Analysis of the Finite Element Approximability of Three-Dimensional Time-Harmonic Electromagnetic Problems Involving Bianisotropic Materials and Metamaterials

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1 Introduction

2 Mathematical description of the problem

In this paper we are interested in electromagnetic problems that involves bianisotropic media under time-harmonic excitation, which was studied in [1]. While the full details of the problem definition and results are available in the reference, here we provide a summary of main points in order to ease the understanding of the present developments.

The problem is formulated in a domain $\Omega \in \mathbb{R}^3$ which has a boundary denoted by Γ. The time harmonic sources imply that all the resulting fields are in turn time-harmonic and the assumed factor $e^{j\omega t}$ is ubiquitous and is suppressed. The media involved in the problem is linear and time-invariant and is considered to be satisfying the following constitutive relations:

$$\begin{cases} \mathbf{D} = (1/c_0)\, P\,\mathbf{E} + L\,\mathbf{B} & \text{in } \Omega, \\ \mathbf{H} = M\,\mathbf{E} + c_0\, Q\,\mathbf{B} & \text{in } \Omega. \end{cases} \tag{1}$$

In the above equation, $\mathbf{E}$, $\mathbf{B}$, $\mathbf{D}$ and $\mathbf{H}$ are complex valued functions defined in Ω and represent, respectively, the electric field, magnetic induction, electric displacement, magnetic field and c_0 is the speed of light in vacuum. The space where we will seek $\mathbf{E}$ and $\mathbf{H}$ is [2] (p. 82; see also p. 69)

$$U = H_{L^2,\Gamma}(\mathrm{curl}, \Omega) = \{\mathbf{v} \in H(\mathrm{curl}, \Omega) \mid \mathbf{v} \times \mathbf{n} \in L_t^2(\Gamma)\}, \tag{2}$$

where [2] (p. 48)

$$L_t^2(\Gamma) = \{\mathbf{v} \in (L^2(\Gamma))^3 \mid \mathbf{v} \cdot \mathbf{n} = 0 \ \text{ almost everywhere on } \Gamma\}. \tag{3}$$

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Based on Maxwell's equations, boundary conditions and constitutive relations, the following variational formulation of the problem can be deduced [3]: given $\omega > 0$, electric and magnetic current densities $\mathbf{J}_e, \mathbf{J}_m \in (L^2(\Omega))^3$ and the known term $\mathbf{f}_R \in L^2_t(\Gamma)$, involved in admittance boundary condition, find $\mathbf{E} \in U$ such that

$$a(\mathbf{E}, \mathbf{v}) = l(\mathbf{v}) \quad \forall \mathbf{v} \in U, \tag{4}$$

where

$$\begin{aligned} a(\mathbf{u}, \mathbf{v}) = c_0 \big(Q \operatorname{curl} \mathbf{u}, \operatorname{curl} \mathbf{v}\big)_{0,\Omega} - \frac{\omega^2}{c_0} \big(P\, \mathbf{u}, \mathbf{v}\big)_{0,\Omega} - j\omega \big(M\, \mathbf{u}, \operatorname{curl} \mathbf{v}\big)_{0,\Omega} \\ - j\omega \big(L \operatorname{curl} \mathbf{u}, \mathbf{v}\big)_{0,\Omega} + j\omega \big(Y\, (\mathbf{n} \times \mathbf{u} \times \mathbf{n}), \mathbf{n} \times \mathbf{v} \times \mathbf{n}\big)_{0,\Gamma} \end{aligned} \tag{5}$$

and

$$l(\mathbf{v}) = -j\omega \big(\mathbf{J}_e, \mathbf{v}\big)_{0,\Omega} - c_0 \big(Q\, \mathbf{J}_m, \operatorname{curl} \mathbf{v}\big)_{0,\Omega} + j\omega \big(L\, \mathbf{J}_m, \mathbf{v}\big)_{0,\Omega} - j\omega \big(\mathbf{f}_R, \mathbf{n} \times \mathbf{v} \times \mathbf{n}\big)_{0,\Gamma}. \tag{6}$$

In [1] we derived certain sufficient conditions that guarantee the well posedness and finite element approximability of the problem. The developed theory was applied to problems involving rotating axisymmetric objects. In this paper we apply the theory to a wider range of problems involving bianisotropic media, demonstrating the generality of the developments and obtaining interesting new solutions. In particular we show how the theory can be applied in the presence of metamaterials by considering the equivalent media of the type discussed in literature [4].

It could be also useful to recall some points on how to check for the sufficient conditions that ensure the well posedness and finite element approximability of the problem. In particular it was observed that most of the hypotheses are easily verified for important practical problems. It turns out that the most critical conditions that need to be verified are conditions (9) and (10) in [1]. These conditions are restated in the following equations.

$$\text{for every } \mathbf{v} \in U, \mathbf{v} \neq 0, \quad \sup_{\mathbf{u} \in U} |a(\mathbf{u}, \mathbf{v})| > 0, \tag{7}$$

$$\text{we can find } \alpha : \inf_{\mathbf{u} \in U,\ \|\mathbf{u}\|_U = 1} \ \sup_{\mathbf{v} \in U,\ \|\mathbf{v}\|_U \leq 1} |a(\mathbf{u}, \mathbf{v})| \geq \alpha > 0. \tag{8}$$

As described in [1], to satisfy the above conditions, we need to verify HM9-HM15 in the paper. Before stating these critical hypotheses a few definitions need to be made. Ω is decomposed into m subdomains Ω_i, $i \in I = \{1, 2, ...m\}$. This decomposition can be made such that $I = I_a \cup I_b$, where I_a is characterized by subdomains where $L = M = 0$. Also, an alternative form of constitutive relations given in (9) is made use of to state some of the hypotheses [5].

$$\begin{cases} \mathbf{E} = \kappa\, \mathbf{D} + \chi\, \mathbf{B} & \text{in } \Omega, \\ \mathbf{H} = \gamma\, \mathbf{D} + \nu\, \mathbf{B} & \text{in } \Omega. \end{cases} \tag{9}$$

The local continuity of the tensors P, Q, L and M can be assumed in most practical problems and which allows the definiton of the following constants.

- $\exists C_L > 0$: $|(L \operatorname{curl} \mathbf{u}, \mathbf{v})_{0,\Omega}| \leq C_L \|\operatorname{curl} \mathbf{u}\|_{0,\Omega} \|\mathbf{v}\|_{0,\Omega}$ for all $\mathbf{u} \in H(\operatorname{curl}, \Omega)$ and $\mathbf{v} \in (L^2(\Omega))^3$,
- $\exists C_M > 0$: $|(M\, \mathbf{u}, \operatorname{curl} \mathbf{v})_{0,\Omega}| \leq C_M \|\mathbf{u}\|_{0,\Omega} \|\operatorname{curl} \mathbf{v}\|_{0,\Omega}$ for all $\mathbf{u} \in (L^2(\Omega))^3$ and $\mathbf{v} \in H(\operatorname{curl}, \Omega)$.

Now the important hypotheses are restated here and are renamed as H1-H7.

H1. $\exists\exists C_{\kappa,d} > 0$, $C_{\nu,d} > 0 : |determinant\,(\kappa)\,| \geq C_{\kappa,d}$, $|determinant\,(\nu)\,| \geq C_{\nu,d}$, $\forall \mathbf{x} \in \overline{\Omega}_i, \forall i \in I$,

H2. $\mathbf{l}_{1,3}^T\,\kappa^{-1}\,\mathbf{l}_{1,3} \neq 0,\ \mathbf{l}_{1,3}^T\,\nu^{-1}\,\mathbf{l}_{1,3} \neq 0\ \forall \mathbf{l}_{1,3} \in \mathbb{R}^3, \mathbf{l}_{1,3} \neq 0,\ \forall \mathbf{x} \in \overline{\Omega}_i,\ \forall i \in I_a,$

H3. $\exists\exists C_{\kappa,r} > 0,\ C_{\nu,r} > 0 :\ |\mathbf{l}_{1,3,n}^T\,\kappa^{-1}\,\mathbf{l}_{1,3,n}| \geq C_{\kappa,r},\ |\mathbf{l}_{1,3,n}^T\,\nu^{-1}\,\mathbf{l}_{1,3,n}| \geq C_{\nu,r}\ \forall \mathbf{l}_{1,3,n} \in \mathbb{R}^3 :$ $\|\mathbf{l}_{1,3,n}\|_2 = 1,\ \forall \mathbf{x} \in \overline{\Omega}_i,\ \forall i \in I_b,$

H4. $\exists\exists C_{\kappa,s} > 0, C_{\nu,s} > 0$:

$$\Big(\sum_{i,j=1}^{3} |\kappa_{ij}|\Big) - \min_{i=1,2,3} |\kappa_{ii}| \leq C_{\kappa,s} \quad \forall \mathbf{x} \in \overline{\Omega}_k, \forall k \in I_b, \tag{10}$$

$$\Big(\sum_{i,j=1}^{3} |\nu_{ij}|\Big) - \min_{i=1,2,3} |\nu_{ii}| \leq C_{\nu,s} \quad \forall \mathbf{x} \in \overline{\Omega}_k, \forall k \in I_b, \tag{11}$$

and κ, χ, γ and ν satisfy

$$\frac{4\left(\big(\sum_{i,j=1}^{3} |\gamma_{ij}|\big) - \min_{i=1,2,3} |\gamma_{ii}|\right)\left(\big(\sum_{i,j=1}^{3} |\chi_{ij}|\big) - \min_{i=1,2,3} |\chi_{ii}|\right)}{\left(-C_{\kappa,s} + \sqrt{C_{\kappa,s}^2 + 4\,C_{\kappa,d}\,C_{\kappa,r}}\right)\left(-C_{\nu,s} + \sqrt{C_{\nu,s}^2 + 4\,C_{\nu,d}\,C_{\nu,r}}\right)} < 1 \tag{12}$$

$\forall \mathbf{x} \in \overline{\Omega}_k,\ \forall k \in I_b.$

H5. We can find $C_{PS} > 0$ such that $|(P\mathbf{u}, \mathbf{u})_{0,\Omega}| \geq C_{PS}\|\mathbf{u}\|_{0,\Omega}^2$ for all $\mathbf{u} \in (L^2(\Omega))^3$.

H6. We can find $C_{QS} > 0$ such that $|(Q\mathrm{curl}\,\mathbf{u}, \mathrm{curl}\,\mathbf{u})_{0,\Omega}| \geq C_{QS}\|\mathrm{curl}\,\mathbf{u}\|_{0,\Omega}^2$ for all $\mathbf{u} \in H(\mathrm{curl}, \Omega)$.

H7. C_{PS}, C_{QS}, C_L and C_M (i.e., all media involved) are such that $C_{QS} - \frac{C_L C_M}{C_{PS}} > 0$.

The section "some hints to apply the developed theory" of [1], provided the guidance to use the theory that was developed. Lemma 1 of the above paper provides a procedure to check conditions H5 and H6 and thus estimate the constants C_{PS} and C_{QS} which may in turn be used to check H7. Let P be decomposed as $P = P_s - jP_{ss}$, and if P_{ss} is uniformly positive definite in some region Ω_{el} and P_s is uniformly positive definite in the complementary region $\Omega \setminus \Omega_{el}$, then H5 can be shown to be true in the following way. We can define $C_1 > 0$ and $C_2 > 0$ as follows.

$$\int_{\Omega_{el}} \mathbf{u}^* P_{ss}\mathbf{u} \geq C_1 \int_{\Omega el} |\mathbf{u}|^2 = C_1||\mathbf{u}||_{0,\Omega_{el}}^2\ \forall \mathbf{u} \in (L^2(\Omega))^3, \tag{13}$$

$$\left|\int_{\Omega\setminus\Omega_{el}} \mathbf{u}^* P_s\mathbf{u}\right| \geq C_2||\mathbf{u}||_{0,\Omega\setminus\Omega_{el}}^2. \tag{14}$$

The continuity of P_s in Ω_{el} allows the definition of $C_3 > 0$ such that

$$\left|\int_{\Omega_{el}} \mathbf{u}^* P_s\mathbf{u}\right| \leq C_3\|\mathbf{u}\|_{0,\Omega_{el}}^2. \tag{15}$$

If $\Omega_{el} = \Omega$, then we can take $C_{PS} = C_1$, if $\Omega_{el} = \emptyset$ then $C_{PS} = C_2$ and in other cases

$$C_{PS} = \frac{1}{\sqrt{2}} \min\left(\sqrt{(1-\alpha)}C_2, \sqrt{C_1^2 + (1 - \frac{1}{\alpha})C_3^2}\right), \tag{16}$$

where α is such that $1 > \alpha > \frac{C_3^2}{C_1^2 + C_3^2} > 0$.

Similar considerations can help to estimate C_{QS} as well.

The values obtained above may not the best possible ones than can be estimated and the condition in H7 can be made less restrictive if the estimates of C_{PS} and C_{QS} can be higher. For example, whenever P_s is uniformly definite in Ω, we have $C_4 > 0$ such that

$$\left| \int_\Omega \mathbf{u}^* P_s \mathbf{u} \right| \geq C_4 ||\mathbf{u}||^2_{0,\Omega}. \tag{17}$$

In this case, the best value for C_{PS} would be the larger among the one in (16) and C_4. Based on the definitions, the constants C_L and C_M can be estimated and the validity of H7 can be checked.

$$C_L = \max_{i \in I_b} \sup_{\mathbf{x} \in \Omega_i} \sqrt{\lambda_{max}(L^* L)} \tag{18}$$

and

$$C_M = \max_{i \in I_b} \sup_{\mathbf{x} \in \Omega_i} \sqrt{\lambda_{max}(M^* M)}, \tag{19}$$

As for the constants involved in H1-H4, the following considerations can be helpful.

$$C_{\kappa,d} = \min_{i \in I_b} \inf_{\mathbf{x} \in \Omega_i} |determinant(\kappa)|, \tag{20}$$

$$C_{\nu,d} = \min_{i \in I_b} \inf_{\mathbf{x} \in \Omega_i} |determinant(\nu)|, \tag{21}$$

$$C_{\kappa,s} = \max_{i \in I_b} \sup_{\mathbf{x} \in \Omega_i} \left((\sum_{i,j=1}^{3} |\kappa_{ij}|) - \min_{i=1,2,3} |\kappa_{ii}| \right), \tag{22}$$

$$C_{\nu,s} = \max_{i \in I_b} \sup_{\mathbf{x} \in \Omega_i} \left((\sum_{i,j=1}^{3} |\nu_{ij}|) - \min_{i=1,2,3} |\nu_{ii}| \right). \tag{23}$$

As for $C_{\kappa,r}$ and $C_{\nu,r}$ the following consideration might be helpful. By definition

$$C_{\kappa,r} = \min_{i \in I_b} \inf_{\mathbf{x} \in \Omega_i} \min_{\mathbf{l}_{1,3,n} \in \mathbb{R}^3 : ||\mathbf{l}_{1,3,n}||_2 = 1} \sqrt{\left(\mathbf{l}^T_{1,3,n} \kappa_{is} \mathbf{l}_{1,3,n}\right)^2 + \left(\mathbf{l}^T_{1,3,n} \kappa_{iss} \mathbf{l}_{1,3,n}\right)^2}, \tag{24}$$

$$C_{\nu,r} = \min_{i \in I_b} \inf_{\mathbf{x} \in \Omega_i} \min_{\mathbf{l}_{1,3,n} \in \mathbb{R}^3 : ||\mathbf{l}_{1,3,n}||_2 = 1} \sqrt{\left(\mathbf{l}^T_{1,3,n} \nu_{is} \mathbf{l}_{1,3,n}\right)^2 + \left(\mathbf{l}^T_{1,3,n} \nu_{iss} \mathbf{l}_{1,3,n}\right)^2}, \tag{25}$$

where κ_{is} and κ_{iss} are the symmetric matrices obtained by the usual decomposition of κ^{-1} and similarly ν_{is} and ν_{iss} are those corresponding to ν^{-1}. If both the symmetric matrices involved in the above expressions are semi-definite, then we can deduce the following lower bounds:

$$C_{\kappa,r} = \min_{i \in I_b} \inf_{\mathbf{x} \in \Omega_i} \sqrt{(\lambda_{min}(\kappa_{is}))^2 + (\lambda_{min}(\kappa_{iss}))^2}, \tag{26}$$

$$C_{\nu,r} = \min_{i \in I_b} \inf_{\mathbf{x} \in \Omega_i} \sqrt{(\lambda_{min}(\nu_{is}))^2 + (\lambda_{min}(\nu_{iss}))^2}. \tag{27}$$

If we also define

$$C_{\chi,s} = \max_{i \in I_b} \sup_{\mathbf{x} \in \Omega_i} \left((\sum_{i,j=1}^{3} |\chi_{ij}|) - \min_{i=1,2,3} |\chi_{ii}| \right), \tag{28}$$

$$C_{\gamma,s} = \max_{i \in I_b} \sup_{\mathbf{x} \in \Omega_i} \left((\sum_{i,j=1}^{3} |\gamma_{ij}|) - \min_{i=1,2,3} |\gamma_{ii}| \right), \tag{29}$$

the sufficient condition for the regularity used for proving uniqueness can be expressed as

$$K_u = \frac{4C_{\chi,s}C_{\gamma,s}}{\left(-C_{\kappa,s} + \sqrt{C_{\kappa,s}^2 + 4C_{\kappa,d}C_{\kappa,r}}\right)\left(-C_{\nu,s} + \sqrt{C_{\nu,s}^2 + 4C_{\nu,d}C_{\nu,r}}\right)} < 1. \tag{30}$$

3 Results and discussion

In this section we apply the theory developed in [1] to several different class to problems which could not be managed with the previous theory like the one in [3]. The conditions are established on the parameters of such problems, under which the well posedness and finite element approximability can be guaranteed. Under such condition, the numerical solutions for the fields are computed for the first time. The details of our finite element simulator is the same as that described in Section 5 of [1]. In particular let us first consider the class of problems involving materials described in Kraft et al. [4].

3.0.1 Plasmonic gratings considered in [4] behaving as bianisotropic metamaterial

In [4], Kraft et. al consider a plasmonic grating which exhibits bianisotropy at visible wavelength. Here we consider scattering problems involving an equivalent medium that can be characterized by the constitutive matrices of the same form given in the paper. The region occupied by the scatterer may be denoted as $\Omega_s \subset \Omega$. We start with the form of the normalized constitutive relation as found in [6] and [7], and obtain the P, Q, L, M matrices [5]. The final form given below is obtained by considering non magnetic material with $\mu_r = 1$ and with an isotropic complex relative permittivity ε_r. The magnetoelectric coupling parameter is denoted ζ_0.

$$P = c_0\varepsilon_0 \begin{bmatrix} \varepsilon_r & 0 & 0 \\ 0 & \varepsilon_r & 0 \\ 0 & 0 & \varepsilon_r - \zeta_0^2 \end{bmatrix} \tag{31}$$

$$Q = \frac{1}{c_0\mu_0} I \tag{32}$$

$$L = M^T = \frac{j\zeta_0}{\mu_0 c_0} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \tag{33}$$

The complementary region $\Omega \setminus \Omega_s$ is occupied by the empty space, which, in our notation, is characterized by $P = c_0\varepsilon_0 I_3$, $Q = \frac{1}{c_0\mu_0} I_3$, $L = M = 0$.

The bianisotropic medium is lossy with the imaginary part $Im(\varepsilon_r) < 0$, where as ζ_0 is assumed real here to avoid some longer calculation. Now Lemma 1 of [1] can be applied to verify hypothesis H5. Inside Ω_s, P can be decomposed as $P = P_s - jP_{ss}$ with

$$P_s = \frac{P + P^*}{2} = c_0\varepsilon_0 \begin{bmatrix} Re(\varepsilon_r) & 0 & 0 \\ 0 & Re(\varepsilon_r) & 0 \\ 0 & 0 & Re(\varepsilon_r) - \zeta_0^2 \end{bmatrix}, \tag{34}$$

and $P_{ss} = \frac{P^* - P}{2j} = -c_0\varepsilon_0 Im(\varepsilon_r) I_3$. Hence we have $\Omega_{el} = \Omega_s$, the lossy region where P_{ss} is uniformly positive definite and the complementary region with free space where P_s is uniformly positive definite. This means that the conditions of relevant Lemma are satisfied and as a result H5 is valid.

From the definitions $C_1 = c_0\varepsilon_0|Im(\varepsilon_r)|$, $C_2 = c_0\varepsilon_0$ and $C_3 = c_0\varepsilon_0 \max(|Re(\varepsilon_r)-\zeta_0^2|, |Re(\varepsilon_r)|)$. To find the minimum of the two expressions in (16), we note that, in the valid range, the value of the first expression decreases monotonically with α where as that of the second expression increases with it. The highest estimate for C_{PS} is obtained when the two expressions have the same value. The value of α at which this happens can be evaluated by equating the two expressions and finding the positive root of the resulting quadratic equation. This value of α, denoted as α_{opt} is given by (35).

$$\alpha_{opt} = \frac{C_2^2 - C_1^2 - C_3^2 + \sqrt{(C_2^2 - C_1^2 - C_3^2)^2 + 4C_2^2C_3^2}}{2C_2^2} \tag{35}$$

Thus we may simply write

$$C_{PS} = \sqrt{\frac{1-\alpha_{opt}}{2}}c_0\varepsilon_0. \tag{36}$$

As mentioned before, this does not mean that a better value of C_{PS} cannot be found. For example, if $Re(\varepsilon_r) - \frac{\zeta_0^2}{\mu_r} > 0$, then P_s is uniformly positive definite in Ω and we can find another candidate for C_{PS}, namely $C_4 = c_0\varepsilon_0 \min(1, |Re(\varepsilon_r -)\zeta_0^2|$. In particular, when $Re(\varepsilon_r) - \zeta_0^2 > 1$, $C_4 = c_0\varepsilon_0$ is always going to give a value for C_{PS} which is higher than that obtained from the Lemma.

The generality of out theory be clearly demonstrated by showing its applicability to bianisotropic metamaterials. Hence in the rest of the subsection the discussion focuses on the cases with $\varepsilon_r < 0$. For this case,$C_3 = c_0\varepsilon_0|Re(\varepsilon_r) - \zeta_0^2|$, and we can directly use the value of C_{PS} in (36). Since the material is assumed to be non magnetic the direct application of definition gives $C_{QS} = \frac{1}{c_0\mu_0}$. Likewise, the continuity constants $C_L = C_M = \frac{|\zeta_0|}{c_0\mu_0}$. Then the inequality in H7 becomes $C_{QS} - \frac{C_LC_M}{C_{PS}} = \frac{1}{c_0\mu_0}(1 - \frac{\zeta_0^2\sqrt{2}}{\sqrt{1-\alpha_{opt}}}) > 0$, which gives (37).

$$|\zeta_0| < \left(\frac{1-\alpha_{opt}}{2}\right)^{1/4}. \tag{37}$$

Since the right hand side of (37) also depends on ζ_0 due to the presence of C_3 in the expression for α_{opt}, we do not have a closed form expression on the limit on ζ_0 below which H7 is satisfied. However a graphical analysis can be done for estimating such a limit on $|\zeta_0|$, as shown in Figure 1. The value of $Re(\varepsilon_r)$ is varied in the range $(-1.0, -5.0)$ whereas $Im(\varepsilon_r)$ assumes values in the range $(-0.1, -0.5)$. It can be observed that for a fixed value of $Im(\varepsilon_r)$, the range of ζ_0 over which H7 is valid steadily decreases as $|Re(\varepsilon_r)|$ increases. As for the dependence on $Im(\varepsilon_r)$, the corresponding range increases when medium becomes more lossy due to higher $|Im(\varepsilon_r)|$, as expected.

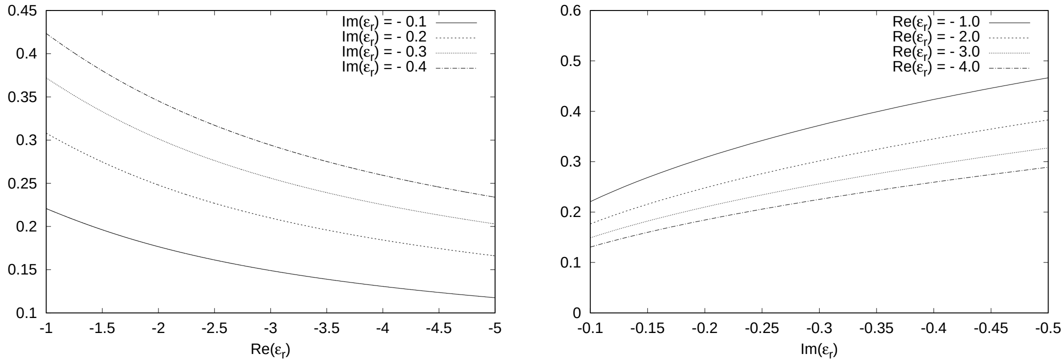

Figure 1: The magnitude of $|\zeta_0|$ below which the condition 8 is satisfied as a function of $Re(\varepsilon_r)$ or $Im(\varepsilon_r)$ for media in Kraft et al. The plot is for real ζ_0 negative values of $Re(\varepsilon_r)$ or $Im(\varepsilon_r)$ and the material is taken to be non magnetic.

We consider the alternative form of constitutive relations , which for the medium inside Ω_s becomes as in equations (38) to (40), for examining the validity of H1-H4 [5].

$$\kappa = \frac{1}{\varepsilon_0 \varepsilon_r} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{\varepsilon_r}{\varepsilon_r - \zeta_0^2} \end{bmatrix} \tag{38}$$

$$\nu = \frac{1}{\mu_0} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{\varepsilon_r}{\varepsilon_r - \zeta_0^2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{39}$$

$$\gamma = -\chi^T = \frac{j\zeta_0 c_0}{\varepsilon_r - \zeta_0^2} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \tag{40}$$

Since H1-H4 needs to hold only locally, and they are trivially valid in the empty space outside the bianisotropic media, we have to just analyze them inside Ω_s occupied by the bianisotropic medium. These constants can be evaluated directly from the definitions. The determinants of κ and ν are, respectively, $\frac{1}{(\varepsilon_0 \varepsilon_r)^3 (1 - \frac{\zeta_0^2}{\varepsilon_r})}$ and $\frac{1}{(\mu_0)^3 (1 - \frac{\zeta_0^2}{\varepsilon_r})}$ which immediately give the values of $C_{\kappa,d}$ and $C_{\nu,d}$.

$$C_{\kappa,d} = \frac{1}{|\varepsilon_0^3 \varepsilon_r^3 (1 - \frac{\zeta_0^2}{\varepsilon_r})|}, \tag{41}$$

$$C_{\nu,d} = \frac{1}{|\mu_0^3 (1 - \frac{\zeta_0^2}{\varepsilon_r})|}. \tag{42}$$

The inverses of the diagonal matrices κ and ν are just the diagonal matrices with the reciprocal entries. Applying equations (26) and (27) gives the values of $C_{\kappa,r}$ and $C_{\nu,r}$.

$$C_{\kappa,r} = |\varepsilon_0 \varepsilon_r (1 - \frac{\zeta_0^2}{\varepsilon_r})|, \tag{43}$$

$$C_{\nu,r} = |\mu_0(1 - \frac{\zeta_0^2}{\varepsilon_r})|. \tag{44}$$

Using equations (26) and (27) we get the $C_{\kappa,s}$ and $C_{\nu,s}$.

$$C_{\kappa,s} = \frac{1}{\varepsilon_0}\max(\frac{2}{|\varepsilon_r|}, \frac{1}{|\varepsilon_r|}(1 + \frac{1}{|1 - \frac{\zeta_0^2}{\varepsilon_r}|})), \tag{45}$$

$$C_{\nu,s} = \frac{1}{\mu_0}\max(2, (1 + \frac{1}{|1 - \frac{\zeta_0^2}{\varepsilon_r}|})), \tag{46}$$

Since we are interested in the case when $Re(\varepsilon_r) < 0$ and real ζ_0, we see that $|1 - \frac{\zeta_0^2}{\varepsilon_r}| > 1$, and the maximum reduces to the values in (47) and (48).

$$C_{\kappa,s} = \frac{2}{|\varepsilon_0\varepsilon_r|}, \tag{47}$$

$$C_{\nu,s} = \frac{2}{\mu_0}. \tag{48}$$

Using equations (28) and (29) we can easily evaluate $C_{\gamma,s}$ and $C_{\chi,s}$.

$$C_{\gamma,s} = C_{\chi,s} = |\frac{\zeta_0 c_0}{\varepsilon_r - \zeta_0^2}|. \tag{49}$$

Using these constants, the value of K_u can be calculated using equation (30). The critical value of $|\zeta_0|$ below which the condition in H4 is satisfied is plotted in Figure 2, with respect to either $Re(\varepsilon_r)$ or $Im(\varepsilon_r)$. The results show that the range of ζ_0 for which H4 holds true increases with the increase in $|Re(\varepsilon_r)|$, while it is practically independent of $Im(\varepsilon_r)$.

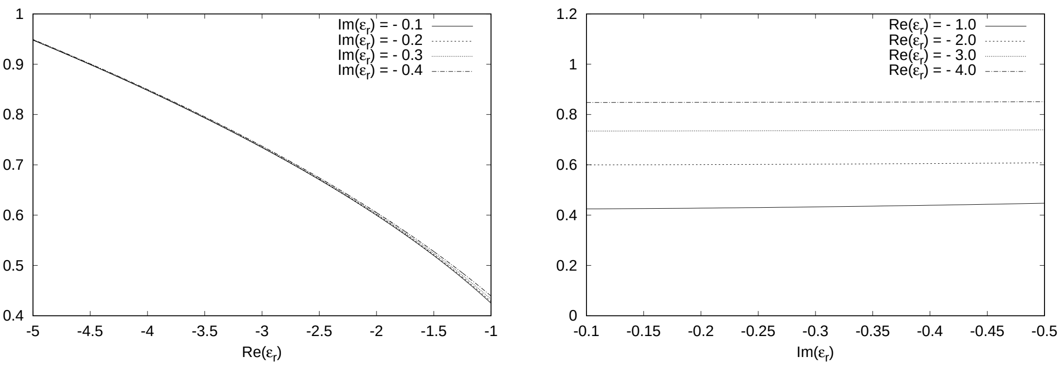

Figure 2: The magnitude of ζ_0 below which the condition 7 is satisfied as a function of $Re(\varepsilon_r)$ or $Im(\varepsilon_r)$ for media in Kraft et al. The plot is for real ζ_0 and the material is taken to be non magnetic.

Let us try to understand the implications of the theory by applying it to the numerical solution of a specific problem involving the medium of interest. We consider the region with the scatterer Ω_s to be a cube filled with homogeneous bianisotropic media. The surrounding region is filled with empty space and the overall domain of numerical investigation, Ω, is of cubic shape as well, and is concentric to Ω_s. In the following Ω is characterized by sides of length 2 μm and Ω_s by those of 0.8 μm. The axes are taken along the sides of the cubes and the excitation is with a plane wave incident along the x axis, with electric field polarized along z axis, having a magnitude of 1 V/m and wavelength of 1 μm.

Inside Ω_s, the medium is characterized by $\varepsilon_r = -1 - j0.4$, $\mu_r = 1$ and $\zeta_0 = -0.41$. This value is such that the hypotheses required for well posedness and finite element approximability are satisfied. In fact, for the ε_r considered, condition 7 is valid for $|\zeta_0| < 0.4235$ and condition 8 is valid for $|\zeta_0| < 0.4393$.

The solutions are obtained with a first order edge element based Galerkin finite element method. The boundary condition is enforced with Y equal to the admittance of vacuum and with an inhomogeneous term $\mathbf{f}_R$, taking into account the incident field.

The domain is discretized uniformly using tetrahedral meshes. The meshing is done by first dividing the domain into small identical cubes, each of which is in turn divided into six tetrahedra. The parameter h denotes the maximum diameter of all the elements of the mesh [8] (p. 131), and in this case it is simply given by the side of the small cube times $\sqrt{3}$. To study the convergence of solution, we consider different levels of refinement of meshes ranked in order of h, ranging from "very coarse" to "very fine". For exmaple the mesh denoted very coarse is characterized by a cubes of sides 800 nm and the resulting mesh has 1331 nodes, 6000 tetrahedral elements and 1200 boundary faces. A summary of the information related to the four different refinements of the meshes that was used is given in Table 1.

Type of Mesh	**Maximum Diameter of the Mesh (h in nm)**	**Number of Nodes**	**Number of Elements**	**Number of Boundary Faces**
Very coarse	$200\sqrt{3}$	1331	6000	1200
Coarse	$100\sqrt{3}$	9261	48000	4800
Fine	$50\sqrt{3}$	68921	384000	19200
Very fine	$25\sqrt{3}$	531441	3072000	76800

Table 1: Details of different meshes used.

The results related to the stability of the simulations are shown in Figure 3. The difference between sucessive refinements progressively decreases and the fine and very fine meshes give solution which are stable. This confirms the well posedness and convergnce result that was predicted using the theory.

The magnitudes and pahses of some for the components of the field along different axis directions are shown in Figures 4 to 7. The solutions obtained for $\zeta_0 = -0.41$ are compared with that for the isotropic case ($\zeta_0 = 0$, $\varepsilon_r = -1 - j0.4$, $\mu_r = 1$). For example we can see in Figures 4 and 7 that there are differences of 10 to 20 percent of the incident field in the magnitudes of the electric fields which are induced by the magneto electric coupling factor ζ_0. Likewise the phases of the fields are similarly affected by the bianisotropy of the medium. These non negligible effects imply that the accuracy of the simulations requires the us to consider the bianisotropy of the medium. Hence the reliability of the finite element silution in the presence of bianisotropy is important for getting good results for this kind of problems. The appliaciton of our theory gives the conditions under which we can guarantee such reliability.

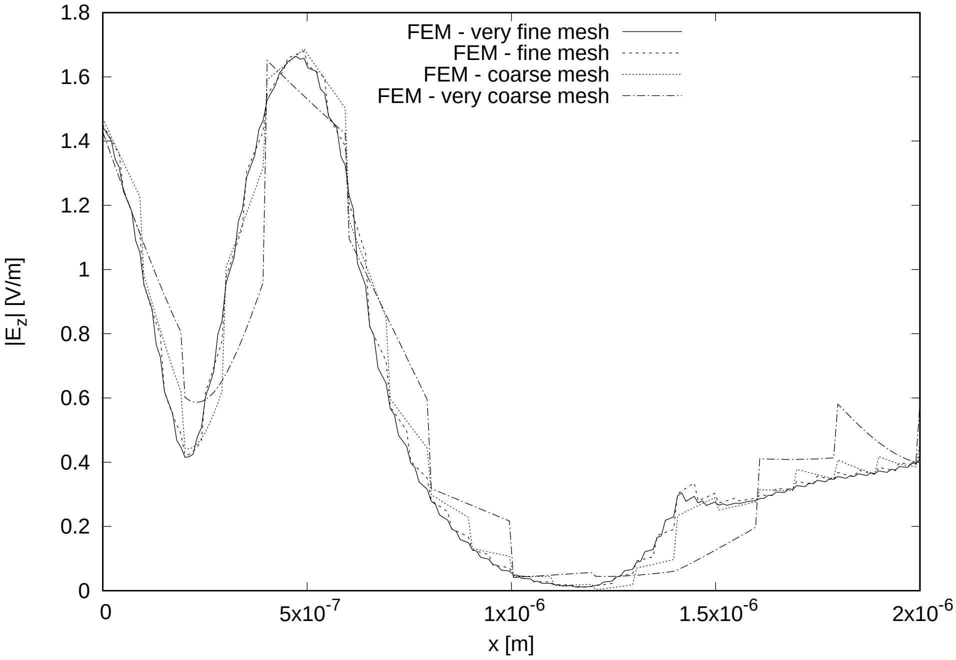

Figure 3: Convergence of the solution for problem involving medium of Kraft et al. The magnitude of the z component of the electric field is plotted along a line parallel to x axis for four different meshes.

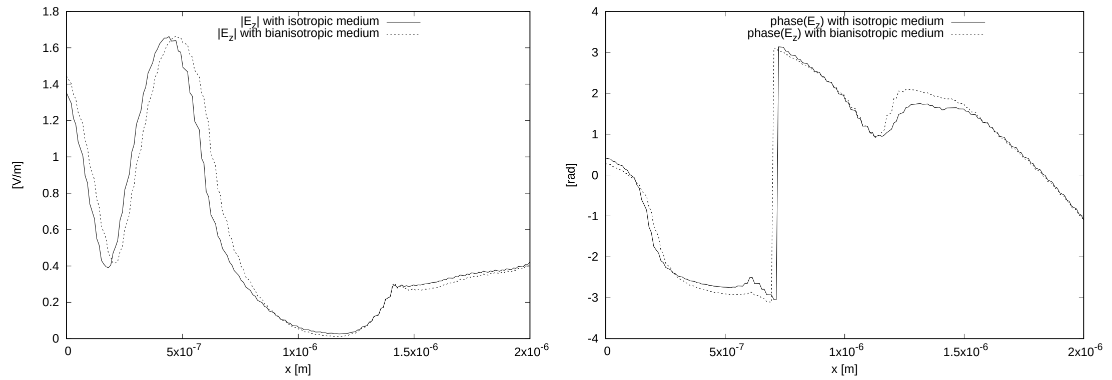

Figure 4: The magnitude and phase of the z component of electric field along a line parallel to x axis and passing though the center of gravity of the domain for problem involving medium of Kraft et al. The plots for bianisotropic case using $\zeta_0 = -0.41$ is compared with the solution obtained in isotropic case using $\zeta_0 = 0$.

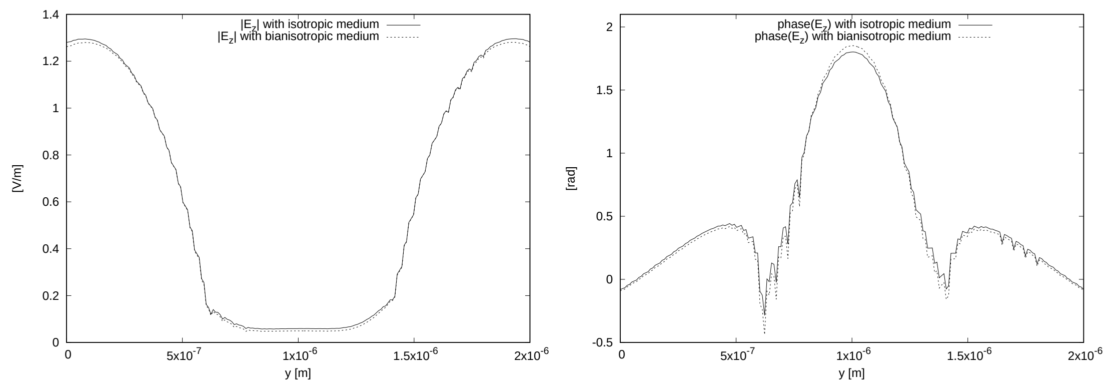

Figure 5: The magnitude and phase of the z component of electric field along a line parallel to y axis and passing though the center of gravity of the domain for problem involving medium of Kraft et al. The plots for bianisotropic case using $\zeta_0 = -0.41$ is compared with the solution obtained in isotropic case using $\zeta_0 = 0$.

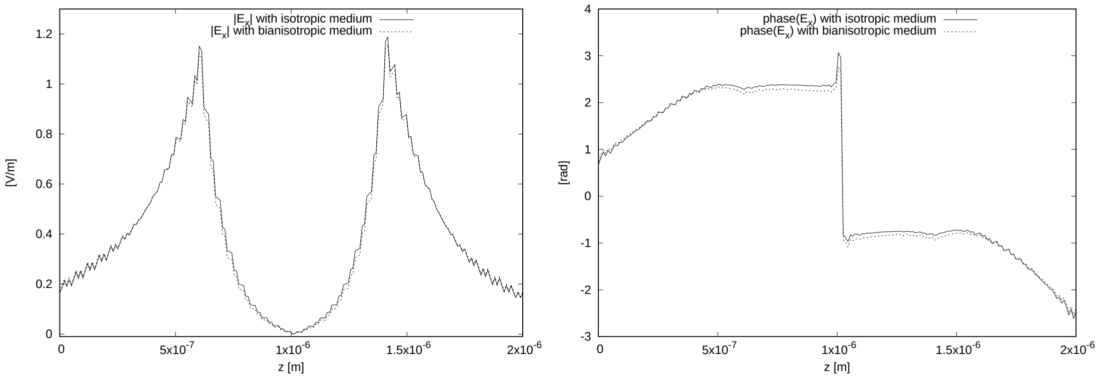

Figure 6: The magnitude and phase of the x component of electric field along a line parallel to z axis and passing though the center of gravity of the domain for problem involving medium of Kraft et al. The plots for bianisotropic case using $\zeta_0 = -0.41$ is compared with the solution obtained in isotropic case using $\zeta_0 = 0$.

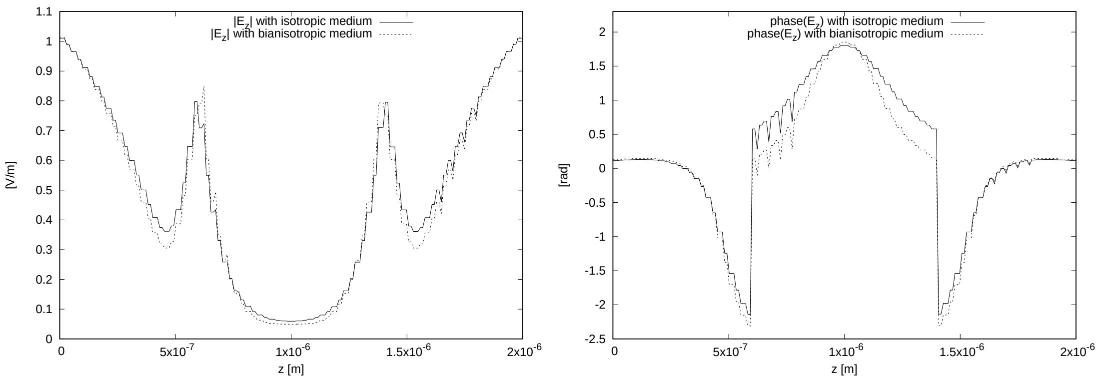

Figure 7: The magnitude and phase of the z component of electric field along a line parallel to z axis and passing though the center of gravity of the domain for problem involving medium of Kraft et al. The plots for bianisotropic case using $\zeta_0 = -0.41$ is compared with the solution obtained in isotropic case using $\zeta_0 = 0$.

3.1 Chirowaveguides considered in [9]

In [9], the authors consider a waveguide partially filled with a chiral medial which is characterized by

$$\left\{ \begin{array}{l} \mathbf{D} = \varepsilon_0 \varepsilon_r I_3 \mathbf{E} - j\xi_c I_3 \mathbf{B} \\ \mathbf{H} = -j\xi_c I_3 \mathbf{E} + \frac{1}{\mu_0 \mu_r} I_3 \mathbf{B} \end{array} \right. \tag{50}$$

4 Conclusions

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