The Adaptive Method of Lines

CDEs Group Presentation

Shian Su, Ria Szeredi and Kenneth Young

University of Melbourne

23rd of May, 2016

Motivation

Method of lines

- Discretise in space (eg. using finite difference) to generate a system of ODEs.
- Solve the system using some solver such as ode15s.
- Temporal accuracy handled by the ODE solver, however spatial accuracy results from the discretisation used.

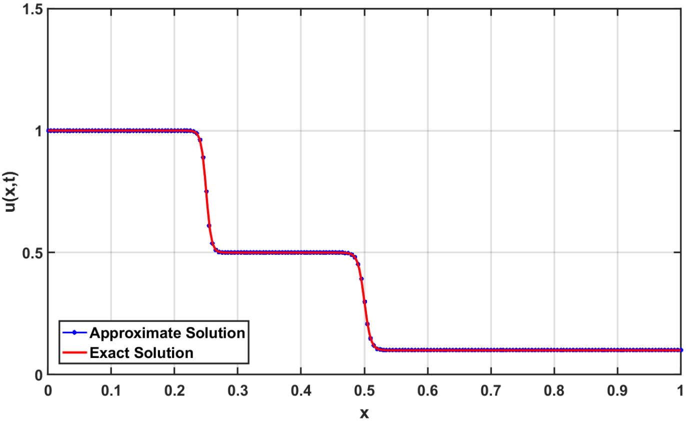
Motivation

Problems with Uniform Mesh

- Method of lines discretises the space uniformly.
- What if our problem has *steep fronts*?
  - Low density uniform mesh doesn't approximate steep fronts well.
  - High density uniform mesh achieves high spatial accuracy, but wastes nodes in regions of low activity.

Method of Lines - Burgers' Equation

- Bad approximation in regions of high spatial activity when grid is uniform. Here we use 201 nodes.



Mesh Adaptation

- Want to adapt the spatial mesh such that nodes are efficiently placed.
  - Static moving mesh.
  - Static grid refinement.
  - Dynamic mesh.

Equidistribution Principle

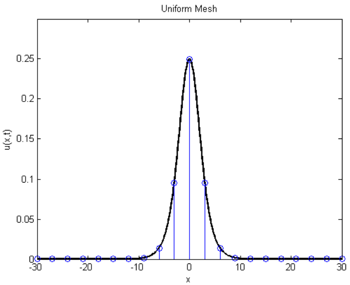
Tracking the action

- Want to assign more nodes to areas with high activity.
- Use some monitor function $m(x)$ to measure the activity in a region.
- Choose the x_i such that for all i,
  $\int_{x_{i-1}}^{x^i} m(x)\,\mathrm{d}x = \int_{x_i}^{x^{i+1}} m(x)\,\mathrm{d}x$ (equidistribution).
- In practice equidistribution is considered optimal, but some suboptimal methods may be used to have roughly equal distribution in order to reduce computational effort.

Choice of Monitor Function

- Arc length $m(x) = \sqrt{\alpha + u_x^2(x)}$.
- Local curvature $m(x) = |u_{xx}(x)|$.
- Other options involving many tuning parameters exist.
- Since the true solution is not known, derivatives must be estimated. Can use natural splines to interpolate current solution and use its derivatives as approximation.

Uniform vs Equidistributed Arclengths (n=21)



Example Problems

- Burgers' Equation
  $u_t + uu_x + u_{xx} = 0$

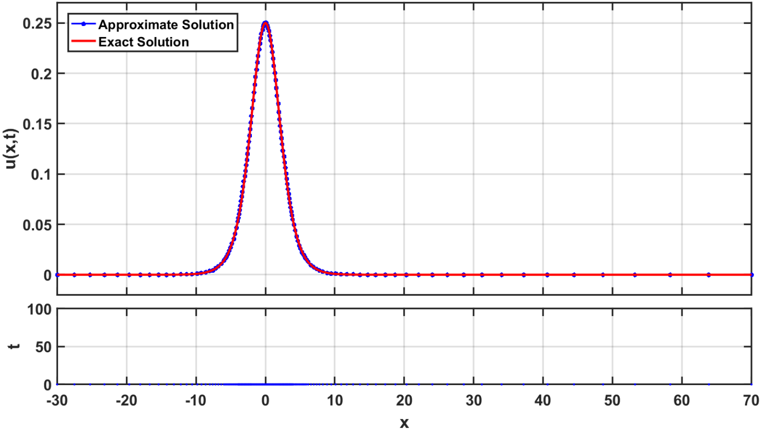
- Korteweg-de Vries (KDV) Equation
  $u_t + u_{xxx} + 6uu_x = 0$

Moving Mesh

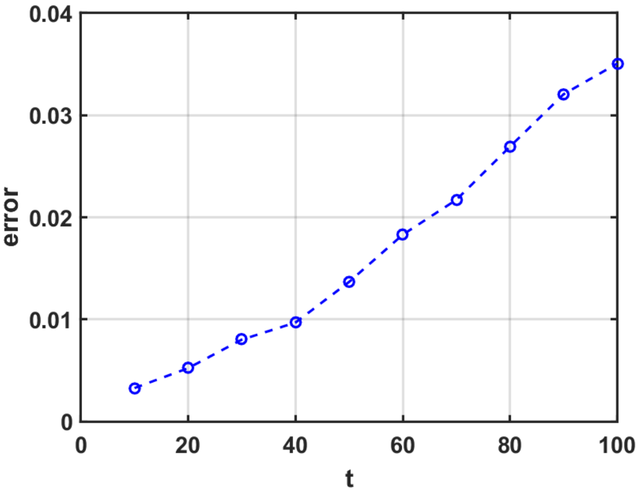
- Fix the number of nodes.
- Pause the PDE solver at a set interval to move the mesh nodes to achieve equidistribution.
- Interpolate the solution from old mesh to generate initial conditions for new mesh.

Moving Mesh - KDV Equation Example 1

Mesh Adaptions	# of Mesh Nodes	Computation Time
10	151	8.9 sec

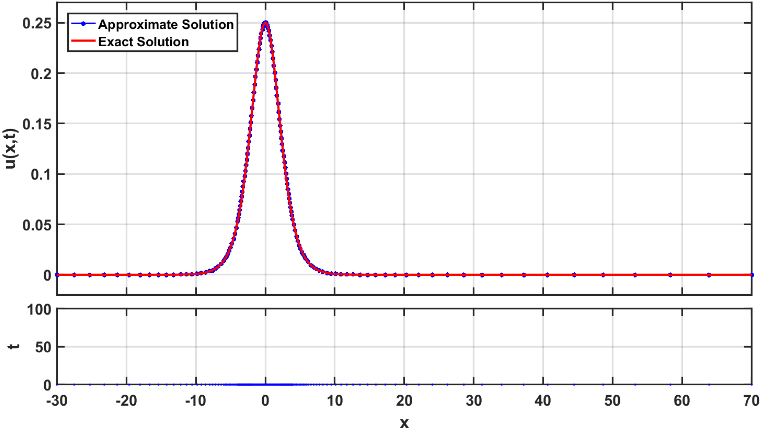


Moving Mesh - KDV Equation Example 1 Error

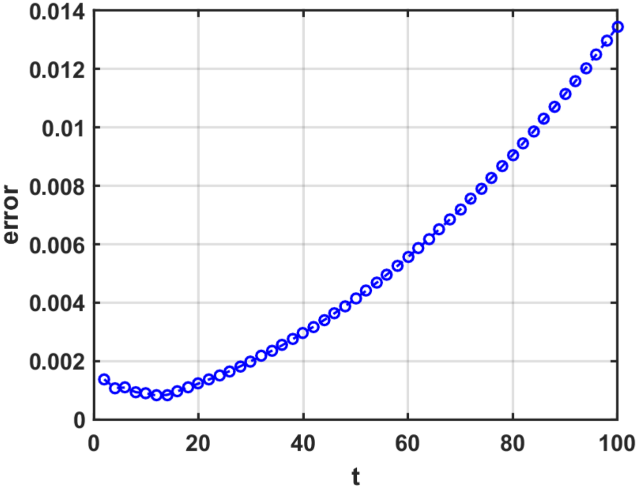


Moving Mesh - KDV Equation Example 2

Mesh Adaptions	# of Mesh Nodes	Computation Time
50	151	19.6 sec

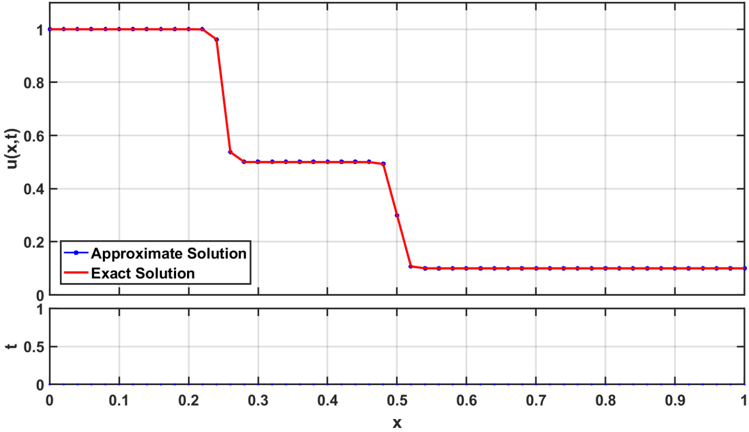


Moving Mesh - KDV Equation Example 2 Error

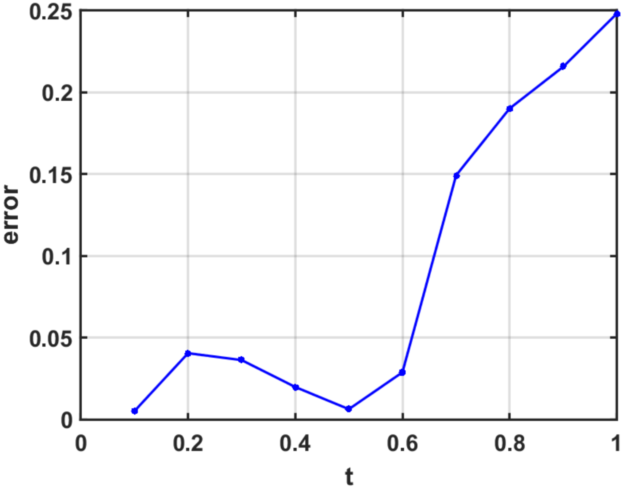


Moving Mesh - Burgers' Equation Example 1

Mesh Adaptions	# of Mesh Nodes	Computation Time
10	375	5.8 sec

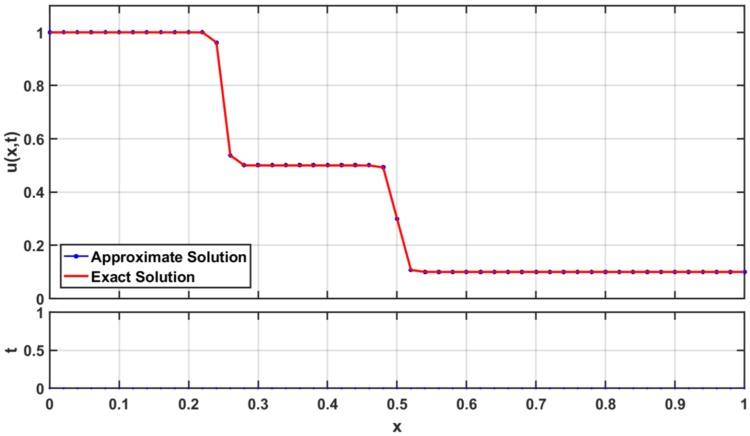


Moving Mesh - Burgers' Equation Example 1 Error

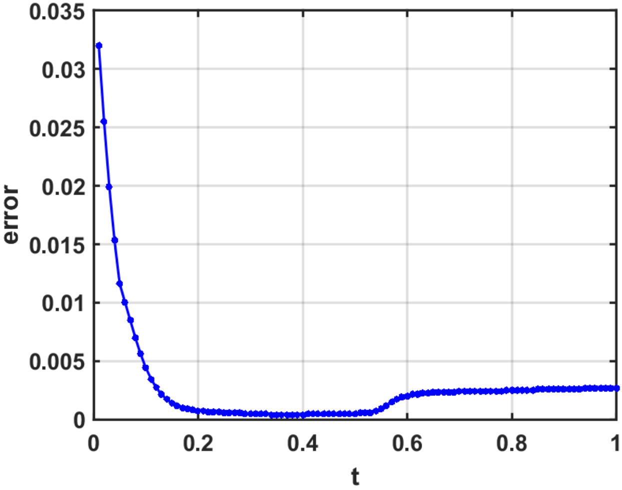


Moving Mesh - Burgers' Equation Example 2

Mesh Adaptions	# of Mesh Nodes	Computation Time
100	251	36.1 sec



Moving Mesh - Burgers' Equation Example 2 Error



Mesh Refinement

- Just as in moving mesh, the mesh is updated at set intervals.
- Procedure for each mesh adaption:
  1. Reset to uniform base grid.
  2. Evaluate monitor function between each pair of nodes.
  3. If threshold exceeded, add nodes with uniform spacing
  4. If below threshold, remove some nodes.
  5. Interpolate and repeat.

Mesh Refinement

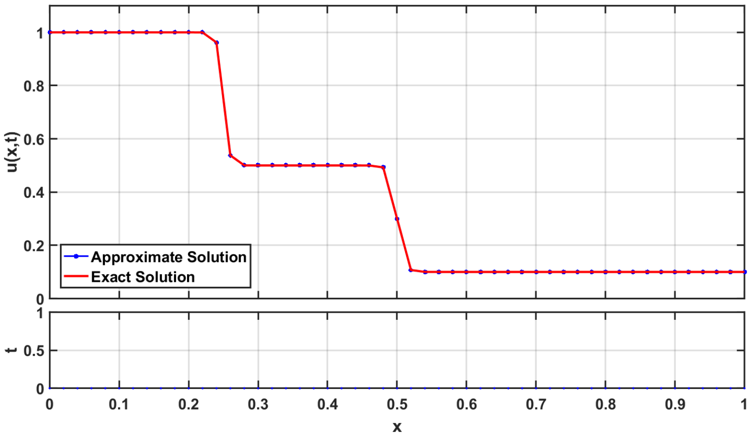
- Just as in moving mesh, the mesh is updated at set intervals.
- Procedure for each mesh adaption:
  1. Reset to uniform base grid.
  2. Evaluate monitor function between each pair of nodes.
  3. If threshold exceeded, add nodes with uniform spacing
  4. If below threshold, remove some nodes.
  5. Interpolate and repeat.

Mesh Refinement

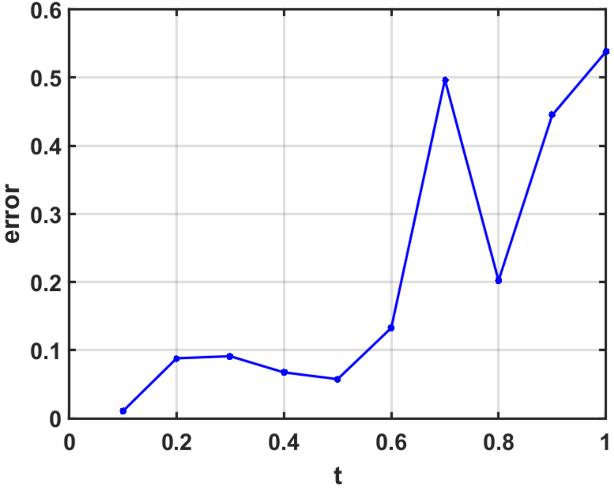
- Just as in moving mesh, the mesh is updated at set intervals.
- Procedure for each mesh adaption:
  1. Reset to uniform base grid.
  2. Evaluate monitor function between each pair of nodes.
  3. If threshold exceeded, add nodes with uniform spacing
  4. If below threshold, remove some nodes.
  5. Interpolate and repeat.

Mesh Refinement - Burgers' Equation Example 1

Mesh Adaptions	# of Mesh Nodes	Computation Time
10	≈ 268.5	6.0 sec

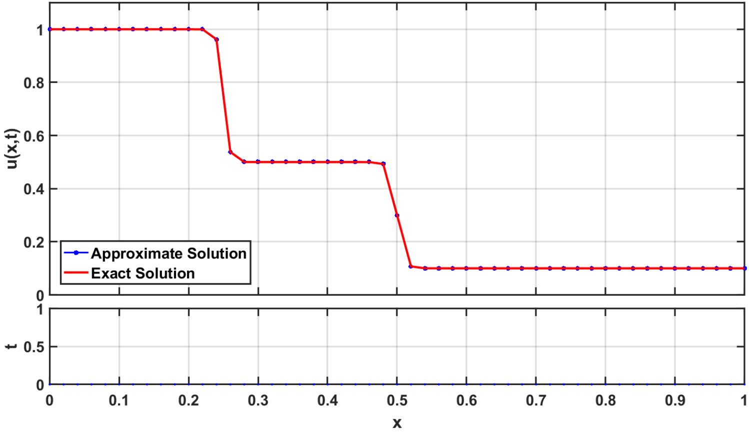


Mesh Refinement - Burgers' Equation Example 1 Error

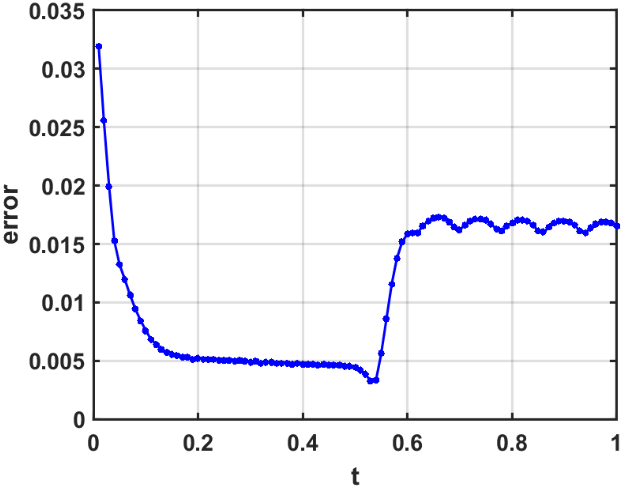


Mesh Refinement - Burgers' Equation Example 2

Mesh Adaption	# of Mesh Nodes	Computation Time
100	≈ 152.5	41.2 sec



Mesh Refinement - Burgers' Equation Example 2 Error



Problems with Static Grid Refinement

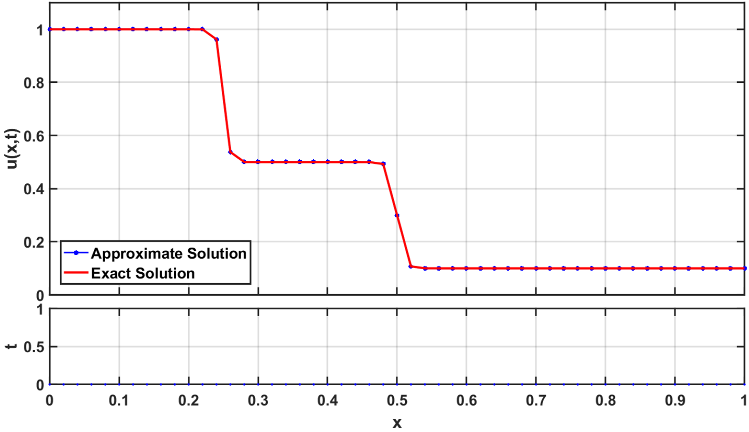
- Infrequent grid refinement can cause the mesh to lag behind the moving solution.
- Refining too often is computationally expensive and negates any benefits from adaptive mesh.
- In dynamic refinement methods, positions of nodes are functions of time.

Dynamic Method

- May be possible to use a priori information to generate functions for the wave to travel along. For example the characteristic curves along which the solution is constant. For a simple moving wave this tracks positions on the wave-front.
- Requires specific a priori knowledge of the problem, not always possible to derive characteristic curves.

Dynamic Method - Burgers' Equation Example

Mesh Adaptions	# of Mesh Points	Computation Time
10	51	15.4 sec



Summary

- Problems arise when uniform grid used in Method of Lines.
- Solution: adapt the grid to account for regions of high activity.
- Static adaptation: refine the grid at discrete time steps.
- Dynamic adaptation: positions of nodes are functions of time.

References

- Vande Wouwer, Saucez and Schiesser, *The Adaptive Method of Lines*, Chapman and Hall, 2001.
- Vande Wouwer, Saucez and Schiesser, *Simulation of ODE/PDE Models with MATLAB, OCTAVE and SCILAB*, Springer 2014.
- Marlow, *Moving Mesh Methods for Solving Parabolic Partial Differential Equations*, Ph. D. Thesis, 2010.

This is the last slide.

Any questions?