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American Options

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1. Introduction/Motivation

- Many popular options allow for "early" exercise, *i.e.*, can be exercised before the option expires.

- Common for exchange-traded options and embedded options

- Less common for OTC options

- Examples:
  - Embedded options
    * callable (putable) bonds
    * sinking funds
    * mortgage-backed securities
  - Bond and bond futures options
  - American swaptions

- Slightly different than American options on stocks $\rightarrow$ stocks don't mature but bonds typically do

2. Problems with Black-Scholes

- Key assumption in deriving Black-Scholes is European option
- If you knew the exact date an American option will be exercised, then it is equivalent to a European options → B-S formula applies
- Value clearly changes as the exercise date changes
- Terminology: "exercise policy" is a rule that specifies the option holder's actions at each time and at each state
- Cash flows of the option depend critically on the option holder's exercise policy
- Is knowledge of the exercise policy sufficient for the use of Black-Scholes?
  - **No**. Since the exercise policy depends on the future state, it does not predict with certainty the date the option will be exercised.
  - → cash flows are "path dependant"

2. Problems... cont'd

- We will assume "optimal exercise," *i.e.*, the option holder chooses the action (exercise or not exercise) that maximizes the option's value
  - Note: As usual, this abstracts from tricky issues like liquidity, incentive issues, *etc.*
- Bottom Line: Black-Scholes can at best be used to place a lower bound on an American option, but not to accurately value the option (even if all of the other assumptions are satisfied)
  - since the holder of an American option always has the option of waiting until expiration to exercise (effectively converting the American option to a European option), an American option can't be worth less than an otherwise identical European option (which can be prices with the B-S formula)

3. Put-Call Non-Parity

- Put-call parity built on the idea of simultaneously buying an call and selling a put on the same underlying with the same strike, K, and maturity
  - this locks in the future price of the underlying at K
- With the possibility of early exercise, this logic breaks down
- Example:
  - 1 year to maturity
  - option to exercise puts and calls in 6 months or 1 year
  - price falls dramatically over the first 6 months
  - $\rightarrow$ induces the holder of the put to exercise and the writer of the put to finance the cash flow
  - no guarantee that the price will rise over the second 6 months to offset this loss with a profit from the call
- $\rightarrow$ put-call strategy with American options is risky

4. Payoff and Price of an American Option

- At each point in time during the life of an American option, the holder can exercise the option or leave it alone

- Optimal exercise implies that they will take the strategy that is worth more

- Cash flow at date τ of an American call with n-period left until expiration:

$$\max\left\{S_\tau - K, C_\tau^{n-1}\right\}$$

  - S is the price of the underlying
  - K is the strike price
  - C^{n-1} is the price of an American call at strike K that expires in $n-1$ periods

- American call price:

$$C_t^n = E_t\left[M_{t,t+1} \max\left\{S_{t+1} - K, C_{t+1}^{n-1}\right\}\right]$$

- American call is found "recursively"

  - $n = 1$: (one-period European)

$$C_{t+n-1}^1 = E_{t+n-1}\left[M_{t+n-1,t+n}\left(S_{t+n} - K\right)^+\right]$$

  - $n = 2$:

$$C_{t+n-2}^2 = E_{t+n-2}\left[M_{t+n-2,t+n-1} \max\left\{S_{t+n-1} - K, C_{t+n-1}^1\right\}\right]$$

  - and so on ...

4. Payoff and Price... cont'd

- Cash flow at date τ of an American put with n-period left until expiration:

$$\max\left\{K - S_\tau, P_\tau^{n-1}\right\}$$

  - S is the price of the underlying
  - K is the strike price
  - P^{n-1} is the price of an American put at strike K that expires in $n-1$ periods

- American put price:

$$P_t^n = E_t\left[M_{t,t+1} \max\left\{S_{t+1} - K, P_{t+1}^{n-1}\right\}\right]$$

- American put is also found "recursively"
  - $n = 1$: (one-period European)

$$P_{t+n-1}^1 = E_{t+n-1}\left[M_{t+n-1,t+n}\left(K - S_{t+n}\right)^+\right]$$

  - $n = 2$:

$$P_{t+n-2}^2 = E_{t+n-2}\left[M_{t+n-2,t+n-1} \max\left\{K - S_{t+n-1}, P_{t+n-1}^1\right\}\right]$$

  - and so on ...

4. Payoff and Price... cont'd

- Note: It is easy to see why Black-Scholes breaks down
- B-S workds for $n = 1$:

$$C^1_{t+n-1} = b^1_{t+n-1} F^1_{t+n-1} \Phi(d) - b^1_{t+n-1} K \Phi(d - \omega)$$

  - recall that $d = \log(F^1_{t+n-1}/K)/\omega + \omega/2$

- this implies that when trying to calculate the expecation at date $t + n - 2$ to calculate C^2_{t+n-2}, we are trying to evaluate an expectation of some highly nonlinear functions of random variables, *e.g.*, $\Phi(d)$
- the "normality" assumptions that help for deriving B-S are of little use here

5. Valuation with Trees

- Backward recursions are very easy to calculate on trees
- Accounts for the popularity of discrete methods in general and binomial models in particular
- Example: 2-year, 5.5% coupon bond
  - Interest rate tree:

			7.8640
		6.9150	
	6.0040		6.1840
5.5400		5.4370	
	4.7210		4.8620
		4.275	
			3.8230

  - 6-month zeros (discount factors):

			0.9622
		0.9666	
	0.9709		0.9700
0.9730		0.9735	
	0.9769		0.9763
		0.9791	
			0.9812

  - Coupon-bond prices (ex-coupon):

			98.863
		98.606	
	99.089		99.668
100.0019		100.021	
	100.955		100.311
		101.155	
			100.823

5. Valuation... cont'd

- Value European call, $n = 3$, $K = 100$
- Value American call, $n = 3$, $K = 100$
- Call Premium?
- **Note:** This methodology is the same for all trees!

6. Example: Callable bond

- The buyer of a callable bond may be viewed as being:
  - long a noncallable bond with the same maturity as the callable one
  - short an option on this bond
- Price of a callable bond

$$P^{(callable)} = P^{(non-call)} - C$$

- In the example, at time-0, the callable bond is worth

$$100.0019 - 0.5003 = 99.5016$$

- Interest rate delta of a callable bond is equal to the delta of the noncallable minus the delta on the option $\rightarrow$ callable bond has less interest rate sensitivity than the noncallable