Universal multi-port interferometers with minimal optical depth

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Universal multi-port interferometers that can be reconfigured to implement any unitary operation are crucial for photonic universal quantum computation, optical neural networks, and boson sampling. Recently, Saygin *et al.* [Phys. Rev. Lett. **124**, 010501 (2020)] have presented numerical evidence that an extremely compact arrangement consisting in d layers of d-dimensional Fourier transforms and $d+1$ layers of configurable phase shifters might be universal in any dimension d. Here, we first present counterexamples to this statement. We show that for $d = 3$ and 4, around 5% of the unitary matrices from a Haar-uniform distribution cannot be recovered with high fidelity using Saygin *et al.*'s scheme. We also identify a family of unitary matrices with elements that cannot be recovered with high fidelity for any $5 \leq d \leq 10$. Secondly, we present numerical evidence that universality is recovered in all dimensions by adding a Fourier transformation and a layer of phase shifters, which preserves the compactness of the scheme and its robustness against imperfections.

Introduction.—A programmable universal multi-port array (PUMA) [1–3] is an interferometer composed of multiple beam splitters and phase-shifters, which can be reconfigured to exactly implement any unitary operation in a d-dimensional complex Hilbert space. Together with single-photon sources and detectors, PUMAs allow for preparing any quantum state, implementing any quantum logic gate, and any arbitrary quantum measurement. Therefore, PUMAs are of fundamental interest for universal quantum computation with photons [4–6], optical neural networks for machine learning [7, 8], boson sampling [9], and the generation of higher-dimensional entanglement [10].

The first PUMA was proposed by Reck *et al.* [1] and consisted of a regular arrangement of $d(d-1)$ $50:50$ beam splitters and d^2 tunable phase shifters. Integrated photonics [11] boosted the practical interest of PUMAs and allowed implementing Reck *et al.*'s scheme in $d = 6$ [12].

Reck *et al.*'s scheme requires that the beam splitters are aligned in $2(2d-3)$ layers; see Fig. 1(a). The number of layers of the interferometric array, or equivalently, the maximum number of beam splitters that a photon must pass through, is known as the optical depth N. The performance of multi-port arrays is reduced by optical loss, which is directly proportional to the optical depth. Therefore, a smaller optical depth implies higher quality of the implemented unitary.

Clements *et al.* [2] noticed that the same number of beam splitters and phase shifters can be rearranged in a configuration with optical depth $2d$ and thus more robust to noise. See Fig. 1(b). The scheme of Clements *et al.* has been recently implemented in $d = 8$ [13].

In parallel, the availability of multi-port beam splitters of d ports [14–17] has stimulated the search for PUMAs with small optical depth based on sequences

$$P_1 T_d P_2 T_d \cdots P_N T_d P_{N+1}, \tag{1}$$

where T_d is the transfer matrix of the multi-port beam splitter and P_j corresponds to a layer of phase shifters introducing a different phase in each mode, that is,

$$P_j = \text{diag}(e^{i\phi_{j_0}}, e^{i\phi_{j_1}}, e^{i\phi_{j_2}}, \ldots, e^{i\phi_{j_{d-1}}}), \tag{2}$$

with $\phi_{j_0} = 0$ for $j \leq N$. Tang, Tanemura, and Nakano [18] numerically showed that, with a particular T_d introduced in [19] and $N > d$, "[any] desired unitary matrix was generated with mean square error smaller than -20 dB for all tested cases." Zhou, Wu, and Hu [20] pointed out that when T_d is the Fourier transform matrix in dimension d (implemented by a "canonical multi-port beam splitter" [14]),

$$(F_d)_{jk} = \frac{1}{\sqrt{d}} e^{2\pi i (j-1)(k-1)/d}, \tag{3}$$

then a sufficiently large N can approximate any arbitrary unitary in dimension d. The question, in both cases, is which is the *minimum* N needed to obtain universality.

Recently, López-Pastor, Lundeen, and Marquardt [3] have proven that universality can be achieved with a sequence of $6d+1$ phase layers and $6d$ Fourier transforms. See Fig. 1(c). This is, to our knowledge, the only analitically proven PUMA with muti-port beam splitters of $d > 2$. However, this design has optical depth $6d$ and is not robust to imperfections, since the circuit is likely to have unbalanced losses for each mode of the interferometer.

Also recently, Saygin *et al.* [21] have proposed a surprisingly compact PUMA consisting on a sequence of $d+1$ phase layers and d layers of a randomly chosen unitary transformation T_d (i.e., with $N = d$ and optical depth d). See Fig. 1(d). Employing numerical simulations in dimensions $d \geq 10$, Saygin *et al.* conclude that their scheme is universal and error insensitive [21]. In contrast to the previous PUMAs [1–3], which can be programmed following an algorithm, the scheme of Saygin *et al.* requires to solve a global optimization problem in order to derive the best settings for the phase shifters.

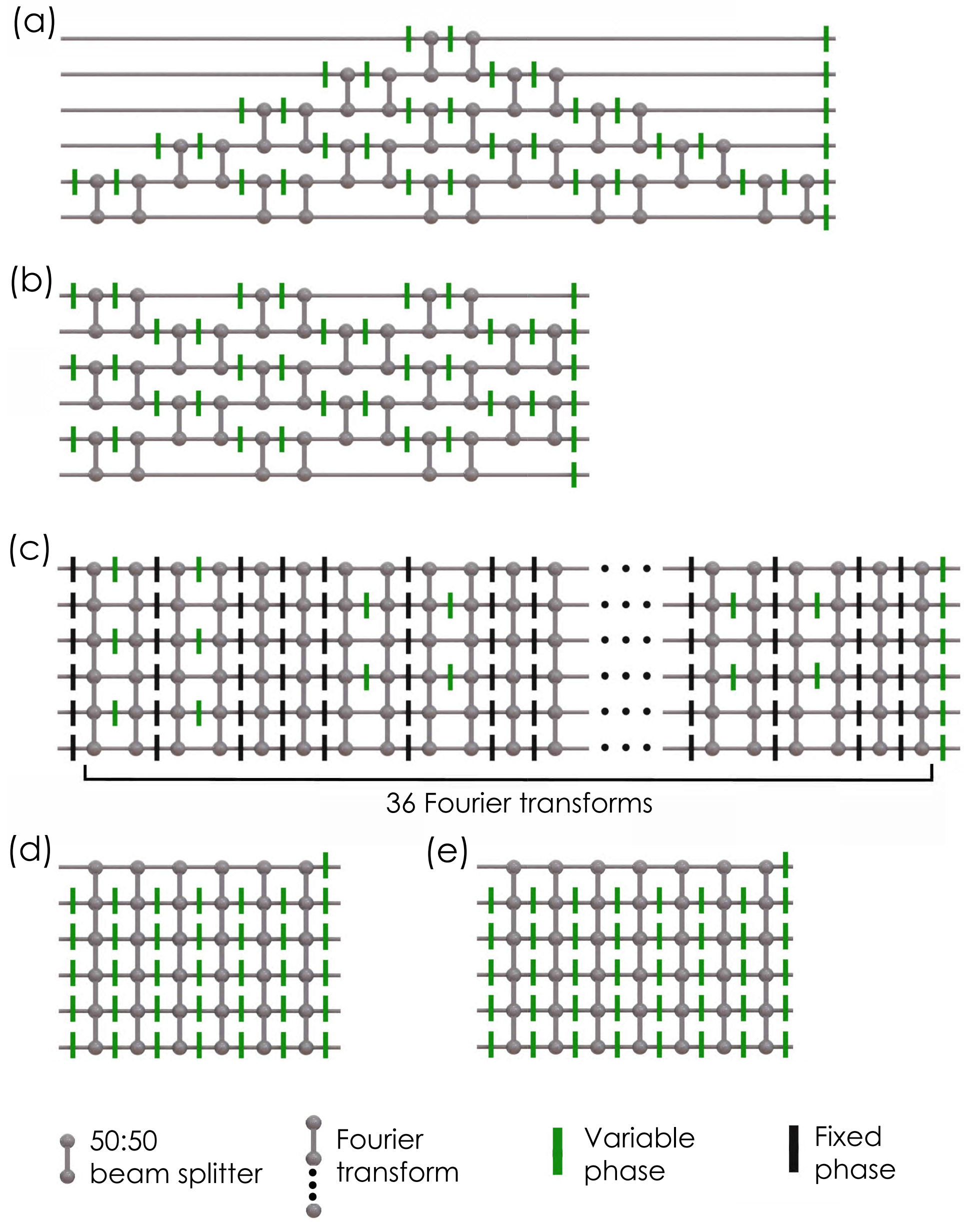

FIG. 1. Comparison between the designs for PUMA for $d = 6$. Every vertical segment with d points represents a d-path Fourier transform. For $d = 2$, a Fourier transform is equivalent to a $50 : 50$ beam splitter. (a) The design of Reck *et al.* [1]. (b) The design of Clements *et al.* [2]. (c) The design of López-Pastor, Lundeen, and Marquardt [3]. (d) The design of Saygon *et al.* [21]. (e) Scheme proposed in this Letter. (a) and (b) have optical depths $2(2d-3)$ and $2d$, respectively. (c) has optical depth is $6d$. (d) and (e) have optical depths d and $d+1$, respectively.

Here, we present two results. First, we prove that the scheme of Saygin *et al.* is not universal and thus the problem of identifying universal multi-port interferometers with minimal optical depth remains open. Then, we present a new PUMA which simultaneously preserves the simplicity of Saygin *et al.*'s scheme, while achieving a high-accuracy implementation of all considered unitary transformations in all studied dimensions. This solution consists in adding to Saygin *et al.*'s scheme a layer of Fourier transforms and phase shifters [see Fig. 1(d)], which minimally increases the optical depth. We also show that the addition of a second layer of Fourier transforms and phase shifters does not lead to a significative improvement in accuracy with respect to the addition of a single layer.

Results.—To prove that Saygin *et al.*'s scheme is not universal, we focus on the lower dimensional regime ($d = 3, \ldots, 10$), since the difficulty for the numerical simulations to produce unitaries which cannot be implemented grows exponentially as the dimension increases. For low dimensions, it is still possible to carry out extensive numerical simulations

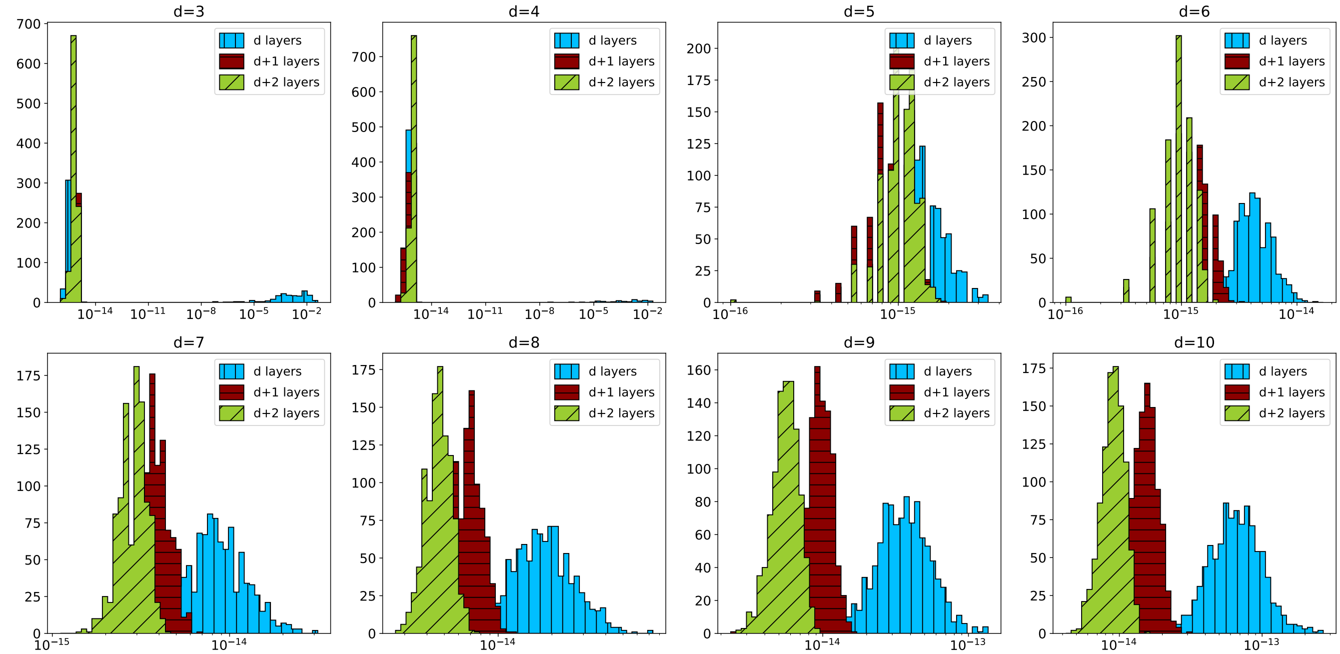

FIG. 2. Histograms of the logarithm of the infidelities obtained when trying to construct random unitary matrices from a Haar-uniform distribution using d layers of Fourier transforms (and $d+1$ layers of phase shifters), as proposed by Saygin *et al.*, $d+1$ layers of Fourier transforms (and $d+2$ layers of phase shifters), as we propose in this Letter, and $d+2$ layers of Fourier transforms (and $d+3$ layers of phase shifters) from dimension $d=3$ to $d=10$. For each dimension, we used 10^3 matrices. Notice that, in $d=3$ and 4, around 5% of the unitary matrices cannot be constructed with high fidelity with the scheme of Saygin *et al.* and that the problem disappeared when we added extra layers of Fourier transforms.

covering a substantial part of the space of unitaries.

We randomly generated thousands of unitary matrices from a Haar-uniform distribution. Then, as Saygin *et al.*, we used optimization methods to find the angles of the phase shifters that best approximate each unitary matrix generated. As a figure of merit, we used the infidelity between two unitary matrices, defined as

$$I(U,V) = 1 - \frac{1}{d^2}|\mathrm{tr}(U^\dagger V)|^2. \tag{4}$$

For each target unitary matrix U, we minimized the infidelity between U and the parametric unitary matrix $V(\vec{\phi})$ given by an arrange like the one by Saygin *et al.* However, we make three different tests. In the first, we used $d+1$ phase layers and d Fourier transforms, as proposed by Saygin *et al.* In the second, we used $d+2$ phase layers and $d+1$ Fourier transforms (i.e., we added an extra layer of Fourier transforms and phase shifters). In the third test we used $d+3$ phase layers and $d+2$ Fourier transforms.

Since the infidelity $I(U,V(\vec{\phi}))$ involves trigonometric functions, it has many local minimums, so a global optimization method has to be used to solve this problem. We adopted a multi-starting strategy where optimization for a given unitary was executed for a large set of different initial conditions in order to find multiple local minimal. The optimization problem was solved via Julia Optim. We explored all dimensions from $d=3$ to $d=10$. For each arrangement in each dimension, we tried 10^3 unitary matrices from a Haar-uniform distribution. In all dimensions 30 randomly generated initial conditions were employed. The results of the optimization are shown in Fig. 2, where histograms for the best-achieved infidelity (in logarithmic scale) are exhibited.

We can see that, in $d=3$ and 4, with the scheme of Saygin *et al.* there is approximately a 5% of matrices which cannot be constructed with high fidelity. In principle, this can be due to two reasons: either the configuration of Saygin *et al.* is not universal in $d=3,4$, or the optimization was not able to find the global minimum. To rule out this last possibility, this 5% of matrices was optimized again with a larger set of initial conditions. However, no significative reduction in infidelity was achieved. Besides, numerical simulations were also implemented employing Matlab GlobalSearch and Python BasinHopping. These maintain the main features exhibited in Fig. 2 but an overall increase in the infidelity for the three tests. This indicates the existence of unitary transformations that cannot be implemented by means of Saygin's *et al.*'s scheme.

Interestingly, it can be seen that for $d+1$ and $d+2$ layers of Fourier transforms (and $d+2$ layers of phase shifters), the implementation of the same matrices is always done with very high fidelity of the same order of magnitude, which is always higher than the one achieved by means of d layers of Fourier transforms. On the other hand, although Fig. 2 does not reveal any problem in using Saygin's *et al.*'s scheme in

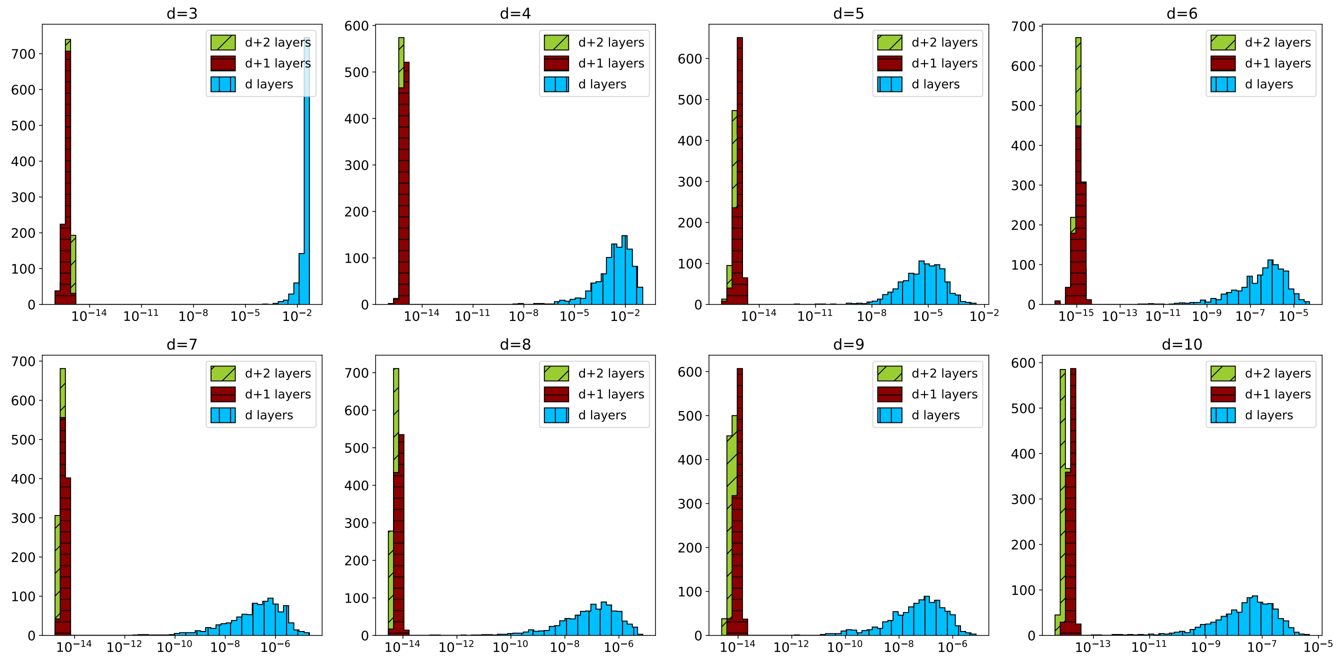

FIG. 3. Histograms of the logarithm of the infidelities obtained when trying to simulate random unitary matrices in the form (5) using d layers of F_d (and $d+1$ layers of phase shifters), as proposed by Saygin *et al.*, ~~and~~ $d+1$ layers of F_d (and $d+2$ layers of phase shifters), as we propose in this Letter, and $d+2$ layers of Fourier transforms (and $d+3$ layers of phase shifters) from dimension $d=3$ to $d=10$. Both matrices U_{d_1} and U_{d_2} are generated from Haar-uniform distribution. For each dimension, a set of 10^3 block-diagonal matrices was employed.

$d = 5, 6, \ldots, 10$, note that simulations with the same number of random unitaries can only cover a portion of the space of unitaries that decreases exponentially as the dimension increases.

To check whether or not Saygin's *et al.*'s scheme is universal in $d \geq 5$, we used what we learned in $d = 3$ and 4 to identify explicit unitary matrices in $d \geq 5$ that cannot be constructed with high fidelity. As pointed out, identifying such matrices is easier in small dimensions such as $d = 3$ and 4, since the probability that finding a random unitary matrix that cannot be constructed decreases exponentially as the dimension increases.

The results in $d = 3$ and 4 lead us to study the performance of Saygin's *et al.*'s scheme (and of the scheme obtained by adding a Fourier transform and a layer of phase shifters) for randomly generated unitary matrices in $d = 5, \ldots, 10$ of the form

$$U = U_{d_1} \oplus U_{d_2}, \tag{5}$$

where U_{d_j} is a unitary matrix of size d_j. It is important to stress that such matrices will never appear in a sample generated from a Haar-uniform distribution, since they span a null measure subspace.

Fig. 3 presents histograms of the infidelity (in logarithmic scale) that result from optimizing 10^3 randomly generated block-diagonal unitaries of the form (5) from a Haar-uniform distribution with $d = 3, \ldots, 10$. The optimization was done following the strategy explained before. In the case of d layers, the histograms are spread along ~~with~~ two to four orders of magnitude of the infidelity and achieve values that are at least two orders of magnitude worse than the case of $d+1$ layers. The infidelities obtained in this latter case are concentrated in a narrow interval of one order of magnitude of the infidelity. Furthermore, the histograms obtained with $d+1$ layers in Figs. 2 and 3 are very similar, which indicates that supplementing Saygin's *et al.*'s scheme with an additional layer allows us to implement all randomly generated matrices within a very narrow interval of high fidelity. In Table I we present the worst-case infidelities from the histograms on Fig. 3. There, one can see that the infidelities achieved through the configuration with d layers are at least eigth orders of magnitude greater than the infidelity obtained with $d+1$ and $d+2$ layers for each dimension, but that this difference remains constant as the dimension increases. Clearly, the addition of an extra layer of Fourier transform and phase shifter to Saygin's *et al.*'s scheme leads to a large increase in accuracy of the implemented unitary transformation. Naturally, this leads to the question of whether adding more layers could lead to even better accuracy. To examine this, we have carried out a third test where the PUMA with $d+1$ layers of Fourier transforms was complemented with an extra layer of Fourier transforms and phase shifter. Numerical simulations with this later configuration, which also considered 10^3 randomly chosen matrices as well as 10^3 block-diagonal matrices and the use of Julia Optim, Matlab GlobalSearch, and Phyton BasinHopping, do not exhibit a significative increase in accuracy with respect to

d	I_d	I_{d+1}	I_{d+2}
3	4.98×10^{-02}	1.21×10^{-15}	1.21×10^{-15}
4	1.06×10^{-01}	1.43×10^{-15}	1.32×10^{-15}
5	3.53×10^{-03}	1.88×10^{-15}	1.65×10^{-15}
6	5.22×10^{-05}	3.43×10^{-15}	2.32×10^{-15}
7	2.24×10^{-05}	6.88×10^{-15}	4.66×10^{-15}
8	7.64×10^{-06}	1.29×10^{-14}	8.44×10^{-15}
9	7.43×10^{-06}	1.81×10^{-14}	9.77×10^{-15}
10	4.76×10^{-06}	2.71×10^{-14}	1.64×10^{-14}

TABLE I. Worst infidelity I_d, I_{d+1}, and I_{d+2} achieved in the implementation of block-diagonal matrices with d, $d+1$ and $d+2$ layers of Fourier transforms, respectively, by Julia Optim.

the PUMA with $d+1$ layers of Fourier transforms. All of this, point out that Saygin *et al.*'s scheme is not universal and leads to the strong conjecture that the PUMA with minimal optical depth may be the one with $d+1$ layers of Fourier transforms and $d+2$ layers of phase shifters.

Let us note that our scheme has $d-1$ more phases than the minimal number of d^2 independent coefficients needed to characterize a unitary transformation in $U(d)$. We have tried three strategies to eliminate $d+1$ phases: (i) deleting one phase, (ii) fixing $d+1$ phases at the inner layers, and (iii) fixing all the phases of the last layer of phase shifters. In all cases, numerical simulations show that each intervention convey the loss of universality in $d=3$ and 4 and an increase in infidelity for higher dimensions.

Conclusions.—For photonic universal quantum computation and many other applications, it is crucial to verify whether Saygin *et al.*'s claim that universality can be achieved with d Fourier transforms and $d+1$ layers of phase shifters is correct. Here we prove that it is not. For many practical purposes, Saygin's *et al.*'s scheme may be good enough, especially in very high dimensions, but it is certainly not universal. The problem of which is the exactly universal configuration with the smallest optical depth remains open. However, here we have provided evidence that the solution might be a similar scheme consisting of $d+1$ Fourier transforms and $d+2$ layers of phase shifters. This scheme has only optical depth $d+1$, which substantially reduces the optical depth of any previous programmable universal multi-port array.

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