Introduction to the Chromatic Complex and Its Application to deRham Cohomology

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Chapter 1

Preliminaries

*The contents of this chapter are not my original work. It is my reproduction of parts of "A Categorification for the Chromatic Polynomial" by Helme-Guizon and Rong which are necessary for the exposition of my efforts which begin in Chapter 2 and in other accompanying documents.

1.1 Review of the Chromatic Complex

1.1.1 Graphs

We will denoted a given graph by G with vertex set $V(G)$ and edge set $E(G)$. We will consider subgraphs of G, denoted G_s, where $V(G_s) = V(G)$ and $E(G_s) = s \subseteq E(G)$. Lastly, we will let k_s denote the number of connected components of the subgraph G_s.

Later on we will need an ordering of the edge set $E(G)$. If we let $|E(G)| = n$, then an ordering of $E(G)$ allows us to represent each $s \subseteq E(G)$ by an ordered n-tuple of ones and zeros, denoted ε, where the j^{th}-element of ε is one if the j^{th}-edge in $E(G)$ is also an element of s, and zero otherwise. Then we can equivalently denote $G_s \equiv G_\varepsilon$ and $k_s \equiv k_\varepsilon$.

Finally, recall that a graph-morphism between two graphs, G and H, is a map $f : V(G) \to V(H)$ such that if $v, w \in V(G)$ are connected by an edge $[v, w] \in E(G)$, then their images $f(v), f(w) \in V(H)$ are connected by an edge $[f(v), f(w)] \in E(H)$.

1.1.2 Chromatic Polynomial

For a given graph G the Chromatic Polynomial, $P_G(\lambda)$, is a function which returns the number of ways to color $V(G)$ with λ-many colors such that no two vertices connected by an edge have the same color.

For the graph with n-vertices and no edges, i.e. where $|V(G)| = n$ and $|E(G)| = 0$, all possible colorings of the n-vertices are permissible. In this case we get $P_G(\lambda) = \lambda^n$. Using this observation as a base case, there is a recursive relation know as the Deletion-Contraction Rule which allows for the construction of the Chromatic Polynomial for a general graph. The Deletion-Contraction Rule is given by:

$$P_G(\lambda) = P_{G\setminus e}(\lambda) - P_{G/e}(\lambda) \tag{1.1}$$

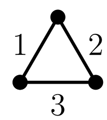
Where G is the given graph, $G \setminus e$ is the subgraph such that $V(G \setminus e) = V(G)$ and $E(G \setminus e) = E(G) \setminus e$ (i.e. the e-edge is deleted from G), and G/e is the resulting graph given by contracting the e-edge in G and identifying the vertices at it's endpoints.

This recursion relation can be resolved into an explicit form called the State Sum Formula which is given by:

$$P_G(\lambda) = \sum_{i \geq 0} (-1)^i \sum_{\substack{s \subseteq E(G) \\ |s| = i}} \lambda^{K_s} \tag{1.2}$$

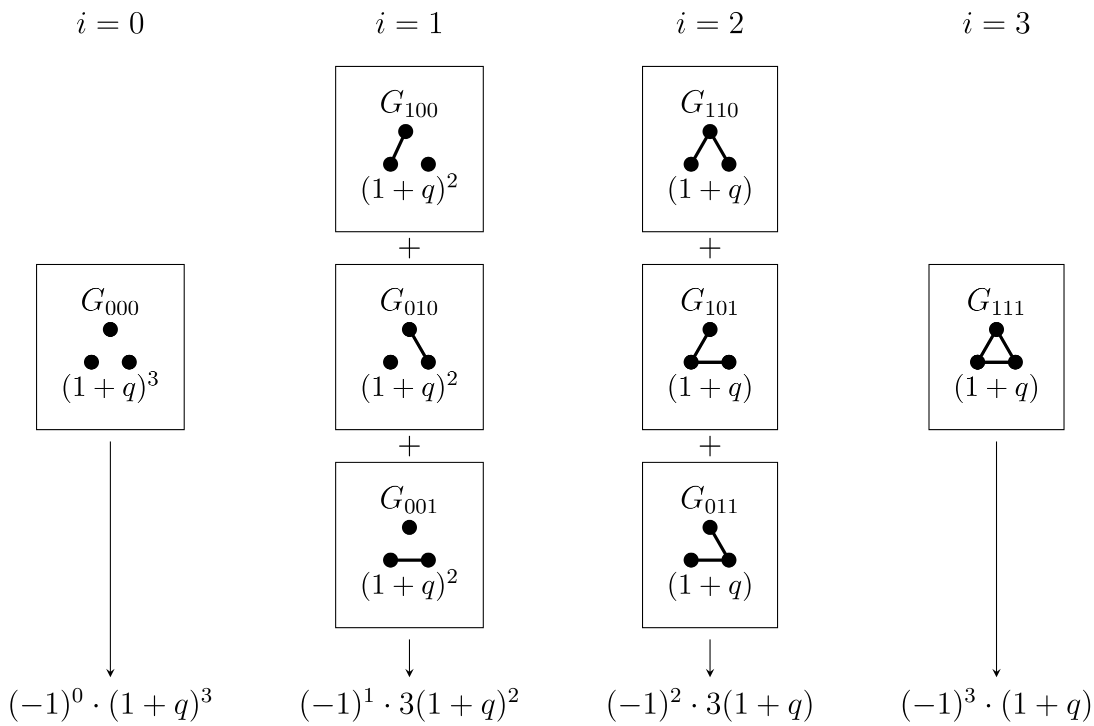
We say two subgraphs G_ε and $G_{\varepsilon'}$ are in the same state if they have the same number of edges. That is they are in the same state if $|\varepsilon| = |\varepsilon'|$. Then we can see that Eq.1.2 is named appropriately as the second sum is over all subgraphs in the same state.

The State Sum Formula can be conveniently expressed diagrammatically. We consider this representation in the following example. Notice that the variable substitution $\lambda = 1 + q$ has been made. This is for reasons which will be left to §1.1.3.

Example 1.1.1. $G =$ 

& $\lambda = 1 + q$

Here is presented the computation of the Chromatic Polynomial for the complete graph on three vertices using the State Sum Formula. The subgraphs have been organized into columns in accordance with their states. Compare this diagram with Eq.1.2.



$$P_G(1+q) = (1+q)^3 - 3(1+q)^2 + 3(1+q) - (1+q) \tag{1.3}$$

1.1.3 Algebra

Let $M = \bigoplus_{i=1}^{n} M^i$ be a graded $\mathbb{Z}$-module. Then the graded-dimension, or q-dimension, is given by

$$\mathrm{qdim}(M) = \sum_{i=0}^{n} q^i \,\mathrm{rank}(M^i) \tag{1.4}$$

where

$$\mathrm{rank}(M_i) = \dim_{\mathbb{Q}}(M^i \otimes \mathbb{Q}) \tag{1.5}$$

Note that the following properties hold for graded $\mathbb{Z}$-modules M and N:

$$\mathrm{qdim}(M \oplus N) = \mathrm{qdim}(M) + \mathrm{qdim}(N) \tag{1.6}$$

$$\text{qdim}(M \otimes N) = \text{qdim}(M) \cdot \text{qdim}(N) \tag{1.7}$$

Now for an example which motivates the change of variables in Ex. 1.1.1.

Example 1.1.2. *Consider the $\mathbb{Z}$-module $M = \mathbb{Z} \oplus \mathbb{Z}\langle x \rangle$. Then the q-dimension of M is given by:*

$$qdim(M) = 1 + q. \tag{1.8}$$

Furthermore, the q-dimension of it's k^{th} tensor power is given by:

$$qdim(M^{\otimes k}) = (1 + q)^k. \tag{1.9}$$

Finally, recall that the Graded Euler Characteristic of a graded $\mathbb{Z}$-module is given by

$$\chi_q(M) = \sum_{i=0}^{n} (-1)^i \text{qdim}(M^i) \tag{1.10}$$

1.1.4 The Construction

The Chain Groups

The diagrammatic representation of the State Sum Formula exemplified in Ex.1.1.1 can be exploited to construct a chain complex which categorifies the Chromatic Polynomial as well as the Deletion Contraction Rule. The Chain groups are given by the following prescriptions.

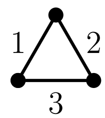
For each G_ε of G, assign the graded $\mathbb{Z}$-module $M^{\otimes k_\varepsilon}$, then direct sum all modules associated to subgraphs of the same state. Then you will have a direct sum of tensor powers of M for each state. These are the chain groups, the i^{th} chain group coming from the i^{th} state.

For motivation, we consider the following:

$$\begin{aligned}
P_G(1+q) &= \sum_{i\geq 0}(-1)^i \sum_{\substack{s\subseteq E(G)\\ |s|=i}} (1+q)^{k_s}\\
&= \sum_{i\geq 0}(-1)^i \sum_{\substack{s\subseteq E(G)\\ |s|=i}} \mathrm{qdim}(M^{\otimes k_s})\\
&= \sum_{i\geq 0}(-1)^i \mathrm{qdim}\left(\bigoplus_{|s|=i} M^{\otimes k_s}\right)\\
&= \chi_q\big(C(G,M)\big)
\end{aligned}$$

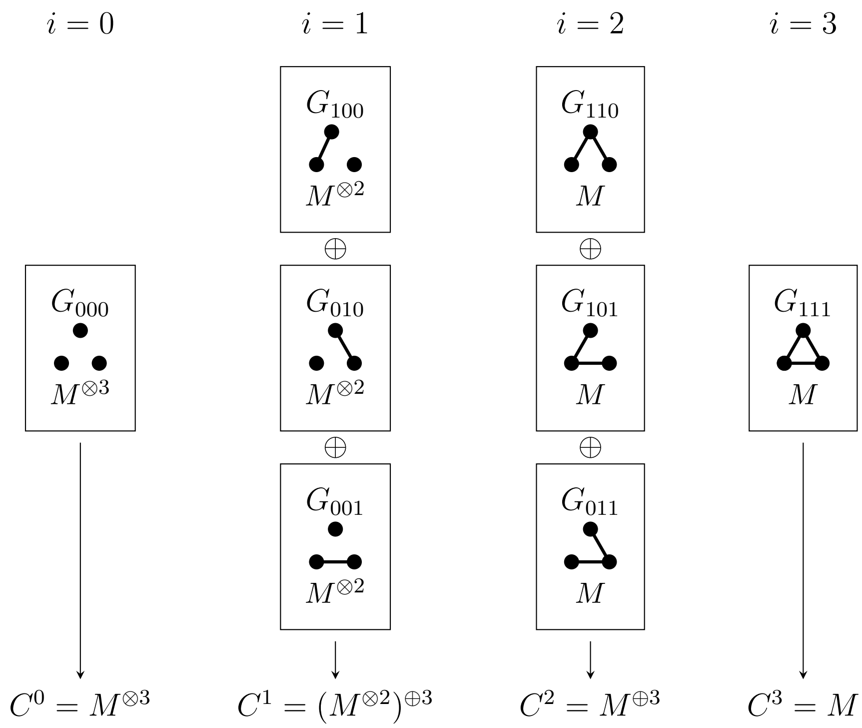
where $C(G,M) = \underset{i}{\oplus} C^i(G,M)$ is the graded $\mathbb{Z}$-module, which depends on G and M, whose i^{th} grading is $C^i(G,M) = \bigoplus_{|s|=i} M^{\otimes K_s}$.

This formulation can be more clearly interpreted in the following example which is a modified reproduction of Ex.1.1.1 in accordance with the new prescriptions.

Example 1.1.3. $G =$ 

$\quad \&\quad M = \mathbb{Z} \oplus \mathbb{Z}\langle x \rangle$

Here is presented the modified computation of the Chromatic Polynomial for the complete graph on three vertices using the State Sum Formula wherein the powers of $(1+q)$ *have been replaced with tensor powers of the* $\mathbb{Z}$*-module,* $M = \mathbb{Z} \oplus \mathbb{Z}\langle x \rangle$.



Now that the chain groups have been defined, it remains to define the differential. This is the project of the following section.

The Differential

We begin with the observation that given a subgraph G_s in the i^{th} state, for each edges $e \in E(G) \setminus s$ we can obtain from G_s an element $G_{s\cup\{e\}}$ in the $(i+1)^{st}$ state by simply including the e-edge into G_s. Note that in the alternative n-tuple notation, this amounts to changing G_ε to $G_{\varepsilon'}$ where ε' is obtain from ε by altering the element corresponding to the e-edge from a zero to a one.

Referring to Ex. 1.1.3, we see that any subgraph in the $i = 1$ state can be obtain from the subgraph in the $i = 0$ state by adding one of the edges in $E(G)$ to G_{000} (producing thereby G_{100}, G_{010}, and G_{001} respectively).

The goal is to define maps from the $\mathbb{Z}$-module assigned to a given G_ε in a state i to the $\mathbb{Z}$-modules assigned to each $G_{\varepsilon'}$ in the state $i+1$ which can be obtained from G_ε by adding an edge. Once this is accomplished, the differential on the level of the chain groups will be given as a signed sum of these maps.

With all this in mind, let $M^{\otimes k_\varepsilon}$ be a $\mathbb{Z}$ module assigned to G_ε in state i, and $M^{\otimes k_{\varepsilon'}}$ be a $\mathbb{Z}$-module assigned to some $G_{\varepsilon'}$ in state $i+1$ which is obtained from G_ε by adding an edge. Then the map $(-1)^{\varepsilon_*}\partial^i_{\varepsilon_*} : M^{\otimes k_\varepsilon} \to M^{\otimes k_{\varepsilon'}}$, where ε_* is the ordered n-tuple ε with the element corresponding to the added edge changed from a zero to an asterisk, is given by the following:

1. $(-1)^{\varepsilon_*}$ is 1 if the number of ones in ε_* before the asterisk is even, and -1 if the number of such ones on odd.

2. If $k_{\varepsilon'} = k_\varepsilon$, i.e. if adding the edge to G_ε does not change the number of connected components, then $\partial^i_{\varepsilon_*} =$ Identity.

3. If $K_{\varepsilon'} = K_\varepsilon - 1$, i.e. if adding the edge to G_ε decreases the number of connected components by one, then $\partial^i_{\varepsilon_*} =$ Identity on all of the tensor

factors except the two corresponding to the components which become connected upon adding the edge. On those two tensor factors $\partial^i_{\varepsilon_*}$ is given by a multiplication $m : M \otimes M \to M$ which maps to the tensor factor in the image $M^{\otimes k_{\varepsilon'}}$ which corresponds to the connected component resulting from adding the edge.

In the case of $M = \mathbb{Z} \oplus \mathbb{Z}[x]$, this multiplication is given on the generators by the following:

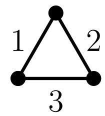
(i) $m(1 \otimes 1) = 1$

(ii) $m(1 \otimes x) = 1 = m(x \otimes 1)$

(iii) $m(x \otimes x) = 0$

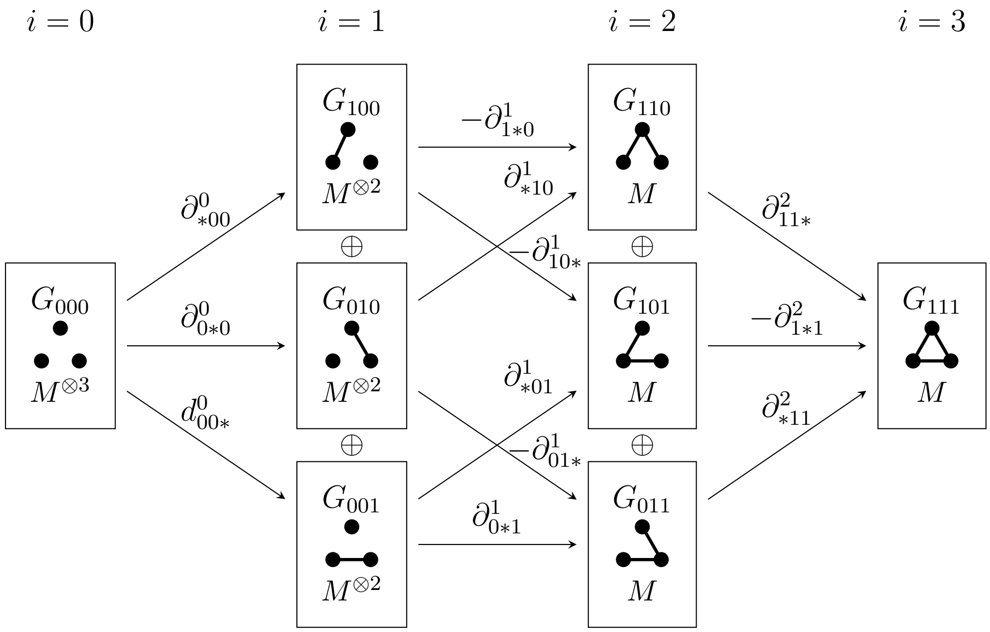
Now that the maps have been defined, the differential $\partial^i : C^i \to C^{i+1}$ is given by the sum of the maps $\partial^i_{\varepsilon_*}$ from state i to state $i+1$. That is, $\partial^i = \sum\limits_{j=i} (-1)^{\varepsilon_*} \partial^j_{\varepsilon_*}$

Again, we update the running example now to include the differential maps with their proper labeling and signs.

Example 1.1.4. $G =$ 

$\&$ $M = \mathbb{Z} \oplus \mathbb{Z}\langle x \rangle$

Here is presented a further modification of Ex.1.1.1 where the maps between states and the composite differentials are depicted.



$$C^0 \xrightarrow{\partial^0} C^1 \xrightarrow{\partial^1} C^2 \xrightarrow{\partial^2} C^3$$

Where:

$$\partial^0 = \partial^0_{*00} + \partial^0_{0*0} + \partial^0_{00*}$$

$$\partial^1 = -\partial^1_{1*0} + \partial^1_{*10} - \partial^1_{10*} + \partial^1_{*01} - \partial^1_{01*} + \partial^1_{0*1}$$

$$\partial^2 = \partial^2_{11*} - \partial^2_{1*1} + \partial^2_{*11}$$

Chapter 2

The Chromatic Complex for Smooth Manifolds

2.1 Modifying the Construction

We begin with the observation that the $\mathbb{Z}$-module considered in the Chromatic Polynomial example of the previous chapter is isomorphic to the Singular Cohomology of $\mathbb{S}^1$ with integral coefficients, i.e. $H^*_{sing}(\mathbb{S}^1;\mathbb{Z}) = \mathbb{Z} \oplus \mathbb{Z}\langle x\rangle$. Furthermore, we can change coefficients from $\mathbb{Z}$ to $\mathbb{R}$ resulting in real vector spaces in place of free $\mathbb{Z}$-modules without any alterations to the theory. In this vein, we consider the deRham Cohomology real coefficients, $H^*_{dr}(\mathbb{S}^1;\mathbb{R}) = \mathbb{R} \oplus \mathbb{R}\langle x\rangle$.

Given a smooth manifold, it's deRham Cohomology may be computed and the resulting graded vector space used to compute the Chromatic Cohomology of a graph. The natural question arises, is there a way to commute these operations. That is, can we compute the Chromatic Cohomology using the cochain complex of differential forms on a given smooth manifold, then compute the deRham Cohomology on the result?

2.1.1 The Chain Complex

We begin by altering the Chromatic Cohomology chain complex via the following prescription. Given a smooth manifold M, we denote the graded real vector space of differential forms on M by Ω^*_M which, when paired with the exterior derivative d, forms the co-chain complex underlying deRham Cohomology. Instead of some free $\mathbb{Z}$-module, we will now assign tensor powers of Ω^*_M to each subgraph G_ε (where the tensor products are over $\mathbb{R}$).

2.1.2 The Differential

Following the construction in § 1.1.4, we begin with per-edge maps from the i^{th} state to the $(i+1)^{st}$ state and define the i^{th} differential map as their signed sum. Since Ω^*_M has an algebra structure given by the wedge product, we can similarly define the per-edge maps in terms of a multiplication. However, the wedge product has some skew-symmetric properties which will need to be addressed via another sign convention.

Let $(\Omega^*_M)^{\otimes k_\varepsilon}$ be the real vector space assigned to G_ε in state i, and $(\Omega^*_M)^{\otimes k_{\varepsilon'}}$ be the real vector space assigned to some $G_{\varepsilon'}$ in state $i+1$ which is obtained from G_ε by adding an edge. Then the map

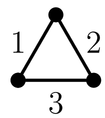
$$(-1)^{\varepsilon_*}\partial^i_{\varepsilon_*} : (\Omega^*_M)^{\otimes k_\varepsilon} \to (\Omega^*_M)^{\otimes k_{\varepsilon'}}$$

is given as follows.

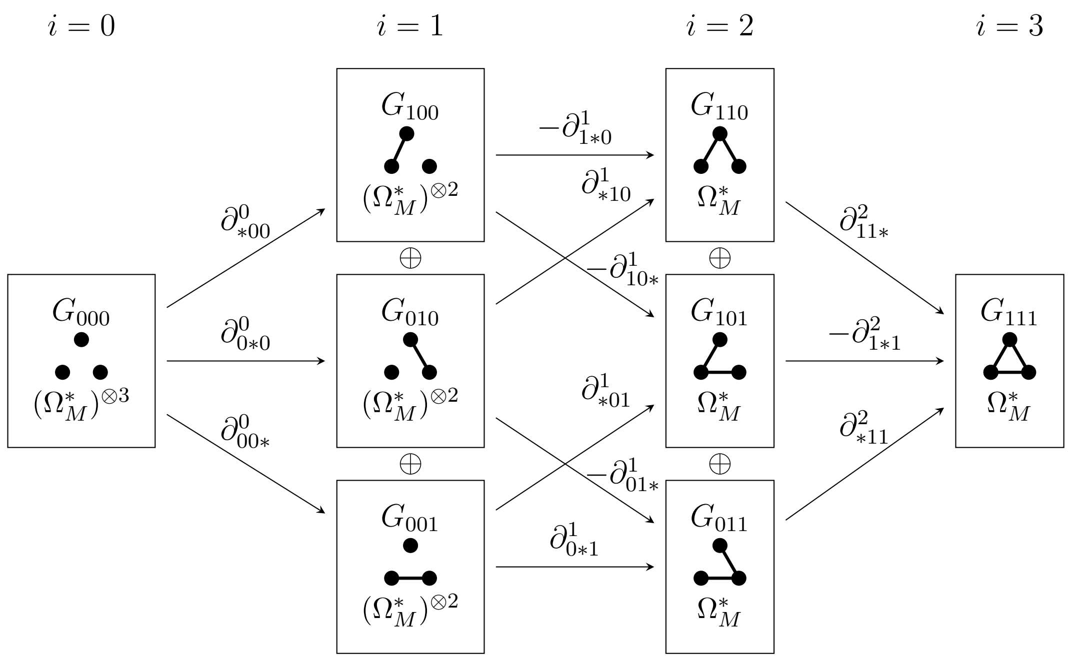
$$\partial_{\varepsilon^*}\left(\bigotimes_{n=1}^{k_\varepsilon} \alpha_n\right) = \left\{ \begin{array}{cc} \bigotimes_{n=1}^{k_\varepsilon} \alpha_n & \text{if } k_{\varepsilon'} = k_\varepsilon \\ \mathrm{s}(i,j)\alpha_1 \otimes ... \otimes \alpha_i \wedge \alpha_j \otimes ...\hat{\alpha_j}... \otimes \alpha_k & \text{if } k_{\varepsilon'} = k_\varepsilon - 1 \end{array} \right\}$$

where: $\mathrm{s}(i,j) = (-1)^{\deg(\alpha_j)\cdot \sum_{n=i+1}^{j-1} \deg(\alpha_n)}$ and where $(-1)^{\varepsilon_*}$ is given as before.

Now our example becomes the following.

Example 2.1.1. $G =$ 

Here is presented Ex.1.1.4 adapted for smooth manifolds.



$$C^0 \xrightarrow{\partial^0} C^1 \xrightarrow{\partial^1} C^2 \xrightarrow{\partial^2} C^3$$

Where:

$$\partial^0 = (-1)^\star \partial^0_{*00} + (-1)^\star \partial^0_{0*0} + (-1)^\star \partial^0_{00*}$$

$$\partial^1 = -(-1)^\star \partial^1_{1*0} + (-1)^\star \partial^1_{*10} - (-1)^\star \partial^1_{10*} + (-1)^\star \partial^1_{*01} - (-1)^\star \partial^1_{01*} + (-1)^\star \partial^1_{0*1}$$

$$\partial^2 = (-1)^\star \partial^2_{11*} - (-1)^\star \partial^2_{1*1} + (-1)^\star \partial^2_{*11}$$

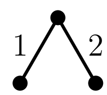
$$C^0 = (\Omega^*_M)^{\otimes 3}$$

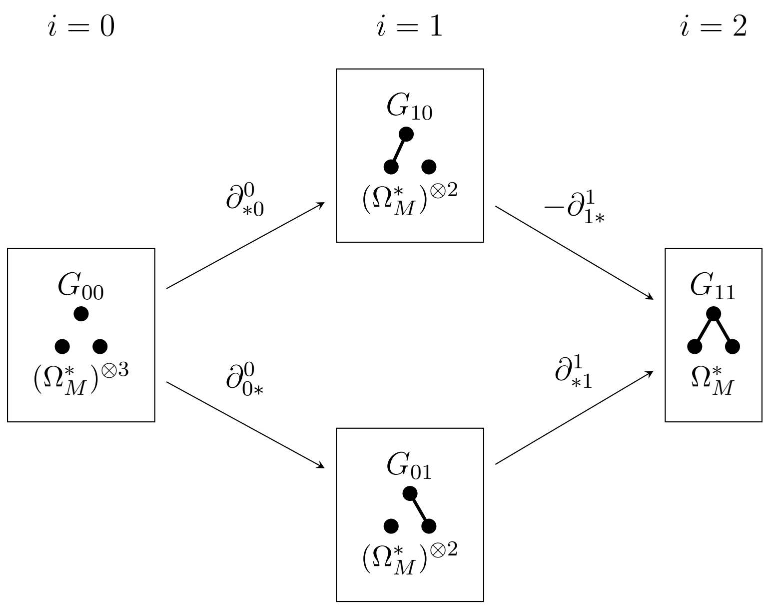
$$C^1 = (\Omega^*_M)^{\otimes 2} \oplus (\Omega^*_M)^{\otimes 2} \oplus (\Omega^*_M)^{\otimes 2}$$

$$C^2 = \Omega^*_M \oplus \Omega^*_M \oplus \Omega^*_M, \quad \& \quad C^3 = \Omega^*_M$$

2.2 A Bigraded Complex

Notice that every grading C^i in the preceeding examples is also a graded space since each C^i consists of tensor powers and direct sums of the graded space Ω_M^*. For simplicity we now chose to omit the third edge from the previous examples. In that case, the Chromatic Complex simplifies to just the top face of the cube given by the subgraphs, $G_{000}, G_{100}, G_{010}$, and G_{110} as follows.

Example 2.2.1. $G =$ 



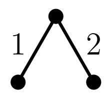
$$C^0 \xrightarrow{\partial^0} C^1 \xrightarrow{\partial^1} C^2$$

Where:

$$\partial^0 = \partial_{*0}^0 + \partial_{0*}^0 \quad \& \quad \partial^1 = -\partial_{1*}^1 + \partial_{*1}^1$$

$$C^0 = (\Omega_M^*)^{\otimes 3} \quad \& \quad C^1 = (\Omega_M^*)^{\otimes 2} \oplus (\Omega_M^*)^{\otimes 2} \quad \& \quad C^2 = \Omega_M^*$$

Now we expand this complex into its explicit bi-complex where the chromatic grading is vertical and the deRham grading is horizontal to obtain the following example.

Example 2.2.2. $G =$ 

$\quad \& \quad M = \mathbb{R}^1$

$$
\begin{array}{ccccccc}
\Omega^0_M & \xrightarrow{d} & \Omega^1_M & \xrightarrow{d} & 0 & \xrightarrow{d} & 0 \\
\partial^1 \Big\uparrow & & \partial^1 \Big\uparrow & & \partial^0 \Big\uparrow & & \partial^0 \Big\uparrow \\
\begin{array}{c}(\Omega^0_M \otimes \Omega^0_M) \\ \oplus \\ (\Omega^0_M \otimes \Omega^0_M)\end{array} & \xrightarrow{d} & \begin{array}{c}(\Omega^0_M \otimes \Omega^1_M) \oplus (\Omega^1_M \otimes \Omega^0_M) \\ \oplus \\ (\Omega^0_M \otimes \Omega^1_M) \oplus \Omega^1_M \otimes \Omega^0_M)\end{array} & \xrightarrow{d} & \begin{array}{c}(\Omega^1_M \otimes \Omega^1_M) \\ \oplus \\ (\Omega^1_M \otimes \Omega^1_M)\end{array} & \xrightarrow{d} & 0 \\
\partial^0 \Big\uparrow & & \partial^0 \Big\uparrow & & \partial^0 \Big\uparrow & & \partial^0 \Big\uparrow \\
(\Omega^0_M \otimes \Omega^0_M \otimes \Omega^0_M) & \xrightarrow{d} & \begin{array}{c}(\Omega^1_M \otimes \Omega^0_M \otimes \Omega^0_M) \\ \oplus \\ (\Omega^0_M \otimes \Omega^1_M \otimes \Omega^0_M) \\ \oplus \\ (\Omega^0_M \otimes \Omega^0_M \otimes \Omega^1_M)\end{array} & \xrightarrow{d} & \begin{array}{c}(\Omega^0_M \otimes \Omega^1_M \otimes \Omega^1_M) \\ \oplus \\ (\Omega^1_M \otimes \Omega^0_M \otimes \Omega^1_M) \\ \oplus \\ (\Omega^1_M \otimes \Omega^1_M \otimes \Omega^0_M)\end{array} & \xrightarrow{d} & (\Omega^1_M \otimes \Omega^1_M \otimes \Omega^1_M)
\end{array}
$$