativeValueModels

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1 Simple yield to maturity based benchmarking

Yield to maturity, also called redemption yield based measures to find relative value in a universe of bonds is the traditional method used by most practitioners in the field.

As is commonly known, yield to maturity assumes that an investor holds the bond to maturity and all the bond's cash flow is reinvested at the computed yield to maturity. It is found by solving for the interest rate that will equate the current price to all cash flows from the bond to maturity. In this sense, it is the same as the internal rate of return (IRR) defined in many finance textbooks (e.g. in Reilly & Brown, 2002). Some technical complications arise from the treatment of accrued interest which according to most market conventions have to be paid upfront by the bond buyer. These and other issues related to bond yields are discussed in specialized resources such as Fabozzi (1999)

The yield to maturity based model shown here has two sub-components, each stored in a separate Excel workbook.

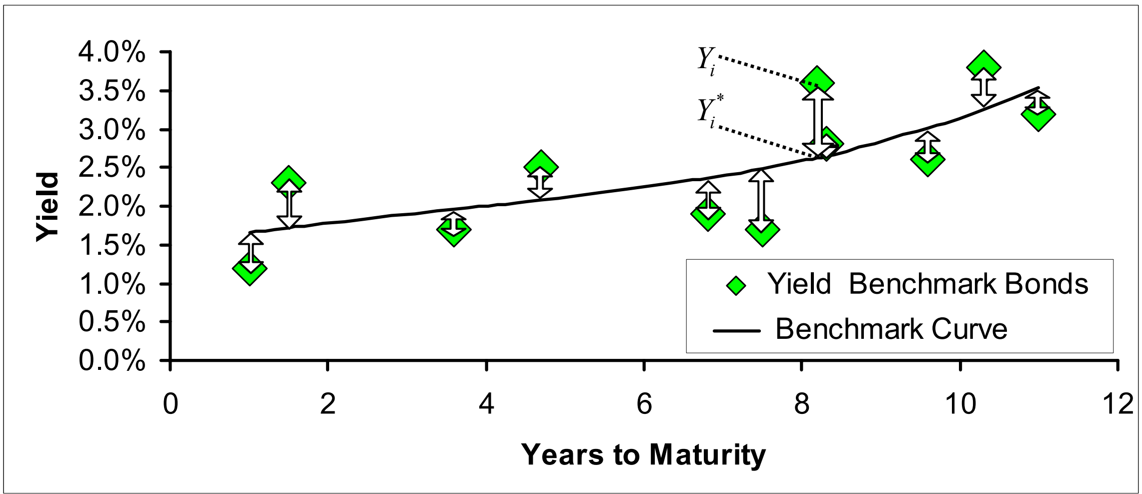
- **ConfBenchmark (Feb04).xls** shows the implementation of a redemption yield based benchmark model for the universe of Swiss government bonds. It is explained in section 1.1.

- **Cheap Rich List (Feb04).xls** produces a cheap/rich analysis for a universe of Swiss domestic corporate bonds using the benchmark curve generated in the Confbenchmark-model. It is explained in section 1.2.

1.1 Benchmark curve

The graphical depiction of the relationship between the yield of the bonds of the same credit quality but different maturities is known as the yield curve. If one was able to subdivide the universe of bonds into homogeneous baskets of bonds, drawing a yield curve for each of them would allow benchmarking within the group. Unfortunately this is not a simple undertaking as will be discussed in the next section. What is generally done, is to compare yields is a well-established liquid benchmark curve. For most markets this would be the government bond market but other rates such as swap rates are also used a benchmarks.

Figure 1: Generic Time / Yield Chart with Bond Yield Curve



There are a number of methods to derive such a yield curve. For a small universe of benchmark bonds, as for instance the one of New Zealand government bonds, one could simply "link" the few bonds in the time/yield chart with straight line connectors which means any intermediate yield would be found through linear interpolation. The model implementation shown here derives a benchmark yield curve from observed bond redemption yields of Swiss government bonds by fitting a polynomial that minimizes the squared error of redemption yields. Many other ways such as fitting cubic splines are employed in financial modelling software. This is not to speak of numerous more advanced methods for yield curve interpolation and fitting for which, as an example, the introduction to Krippner (2002) lists the leading references.

The following paragraph derives the calculation of the polynomial coefficients.

Defining …

Redemption yield bond i: Y_i with i = 1,2, … to n (number of bonds in basket)

Interpolated yield bond i: $Y*_i = a_m t_i^m + a_{m-1} t_i^{m-1} + ... + a_1 t_i + a_0$

where

t_i : time to maturity of bond i

$a_0, a_1, ..., a_m$: coefficients of m degree polynomial

… we need to minimize $\sum_{i=1}^{n} (Y*_i - Y_i)^2 \Rightarrow \frac{\partial \sum_{i=1}^{n} (Y*_i - Y_i)^2}{\partial a_k} = 0$ for k = 0,1,2 up to the desired dimension of the polynomial m. These are m+1 equations for the unknown coefficients $a_0, a_1, ..., a_m$.

Some algebra shows that $a_0, a_1, ..., a_m$ must be a solution of the following system of linear equations:

$$\begin{bmatrix} n & \sum_n t_i & \sum_n t_i^2 & .. & .. & \sum_n t_i^m \\ \sum_n t_i & \sum_n t_i^2 & \sum_n t_i^3 & .. & .. & \sum_n t_i^{m+1} \\ \sum_n t_i^2 & \sum_n t_i^3 & \sum_n t_i^4 & .. & . & \sum_n t_i^{m+2} \\ : & : & : & & . & : \\ : & : & . & . & & : \\ \sum_n t_i^m & \sum_n t_i^{m+1} & \sum_n t_i^{m+2} & .. & .. & \sum_n t_i^{2m} \end{bmatrix} \times \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ : \\ : \\ a_m \end{bmatrix} = \begin{bmatrix} \sum_n Y_i \\ \sum_n Y_i t_i \\ \sum_n Y_i t_i^2 \\ : \\ : \\ \sum_n Y_i t_i^m \end{bmatrix}$$

This system is typically solved numerically by one of many well known algorithms such as the ones described in Press et al. (1992). Yet because a coefficient like $\sum_n t_i^{2m}$ becomes an extremely large number for higher values of m, algorithms will lose accuracy. However, this is not an issue in most cases because meaningful interpolations will not exceed 3rd to 4th order polynomials. The VBA implementation in the model uses the so-called LU decomposition to solve the system as described in Press et. al (1992) p.43.

1.2 Redemption yield based cheap rich analysis

Corporate bonds are not homogeneous in their credit quality so bond analysts compare their yield to some well established benchmark, e.g. government bond curve. This yield differential is then measured in basis points (bps) and gives an indication of the credit risk priced into the corporate bond.

An example: a five year government bonds yields 4.5%, a corresponding corporate bond 5.2%. The yield differential or credit spread is 0.7% which equals 70 bps.

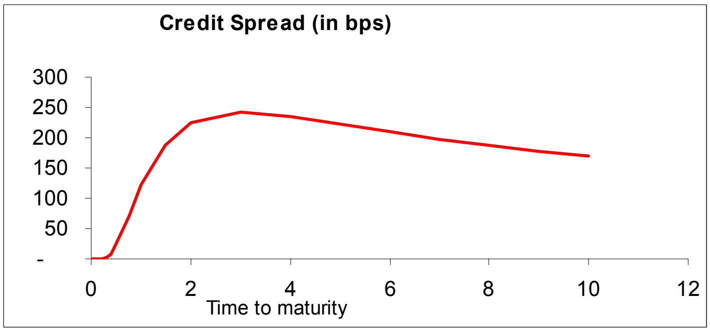
This section first discusses the nature of these credit spreads, followed by a description how these have been modelled here. It concludes with a simplified numerical example of cheap-rich analysis.

1.2.1 The nature of credit spreads

What should be the proper credit spread for a particular bond?
Practitioners typically compare a particular credit spread of with the ones mapped for bonds of comparable credit quality. Many bonds now have credit ratings by one of the leading agencies so they might base this comparison on averages observed for each credit quality. Deviations from these averages are then used to generate a cheap/rich lists of corporate bonds. This simple method corresponds to a parallel shift of the benchmark yield curve to account for the credit risk of a particular rating category.

Beyond this, there is obviously the whole body of credit risk modelling literature that tries to develop approaches to pricing claims subject to credit risk. A result of such modelling is often a so-called “term structure of credit spreads” which plots the credit spread as a function of time to maturity of the risky bond.

Figure 2: Credit Spreads: Typical Shape

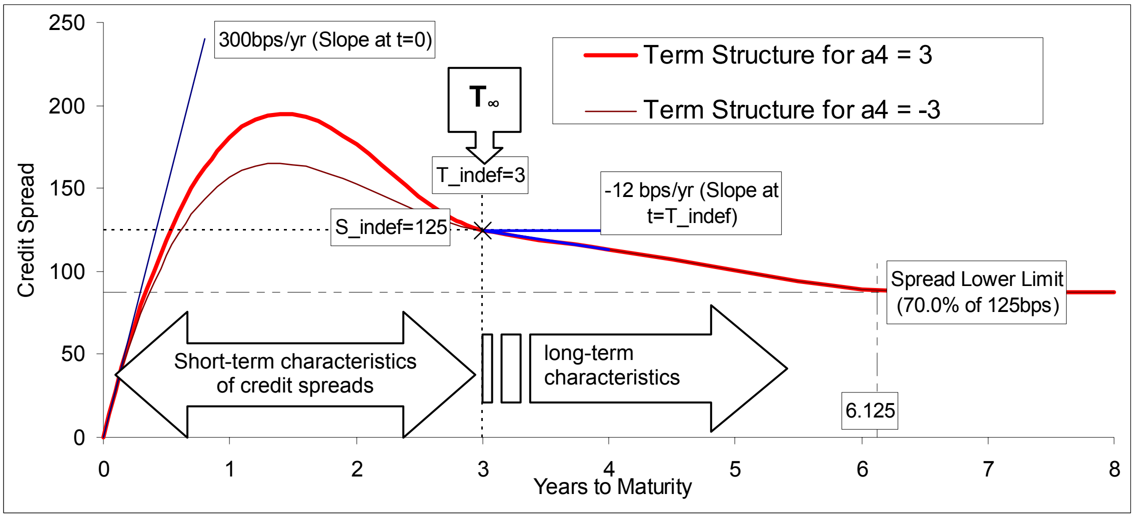


The small chart to the right plots such a term structure derived by a model of Longstaff & Schwartz (1995) which is explained in more detail in Appendix 1. What is more important here is that it reflects a peculiarity of credit spreads often observed in the markets. They are not constant as assumed in the simple approach above but for short-term bonds the spreads become tiny because the likelihood of events pushing weaker credits into bankruptcy becomes minute. There is also the typical “hump” which indicates higher spreads for medium term than for long-term bonds. The hump becomes very pronounced for lower non-investment grade quality (e.g B-rated bond) since bonds are then no longer traded on a yield but on break-up value basis. On the other hand, it can hardly be observed for high-grade bonds.

1.2.2 Defining the term structure of credit spreads in the model

The model presented here lets the user choose the desired term structure of credit spreads for each rating category by means of shape parameters. This is illustrated through a numerical example in Figure 3 below. The term structure is basically broken down into two sub-periods. A short- to medium term period to T_∞ (T_indef) which is followed by the long-term characteristics of the credit spread.

Figure 3: Explanation Shape Parameters Term Structure of Credit Spreads



As to the first period, a fourth order polynomial is fitted between zero and T_∞.

$$Spread(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3 + a_4 t^4$$

It must meet the following three boundary conditions:

- must be zero a time = 0 which means $a_0 = 0$
- must reach S_∞(S_indef) at time $t=T_\infty$.
- slope at point S_∞/T_∞ must equal the slope of the long-term curve beyond T_∞. (more details on this slope are below)

The user affects the shape of the polynomial with two parameters.

- The initial slope (in bps per year) at time = zero may be specified
- The shape of the hump is also affected by parameter a4 which corresponds to the fourth order coefficient of the polynomial. Figure 3 shows the term structure for a4= 3, respectively minus 3.

In the long-term horizon, the user can specify a slope for the further development of credit spreads. In most cases it will be set to or close to zero to obtain constant credit spreads beyond time T_∞. There is also the option to specify a lower limit below which credit spread may never decline. This parameter is set as a percentage of S_∞. If this lower limit is specified as a number greater than one, it actually becomes an upper limit specification.

With above parameters, a wide range of shape specifications becomes possible. Figure 4 lists a range of potential shapes including some comments as the circumstances these could be appropriate.

Figure 4: Selection of Term Structure Shapes

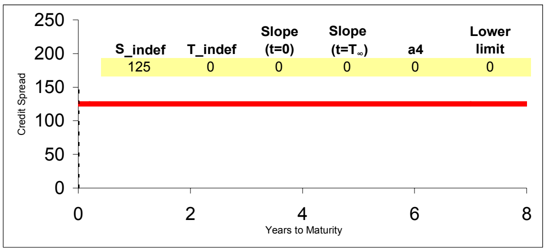


Chart 1: Simple, constant spread

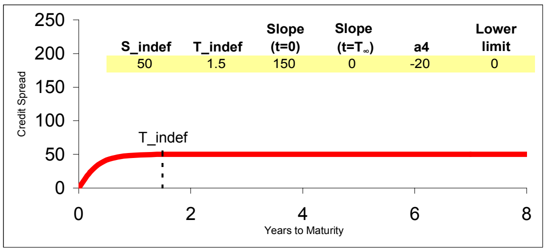


Chart 2: No hump, suitable for high quality bonds

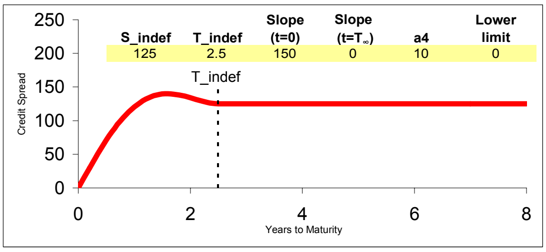


Chart 3:Small hump, suitable for medium quality bonds

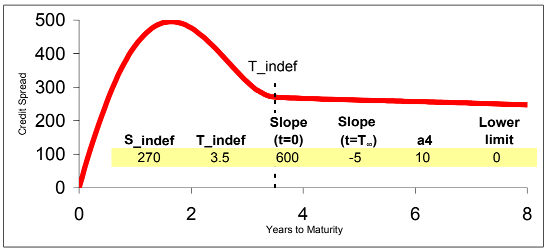


Chart 4:Pronounced hump, suitable for non-investment grade bonds

There is a final question not answered yet? How would the user actually determine appropriate parameters for the specific market conditions?

An academic approach would entail estimating them based on observed market spreads yet, unfortunately, in practice this is often not feasible. As an example, the universe of say BB-minus bonds might consist of just four securities which all happen to mature at almost the same time. A pragmatic solution consists of setting S_∞ to the approximate credit spread of the day but to leave the remaining shape parameters unchanged. They would only be reviewed if patterns strongly change. The analyst may even decide to set the spread S_∞ not in line with the current market but with his/her own subjective expectations regarding its future development. S_∞ could then be called a target spread and it is also labelled as such in the practical implementation of the model.

1.2.3 Detecting cheap/rich bonds: a simplified numerical example

For illustration, the following tables (Table 1 and 2) and figures (Figure 5) present a simplified numerical example of cheap/rich analysis. There are three bonds analysed. Bond I has an AA rating while bonds IIa and IIb are rated BBB. with A. A buy [sell] signal is generated if the model price based on the target spread is above [below] the market price.

Table 1: Data Example Bonds

Bond	I	IIa	IIb
Rating	AA	BBB	BBB
Price ($ per $face value)*	102	99	106
Coupon (paid once annually)	4.00%	4.00%	4.00%
Time to Maturity	2.79 yrs	5.20 yrs	7.79 yrs
Yield to Maturity	3.24%	4.22%	3.12%
Spread to Benchmark	136 bps	208 bps	58 bps
Target Spread (derived from parameters in Table 2)	50 bps	122 bps	120 bps
Model Yield	2.37%	3.36%	3.73%
Model Price (MP)	104.338	103.016	101.761
	MP > Price	MP > Price	MP < Price
Recommendation	Cheap Bond: **Buy**	Cheap Bond: **Buy**	Rich Bond: **Sell**

* Price excluding accrued interest

Figure 5: Generic Yield to Maturity Based Cheap Rich Analysis

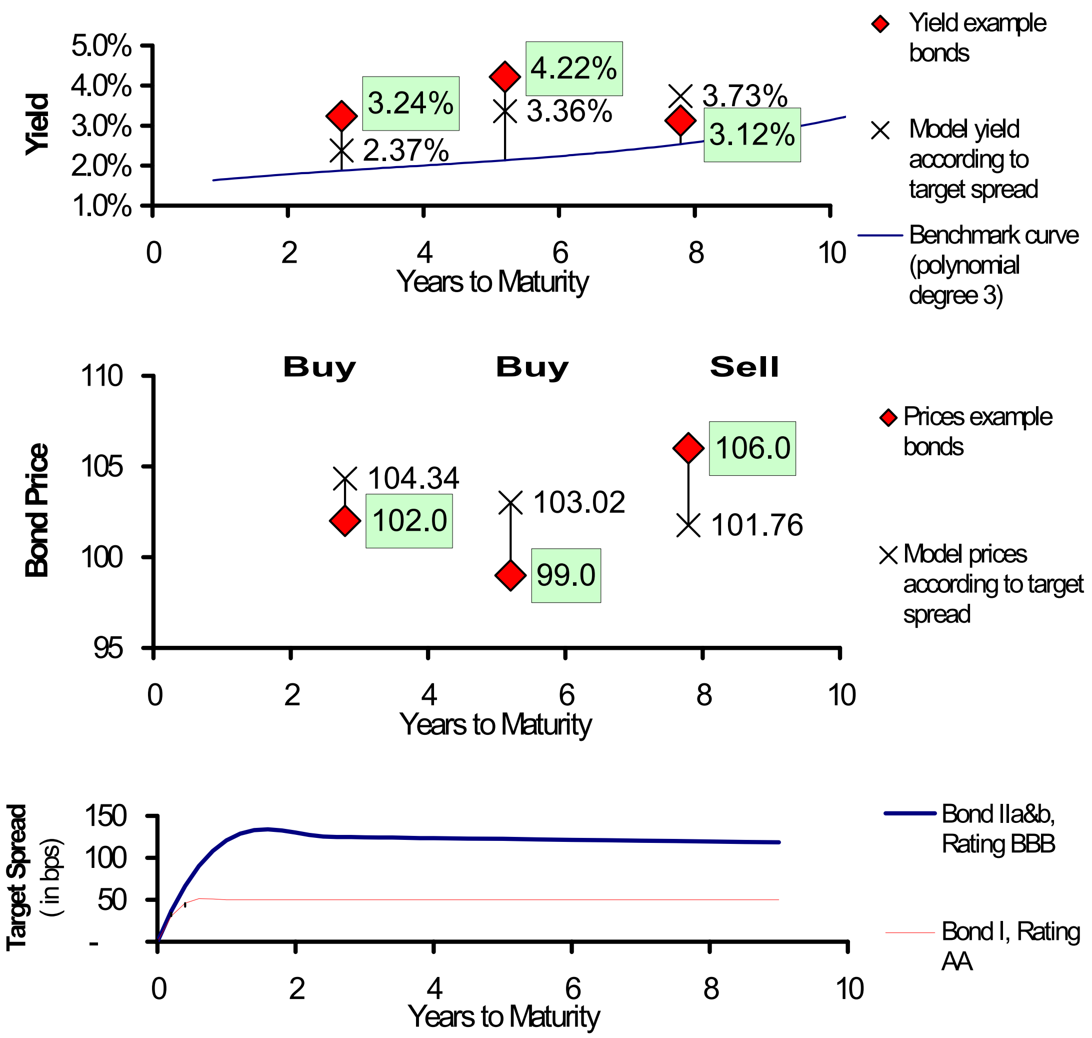


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Table 2: Term Structure of Credit Spread Parameters

Rating	AA	BBB
S_∞ (in bps)	50	125
T_∞	1	2.5
Slope (t=0)	200	200
Slope ($t=T_\infty$)	0	-1
"Hump parameter" a4	-6	2
Lower limit (as % of S_∞)	0.8	0.8

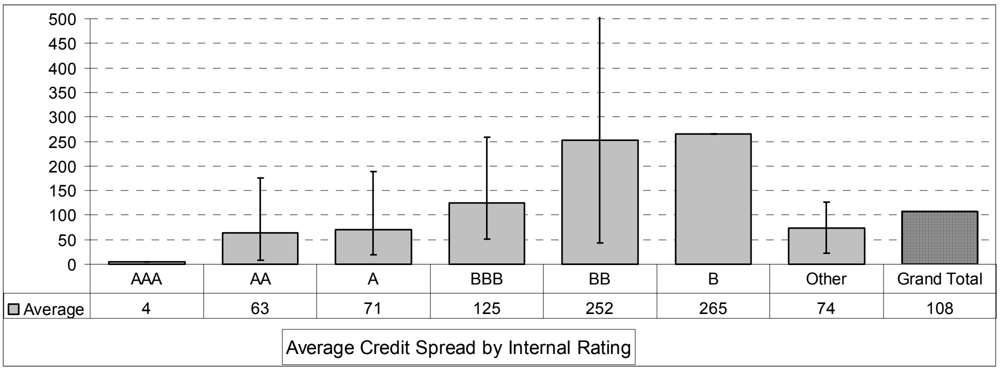
Compared to the main model, the version above is simplified in these aspects.

- The illustrative example does not take bid/ask spreads into consideration when generating buy/sell signals.
- To limit the number of recommendations the user may specify a sensitivity parameter to suppress recommendations where the price is very close to the model price. The sensitivity chosen could, for instance, take transaction charges into account.
- Similarly, for shorter bonds typical price changes in less liquid markets may lead to very erratic yield moves that do translate into meaningful buy/sell signals. The user may thus exclude the analysis for very short-term bonds.
- Finally, the main model provides statistics on the credit spreads observed in the basket. An example is shown in Figure 6 below.

Figure 6: Generic Yield to Maturity Based Cheap Rich Analysis

Analysis credit spreads Swiss industrial bonds 26/02/1999 0:00

Internal Rating	AAA	AA	A	BBB	BB	B	Other	Grand Total
Number of bonds	1	9	36	31	10	1	10	98
Average spread to Swiss Govt (bps)	4	63	71	125	252	265	74	108
Min of spread (bps)	4	8	20	51	44	265	23	4
Max of spread (bps)	4	175	189	259	915	265	127	915
Years to maturity/ next call (average)	1.8	5.0	4.2	4.5	3.9	6.3	5.9	4.5



2 JP Morgan discount factor model

Redemption yield based models have their limitations. In line with the underlying assumptions, coupons need to be reinvested at the computed internal rate of return. To see this, consider the following two hypothetical five year government bonds A and B. The two bonds differ in their coupon rate, which is 10% for A and 2% for B. A pure yield to maturity based analysis would call for identical redemption yields yet there would be a great deal higher reinvestment risk for bond A. For instance, assuming annual coupon payments, the coupon of bond A, respectively B in year four would need to be reinvested for one year at that same five year rate. With yield curves generally upward sloping, this rate is likely to be lower and so the high-coupon bond A will probably achieve a lower return.

Having said this, yield to maturity based models should nonetheless not be completely discounted. They are indeed appropriate for many bond markets and instruments that have following properties:

- Bond coupons are comparably uniform and low (less than 5%)
  Since new issues are usually placed at or around par, this is usually the case when the interest environment has been comparably stable over the preceding few years.
- The market is not too liquid with comparably wide bid/ask price spreads.
  This means the models need be somewhat less discerning to detect arbitrage in the markets.

Conversely, for liquid bond markets such as the one for US Treasuries, the shortcomings of pure yield to maturity analysis are too great. Here the proper approach is to think about bonds as a package of cash flows, more specifically as packages of zero-coupon instruments. To find, respectively explain the price of the whole package, each of this zero bonds is then discounted at a unique interest rate appropriate for the time period in which the cash flow will be received.

The model shown here is known under the name “JP Morgan (JPM) Discount Factor Model”. A preliminary literature search revealed that a description of this model could not be found in standard academic data bases. Having said this, it might have been published under a different name.

The JPM Discount Factor Model refrains from modelling the unique interest rates at each point in time because these are, in colloquial terms, “not very well behaved”. Experience shows that interest term structures come in all kinds of shapes. There are not just upward or downward sloping curves but they often have “humps” and “kinks” that make the formulation of an suitable model very demanding. On the other hand, modelling the discount factors is an easier approach. This function has a clear boundary at time zero and is monotonously declining over time. In the JPM model it is modelled as a polynomial with coefficients determined such that the sum of least square errors of market price minus model price is minimized.

The following first shows the mathematical derivation of the model which is followed by a brief description of its implementation for the small basket of NZ government bonds.

2.1 Mathematical derivation

2.1.1 Fitting the coefficients

Bond universe with n bonds with market prices P = $[p_1, p_2, \ldots, p_n]^T$ which all must be equal to the present value of expected cash flows:

$$
\begin{array}{llllll}
c_1 d_{t_{1,1}} + & c_1 d_{t_{1,2}} + & (1+c_1) d_{t_{1,3}} & & & = p_1 \\
c_2 d_{t_{2,1}} + & c_2 d_{t_{2,2}} + & c_2 d_{t_{2,3}} + & (1+c_2) d_{t_{2,4}} & & = p_2 \\
 & \vdots & \vdots & \vdots & \vdots & \vdots \\
c_n d_{t_{n,1}} + & c_n d_{t_{n,2}} + \ldots & \ldots + c_n d_{t_{n,j}} + \ldots & & \ldots + (1+c_n) d_{t_{n,n}} & = p_n
\end{array}
$$

with

c_i : coupon rate i of bond 1…n.

$d_{t_{i,j}}$: discount factor at the time of the j[th] coupon if bond i.

$d_{t_{i,j}}$ is set to the polynomial $a_m t_{i,j}^m + a_{m-1} t_{i,j}^{m-1} + \ldots + a_1 t_{i,j} + a_0$.

The method finds the vector of coefficients $A = [a_0, a_1 \ldots a_m]^T$ that minimizes the sum of the squared difference between market price vector P and model price vector $PM = [pm_0, pm_1 \ldots pm_m]^T$

The solution of this least square problem is derived for a three dimensional case m=3:

$$
\begin{bmatrix}
c_1 & c_1 \sum_j t_{1,j} & c_1 \sum_j t_{1,j}^2 & c_1 \sum_j t_{1,j}^3 \\
c_2 & c_2 \sum_j t_{2,j} & c_2 \sum_j t_{2,j}^2 & c_2 \sum_j t_{2,j}^3 \\
c_3 & c_3 \sum_j t_{3,j} & c_3 \sum_j t_{3,j}^2 & c_3 \sum_j t_{3,j}^3 \\
\vdots & \vdots & \vdots & \vdots \\
c_n & c_n \sum_j t_{n,j} & c_n \sum_j t_{n,j}^2 & c_n \sum_j t_{n,j}^3
\end{bmatrix}
\times
\begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix}
=
\begin{bmatrix} pm_1 \\ pm_1 \\ pm_3 \\ \vdots \\ pm_n \end{bmatrix}
$$

Minimize $\sum_{i=1}^{n} (pm_i - p_i)^2 \Rightarrow \frac{\partial \sum_{i=1}^{n} (pm_i - p_i)^2}{\partial a_k}$ = 0 for k = 0,1,2,3 yields four equations for the four unknown a_0, a_1, a_2, a_3.

Defining C= $\begin{bmatrix} C_{1,1} & C_{1,2} & C_{1,3} & C_{1,4} \\ C_{2,1} & C_{2,2} & C_{2,3} & C_{2,4} \\ \vdots & \vdots & \vdots & \vdots \\ C_{n,1} & C_{n,2} & C_{n,3} & C_{n,4} \end{bmatrix} = \begin{bmatrix} c_1 & c_1 \sum_j t_{1,j} & c_1 \sum_j t_{1,j}^2 & c_1 \sum_j t_{1,j}^3 \\ c_2 & c_2 \sum_j t_{2,j} & c_2 \sum_j t_{2,j}^2 & c_2 \sum_j t_{2,j}^3 \\ \vdots & \vdots & \vdots & \vdots \\ c_n & c_n \sum_j t_{n,j} & c_n \sum_j t_{n,j}^2 & c_n \sum_j t_{n,j}^3 \end{bmatrix}$,

… one must solve the system of following linear equations to find the coefficient vector A:

$$C^T C \times A = C^T \times P \quad \Rightarrow \quad A = \left(C^T C\right)^{-1} \times \left(C^T \times P\right)$$

The model prices are then found by multiplying C with the coefficient vector A:

$$C \times A = PM$$

Bonds priced below [above] their corresponding model price are “cheap” [“rich”].
The vector of (under pricing)/over pricing $R = [r_0, r_1 ... r_m]^T$ is then:

$$R = PM - P = C \times A - P$$

2.1.2 Boundary conditions

Often constraints are introduced to force the discount factor at time zero to one (which is a most reasonable assumption), respectively one sets the first derivative at time zero to fit a short-term rate, e.g. overnight bank rate observed in the market.

For the restriction d (t=0) =1 a_o is set to 1.

For an assumed short-term rate r, a_1 becomes minus the continuously compounded equivalent of the short rate, respectively in terms of the annual equivalent rate:

a_1= - r_c = - ln(1 + r_{ann})

where
r_{ann} is the short-term rate expressed as an annual equivalent yield
r_c is the short-term rate expressed as an continuously compounded yield

Derivation:
The discount factor at time $(t_0 + \Delta t)$ for r_c continuously compounded yield with Taylor expansion of exponential function

$$a_0 + a_1(t_0 + \Delta t) + a_2(t_0 + \Delta t)^2 + ... = e^{-(t_0 + \Delta t) r_c} = e^{-t_0 r_c} - r_c e^{-t_0 r_c} \Delta t + \frac{1}{2} r_c^2 e^{-t_0 r_c} \Delta t^2 + ...$$

for $t_0 = 0$, $a_0 = 1$ and omitting second and higher order terms of Δt:

$$1 + a_1 \Delta t + a_2 \Delta t^2 + ... = 1 - r_c \Delta t + \frac{1}{2} r_c \Delta t^2 + ... \Rightarrow a_1 = -r_c = -\ln(1 + r_{ann})$$

2.2 Description EXCEL/VBA model parameters

The JPM model is saved under the name

Term structure JP Morgan Model (Feb04).xls

It applies the model to the small universe of NZD government bonds. It could easily be adapted to determine cheap, respectively rich bonds for the universe of any basket of uniform bonds.

JP Morgan Discount Factor Modelling

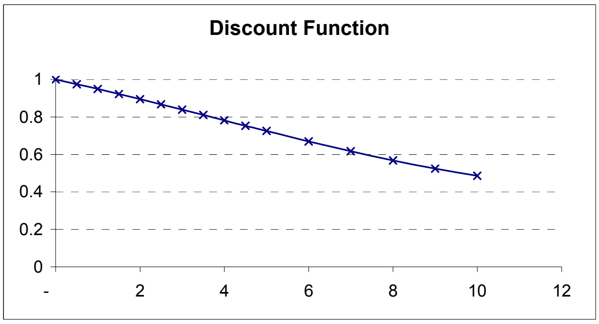
Settlement date

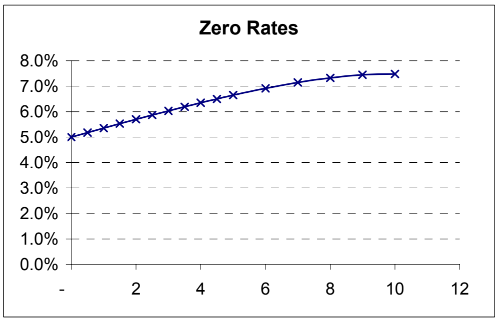
14-Feb-99

Coupon	Maturity	Bid	Ask	Mid	JPM Fair Price	(cheap) / rich	JPM Coefficients	
6.50%	15-Feb-00	100.563	100.583	100.57%	101.17%	(0.60%)	a0	1
8.00%	15-Feb-01	102.786	102.854	102.82%	104.48%	(1.66%)	a1	-0.04879016
10.00%	15-Mar-02	108.406	108.526	108.47%	111.34%	(2.87%)	a2	-0.00222866
5.50%	15-Apr-03	96.673	96.827	96.75%	97.27%	(0.52%)	a3	0.000197076
8.00%	15-Apr-04	105.034	105.234	105.13%	106.56%	(1.43%)	a4	#N/A
8.00%	15-Nov-06	106.518	106.809	106.66%	106.06%	0.61%	a5	#N/A
7.00%	15-Jul-09	100.549	100.903	100.73%	98.91%	1.81%	a6	#N/A
6.00%	15-Nov-11	91.666	92.049	91.86%	92.83%	(0.97%)	a7	#N/A

Model Parameters

deg	3	Degree JP Morgan polynomial
restr	0.05	0: no restrictions, 1:DF(t=0) =1, other values: short rate
frequency	2	Number of coupon payments per year





Parameters as well prices are set in yellow shaded areas.
The major parameters are ..

deg: This allows the choice of the degree of the discount function polynomial. It is not meaningful to choose a much larger value for the small bond universe in the example.

restr: **0** means no restrictions,
1 means discount function is forced to one for t=0 which means coefficient a_o is set to one.
any other value is assumed to be the short-term rate (annual effective yield, i.e. non-continuous yield). a_1 is calculated with the formula given earlier in section “Boundary conditions”. At the same time, the discount function and thus a_o is also forced to one.

Price data would have to be linked to a real time data provider and need to be specified without accrued interest. In the New Zealand market bonds are quoted on a yield to maturity basis. Such yields would first need to be converted into prices using the RBNZ formula:

$$P = \frac{1}{(1+i)^{a/b}} \left[\left[\sum_{k=0}^{n} \frac{C}{(1+i)^k} \right] + \frac{FV}{(1+i)^n} \right]$$

where

P:	Market Value of Bond
FV:	Nominal or face value of bond
i:	Annual market yield / 2 (in %)
c:	Annual coupon rate in %
C:	Coupon Payment (= c/2 * FV) – semi-annual coupon
n:	number of full coupon periods remaining until maturity
a:	Number of days from settlement to next coupon date
b:	Number of days from last to next coupon date

3 References

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<u>Appendix 1</u>: Longstaff & Schwartz 1995 Model

In their 1995 Journal of Finance paper, Francis A. Longstaff and Eduardo S. Schwartz (L&S 95) show a simple approach to value risky debt subject to both default and interest rate risk. In line with the traditional Black-Scholes (1973) and Merton (1974) contingent claims-based framework, default risk is modelled using option pricing theory. This means default occurs if the level of asset of a firm (V) falls below a bankruptcy threshold (K). V is assumed the follow the following stochastic process

$$dV = \mu V dt + \sigma V dZ_1$$

where σ is the instantaneous standard deviation of the asset process (constant) and dZ_1 is a standard Wiener process.

Similarly, interest rates are assumed to follow a standard Vasicek process (Vasicek 1997) as follows:

$$dr = (\zeta - \beta r)dt + \eta dZ_2$$

where

- ζ is the long-term equilibrium of mean reverting process (constant)
- β is the "pull-back" factor - speed of adjustment (constant)
- η is the spot rate volatility (constant)
- dZ_2 is a standard Wiener process.

The instantaneous correlation between dZ_1 and dZ_2 is ρdt

L&S 95 derive the value of a risky discount bond as

$$P(X,r,T) = D(r,T) - wD(r,T)Q(X,r,T)$$

Thus the price of this bond is a function of X, which corresponds to the ratio of V/K, the interest rate r, and the time to maturity T. The terms to be calculated are explained below.

$D(r,T) = e^{A(T)-B(T)r)}$ is the value of riskfree (no credit risk) discount bond according to Vasicek (1977) with

$$A(T) = \left(\frac{\eta^2}{2\beta^2} - \frac{\alpha}{\beta} \right) T + \left(\frac{\eta^2}{\beta^3} - \frac{\alpha}{\beta^2} \right) \left(e^{-\beta T} - 1 \right) - \left(\frac{\eta^2}{4\beta^3} \right) \left(e^{-2\beta T} - 1 \right)$$

$$B(T) = \frac{1 - e^{-\beta T}}{\alpha}$$

Here α represents the sum of the parameter ζ plus a constant representing the market price of risk, β is defined above

The Q(X,r,T) term can be interpreted as probability - under risk neutral measure - that default occurs. It is the limit of $Q(X,r,T,n)$ as $n \to \infty$.
$Q(X,r,T,n)$ is calculated as follows:

$$Q(X,r,T,n)=\sum_{i=1}^{n} q_i \text{ with } \begin{array}{l} q_1 = N(a_1) \\ q_i = N(a_i) - \sum_{j=1}^{i-1} q_j N(b_{ij}),\ i = 2,3,\ldots.,n \end{array}$$

where N(.) denotes the cumulative standard normal distribution and

$$a_i = \frac{-\ln X - M(iT/n,T)}{\sqrt{S(iT/n)}} \qquad b_{ij} = \frac{M(jT/n,T) - M(iT/n,T)}{\sqrt{S(iT/n) - S(jT/n)}}$$

Expressions $M(t,T)$ and $S(T)$ are defined as follows:

$$\begin{aligned} M(t,T) &= \left(\frac{\alpha-\rho\sigma\eta}{\beta} - \frac{\eta^2}{\beta^2} - \frac{\sigma^2}{2}\right)t \\ &+ \left(\frac{\rho\sigma\eta}{\beta^2} + \frac{\eta^2}{2\beta^3}\right)\exp(-\beta T)(\exp(\beta t)-1) \\ &+ \left(\frac{r}{\beta} - \frac{\alpha}{\beta^2} + \frac{\eta^2}{\beta^3}\right)(1-\exp(-\beta t)) \\ &- \left(\frac{\eta^2}{2\beta^3}\right)\exp(-\beta T)(1-\exp(-\beta t)) \end{aligned}$$

and

$$\begin{aligned} S(T) &= \left(\frac{\rho\sigma\eta}{\beta} + \frac{\eta^2}{\beta^2} + \sigma^2\right)t \\ &- \left(\frac{\rho\sigma\eta}{\beta^2} + \frac{2\eta^2}{\beta^3}\right)(1-\exp(-\beta t)) \\ &+ \left(\frac{\eta^2}{2\beta^3}\right)(1-\exp(-2\beta t)) \end{aligned}$$

As a reminder, ρ is the instantaneous correlation between the asset and interest rate processes.

Finally, the constant parameter w is the write-down in case of a default in percent of the face value. In other words, it is one minus the recovery rate in case of a default.

Once the value of a pure discount bond is found, the value of a coupon bond is simply valued as series of discount bonds consisting of coupons and principal repayment. Note that L&S 95 also derive a closed form solution for perpetual floating rate debt in a similar fashion.

The authors conclude their work with an empirical model validation. They conduct a regression analysis of how historically observed spreads (sourced from Moody's) have correlated with the return of share indices as a proxy for the asset process, respectively change in interest rates. They indeed find significant negative correlations in most cases with both interest rate changes and development of asset prices. Just for high grade bonds (AAA and AA bond) they determined less significance for the asset correlation coefficient. This may be expected though because well cushioned high grade credits will be less affected by downturns in the share market.

As an illustration of the L&S 95 model outputs, the charts on the following page show the value and yield of a discount bond as a function of time to maturity T for the parameters in Table A1.

Chart A1: Risky discount bond prices as a function of bond tenor (time to maturity)

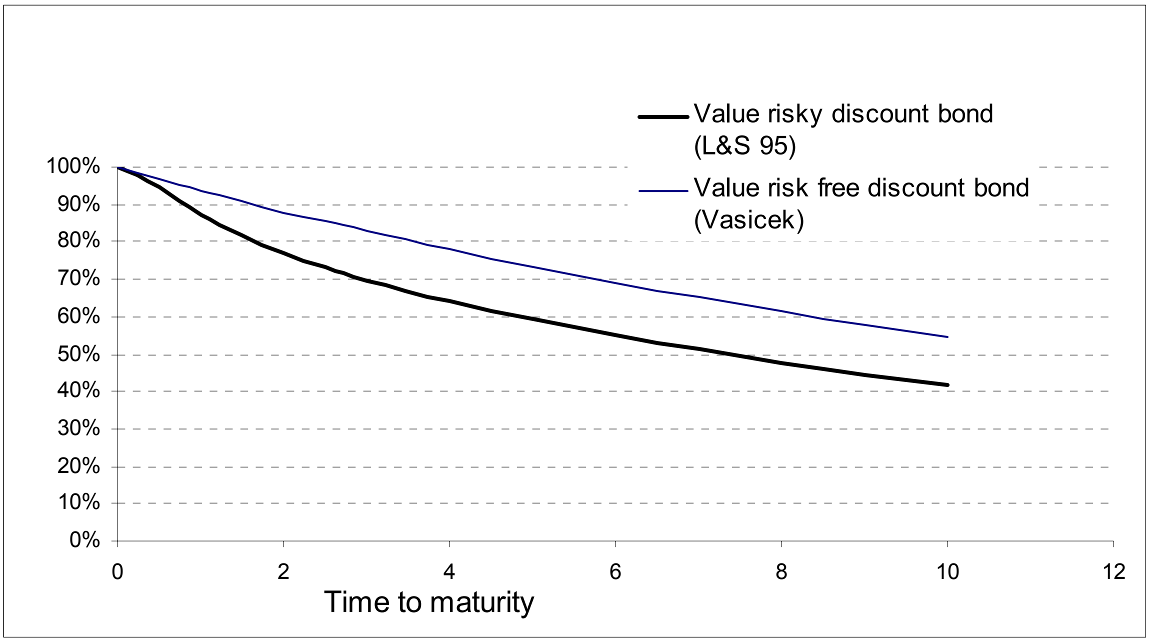

Chart A2: Term Structure of Interest Risky Discount Bonds

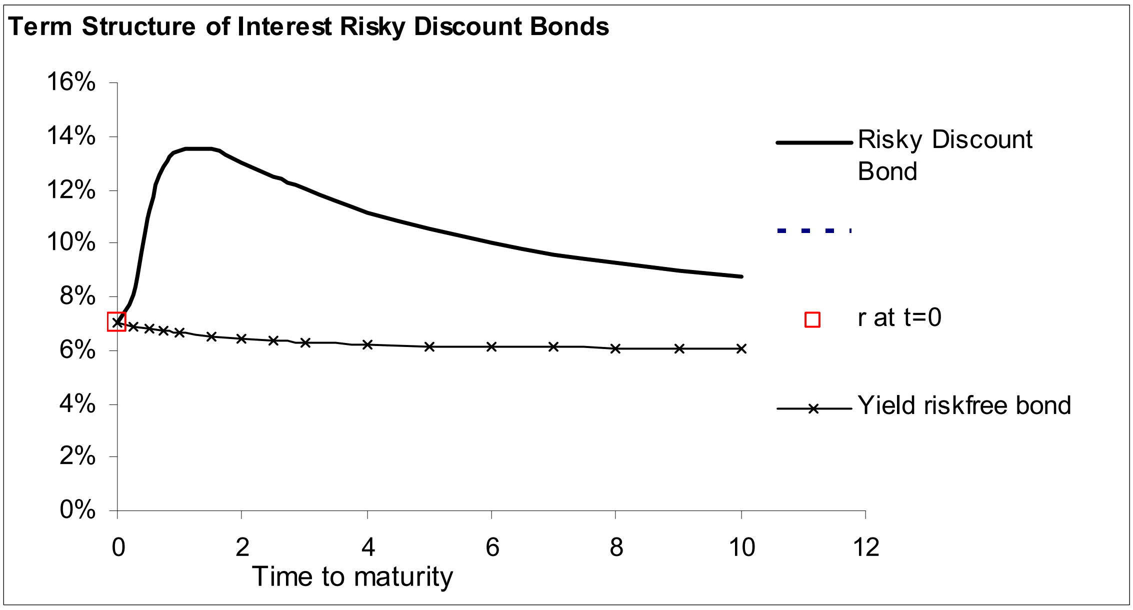

Table A1: Parameters L&S 95 Model Example

Parameter description	Symbol	Value
Rate r_0 at t=0	r	7.0%
"Pullback" factor interest rate	β	1.00
Instantaneous std dev. of short rate	η	3.162%
	η^2	0.0010
α in L&S = ζ + constant	α	0.060
V_0/K =X (measure of initial credit quality)	X	1.30
Writedown = 1 - Recovery Rate	w	0.50
Volatility of asset value process	σ	20.00%
	σ^2	0.0400
Instantaneous correl. asset/interest rate	ρ	- 0.25
Iterations for Q	n	100