Differentials for Composite deRham and Chromatic Cohomologies

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Abstract: Here we define the differentials for the composite deRham and Chromatic complexes for a graph G and smooth manifold M. We then show that these differentials square to zero and commute.

1. Definitions

A basic element of a chain group in the composite complex is a formal tensor product over $\mathbb{R}$ of differential forms over M of non-necessarily equal degrees and is denoted by, $\overset{k}{\underset{n=1}{\otimes}} \alpha_n$. It will be enough to check the desired properties on a basic element. The more general properties will then follow from linearity.

Recall that the chromatic differential, ∂, is given by a signed sum of per edge maps ∂_{ε^*} which are given on a basis element by

$$\partial_{\varepsilon^*}\left(\overset{k}{\underset{n=1}{\otimes}} \alpha_n\right) = \left\{ \begin{array}{cc} \overset{k}{\underset{n=1}{\otimes}} \alpha_n & \text{if } k_{\varepsilon'} = k_\varepsilon \\ \mathrm{s}(i,j)\alpha_1 \otimes ... \otimes \alpha_i \wedge \alpha_j \otimes ...\hat{\alpha_j} ... \otimes \alpha_k & \text{if } k_{\varepsilon'} = k_\varepsilon - 1 \end{array} \right\}$$

$$\text{where: } \mathrm{s}(i,j) = (-1)^{\deg(\alpha_j)\cdot \sum\limits_{n=i+1}^{j-1} \deg(\alpha_n)}$$

where in the first case the edge being added preserves the number of connected components of the subgraph ($k_{\varepsilon'} = k_\varepsilon$) and thus the per edge map is the identity map. In the second case, the edge being added reduces the number of connected components by 1 by connecting the i^{th} component to the j^{th} component.

The deRham differential of a basic element in the formal tensor product is given by the extended Leibniz rule as follows

$$\mathrm{d}\left(\overset{k}{\underset{n=1}{\otimes}} \alpha_n\right) = \sum_{n=1}^{k} (-1)^{\sum\limits_{m=1}^{n-1} \deg(\alpha_m)} \alpha_1 \otimes ... \otimes \mathrm{d}\alpha_n \otimes ... \otimes \alpha_k$$

2. $\mathbf{d}\partial = \partial\mathbf{d}$

Since the chromatic differential is a signed sum of the per-edge maps, it is enough to show that the deRham differential commutes with each per-edge map. Clearly it commutes when $k_{\varepsilon'} = k_{\varepsilon}$. So let's consider the other case.

First consider $\mathrm{d} \circ \mathrm{d}_{\varepsilon^*}$:

$$\mathrm{d}\circ\mathrm{d}_{\varepsilon^*}\left(\bigotimes_{n=1}^{k}\alpha_n\right) = \mathrm{d}\left((-1)^{\deg(\alpha_j)\cdot\sum\limits_{n=i+1}^{j-1}\deg(\alpha_n)}\alpha_1\otimes ...\otimes\alpha_i\wedge\alpha_j\otimes ...\hat{\alpha_j}...\otimes\alpha_k\right)$$

$$= (-1)^{\deg(\alpha_j)\cdot\sum\limits_{n=i+1}^{j-1}\deg(\alpha_n)}\left(\sum_{n=1}^{i-1}(-1)^{\sum\limits_{m=1}^{n-1}\deg(\alpha_m)}\alpha_1\otimes ...\otimes \mathrm{d}(\alpha_n)\otimes ...\otimes\alpha_i\wedge\alpha_j\otimes ....\hat{\alpha_j}...\otimes\alpha_k\right.$$

$$+ (-1)^{\sum\limits_{m=1}^{i-1}\deg(\alpha_m)}\alpha_1\otimes ...\otimes \mathrm{d}(\alpha_i\wedge\alpha_j)\otimes ....\hat{\alpha_j}...\otimes\alpha_k$$

$$+ \sum_{n=i+1}^{j-1}(-1)^{\deg(\alpha_j)+\sum\limits_{m=1}^{n-1}\deg(\alpha_m)}\alpha_1\otimes ...\otimes\alpha_i\wedge\alpha_j\otimes ....\otimes \mathrm{d}(\alpha_n)\otimes ...\hat{\alpha_j}...\otimes\alpha_k$$

$$\left.+ \sum_{n=j+1}^{k}(-1)^{\sum\limits_{m=1}^{n-1}\deg(\alpha_m)}\alpha_1\otimes ...\otimes\alpha_i\wedge\alpha_j\otimes ...\hat{\alpha_j}...\otimes \mathrm{d}(\alpha_n)\otimes ...\otimes\alpha_k\right)$$

Note that the term

$$(-1)^{\sum\limits_{m=1}^{i-1}\deg(\alpha_m)}\alpha_1\otimes ...\otimes \mathrm{d}(\alpha_i\wedge\alpha_j)\otimes ....\hat{\alpha_j}...\otimes\alpha_k$$

expands under the product rule to become

$$(-1)^{\sum\limits_{m=1}^{i-1}\deg(\alpha_m)}\alpha_1\otimes...\otimes \mathrm{d}(\alpha_i)\wedge\alpha_j\otimes....\hat{\alpha_j}...\otimes\alpha_k + (-1)^{\sum\limits_{m=1}^{i}\deg(\alpha_m)}\alpha_1\otimes...\otimes\alpha_i\wedge \mathrm{d}(\alpha_j)\otimes....\hat{\alpha_j}...\otimes\alpha_k$$

Therefore, the composition $\mathrm{d}\circ\mathrm{d}_{\varepsilon^*}\left(\bigotimes\limits_{n=1}^{k}\alpha_n\right)$ becomes

$$= (-1)^{\deg(\alpha_j)\cdot\sum\limits_{n=i+1}^{j-1}\deg(\alpha_n)}\left(\sum_{n=1}^{i-1}(-1)^{\sum\limits_{m=1}^{n-1}\deg(\alpha_m)}\alpha_1\otimes ...\otimes \mathrm{d}(\alpha_n)\otimes ...\otimes\alpha_i\wedge\alpha_j\otimes ....\hat{\alpha_j}...\otimes\alpha_k\right.$$

$$+ (-1)^{\sum\limits_{m=1}^{i-1}\deg(\alpha_m)}\alpha_1\otimes ...\otimes \mathrm{d}(\alpha_i)\wedge\alpha_j\otimes ....\hat{\alpha_j}...\otimes\alpha_k$$

$$+ (-1)^{\sum\limits_{m=1}^{i}\deg(\alpha_m)}\alpha_1\otimes ...\otimes\alpha_i\wedge \mathrm{d}(\alpha_j)\otimes ....\hat{\alpha_j}...\otimes\alpha_k$$

$$+ \sum_{n=i+1}^{j-1}(-1)^{\deg(\alpha_j)+\sum\limits_{m=1}^{n-1}\deg(\alpha_m)}\alpha_1\otimes ...\otimes\alpha_i\wedge\alpha_j\otimes ....\otimes \mathrm{d}(\alpha_n)\otimes ...\hat{\alpha_j}...\otimes\alpha_k$$

$$\left.+ \sum_{n=j+1}^{k}(-1)^{\sum\limits_{m=1}^{n-1}\deg(\alpha_m)}\alpha_1\otimes ...\otimes\alpha_i\wedge\alpha_j\otimes ...\hat{\alpha_j}...\otimes \mathrm{d}(\alpha_n)\otimes ...\otimes\alpha_k\right)$$

Now we compute the composition $\partial_{\varepsilon^*} \circ \mathrm{d}\left(\bigotimes_{n=1}^{k} \alpha_n\right)$ and show it is equivalent.

$$\partial_{\varepsilon^*} \circ \mathrm{d}\left(\bigotimes_{n=1}^{k} \alpha_n\right) = \partial_{\varepsilon^*}\left(\sum_{n=1}^{k}(-1)^{\sum_{m=1}^{n-1}\deg(\alpha_m)}\alpha_1 \otimes \ldots \otimes \mathrm{d}(\alpha_n) \otimes \ldots \otimes \alpha_k\right)$$

$$\begin{aligned}
&= (-1)^{\deg(\alpha_j)\cdot\sum_{n=i+1}^{j-1}\deg(\alpha_n)}\sum_{n=1}^{i-1}(-1)^{\sum_{m=1}^{n-1}\deg(\alpha_m)}\alpha_1 \otimes \ldots \otimes \mathrm{d}(\alpha_n) \otimes \ldots \otimes \alpha_i \wedge \alpha_j \otimes \ldots\hat{\alpha_j}\ldots \otimes \alpha_k\\
&\quad + (-1)^{\deg(\alpha_j)\cdot\sum_{n=i+1}^{j-1}\deg(\alpha_n)}(-1)^{\sum_{m=1}^{i-1}\deg(\alpha_m)}\alpha_1 \otimes \ldots \otimes \mathrm{d}(\alpha_i) \wedge \alpha_j \otimes \ldots\hat{\alpha_j}\ldots \otimes \alpha_k\\
&\quad + (-1)^{\deg(\alpha_j)\cdot\left(1+\sum_{n=i+1}^{j-1}\deg(\alpha_n)\right)}\sum_{n=i+1}^{j-1}(-1)^{\sum_{m=1}^{n-1}\deg(\alpha_m)}\alpha_1 \otimes \ldots \otimes \alpha_i \wedge \alpha_j \otimes \ldots \otimes \mathrm{d}(\alpha_n) \otimes \ldots\hat{\alpha_j}\ldots \otimes \alpha_k\\
&\quad + (-1)^{(\deg(\alpha_j)+1)\cdot\sum_{n=i+1}^{j-1}\deg(\alpha_n)}(-1)^{\sum_{m=1}^{j-1}\deg(\alpha_m)}\alpha_1 \otimes \ldots \otimes \alpha_i \wedge \mathrm{d}(\alpha_j) \otimes \ldots\hat{\alpha_j}\ldots \otimes \alpha_k\\
&\quad + (-1)^{\deg(\alpha_j)\cdot\sum_{n=i+1}^{j-1}\deg(\alpha_n)}\sum_{n=j+1}^{k}(-1)^{\sum_{m=1}^{n-1}\deg(\alpha_m)}\alpha_1 \otimes \ldots \otimes \alpha_i \wedge \alpha_j \otimes \ldots\hat{\alpha_j}\ldots \otimes \mathrm{d}(\alpha_n) \otimes \ldots \otimes \alpha_k\\
&= (-1)^{\deg(\alpha_j)\cdot\sum_{n=i+1}^{j-1}\deg(\alpha_n)}\left(\sum_{n=1}^{i-1}(-1)^{\sum_{m=1}^{n-1}\deg(\alpha_m)}\alpha_1 \otimes \ldots \otimes \mathrm{d}(\alpha_n) \otimes \ldots \otimes \alpha_i \wedge \alpha_j \otimes \ldots\hat{\alpha_j}\ldots \otimes \alpha_k\right.\\
&\quad + (-1)^{\sum_{m=1}^{i-1}\deg(\alpha_m)}\alpha_1 \otimes \ldots \otimes \mathrm{d}(\alpha_i) \wedge \alpha_j \otimes \ldots\hat{\alpha_j}\ldots \otimes \alpha_k\\
&\quad + (-1)^{\deg(\alpha_j)}\sum_{n=i+1}^{j-1}(-1)^{\sum_{m=1}^{n-1}\deg(\alpha_m)}\alpha_1 \otimes \ldots \otimes \alpha_i \wedge \alpha_j \otimes \ldots \otimes \mathrm{d}(\alpha_n) \otimes \ldots\hat{\alpha_j}\ldots \otimes \alpha_k\\
&\quad + (-1)^{\sum_{n=i+1}^{j-1}\deg(\alpha_n)}(-1)^{\sum_{m=1}^{j-1}\deg(\alpha_m)}\alpha_1 \otimes \ldots \otimes \alpha_i \wedge \mathrm{d}(\alpha_j) \otimes \ldots\hat{\alpha_j}\ldots \otimes \alpha_k\\
&\quad \left. + \sum_{n=j+1}^{k}(-1)^{\sum_{m=1}^{n-1}\deg(\alpha_m)}\alpha_1 \otimes \ldots \otimes \alpha_i \wedge \alpha_j \otimes \ldots\hat{\alpha_j}\ldots \otimes \mathrm{d}(\alpha_n) \otimes \ldots \otimes \alpha_k\right)
\end{aligned}$$

Notice that the following product simplifies:

$$\begin{aligned}
(-1)^{\sum_{n=i+1}^{j-1}\deg(\alpha_n)}(-1)^{\sum_{m=1}^{j-1}\deg(\alpha_m)} &= (-1)^{\sum_{n=i+1}^{j-1}\deg(\alpha_n)+\sum_{m=1}^{j-1}\deg(\alpha_m)}\\
&= (-1)^{\sum_{n=i+1}^{j-1}\deg(\alpha_n)+\sum_{m=i+1}^{j-1}\deg(\alpha_m)+\sum_{m=1}^{i}\deg(\alpha_m)}\\
&= (-1)^{2\sum_{n=i+1}^{j-1}\deg(\alpha_n)+\sum_{m=1}^{i}\deg(\alpha_m)}\\
&= (-1)^{\sum_{m=1}^{i}\deg(\alpha_m)}
\end{aligned}$$

Therefore we find the composition becomes:

$$
\begin{aligned}
&= (-1)^{\deg(\alpha_j)\cdot\sum\limits_{n=i+1}^{j-1}\deg(\alpha_n)} \Bigg(\sum_{n=1}^{i-1} (-1)^{\sum\limits_{m=1}^{n-1}\deg(\alpha_m)} \alpha_1 \otimes ... \otimes \mathrm{d}(\alpha_n) \otimes ... \otimes \alpha_i \wedge \alpha_j \otimes ...\hat{\alpha_j}... \otimes \alpha_k \\
&\quad + (-1)^{\sum\limits_{m=1}^{i-1}\deg(\alpha_m)} \alpha_1 \otimes ... \otimes \mathrm{d}(\alpha_i) \wedge \alpha_j \otimes ...\hat{\alpha_j}... \otimes \alpha_k \\
&\quad + \sum_{n=i+1}^{j-1} (-1)^{\deg(\alpha_j)+\sum\limits_{m=1}^{n-1}\deg(\alpha_m)} \alpha_1 \otimes ... \otimes \alpha_i \wedge \alpha_j \otimes ... \otimes \mathrm{d}(\alpha_n) \otimes ...\hat{\alpha_j}... \otimes \alpha_k \\
&\quad + (-1)^{\sum\limits_{m=1}^{i}\deg(\alpha_m)} \alpha_1 \otimes ... \otimes \alpha_i \wedge \mathrm{d}(\alpha_j) \otimes ...\hat{\alpha_j}... \otimes \alpha_k \\
&\quad + \sum_{n=j+1}^{k} (-1)^{\sum\limits_{m=1}^{n-1}\deg(\alpha_m)} \alpha_1 \otimes ... \otimes \alpha_i \wedge \alpha_j \otimes ...\hat{\alpha_j}... \otimes \mathrm{d}(\alpha_n) \otimes ... \otimes \alpha_k \Bigg)
\end{aligned}
$$

Comparing this to what was computed earlier, we see they are equivalent. Thus $\mathrm{d} \circ \partial_{\varepsilon^*} = \partial_{\varepsilon^*} \circ \mathrm{d}$ from which it follows that $\mathrm{d} \circ \partial = \partial \circ \mathrm{d}$.

3. dd= 0

Here we show that the deRham differential as defined over the formal tensor product $(\Omega^*(M))^{\otimes k}$ still squares to zero.

$$\mathrm{d}\circ\mathrm{d}\left(\bigotimes_{n=1}^{k}\alpha_n\right)=\mathrm{d}\left(\sum_{n=1}^{k}(-1)^{\sum_{m=1}^{n-1}\deg(\alpha_m)}\alpha_1\otimes\ldots\otimes\mathrm{d}(\alpha_n)\otimes\ldots\otimes\alpha_k\right)$$
$$=\sum_{n=1}^{k}(-1)^{\sum_{m=1}^{n-1}\deg(\alpha_m)}\mathrm{d}\left(\alpha_1\otimes\ldots\otimes\mathrm{d}(\alpha_n)\otimes\ldots\otimes\alpha_k\right)$$

$$=\sum_{n=1}^{k}(-1)^{\sum_{m=1}^{n-1}\deg(\alpha_m)}\left(\sum_{i=1}^{n-1}(-1)^{\sum_{j=1}^{i-1}\deg(\alpha_j)}\alpha_1\otimes\ldots\otimes\mathrm{d}(\alpha_i)\otimes\ldots\otimes\mathrm{d}(\alpha_n)\otimes\ldots\otimes\alpha_k\right.$$
$$+(-1)^{\sum_{j=1}^{n-1}\deg(\alpha_j)}\alpha_1\otimes\ldots\otimes\mathrm{d}\circ\mathrm{d}(\alpha_n)\otimes\ldots\otimes\alpha_k$$
$$\left.+\sum_{i=n+1}^{k}(-1)^{1+\sum_{j=1}^{i-1}\deg(\alpha_j)}\alpha_1\otimes\ldots\otimes\mathrm{d}(\alpha_n)\otimes\ldots\otimes\mathrm{d}(\alpha_i)\otimes\ldots\otimes\alpha_k\right)$$

Since $\mathrm{d}\circ\mathrm{d}(\alpha_n)=0$ we get the following:

$$=\sum_{n=1}^{k}(-1)^{\sum_{m=1}^{n-1}\deg(\alpha_m)}\left(\sum_{i=1}^{n-1}(-1)^{\sum_{j=1}^{i-1}\deg(\alpha_j)}\alpha_1\otimes\ldots\otimes\mathrm{d}(\alpha_i)\otimes\ldots\otimes\mathrm{d}(\alpha_n)\otimes\ldots\otimes\alpha_k\right.$$
$$\left.-\sum_{i=n+1}^{k}(-1)^{\sum_{j=1}^{i-1}\deg(\alpha_j)}\alpha_1\otimes\ldots\otimes\mathrm{d}(\alpha_n)\otimes\ldots\otimes\mathrm{d}(\alpha_i)\otimes\ldots\otimes\alpha_k\right)$$
$$=\sum_{n=1}^{k}(-1)^{\sum_{m=1}^{n-1}\deg(\alpha_m)}\sum_{i=1}^{n-1}(-1)^{\sum_{j=1}^{i-1}\deg(\alpha_j)}\alpha_1\otimes\ldots\otimes\mathrm{d}(\alpha_i)\otimes\ldots\otimes\mathrm{d}(\alpha_n)\otimes\ldots\otimes\alpha_k$$
$$-\sum_{n=1}^{k}(-1)^{\sum_{m=1}^{n-1}\deg(\alpha_m)}\sum_{i=n+1}^{k}(-1)^{\sum_{j=1}^{i-1}\deg(\alpha_j)}\alpha_1\otimes\ldots\otimes\mathrm{d}(\alpha_n)\otimes\ldots\otimes\mathrm{d}(\alpha_i)\otimes\ldots\otimes\alpha_k$$

Now we re-order the second summation to obtain:

$$=\sum_{n=1}^{k}(-1)^{\sum_{m=1}^{n-1}\deg(\alpha_m)}\sum_{i=1}^{n-1}(-1)^{\sum_{j=1}^{i-1}\deg(\alpha_j)}\alpha_1\otimes\ldots\otimes\mathrm{d}(\alpha_i)\otimes\ldots\otimes\mathrm{d}(\alpha_n)\otimes\ldots\otimes\alpha_k$$
$$-\sum_{i=1}^{k}(-1)^{\sum_{j=1}^{i-1}\deg(\alpha_j)}\sum_{n=1}^{i-1}(-1)^{\sum_{m=1}^{n-1}\deg(\alpha_m)}\alpha_1\otimes\ldots\otimes\mathrm{d}(\alpha_n)\otimes\ldots\otimes\mathrm{d}(\alpha_i)\otimes\ldots\otimes\alpha_k$$

Now we re-index the second double sum by interchanging $i \leftrightarrow n$ and $j \leftrightarrow m$ to get:

$$
\begin{aligned}
&= \sum_{n=1}^{k} (-1)^{\sum\limits_{m=1}^{n-1} \deg(\alpha_m)} \sum_{i=1}^{n-1} (-1)^{\sum\limits_{j=1}^{i-1} \deg(\alpha_j)} \alpha_1 \otimes ... \otimes \mathrm{d}(\alpha_i) \otimes ... \otimes \mathrm{d}(\alpha_n) \otimes ... \otimes \alpha_k \\
&\quad - \sum_{n=1}^{k} (-1)^{\sum\limits_{m=1}^{n-1} \deg(\alpha_m)} \sum_{i=1}^{n-1} (-1)^{\sum\limits_{j=1}^{i-1} \deg(\alpha_j)} \alpha_1 \otimes ... \otimes \mathrm{d}(\alpha_i) \otimes ... \otimes \mathrm{d}(\alpha_n) \otimes ... \otimes \alpha_k \\
&= 0
\end{aligned}
$$

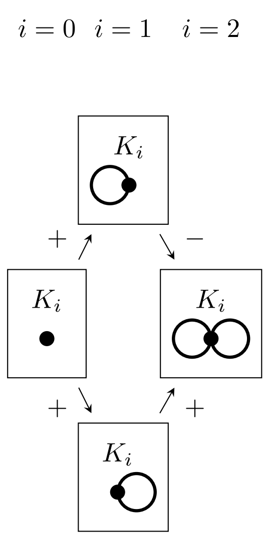
Therefore, $\mathrm{d} \circ \mathrm{d} = 0$.

4. $\partial \circ \partial = 0$

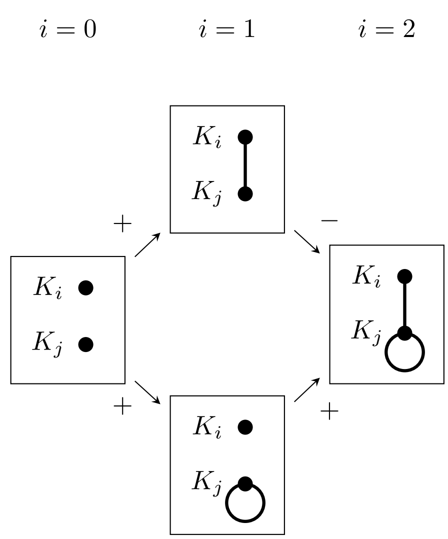
A per-edge map in the chromatic complex corresponds to adding and edge to a given subgraph. Therefore the chromatic differential is a signed sum over all possible ways to add an edge to all of the subgraphs in a given state. It follows that the square of the chromatic differential corresponds to the signed sum over all possible ways to add pairs of edges to every subgraph in a given state.

Notice that adding two edges to a given subgraph determines a face of the n-cube, $\{0,1\}^n$, in the chromatic complex. Thus the square of the chromatic differential can be written as the sum over each face of the n-cube which starts at the given state. Therefore, it is enough to show that the chromatic differential squares to zero on every face of the n-cube. Furthermore, given the sign convention for the per-edge maps of the chromatic complex, all but one of the edges in any given face have the same sign. Without loss of generality, we will assume that the face has three positive edges and one negative edge.

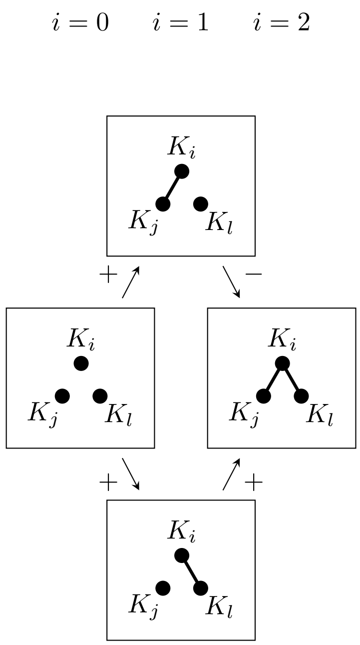
There are four distinct types faces of the n-cube corresponding to the four ways to add two edges to a given subgraph. They are classified by how they connect the components of the subgraph. In fact, since the chromatic differential is identity on the unaffected components it is sufficient to consider at most four components, K_i, K_j, K_l, and K_m where $i < j < l$. Then the four ways to connect components of a subgraph by adding two edges are as follows:



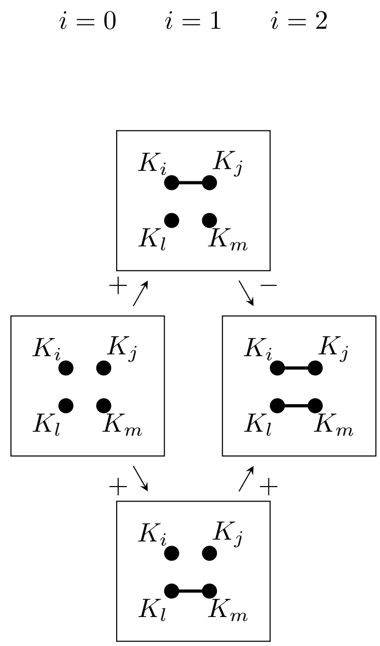
1. K_i to itself twice



2. K_i to itself and K_j



3. K_i to K_j to K_l



4. K_i to K_j and K_l to K_m

In the first case, the number of connected components remains constant as the edges are added. Therefore all of the per-edge maps are identity. Therefore the square of the differential is zero is a result of the per-edge sign assignment. Similarly, in the second case, connecting K_i to itself preserves the number of connected components. Therefore, two of the per-edge maps in the second case are identity. Since the identity commutes with the non-identity per-edge maps, the square of this differential is also zero as a result of the per-edge sign assignment. Therefore, we only need to investigate cases three and four.

So, we assume that the subgraph at the left vertex of the square has k components and that the result of adding some pair of edges is to connect the components K_i, K_j and K_l. The per-edge map which corresponds to adding the edge which connects K_i to K_j will be denoted ∂_{ij}. Similarly, per-edge map which corresponds to adding the edge which connects K_i to K_l will be denoted ∂_{il}. Then the chromatic differential restricted to this face of the state cube is given by $\partial_{ij} \circ \partial_{il} - \partial_{il} \circ \partial_{ij}$.

They are given from the definition by the formulas:

$$\partial_{ij}\left(\bigotimes_{n=1}^{k} \alpha_n\right) = (-1)^{\deg(\alpha_j)\cdot \sum\limits_{n=i+1}^{j-1} \deg(\alpha_n)} \alpha_1 \otimes ... \otimes \alpha_i \wedge \alpha_j \otimes ...\hat{\alpha_j}... \otimes \alpha_k$$

$$\partial_{il}\left(\bigotimes_{n=1}^{k}\alpha_n\right)=(-1)^{\deg(\alpha_l)\cdot\sum\limits_{n=i+1}^{l-1}\deg(\alpha_n)}\alpha_1\otimes...\otimes\alpha_i\wedge\alpha_l\otimes...\hat{\alpha_l}...\otimes\alpha_k$$

Therefore, we can compute the following:

$$(\partial_{ij}\circ\partial_{il}-\partial_{il}\circ\partial_{ij})\left(\bigotimes_{n=1}^{k}\alpha_n\right)=\partial_{ij}\circ\partial_{il}\left(\bigotimes_{n=1}^{k}\alpha_n\right)-\partial_{il}\circ\partial_{ij}\left(\bigotimes_{n=1}^{k}\alpha_n\right)$$

$$\begin{aligned}
&=\partial_{ij}\left((-1)^{\deg(\alpha_l)\cdot\sum\limits_{n=i+1}^{l-1}\deg(\alpha_n)}\alpha_1\otimes...\otimes\alpha_i\wedge\alpha_l\otimes...\hat{\alpha_l}...\otimes\alpha_k\right)\\
&\quad-\partial_{il}\left((-1)^{\deg(\alpha_j)\cdot\sum\limits_{n=i+1}^{j-1}\deg(\alpha_n)}\alpha_1\otimes...\otimes\alpha_i\wedge\alpha_j\otimes...\hat{\alpha_j}...\otimes\alpha_k\right)\\
&=(-1)^{\deg(\alpha_j)\cdot\sum\limits_{n=i+1}^{j-1}\deg(\alpha_n)}(-1)^{\deg(\alpha_l)\cdot\sum\limits_{n=i+1}^{l-1}\deg(\alpha_n)}\alpha_1\otimes...\otimes\alpha_i\wedge\alpha_l\wedge\alpha_j\otimes...\hat{\alpha_j}...\hat{\alpha_l}...\otimes\alpha_k\\
&\quad-(-1)^{\deg(\alpha_l)\cdot\left(\sum\limits_{n=i+1}^{l-1}\deg(\alpha_n)-\deg(\alpha_j)\right)}(-1)^{\deg(\alpha_j)\cdot\sum\limits_{n=i+1}^{j-1}\deg(\alpha_n)}\alpha_1\otimes...\otimes\alpha_i\wedge\alpha_j\wedge\alpha_l\otimes...\hat{\alpha_j}...\hat{\alpha_l}...\otimes\alpha_k\\
&=\mathrm{s}(i,j)\mathrm{s}(i,l)\alpha_1\otimes...\otimes\alpha_i\wedge\alpha_l\wedge\alpha_j\otimes...\hat{\alpha_j}...\hat{\alpha_l}...\otimes\alpha_k\\
&\quad-(-1)^{-\deg(\alpha_l)\deg(\alpha_j)}\mathrm{s}(i,l)\mathrm{s}(i,j)\alpha_1\otimes...\otimes\alpha_i\wedge\alpha_j\wedge\alpha_l\otimes...\hat{\alpha_j}...\hat{\alpha_l}...\otimes\alpha_k
\end{aligned}$$

Note that since the wedge product is skew-symmetric,

$$\alpha_1\otimes...\otimes\alpha_i\wedge\alpha_l\wedge\alpha_j\otimes...\hat{\alpha_j}...\hat{\alpha_l}...\otimes\alpha_k=(-1)^{\deg(\alpha_l)\deg(\alpha_j)}\alpha_1\otimes...\otimes\alpha_i\wedge\alpha_j\wedge\alpha_l\otimes...\hat{\alpha_j}...\hat{\alpha_l}...\otimes\alpha_k$$

Furthermore, since $(-1)^k=(-1)^{-k}$, we find that:

$$\begin{aligned}
&=\mathrm{s}(i,j)\mathrm{s}(i,l)(-1)^{\deg(\alpha_l)\deg(\alpha_j)}\alpha_1\otimes...\otimes\alpha_i\wedge\alpha_j\wedge\alpha_l\otimes...\hat{\alpha_j}...\hat{\alpha_l}...\otimes\alpha_k\\
&\quad-(-1)^{\deg(\alpha_l)\deg(\alpha_j)}\mathrm{s}(i,l)\mathrm{s}(i,j)\alpha_1\otimes...\otimes\alpha_i\wedge\alpha_j\wedge\alpha_l\otimes...\hat{\alpha_j}...\hat{\alpha_l}...\otimes\alpha_k\\
&=0
\end{aligned}$$

Therefore, $(\partial_{ij}\circ\partial_{il}-\partial_{il}\circ\partial_{ij})\left(\bigotimes_{n=1}^{k}\alpha_n\right)=0$. Thus the square of the chromatic differential is zero on any such faces.

The fourth case, in which four components are connected into two components, has three subcases depending on where the K_m component is in the ordering $i<j<l$. That is, we could have $m<i<k<l$, $i<m<k<l$, or $i<k<m<l$. In each case, basic element of the k-fold tensor product looks like

$$\begin{aligned}
&m<i<k<l\Rightarrow\alpha_1\otimes...\otimes\alpha_m\otimes...\otimes\alpha_i\otimes...\otimes\alpha_j\otimes...\otimes\alpha_l\otimes...\otimes\alpha_k\\
&i<m<k<l\Rightarrow\alpha_1\otimes...\otimes\alpha_i\otimes...\otimes\alpha_m\otimes...\otimes\alpha_j\otimes...\otimes\alpha_l\otimes...\otimes\alpha_k\\
&i<j<m<l\Rightarrow\alpha_1\otimes...\otimes\alpha_i\otimes...\otimes\alpha_j\otimes...\otimes\alpha_m\otimes...\otimes\alpha_l\otimes...\otimes\alpha_k
\end{aligned}$$

In the first and third case, it is clear that the per-edge maps ∂_{ij} and ∂_{ml} commute. Thus, the square of the differential is zero, $(\partial_{ij} \circ \partial_{ml} - \partial_{ml} \circ \partial_{ij}) = 0$. Therefore it is only necessary to check that second case.

$$
\begin{aligned}
&(\partial_{ij} \circ \partial_{ml} - \partial_{ml} \circ \partial_{ij})\,(\alpha_1 \otimes ... \otimes \alpha_i \otimes ... \otimes \alpha_m \otimes ... \otimes \alpha_j \otimes ... \otimes \alpha_l \otimes ... \otimes \alpha_k) \\
&= \partial_{ij} \circ \partial_{ml}\,(\alpha_1 \otimes ... \otimes \alpha_i \otimes ... \otimes \alpha_m \otimes ... \otimes \alpha_j \otimes ... \otimes \alpha_l \otimes ... \otimes \alpha_k) \\
&\quad - \partial_{ml} \circ \partial_{ij}\,(\alpha_1 \otimes ... \otimes \alpha_i \otimes ... \otimes \alpha_m \otimes ... \otimes \alpha_j \otimes ... \otimes \alpha_l \otimes ... \otimes \alpha_k) \\
&= \partial_{ij}\,(\mathrm{s}(m,l)\alpha_1 \otimes ... \otimes \alpha_i \otimes ... \otimes \alpha_m \wedge \alpha_l \otimes ... \otimes \alpha_j \otimes ... \hat{\alpha_l} ... \otimes \alpha_k) \\
&\quad - \partial_{ml}\,(\mathrm{s}(i,j)\alpha_1 \otimes ... \otimes \alpha_i \wedge \alpha_j \otimes ... \otimes \alpha_m \otimes ... \hat{\alpha_j} ... \otimes \alpha_l \otimes ... \otimes \alpha_k) \\
&= (-1)^{\deg(\alpha_j)\deg(\alpha_l)}\mathrm{s}(i,j)\mathrm{s}(m,l)\alpha_1 \otimes ... \otimes \alpha_i \wedge \alpha_j \otimes ... \otimes \alpha_m \wedge \alpha_l \otimes ... \hat{\alpha_j} ... \hat{\alpha_l} ... \otimes \alpha_k \\
&\quad - (-1)^{-\deg(\alpha_j)\deg(\alpha_l)}\mathrm{s}(i,j)\mathrm{s}(m,l)\alpha_1 \otimes ... \otimes \alpha_i \wedge \alpha_j \otimes ... \otimes \alpha_m \wedge \alpha_l \otimes ... \hat{\alpha_j} ... \hat{\alpha_l} ... \otimes \alpha_k \\
&= (-1)^{\deg(\alpha_j)\deg(\alpha_l)}\mathrm{s}(i,j)\mathrm{s}(m,l)\alpha_1 \otimes ... \otimes \alpha_i \wedge \alpha_j \otimes ... \otimes \alpha_m \wedge \alpha_l \otimes ... \hat{\alpha_j} ... \hat{\alpha_l} ... \otimes \alpha_k \\
&\quad - (-1)^{\deg(\alpha_j)\deg(\alpha_l)}\mathrm{s}(i,j)\mathrm{s}(m,l)\alpha_1 \otimes ... \otimes \alpha_i \wedge \alpha_j \otimes ... \otimes \alpha_m \wedge \alpha_l \otimes ... \hat{\alpha_j} ... \hat{\alpha_l} ... \otimes \alpha_k \\
&= 0
\end{aligned}
$$

Therefore, $(\partial_{ij} \circ \partial_{ml} - \partial_{ml} \circ \partial_{ij}) \left(\bigotimes_{n=1}^{k} \alpha_n \right) = 0$. Thus the square of the chromatic differential is zero on any such faces.

The square chromatic differential is thereby zero on every face of the chromatic complex and thus is zero everywhere, $\partial \circ \partial = 0$.