An Example Computation for the Composite deRham and Chromatic Cohomologies

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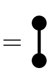
February 20, 2019

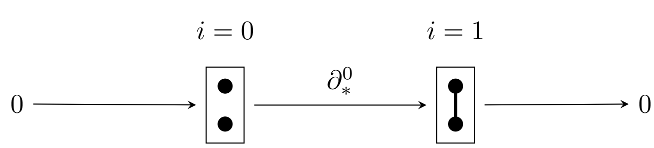
Abstract: Here is presented a detailed computation of the composite deRham and Chromatic Cohomologies, using the spectral sequence of the bicomplex, in which the graph consists of two vertices connected by one edge and the manifold is $\mathbb{R}^1$. Furthermore, we consider the deRham complex with compact supports, $\Omega_c^*(M)$.

1. Construction of the $\mathbf{E_0^{**}}$ Page

As the simplest non-trivial computation, we consider the graph $G = P_1$ and $M = \mathbb{R}^1$. Notice that simpler graphs, e.g. multiple vertices and no edges, reduce to deRham Cohomology when there is only one vertex and tensor products thereof for multiple vertices which simplify by the Kunneth Formula.

We begin by considering the Chromatic Complex coming from the graph $G = P_1$ as detailed in the following example.

Example 0.1. $G =$ 



$$\begin{tikzcd} 0 \arrow[r] & \Omega_c^* \otimes \Omega_c^* \arrow[r, "\partial^0"] & \Omega_c^* \arrow[r] & 0 \end{tikzcd}$$

Where:

$$\partial^0(\alpha \otimes \beta) = \alpha \wedge \beta$$

Now, the deRham Complex of $M = \mathbb{R}^1$ with compact support is detailed in the following example where we recall that

$$\Omega_c^*(M) = \Omega_c^0(M) \oplus \Omega_c^1(M)$$

is the direct sum of the space of compactly supported smooth functions on M with the space of compactly supported one-forms on M. As can be seen in the proceeding example, we also need consider the tensor product

$$\Omega_c^*(M) \otimes \Omega_c^*(M) = \Big(\Omega_c^0(M) \otimes \Omega_c^0(M)\Big) \oplus \begin{pmatrix} \Omega_c^0(M) \otimes \Omega_c^1(M) \\ \oplus \\ \Omega_c^1(M) \otimes \Omega_c^0(M) \end{pmatrix} \oplus \Big(\Omega_c^1(M) \otimes \Omega_c^1(M)\Big)$$

Example 0.2. $M = \mathbb{R}^1$

$$\underline{\Omega_c^*(M)}$$

$$0 \longrightarrow \Omega_c^0(M) \xrightarrow{d} \Omega_c^1(M) \longrightarrow 0$$

$$\underline{\Omega_c^*(M) \otimes \Omega_c^*(M)}$$

$$0 \longrightarrow \Omega_c^0(M) \otimes \Omega_c^0(M) \xrightarrow{d} \begin{matrix} \Omega_c^0(M) \otimes \Omega_c^1(M) \\ \oplus \\ \Omega_c^1(M) \otimes \Omega_c^0(M) \end{matrix} \xrightarrow{d} \Omega_c^1(M) \otimes \Omega_c^1(M) \longrightarrow 0$$

Where:

d is the usual exterior derivative

The following digram depicts the E_0^{**} page of the spectral sequence in the bicomplex where the Chromatic differential moves vertically while the deRham differential moves horizontally.

Example 0.3. E_0^{**}

$G =$  *&* $\quad M = \mathbb{R}^1$

$$\begin{tikzcd}
 & 0 & 0 & 0 & \\
0 \arrow[r] & \Omega_c^0(M) \arrow[u] \arrow[r, "d"] & \Omega_c^1(M) \arrow[u] \arrow[r] & 0 \arrow[u] \arrow[r] & 0 \\
0 \arrow[r] & \Omega_c^0(M) \otimes \Omega_c^0(M) \arrow[u, "\partial"'] \arrow[r, "d"] & \begin{matrix} \Omega_c^0(M)\otimes\Omega_c^1(M) \\ \bigoplus \\ \Omega_c^1(M)\otimes\Omega_c^0(M) \end{matrix} \arrow[u, "\partial"'] \arrow[r, "d"] & \Omega_c^1(M) \otimes \Omega_c^1(M) \arrow[u] \arrow[r] & 0 \\
 & 0 \arrow[u] & 0 \arrow[u] & 0 \arrow[u] &
\end{tikzcd}$$

2. Computing the $\mathbf{E_1^{**}}$ Page

<u>$\mathbf{E_1^{00}}$</u>

Note that E_1^{00} is given by the kernel of

$$\partial : \Omega_c^0(M) \otimes \Omega_c^0(M) \to \Omega_c^0(M)$$

That is
$$E_1^{00} = \mathrm{Ker}(\partial) = \{f \otimes g \in \Omega_c^0(M) \otimes \Omega_c^0(M) \mid f \wedge g = 0\}$$

Note that since f and g are smooth compactly supported functions on $\mathbb{R}^1$, the wedge product is just the point-wise product, $f \wedge g = f \cdot g$ which is equivalent to f and g having disjoint support. Therefore, we may equivalently describe E_1^{00} in the following way,

$$\begin{aligned}
E_1^{00} &= \{f \otimes g \in \Omega_c^0(M) \otimes \Omega_c^0(M) \mid f \wedge g = 0\} \\
&= \{f \otimes g \in \Omega_c^0(M) \otimes \Omega_c^0(M) \mid f \cdot g = 0\} \\
&= \{f \otimes g \in \Omega_c^0(M) \otimes \Omega_c^0(M) \mid \mathrm{supp}(f) \cap \mathrm{supp}(g) = \emptyset\}
\end{aligned}$$

<u>$\mathbf{E_1^{01}}$</u>

Now, $E_1^{01} = {}^{\Omega_c^0(M)}/_{\mathrm{Img}(\partial)}$, where again $\partial : \Omega_c^0(M) \otimes \Omega_c^0(M) \to \Omega_c^0(M)$. The image is given by $\mathrm{Img}(\partial) = \{h \in \Omega_c^0(M) \mid \exists f, g \in \Omega_c^0(M) \text{ s.t. } h = f \cdot g\}$.

We claim that this map is surjective. Let $h \in \Omega_c^0(M)$ and $U \subset M$ be open and such that $\mathrm{supp}(h) \subset U$. Then there exists a bump function, ρ_U, with compact support in U such that its restriction to $\mathrm{supp}(h)$ is the constant 1.

Therefore, $h \otimes \rho_U \in \Omega_c^0(M) \otimes \Omega_c^0(M)$ is such that $\partial(h \otimes \rho_U) = h \cdot \rho_U = h$. Since ∂ is surjective, the quotient is zero.

$$E_1^{01} = 0$$

$\mathbf{E_1^{10}}$

Again E_1^{10} is given by the kernel of the map

$$\partial : \Big(\Omega_c^1(M) \otimes \Omega_c^0(M)\Big) \oplus \Big(\Omega_c^0(M) \otimes \Omega_c^1(M)\Big) \to \Omega_c^1$$

That is

$$E_1^{10} = \mathrm{Ker}(\partial) = \{(\alpha \otimes s \ , \ t \otimes \beta) \mid \alpha \wedge s + t \wedge \beta = 0\}$$

Since $s, t \in \Omega_c^0$ these wedge products again reduce to point-wise products which means it can be equivalently characterized as follows,

$$\begin{aligned} E_1^{10} &= \Big\{(\alpha \otimes s \ , \ t \otimes \beta) \in \Big(\Omega_c^1(M) \otimes \Omega_c^0(M)\Big) \oplus \Big(\Omega_c^0(M) \otimes \Omega_c^1(M)\Big) \mid \alpha \wedge s + t \wedge \beta = 0\Big\} \\ &= \Big\{(\alpha \otimes s \ , \ t \otimes \beta) \in \Big(\Omega_c^1(M) \otimes \Omega_c^0(M)\Big) \oplus \Big(\Omega_c^0(M) \otimes \Omega_c^1(M)\Big) \mid \alpha \cdot s + t \cdot \beta = 0\Big\} \end{aligned}$$

$\mathbf{E_1^{11}}$

Now, $E_1^{11} = \ {}^{\Omega_c^1(M)}/_{\mathrm{Img}(\partial)}$, where again

$$\partial : \Big(\Omega_c^1(M) \otimes \Omega_c^0(M)\Big) \oplus \Big(\Omega_c^0(M) \otimes \Omega_c^1(M)\Big) \to \Omega_c^1$$

The image is therefore given by

$$\mathrm{Img}(\partial) = \{w \in \Omega_c^1 \mid \exists \alpha, \beta \in \Omega_c^1(M) \ \& \ s, t \in \Omega_c^0(M) \text{ s.t. } w = \alpha \cdot s + t \cdot \beta\}$$

As in the computation for E_1^{01}, we claim that the map is surjective. Let $w \in \Omega_c^1(M)$ and $U \subset M$ be open and such that $\mathrm{supp}(w) \subset U$. Then there exists a bump function, ρ_U, with compact support in U such that its restriction to $\mathrm{supp}(w)$ is the constant 1. Therefore

$$(w \otimes \rho_U \ , \ 0 \otimes 0) \in \Big(\Omega_c^1(M) \otimes \Omega_c^0(M)\Big) \oplus \Big(\Omega_c^0(M) \otimes \Omega_c^1(M)\Big)$$

is such that $\partial\big((w \otimes \rho_U \ , \ 0 \otimes 0)\big) = w \cdot \rho_U + 0 \cdot 0 = w$. Since ∂ is surjective, the quotient is zero.

$$E_1^{11} = 0$$

$\mathbf{E_1^{20}}$

Here the vertical maps coming into and out of $\Omega_c^1(M) \otimes \Omega_c^1(M)$ are both zero maps. Therefore,

$$E_1^{20} = \Omega_c^1(M) \otimes \Omega_c^1(M)$$

We therefore get the following E_1^{**} page.

Example 0.4. E_1^{**}

$G =$  *&* $M = \mathbb{R}^1$

$$
\begin{tikzcd}
 & 0 & 0 & 0 & \\
0 \arrow[r] & 0 \arrow[u] \arrow[r] & 0 \arrow[u] \arrow[r] & 0 \arrow[u] \arrow[r] & 0 \\
0 \arrow[r] & E_1^{00} \arrow[u] \arrow[r, "d"] & E_1^{10} \arrow[u] \arrow[r, "d"] & E_1^{20} \arrow[u] \arrow[r] & 0 \\
 & 0 \arrow[u] & 0 \arrow[u] & 0 \arrow[u] &
\end{tikzcd}
$$

Where:

$E_1^{00} = \{f \otimes g \in \Omega_c^0(M) \otimes \Omega_c^0(M) \mid f \cdot g = 0\}$

$E_1^{10} = \Big\{(\alpha \otimes s \ , \ t \otimes \beta) \in \Big(\Omega_c^1(M) \otimes \Omega_c^0(M)\Big) \oplus \Big(\Omega_c^0(M) \otimes \Omega_c^1(M)\Big) \ \Big| \ \alpha \cdot s + t \cdot \beta = 0\Big\}$

$E_1^{20} = \Omega_c^1(M) \otimes \Omega_c^1(M)$

3. Computing the $\mathbf{E_2^{**}}$ Page

$\underline{\mathbf{E_2^{00}}}$

Note that E_2^{00} is given by the kernel of

$$\mathrm{d} : E_1^{00} \to E_1^{10}$$

That is

$$E_2^{00} = \mathrm{Ker}(\mathrm{d}) = \{f \otimes g \in \Omega_c^0(M) \otimes \Omega_c^0(M) \mid f \cdot g = 0 \ \& \ \big(\mathrm{d}f \otimes g \ , \ f \otimes \mathrm{d}g\big) = (0,0)\}$$

Note that the condition that $\big(\mathrm{d}f \otimes g \ , \ f \otimes \mathrm{d}g\big) = (0,0)$ is equivalent to $\mathrm{d}f = 0 = \mathrm{d}g$ which requires that f and g be constant functions. However, since $f, g \in \Omega_c^0(M)$, this implies that $f = 0 = g$. Thus the kernel of the map is zero.

$$E_2^{00} = 0$$

$\underline{\mathbf{E_2^{10}}}$

E_2^{10} is given by the quotient of the kernel of $\mathrm{d} : E_1^{10} \to E_1^{20}$ by the image of $\mathrm{d} : E_1^{00} \to E_1^{10}$. More explicitly,

$$\mathrm{Ker}(\mathrm{d}) = \{(\alpha \otimes s \ , \ t \otimes \beta) \mid \alpha \cdot s + t \cdot \beta = 0 \ \& \ -\alpha \otimes \mathrm{d}s + \mathrm{d}t \otimes \beta = 0\}$$

$$\mathrm{Img(d)} = \{(\mathrm{d}f \otimes g \ , \ f \otimes \mathrm{d}g) \mid f \cdot g = 0\}$$

By examining the conditions on the kernel it can be show that it is a subset of the image in the following manner. First, the condition that $-\alpha \otimes \mathrm{d}s + \mathrm{d}t \otimes \beta = 0$ means that $\alpha \otimes \mathrm{d}s = \mathrm{d}t \otimes \beta$. However, since the tensor product is over $\mathbb{R}$, this implies that $\alpha = \lambda \mathrm{d}t$ and $\mathrm{d}s = \frac{1}{\lambda}\beta$ for some non-zero constant, λ. Therefore, we can rewrite $(\alpha \otimes s \ , \ t \otimes \beta) = (\lambda \mathrm{d}t \otimes s \ , \ t \otimes \lambda \mathrm{d}s)$.

In this light, the condition that $\alpha \cdot s + t \cdot \beta = 0$ becomes $\lambda \mathrm{d}t \cdot s + t \cdot \lambda \mathrm{d}s = 0$. By the product rule, this is equivalent to $\lambda \mathrm{d}(s \cdot t) = 0$ implying that $s \cdot t = \text{const.}$. Since $s, t \in \Omega^0_c(M)$ the constant must be zero, i.e. $s \cdot t = 0$. Thus the kernel is contained in the image.

$$E_2^{10} = 0$$

$\underline{\mathbf{E_2^{20}}}$

Now, $E_2^{20} = {}^{E_1^{20}}/_{\mathrm{Img(d)}}$, where $\mathrm{d} : E_1^{10} \to E_1^{20}$. More explicitly.

$$E_1^{20} = \Omega^1_c(M) \otimes \Omega^1_c(M)$$

$$\mathrm{Img(d)} =$$

$$\{-\alpha \otimes \mathrm{d}s + \mathrm{d}t \otimes \beta \mid \exists \alpha, \beta \in \Omega^1_c(M) \ \& \ s, t \in \Omega^0_c(M) \text{ s.t } \alpha \cdot s + t \cdot \beta = 0\}$$

We therefore get the following E_2^{**} page.

Example 0.5. E_2^{**}

$G =$  *&* $\quad M = \mathbb{R}^1$

$$\begin{tikzcd}
 & 0 & 0 & 0 & \\
0 \arrow[r] & 0 \arrow[u] \arrow[r] & 0 \arrow[u] \arrow[r] & 0 \arrow[u] \arrow[r] & 0 \\
0 \arrow[r] & 0 \arrow[u] \arrow[r] & 0 \arrow[u] \arrow[r] & E_2^{20} \arrow[u] \arrow[r] & 0 \\
 & 0 \arrow[u] & 0 \arrow[u] & 0 \arrow[u] &
\end{tikzcd}$$

Where:

$$E_2^{20} = \frac{\Omega^1_c(M) \otimes \Omega^1_c(M)}{Img(d)}$$