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# Metamaterials at zero frequency

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**Abstract**
We investigate the problem of designing metamaterial structures which operate at very low frequencies. As an example, we consider the case of a DC magnetic cloak, which requires a variable, anisotropic magnetic permeability with both paramagnetic and diamagnetic components. We show that a structure based on superconducting components is the key to diamagnetism at low frequencies, and present a metamaterial design which meets the requirements of the cloak.

(Some figures in this article are in colour only in the electronic version)

## 1. Introduction

A new class of electromagnetic materials, metamaterials, has recently been the centre of much attention. Just as conventional materials owe their properties to the average response of millions of individual atoms and molecules, so the response of a metamaterial is dictated by microstructure carefully engineered on a scale much less than the wavelength. The additional design flexibility has led to the realization of properties not found in nature, such as negative refraction [1] and to highly anisotropic properties that can be exploited to control and direct radiation. Recent papers have described the cloaking of objects from microwaves using metamaterials [2–4]. The subject of metamaterials has been reviewed in several papers [5–7].

All of these applications have been realized at microwave frequencies and above. In this paper we examine the challenges of creating metamaterials to operate at near-zero frequencies, and as an example we consider the problem of cloaking an object from a DC magnetic field so that an object contained in the cloak experiences zero field, but outside the cloak the magnetic field is undisturbed. Theory [3] gives the recipe for such a cloak and requires that

$$\mu_r = \frac{R_2}{R_2 - R_1}\frac{(r - R_1)^2}{r^2}, \qquad \mu_\theta = \frac{R_2}{R_2 - R_1}, \qquad \mu_\phi = \frac{R_2}{R_2 - R_1}, \tag{1}$$

where the components of the permeability tensor $\mu$ are diagonal in spherical polar coordinates; $R_1$ is the radius of the cloaked region while $R_2$ is the outer radius of the cloak. Had we also wished to screen out electric fields, we would additionally require

$$\varepsilon_r = \frac{R_2}{R_2 - R_1}\frac{(r - R_1)^2}{r^2}, \qquad \varepsilon_\theta = \frac{R_2}{R_2 - R_1}, \qquad \varepsilon_\phi = \frac{R_2}{R_2 - R_1}, \tag{2}$$

  

where $\varepsilon$ is the electrical permittivity tensor. Here we concentrate on magnetic screening for which $\varepsilon$ is of course irrelevant for static fields.

Equation (1) demonstrates a characteristic of all cloaks: components of $\mu$ in the direction of compression of the cloak (the radial direction in this example) are decreased, whereas components in orthogonal directions are increased:

$$\mu_r < 1, \qquad \mu_\theta = \mu_\phi > 1. \tag{3}$$

A permeability of less than one presents particular problems at zero frequency for the following reason. There is a general requirement that the group velocity should not exceed the velocity of light in free space,

$$v_g = \left(\frac{\mathrm{d}k}{\mathrm{d}\omega}\right)^{-1} = \left(\frac{\mathrm{d}}{\mathrm{d}\omega}\frac{\omega\sqrt{\varepsilon\mu}}{c_0}\right)^{-1} = c_0\left(\sqrt{\varepsilon\mu} + \omega\frac{\mathrm{d}\sqrt{\varepsilon\mu}}{\mathrm{d}\omega}\right)^{-1} < c_0. \tag{4}$$

For a wave propagating perpendicular to the radius with magnetic component of the field aligned with the radius, this amounts to

$$\sqrt{\varepsilon_\theta\mu_r} + \omega\frac{\mathrm{d}\sqrt{\varepsilon_\theta\mu_r}}{\mathrm{d}\omega} > 1. \tag{5}$$

If we require $\mu_r < 1$, then either we must also require

$$\varepsilon_\theta > \mu_r^{-1} \tag{6}$$

or the metamaterial must be dispersive. The latter possibility is not available to us at $\omega = 0$ because of the factor of $\omega$ prefacing the second dispersive term in (5). Hence, any metamaterial designed to screen magnetism for static fields will inevitably have a large positive electrical response.

Next we must consider the issue of diamagnetism at $\omega = 0$. Materials magnetize either because electron spins align themselves with the field, giving rise to paramagnetic polarization with $\mu > 1$, or by supporting induced internal currents that oppose the applied field and thus give rise to diamagnetism with $\mu < 1$. We are therefore inevitably led to the use of superconductors (or at least very good conductors) which are able to sustain the constant induced currents required for the desired low-frequency response.

It is often remarked that superconductors behave like perfect diamagnets and can be described as having $\mu = 0$. However, this description is erroneous. More correctly, at least at the level of the London equations, a superconductor should be described as a lossless plasma with

$$\varepsilon = 1 - \frac{\omega_p^2}{\omega^2}, \qquad \mu = 1. \tag{7}$$

The divergence at zero frequency as $\omega^{-2}$ distinguishes it from a perfect conductor and ensures that the fields inside the superconductor are not zero but die off exponentially, as can be deduced by solving Maxwell's equations for the complex wavevector,

$$\lim_{\omega\to 0} k = \lim_{\omega\to 0}\frac{\omega}{c_0}\sqrt{\varepsilon\mu} = \mathrm{i}\frac{\omega_p}{c_0} = \mathrm{i}\frac{1}{\Lambda_L} \tag{8}$$

where $\Lambda_L$ is the London penetration depth.

However, cutting up a superconductor into pieces to make a metamaterial gives quite different properties, as we shall see. We shall show how to exploit them to construct diamagnetic metamaterials with a high degree of anisotropy.

Finally, we comment that we have not found a way to make metamaterials with

$$\lim_{\omega\to 0}\varepsilon < 1. \tag{9}$$

This would require a magnetic 'current' of circulating magnetic poles. Since magnetic poles do not exist as far as we know we have to conclude that it is impossible to realize (9).

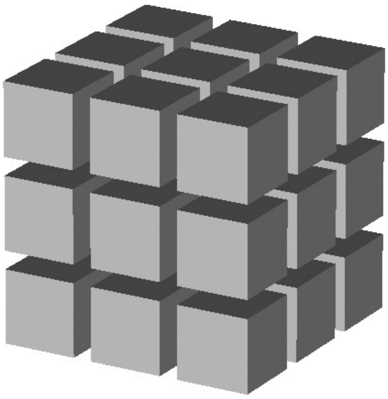

**Figure 1.** A lattice of superconducting cubes.

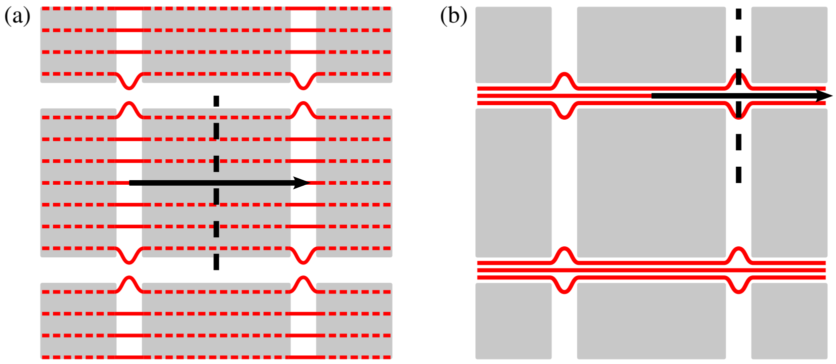


**Figure 2.** A slice through the superconducting cube structure subjected to uniform (a) electric and (b) magnetic fields. The dotted lines in (a) represent the $\boldsymbol{D}$ field in the region where the $\boldsymbol{E}$ field is zero. The thicker unbroken lines marked with arrows denote the integral paths (used for averaging $\boldsymbol{E}$ and $\boldsymbol{H}$), while the dashed lines perpendicular to these show a section through the integration surfaces (used for averaging $\boldsymbol{D}$ and $\boldsymbol{B}$).

## 2. Lattice of cubes

We begin by investigating the structure illustrated in figure 1: solid cubes of superconducting material are arranged on a simple cubic lattice. In the limit where the cube size $d$ approaches the lattice constant $l$, we can obtain an estimate for the effective permittivity and permeability.

We first consider the case of an electric field applied parallel to one of the cube edges. The $\boldsymbol{E}$ field is then largely confined to the gaps between those faces perpendicular to the field orientation, as shown in figure 2(a). To extract the effective permittivity, we need an averaging procedure; we follow Pendry *et al* [8] by using a line integral for averaging $\boldsymbol{E}$ and a surface integral for $\boldsymbol{D}$. This method has been reviewed in detail by Smith and Pendry [9]. Figure 2(a) shows the relevant line and surface.

Assuming that the field is uniform (of strength $D_0/\varepsilon_0$) in the gaps between the faces, we obtain the averaged fields

$$\varepsilon_0 \bar{E} = \frac{1}{l}(l - d)D_0 \tag{10}$$

and

$$\bar{D} = \frac{d^2}{l^2} D_0. \tag{11}$$

We define the effective permittivity using the ratio of these quantities:

$$\varepsilon_{\mathrm{eff}} = \frac{\bar{D}}{\varepsilon_0 \bar{E}} = \frac{d^2}{l(l-d)}. \tag{12}$$

The method for extracting the effective permeability is very similar. We consider a magnetic field applied in the direction of one of the cube edges. This time, the field is confined to the gaps between those faces *parallel* to the field direction, as shown in figure 2(b). We take the average of $\boldsymbol{H}$ along a line in one of these gaps, and the average of $\boldsymbol{B}$ over a surface perpendicular to this line. As with the electric field, we make the approximation that the field is uniform in the relevant region; taking the magnetic induction to be $B_0$, we have

$$\begin{aligned} \mu_0 \bar{H} &= B_0, \\ \bar{B} &= \frac{l^2 - d^2}{l^2} B_0. \end{aligned} \tag{13}$$

This gives us an effective magnetic permeability

$$\mu_{\mathrm{eff}} = \frac{\bar{B}}{\mu_0 \bar{H}} = \frac{l^2 - d^2}{l^2}. \tag{14}$$

We expect (12) and (14) to be more accurate as $d$ approaches $l$; our assumption that the fields are entirely confined to the gaps and uniform within them then becomes more realistic.

The model predicts that $0 < \mu_{\mathrm{eff}} < 1$ and $\varepsilon_{\mathrm{eff}} > 1$. We should ensure that these parameters combine to give a predicted speed of light in our metamaterial which does not exceed $c_0$. The phase velocity (which is equal to the group velocity when $\omega = 0$) is given by

$$v_{\mathrm{ph}} = \frac{c_0}{\sqrt{\varepsilon_{\mathrm{eff}} \mu_{\mathrm{eff}}}} = c_0 \sqrt{\frac{l^3}{d^2(l+d)}}, \tag{15}$$

which is less than $c_0$ for $d > 0.755l$. In the limit $d \to l$, the velocity tends to $c_0/\sqrt{2}$.

We can test these predictions by running numerical simulations. Our approach here is to calculate the dispersion relation for the infinite metamaterial. To this end, we take a unit cell (containing a single cube) and apply periodic boundary conditions with a specific phase shift across the cell in one or more directions; for example, if the phase shift in the $x$ direction is $\phi_x$ then all the field components must satisfy the condition

$$f(\boldsymbol{r} + l\hat{\boldsymbol{x}}) = \exp(\mathrm{i}\phi_x) f(\boldsymbol{r}). \tag{16}$$

The phase shift $\phi_x$ is therefore naturally associated with the Bloch wavevector $k_x = \phi_x / l$. We use a commercial solver package to determine the eigenfrequencies of the system for a range of phase shifts; this gives us the band structure of our metamaterial. Figure 3 shows an example.

The gradient of the dispersion relation at $\omega = 0$ for this particular configuration corresponds to a group velocity of $0.768c_0$. This is less than the value predicted by (15); however, as we make the cubes larger, our approximations get better and the group velocity approaches the predicted limiting value of $c_0/\sqrt{2}$ (see table 1).

We cannot extract $\varepsilon_{\mathrm{eff}}$ and $\mu_{\mathrm{eff}}$ directly from the dispersion relation, but this result gives us confidence that the predictions in (12) and (14) are reasonably accurate.

We have therefore established that the lattice of superconducting cubes can provide us with an effective permeability in the desired range. However, this system fails to meet our other criterion: that the permeability be anisotropic. We will address this shortcoming in the next section.

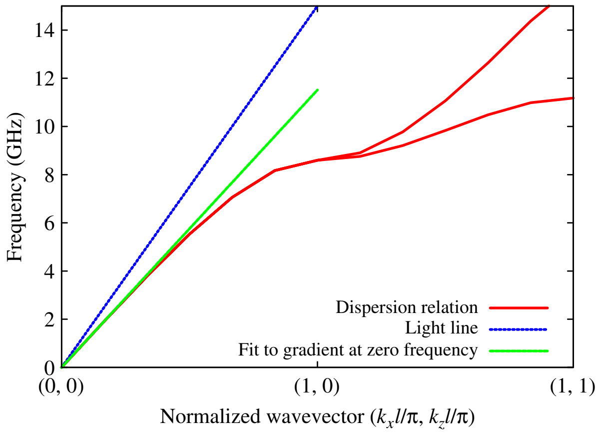


**Figure 3.** The dispersion relation for the system of superconducting cubes with $l = 10$ and $d = 8$ mm.

**Table 1.** Phase velocities calculated from the dispersion relation for superconducting cube structures with lattice constant $l = 10$ mm and a range of cube lengths $d$.

| $d$ (mm) | Phase velocity (units of $c_0$) |
|---|---|
| 8 | 0.768 |
| 9 | 0.734 |
| 9.5 | 0.719 |
| 9.8 | 0.711 |
| $d \to l$ | 0.707 |

## 3. Lattice of plates

One way to make the system anisotropic is to flatten the cubes so that they become plates. If the plates are very thin, there will be no appreciable electric response in the direction perpendicular to the plates; we can therefore assume that $\varepsilon_\perp = 1$. Parallel to the plates, we expect an electric response similar to the cubes, though weaker in magnitude. For the magnetic response, the situation is reversed; we expect little response to a field parallel to the plates, and a weakened diamagnetic effect when the field is normal to them.

Once again, numerical simulations allow us to put these predictions to the test. The lower symmetry of the lattice of plates means that modes which were degenerate in the cube system are now distinct in energy, and provide us with more information; starting only with the assumption that $\varepsilon_\perp = 1$, we can directly obtain $\mu_\parallel$, $\varepsilon_\parallel$ and $\mu_\perp$ from the band structure, which we calculate as before. An example is shown in figure 4.

When the wavevector is normal to the plates, there are two degenerate modes with phase velocity $c_0/\sqrt{\varepsilon_\parallel \mu_\parallel}$. This can be seen in the rightmost quadrant of figure 4. The degeneracy is lifted when the wavevector lies in the plane of the plates. In the leftmost quadrant of the figure, the wavevector lies parallel to one set of the plate edges; the phase velocity is then either $c_0/\sqrt{\varepsilon_\parallel \mu_\perp}$ or $c_0/\sqrt{\varepsilon_\perp \mu_\parallel}$, depending on the polarization. With the assumption that $\varepsilon_\perp = 1$, measuring these three phase velocities gives us all the information necessary to determine the three unknowns $\mu_\parallel$, $\varepsilon_\parallel$ and $\mu_\perp$. In table 2 we present effective medium parameters calculated in this way.

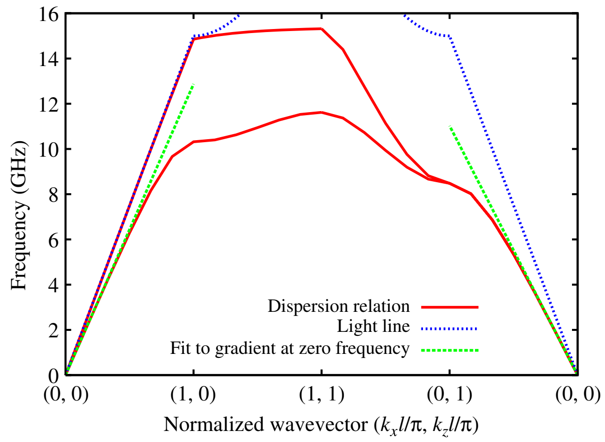


**Figure 4.** The dispersion relation for the lattice of superconducting plates. The lattice constant $l$ is 10 mm; the plates are 8 mm wide and 0.5 mm thick. The $z$ axis is normal to the plates; the sides of the plate are aligned with the $x$ and $y$ axes.

**Table 2.** Effective medium parameters for various systems. The lattice constant is 10 mm; in the case of the tetragonal lattice, this is reduced to $s$ in the direction normal to the plates.

| System | $d$ (mm) | $s$ (mm) | $\mu_\parallel$ | $\varepsilon_\parallel$ | $\mu_\perp$ |
|---|---|---|---|---|---|
| Plates (cubic lattice) | 8 | — | 1.00 | 1.84 | 0.74 |
| | 9.5 | — | 1.00 | 3.45 | 0.47 |
| | 9.8 | — | 1.00 | 5.47 | 0.32 |
| Plates (tetragonal lattice) | 8 | 6 | 1.01 | 2.32 | 0.64 |
| | 8 | 4 | 1.01 | 2.73 | 0.58 |
| | 9.8 | 6 | 1.00 | 8.07 | 0.23 |
| Discs (cubic lattice) | 8 | — | 1.00 | 1.43 | 0.83 |

The in-plane permeability $\mu_\parallel$ is always equal to or very close to unity, while that normal to the plates is lower. We can drive $\mu_\perp$ to smaller values by making the plates larger (increasing $d$) or by squashing the lattice in the direction normal to the plates (so that it becomes tetragonal). Combining these approaches gives us a predicted practical range of approximately $0.2 < \mu_\perp < 1$.

A simple variation on the structure of cubes is to place superconducting spheres on a lattice. The effective medium is then isotropic, like the lattice of cubes; with a lattice spacing of 10 mm and spheres of diameter 8 mm, the dispersion relation looks very much like figure 3, while the phase velocity at $\omega = 0$ is $0.853c_0$. By squashing the spheres into discs, we return to an anisotropic system, and are once again able to extract the effective medium parameters from the dispersion relation. These are listed in table 2; the system behaves qualitatively like the lattice of plates, although the effect is less pronounced.

It is also interesting to investigate what happens when we put more plates into the unit cell. We first add one additional plate, at right angles to the first, as shown in figure 5(a); this should give us a uniaxial anisotropic metamaterial, like the single-plate system. Finally, we add a third plate, perpendicular to the first two; this system is now isotropic, and we expect that it will behave in a similar way to the structure of solid cubes. The unit cell of this system is illustrated in figure 5(b).

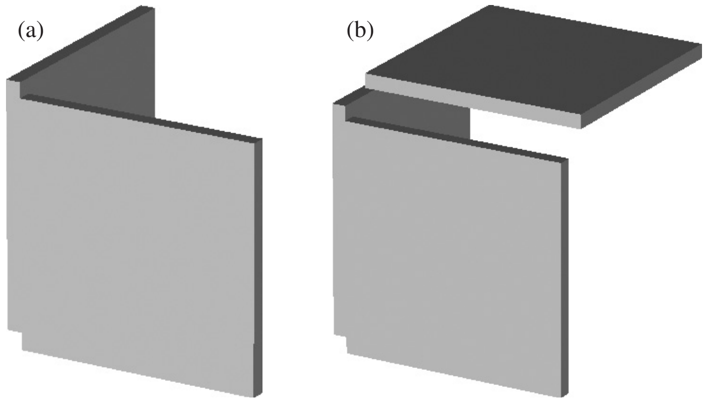


**Figure 5.** Unit cells for the (a) two-plate and (b) three-plate systems.

The band structure of the anisotropic two-plate structure is similar to that of the single-plate system shown in figure 3. The main difference is that there is no mode which follows the light line; for any orientation of the electric field, there is always an appropriately aligned plate which can be polarized.

The results for the single-plate systems demonstrate that the plates have very little effect on an in-plane magnetic field. If we assume that this remains true for the two-plate system then we may assume that $\mu_\perp = 1$. We use the 'perpendicular' label to refer to the direction of the four-fold symmetry axis of the metamaterial; this is no longer normal to the plates themselves. With this assumption, we again have enough information to extract the remaining effective medium parameters. We obtain $\varepsilon_\parallel = 1.76$, $\mu_\parallel = 0.79$, and $\varepsilon_\perp = 2.40$ when the dimensions are $d = 8$ mm, $l = 10$ mm. This is exactly the response that one would predict by summing the contributions of the two sets of plates and neglecting any interaction between them; $\varepsilon_\parallel$ and $\mu_\parallel$ are approximately equal to the corresponding single-plate parameters ($\varepsilon_\parallel$ and $\mu_\perp$ respectively), while $\varepsilon_\perp$ represents a polarization close to twice that of $\varepsilon_\parallel$, because both sets of plates can now contribute.

For the three-plate structure with dimensions $d = 8$ mm, $l = 10$ mm, the extracted group velocity is $0.744c_0$, which is very close to the value obtained for the solid cube.

In our simulations, we treated the superconducting material as a perfect electrical conductor rather than a perfect plasma. This should not affect the results at the microwave frequencies we have been considering. Of course, this also ignores the possibility of tunnelling between the plates (the Josephson effect).

## 4. Applications

There are several approaches to cloaking from electromagnetic radiation. Alu and Engheta use plasmonic resonances [10, 11] to make a subwavelength object invisible. Milton and Nicorovici also use resonances of a more sophisticated kind [12], but their prescription will only screen point objects from view; extended objects remain visible, and the scheme is therefore of limited applicability. However, by employing metamaterials it is possible to specify a more complete cloaking scheme. Leonhardt [4] has given a recipe that cloaks within the ray approximation. A similar but more complete formula based on the full Maxwell equations gives an exact cloak [2], albeit with some complicated specifications for the metamaterials concerned. The

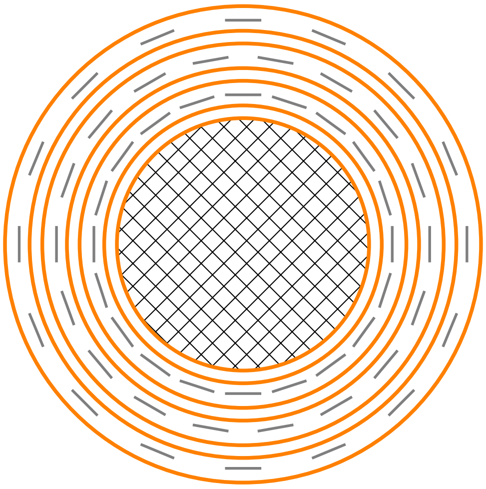

**Figure 6.** The proposed magnetic cloak; the shaded region in the centre is hidden from external magnetic fields. The plates form broken circles (in cross section); the full circles show the ferrite or amorphous metal.

latter approach is relevant to this paper as it encompasses the screening of a static magnetic field; this is something that none of the other treatments can address.

To build a magnetic cloak, we need an anisotropic material which is diamagnetic or paramagnetic depending on the field direction. Our superconducting metamaterials are diamagnetic in the direction perpendicular to the plates; we can tune the effective permeability in this direction over a range of values between 0.2 and 1 by adjusting the plate size and spacing. We need to incorporate a component which will provide an anisotropic paramagnetic response. This is easily achieved at low frequencies by a simple layered structure.

A metamaterial composed of alternating layers of thickness $d_1$ and $d_2$ with permeability $\mu_1$ and $\mu_2$ has an effective permeability

$$\mu_{\parallel} = \frac{\mu_1 d_1 + \mu_2 d_2}{d_1 + d_2}, \tag{17}$$

$$\mu_{\perp} = \left( \frac{\mu_1^{-1} d_1 + \mu_2^{-1} d_2}{d_1 + d_2} \right)^{-1}. \tag{18}$$

We would like $\mu_{\perp}$ to be close to unity; if we want the cloak to be thin, we need $\mu_{\parallel}$ to be large. The obvious way to achieve this is to have one material non-magnetic ($\mu_1 = 1$) and the other highly paramagnetic ($\mu_2 \gg 1$), with $d_1 \gg d_2$. Both materials need to be non-conducting, to avoid spoiling the superconductor-induced diamagnetism. Some suitable candidates for the paramagnetic material are to be found among the ferrites, which combine large values of $\mu$ with high resistivity and low hysteresis, as do certain amorphous metals.

As a test, we added two thin (1 mm) layers of a hypothetical non-conducting magnetic material with $\mu = 10$ to our single-plate unit cell, and carried out the usual simulations. With $d = 9.5$ mm, the effective medium parameters are modified in exactly the way we expect: the material becomes paramagnetic in the plane of the plates ($\mu_{\parallel} = 1.81$), while the out-of-plane diamagnetism is affected only slightly ($\mu_{\perp} = 0.53$ instead of 0.47). Our metamaterial is now complete.

The cloak described in (1) is spherical, while our system of superconducting plates is based on a cubic lattice. However, we can get away with wrapping the cubic system around the surface of a sphere as long as the unit cell size is significantly smaller than the radius. By decreasing the size of the plates or making the gaps larger as we move outwards, we can achieve the desired variation in $\mu_r$. Figure 6 illustrates a section through the cloaking structure.

We have taken up the challenge of designing metamaterials that function at very low frequencies. We have shown that by using superconducting components, it is possible to create and control anisotropic diamagnetism in this regime; this is one of the essential ingredients of a magnetic cloak, for which we have presented a practical design.

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