Numerical results section

Praveen Kalarickel Ramakrishnan*   Mirco Raffetto†

February 13, 2020

1 Practical Implications

1.1 Other bianisotropic media

The theory developed in our paper can be used to prove the reliability of finite element solver for many problems involving bianisotropic media. This includes many scenarios which can not be managed by previous results [1]. In particular let us consider the class of problems involving materials described in Kraft et al. [2].

1.1.1 Bianisotropic metamaterial considered by Kraft et al.

In [2], Kraft et al. consider an equivalent bianisotropic media which can be characterized by the constitutive matrices which is lossy and bianisotropic. To illustrate the main principles, let us limit to the case where ε_r and μ_r are isotropic and hence represented by complex numbers. We adopt normalized quantities as in [3] and [4]. The resulting constitutive matrices are given in equations 70 to 72.

$$P = c_0 \varepsilon_0 \varepsilon_r \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 - \frac{\zeta_0^2}{\varepsilon_r \mu_r} \end{bmatrix} \tag{70}$$

$$Q = \frac{1}{c_0 \mu_0 \mu_r} I \tag{71}$$

$$L = M^T = \frac{j\zeta_0}{\mu_0 \mu_r c_0} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \tag{72}$$

The symmetric parts of the P matrix are given by

$$P_s = c_0 \varepsilon_0 \begin{bmatrix} Re(\varepsilon_r) & 0 & 0 \\ 0 & Re(\varepsilon_r) & 0 \\ 0 & 0 & Re(\varepsilon_r - \frac{\zeta_0^2}{\mu_r}) \end{bmatrix} \tag{73}$$

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*Department of Electrical, Electtronic, Telecommunications Engineering and Naval Architecture, University of Genoa, Via Opera Pia 11a, I-16145, Genoa, Italy, email:pravin.nitc@gmail.com

†Department of Electrical, Electtronic, Telecommunications Engineering and Naval Architecture, University of Genoa, Via Opera Pia 11a, I-16145, Genoa, Italy, email:raffetto@dibe.unige.it

$$P_{ss} = c_0\varepsilon_0 \begin{bmatrix} Im(\varepsilon_r) & 0 & 0 \\ 0 & Im(\varepsilon_r) & 0 \\ 0 & 0 & Im(\varepsilon_r - \frac{\zeta_0^2}{\mu_r}) \end{bmatrix} \tag{74}$$

The materials considered do satisfy $Im(\varepsilon_r) \leq 0$, $Im(\mu_r) \leq 0$ and $Im(\zeta_0) \leq 0$ and are passive as is confirmed from Figure 5 of [2] (the authors there use the opposite time convention). Hence if we have some loss due to any of these parameters, then P_{ss} is negative definite. Let us consider scattering problems in which such a medium is surrounded by air. In this case the region occupied by the material of this type may be denoted Ω_{el} and $\Omega \setminus \Omega_{el}$ has obviously got a P_s which is positive definite.

We may apply Lemma 1 (*Need to modify Lemma 1 to include the case when P_s is positive def over whole region*) to obtain the constant C_{PS}. P_s is positive definite inside Ω_{el} if $Re(\varepsilon_r) > 0$ and $Re(\varepsilon_r - \frac{\zeta_0^2}{\mu_r}) > 0$. In this case C_{PS} can simply be written as

$$C_{PS} = c_0\varepsilon_0 \min(1, Re(\varepsilon_r), Re(\varepsilon_r - \frac{\zeta_0^2}{\mu_r})) \text{ if } Re(\varepsilon_r) > 0 \text{ and } Re(\varepsilon_r - \frac{\zeta_0^2}{\mu_r}) > 0. \tag{75}$$

Here $K_0 = \min(1, Re(\varepsilon_r), Re(\varepsilon_r - \frac{\zeta_0^2}{\mu_r}))$, $K_1 = \min(|Im(\varepsilon_r - \frac{\zeta_0^2}{\mu_r})|, |Im(\varepsilon_r)|)$. When $Re(\varepsilon_r) < 0$ and $Re(\varepsilon_r - \frac{\zeta_0^2}{\mu_r}) < 0$, P_s is negative definite inside Ω_{el} and positive definite outside. In this case we have

$$C_1 = c_0\varepsilon_0 \min(|Im(\varepsilon_r - \frac{\zeta_0^2}{\mu_r})|, |Im(\varepsilon_r)|), \tag{76}$$

$$C_3 = c_0\varepsilon_0 \max(|Re(\varepsilon_r - \frac{\zeta_0^2}{\mu_r})|, |Re(\varepsilon_r)|), \tag{77}$$

$$C_5 = c_0\varepsilon_0. \tag{78}$$

We can denote $K_1 = \min(|Im(\varepsilon_r - \frac{\zeta_0^2}{\mu_r})|, |Im(\varepsilon_r)|)$ and $K_3 = \max(|Re(\varepsilon_r - \frac{\zeta_0^2}{\mu_r})|, |Re(\varepsilon_r)|)$. Then using equation (25) we can find the value of C_{PS} as in equation (78).

$$C_{PS} = \frac{1}{\sqrt{2}} c_0\varepsilon_0 \min\left(\sqrt{1-\alpha}, \sqrt{K_1^2 + (1-\frac{1}{\alpha})K_3^2}\right) \text{ if } Re(\varepsilon_r) < 0 \text{ and } Re(\varepsilon_r - \frac{\zeta_0^2}{\mu_r}) < 0. \tag{79}$$

The first entry in the above expression for the minimum decreases with α and the second one decreases with α over the range of interest. The optimal value is obtained when the two quantities are equal which can be calculated as the positive root of the resulting quadratic equation and is given by

$$\alpha_{opt} = \frac{(1 - (K_1^2 + K_3^2)) + \sqrt{(1 - (K_1^2 + K_3^2))^2 + 4K_3^2}}{2} \tag{80}$$

If we define η_{opt} as

$$\eta_{opt} = \begin{cases} \min(1, Re(\varepsilon_r), Re(\varepsilon_r - \frac{\zeta_0^2}{\mu_r})) & \text{if } Re(\varepsilon_r) > 0 \text{ and } Re(\varepsilon_r - \frac{\zeta_0^2}{\mu_r}) > 0, \\ \sqrt{\frac{1-\alpha_{opt}}{2}} & \text{if } Re(\varepsilon_r) < 0 \text{ and } Re(\varepsilon_r - \frac{\zeta_0^2}{\mu_r}) < 0. \end{cases} \tag{81}$$

the above results for C_{PS} can be summarized as

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$$C_{PS} = c_0 \varepsilon_0 \eta_{opt} \tag{82}$$

We can assume that $Re(\mu_r) > 1$ as is the case for the medium considered in [2], and then the other constants are easy to calculate. We get $C_{QS} = \frac{1}{c_0 \mu_0 |\mu_r|}$ and $C_L = C_M = \frac{|\zeta_0|}{c_0 \mu_0 |\mu_r|}$. Then the inequality in HM15 for proving condition (10) can be written as $C_{QS} - \frac{C_L C_M}{C_{PS}} = \frac{1}{c_0 \mu_0 |\mu_r|}(1 - \frac{|\zeta_0|^2}{\eta_{opt}}) > 0$ which becomes

$$\frac{|\zeta_0|}{\sqrt{\eta_{opt}}} < 1 \tag{83}$$

In general, since η_{opt} is itself dependent on ζ_0, it is not easy to immediately obtain a bound for ζ_0 in closed form. But to get some indication, we consider some particular cases. For this we take ζ_0 to be real and $\mu_r = 1$ and study the maximum allowable $|\zeta_0|$ for which condition (10) is guaranteed, for different values of ϵ_r. In this case if $Re(\varepsilon_r) > 0$ and $Re(\varepsilon_r - \zeta_0^2) = Re(\varepsilon_r) - \zeta_0^2 > 0$, equation (83) gives two conditions. First we get that $|\zeta_0| < 1$ if $\varepsilon_r > 2$ (in this case both $\varepsilon_r > 1$ and $\varepsilon_r - \zeta_0^2 > 1$). Second if $0 < Re(\varepsilon_r) < 2$ then the condition becomes $|\zeta_0| < \sqrt{\frac{Re(\varepsilon_r)}{2}}$. Finally if $Re(\varepsilon_r) < 0$, $K_1 = |Im(\varepsilon_r)|$ and $K_3 = |Re(\varepsilon_r)| + \zeta_0^2$. Then the dependence of maximum $|\zeta_0|$ on ε_r can be evaluated numerically and the result is shown in Figure 1 which follow expected trends. For fixed value of $Im(\varepsilon_r)$, as $|Re(\varepsilon_r)|$ increases, the range of ζ_0 for which condition (10) is guaranteed decreases. On the other hand, for a fixed value of $Re(\varepsilon_r)$, the magnitude of ζ_0 for which the condition is satisfied clearly increases with $|Im(\varepsilon_r)|$.
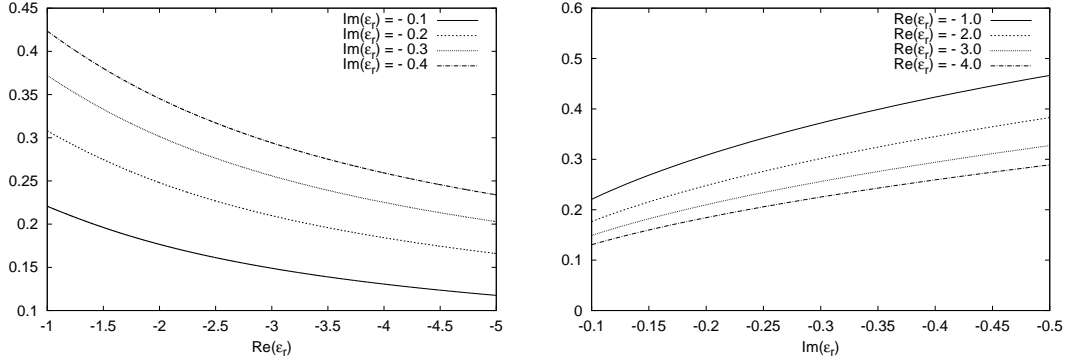

Figure 1: The magnitude of ζ_0 below which the condition (10) is satisfied as a function of $Re(\varepsilon_r)$ or $Im(\varepsilon_r)$ for media in Kraft et al. The plot is for real ζ_0 negative values of $Re(\varepsilon_r)$ or $Im(\varepsilon_r)$ and the material is taken to be non magnetic.

For checking condition (9) let us represent the constitutive matrices in the alternative form in equations (84), (85) and (86).

$$\kappa = \frac{1}{\varepsilon_0 \varepsilon_r} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{\varepsilon_r \mu_r}{\varepsilon_r \mu_r - \zeta_0^2} \end{bmatrix} \tag{84}$$

$$\nu = \frac{1}{\mu_0 \mu_r} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{\varepsilon_r \mu_r}{\varepsilon_r \mu_r - \zeta_0^2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{85}$$

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$$\gamma = -\chi^T = \frac{j\zeta_0 c_0}{\varepsilon_r \mu_r - \zeta_0^2} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \tag{86}$$

The following constants can be evaluated directly using the definitions:

$$C_{\kappa,d} = \frac{1}{\varepsilon_0^3} \min(1, \frac{1}{|\varepsilon_r^3(1 - \frac{\zeta_0^2}{\varepsilon_r \mu_r})|}), \tag{87}$$

$$C_{\nu,d} = \frac{1}{\mu_0^3} \min(1, \frac{1}{|\mu_r^3(1 - \frac{\zeta_0^2}{\varepsilon_r \mu_r})|}), \tag{88}$$

$$C_{\kappa,r} = \varepsilon_0 \min(1, |\varepsilon_r(1 - \frac{\zeta_0^2}{\varepsilon_r \mu_r})|), \tag{89}$$

$$C_{\nu,r} = \mu_0 \min(1, |\mu_r(1 - \frac{\zeta_0^2}{\varepsilon_r \mu_r})|), \tag{90}$$

$$C_{\kappa,s} = \frac{1}{\varepsilon_0} \max(2, \frac{2}{|\varepsilon_r|}, \frac{1}{|\varepsilon_r|}(1 + \frac{1}{|1 - \frac{\zeta_0^2}{\varepsilon_r \mu_r}|})), \tag{91}$$

$$C_{\nu,s} = \frac{1}{\mu_0} \max(2, \frac{2}{|\mu_r|}, \frac{1}{|\mu_r|}(1 + \frac{1}{|1 - \frac{\zeta_0^2}{\varepsilon_r \mu_r}|})), \tag{92}$$

$$C_{\gamma,s} = C_{\chi,s} = |\frac{\zeta_0 c_0}{\varepsilon_r \mu_r - \zeta_0^2}|. \tag{93}$$

After evaluating these constants, equation (41) gives the criteria for satisfying condition (9). To get an indication of the trends, let us specialize these evaluations for the case when ζ_0 is real and $\mu_r = 1$. The dependence of the maximum allowable magnitude of ζ_0 that guarantees condition (9) on ε_r is shown in Figure 2. For $|Re(\varepsilon_r)| > 1$, the allowable ζ_0 value falls as $|Re(\varepsilon_r)|$ increases and is largely independent of $Im(\varepsilon_r)$. The maximum permissible $|\zeta_0|$ of 0.4126 is reached when $Re(\varepsilon_r) = -1$ on the negative side, where as on the positive range of $Re(\varepsilon_r)$ the maximum value is 0.3692 obtained when $Re(\varepsilon_r) = 1$. When $|Re(\varepsilon_r)| < 1$ again the allowable value of $|\zeta_0|$ falls as we move away from $|Re(\varepsilon_r)| = 1$ and there is some dependence on $Im(\varepsilon_r)$ in this case.
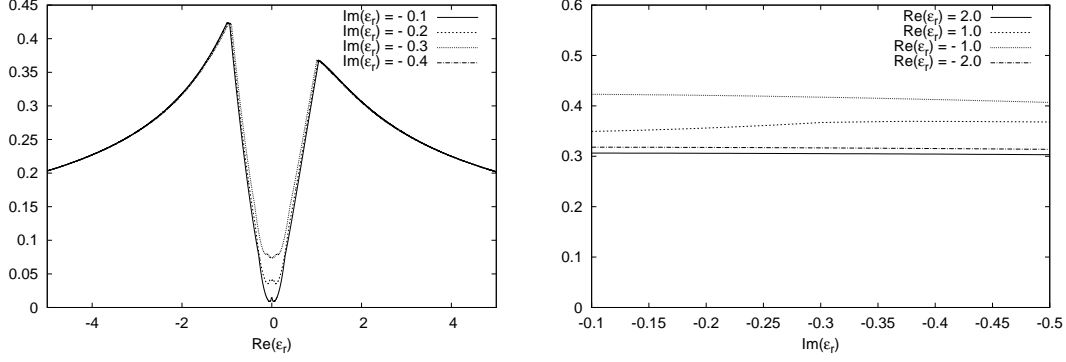
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Figure 2: The magnitude of ζ_0 below which the condition (9) is satisfied as a function of $Re(\varepsilon_r)$ or $Im(\varepsilon_r)$ for media in Kraft et al. The plot is for real ζ_0 and the material is taken to be non magnetic.

Let us now turn to a specific numerical example and obtain some results for the fields in the presence of this type of media. A cube shaped scatterer is considered which is filled with bianisotropic medium of interest and is surrounded by air. The incident plane wave is travelling along x axis with the electric field polarization along z and having magnitude 1 V/m and the wavelength is 1 micrometer. The scatterer is characterized by a cube of side 800 nm and has $\varepsilon_r = -1 - j0.4$, $\mu_r = 1$ and $\zeta_0 = -0.41$. This satisfies both conditions required for well posedness since for the given ε_r and μ_r, condition (9) requires $|\zeta_0| < 0.4126$ and condition (10) requires $|\zeta_0| < 0.4235$. The numerical domain considered is a cube of side 2 micrometer and the scatterer is placed at the center of the domain. The finite element solver is based on Galerkin's method with first order edge elements and with tetrahedral mesh.

The convergence behaviour of the solution is examined by considering different sizes of the meshes. The meshing is done by discretizing the domain with small cubes which in turn are divided into six tetrahedra. The four successive meshes are characterized by cubes of sides 200 nm, 100 nm, 50 nm and 25 nm which are referred as very coarse, coarse, fine and very fine mesh respectively. The very coarse mesh has 1331 nodes and 6000 elements, coarse mesh has 9261 nodes and 48000 elements, the fine mesh has 68921 nodes and 384000 elements and finally the very fine mesh has got 531441 nodes and 3072000 elements. The result for the z component of the electric field along a line parallel to x axis and passing through the center of gravity of the domain ($y = z = 1$ micrometer) is shown in Figure 3. As our theory predicts, the convergence of the solution is obtained and the result with fine mesh is reliable.

To make sense of the bianisotropic effects on the fields due to this kind of media, let us compare the solution obtained with that in isotropic case when $\zeta_0 = 0$ but with the same ε_r and μ_r. Some of the components of the fields which help to understand such effects are shown in Figures 4 to 7. For example there are differences of 10 to 20 percent of incident field in the magnitudes of electric fields which are induced due to the magnetoelectric coupling factor ζ_0, as can be seen from the magnitude plots of Figures 4 and 7. Likewise phases are also affected by the bianisotropy of the medium. Such results show non negligible influence induced due to the values of ζ_0 that fall within the ranges given by our conditions and hence demonstrate the relevance of our theory for the reliability of such solutions.
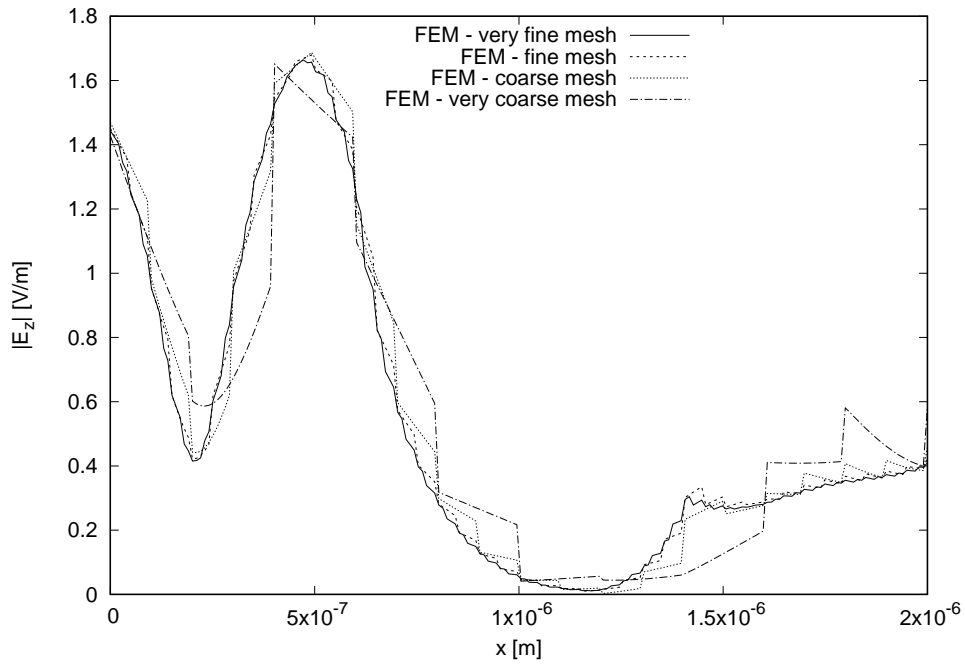
Figure 3: Convergence of the solution for problem involving medium of Kraft et al. The magnitude of the z component of the electric field is plotted along a line parallel to x axis for four different meshes.
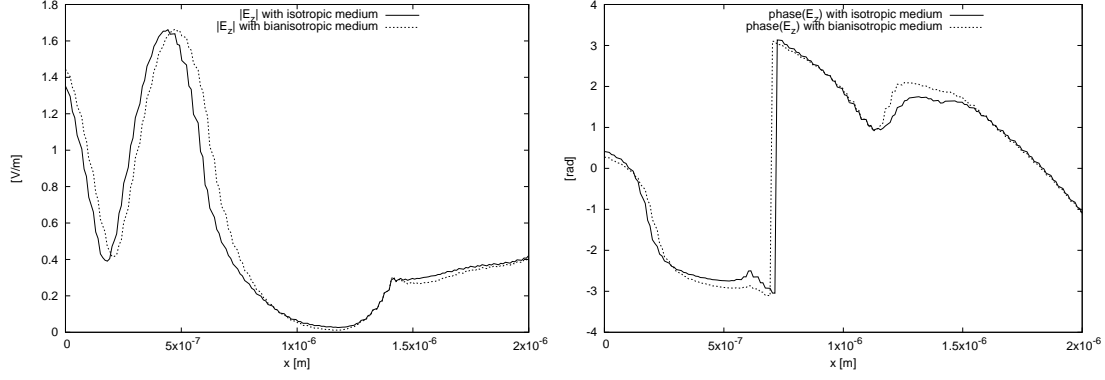
Figure 4: The magnitude and phase of the z component of electric field along a line parallel to x axis and passing though the center of gravity of the domain for problem involving medium of Kraft et al. The plots for bianisotropic case using $\zeta_0 = -0.41$ is compared with the solution obtained in isotropic case using $\zeta_0 = 0$.
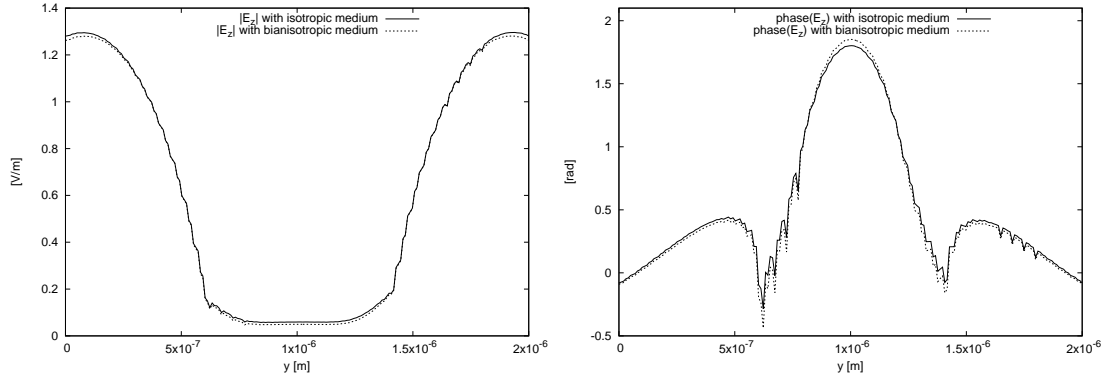

Figure 5: The magnitude and phase of the z component of electric field along a line parallel to y axis and passing though the center of gravity of the domain for problem involving medium of Kraft et al. The plots for bianisotropic case using $\zeta_0 = -0.41$ is compared with the solution obtained in isotropic case using $\zeta_0 = 0$.
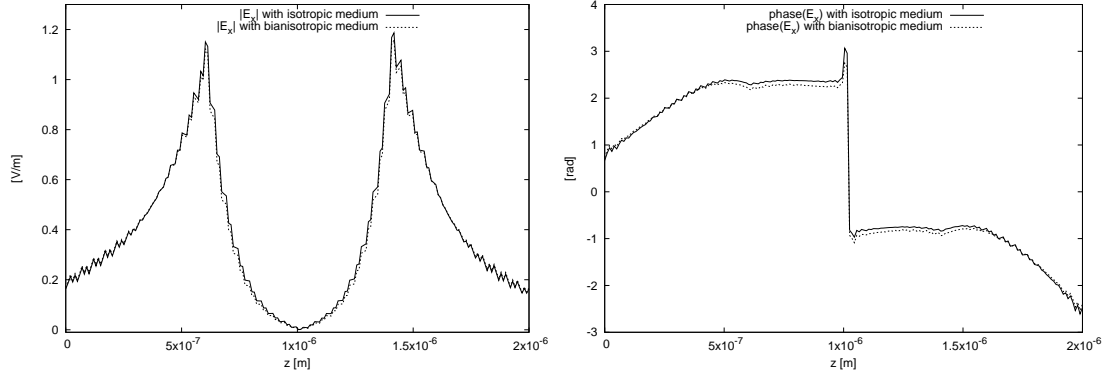
Figure 6: The magnitude and phase of the x component of electric field along a line parallel to z axis and passing though the center of gravity of the domain for problem involving medium of Kraft et al. The plots for bianisotropic case using $\zeta_0 = -0.41$ is compared with the solution obtained in isotropic case using $\zeta_0 = 0$.
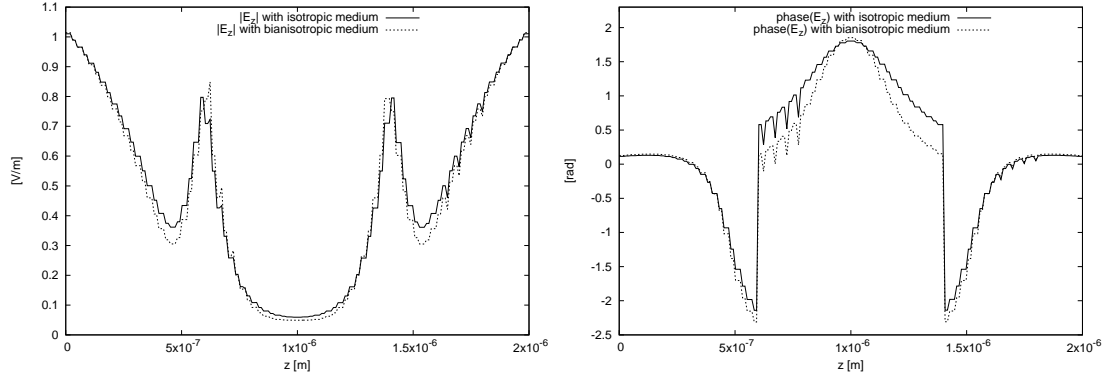

Figure 7: The magnitude and phase of the z component of electric field along a line parallel to z axis and passing though the center of gravity of the domain for problem involving medium of Kraft et al. The plots for bianisotropic case using $\zeta_0 = -0.41$ is compared with the solution obtained in isotropic case using $\zeta_0 = 0$.

References

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