The Chromatic Differential Factors

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Abstract: Here we show that the Chromatic Differential factors as the composition of the Kunneth Homorphism and the Diagonal Map on the deRham complex.

1. The Kunneth Homorphism on the deRham Complex

Let $\pi : M \times M \to M$ be given by projection onto the first factor and $\rho : M \times M \to M$ be given by projection onto the second factor. Then we get induced maps on the deRham complex given by $\pi^* : \Omega_c^*(M) \to \Omega_{M\times M}^*$ and $\rho^* : \Omega_c^*(M) \to \Omega_{M\times M}^*$. Note that the pullback of a compact form under a projection is not necessarily compact.

Now the Kunneth Homorphism, $\psi : \Omega_c^*(M) \otimes \Omega_c^*(M) \to \Omega_c^*(M \times M)$ is given by $\omega \otimes \phi \mapsto \pi^*(\omega) \wedge \rho^*(\phi)$. Since $supp\big(\pi^*(\omega) \wedge \rho^*(\phi)\big) = supp(\omega) \times supp(\phi)$ we get that $supp(\pi^*(\omega) \wedge \rho^*(\phi)) \subset M \times M$ is compact because the finite product of compact sets is compact. This map then induces the familiar isomorphism on compactly supported cohomology, $\psi^* : H_c^*(M) \otimes H_c^*(M) \to H_c^*(M \times M)$ under the right assumptions on M.

2. The Diagonal Map

Now consider the embedding of the diagonal into the product denoted by $D : M \to M \times M$. Then we get an induced map on the deRham complex $D^* : \Omega_c^*(M \times M) \to \Omega_c^*(M)$ where $D^*(\omega)$ is just the restriction of ω to the diagonal $D(M) \subset M \times M$.

Recall that the compact subset of a Hausdorff space is closed. Therefore $supp(\omega)$ is closed in $M \times M$. It follows that $supp(D^*(\omega)) = supp(\omega) \cap D(M)$ is closed in $M \times M$ since it is the intersection of two closed sets in $M \times M$. Next, $supp(D^*(\omega)) \subset M \times M$ is a closed subset of $supp(\omega) \subset M \times M$ which is compact. Since the closed subset of a compact set is also compact, we get that $supp(D^*(\omega))$ is compact in $M \times M$. Finally, since $D(M)$ is a closed subspace of $M \times M$, it follows that $supp(D^*(\omega))$ is compact in the diagonal $D(M)$.

3. The Chromatic Differential

From these two maps we get the following diagram,

$$\begin{tikzcd}
\Omega_c^*(M) \otimes \Omega_c^*(M) \arrow[d, "\psi"'] \arrow[dr, "\partial = D^* \circ \psi"] & \\
\Omega_c^*(M \times M) \arrow[r, "D^*"'] & \Omega_c^*(M)
\end{tikzcd}$$

Now we see that the composition $D^* \circ \psi : \Omega_c^*(M) \otimes \Omega_c^*(M) \to \Omega_c^*(M)$ is given by $\omega \otimes \phi \mapsto \left(\pi^*(\omega) \wedge \rho^*(\phi)\right)\Big|_{D(M)}$ which is exactly action of the chromatic differential, ∂, in the chromatic complex.

The nice thing about this factorization is that it makes explicit how the chromatic complex is encoding information about the diagonals in the graph configuration space. Namely, the kernel of the chromatic differential is generated by tensor products of forms, which when wedged together, restrict to zero on the diagonal.