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Technical Note on the Black-Scholes Formula

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The Black-Scholes formula has been derived many ways. One we find appealing was suggested by Rubinstein (1976): assume that the underlying and the pricing kernel are jointly log-normal. Unlike the original, this does not require deterministic interest rates and is therefore easily applied to fixed income. The analysis consists of evaluating integrals of log-normal random variables. It's tedious at times, but involves no particularly advanced mathematics.

1 Option Overview

General theory. Consider the price at date t of a European option with maturity n periods and strike price K. (The length of a period is h years, but that need not concern us now.) If S is the spot price of the underlying, the call generates a cash flow at $t+n$ of

$$(S_{t+n} - K)^+ ,$$

where $x^+ \equiv \max(x, 0)$. The call price satisfies

$$c_t^n \;=\; E_t \left[M_{t,t+n} \left(S_{t+n} - K\right)^+ \right] . \tag{1}$$

This formula is completely general, in the following sense: in any arbitrage-free environment, there exists a positive random variable M making it true.

Put-call parity. A put option with the same strike K generates a cash flow at $t+n$ of

$$(K - S_{t+n})^+ .$$

Its price p satisfies

$$c_t^n - p_t^n \;=\; b_t^n (F - K). \tag{2}$$

All the relevant features of put prices can be derived from calls using this relation. Logic: We can lock in a fixed price at $t+n$ two ways. The first is to buy a forward contract, which costs F_t^n at $t+n$ or $b_t^n F_t^n$ at t. The second is to buy a call and sell a put at the same strike. The cost is $p_t^n - c_t^n$ plus the present value of the strike price, namely $b_t^n K$. If the two approaches cost the same (an arbitrage result), put-call parity is the result.

The formula. The Black-Scholes formula and its many relatives are special cases of (1). Our favorite version is

$$c_t^n \;=\; b_t^n F_t^n \Phi(d) - b_t^n K \Phi(d - w) \tag{3}$$

where

$$
\begin{aligned}
b_t^n &= \text{n-period discount factor} \\
F_t^n &= \text{forward price of the underlying} \\
\Phi &= \text{cumulative normal distribution function} \\
w^2 &= Var_t\,(\log S_{t+n}) \\
d &= \frac{\log(F/K) + w^2/2}{w}
\end{aligned}
$$

This is a little different from the original, but captures the same information.

2 Log-Normal Formulas

Normal random variables

If $x \sim \mathrm{N}(\mu, \sigma^2)$ (we say: "x is normal with mean μ and variance σ^2"), then its density function is

$$f(x) \;=\; (2\pi\sigma^2)^{-1/2} \exp\left[-(x-\mu)^2/2\sigma^2\right]$$

Normal random variables like x have positive probability over all real numbers, including negative ones. Note that f is a legimate density since it's positive and integrates to one:

$$\int_{-\infty}^{\infty} f(x)dx \;=\; 1.$$

The "standardized" variable $z = (x-\mu)/\sigma$ is N(0,1), so it has density function

$$f(z) \;=\; (2\pi)^{-1/2} \exp(-z^2/2).$$

Graphs of both functions are examples of the familiar bell-shaped curve.

Suppose we are interested in the probability that $x \leq x^*$ for some arbitrary choice of x^*. Given the relation between x and z, this is the same as $z \leq z^* = (x^* - \mu)/\sigma$. (If σ were negative, we'd have to flip the inequality.) Its value is given by

$$\Phi(z^*) \;=\; (2\pi)^{-1/2} \int_{-\infty}^{z^*} \exp(-z^2/2)dz.$$

There's no simple formula for Φ, but it comes up often enough that we give it a letter to identify it. In any case, its value is easily computed for any given z^*. Since $f(z)$ is symmetric $[f(z) = f(-z)]$, the probability that $z \geq z^*$ is

$$\int_{z^*}^{\infty} f(z)dz \;=\; 1 - \Phi(z^*) \;=\; \Phi(-z^*).$$

(Draw a picture of the area under the density to see how this works.)

Mean of a log-normal random variable

Suppose $x = \log X \sim \mathrm{N}(\mu, \sigma^2)$. Then we say X is log-normal (short for "the log of X is normal"). Log-normality is useful in many contexts, since it has the same convenience as normality yet ensures that X is positive (think of an asset price).

Formula 1. What is the mean of $X = e^x$? The answer is

$$E(X) \;=\; e^{\mu + \sigma^2/2}.$$

Let's show why. The expectation involves the integral

$$E\left(e^x\right) \;=\; (2\pi\sigma^2)^{-1/2} \int_{-\infty}^{\infty} e^x e^{-(x-\mu)^2/2\sigma^2} dx.$$

We can rearrange the exponents as follows:

$$x - \frac{(x-\mu)^2}{2\sigma^2} \;=\; \frac{2\sigma^2 x - (x-\mu)^2}{2\sigma^2} \;=\; \frac{[x-(\mu+\sigma^2)]^2}{2\sigma^2} + \mu + \frac{\sigma^2}{2}$$

(This is an old trick called "completing the square.") Then the integral becomes

$$\begin{aligned} E\left(e^x\right) &= e^{\mu + \sigma^2/2} \times (2\pi\sigma)^{-1/2} \int_{-\infty}^{\infty} e^{-(x-\mu-\sigma^2/2)^2/2\sigma^2} dx \\ &= e^{\mu + \sigma^2/2}. \end{aligned} \tag{4}$$

(The part that follows the "$\times$" is the integral of a normal density, namely one.)

Multivariate normal random variables

Suppose x is a vector random variable of dimension n. If $x \sim \mathrm{N}(\mu, \Sigma)$, then

$$f(x) \;=\; (2\pi|\Sigma|)^{-n/2} \exp\left[-(x-\mu)^\top \Sigma^{-1}(x-\mu)/2\right].$$

You'll need to have some familiarity with basic linear algebra to follow this. Here μ is a vector, Σ a matrix, and $|\Sigma|$ the determinant of Σ. If x is two-dimensional, we might write

$$\mu \;=\; \begin{bmatrix} \mu_1 \\ \mu_2 \end{bmatrix}, \quad \Sigma \;=\; \begin{bmatrix} \sigma_1^2 & \sigma_{12} \\ \sigma_{12} & \sigma_2^2 \end{bmatrix}.$$

Then

$$\begin{aligned} |\Sigma| &= \sigma_1^2\sigma_2^2 - (\sigma_{12})^2 \\ \Sigma^{-1} &= [\sigma_1^2\sigma_2^2 - (\sigma_{12})^2]^{-1} \begin{bmatrix} \sigma_2^2 & -\sigma_{12} \\ -\sigma_{12} & \sigma_1^2 \end{bmatrix}. \end{aligned}$$

Two more integral formulas

Formula 2. Suppose $X = (X_1, X_2)$ is bivariate log-normal, meaning $x = \log X$ is bivariate normal. Consider an integral like

$$E(X_1|X_2 \geq K) \;=\; (2\pi|\Sigma|)^{-n/2} \int_{x_1=-\infty}^{\infty} \int_{x_2=\log K}^{\infty} e^{x_1} \exp\left[-(x-\mu)^\top \Sigma^{-1}(x-\mu)/2\right] dx_2 dx_1.$$

(This should look vaguely like (1), which is where we'll put it to work.) The answer is

$$E(X_1|X_2 \geq K) \;=\; E(X_1)\Phi(d), \tag{5}$$

where

$$d \;=\; \frac{\mu_2 - \log K + \sigma_{12}}{\sigma_2}.$$

Intuition: If X_1 and X_2 were independent, the answer would be the product of $E(X_1)$ and the probability that $X_2 \geq K$. If $\sigma_{12} = 0$, that's what we have here. The more general case covers the correlation between X_1 and X_2.

Deriving Formula 2. We approach this as we did $E(X)$, but it's considerably more tedious. Note that the joint density can be expressed

$$f(x_1, x_2) \;=\; f(x_1|x_2) f(x_2). \tag{6}$$

In this case all the relevant densities are normal: $x_2 \sim \mathrm{N}(\mu_2, \sigma_2^2)$ and $(x_1|x_2) \sim \mathrm{N}(a_1, b_1)$ with

$$\begin{aligned} a_1 &= \mu_1 + \left(\frac{\sigma_{12}}{\sigma_2^2}\right)(x_2 - \mu_2) \\ b_1 &= \left(1 - \rho^2\right)\sigma_1^2 \\ \rho &= \frac{\sigma_{12}}{\sigma_1 \sigma_2}. \end{aligned}$$

The first can be thought of as the regression line of x_1 on x_2, the second the variance of the residuals from the regression. They can be verified by showing that the densities satisfy equation (6). Note that when $\sigma_{12} = 0$ (x_1 and x_2 are uncorrelated), $(a_1, b_1) = (\mu_1, \sigma_1^2)$, which sounds right.

Now consider the integral:

$$E(X_1|X_2 \geq K) \;=\; \int_{x_2=\log K}^{\infty} f(x_2) \left[\int_{x_1=-\infty}^{\infty} e^{x_1} f(x_1|x_2) dx_1\right] dx_2$$

We can apply Formula 1 to the expression in brackets:

$$\begin{aligned}
\int_{x_1=-\infty}^{\infty} e^{x_1} f(x_1|x_2)dx_1 &= \exp(a_1 + b_1/2) \\
&= \exp[\mu_1 + (1-\rho^2)\sigma_1^2/2 + (\sigma_{12}/\sigma_2^2)(x_2-\mu_2)] \\
&= \exp[\mu_1 + \sigma_1^2/2 - \rho^2\sigma_1^2/2 + (\sigma_{12}/\sigma_2^2)(x_2-\mu_2)] \\
&= E(X_1)\exp[-\rho^2\sigma_1^2/2 + (\sigma_{12}/\sigma_2^2)(x_2-\mu_2)].
\end{aligned}$$

The integral is therefore

$$E(X_1) \times (2\pi\sigma_2^2)^{-1/2} \int_{x_2=\log K}^{\infty} \exp[(\sigma_{12}/\sigma_2^2)(x_2-\mu_2) - \sigma_{12}^2/\sigma_2^2 - (x_2-\mu_2)^2/2\sigma_2^2]dx_2$$

The exponent can be rewritten as

$$(\sigma_{12}/\sigma_2^2)(x_2-\mu_2) - \sigma_{12}^2/\sigma_2^2 - (x_2-\mu_2)^2/2\sigma_2^2 \;=\; \frac{(x_2-\mu_2-\sigma_{12})^2}{2\sigma_2^2},$$

which allows us to write the integral as

$$E(X_1) \times (2\pi\sigma_2^2)^{-1/2} \int_{x_2=\log K}^{\infty} \exp[-(x_2-\mu_2-\sigma_{12})^2/2\sigma_2^2]dx_2.$$

If we define $z = (x_2 - \mu_2 - \sigma_{12})/\sigma_2$, this becomes

$$E(X_1) \times (2\pi)^{-1/2} \int_{z=z^*}^{\infty} e^{-z^2/2} dz$$

with $z^* = (\log K - \mu_2 - \sigma_{12})/\sigma_2$. The expression after the "$\times$"is

$$Prob(z \geq z^*) \;=\; 1 - \Phi(z^*) \;=\; \Phi(-z^*),$$

the last step following from the symmetry of the normal distribution. Since $d = -z^*$, that leaves us with Formula 2.

Formula 3. For the same setting,

$$E\left(X_1X_2|X_2 \geq K\right) \;=\; E(X_1X_2)\Phi(d), \tag{7}$$

where

$$d \;=\; \frac{\mu_2 - \log K + \sigma_{12} + \sigma_2^2}{\sigma_2}.$$

This follows from (5) with the change in variables $X_1 = X_1X_2$. Eg, we replace $Cov(x_1, x_2) = \sigma_{12}$ with $Cov(x_1 + x_2, x_2) = \sigma_{12} + \sigma_2^2$.

3 Deriving the Black-Scholes Formula

The Black-Scholes formula (3) follows from (1) if $(M_{t,t+n}, S_{t+n})$ is (conditionally) bivariate log-normal. We show this by applying the three formulas above.

Forward prices

A forward contract arranged at t specifies the exchange at a future date $t+n$ of an asset at a price F_t^n set at t. The price satisfies

$$0 \;=\; E_t \left[M_{t,t+n} \left(S_{t+n} - F_t^n \right) \right]$$

or

$$E_t \left(M_{t,t+n} \right) F_t^n \;=\; b_t^n F_t^n \;=\; E_t \left(M_{t,t+n} S_{t+n} \right).$$

Applying Formula 1 for expectations of log-normals, equation (4), we get

$$\begin{aligned}
b_t^n F_t^n &= E_t \left(M_{t,t+n} S_{t+n} \right) \;=\; \exp(\mu_1 + \mu_2 + [\sigma_1^2 + \sigma_2^2 + 2\sigma_{12}]/2) \\
b_t^n &= E_t \left(M_{t,t+n} \right) \;=\; \exp(\mu_1 + \sigma_1^2/2) \\
F_t^n &= E_t \left(M_{t,t+n} S_{t+n} \right) / E_t \left(M_{t,t+n} \right) \;=\; \exp(\mu_2 + \sigma_{12} + \sigma_2^2/2).
\end{aligned}$$

The first and third of these are used below.

Expectations

Turn now to the two expectations in (1). The first is

$$E_t \left(M_{t,t+n} S_{t+n} | S_{t+n} \geq K \right) \;=\; b_t^n F_t^n \Phi(d)$$

with

$$d \;=\; \frac{\mu_2 - \log K + \sigma_{12} + \sigma_2^2}{\sigma_2} \;=\; \frac{\log(F_t^n/K) + \sigma_2^2/2}{\sigma_2}.$$

This follows from Formula 3 with $(M_{t,t+n}, S_{t+n}) = (X_1, X_2)$. The second is

$$E_t \left(M_{t,t+n} | S_{t+n} \geq K \right) K \;=\; b_t^n K \Phi(d),$$

with

$$d \;=\; \frac{\mu_2 - \log K + \sigma_{12}}{\sigma_2} \;=\; \frac{\log(F_t^n/K) - \sigma_2^2/2}{\sigma_2}.$$

This follows from Formula 2 with $(M_{t,t+n}, S_{t+n}) = (X_1, X_2)$. If we equate w with σ_2, the Black-Scholes formula (3) follows immediately.

4 Annualized Volatility

Industry practice differs in one respect: volatility is typically expressed as an annual percentage. If the maturity of the option is $N = hn$ years, then standard practice is to report annualized volatility v:

$$w^2 = N v^2.$$

In most versions of the formula, volatility w is replaced by it annualized version, $N^{1/2} v$ and multiplied by 100 to make it a percentage.

References

Mark Rubinstein, "The valuation of uncertain income streams and the pricing of options," *Bell Journal of Economics* 7, 407-425.