Kalman Filter with Constant Acceleration Model

In [1]:

```
import numpy as np
%matplotlib inline
import matplotlib.pyplot as plt
from scipy.stats import norm
```

Situation covered: You have an acceleration sensor (in 2D: $\ddot x$ and $\ddot y$) and try to calculate velocity ($\dot x$ and $\dot y$) as well as position (x and y) of a person holding a smartphone in his/her hand.\n

State Vector - Constant Acceleration

Constant Acceleration Model for Ego Motion in Plane

$$x_k = \begin{bmatrix} x \\ y \\ \dot x \\ \dot y \\ \ddot x \\ \ddot y \end{bmatrix}$$

Formal Definition (Motion of Law):

$$x_{k+1} = \mathbf{A} \cdot x_k + B \cdot u$$

Hence, we have no control input u:

$$y = H \cdot x$$

$$x_{k+1} = \begin{bmatrix} 1 & 0 & \Delta t & 0 & \frac{1}{2}\Delta t^2 & 0 \\ 0 & 1 & 0 & \Delta t & 0 & \frac{1}{2}\Delta t^2 \\ 0 & 0 & 1 & 0 & \Delta t & 0 \\ 0 & 0 & 0 & 1 & 0 & \Delta t \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \\ \dot x \\ \dot y \\ \ddot x \\ \ddot y \end{bmatrix}_k$$

Just the acceleration ($\ddot x$ & $\ddot y$) is measured.

Observation Model:

$$y = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \cdot x$$

Initial State

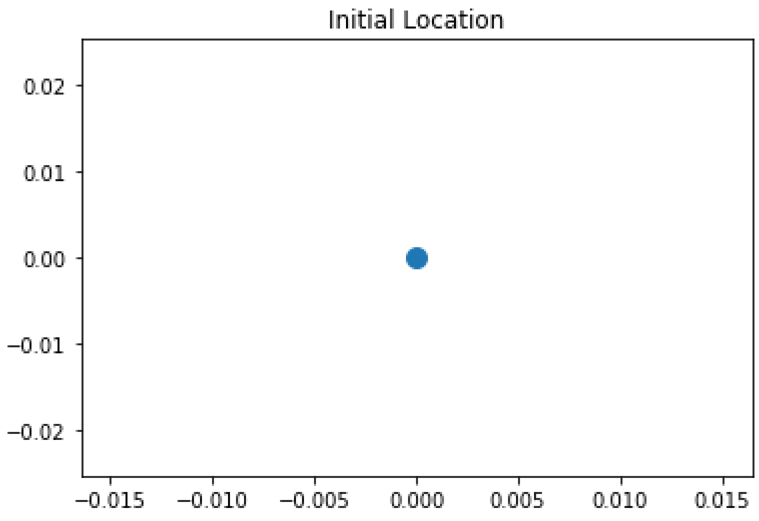
In [2]:

```
x = np.matrix([[0.0, 0.0, 0.0, 0.0, 0.0, 0.0]]).T
print(x, x.shape)
n=x.size # States
plt.scatter(float(x[0]),float(x[1]), s=100)
plt.title('Initial Location')
```

```
[[0.]
 [0.]
 [0.]
 [0.]
 [0.]
 [0.]] (6, 1)
```

Out[2]:

```
Text(0.5, 1.0, 'Initial Location')
```



Initial Uncertainty

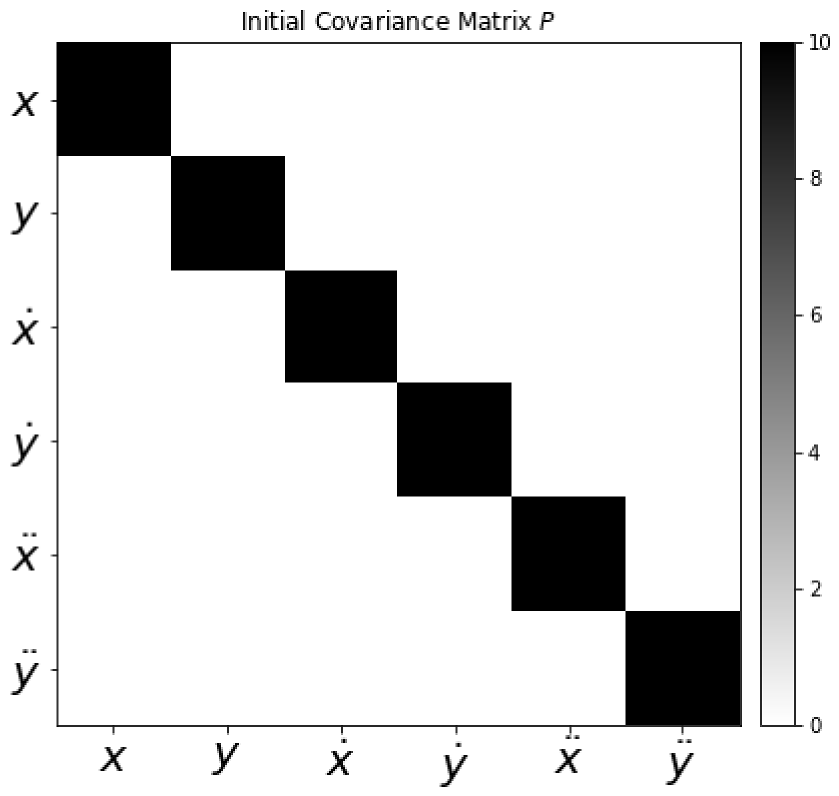
In [3]:

```
P = np.matrix([[10.0, 0.0, 0.0, 0.0, 0.0, 0.0],
              [0.0, 10.0, 0.0, 0.0, 0.0, 0.0],
              [0.0, 0.0, 10.0, 0.0, 0.0, 0.0],
              [0.0, 0.0, 0.0, 10.0, 0.0, 0.0],
              [0.0, 0.0, 0.0, 0.0, 10.0, 0.0],
              [0.0, 0.0, 0.0, 0.0, 0.0, 10.0]])
print(P, P.shape)
```

```
[[10.  0.  0.  0.  0.  0.]
 [ 0. 10.  0.  0.  0.  0.]
 [ 0.  0. 10.  0.  0.  0.]
 [ 0.  0.  0. 10.  0.  0.]
 [ 0.  0.  0.  0. 10.  0.]
 [ 0.  0.  0.  0.  0. 10.]] (6, 6)
```

In [4]:

```
fig = plt.figure(figsize=(6, 6))
im = plt.imshow(P, interpolation="none", cmap=plt.get_cmap('binary'))
plt.title('Initial Covariance Matrix $P$')
ylocs, ylabels = plt.yticks()
# set the locations of the yticks
plt.yticks(np.arange(7))
# set the locations and labels of the yticks
plt.yticks(np.arange(6),('$x$', '$y$', '$\dot x$', '$\dot y$', '$\ddot x$', '$\ddot

xlocs, xlabels = plt.xticks()
# set the locations of the yticks
plt.xticks(np.arange(7))
# set the locations and labels of the yticks
plt.xticks(np.arange(6),('$x$', '$y$', '$\dot x$', '$\dot y$', '$\ddot x$', '$\ddot

plt.xlim([-0.5,5.5])
plt.ylim([5.5, -0.5])

from mpl_toolkits.axes_grid1 import make_axes_locatable
divider = make_axes_locatable(plt.gca())
cax = divider.append_axes("right", "5%", pad="3%")
plt.colorbar(im, cax=cax)


plt.tight_layout()
```



Dynamic Matrix

It is calculated from the dynamics of the Egomotion.

$$\begin{aligned}
x_{k+1} &= x_k + \dot{x}_k \cdot \Delta t + \ddot{x}_k \cdot \frac{1}{2}\Delta t^2 \\
y_{k+1} &= y_k + \dot{y}_k \cdot \Delta t + \ddot{y}_k \cdot \frac{1}{2}\Delta t^2 \\
\dot{x}_{k+1} &= \dot{x}_k + \ddot{x} \cdot \Delta t \\
\dot{y}_{k+1} &= \dot{y}_k + \ddot{y} \cdot \Delta t \\
\ddot{x}_{k+1} &= \ddot{x}_k \\
\ddot{y}_{k+1} &= \ddot{y}_k
\end{aligned}$$

In [5]:

```
dt = 0.5 # Time Step between Filter Steps

A = np.matrix([[1.0, 0.0, dt, 0.0, 1/2.0*dt**2, 0.0],
              [0.0, 1.0, 0.0, dt, 0.0, 1/2.0*dt**2],
              [0.0, 0.0, 1.0, 0.0, dt, 0.0],
              [0.0, 0.0, 0.0, 1.0, 0.0, dt],
              [0.0, 0.0, 0.0, 0.0, 1.0, 0.0],
              [0.0, 0.0, 0.0, 0.0, 0.0, 1.0]])
print(A, A.shape)
```

```
[[1.    0.    0.5   0.    0.125 0.   ]
 [0.    1.    0.    0.5   0.    0.125]
 [0.    0.    1.    0.    0.5   0.   ]
 [0.    0.    0.    1.    0.    0.5  ]
 [0.    0.    0.    0.    1.    0.   ]
 [0.    0.    0.    0.    0.    1.   ]] (6, 6)
```

Measurement Matrix

Here you can determine, which of the states is covered by a measurement. In this example,the acceleration is measured ($\ddot{x}$ and $\ddot{y}$).

In [6]:

```
H = np.matrix([[0.0, 0.0, 0.0, 0.0, 1.0, 0.0],
              [0.0, 0.0, 0.0, 0.0, 0.0, 1.0]])
print(H, H.shape)
```

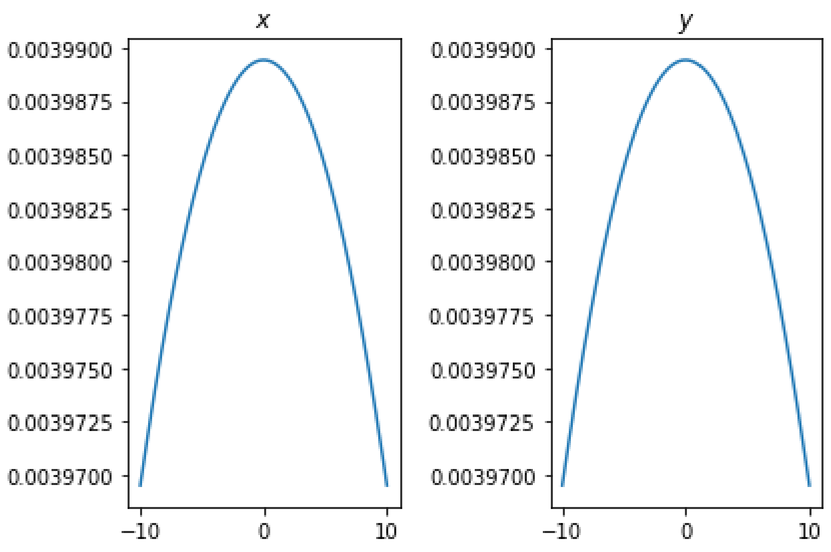
```
[[0. 0. 0. 0. 1. 0.]
 [0. 0. 0. 0. 0. 1.]] (2, 6)
```

Measurement Noise Covariance

In [7]:

```
ra = 10.0**2
R = np.matrix([[ra, 0.0],
               [0.0, ra]])
print(R, R.shape)

# Plot between -10 and 10 with .001 steps.
xpdf = np.arange(-10, 10, 0.001)
plt.subplot(121)
plt.plot(xpdf, norm.pdf(xpdf,0,R[0,0]))
plt.title('$x$')

plt.subplot(122)
plt.plot(xpdf, norm.pdf(xpdf,0,R[1,1]))
plt.title('$y$')


plt.tight_layout()
```

```
[[100.   0.]
 [  0. 100.]] (2, 2)
```



Process Noise Covariance Matrix Q for CA Model

The Position of an object can be influenced by a force (e.g. wind), which leads to an acceleration disturbance (noise). This process noise has to be modeled with the process noise covariance matrix Q.

$$Q = \begin{bmatrix} \sigma_x^2 & \sigma_{xy} & \sigma_{x\dot x} & \sigma_{x\dot y} & \sigma_{x\ddot x} & \sigma_{x\ddot y} \\ \sigma_{yx} & \sigma_y^2 & \sigma_{y\dot x} & \sigma_{y\dot y} & \sigma_{y\ddot x} & \sigma_{y\ddot y} \\ \sigma_{\dot x x} & \sigma_{\dot x y} & \sigma_{\dot x}^2 & \sigma_{\dot x\dot y} & \sigma_{\dot x\ddot x} & \sigma_{\dot x\ddot y} \\ \sigma_{\dot y x} & \sigma_{\dot y y} & \sigma_{\dot y\dot x} & \sigma_{\dot y}^2 & \sigma_{\dot y\ddot x} & \sigma_{\dot y\ddot y} \\ \sigma_{\ddot x x} & \sigma_{\ddot x y} & \sigma_{\ddot x\dot x} & \sigma_{\ddot x\dot y} & \sigma_{\ddot x}^2 & \sigma_{\ddot x\ddot y} \\ \sigma_{\ddot y x} & \sigma_{\ddot y y} & \sigma_{\ddot y\dot x} & \sigma_{\ddot y\dot y} & \sigma_{\ddot y\ddot x} & \sigma_{\ddot y}^2 \end{bmatrix}$$

To easily calcualte Q, one can ask the question: How the noise effects my state vector? For example, how the acceleration change the position over one timestep dt.

One can calculate Q as

$$Q = G \cdot G^T \cdot \sigma_a^2$$

with $G = \begin{bmatrix}0.5dt^2 & 0.5dt^2 & dt & dt & 1.0 & 1.0\end{bmatrix}^T$ and σ_a as the acceleration process noise.

Symbolic Calculation

In [8]:

```
from sympy import Symbol, Matrix
from sympy.interactive import printing
printing.init_printing(use_latex=True)
dts = Symbol('\Delta t')
Qs = Matrix([[0.5*dts**2],[0.5*dts**2],[dts],[dts],[1.0],[1.0]])
```

In [9]:

```
Qs*Qs.T
```

Out[9]:

$$\begin{bmatrix} 0.25\Delta t^4 & 0.25\Delta t^4 & 0.5\Delta t^3 & 0.5\Delta t^3 & 0.5\Delta t^2 & 0.5\Delta t^2 \\ 0.25\Delta t^4 & 0.25\Delta t^4 & 0.5\Delta t^3 & 0.5\Delta t^3 & 0.5\Delta t^2 & 0.5\Delta t^2 \\ 0.5\Delta t^3 & 0.5\Delta t^3 & \Delta t^2 & \Delta t^2 & 1.0\Delta t & 1.0\Delta t \\ 0.5\Delta t^3 & 0.5\Delta t^3 & \Delta t^2 & \Delta t^2 & 1.0\Delta t & 1.0\Delta t \\ 0.5\Delta t^2 & 0.5\Delta t^2 & 1.0\Delta t & 1.0\Delta t & 1.0 & 1.0 \\ 0.5\Delta t^2 & 0.5\Delta t^2 & 1.0\Delta t & 1.0\Delta t & 1.0 & 1.0 \end{bmatrix}$$

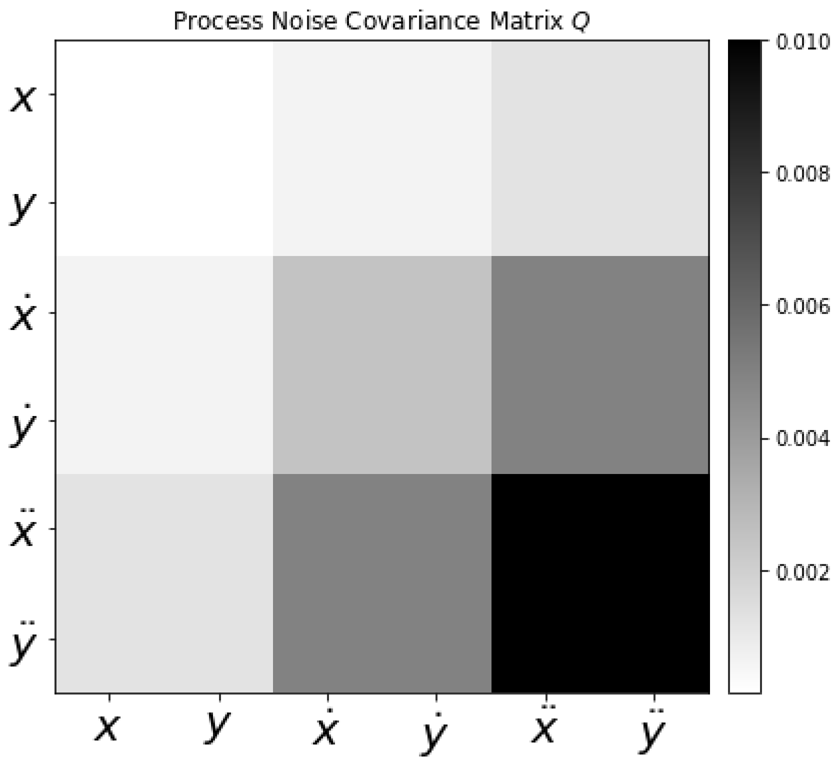
In [10]:

```
sa = 0.1
G = np.matrix([[1/2.0*dt**2],
               [1/2.0*dt**2],
               [dt],
               [dt],
               [1.0],
               [1.0]])
Q = G*G.T*sa**2

print(Q, Q.shape)
```

```
[[0.00015625 0.00015625 0.000625   0.000625   0.00125    0.00125   ]
 [0.00015625 0.00015625 0.000625   0.000625   0.00125    0.00125   ]
 [0.000625   0.000625   0.0025     0.0025     0.005      0.005     ]
 [0.000625   0.000625   0.0025     0.0025     0.005      0.005     ]
 [0.00125    0.00125    0.005      0.005      0.01       0.01      ]
 [0.00125    0.00125    0.005      0.005      0.01       0.01      ]]
(6, 6)
```

In [11]:

```
fig = plt.figure(figsize=(6, 6))
im = plt.imshow(Q, interpolation="none", cmap=plt.get_cmap('binary'))
plt.title('Process Noise Covariance Matrix $Q$')
ylocs, ylabels = plt.yticks()
# set the locations of the yticks
plt.yticks(np.arange(7))
# set the locations and labels of the yticks
plt.yticks(np.arange(6),('$x$', '$y$', '$\dot x$', '$\dot y$', '$\ddot x$', '$\ddot

xlocs, xlabels = plt.xticks()
# set the locations of the yticks
plt.xticks(np.arange(7))
# set the locations and labels of the yticks
plt.xticks(np.arange(6),('$x$', '$y$', '$\dot x$', '$\dot y$', '$\ddot x$', '$\ddot

plt.xlim([-0.5,5.5])
plt.ylim([5.5, -0.5])

from mpl_toolkits.axes_grid1 import make_axes_locatable
divider = make_axes_locatable(plt.gca())
cax = divider.append_axes("right", "5%", pad="3%")
plt.colorbar(im, cax=cax)

plt.tight_layout()
```



In [12]:

```
I = np.eye(n)
print(I, I.shape)
```

```
[[1. 0. 0. 0. 0. 0.]
 [0. 1. 0. 0. 0. 0.]
 [0. 0. 1. 0. 0. 0.]
 [0. 0. 0. 1. 0. 0.]
 [0. 0. 0. 0. 1. 0.]
 [0. 0. 0. 0. 0. 1.]] (6, 6)
```

Measurement

They are generated synthetically

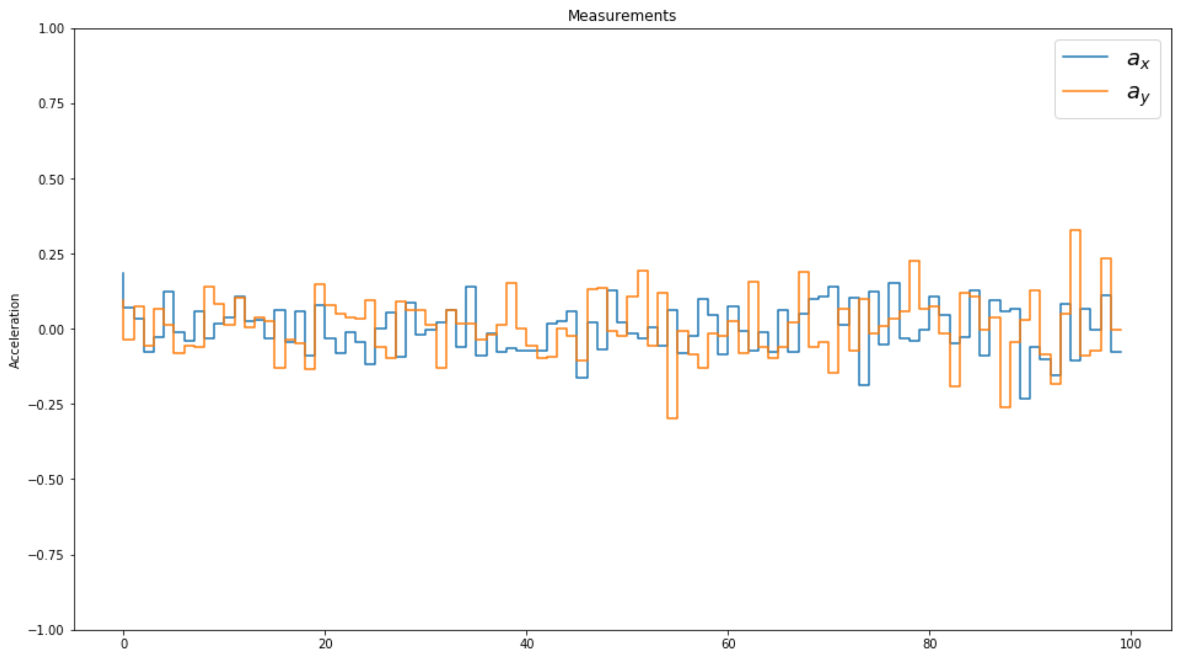
In [13]:

```
m = 100 # Measurements

# Acceleration
sa= 0.1 # Sigma for acceleration
ax= 0.0 # in X
ay= 0.0 # in Y

mx = np.array(ax+sa*np.random.randn(m))
my = np.array(ay+sa*np.random.randn(m))

measurements = np.vstack((mx,my))

print(measurements.shape)
print('Standard Deviation of Acceleration Measurements=%.2f' % np.std(mx))
print('You assumed %.2f in R.' % R[0,0])
```

```
(2, 100)
Standard Deviation of Acceleration Measurements=0.08
You assumed 100.00 in R.
```

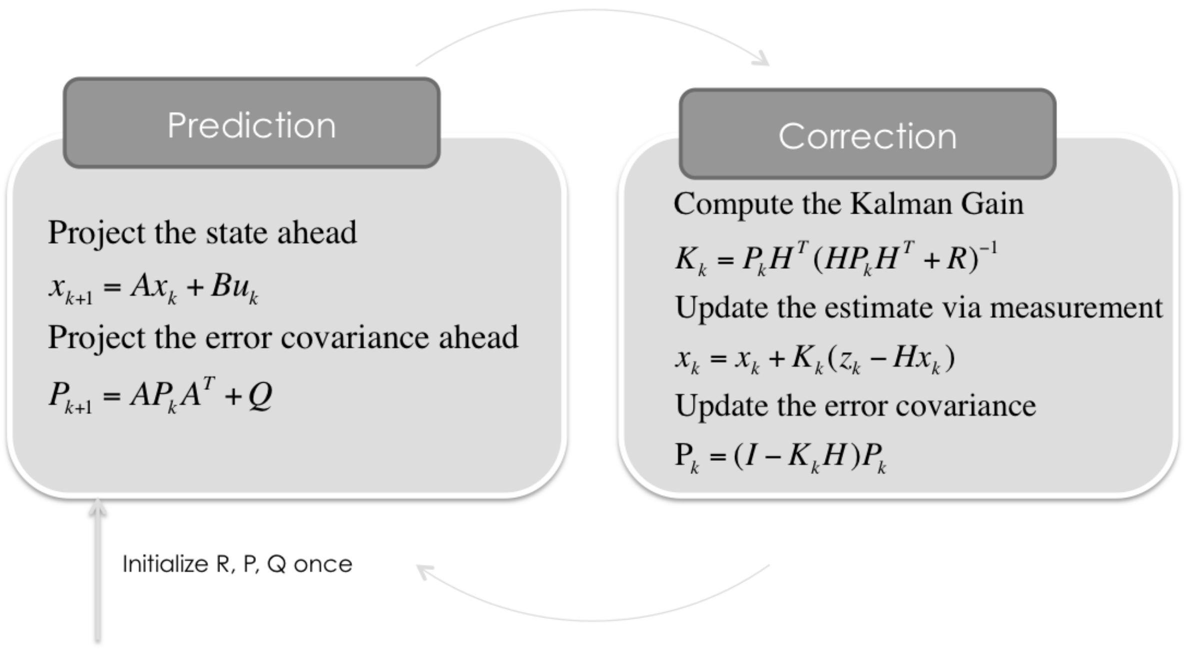
In [14]:

```
fig = plt.figure(figsize=(16,9))
plt.step(range(m),mx, label='$a_x$')
plt.step(range(m),my, label='$a_y$')
plt.ylabel('Acceleration')
plt.title('Measurements')
plt.ylim([-1, 1])
plt.legend(loc='best',prop={'size':18})
plt.savefig('Kalman-Filter-CA-Measurements.png', dpi=72, transparent=True, bbox_inc
```



In [15]:

```
# Preallocation for Plotting
xt = []
yt = []
dxt= []
dyt= []
ddxt=[]
ddyt=[]
Zx = []
Zy = []
Px = []
Py = []
Pdx= []
Pdy= []
Pddx=[]
Pddy=[]
Kx = []
Ky = []
Kdx= []
Kdy= []
Kddx=[]
Kddy=[]
```


Prediction
Project the state ahead
$x_{k+1} = Ax_k + Bu_k$
Project the error covariance ahead
$P_{k+1} = AP_kA^T + Q$
Correction
Compute the Kalman Gain
$K_k = P_kH^T(HP_kH^T + R)^{-1}$
Update the estimate via measurement
$x_k = x_k + K_k(z_k - Hx_k)$
Update the error covariance
$\mathrm{P}_k = (I - K_kH)P_k$
Initialize R, P, Q once

In [16]:

```
for n in range(m):

    # Time Update (Prediction)
    # ========================
    # Project the state ahead
    x = A*x

    # Project the error covariance ahead
    P = A*P*A.T + Q


    # Measurement Update (Correction)
    # ===============================
    # Compute the Kalman Gain
    S = H*P*H.T + R
    K = (P*H.T) * np.linalg.pinv(S)


    # Update the estimate via z
    Z = measurements[:,n].reshape(H.shape[0],1)
    y = Z - (H*x)                            # Innovation or Residual
    x = x + (K*y)

    # Update the error covariance
    P = (I - (K*H))*P



    # Save states for Plotting
    xt.append(float(x[0]))
    yt.append(float(x[1]))
    dxt.append(float(x[2]))
    dyt.append(float(x[3]))
    ddxt.append(float(x[4]))
    ddyt.append(float(x[5]))
    Zx.append(float(Z[0]))
    Zy.append(float(Z[1]))
    Px.append(float(P[0,0]))
    Py.append(float(P[1,1]))
    Pdx.append(float(P[2,2]))
    Pdy.append(float(P[3,3]))
    Pddx.append(float(P[4,4]))
    Pddy.append(float(P[5,5]))
    Kx.append(float(K[0,0]))
    Ky.append(float(K[1,0]))
    Kdx.append(float(K[2,0]))
    Kdy.append(float(K[3,0]))
    Kddx.append(float(K[4,0]))
    Kddy.append(float(K[5,0]))
```

Plots

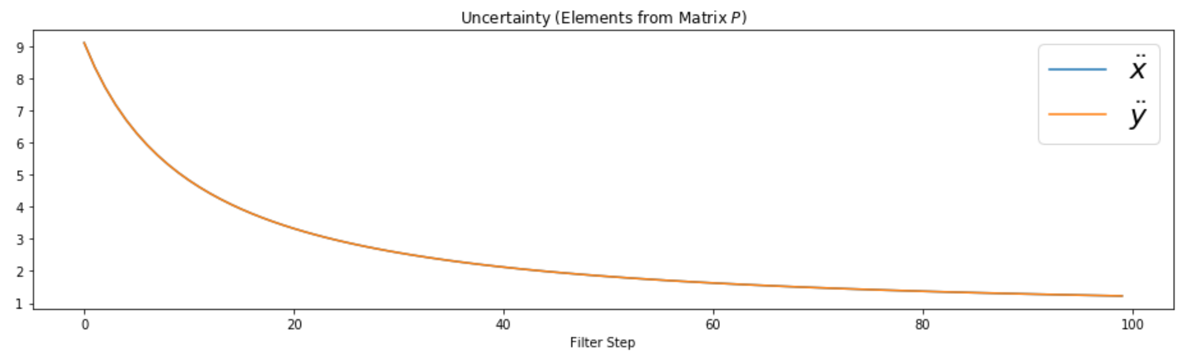
Uncertainty

In [17]:

```
fig = plt.figure(figsize=(16,4))
#plt.plot(range(len(measurements[0])),Px, label='$x$')
#plt.plot(range(len(measurements[0])),Py, label='$y$')
plt.plot(range(len(measurements[0])),Pddx, label='$\ddot x$')
plt.plot(range(len(measurements[0])),Pddy, label='$\ddot y$')

plt.xlabel('Filter Step')
plt.ylabel('')
plt.title('Uncertainty (Elements from Matrix $P$)')
plt.legend(loc='best',prop={'size':22})
```

Out[17]:

```
<matplotlib.legend.Legend at 0x22e8b307550>
```



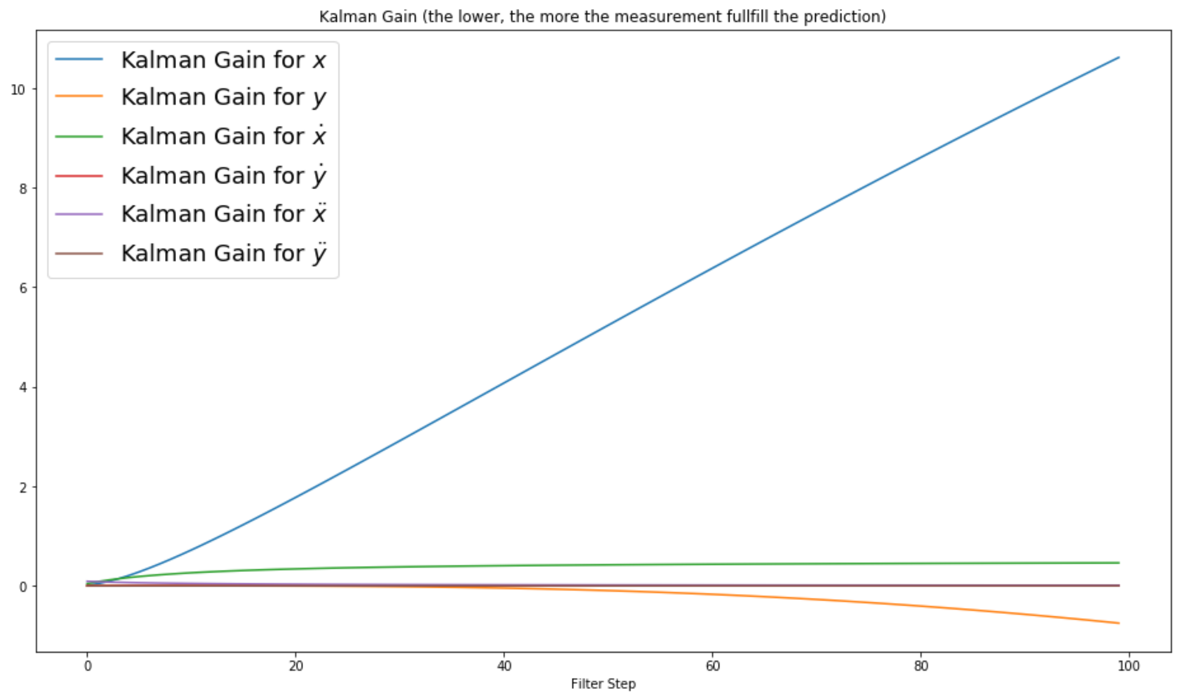
Kalman Gains

In [18]:

```
fig = plt.figure(figsize=(16,9))
plt.plot(range(len(measurements[0])),Kx, label='Kalman Gain for $x$')
plt.plot(range(len(measurements[0])),Ky, label='Kalman Gain for $y$')
plt.plot(range(len(measurements[0])),Kdx, label='Kalman Gain for $\dot x$')
plt.plot(range(len(measurements[0])),Kdy, label='Kalman Gain for $\dot y$')
plt.plot(range(len(measurements[0])),Kddx, label='Kalman Gain for $\ddot x$')
plt.plot(range(len(measurements[0])),Kddy, label='Kalman Gain for $\ddot y$')

plt.xlabel('Filter Step')
plt.ylabel('')
plt.title('Kalman Gain (the lower, the more the measurement fullfill the prediction
plt.legend(loc='best',prop={'size':18})
```

Out[18]:

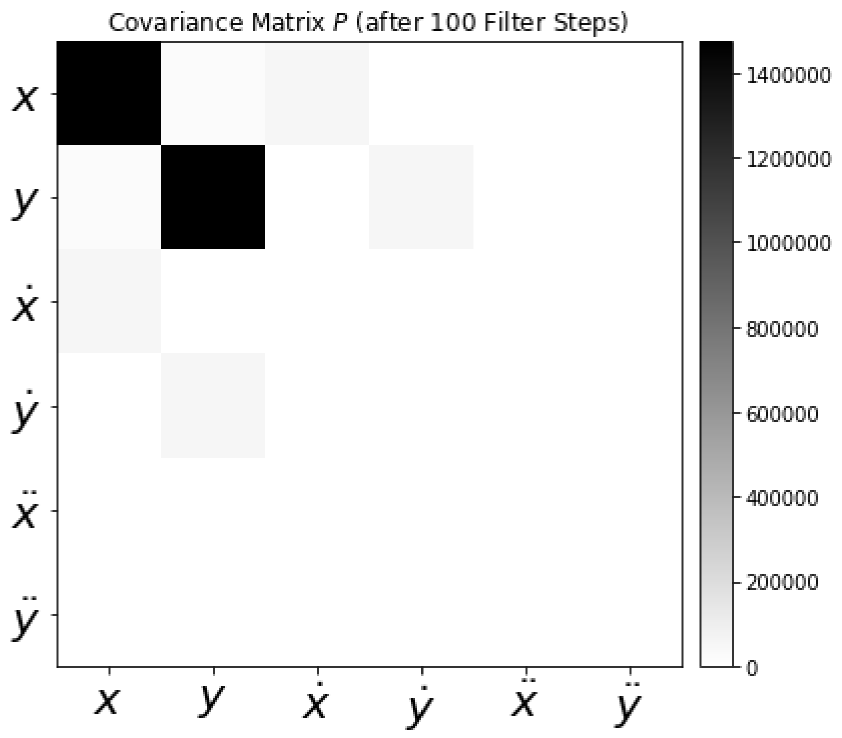
```
<matplotlib.legend.Legend at 0x22e8aa8f080>
```



Covariance Matrix

In [19]:

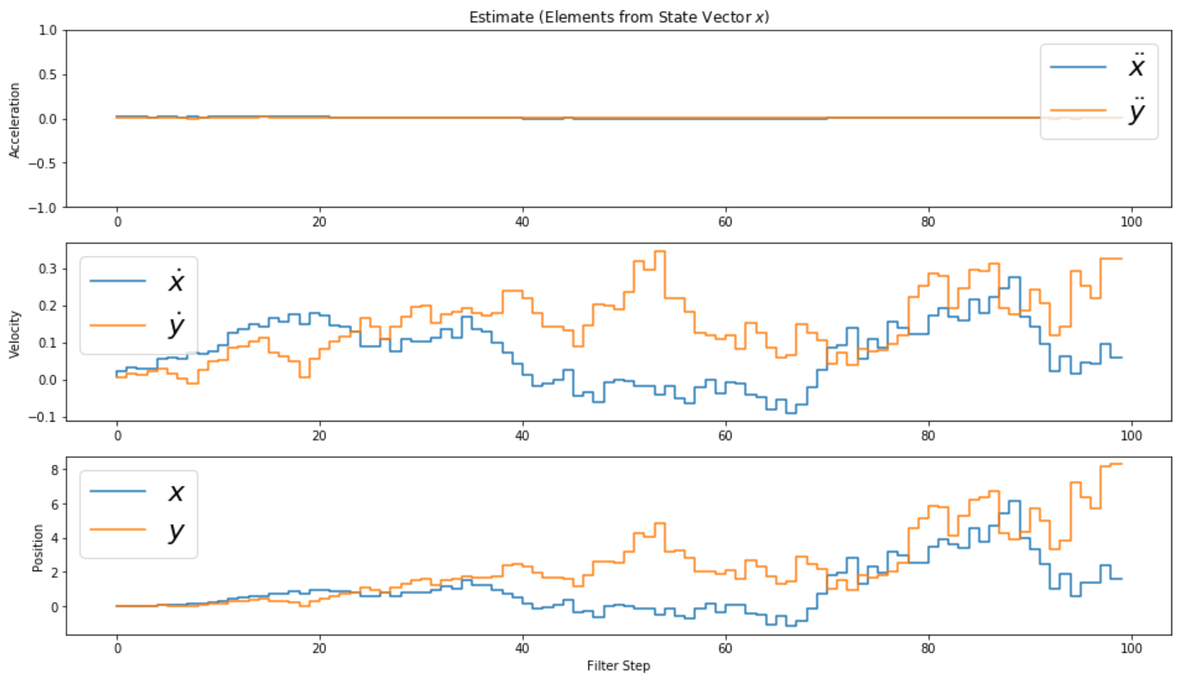
```
fig = plt.figure(figsize=(6, 6))
im = plt.imshow(P, interpolation="none", cmap=plt.get_cmap('binary'))
plt.title('Covariance Matrix $P$ (after %i Filter Steps)' % (m))
ylocs, ylabels = plt.yticks()
# set the locations of the yticks
plt.yticks(np.arange(7))
# set the locations and labels of the yticks
plt.yticks(np.arange(6),('$x$', '$y$', '$\dot x$', '$\dot y$', '$\ddot x$', '$\ddot

xlocs, xlabels = plt.xticks()
# set the locations of the yticks
plt.xticks(np.arange(7))
# set the locations and labels of the yticks
plt.xticks(np.arange(6),('$x$', '$y$', '$\dot x$', '$\dot y$', '$\ddot x$', '$\ddot

plt.xlim([-0.5,5.5])
plt.ylim([5.5, -0.5])

from mpl_toolkits.axes_grid1 import make_axes_locatable
divider = make_axes_locatable(plt.gca())
cax = divider.append_axes("right", "5%", pad="3%")
plt.colorbar(im, cax=cax)


plt.tight_layout()
plt.savefig('Kalman-Filter-CA-CovarianceMatrix.png', dpi=72, transparent=True, bbox
```



State Vector

In [20]:

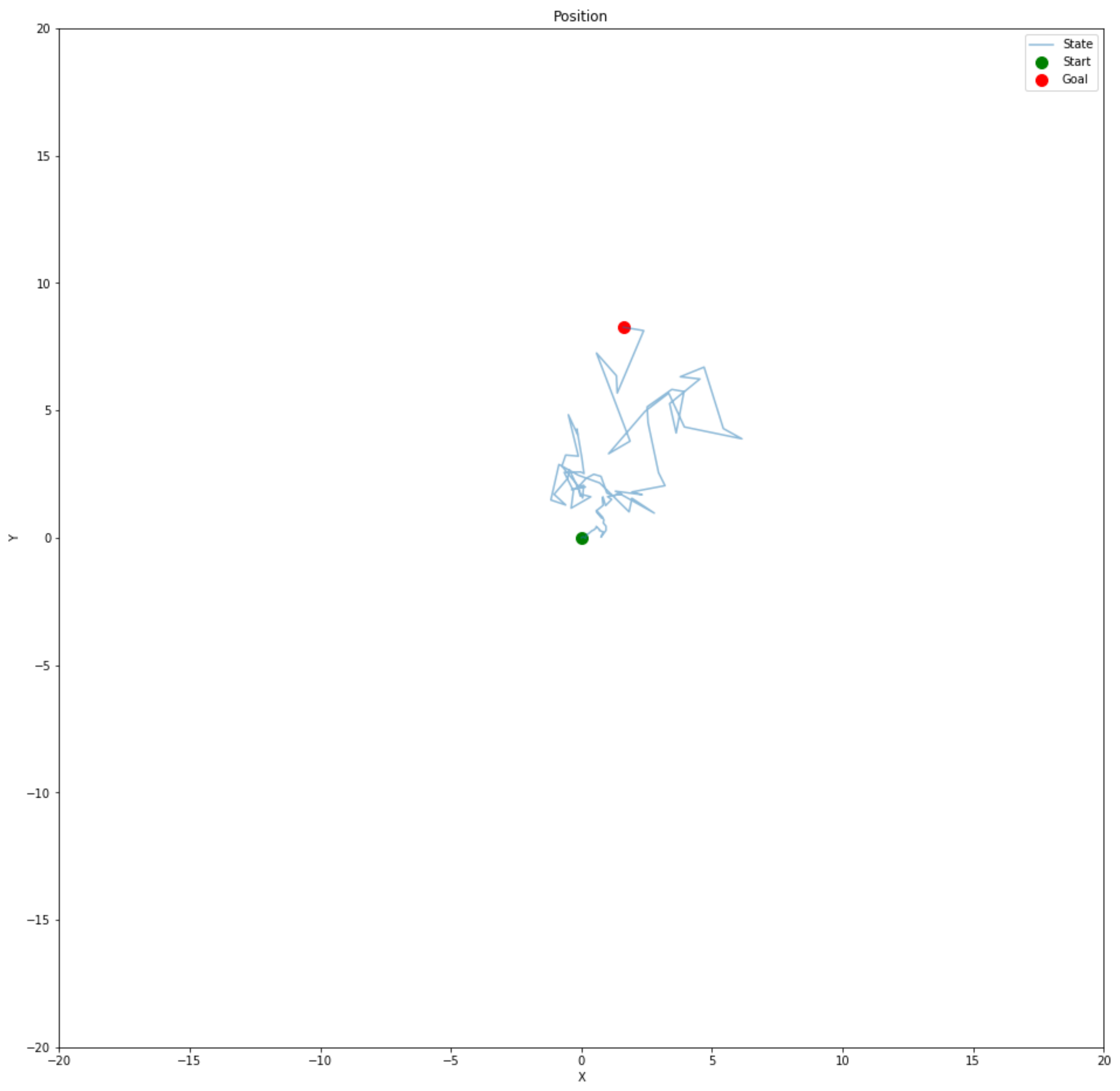
```
fig = plt.figure(figsize=(16,9))

plt.subplot(311)
plt.step(range(len(measurements[0])),ddxt, label='$\ddot x$')
plt.step(range(len(measurements[0])),ddyt, label='$\ddot y$')

plt.title('Estimate (Elements from State Vector $x$)')
plt.legend(loc='best',prop={'size':22})
plt.ylabel('Acceleration')
plt.ylim([-1,1])

plt.subplot(312)
plt.step(range(len(measurements[0])),dxt, label='$\dot x$')
plt.step(range(len(measurements[0])),dyt, label='$\dot y$')

plt.ylabel('')
plt.legend(loc='best',prop={'size':22})
plt.ylabel('Velocity')

plt.subplot(313)
plt.step(range(len(measurements[0])),xt, label='$x$')
plt.step(range(len(measurements[0])),yt, label='$y$')

plt.xlabel('Filter Step')
plt.ylabel('')
plt.legend(loc='best',prop={'size':22})
plt.ylabel('Position')
plt.savefig('Kalman-Filter-CA-StateEstimated.png', dpi=72, transparent=True, bbox_i
```



Position x/y

In [21]:

```
fig = plt.figure(figsize=(16,16))
plt.scatter(xt[0],yt[0], s=100, label='Start', c='g')
plt.scatter(xt[-1],yt[-1], s=100, label='Goal', c='r')
plt.plot(xt,yt, label='State',alpha=0.5)

plt.xlabel('X')
plt.ylabel('Y')
plt.title('Position')
plt.legend(loc='best')
plt.xlim([-20, 20])
plt.ylim([-20, 20])
plt.savefig('Kalman-Filter-CA-Position.png', dpi=72, transparent=True, bbox_inches=
```



Conclusion

In [22]:

```
dist=np.cumsum(np.sqrt(np.diff(xt)**2 + np.diff(yt)**2))
print('Your drifted {} units from origin.'.format(dist[-1]))
```

```
Your drifted 65.71758155005215 units from origin.
```

As you can see, bad idea just to measure the acceleration and try to get the position. The errors integrating up, so your position estimation is drifting.

Reference :

- https://github.com/balzer82/Kalman/blob/master/Kalman-Filter-CA.ipynb?create=1 (https://github.com/balzer82/Kalman/blob/master/Kalman-Filter-CA.ipynb?create=1)
- https://balzer82.github.io/Kalman/ (https://balzer82.github.io/Kalman/)