Hogstadiet's Mathematics crop 2023/24

Final breeding 27 January 2024

Solution proposal

#1.

Suggested solution: Let's start by calling the placements A through F to do our solution more light-weight.

	L¨attast A	B	C	D	AND	Svÿarast F
Erland	3	2	4	1	5	6
Rebecca	2	1	5	3	6	4
Fire	4	2	5	3	6	1

a) We start by noting that everyone got the same number of rights. Of the total of 18 guesses (6 guesses were for 3 people), then the total number of correct guesses must be divisible by 3. In addition, at least 6 must be correct guesses because all placements had at least one correct guess.

Locations A and F were all guessed differently, so there is exactly one guess that is correctly. At location B to E, two people guessed the same problem, so can there are either one or two correct guesses. Maximum it can therefore there are 1 + 2 + 2 + 2 + 2 + 1 = 10 correct guesses.

Now we know that there were at least 6 correct guesses, a maximum of 10 correct guesses, and that the number of correct guesses is evenly divisible by 3. This made it either 6 or 9 correct guesses.

In order for there to be only 6 correct guesses, exactly one guess must be made first each placement be correct. However, this means that problem 1 is both at location B and location D, which is not possible. That is the only possibility that there were 9 correct guesses in total, three per person.

Answer: Each person had 3 rights.

b) At location A and F respectively, only one person will have the right. To be able to reach 9 correct guesses then three of the four remaining placements (BE) have two who guessed correctly.

Let's look at problem 5. Since someone guessed correctly at each location must problem 5 either have location C or E. Suppose problem 5 is in location E, then only one person has guessed E correctly. This also means that problem 5 cannot be in location C, so even there only one person was right. This however, contradicts that only one of the placements B to E had a correct guess. Therefore, our assumption was wrong that problem 5 was at location E. Instead, we must problem 5 be at location C, and it follows that problem 6 must be at location E.

Therefore, Erland was wrong in choosing C as E. In addition, he was wrong in the placement F, because we know that problem 6 was at location E. He must therefore have right on all three remaining locations (A, B and D). Now we know that problem 3 was correct at location A, problem 2 was correct at location B, and problem 1 right at location D.

We now know that the problems come in the order 3, 2, 5, 1, 6, and the hardest The problem must therefore have been the only remaining one, problem 4.

The correct row thus looks like in the table, and now all that remains is to double check that there were exactly 9 correct guesses, three for each person.

	L¨attast					Svÿarast
	A	B	C	D	AND	F
Correct	3	2	5	1	6	4
Erland	$\boxed{3}$	$\boxed{2}$	4	$\boxed{1}$	5	6
Rebecca	2	1	$\boxed{5}$	3	$\boxed{6}$	$\boxed{4}$
Fire	4	$\boxed{2}$	$\boxed{5}$	3	$\boxed{6}$	1

#2.

Suggested solution: We begin by stating that since $a > 0$, then $a^2 > 0$. It gives that $b^2 = 1 ÿ a^2 < 1$, which means that $b < 1$. In the same way we can state that $a < 1$, i.e. $0 < a, b < 1$.

Specifically, we know that $1ÿa > 0$ and $1ÿb > 0$. This means that the fractions are defined, and that the difference does not change direction if we wanted to multiply by these factors.

Now consider the expression in the middle

$$\frac{a^2}{1 ÿ b} + \frac{b^2}{1 ÿ a}$$

We can write about $a^2 = 1 ÿ b^2$ and $b^2 = 1 ÿ a^2$.

$$\frac{a^2}{1 ÿ b} + \frac{b^2}{1 ÿ a} = \frac{1 ÿ b^2}{1 ÿ b} + \frac{1 ÿ a^2}{1 ÿ a}$$

According to the conjugate rule, ¨ar $1 ÿ b^2 = (1 ÿ b)(1 + b)$, and the equivalent for a. That gives us

$$\frac{1 ÿ b^2}{1 ÿ b} + \frac{1 ÿ a^2}{1 ÿ a} = \frac{(1 + b)(1 ÿ b)}{1 ÿ b} + \frac{(1 + a)(1 ÿ a)}{1 ÿ a}$$

Finally, we can abbreviate with $(1 ÿ b)$ and $(1 ÿ a)$ respectively, which are found in both the numerator and names. We get

$$\frac{(1 + b)(1 ÿ b)}{1 ÿ b} + \frac{(1 + a)(1 ÿ a)}{1 ÿ a} = 1 + b + 1 + a = 2 + a + b$$

We have thus concluded that the expression in the middle can be written as $2 + a + b$.

If we insert this new expression into our original inequality, we get

$$3 ÿ 2 + a + b ÿ 4$$

i.e

$$1 ÿ a + b ÿ 2$$

We have already established that a < 1 and b < 1, and then a + b < 2. We have thus shown the highest inequality.

Now it remains to show the left-hand inequality 1 ÿ a + b.

Since both sides of the inequality are positive, we can square the expressions with preserved inequality, i.e.

$$1 ÿ a + b ÿ 1^2 ÿ (a + b)^2$$

If we develop the upper management, we get

$$(a + b)^2 = a^2 + 2ab + b^2 = (a^2 + b^2) + 2ab = 1 + 2ab$$

because it is given in the problem that a is 2ab > 0, $a^2 + b^2 = 1$. Since both a and b are positive > 1.
i.e. 1 + 2ab > 1, in other words $(a + b)^2$

Thus we have also shown the left inequality.

#3.

Suggested solution: Let's draw the figure and name the so far unnamed points of intersection as S, T and F. Then we draw the auxiliary lines BD and ST, as well as the height h1 in the triangle ÿ SMT and h2 in triangle ÿDMB. See figure 1.

Let's start by examining the line ST. Since

$$\frac{|AS|}{|AB|} = \frac{|AT|}{|AD|} = \frac{1}{2}$$

ÿSAT is an apex triangle in ÿBAD, and thus ST is parallel to BD. In addition, the length ratio is |BD|.
$\frac{1}{2}$, your |ST| = $\frac{1}{2}$

Since ÿSMT and ÿDMB are vertical angles, and ST is parallel to BD, then the triangles ÿSMT and ÿDMB are congruent. Furthermore, we know that |ST| = |BD|, i.e. the length ratio is 1 : 2. This also means that h1 $\frac{1}{2}$
$\frac{1}{2}$ h2.

Since ST, EF and BD are all parallel, this gives that

$$\frac{|SE|}{|EB|} = \frac{h1}{h2} = \frac{1}{2}$$

i.e. E divides the distance SB in the ratio 1 : 2.

The problem requires the length of the stretch AE

$$|AE| = |AS| + |SE|$$

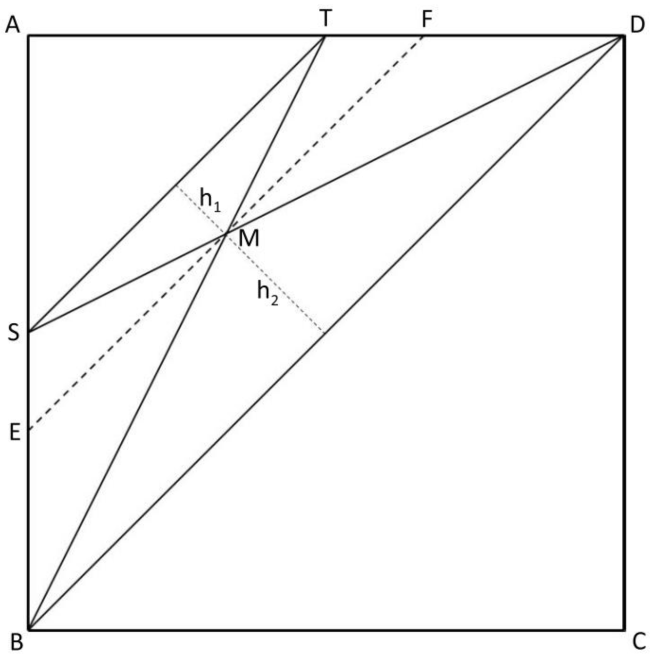

Figure 1: Problem 3

AS, we know half the side of the square, so it is 6 cm long. SB cuts the other half, also 6 cm long. If E divides SB in the ratio 1 : 2, this means that |SE| = 2 cm.

$$|AE| = |AS| + |SE| = 6 \text{ cm} + 2 \text{ cm} = 8 \text{ cm}$$

Svar: Str¨ackan Ae¨ar 8 cm.

#4.

Suggested solution: Let us denote the number of guests by n. After each game round, Aunt Ester gives out a total of 1 + 2 + · · · + n coins. This arithmetic sum can be calculated as

$$1 + 2 + \cdots + n = \frac{n(n+1)}{2}$$

After a total of k game rounds, Aunt Ester has distributed a total of 2024 coins. So we know that

$$k \cdot \frac{n(n+1)}{2} = 2024$$

Now let's prime factorize 2024:

$$2024 = 2 \cdot 2 \cdot 2 \cdot 11 \cdot 23$$

Alltsÿa g¨aller

$$k \cdot \frac{n(n+1)}{2} = 2 \cdot 2 \cdot 2 \cdot 11 \cdot 23$$

i.e

$$k \cdot n(n + 1) = 2 \cdot 2 \cdot 2 \cdot 2 \cdot 11 \cdot 23$$

Since we both have the factor n and n + 1, this means that we must be able to create two consecutive numbers from the prime factors. One of these numbers must be odd, which can only be 1, 11, 23 or 11 · 23 = 253.

It cannot be 253 because the product n(n + 1) will then be greater than 2024. Nor can it be 11 because with the other factors we must be able to form either 10 or 12, which doesn't work because we don't have a 5th or a 3rd. Finally, it cannot be 1 either because we know that Aunt Ester had at least four guests (the priests from Manchester).

This leaves that n or n + 1 must be 23. If n = 23, we must be able to factor 24, which is impossible because we have no 3rd prime factorization of 2024. On the other hand, it is possible to form 22 = 2 · 11, which gives that n = 22 and Aunt Ester thus had 22 guests.

Answer: Aunt Ester had 22 guests.

#5.

Suggested solution: Let's start by setting designations and drawing two radii as in figure 2

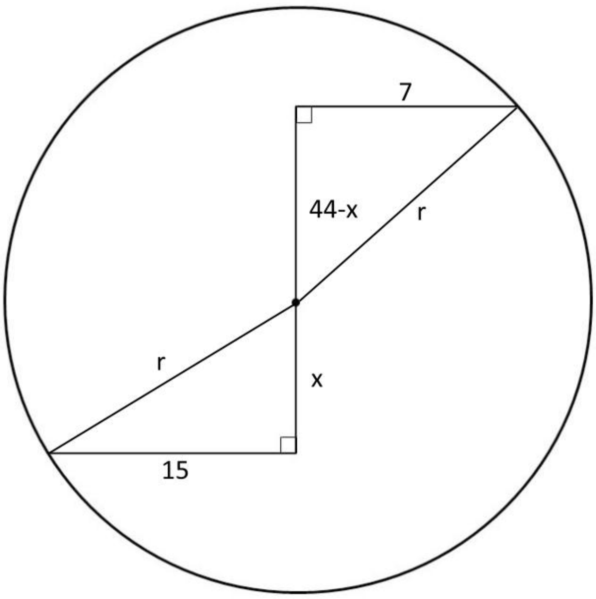

Figure 2: Problem 5

The Pythagorean theorem for the lower triangle then gives that the radius of the circle, the hypotenuse, can be written as

$$r^2 = 15^2 + x^2 = 225 + x^2$$

From the upper triangle we get that the hypotenuse can also be written as

$$r^2 = 72 + (44 ÿ x)^2 = 49 + 442 ÿ 88x + x^2$$

If we put these two expressions together, we get

$$225 + x^2 = 49 + 442 ÿ 88x + x^2$$

$$176 = 442 ÿ 88x$$

$$442 ÿ 176 = 88x$$

Since both 176 and 88 are divisible by 44, we can divide all terms by 44:

$$44 ÿ 4 = 2x$$

$$x = 20$$

If we insert this into the expression $r^2 = 225 + x^2$ we get

$$r^2 = 225 + 202 = 225 + 400 = 625$$

i.e. r = 25, which means that the diameter of the circle is 50.

#6.

Suggested solution: Let us denote the total sum of all the numbers in the grid by S.

The largest possible total sum Smax for a grid of size 2024 × 2024 where all numbers are at most 2023 is 20242 · 2023. Similarly, the smallest possible sum Smin = 20242 · (ÿ2023) = ÿSmax.

If there are still some rows or columns with a negative sum, then choose one of them, any one, and perform the operation on the numbers in it (we can call this a smart move). If this row or column previously had sum ÿn, then it sums +n after all numbers are multiplied by ÿ1. This makes Searching with 2n, and since n ÿ 1 it makes Searching with at least 2.

Let's only make smart moves.

Since each smart move increases the total by at least 2, the most we can do

$$\frac{Smax ÿ Smin}{2} = \frac{Smax ÿ (ÿSmax)}{2} = \frac{2Smax}{2} = Smax$$

smart moves in a row. This means that sooner or later (after at most Smax smart moves) we cannot have any more smart moves available. The fact that there is no smart move available means that there is no negative sum row or column to choose.

So we have shown that it is possible to achieve a grid without negative row and column sums.