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Factorization of unitary matrices

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Abstract

Factorization of $n \times n$ unitary matrices as a product of n diagonal phase matrices interlaced with $n-1$ orthogonal matrices, each one generated by a real vector, is provided. As a byproduct an explicit form for the Weyl factorization of unitary matrices is given. The results can be used at the parametrization of complex Hadamard matrices and in finding the Laplace–Beltrami operators on unitary groups.

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1. Introduction

Matrix factorization is a live subject of linear algebra. It seems that no general theory is yet available although many results appear almost every day. However, our goal will not be so ambitious to present a general theory of matrix factorizations but to tackle the problem of factorization of unitary matrices. Unitary matrices are a first hand tool in solving many problems in mathematical and theoretical physics and the diversity of the problems necessitates to keep improving it. In fact, the matrix factorization is closely related to the parametrization of unitary matrices, and the classical result by Murnagham (1962) on parametrization of the n-dimensional unitary group $U(n)$ is the following: an arbitrary $n \times n$ unitary matrix is the product of a diagonal matrix containing n phases and $n(n-1)/2$ matrices whose main building block has the form

$$U = \begin{pmatrix} \cos\theta & -\sin\theta\, \mathrm{e}^{-\mathrm{i}\varphi} \\ \sin\theta\, \mathrm{e}^{\mathrm{i}\varphi} & \cos\theta \end{pmatrix}. \tag{1}$$

The parameters entering the parametrization are $n(n-1)/2$ angles θ_i and $n(n+1)/2$ phases φ_i.

A selection of a specific set of angles and/or phases has no theoretical significance because all the choices are mathematically equivalent; however, a clever choice may shed some light on important qualitative issues. Because in any group the product of two arbitrary elements is again an element of the group, there is a freedom in choosing the ‘building’ blocks to be used in a definite application. In some sense the application we have in view imposes the factorization.

To our knowledge one of the first such problems is raised by Reck *et al* (1994), who describe an experimental realization of any discrete unitary operator. Such devices will find practical applications in quantum cryptography and in quantum teleportation. Starting from Murnagham parametrization they show that any $n \times n$ unitary matrix A_n can be written as a product $A_n = B_n C_{n-1}$ where $B_n \in U(n)$ is at its turn a product of $n-1$ unitary matrices containing each one a block of the form (1) and C_{n-1} is a $U(n-1)$ matrix. Consequently, the experimental realization of an $n \times n$ unitary operator is reduced to the realization of two unitary operators one of which is a little bit simpler. The experimental realization of a $U(3)$ matrix, sketched in their figure 2, suggests that it would be preferable that phases entering the parametrization should be factored out, the device becoming simpler and the phase shifters, their terminology for phases, being placed at the input and output ports, respectively.

A mixing of the Murnagham factorization and that of Reck *et al* is proposed by Rowe *et al* (1999) in their study on the representations of Weyl group and Wigner functions for $SU(3)$. The last parametrization is also used by Nemoto (2000) in his attempt to develop generalized coherent states for $SU(n)$ systems.

Another kind of factorization is suggested by Chaturvedi and Mukunda (2001), whose aim is to obtain a more 'suitable' parametrization of the Cabibbo–Kobayashi–Maskawa matrix appearing in particle theory. Although the proposed forms for $n = 3, 4$ are awfully complicated by comparison with other parametrizations existing in the literature, and for this reason this factorization cannot be easily extended to cases $n \geqslant 5$, the paper contains a novel idea namely that an $SU(n)$ matrix can be parametrized by a sequence of $n-1$ complex vectors of dimensions $2, 3, \ldots, n$. Fortunately, there is an alternative simpler construction as it may be inferred from the construction of an $SU(3)$ matrix as a product of two matrices, each of them generated by three- and two-dimensional complex vectors, respectively. (Mathur and Sen 2001).

The aim of this paper is to elaborate this alternative construction in order to obtain a factorization of $n \times n$ unitary matrices as a product of n diagonal matrices containing the phases and $n-1$ orthogonal matrices, each of them generated by real vectors of dimensions $2, 3, \ldots, n$. The main result of the paper is the following.

Any element $A_n \in U(n)$ can be factored into an ordered product of $2n-1$ matrices of the following form

$$A_n = d_n \mathcal{O}_n d_{n-1}^1 \mathcal{O}_{n-1}^1 \cdots d_2^{n-2} \mathcal{O}_2^{n-1} d_1^{n-1}$$

where d_{n-k}^k are diagonal phase matrices and $\mathcal{O}_{n-k}^k$ are orthogonal matrices whose columns are generated by real $(n-k)$-dimensional unit vectors.

The idea behind such a factorization is to look for a variety on which $U(n)$ acts transitively which in our case is the complex unit sphere of $\mathbf{C}^n$. The transitivity property allows the factorization of an arbitrary element $A_n \in U(n)$ under the form $A_n = B_n C_{n-1}$ where C_{n-1} is an arbitrary element of $U(n-1)$ and B_n is a special element of $U(n)$ which has the remarkable property that it is parametrized by *an arbitrary point* of the complex variety. The second step of the algorithm is to provide an explicit construction of this special element of $U(n)$. In our case the construction reduces to the completion of a unitary matrix whose first column is explicitly known, without introducing supplementary parameters. For doing that we used some elementary facts from contraction theory and spectral theory of symmetric operators.

As a byproduct of our factorization, we obtain the Weyl form (Weyl 1946) of a unitary matrix $W = w^* d w$, where w is a unitary matrix, w^* is its adjoint and d is a diagonal matrix containing n phases. The Weyl factorization was the key ingredient in finding the 'radial' part of the Laplace–Beltrami operator on $U(n)$ and $SU(n)$ (Wadia 1980, Menotti and Onofri 1981) and this explicit form could help in finding completely the Laplace–Beltrami operator on unitary groups. The applications we have in view concern the construction of Laplace–Beltrami operators for unitary groups and the parametrization of complex Hadamard matrices. We recall that one does not know the Laplacian even for the $SU(3)$ group, the only partial result being that obtained by Bég and Ruegg (1965). The parametrization of the complex Hadamard matrices is very important in the quantum theory of information (Werner 2000), the complex Hadamard matrices being those unitary matrices whose entries moduli equal $1/\sqrt{n}$.

The paper is organized as follows: in section 2 we derive a factorization of $n \times n$ unitary matrices as a product of n diagonal matrices interlaced with $n-1$ orthogonal matrices generated by real vectors of dimensions $2, 3, \ldots, n-1$. The explicit form of the orthogonal matrices entering the factorization is found in section 3 and the paper ends with concluding remarks.

2. Factorization of unitary matrices

The unitary group $U(n)$ is the group of automorphisms of the Hilbert space $(\mathbf{C}^n, \langle\cdot|\cdot\rangle)$ where $\langle\cdot|\cdot\rangle$ is the Hermitian scalar product $\langle x|y\rangle = \sum_{i=1}^{i=n} \overline{x_i} y_i$. If $A_n \in U(n)$ by A_n^*, we will denote the adjoint matrix and then $A_n^* A_n = I_n$, where I_n is the $n \times n$ unit matrix. It follows that $\det A_n = \mathrm{e}^{\mathrm{i}\varphi}$, where φ is a phase, and $\dim_{\mathbf{R}} U(n) = n^2$.

First of all we want to introduce some notation that will be useful in the following. The product of two unitary matrices being again a unitary matrix, it follows that the multiplication of a row or a column by an arbitrary phase does not affect the unitarity property. Indeed, the multiplication of the jth row by $\mathrm{e}^{\mathrm{i}\varphi_j}$ is equivalent to the left multiplication by a diagonal matrix whose all diagonal entries but the jth ones are equal to unity and $a_{jj} = \mathrm{e}^{\mathrm{i}\varphi_j}$. The first building blocks appearing in the factorization of unitary matrices are diagonal matrices written in the form $d_n = (\mathrm{e}^{\mathrm{i}\varphi_1}, \ldots, \mathrm{e}^{\mathrm{i}\varphi_n})$ with $\varphi_j \in [0, 2\pi)$, $j = 1, \ldots, n$, arbitrary phases, and all off-diagonal entries zero. We also introduce the notation $d_k^{n-k} = (1_{n-k}, \mathrm{e}^{\mathrm{i}\psi_1}, \ldots, \mathrm{e}^{\mathrm{i}\psi_k})$, $k < n$, where 1_{n-k} means that the first $(n-k)$ diagonal entries are equal to unity, i.e. it can be obtained from d_n by making the first $n-k$ phases zero. Multiplying at left by d_n, an arbitrary unitary matrix, the first row will be multiplied by $\mathrm{e}^{\mathrm{i}\varphi_1}$, the second by $\mathrm{e}^{\mathrm{i}\varphi_2}$, etc and the last one by $\mathrm{e}^{\mathrm{i}\varphi_n}$. Multiplying at right by d_k^{n-k} the first $n-k$ columns remain unmodified and the other ones are multiplied by $\mathrm{e}^{\mathrm{i}\psi_1}, \ldots, \mathrm{e}^{\mathrm{i}\psi_k}$, respectively. A consequence of this property is the following: the phases of the elements of an arbitrary row and/or column can be taken zero or π and a convenient choice is to take the elements of first column non-negative numbers and those of the first row real numbers. We can also interchange any two columns (rows) and the new matrix is again unitary. This follows from the equivalence between the permutation of the ith and jth rows (columns) with the left (right) multiplication by the unitary matrix P_{ij} whose all diagonal entries but a_{ii} and a_{jj} are equal to unity, $a_{ii} = a_{jj} = 0$, $a_{ij} = a_{ji} = 1$, $i \neq j$, and all the other entries vanish. In conclusion, an arbitrary $A_n \in U(n)$ can be written as a product of two matrices, the first one diagonal, in the form

$$A_n = d_n \tilde{A}_n \qquad (2)$$

where $\tilde{A}_n$ is a matrix with the first column elements non-negative numbers.

Other building blocks that will appear in factorization of $\tilde{A}_n$ are the two-dimensional rotations which operate in the $i, i+1$ plane of the form

$$J_{i,i+1} = \begin{pmatrix} I_{i-1} & 0 & 0 \\ 0 & \begin{matrix} \cos\theta_i & -\sin\theta_i \\ \sin\theta_i & \cos\theta_i \end{matrix} & 0 \\ 0 & 0 & I_{n-i-1} \end{pmatrix} \qquad i = 1, \ldots, n-1 \tag{3}$$

where I_k denotes the k-dimensional unit matrix.

Let v be the vector $v = (1, 0, \ldots, 0)^t \in S_{2n-1} \in \mathbf{C}^n$ where t denotes the transpose and S_{2n-1} is the unit sphere of the Hilbert space $\mathbf{C}^n$ whose real dimension is $2n - 1$. By applying $A_n \in U(n)$ to the vector v, we find

$$A_n v = a = (a_{11} \ldots a_{n1})^t$$

where $a \in S_{2n-1}$ because A_n is unitary. The vector a is completely determined by the first column of the matrix A_n. Conversely, given an arbitrary vector of the unit sphere $w \in S_{2n-1}$ this point determines a unique first row of a unitary matrix which maps w to the vector v. Therefore, $U(n)$ acts transitively on S_{2n-1}. The subgroup of $U(n)$ which leaves v invariant is $U(n-1)$ on the last $n-1$ dimensions such that

$$S_{2n-1} = \textit{coset space } U(n)/U(n-1).$$

Thus, apart from global matching problems or ambiguities on a subset of measure zero, we expect that any element of $U(n)$ should be uniquely specified by a pair of a vector $b \in S_{2n-1}$ and an arbitrary element of $U(n-1)$. Thus, we are looking for a factorization of an arbitrary element $A_n \in U(n)$ in the form

$$A_n = B_n \begin{pmatrix} 1 & 0 \\ 0 & A_{n-1} \end{pmatrix} \tag{4}$$

where $B_n \in U(n)$ is a unitary matrix whose first column is uniquely defined by a vector $b \in S_{2n-1}$, but the other columns for the moment are still arbitrary and A_{n-1} is an arbitrary element of $U(n-1)$. For the $SU(3)$ group such a factorization was obtained recently (Chaturvedi and Mukunda 2001, Mathur and Sen 2001). Iterating the previous equation we arrive at the conclusion that an element of $U(n)$ can be written as a product of n unitary matrices

$$A_n = B_n \cdot B_{n-1}^1 \cdots B_1^{n-1} \tag{5}$$

where

$$B_{n-k}^k = \begin{pmatrix} I_k & 0 \\ 0 & B_{n-k} \end{pmatrix}.$$

$B_k, k = 1, \ldots, n-1$, are $k \times k$ unitary matrices whose first column is generated by vectors $b_k \in S_{2k-1}$; for example, B_1^{n-1} is the diagonal matrix $(1, \ldots, 1, \mathrm{e}^{\mathrm{i}\varphi_{n(n+1)}})$.

The still arbitrary columns of B_k will be chosen in such a way that we should obtain a simple form for the matrices B_k^{n-k}, and we require that B_k should be completely specified by the parameters entering the vector b_k and nothing else. In the following, we show that such a parametrization does exist and then $A_n \in U_n$ in (5) will be written as a product of n unitary matrices each one parametrized by $2k-1$, $k = 1, \ldots, n$, real parameters such that the number of independent parameters entering A_n will be $1 + 3 + \cdots + 2n - 1 = n^2$ as it should be.

In other words, our problem is to complete an $n \times n$ matrix whose first column is given by a vector $b_n \in S_{2n-1}$ to a unitary matrix and we have to do it without introducing supplementary parameters. For $n = 3$ this was found by us (Diţă 1994) in an other context and here we give the construction for arbitrary n.

If we take into account the property (2), the problem simplifies a little bit since then

$$B_n = d_n \tilde{B}_n$$

where the first column of $\tilde{B}_n$ has non-negative entries. Denoting this column by v_1 we will use the parametrization

$$v_1 = (\cos\theta_1, \cos\theta_2 \sin\theta_1, \ldots, \sin\theta_1 \ldots \sin\theta_{n-1})^t \tag{6}$$

where $\theta_i \in [0, \pi/2]$, $i = 1, \ldots, n-1$, are the angles. Thus, B_n will be parametrized by n phases and $n-1$ angles. According to the above factorization, $\tilde{B}_n$ is nothing else than the orthogonal matrix generated by the vector v_1. Thus, with no loss of generality $B_n = d_n \mathcal{O}_n$ with $\mathcal{O}_n \in SO(n)$. In this way the factorization of A_n will be

$$A_n = d_n \mathcal{O}_n d_{n-1}^1 \mathcal{O}_{n-1}^1 \cdots d_2^{n-2} \mathcal{O}_2^{n-1} d_1^{n-1}$$

where $\mathcal{O}_{n-k}^k$ has the same structure as B_{n-k}^k, i.e.

$$\mathcal{O}_{n-k}^k = \begin{pmatrix} I_k & 0 \\ 0 & \mathcal{O}_{n-k} \end{pmatrix}.$$

In conclusion the factorization of unitary matrices reduces to the parametrization of orthogonal matrices generated by unit vectors of the *real* sphere of dimensions $2, \ldots, n$ and in the next section we show how to do it.

3. Parametrization of orthogonal matrices $\mathcal{O}_n$

For the explicit construction of the orthogonal matrix $\mathcal{O}_n$ we need some notions from contraction theory. An operator T applying the Hilbert space $\mathcal{H}$ in the Hilbert space $\mathcal{H}'$ is a contraction if for any $v \in \mathcal{H}$, $\|Tv\|_{\mathcal{H}'} \leqslant \|v\|_{\mathcal{H}}$, i.e. $\|T\| \leqslant 1$ (Sz-Nagy and Foias 1967). For any contraction we have $T^*T \leqslant I_{\mathcal{H}'}$ and $TT^* \leqslant I_{\mathcal{H}}$ and the defect operators

$$D_T = (I_{\mathcal{H}} - T^*T)^{1/2} \qquad D_{T^*} = (I_{\mathcal{H}'} - TT^*)^{1/2}$$

are Hermitian operators in $\mathcal{H}$ and $\mathcal{H}'$, respectively. They have the property

$$TD_T = D_{T^*}T \qquad T^*D_{T^*} = D_T T^*. \tag{7}$$

In the following we are interested in a contraction of a special form, namely that generated by an n-dimensional real vector $b \in R^n$, i.e. $T = (b_1, \ldots, b_n)^t$, where b_i are the coordinates of b; its norm is $\|T\| = (b, b)$ and T will be a contraction iff $(b, b) \leqslant 1$, i.e. if b is a point within the unit ball of R^n. If $(b, b) = 1$, that is the case we are interested in, T is an isometry, i.e.

$$T^*T = 1 \qquad \text{and} \qquad D_T = 0$$

and in this case D_{T*} is an orthogonal projection. A direct calculation shows that $\det\big(\lambda I_n - D_{T^*}^2\big) = \lambda(\lambda - 1)^{n-1}$ such that the eigenvalue $\lambda = 0$ is simple and the eigenvalue $\lambda = 1$ is degenerated. From the first relation (7) we have

$$D_{T^*}T = D_{T^*}b = TD_T = bD_T = 0$$

i.e. b is the eigenvector of D_{T^*} which corresponds to the $\lambda = 0$ eigenvalue.

The orthogonal matrix $\mathcal{O}_n$ which brings the operator D_{T^*} to a diagonal form

$$\mathcal{O}_n^t D_{T^*} \mathcal{O}_n = \begin{pmatrix} 0 & 0 \\ 0 & I_{n-1} \end{pmatrix}$$

is the orthogonal matrix we are looking for because it is generated by an arbitrary n-dimensional real vector of unit norm. The multiplicity of the $\lambda = 1$ eigenvalue being $n-1$, the form of the matrix $\mathcal{O}_n$ is not uniquely defined. In this situation we have to make a choice between the possible bases. Our criterion was that the resulting orthogonal matrix $\mathcal{O}_n$ should have as many as possible vanishing entries. We found such a matrix that has $(n-1)(n-2)/2$ zero entries in the upper right corner and the result is expressed by the following lemma.

Lemma 1. *The orthonormalized eigenvectors of the eigenvalue problem*

$$D_{T^*} v_k = \lambda_k v_k \qquad k = 1, \ldots, n$$

are the columns of the orthogonal matrix $\mathcal{O}_n \in SO(n)$ and are generated by the vector v_1, as in (6), as

$$v_1 = \begin{pmatrix} \cos\theta_1 \\ \sin\theta_1 \cos\theta_2 \\ \cdot \\ \cdot \\ \cdot \\ \sin\theta_1 \cdots \sin\theta_{n-1} \end{pmatrix}$$

and

$$v_{k+1} = \frac{\mathrm{d}}{\mathrm{d}\theta_k} v_1(\theta_1 = \cdots = \theta_{k-1} = \pi/2) \qquad k = 1, \ldots, n-1$$

where in the above formula one calculates first the derivative and afterwards the restriction to $\pi/2$.

Proof. Elementary calculations show that $(v_i, v_j) = \delta_{ij}, i, j = 1, \ldots, n$, and thus v_k are linearly independent. Because the multiplicity of the zero eigenvalue is unity, it follows that v_k, $k = 2, \ldots, n$, are orthogonal eigenvectors corresponding to the $\lambda = 1$ eigenvalue. □

The orthogonal matrices $\mathcal{O}_n$ $\left(\mathcal{O}^k_{n-k}\right)$ can be written as products involving only the two-dimensional rotations defined by (3). Thus we have.

Lemma 2. *The orthogonal matrices $\mathcal{O}_n$ $\left(\mathcal{O}^k_{n-k}\right)$ at their turn can be factored into a product of $n-1$ $(n-k-1)$ matrices of the form $J_{i,i+1}$; e.g., we have*

$$\mathcal{O}_n = J_{n-1,n} J_{n-2,n-1} \cdots J_{1,2} \tag{8}$$

where $J_{i,i+1}$ are $n \times n$ rotations introduced by equation (3).

Putting together all the preceding information one obtains the following result.

Theorem 1. *Any element $A_n \in U(n)$ can be factored into an ordered product of $2n-1$ matrices of the form*

$$A_n = d_n \mathcal{O}_n d^1_{n-1} \mathcal{O}^1_{n-1} \cdots d^{n-2}_2 \mathcal{O}^{n-1}_2 d^{n-1}_1 \tag{9}$$

where d^k_{n-k} are diagonal phase matrices and $\mathcal{O}^k_{n-k}$ are orthogonal matrices whose columns are generated by real $(n-k)$-dimensional unit vectors according to lemma 1. By using the factorization (8) the above formula can be written as a product of n diagonal phase matrices and $n(n-1)/2$ two-dimensional rotations $J_{k,k+1}$.

The condition $\sum_{i=1}^{n(n+1)/2} \varphi_i = 0$, imposed on φ_i the arbitrary phases entering the parametrization of A_n, gives the factorization of $SU(n)$ matrices.

If $w_n = \mathcal{O}_n d^1_{n-1} \mathcal{O}^1_{n-1} \cdots d^{n-2}_2 \mathcal{O}^{n-1}_2 d^{n-1}_1 = \mathcal{O}_n d^1_{n-1} w_{n-1}$, then

$$W_n = w^*_n d_n w_n \tag{10}$$

is one (of the many possible) Weyl representation of unitary matrices.

If all the phases entering A_n are zero or π, $\varphi_i = 0, \pi$, $i = 1, \ldots, n(n+1)/2$, one gets the factorization of the rotation group $O(n)$; the factorization of the special group $SO(n)$ is obtained when an even number of phases take the value π.

Remark 1. The above factorization is not unique and we propose it as the standard (and simplest) representation. Equivalent factorizations (parametrizations) can be obtained by inserting matrices such as P_{ij} as factors in formulae (8)–(10) since the number of parameters remains the same and only the final form of the matrices is different. As concerns equation (10) we made the choice that leads to the simplest form for the matrix elements of W_n as polynomial functions of sines and cosines which enter the parametrization of orthogonal matrices. For example, instead of $w_n = \mathcal{O}_n d^1_{n-1} w_{n-1}$ we could take $w_n = \mathcal{O}_n W_{n-1}$, where W_{n-1} is at its turn given by a formula such as (10) and so on.

Remark 2. If in our formula for $A_n \in O(n)$ we factorize all the two-dimensional rotations $J_{i,i+1}$ as

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} = \begin{pmatrix} 1 & -\tan\theta/2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \sin\theta & 1 \end{pmatrix} \begin{pmatrix} 1 & -\tan\theta/2 \\ 0 & 1 \end{pmatrix}$$

we obtain a factorization of rotations similar to that found by Strang (1997) and Toffoli (1997).

Examples. An element $A_4 \in U(4)$ factors as

$$A_4 = d_4 \mathcal{O}_4 d^1_3 \mathcal{O}^1_3 d^2_2 \mathcal{O}^2_2 d^3_1$$

where $d_4 = (\mathrm{e}^{\mathrm{i}\varphi_1}, \mathrm{e}^{\mathrm{i}\varphi_2}, \mathrm{e}^{\mathrm{i}\varphi_3}, \mathrm{e}^{\mathrm{i}\varphi_4})$, $d^1_3 = (1, \mathrm{e}^{\mathrm{i}\varphi_5}, \mathrm{e}^{\mathrm{i}\varphi_6}, \mathrm{e}^{\mathrm{i}\varphi_7})$, $d^2_2 = (1, 1, \mathrm{e}^{\mathrm{i}\varphi_8}, \mathrm{e}^{\mathrm{i}\varphi_9})$, $d^3_1 = (1, 1, 1, \mathrm{e}^{\mathrm{i}\varphi_{10}})$, and $\mathcal{O}_4$, $\mathcal{O}^1_3$ and $\mathcal{O}^2_2$ are the following matrices:

$$\mathcal{O}_4 = \begin{pmatrix} \cos\theta_1 & -\sin\theta_1 & 0 & 0 \\ \sin\theta_1\cos\theta_2 & \cos\theta_1\cos\theta_2 & -\sin\theta_2 & 0 \\ \sin\theta_1\sin\theta_2\cos\theta_3 & \cos\theta_1\sin\theta_2\cos\theta_3 & \cos\theta_2\cos\theta_3 & -\sin\theta_3 \\ \sin\theta_1\sin\theta_2\sin\theta_3 & \cos\theta_1\sin\theta_2\sin\theta_3 & \cos\theta_2\sin\theta_3 & \cos\theta_3 \end{pmatrix}$$

$$\mathcal{O}^1_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta_4 & -\sin\theta_4 & 0 \\ 0 & \sin\theta_4\cos\theta_5 & \cos\theta_4\cos\theta_5 & -\sin\theta_5 \\ 0 & \sin\theta_4\sin\theta_5 & \cos\theta_4\sin\theta_5 & \cos\theta_5 \end{pmatrix}$$

$$\mathcal{O}^2_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos\theta_6 & -\sin\theta_6 \\ 0 & 0 & \sin\theta_6 & \cos\theta_6 \end{pmatrix}.$$

Formula (8) for $\mathcal{O}_4$ takes the form

$$\mathcal{O}_4 = J_{3,4} \cdot J_{2,3} \cdot J_{1,2}$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos\theta_3 & -\sin\theta_3 \\ 0 & 0 & \sin\theta_3 & \cos\theta_3 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta_2 & -\sin\theta_2 & 0 \\ 0 & \sin\theta_2 & \cos\theta_2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\theta_1 & -\sin\theta_1 & 0 & 0 \\ \sin\theta_1 & \cos\theta_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

The Weyl form of a 2×2 unitary matrix is

$$W_2 = w_2^* d_2 w_2 = d_1^{1*} \mathcal{O}_2^t d_2 \mathcal{O}_2 d_1^1$$
$$= \begin{pmatrix} e^{i\varphi_1} \cos^2\theta + e^{i\varphi_2} \sin^2\theta & \cos\theta \sin\theta \, e^{i\varphi_3}(e^{i\varphi_1} - e^{i\varphi_2}) \\ \cos\theta \sin\theta \, e^{i\varphi_3}(e^{i\varphi_1} - e^{i\varphi_2}) & e^{i\varphi_2} \cos^2\theta + e^{i\varphi_1} \sin^2\theta \end{pmatrix}$$

where $d_2 = (e^{i\varphi_1}, e^{i\varphi_2})$, $d_1^1 = (1, e^{i\varphi_3})$ and $\mathcal{O}_2$ is the two-dimensional rotation matrix.

4. Concluding remarks

In this paper we proposed a new factorization of unitary matrices which can be useful in many domains of mathematical and theoretical physics. We suggest it to become the standard factorization since its form is given in terms of two-dimensional rotations and diagonal phase matrices and taking into account its recursive nature it will be more appropriate to design and implement software packages necessary for solving definite problems.

To see how the recursive property helps let us consider the simplest problem of the Laplace–Beltrami operator on the complex sphere S_{2n-1}. We start with the unit vector $|v_n\rangle = (e^{i\varphi_1}\cos\theta_1, \ldots, e^{i\varphi_n}\sin\theta_1 \cdots \sin\theta_{n-1}) \in S_{2n-1}$ and write it as

$$|v_n\rangle = e^{i\varphi_1}\cos\theta_1|1\rangle + \sin\theta_1|v_{n-1}\rangle$$

where $|1\rangle = (1, 0, \ldots, 0)$ and it is orthogonal to $|v_{n-1}\rangle$. We compute the Lagrangian

$$\mathcal{L}_n = \frac{1}{2}\sum_{i=1}^{n}(\dot{v}_n)_i(\bar{\dot{v}}_n)_i = \frac{1}{2}\langle \dot{v}_n|\dot{v}_n\rangle$$

where dot denotes the total derivative with respect to 'time' and bar denotes the complex conjugation, and we obtain

$$\mathcal{L}_n = \dot{\theta}_1^2 + \dot{\varphi}_1^2\cos^2\theta_1 + \sin^2\theta_1\mathcal{L}_{n-1}$$

and by iterating it one gets the Lagrangian and afterwards the Laplace operator that has the form

$$\Delta = \sum_{k=1}^{n-1}\frac{1}{\sin^2\theta_1\cdots\sin^2\theta_{k-1}\cos\theta_k\sin^{2(n-k)-1}\theta_k}\frac{\partial}{\partial\theta_k}\left(\cos\theta_k\sin^{2(n-k)-1}\theta_k\frac{\partial}{\partial\theta_k}\right)$$
$$+\sum_{k=1}^{n-1}\frac{1}{\sin^2\theta_1\cdots\sin^2\theta_{k-1}\cos^2\theta_k}\frac{\partial^2}{\partial\varphi_k^2} + \frac{1}{\sin^2\theta_1\cdots\sin^2\theta_{n-1}}\frac{\partial^2}{\partial\varphi_n^2}.$$

By using relation (8) written now in the form

$$\mathcal{O}_n = \mathcal{O}_{n-1}^1 J_{1,2}$$

and doing a similar calculation, we obtain the Laplace operator on the variety $\mathcal{O}_n$

$$\Delta = \sum_{k=1}^{n-1}\frac{\partial^2}{\partial\theta_k^2}$$

whose form shows that on this variety the Laplacian is diagonal. On the $SO(n)$ group the problem is more complicated since there are correlations between terms arising from different vectors parametrizing it, but within each vector the Laplacian is diagonal. We did not find the formula for arbitrary n but solved only the cases $SO(3)$ and $SO(4)$. As concerns the unitary group $U(n)$ the same problem appears even more complicated because of the phases and we solved only the $U(3)$ case. A complete treatment of such problems will be given elsewhere.

As concerns the complex Hadamard matrices by using our formalism we got for $n = 6$ three new nonequivalent matrices, the most interesting being

$$\frac{1}{\sqrt{6}}\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & \mathrm{i} & -\mathrm{i} & -\mathrm{i} & \mathrm{i} \\ 1 & \mathrm{i} & -1 & \mathrm{e}^{\mathrm{i}t} & -\mathrm{e}^{\mathrm{i}t} & -\mathrm{i} \\ 1 & -\mathrm{i} & -\mathrm{e}^{-\mathrm{i}t} & -1 & \mathrm{i} & \mathrm{e}^{-\mathrm{i}t} \\ 1 & -\mathrm{i} & \mathrm{e}^{-\mathrm{i}t} & \mathrm{i} & -1 & -\mathrm{e}^{-\mathrm{i}t} \\ 1 & \mathrm{i} & -\mathrm{i} & -\mathrm{e}^{\mathrm{i}t} & \mathrm{e}^{\mathrm{i}t} & -1 \end{pmatrix}$$

which depends on an arbitrary phase (Diţă 2002).

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