The Adaptive Method of Lines

CDEs Group Presentation

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23rd of May, 2016

Motivation

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- Solve the system using some solver such as ode15s.
- Temporal accuracy handled by the ODE solver, however spatial accuracy results from the discretisation used.

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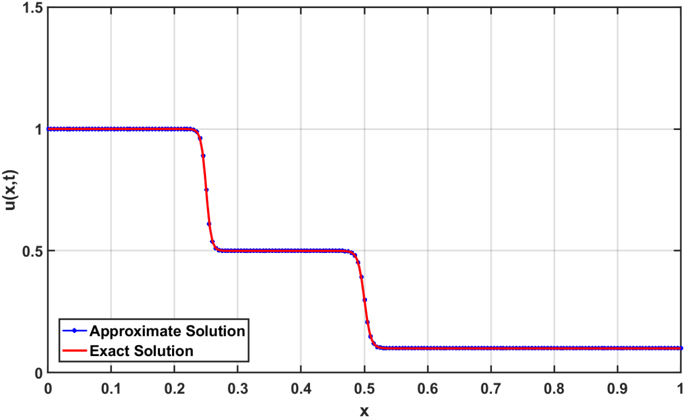
- Method of lines discretises the space uniformly.
- What if our problem has *steep fronts*?
  - Low density uniform mesh doesn't approximate steep fronts well.
  - High density uniform mesh achieves high spatial accuracy, but wastes nodes in regions of low activity.

Method of Lines - Burgers' Equation

- Bad approximation in regions of high spatial activity when grid is uniform. Here we use 201 nodes.



Mesh Adaptation

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  - Static moving mesh.
  - Static grid refinement.
  - Dynamic mesh.

Equidistribution Principle

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- In theory exact equidistribution is optimal, but in practice we may use suboptimal methods that are faster.

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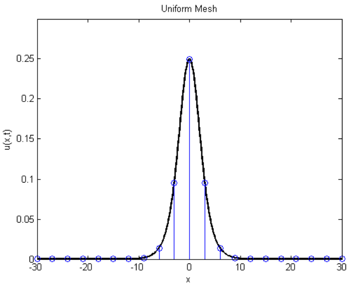
- Other options involving many tuning parameters exist.
- Since the true solution is not known, derivatives must be estimated, eg. using spline interpolation.

Uniform vs Equidistributed Arclengths (n=21)



Example Problems

- Burgers' Equation
  $u_t + uu_x + u_{xx} = 0$

- Korteweg-de Vries (KDV) Equation
  $u_t + u_{xxx} + 6uu_x = 0$

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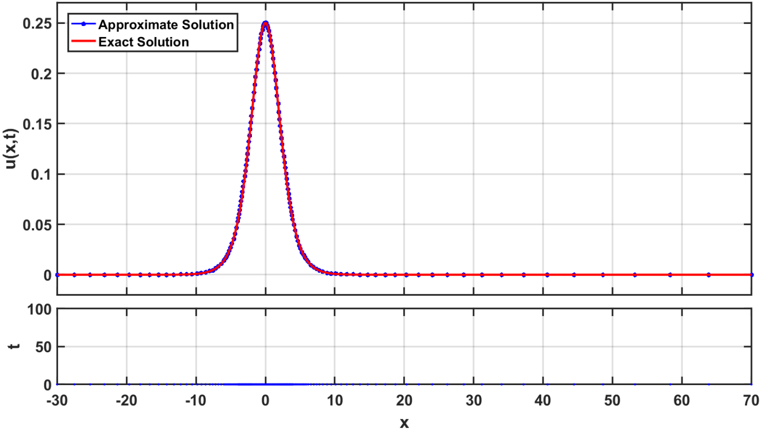
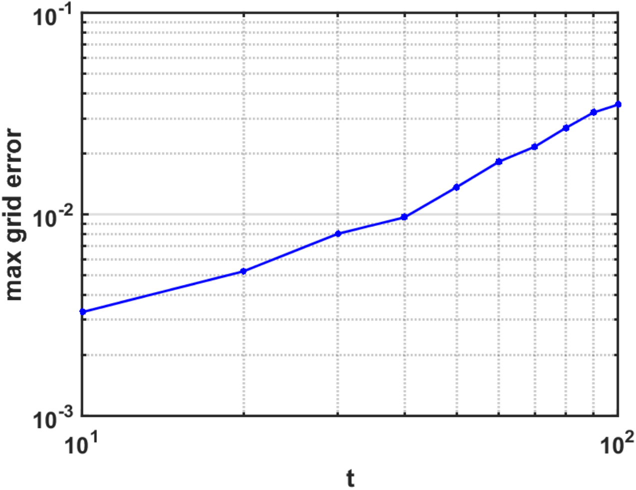
- Interpolate the solution from old mesh to generate initial conditions for new mesh.

Moving Mesh - KDV Equation Example 1

Mesh adaptations	# of Mesh Nodes	Computation Time
10	151	8.9 sec

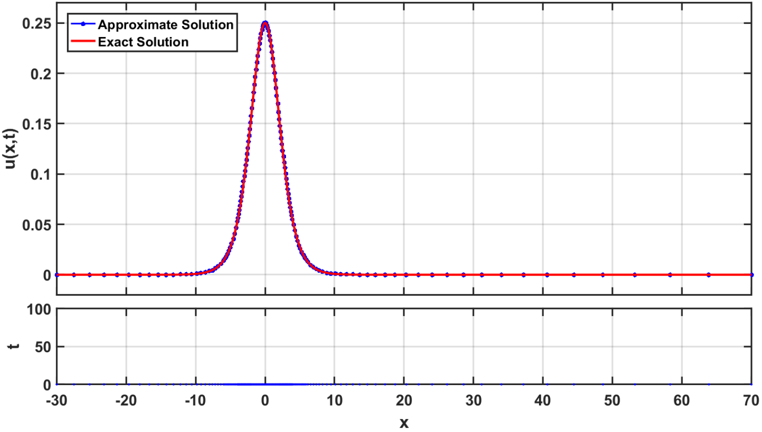


Moving Mesh - KDV Equation Example 1 Error

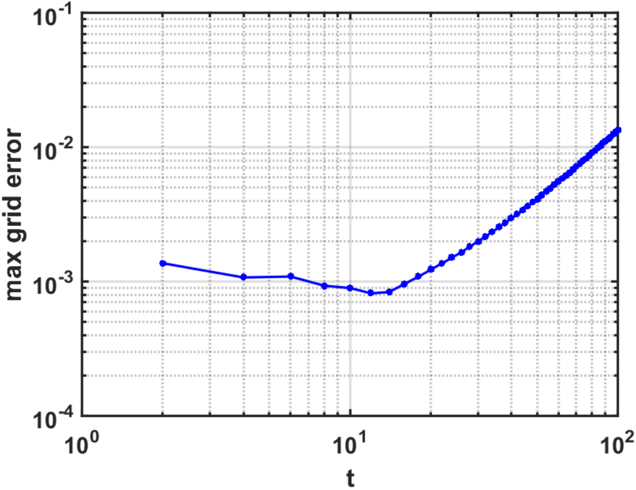


Moving Mesh - KDV Equation Example 2

Mesh adaptations	# of Mesh Nodes	Computation Time
50	151	19.6 sec

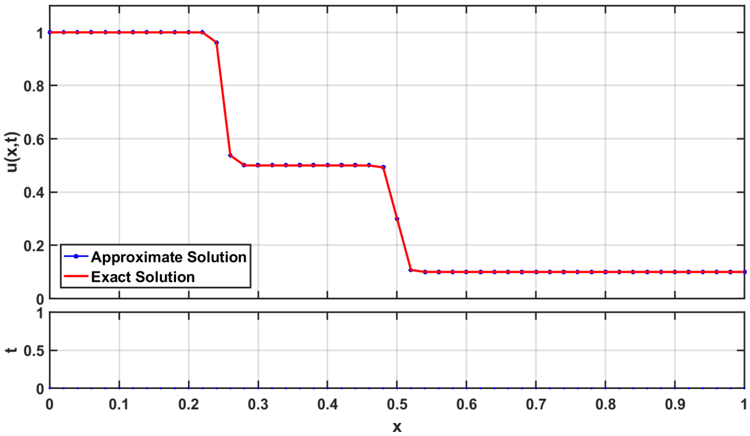


Moving Mesh - KDV Equation Example 2 Error

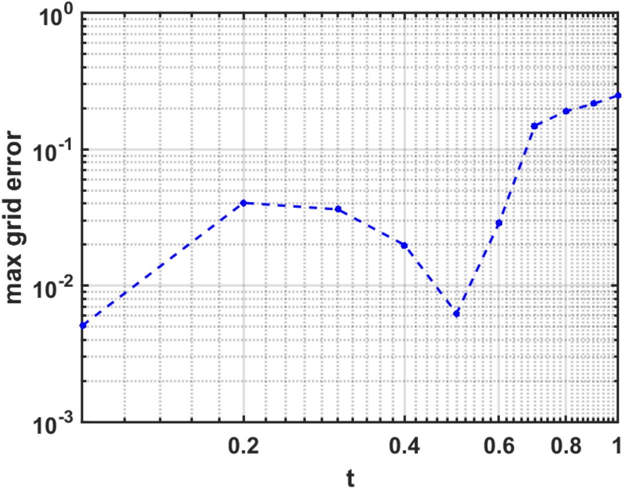


Moving Mesh - Burgers' Equation Example 1

Mesh adaptations	# of Mesh Nodes	Computation Time
10	375	5.8 sec

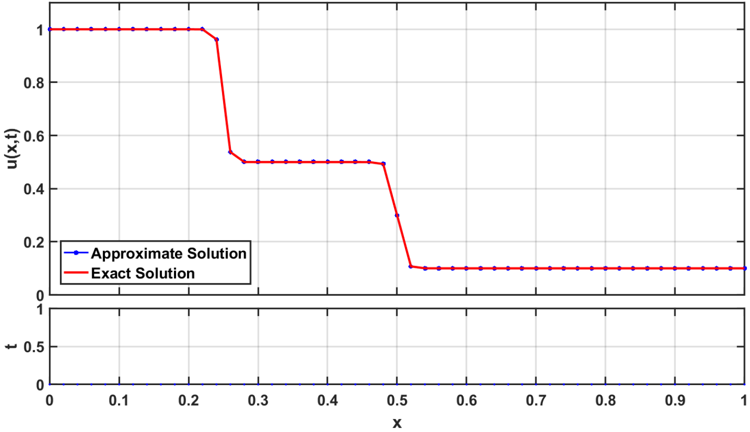


Moving Mesh - Burgers' Equation Example 1 Error

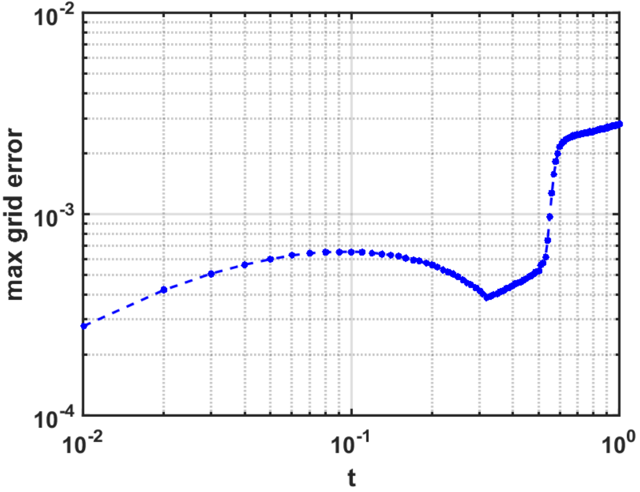


Moving Mesh - Burgers' Equation Example 2

Mesh adaptations	# of Mesh Nodes	Computation Time
100	251	36.1 sec



Moving Mesh - Burgers' Equation Example 2 Error



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- Thresholds, τ_j, on the monitor function dictate how many nodes are added or removed from the grid.

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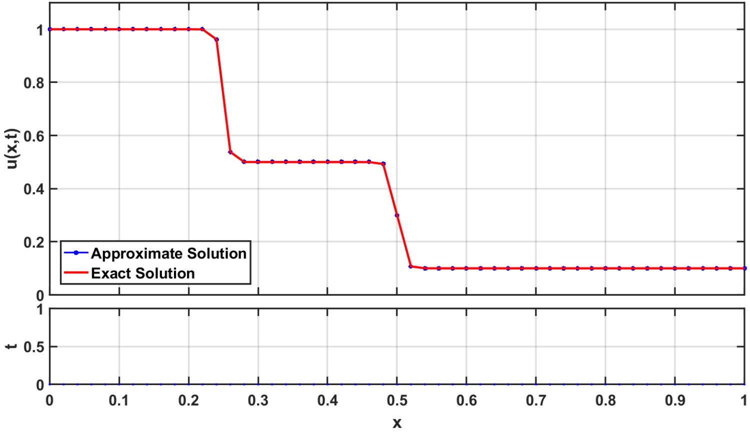
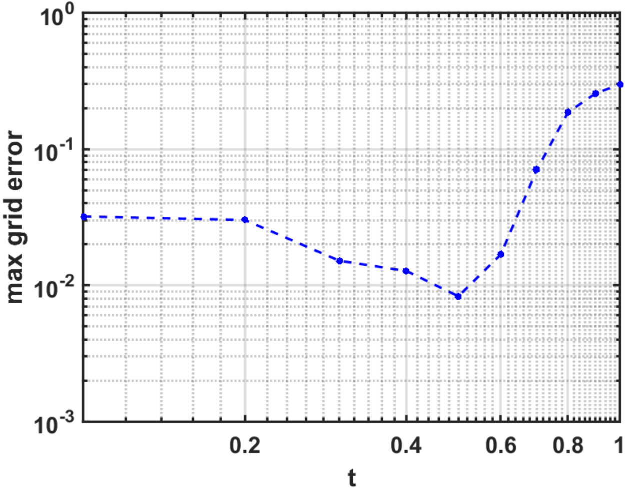
Mesh Refinement - Burgers' Equation Example 1

Mesh adaptations	# of Mesh Nodes	Computation Time
10	≈ 268.5	6.0 sec

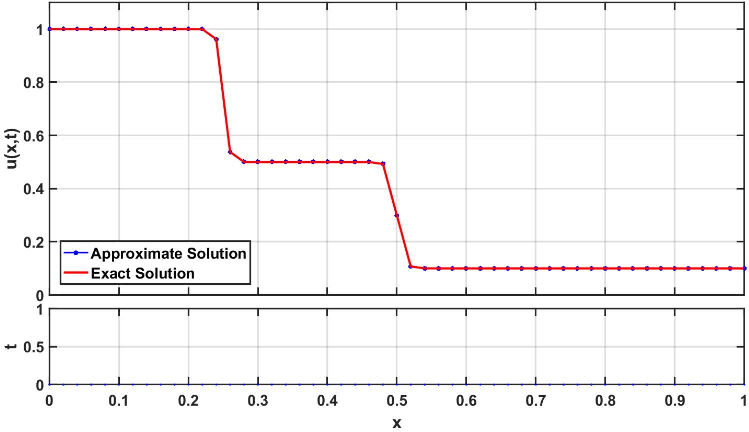


Mesh Refinement - Burgers' Equation Example 1 Error

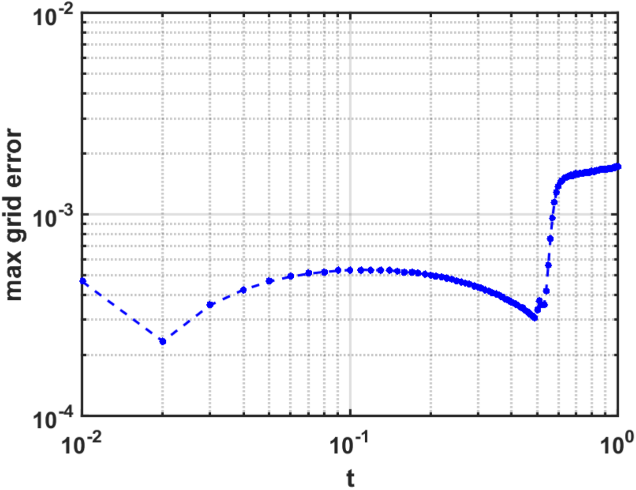


Mesh Refinement - Burgers' Equation Example 2

Mesh adaptation	# of Mesh Nodes	Computation Time
100	$\approx$ 152.5	41.2 sec



Mesh Refinement - Burgers' Equation Example 2 Error



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- Refining too often is computationally expensive and negates any benefits from adaptive mesh.
- In dynamic refinement methods, positions of nodes are functions of time.

Dynamic Method

- May be possible to use a priori information to generate functions for the wave to travel along.
    - Characteristic curves.
    - Moving Mesh PDE (MMPDE).

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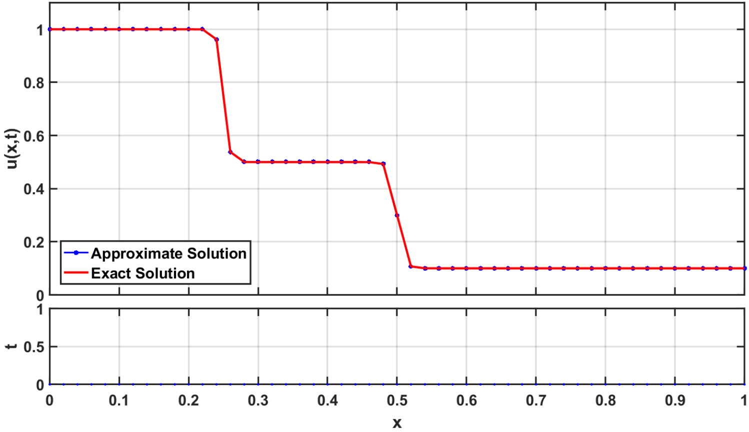
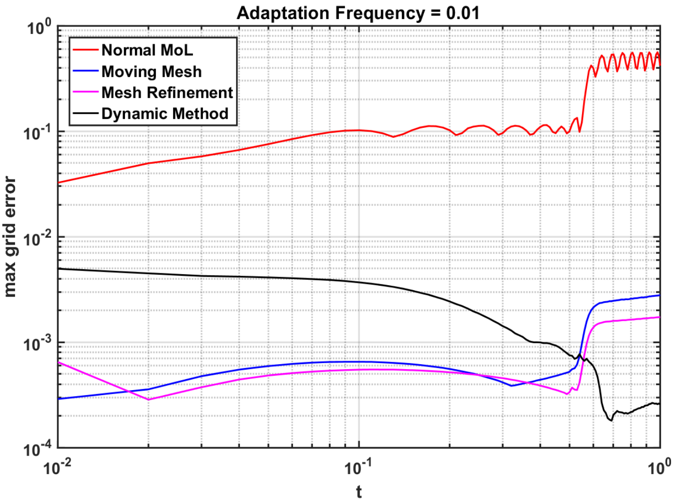
- Characteristic curves may not always be useful.
- Moving mesh requires another PDE to be solved, but are highly flexible.

Dynamic Method - Burgers' Equation Example

Mesh adaptations	# of Mesh Points	Computation Time
10	51	15.4 sec



Error Comparison



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Summary

- Problems arise when uniform grid used in Method of Lines for problems with steep moving fronts.
- Solution: adapt the grid to account for regions of high activity.
    - Static adaptation: refine the grid at discrete time steps.
    - Dynamic adaptation: positions of nodes are functions of time.
- While the adaptive MoL is harder to implement, it is expected to perform better than the standard MoL.

References

- Vande Wouwer, Saucez and Schiesser, *The Adaptive Method of Lines*, Chapman and Hall, 2001.
- Vande Wouwer, Saucez and Schiesser, *Simulation of ODE/PDE Models with MATLAB, OCTAVE and SCILAB*, Springer 2014.
- Marlow, *Moving Mesh Methods for Solving Parabolic Partial Differential Equations*, Ph. D. Thesis, 2010.

Any questions?