Hogstadiet's Mathematics crop 2023/24

Qualification board 8-14 November 2023

Solution proposal and request template

Each task gives 0–3 points. Only answers without justification earn 0 points unless otherwise stated the rectification instructions are indicated. Completely correct solution gives 3 points. Only full points are given.

The tasks can often be solved in many different ways and it is likely that the students will find other solution methods than those suggested below. The assessment template shows the partial points given for different steps in the proposed solutions, and these points must be added. If the student has achieved another solution or partial solution serves the assessment template as a starting point for the assessment.

Thank you for your participation!

#1.

Suggested solution: Let's start by denoting an arbitrary two-digit number with a tens digit a and singular number b. The two-digit number can then be expressed as 10a + bd¨ar a ¨ar which number preferably except zero, and b can be any number.

The desired number, or the number, is equal to the sum of its digit product (a · b) and its digit sum (a + b). This means that we can draw the connection

$$a \cdot 10 + b = a \cdot b + (a + b)$$

If we simplify this we get

$$10a = ab + a$$

Now we note that a ÿ= 0, which means that we can divide by ai both the right-hand side and the left-hand side. We'll get that

$$10 = b + 1$$

$$b = 9$$

The relationship therefore applies to all two-digit numbers with the singular digit 9, regardless of the value of the tens digit a.

The numbers sought are thus 19, 29, 39, 49, 59, 69, 79, 89, and 99. That all these work follows from that we had equivalences in all stages of the solution.

(If we had had a different solution method, we might have had to verify that all these nine number satisfies the condition.)

Answers: 19, 29, 39, 49, 59, 69, 79, 89, 99.

Po¨ang:

Only answers give no points. Justifications are required.

Expresses the relationship "the numbers are equal to the sum of their digit product." and its digit sum" in correct equation +1p

Excludes all numbers that do not end in 9 with good justification (for example solved equation) +1p

Determines all nine numbers (and verifies them, if required by the student's chosen solution method) +1p

#2.

Suggested solution: Let's start by drawing a figure and label the angles of the triangle with x, y and z. Then the three exterior angles become 180ÿ ÿ x, 180ÿ ÿ y and 180ÿ ÿ z.

The sum of all exterior angles becomes

(180ÿ ÿ x) + (180ÿ ÿ y) + (180ÿ ÿ z) = 540ÿ ÿ (x + y + z)

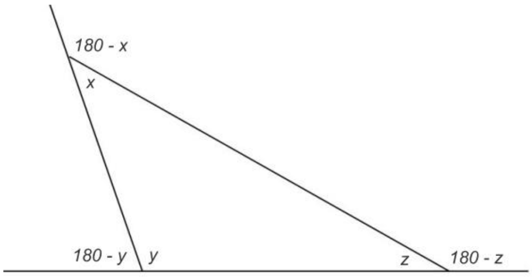

Figure 1:

But x+y+z is exactly the sum of the angles in the triangle, and the sum of the angles in a triangle is always 180ÿ . Thus we get that the sum of the exterior angles is

540ÿ ÿ (x + y + z) = 540ÿ ÿ 180ÿ = 360ÿ

We know that the exterior angles are 7:9:14, i.e. the exterior angles can be written as 7m, 9m and 14m for some unknown number m. Our first measure to determine this m.

We can now calculate the sum of the exterior angles again, since we know that their sum is 360ÿ :

7m + 9m + 14m = 30m = 360ÿ

i.e

m = 360ÿ / 30 = 12ÿ

This means that the exterior angles are

7m = 7 · 12ÿ = 84ÿ

9m = 9 · 12ÿ = 108ÿ

14m = 14 · 12ÿ = 168ÿ

Each of the angles in the triangle is 180ÿ minus its exterior angle, i.e. the corresponding interior angles of the triangle are

180ÿ ÿ 84ÿ = 96ÿ

180ÿ ÿ 108ÿ = 72ÿ

180ÿ ÿ 168ÿ = 12ÿ

The smallest angle of the triangle is therefore 12ÿ .

Answer: The smallest angle of the triangle is 12ÿ

Points:
Answers only give no points. Justifications are required.

Assumes that the triangle has certain defined angles, or some defined shape (e.g. isosceles or right-angled), and then begins to perform calculations on this. This will not lead to a complete solution, but the first two points below cannot be awarded either. 0p

Draws a figure and designates angles and exterior angles in the triangle generally applicable +1p
Determines the sum of all exterior angles to 360ÿ +1p
Correctly determines the smallest angle +1p

#3.

Suggested solution 1: Let Aunt Ester's age be m. Then the average age of all the guests is m. This means that the sum of the ages of all 26 guests is 26m.

When Aunt Ester is in the living room, the average age is m + 3. Since there are 15 guests and Aunt Ester in the room, there are 16 people. This means that the sum of everyone's ages is

$$16(m + 3) = 16m + 48$$

If we exclude Aunt Ester, whose age is m, then the sum of the ages of the other guests

$$(16m + 48) ÿ m = 15m + 48$$

The other 11 guests are now sitting in the kitchen. The sum of their ages is now the total guest age (26m) minus the total age of all those sitting in the living room (15m + 48), i.e.

$$26m ÿ (15m + 48) = 11m ÿ 48$$

Since Aunt Ester went into the kitchen, there are 12 people with a total age

$$(11m ÿ 48) + m = 12m ÿ 48$$

The average age of the 12 people is

$$\frac{12m ÿ 48}{12} = m ÿ 4$$

The average age is also 4 years less than Aunt Ester's age.

Suggested solution 2: Instead of Aunt Ester walking between the living room and the kitchen, we can imagine that she has also invited her twin sister, and keeps her sister ok while she herself is standing in the living room. Since the sister is the same age as aunt Ester, this does not change the mean values.

Let Aunt Ester's age be m. This is also the average age of all guests m. This means that the sum of the ages of all 26 guests is 26m.

The average age of everyone in the living room, and even if we count Aunt Ester and her imaginary twin sister, is

$$\frac{26m + m + m}{28} = m$$

i.e. still m, Aunt Ester's age.

In the living room, we calculate the average age of 16 people, the 15 guests and Aunt Ester.
In the kitchen, we calculate the average age of 12 people (11 guests and Aunt Ester's twin sister).
Medelÿaldern ber¨aknas alltsÿa ¨over $\frac{16}{12} = \frac{4}{3}$ more in the living room.

Since the average age of all these 28 people (26 guests, Aunt Ester, and her twin sister) is Aunt Ester's age, the average age relative to Aunt Ester's age must be lower than the are more in the living room. That is, since the a$\frac{4}{3}$verage age is 3 more than Aunt Esther's age in the living room, it is

$\frac{4}{3} \cdot 3 = 4$ less odd. 3

Answer: 4 years less.

Points:
Answers only give no points. Justifications are required.

Understands that the sum of all ages should be calculated, and that the age in the living room is calculated over 16 people and the kitchen over 12 people. +1 p
Correctly expresses the total age of the 15 guests sitting in the living room, or sets up some relevant relationship between the average age of the kitchen and the living room +1p Correctly determines that the average age of the kitchen 4 years younger than Aunt Ester's age +1p

#4.

Suggested solution: Note that only answers are required for this problem. Here are some different ways to express the three answers with all six given numbers. Note that there may be more ways, which of course are also correct, as long as all six given numbers are used.

$$2023 = 2 \cdot 10 \cdot 100 + 25 + 5 \text{ ÿ } 7$$

$$= 25 \cdot 5 \cdot (10 + 7) \text{ ÿ } 100 \text{ ÿ } 2$$

$$2024 = (25 \text{ ÿ } 2)(100 \text{ ÿ } 10 + 5 \text{ ÿ } 7)$$

$$= 100 \cdot (25 \text{ ÿ } 5) + 2 \cdot 7 + 10 = (\frac{25}{5} + 7 + 100 \cdot 10) \cdot 2$$

$$2025 = (100 \text{ ÿ } 25)(5 \cdot 7 + 2 \text{ ÿ } 10)$$

$$= 25 \cdot (100 \text{ ÿ } 2 \cdot 7 \text{ ÿ } 10 + 5) = (10 + 7 + 2) \cdot 100 + 5 \cdot 25 = (10 \cdot (7 + 2) \text{ ÿ } 5) \cdot 25 \text{ ÿ } 100$$

Points:
Only answers with expressions are sufficient.

Expresses 2023 with all given numbers and operations	+1p
Expresses 2024 with all given numbers and operations	+1p
Expresses 2025 with all given numbers and operations	+1p

#5.

Solution proposal 1: Our strategy for this solution is to first prove that three weights are not enough. After that we find a setup with four weights which shows that it is possible to create all whole tones from 3 to 9.

Suppose we have three weights A, B and C, with which it is possible to create all seven weights 3 ÿ 9 tons. We can also assume that A ÿ B ÿ C. With these three weights, we can then form the following seven sums:

A
B
C
A + B
A + C
B + C
A + B + C

These seven sums must now correspond exactly to the seven weights 3 ÿ 9 tons.

The lightest weight that can be created is the one that contains only weight A (since A is the lightest of the three weights). This must then be the smallest of the seven weights 3 ÿ 9, i.e. 3 tons, since each of the seven sums corresponds to one of the seven weights.

But if A = 3, we know that B ÿ 3 and C ÿ 3, i.e. all combinations of two weights will weigh at least 3 + 3 = 6 tons. This means that in order to obtain the weights 4 and 5 tons, we must have B = 4 and C = 5.

This now gives that A + B = 3 + 4 = 7, A + C = 3 + 5 = 8, B + C = 4 + 5 = 9 and A + B + C = 3 + 4 + 5 = 12 tons. We can therefore not create the weight 6 tons under these assumptions, which means that three weights are not enough.

Finally, a construction that works with 4 numbers: 3, 4, 5 and 6, because these four numbers can be formed with single weights and the remaining ones can be written as 7 = 3 + 4, 8 = 3 + 5 and 9 = 3 + 6.

Examples of other sets of four numbers that work: {1,2,3,4}, {1,2,3,5}, {2,2,3,4}, {1,2,3 ,7}, {1,2,4,8}, {1,1,3,6}, {1,3,4,5}, {1,3,5,7}. . .

Suggested solution 2: With k numbers you can form (at most) 2k different sums. However, one of these must be 0 (the sum that does not contain any terms), so at most 2k ÿ 1 of them can be the seven sums from 3 to 9. This means that at least 3 numbers must be used (23 ÿ 1 = 7).

If the 3 numbers are to be able to form 7 different non-zero sums, they must be different. They must also be greater than or equal to 3, because we cannot waste any nonzero sum on being less than 3. So the three numbers that at least 3, 4 and 5, but then the largest sum becomes 3+ 4+ 5 = 12 > 9. One of the sums thus creates a weight that is not between 3 and 9, and thus we are forced "waste away" one of the seven sums on this weight. Six sums are now left to cover seven weights. This is of course impossible. So it is not possible to do this with 3 (or fewer) numbers.

Finally, a construction that works with 4 numbers: 1, 3, 5 and 7.

Answer: Viktualia must have at least four weights.

Points:
Answers only give no points. Justifications are required.

Tries a few different setups with 3 numbers, and concludes that it doesn't work	0p
Justifies with (albeit vague) reasoning that 3 numbers are not enough	+1p
Justifies very well that 3 numbers are not enough	+1p
Shows that 4 numbers are enough, by giving a set of numbers, and verifying that all weights can be created	+1 p

#6.

Suggested solution 1: The sum of all the numbers in the grid is

$$(2 + 4) \cdot 3 + 6 \cdot 3 + 21 + (2 + 3 + 4) \cdot 2 + 6 \cdot 1 + (4 + 2) \cdot 3 =$$

$$= 18 + 18 + 21 + 18 + 6 + 18 = 99$$

The sum of the weights must be 30, i.e. the sum of the two parts must be 99 ÿ 30 = 69. However, both parts must have the same sum, so their joint sum must be an even number. This is a contradiction. There are thus no possible paths.

Suggested solution 2: The sum of all the numbers in the grid is odd, because the sums of the rows are

$$j + j + u + j + j + j = u$$

The sum of the winding paths is equal to (30). Thus the sum of the two parts must be u ÿ j = u.
But the sum of two equal numbers never becomes odd, which means that the condition can never be fulfilled.

Answer: There are no solutions because a tortuous path of value 30 always divides the grid into an odd and an even part.

Points:
Answers only give no points. Justifications are required.

Draw some winding roads and see that in these cases the upper and lower parts are different. Then concludes, without general reasoning, that it is not possible.	0p

Calculates the grid sum to 99, or realizes that the total is odd	+1 p
Draws correct conclusion (that it is impossible), with correct (if vague) general reasoning	+1 p
Very good justification	+1 p