Kalman Filter Analysis

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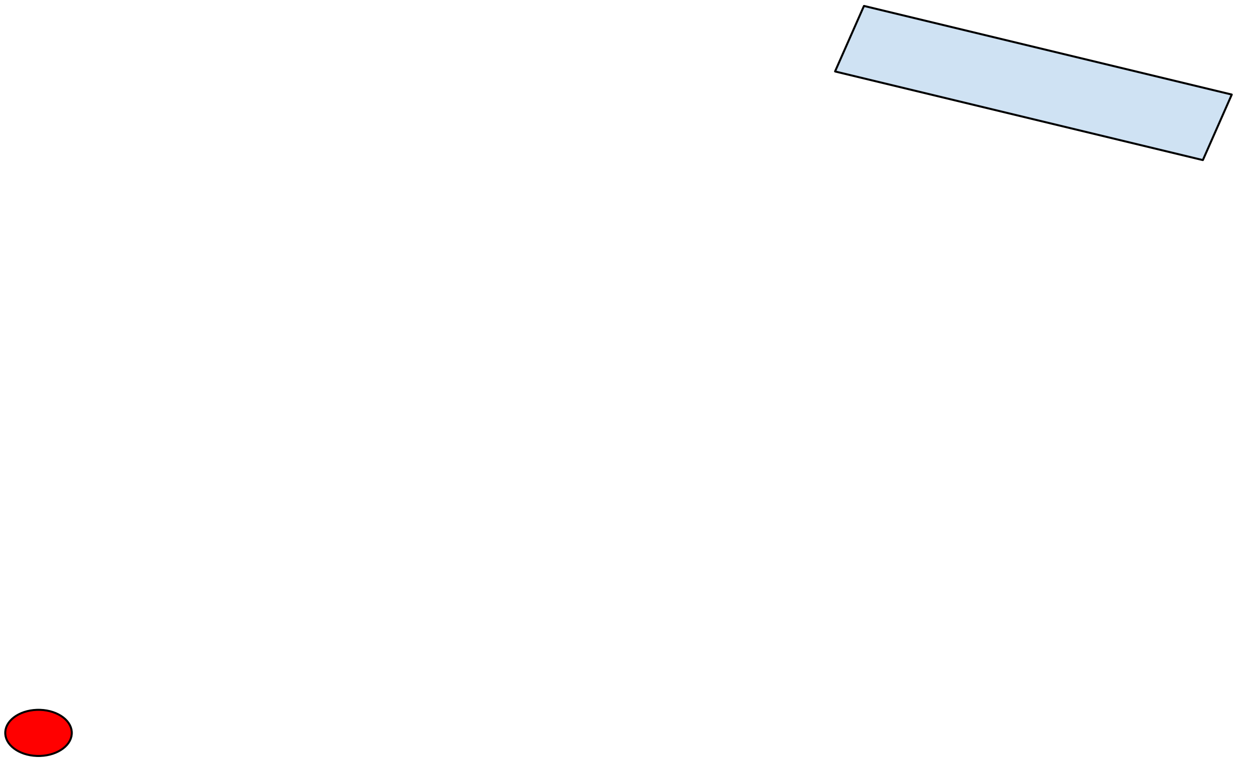
Outline

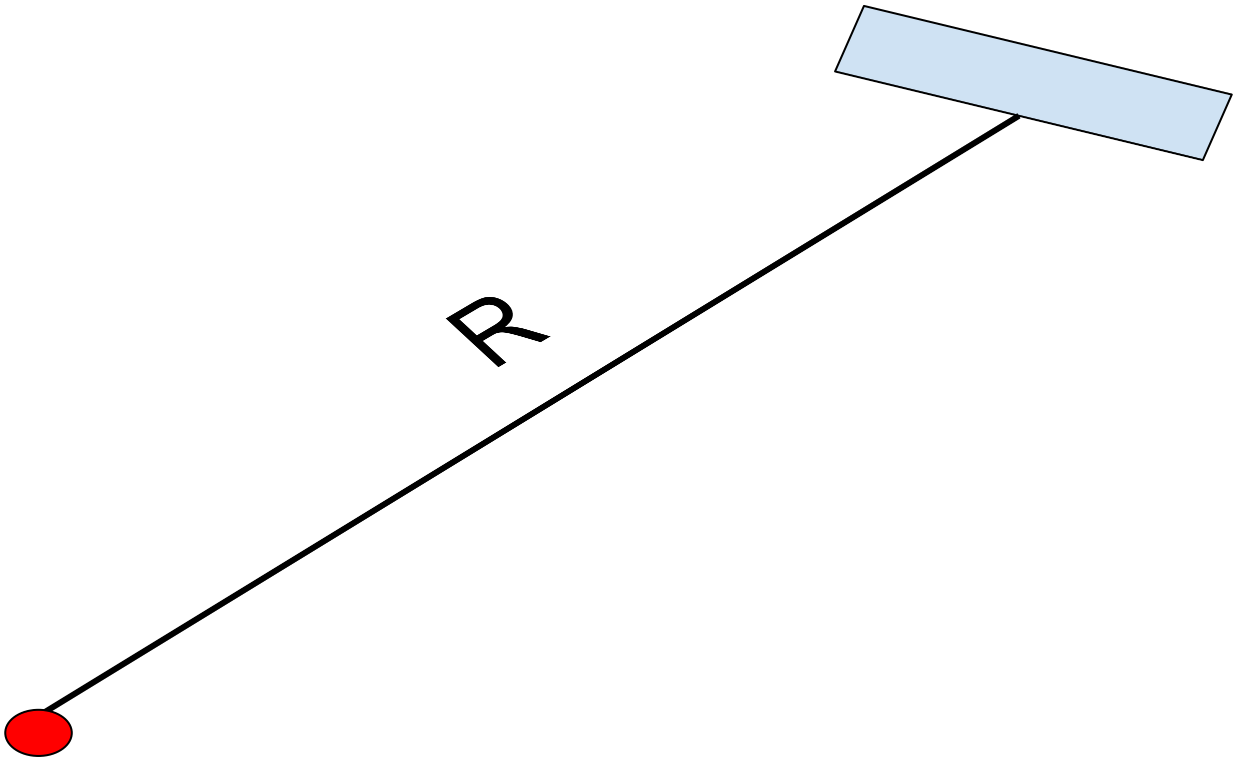
- Study Case 1: Kalman Filter
  - What happens when the observations changes drastically
  - Model
  - Study case setup
  - Results
- Study Case 2: Extended Kalman Filter
  - Model
  - Study case setup
  - Results
- Conclusions

Study Case 1

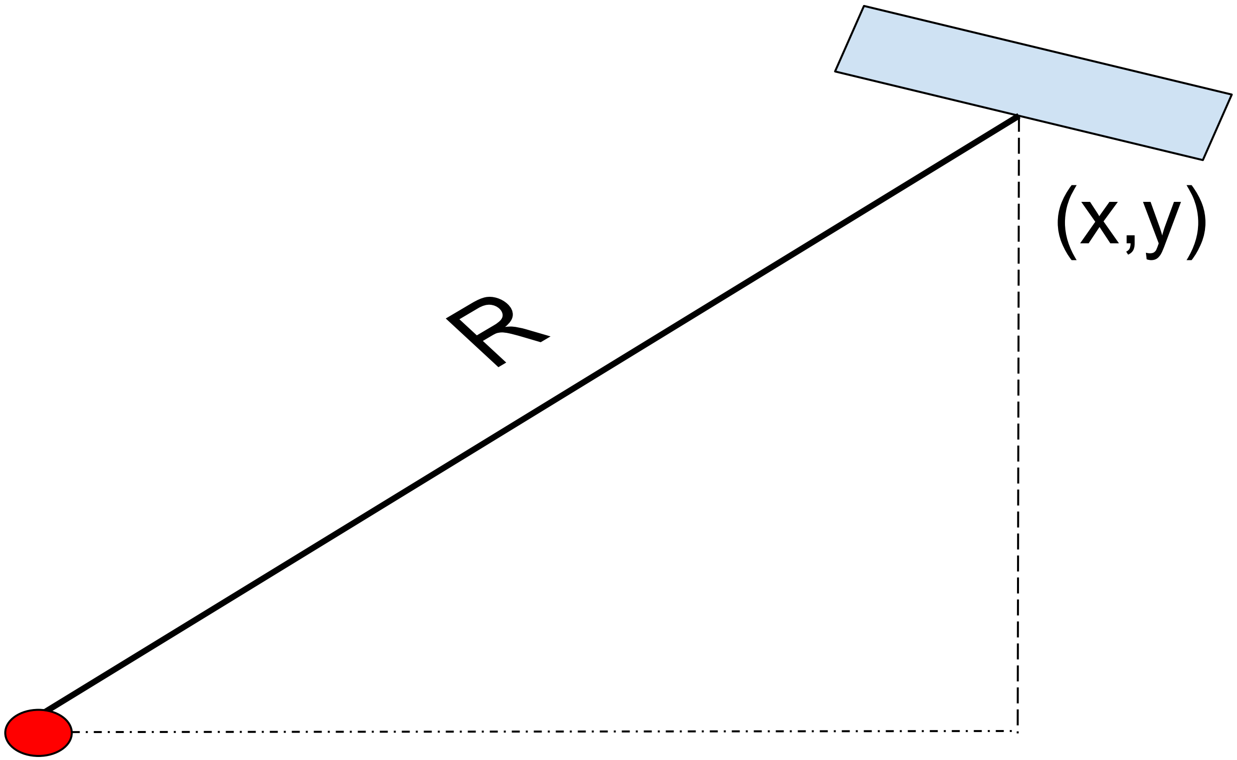
Study Case 1



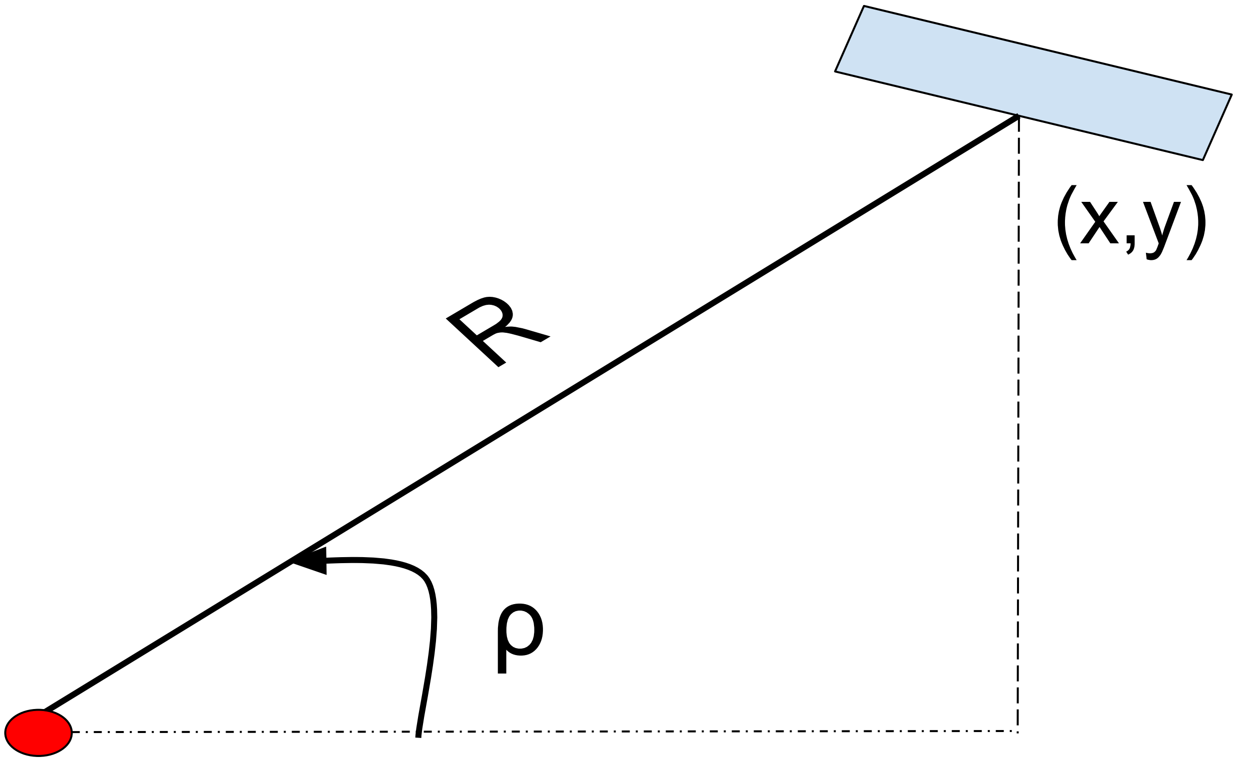
Study Case 1



Study Case 1



Study Case 1



Study Case 1: Model

$$
\begin{aligned}
\rho_m[k] &= \rho[k] + \nu_\rho[k] \\
R_m[k] &= R[k] + \nu_R[k]
\end{aligned}
$$

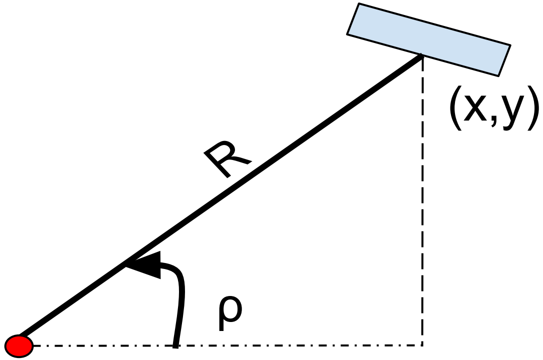
Study Case 1: Model

$$\rho_m[k] = \rho[k] + \nu_\rho[k]$$

$$R_m[k] = R[k] + \nu_R[k]$$

Observation Real Zero Mean White Noise

Study Case 1



Given:

Rm, ρm

Estimate:

R, ρ and noise

$$\rho_m[k] = \rho[k] + \nu_\rho[k]$$

$$R_m[k] = R[k] + \nu_R[k]$$

Study Case 1: Data Generation

$$
\begin{aligned}
x[k+1] \quad &= x[k] + V\sin(\phi[k]\pi/180)\Delta t \\
y[k+1] \quad &= y[k] + V\cos(\phi[k]\pi/180)\Delta t \\
\phi[k+1] \quad &= if(k < N/2) \quad 330 \quad else \quad 200
\end{aligned}
$$

V = 600 m/s , Δt = 1 s, N = 2000

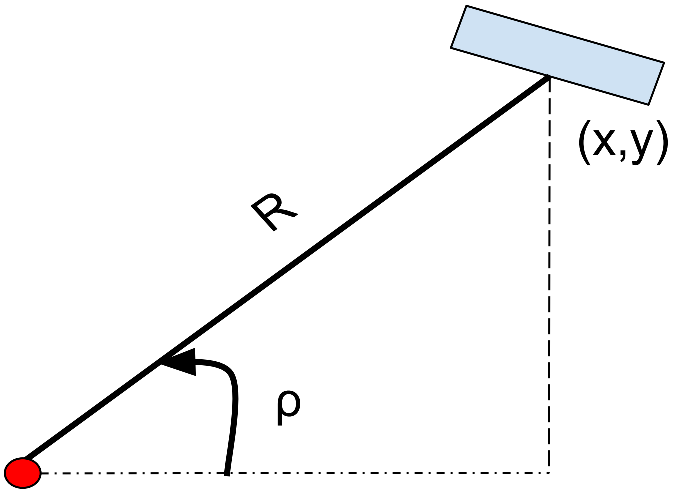
Study Case 1: Data Generation

$$x[0] \quad = R[0] \sin(\rho[0]\pi/180)$$

$$y[0] \quad = R[0] \cos(\rho[0]\pi/180)$$

$\boxed{R[0] = 400000 \quad \rho[0] = 150 \text{ degrees}}$

Study Case 1: Data Generation



$$
\begin{aligned}
\rho_m[k] &= \rho[k] + \nu_\rho[k] \\
R_m[k] &= R[k] + \nu_R[k]
\end{aligned}
$$

Study Case 1: Data Generation

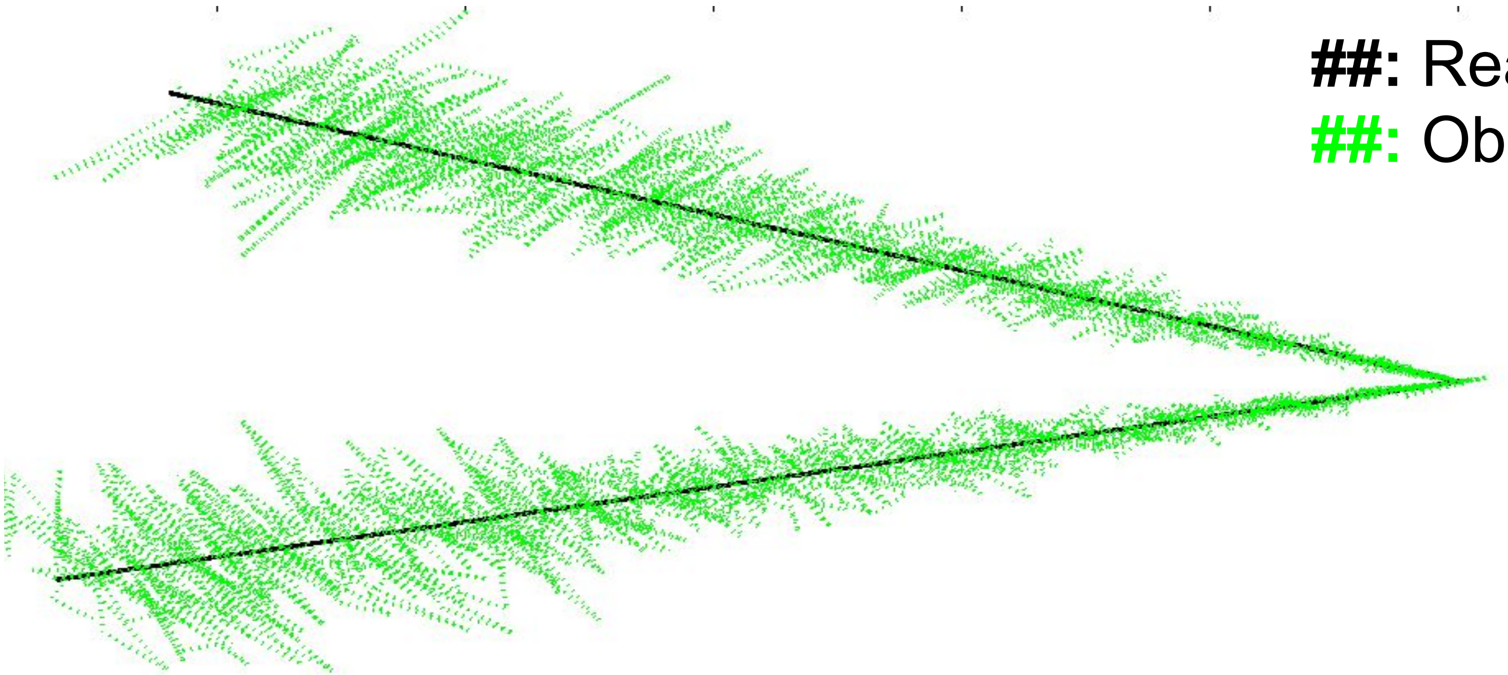
$$R[k] = \sqrt{x^2[k] + y^2[k]}$$

$$\rho[k] = \frac{180}{\pi}\arctan(y[k], x[k])$$

v_R = 1000 m

v_ρ = 5 degree

Study Case 1: Data Generation



##: Real Data
##: Observation

Study Case 1: Kalman filter - Prediction

$$
\begin{aligned}
\hat{\mathbf{x}}_k &= \mathbf{F}_k \hat{\mathbf{x}}_{k-1} \\
\mathbf{P}_k &= \mathbf{F_k} \mathbf{P}_{k-1} \mathbf{F}_k^T + \mathbf{Q}_k
\end{aligned}
$$

Study Case 1: Kalman filter - Prediction

$$
\begin{aligned}
\hat{\mathbf{x}}_k &= \mathbf{F}_k \hat{\mathbf{x}}_{k-1} \\
\mathbf{P}_k &= \mathbf{F_k} \mathbf{P}_{k-1} \mathbf{F}_k^T + \mathbf{Q}_k
\end{aligned}
$$

From model:

$$
\begin{aligned}
\rho_m[k] &= \rho[k] + \nu_\rho[k] \\
R_m[k] &= R[k] + \nu_R[k]
\end{aligned}
$$

Study Case 1: Kalman filter - Prediction

$$\hat{\mathbf{x}}_k = \mathbf{F}_k \hat{\mathbf{x}}_{k-1}$$

$$\mathbf{P}_k = \mathbf{F_k} \mathbf{P}_{k-1} \mathbf{F}_k^T + \mathbf{Q}_k$$

$$\begin{bmatrix} R[k] \\ \rho[k] \\ \nu_R[k] \\ \nu_\rho[k] \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} R[k-1] \\ \rho[k-1] \\ \nu_R[k-1] \\ \nu_\rho[k-1] \end{bmatrix}$$

Study Case 1: Kalman filter - Prediction

$$\hat{\mathbf{x}}_k = \mathbf{F}_k \hat{\mathbf{x}}_{k-1}$$

$$\mathbf{P}_k = \mathbf{F_k} \mathbf{P}_{k-1} \mathbf{F}_k^T + \mathbf{Q}_k$$

$$\mathbf{Q}_k = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \sigma_u^2 & 0 \\ 0 & 0 & 0 & \sigma_u^2 \end{bmatrix}$$

$$\sigma_u^2 = 0.1$$

$$\mathbf{P}[0] = \mathbf{I}$$

Study Case 1: Kalman filter - Update

$$\mathbf{K} = \mathbf{P}_k \mathbf{H}_k^T (\mathbf{H}_k \mathbf{P}_k \mathbf{H}_k^T + \mathbf{R}_k)^{-1}$$

$$\hat{\mathbf{x}}_{k+1} = \hat{\mathbf{x}}_k + \mathbf{K}(\mathbf{z}_k - \mathbf{H}_k \hat{\mathbf{x}}_k)$$

$$\mathbf{P}_{k+1} = \mathbf{P}_k - \mathbf{K}\mathbf{H}_k \mathbf{P}_k$$

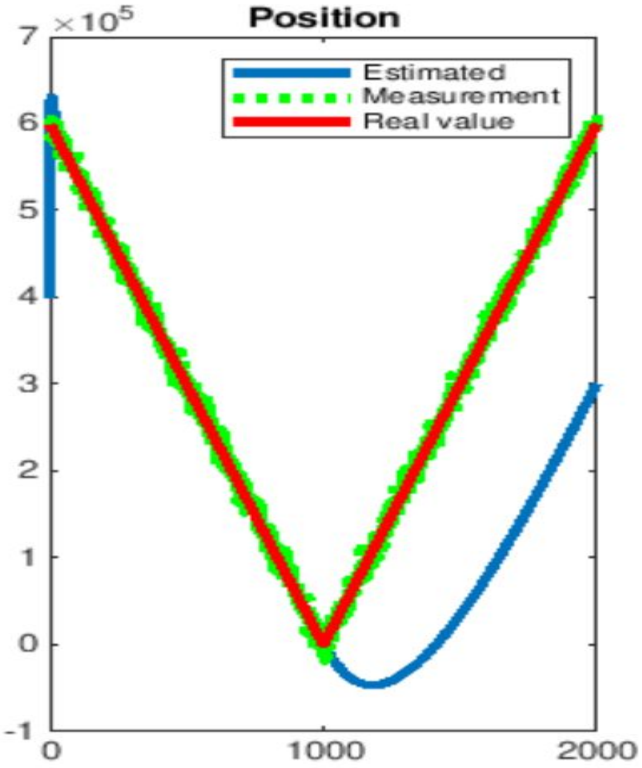
Study Case 1: Kalman filter - Update

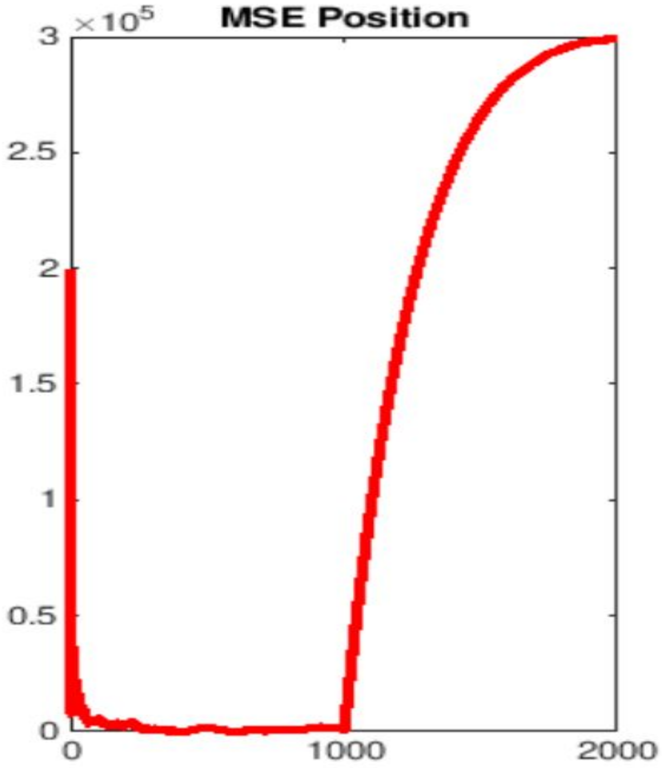
$$
\begin{aligned}
\hat{\mathbf{x}}_{k+1} &= \hat{\mathbf{x}}_k + \mathbf{K}(\mathbf{z}_k - \mathbf{H}_k\hat{\mathbf{x}}_k) \\
\mathbf{P}_{k+1} &= \mathbf{P}_k - \mathbf{K}\mathbf{H}_k\mathbf{P}_k \\
\mathbf{K} &= \mathbf{P}_k\mathbf{H}_k^T(\mathbf{H}_k\mathbf{P}_k\mathbf{H}_k^T + \mathbf{R}_k)^{-1}
\end{aligned}
$$

$$
z_k = \begin{bmatrix} R[k] \\ \rho[k] \end{bmatrix} \qquad \mathbf{H}_k = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}
$$

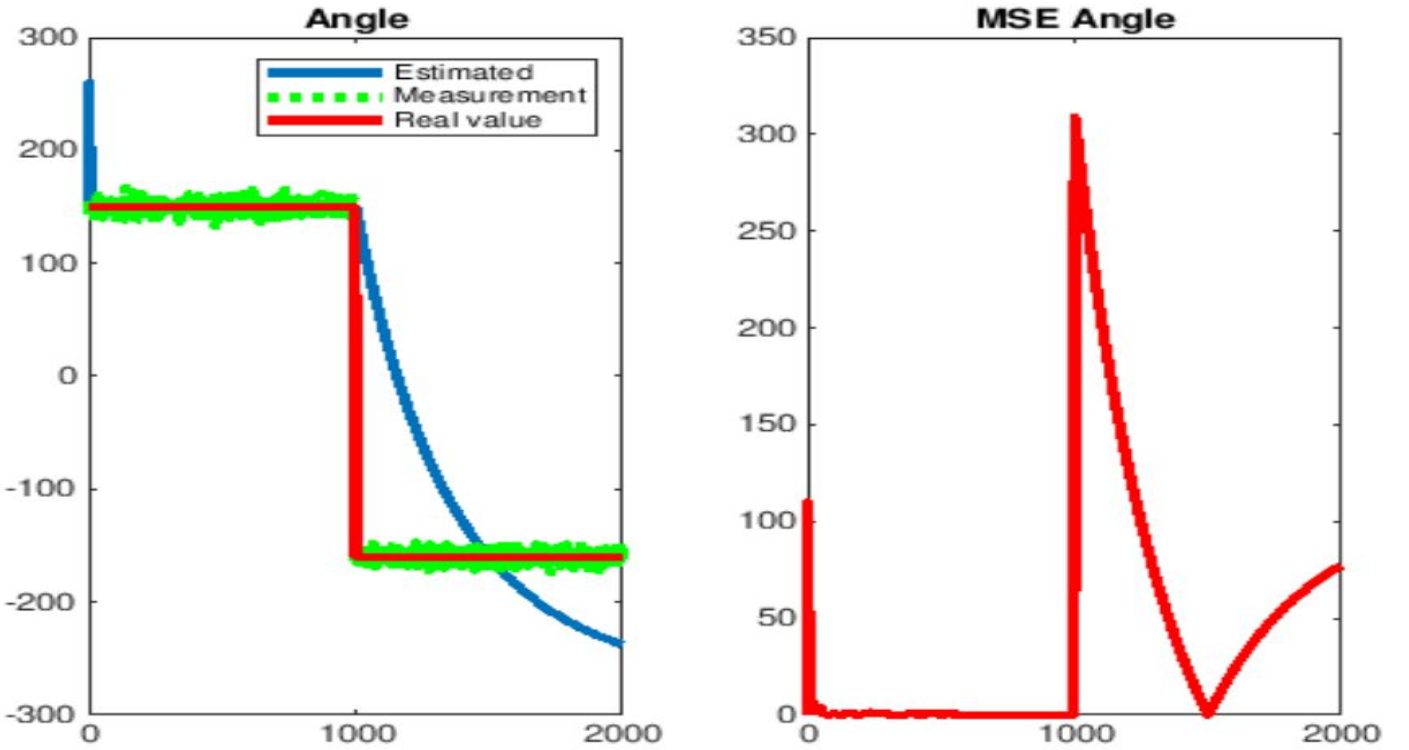
$$
\mathbf{R}_k = \begin{bmatrix} \nu_R^2 & 0 \\ 0 & \nu_\rho^2 \end{bmatrix} = \begin{bmatrix} 1e8 & 0 \\ 0 & 25 \end{bmatrix}
$$

Study Case 1: Kalman filter - Results: Range

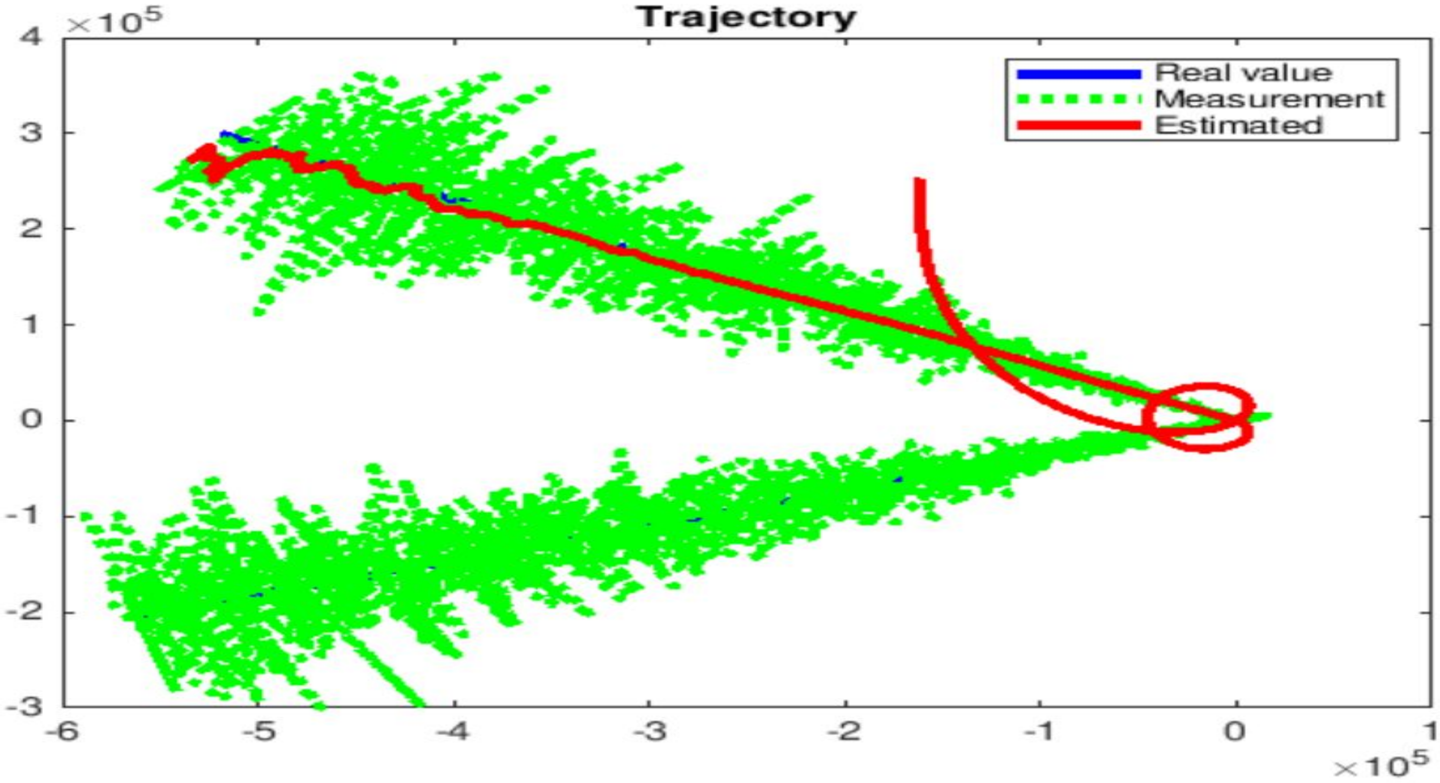




Study Case 1: Kalman filter - Results: Angle



Study Case 1: Kalman filter - Results



Study Case 1: Kalman filter

What would it happen, if somehow we reset the kalman filter when the deviation is too big…

Study Case 1: Kalman filter

What would it happen, if somehow we reset the kalman filter when the deviation is too big…

If

    dist(Measurements, Estimation) > ε

Then

    Reset Filter

Study Case 1: Kalman filter

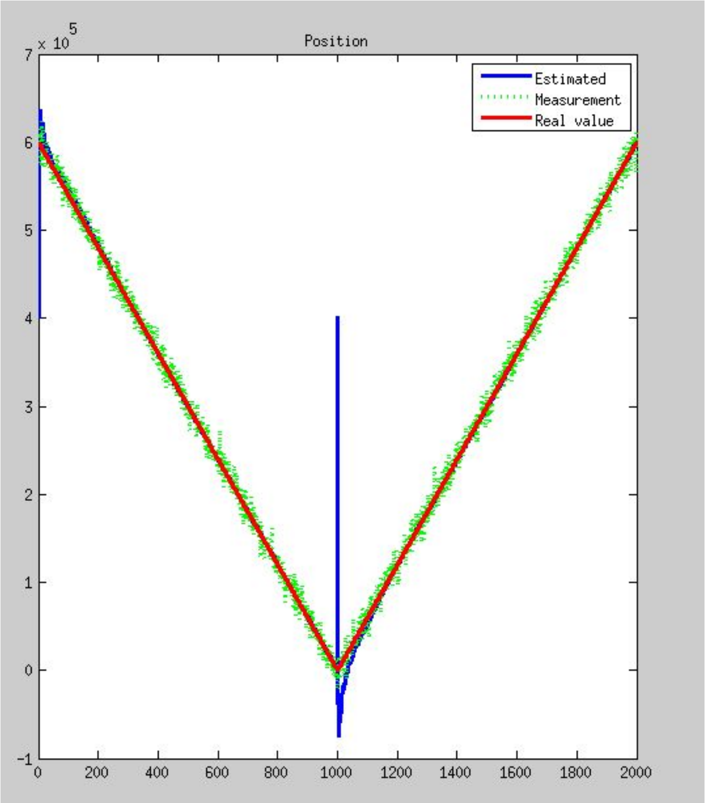
If

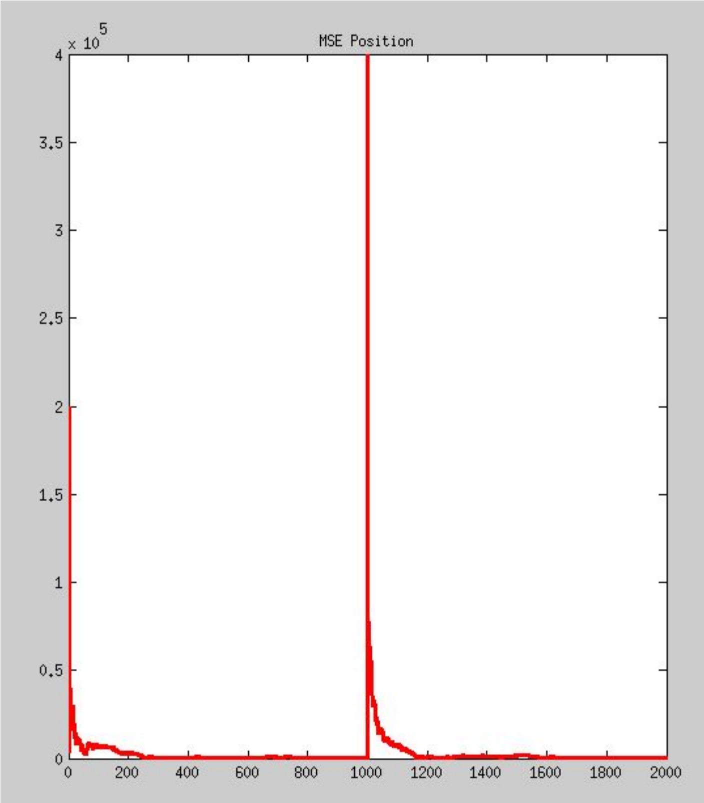
$\quad \text{dist}(\rho_m, \rho_{est}) > 4\sigma_\rho^2$

Then

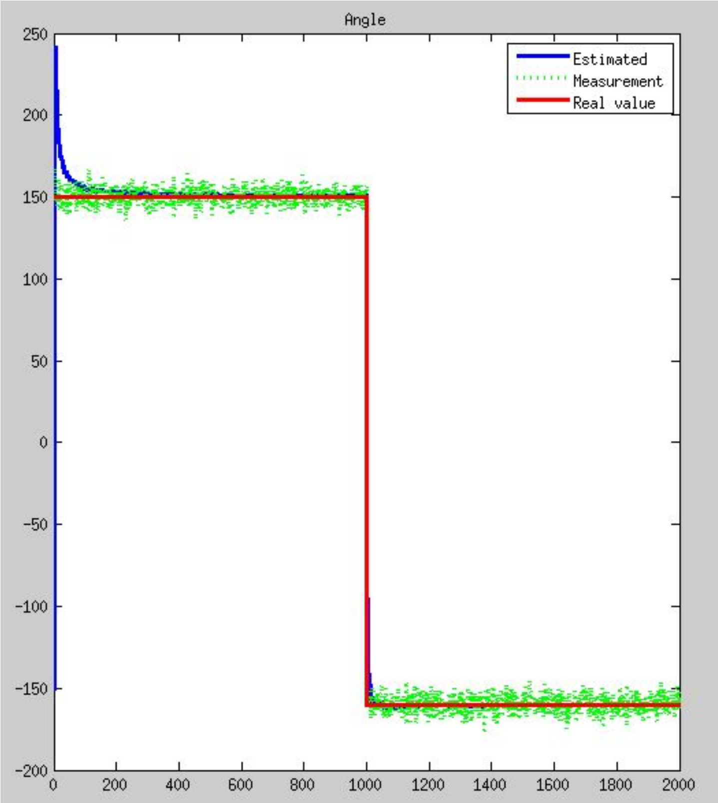
$\quad$ Reset Filter

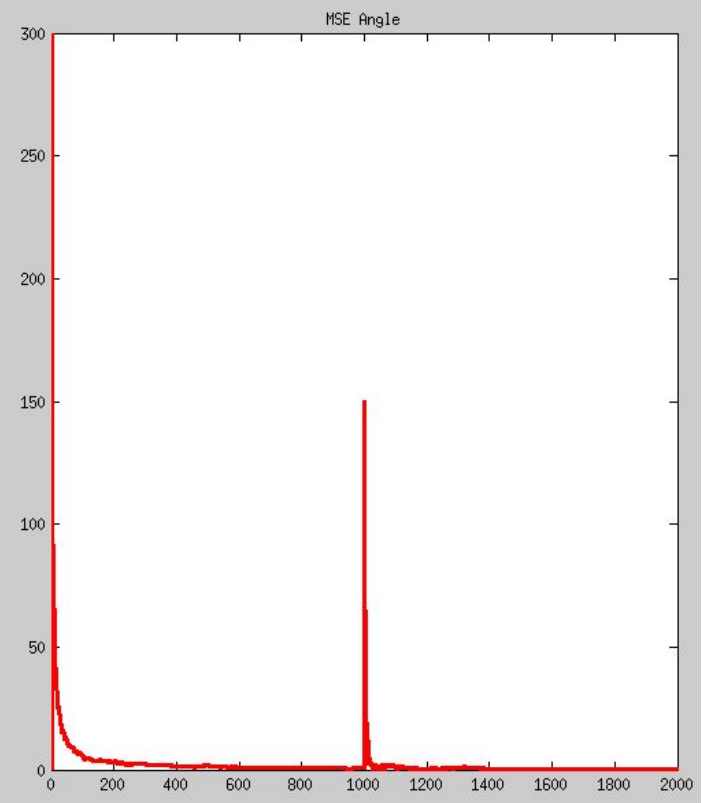
Study Case 1: Kalman filter



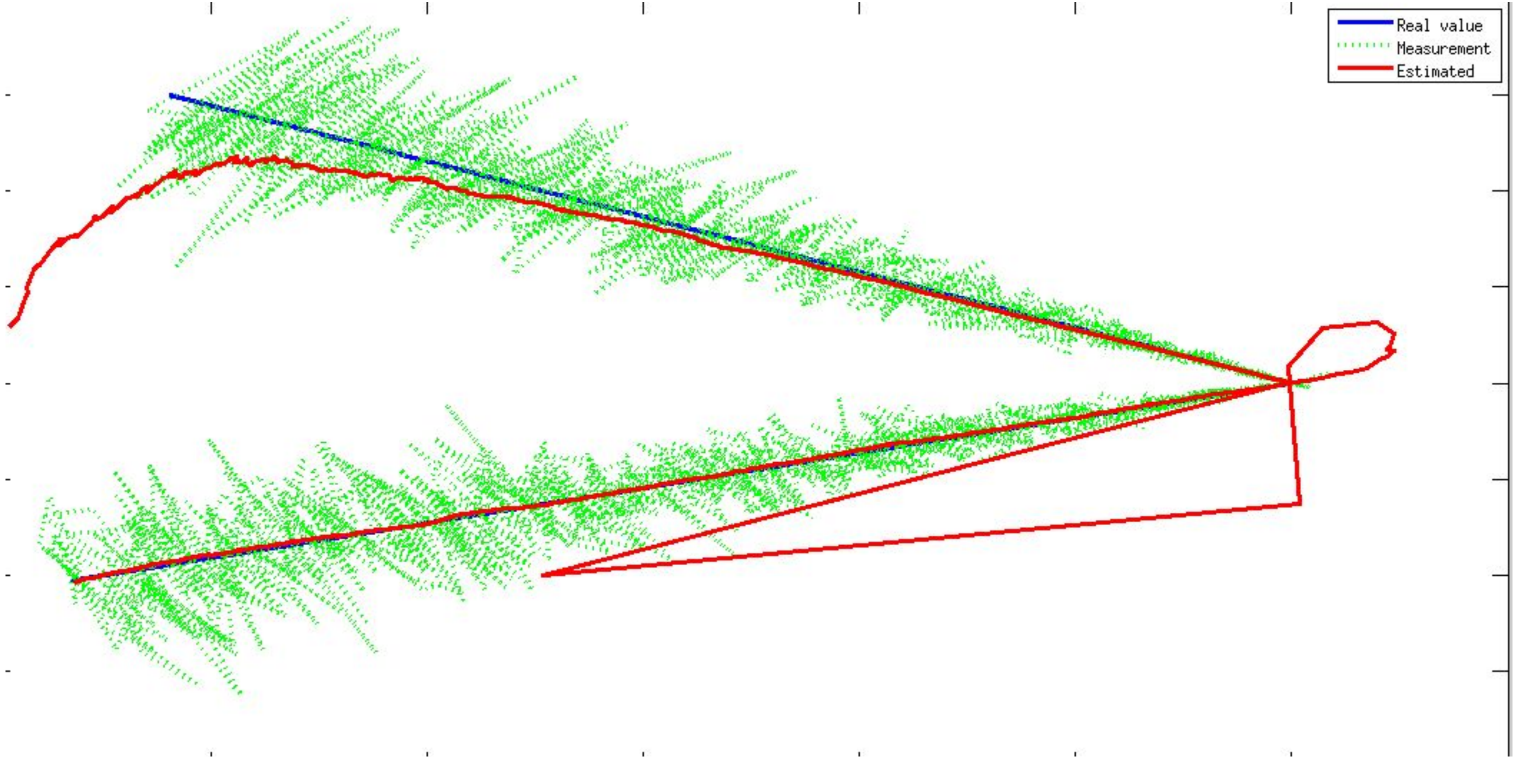


Study Case 1: Kalman filter





Study Case 1: Kalman filter



Study Case 2: Extended Kalman Filter

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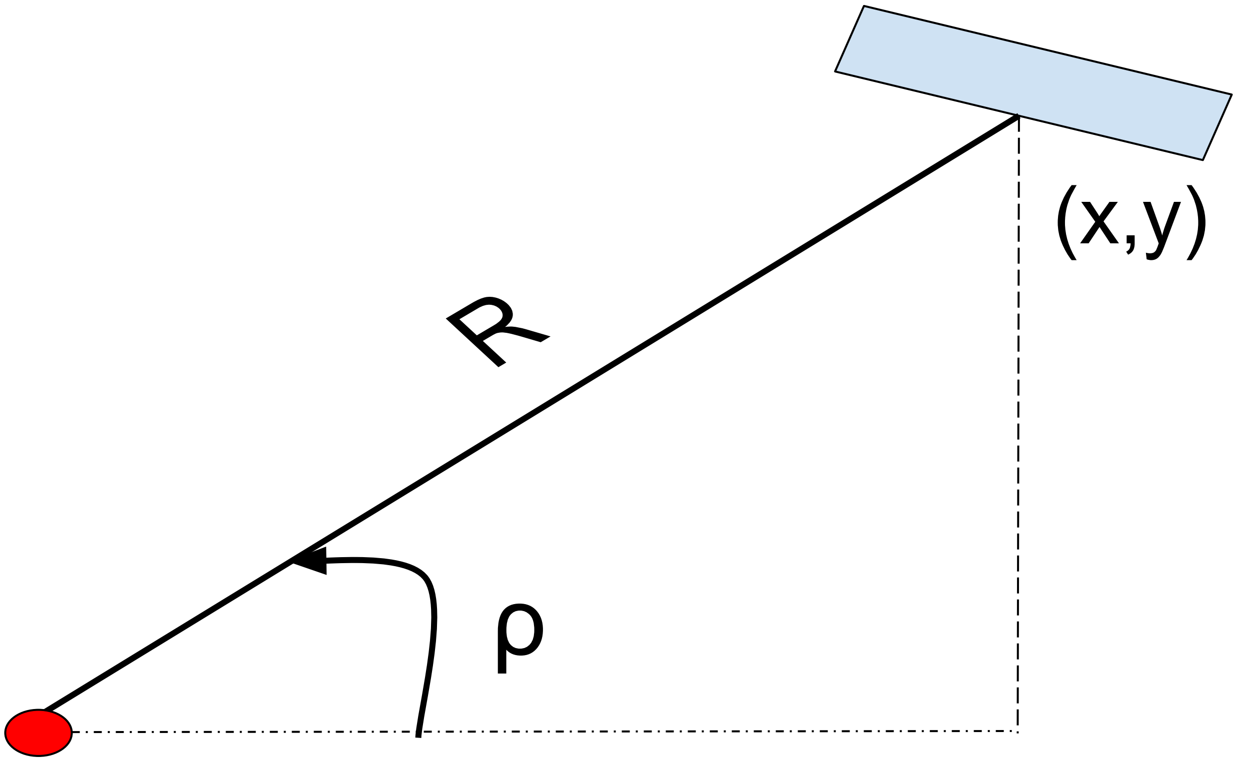
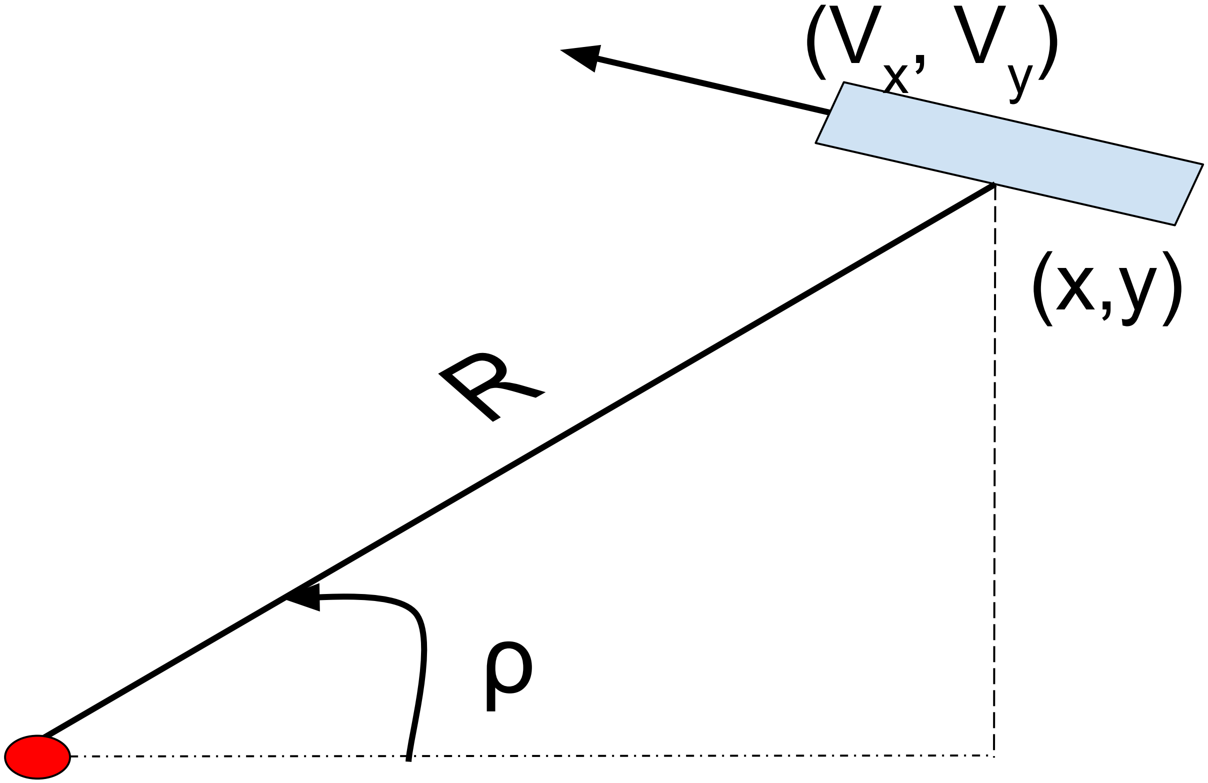
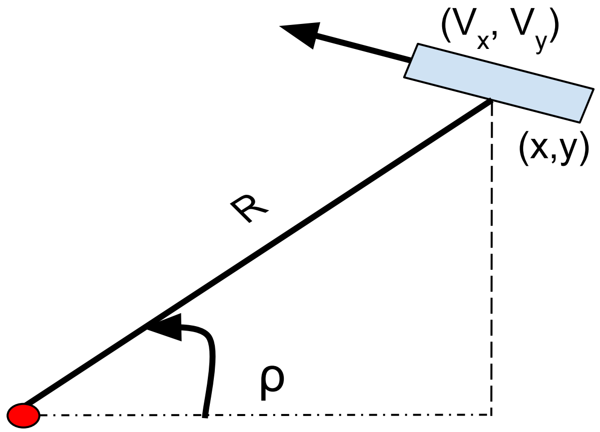


Study Case 2: Extended Kalman Filter



Study Case 2: Extended Kalman Filter



Given:

R and ρ

Estimate:

Speed and Position

Study Case 2: Data Generation: Model

$$x[k+1] \quad = x[k] + V_x \Delta t$$

$$y[k+1] \quad = y[k] + V_y \Delta t$$

$x[0] = 10$ m $y[0] = 5$ m

$V_x = -0.2$ m/s $V_y = 0.2$ m/s

Study Case 2: Data Generation

$$
\begin{aligned}
R[k] &= \sqrt{x^2[k] + y^2[k]} \\
\rho[k] &= \arctan(y[k], x[k])
\end{aligned}
$$

Study Case 2: Data Generation

$$
\begin{aligned}
R[k] &= \sqrt{x^2[k] + y^2[k]} \\
\rho[k] &= \mathrm{arctan}(y[k], x[k])
\end{aligned}
$$

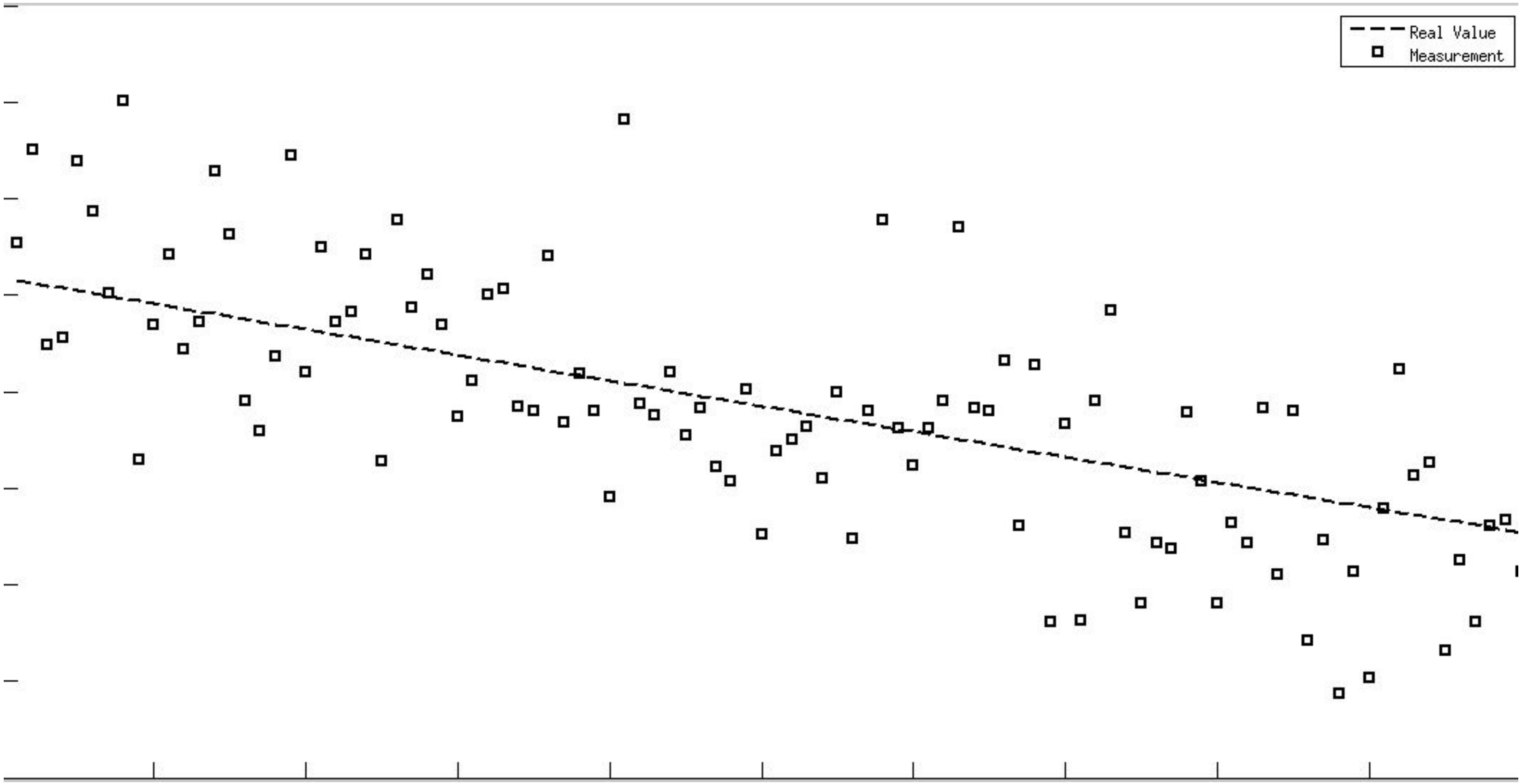
$$
\begin{aligned}
\rho_m[k] &= \rho[k] + \nu_\rho[k] \\
R_m[k] &= R[k] + \nu_R[k]
\end{aligned}
$$

Study Case 2: Data Generation: Observations

$$\begin{aligned} \rho_m[k] &= \rho[k] + \nu_\rho[k] \\ R_m[k] &= R[k] + \nu_R[k] \end{aligned}$$

v_R: Noise with $\sigma_R^2 =$ 1 m
v_ρ : Noise with $\sigma_\rho^2 =$ 0.01 rad

Study Case 2: Data Generation: Range



Study Case 2: Kalman filter - Prediction

$$\hat{\mathbf{x}}_k = \mathbf{F}_k \hat{\mathbf{x}}_{k-1}$$

$$\mathbf{P}_k = \mathbf{F_k} \mathbf{P}_{k-1} \mathbf{F}_k^T + \mathbf{Q}_k$$

Study Case 2: Kalman filter - Prediction

$$\hat{\mathbf{x}}_k = \mathbf{F}_k \hat{\mathbf{x}}_{k-1}$$

$$\mathbf{P}_k = \mathbf{F_k} \mathbf{P}_{k-1} \mathbf{F}_k^T + \mathbf{Q}_k$$

From model:

$$x[k+1] = x[k] + V_x \Delta t$$

$$y[k+1] = y[k] + V_y \Delta t$$

Study Case 2: Kalman filter - Prediction

$$\hat{\mathbf{x}}_k = \mathbf{F}_k \hat{\mathbf{x}}_{k-1}$$

$$\mathbf{P}_k = \mathbf{F_k} \mathbf{P}_{k-1} \mathbf{F}_k^T + \mathbf{Q}_k$$

$$\begin{bmatrix} x[k] \\ y[k] \\ V_x[k] \\ V_y[k] \end{bmatrix} = \begin{bmatrix} 1 & 0 & \Delta t & 0 \\ 0 & 1 & 0 & \Delta t \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x[k-1] \\ y[k-1] \\ V_x[k-1] \\ V_y[k-1] \end{bmatrix}$$

Study Case 2: Kalman filter - Prediction

$$\hat{\mathbf{x}}_k = \mathbf{F}_k \hat{\mathbf{x}}_{k-1}$$

$$\mathbf{P}_k = \mathbf{F_k} \mathbf{P}_{k-1} \mathbf{F}_k^T + \mathbf{Q}_k$$

$$\mathbf{Q}_k = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \sigma_u^2 & 0 \\ 0 & 0 & 0 & \sigma_u^2 \end{bmatrix}$$

$$\sigma_u^2 = 0.1$$

$$\mathbf{P}[0] = \mathbf{I}$$

Study Case 2: Kalman filter - Update

$$\mathbf{K} = \mathbf{P}_k \mathbf{H}_k^T (\mathbf{H}_k \mathbf{P}_k \mathbf{H}_k^T + \mathbf{R}_k)^{-1}$$

$$\hat{\mathbf{x}}_{k+1} = \hat{\mathbf{x}}_k + \mathbf{K}(\mathbf{z}_k - \mathbf{H}_k \hat{\mathbf{x}}_k)$$

$$\mathbf{P}_{k+1} = \mathbf{P}_k - \mathbf{K}\mathbf{H}_k \mathbf{P}_k$$

Study Case 2: Kalman filter - Update

$$\mathbf{K} = \mathbf{P}_k \mathbf{H}_k^T (\mathbf{H}_k \mathbf{P}_k \mathbf{H}_k^T + \mathbf{R}_k)^{-1}$$

$$\hat{\mathbf{x}}_{k+1} = \hat{\mathbf{x}}_k + \mathbf{K}(\mathbf{z}_k - \mathbf{h}_k(\hat{x}_k))$$

$$\mathbf{P}_{k+1} = \mathbf{P}_k - \mathbf{K}\mathbf{H}_k\mathbf{P}_k$$

Non-linear factor

Study Case 2: Kalman filter - Update

$$\begin{aligned}\mathbf{K} &= \mathbf{P}_k \mathbf{H}_k^T (\mathbf{H}_k \mathbf{P}_k \mathbf{H}_k^T + \mathbf{R}_k)^{-1} \\ \hat{\mathbf{x}}_{k+1} &= \hat{\mathbf{x}}_k + \mathbf{K}(\mathbf{z}_k - \mathbf{h}_k(\hat{x}_k)) \\ \mathbf{P}_{k+1} &= \mathbf{P}_k - \mathbf{K}\mathbf{H}_k \mathbf{P}_k\end{aligned}$$

$$z_k = \begin{bmatrix} R[k] \\ \rho[k] \end{bmatrix} \qquad \mathbf{R}_k = \begin{bmatrix} \sigma_R^2 & 0 \\ 0 & \sigma_\rho^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0.01 \end{bmatrix}$$

Study Case 2: Kalman filter - Update

$$
\begin{aligned}
\mathbf{K} &= \mathbf{P}_k \mathbf{H}_k^T (\mathbf{H}_k \mathbf{P}_k \mathbf{H}_k^T + \mathbf{R}_k)^{-1} \\
\hat{\mathbf{x}}_{k+1} &= \hat{\mathbf{x}}_k + \mathbf{K}(\mathbf{z}_k - \mathbf{h}_k(\hat{x}_k)) \\
\mathbf{P}_{k+1} &= \mathbf{P}_k - \mathbf{K}\mathbf{H}_k \mathbf{P}_k
\end{aligned}
$$

We know that:

$$
\begin{aligned}
R[k] &= \sqrt{x^2[k] + y^2[k]} \\
\rho[k] &= \arctan(y[k], x[k])
\end{aligned}
$$

Study Case 2: Kalman filter - Update

$$
\begin{aligned}
\mathbf{K} &= \mathbf{P}_k \mathbf{H}_k^T (\mathbf{H}_k \mathbf{P}_k \mathbf{H}_k^T + \mathbf{R}_k)^{-1} \\
\hat{\mathbf{x}}_{k+1} &= \hat{\mathbf{x}}_k + \mathbf{K}(\mathbf{z}_k - \mathbf{h}_k(\hat{x}_k)) \\
\mathbf{P}_{k+1} &= \mathbf{P}_k - \mathbf{K}\mathbf{H}_k \mathbf{P}_k
\end{aligned}
$$

Thus:

$$
\mathbf{h}_k(\hat{x}_k) = \begin{bmatrix} \sqrt{(}\hat{x}_k[2]^2 + \hat{x}_k[1]^2) \\ \arctan(\hat{x}_k[2], \hat{x}_k[1]) \end{bmatrix}
$$

Study Case 2: Kalman filter - Update

$$
\begin{aligned}
\mathbf{K} &= \mathbf{P}_k \mathbf{H}_k^T (\mathbf{H}_k \mathbf{P}_k \mathbf{H}_k^T + \mathbf{R}_k)^{-1} \\
\hat{\mathbf{x}}_{k+1} &= \hat{\mathbf{x}}_k + \mathbf{K}(\mathbf{z}_k - \mathbf{h}_k(\hat{x}_k)) \\
\mathbf{P}_{k+1} &= \mathbf{P}_k - \mathbf{K}\mathbf{H}_k\mathbf{P}_k
\end{aligned}
$$

Thus:

$$
\mathbf{H}_k = \begin{bmatrix} \frac{\hat{x}_k[1]}{R_k} & \frac{\hat{x}_k[2]}{R_k} & 0 & 0 \\ \frac{-\hat{x}_k[2]}{R_k} & \frac{\hat{x}_k[1]}{R_k} & 0 & 0 \end{bmatrix}
$$

$$
\mathbf{R}_k = \sqrt{\hat{x}_k[2]^2 + \hat{x}_k[1]^2)}
$$

Study Case 2: Kalman filter - Update

$$\begin{aligned}\mathbf{K} &= \mathbf{P}_k \mathbf{H}_k^T (\mathbf{H}_k \mathbf{P}_k \mathbf{H}_k^T + \mathbf{R}_k)^{-1} \\ \hat{\mathbf{x}}_{k+1} &= \hat{\mathbf{x}}_k + \mathbf{K}(\mathbf{z}_k - \mathbf{h}_k(\hat{x}_k)) \\ \mathbf{P}_{k+1} &= \mathbf{P}_k - \mathbf{K}\mathbf{H}_k\mathbf{P}_k\end{aligned}$$

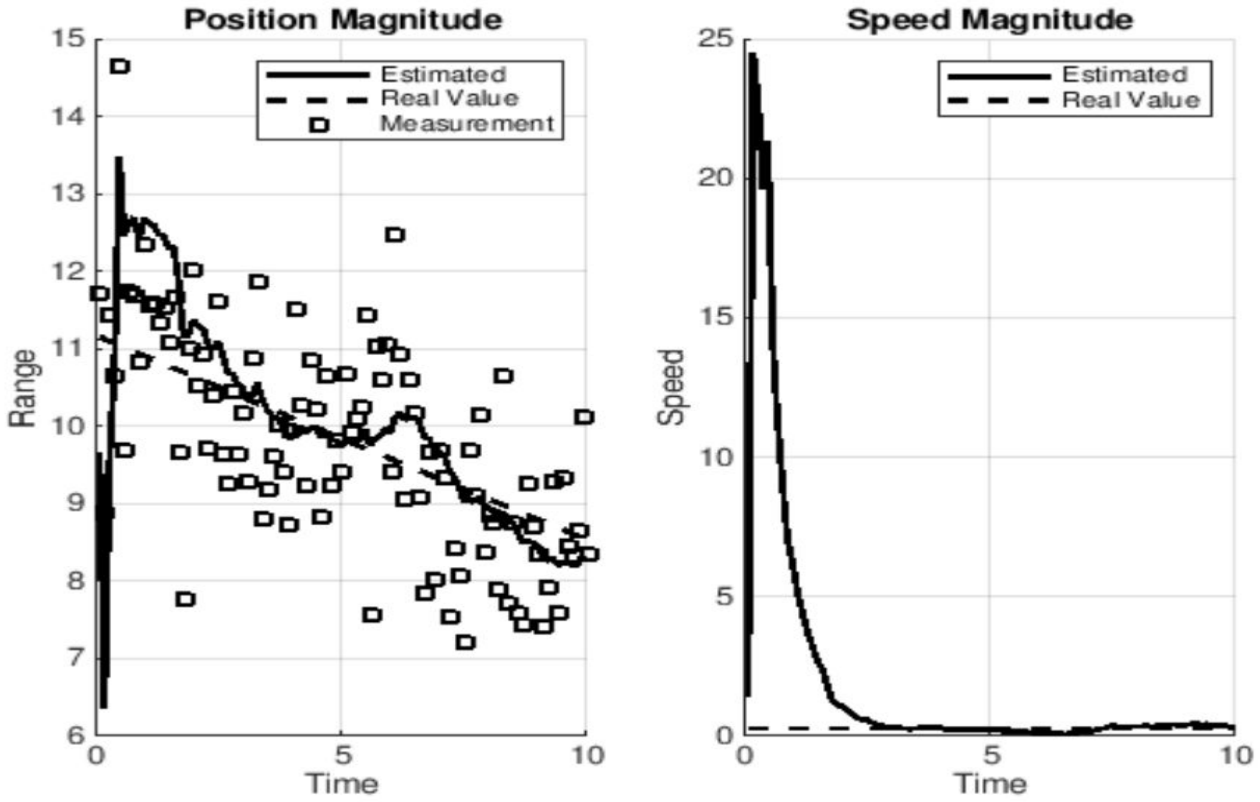
Thus:

$$\mathbf{H}_k = \begin{bmatrix} \frac{\hat{x}_k[1]}{R_k} & \frac{\hat{x}_k[2]}{R_k} & 0 & 0 \\ \frac{-\hat{x}_k[2]}{R_k} & \frac{\hat{x}_k[1]}{R_k} & 0 & 0 \end{bmatrix}$$

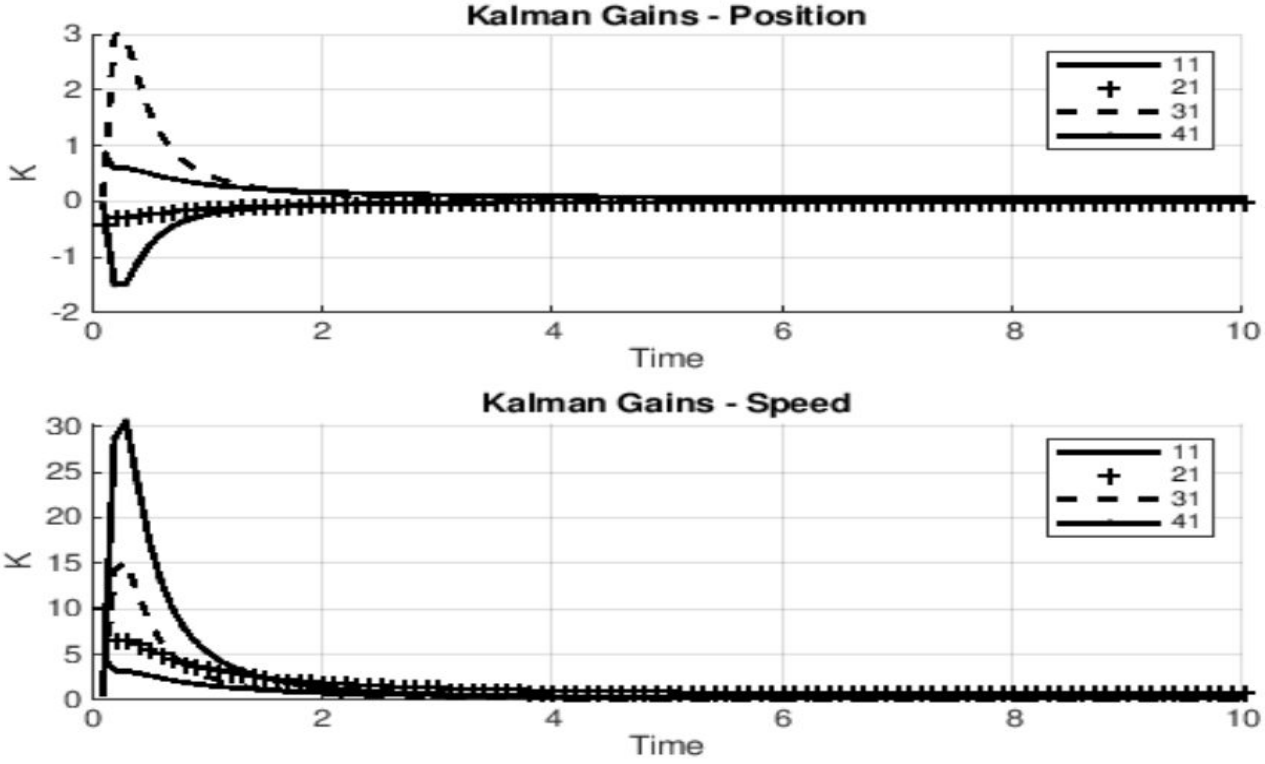
Jacobian

$$\mathbf{R}_k = \sqrt{\hat{x}_k[2]^2 + \hat{x}_k[1]^2)}$$

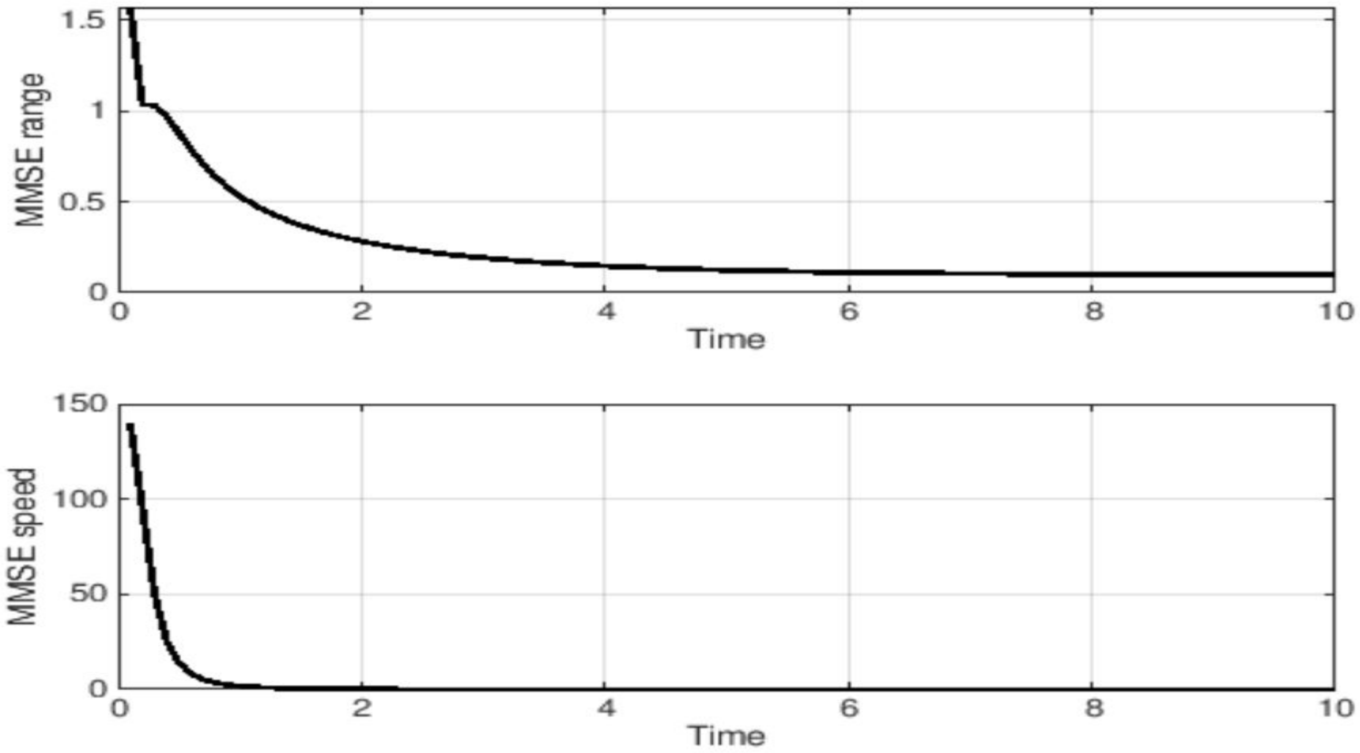
Study Case 2: Kalman filter - Results



Study Case 2: Kalman filter - Results



Study Case 2: Kalman filter - Results



Conclusions

Conclusions

- Easy to implement, but we must calculate a inverse of a matrix in each iteration
- Requires a good model regardless if it linear or not
- If the models “fails” there should be a self-healing or self-reset mechanism
- Sensible to initial guess
- Need to have a good noise estimation

Questions ?