Regularity of time-harmonic electromagnetic fields in the interior of bianisotropic materials and metamaterials

Paolo Fernandes[*] Marina Ottonello[†] Mirco Raffetto[‡]

July 28, 2010

Abstract

The regularity of the four time-harmonic vector fields composing any strong solution of the system obtained from Maxwell's curl equations and the constitutive relations in the interior of an inhomogeneous bianisotropic material is investigated. The results are given as interior Sobolev or Hölder regularity. Possible local C^∞ regularity or local analyticity of the four vector fields are discussed, too. Each of these regularity results is obtained under specific conditions on the impressed current densities and on the constitutive parameters of the bianisotropic material considered, but it is shown that such conditions do not significantly limit the coverage of our analysis in terms of applications.

Keywords: Time-harmonic Maxwell's equations; interior Sobolev regularity of electromagnetic fields; interior Hölder regularity of electromagnetic fields; analyticity of electromagnetic fields; anisotropic media and metamaterials; bianisotropic media and metamaterials.

1 Introduction

The studies on the regularity of time-harmonic electromagnetic fields can be divided, as it is usually the case for studies related to the regularity of the solutions of systems of partial differential equations [1], [2], [3], into two classes: those dedicated to the global regularity of the solutions of electromagnetic boundary value problems and those addressing the question of interior regularity of electromagnetic fields.

The first important contributions on this topic date back to Müller [4] and Leis [5]. Other more recent results belonging to the second class can be found in [6], for $(H^1_{\rm loc})^3$ regularity of the electric and magnetic fields, and in [7] and [8], for their local Hölder continuity. All these contributions require that all linear materials involved are at most anisotropic with symmetric and positive definite three-by-three matrix-valued real functions as constitutive parameters.

As far as the first class is concerned, some global regularity results can be found in [9], [10], [11]. Furthermore, [12] provides global regularity for electromagnetic initial boundary value problems. In [13], a detailed investigation of the singular behaviour of time-harmonic electromagnetic fields at corners and edges of both the problem boundary and the interfaces between different materials is worked out. Its main theorem is given as a *piecewise* H^s regularity of the fields and, thus, for $s \geq 1/2$ is finer than usual global H^s regularity results. Due to the presence of the possible additional boundary singularities, global regularity results cannot provide stronger regularity results than those obtained from interior analyses. In particular, unless very specific conditions are satisfied,

[*]Istituto di Matematica Applicata e Tecnologie Informatiche del Consiglio Nazionale delle Ricerche, Via De Marini 6, I–16149 Genoa, Italy, email: fernandes@ge.imati.cnr.it

[†]Department of Biophysical and Electronic Engineering, University of Genoa, Via Opera Pia 11a, I–16145, Genoa, Italy, email: ottonello@dibe.unige.it

[‡]Department of Biophysical and Electronic Engineering, University of Genoa, Via Opera Pia 11a, I–16145, Genoa, Italy, email: raffetto@dibe.unige.it

all these results state the global H^s regularity of the components of the electric and magnetic fields, for some $s < 1$. In all these cases, too, the media are assumed to be at most anisotropic, with a positive definite symmetric part of the three-by-three matrix-valued functions describing the constitutive parameters.

In the last decade, however, several innovative materials, like the so-called metamaterials [14], [15], [16], a kind of artificially made materials that do not exist in nature [17], or the bianisotropic media [18], [19], [20], have been considered, designed and obtained. Bianisotropic materials have fundamental scientific significance and technological applications and have attracted the attention of many researchers [21], [22], [23], [24], [25], [26], [27], [28], [29], [30], [31]. They are materials in which a single applied field, either electric or magnetic, induces both an electric and a magnetic polarization, in a way that, moreover, is not invariant under rotation of the material. Metamaterials, for which a similar interest for potential applications does exist [16], are materials that behave as if their electric permittivity and/or magnetic permeability had negative real parts [16]. Many of them are highly inhomogeneous artificial materials which, as a side effect of that, exhibit an anisotropic [32], [33], [34] or even bianisotropic [35], [36] (effective) behaviour.

For these reasons, generalizing to these innovative materials known results concerning time-harmonic electromagnetic boundary value problems is of interest. In this sense, for the sake of completeness, it can be useful to point out that, with an appropriate combination of the losses and of the symmetric parts of the matrices defining the constitutive parameters, well-posedness results can be found in [37], for anisotropic metamaterials, and in [38], for bianisotropic media. In [39], [40] and [41] the same type of results can be found in the presence of isotropic metamaterials with appropriate discontinuities of the constitutive parameters.

However, to the best of authors' knowledge, neither global nor local regularity results are available for these innovative materials. For these reasons, in this work we try to asses whether the more general structure of the constitutive relations by which the aforementioned innovative materials are modeled sets by itself specific limitations to the interior regularity of the solutions of time-harmonic Maxwell equations also inside a single material. Moreover, we consider the constitutive relations of bianisotropic materials, without assuming positive definiteness of the symmetric part of any of the matrices involved in them, in order to include general metamaterials, too. In particular, we deduce that, under some regularity assumptions on the constitutive parameters and on the impressed current densities which do not significantly affect the coverage of our analysis in terms of applications, the four vector fields defining any strong solution of the system determined by the time-harmonic Maxwell's curl equations and the constitutive relations have Cartesian components belonging to $W^{n,p}_{\text{loc}}(\Omega)$ or to $C^{n,\lambda}(\Omega)$, for $n \in \mathbb{N}, n > 0, p \in \mathbb{R}, p \geq 2$ and $\lambda \in (0, 1]$, if the domain Ω is occupied by a single and possibly inhomogeneous bianisotropic material. Conditions implying the $C^\infty(\Omega)$ regularity of these components or even their interior analyticity are also provided.

Regularity results concerning the fields inside a single material can be useful, for example, to prove uniqueness of the solution of time-harmonic electromagnetic driven problems in more complicate configurations of materials [4], [6].

Even though it does not seem physically sensible to consider enforced electric and magnetic current densities in the interior of the media of interest, for the sake of generality in our development we will consider the presence of solenoidal current densities. Solenoidality is a necessary requirement since our technical developments are based on the use of vector potentials for the electric displacement and the magnetic induction. It can be useful to point out that in some of the works considered above such a solenoidality hypothesis on the impressed sources is not present. However, also in our setting this hypothesis is not required if the electromagnetic field is generated by sources external to the considered material.

Finally, as already pointed out, the results of interior regularity of time-harmonic electromagnetic fields are deduced for many anisotropic and bianisotropic materials of practical interest. However, it is also pointed out that for a few models of media considered in the open literature our results do not apply and, moreover, that for other hypothetical models of metamaterials and sources not even a very weak H^1_{loc} regularity result can be achieved.

The paper is organized as follows. In Section 2 we introduce two solenoidal vector potentials and define a first system of partial differential equations satisfied by them. In Section 3 an alternative formulation, more suited to assess the regularity of the vector potentials, is introduced. Some local regularity results for the solutions of elliptic systems of partial differential equations are then used in Section 4 to deduce the corresponding results for the aforementioned vector potentials. The different forms of the constitutive relations for bianisotropic media and the relationships among them are considered in Section 5. In Section 6 several regularity results for time-harmonic electromagnetic fields in the interior of many anisotropic media and metamaterials of practical interest are deduced. Finally, an analogous set of results is obtained in Section 7 for many bianisotropic media and metamaterials of practical interest.

2 Problem definition

Consider an open domain $\Omega \subset \mathbb{R}^3$, filled by a general bianisotropic material. Since we are interested in the local regularity of the electromagnetic fields inside a single material, we can assume, without any loss of generality, that Ω is also bounded and connected.

The electromagnetic fields are generated by the impressed electric and magnetic current densities $\mathbf{J}_e$ and $\mathbf{J}_m$ [42] (p. 7 and p. 19), and we assume (in the following we use the acronyms HSn, HMn or HAn to mean, respectively, Hypothesis on the Sources #n, Hypothesis on the Media #n or Hypothesis on the Auxiliary scalar fields #n)

HS1. $\mathbf{J}_e, \mathbf{J}_m \in H(\mathrm{div}^0, \Omega)$

where [43]

$$H(\mathrm{div}^0, \Omega) = \{\mathbf{v} \in (L^2(\Omega))^3 : \nabla \cdot \mathbf{v} = 0\}. \tag{1}$$

In this setting, we will investigate the local regularity of any possible solution $(\mathbf{E}, \mathbf{B}, \mathbf{H}, \mathbf{D}) \in H(\mathrm{curl}, \Omega) \times H(\mathrm{div}^0, \Omega) \times H(\mathrm{curl}, \Omega) \times H(\mathrm{div}^0, \Omega)$ (i. e., strong solutions [44] (p. 578)) of time-harmonic Maxwell's equations

$$\begin{cases} \nabla \times \mathbf{H} = j\omega\, \mathbf{D} + \mathbf{J}_e & \text{in } \Omega \\ \nabla \times \mathbf{E} = -j\omega\, \mathbf{B} - \mathbf{J}_m & \text{in } \Omega \end{cases} \tag{2}$$

when $\mathbf{E}$, $\mathbf{B}$, $\mathbf{H}$ and $\mathbf{D}$ are linked together by the following frequency-domain form of the constitutive relations of a bianisotropic material [19] (pp. 8-9)

$$\begin{cases} \mathbf{E} = \kappa\, \mathbf{D} + \chi\, \mathbf{B} & \text{in } \Omega \\ \mathbf{H} = \gamma\, \mathbf{D} + \nu\, \mathbf{B} & \text{in } \Omega, \end{cases} \tag{3}$$

where [43]

$$H(\mathrm{curl}, \Omega) = \{\mathbf{v} \in (L^2(\Omega))^3 : \nabla \times \mathbf{v} \in (L^2(\Omega))^3\}, \tag{4}$$

$\omega \in \mathbb{R}$, $\omega > 0$, is the angular frequency, and κ, χ, γ and ν are four 3-by-3 matrix-valued complex functions of the space point only, representing the effective [45], [16] constitutive parameters.

Since Ω is filled by a single material, even though possibly inhomogeneous, we assume at least that (see [43], p. 2, for the definition of $L^\infty(\Omega)$)

HM1. $\kappa, \chi, \gamma, \nu \in (C^0(\Omega) \cap L^\infty(\Omega))^{3\times 3}$

and will investigate whether stronger local regularity of the fields holds true under suitable additional assumptions on κ, χ, γ, ν, $\mathbf{J}_e$ and $\mathbf{J}_m$.

Remark 1. *Notice that, according to the previous considerations, for a highly inhomogeneous artificial material κ, χ, γ and ν should be considered as effective parameters, that is to say those of an ideal material by which we model the overall behaviour of the real one, neglecting fine details. Hence, in this case, regularity does not concern the real fields, but is rather a feature of the fields in the model, which may be useful, however, for the mathematical analysis and numerical approximation of the model itself.*

Since we are looking for regularity properties that hold locally in Ω, namely for any $\mathcal{O}$ open and bounded such that $\overline{\mathcal{O}} \subset \Omega$, we can restrict ourselves to prove them just in the case in which Ω is an open ball. Then, by standard arguments (e. g., [1] and Chapter 3 of [46]), they will hold true for a general Ω.

If Ω is an open ball, for any strong solution $(\mathbf{E}, \mathbf{B}, \mathbf{H}, \mathbf{D})$ of (2)-(3), since $\mathbf{B}$ and $\mathbf{D}$ belong to $H(\mathrm{div}^0, \Omega)$, Theorem 3.4, p. 45, and Remark 3.10, p. 47, of [43] ensure that there is a not uniquely determined pair of vector potentials $(\mathbf{A}_1, \mathbf{A}_2) \in (H^1(\Omega))^3 \times (H^1(\Omega))^3$ such that $\nabla \cdot \mathbf{A}_1 = 0$, $\nabla \cdot \mathbf{A}_2 = 0$ and

$$\mathbf{D} = \nabla \times \mathbf{A}_1, \ \mathbf{B} = \nabla \times \mathbf{A}_2. \tag{5}$$

Since $(\mathbf{E}, \mathbf{B}, \mathbf{H}, \mathbf{D})$ is a strong solution of (2)-(3), the pair $(\mathbf{A}_1, \mathbf{A}_2)$ will satisfy

$$\begin{cases} \mathbf{E} = \kappa\,(\nabla \times \mathbf{A}_1) + \chi\,(\nabla \times \mathbf{A}_2) & \text{in } \Omega \\ \mathbf{H} = \gamma\,(\nabla \times \mathbf{A}_1) + \nu\,(\nabla \times \mathbf{A}_2) & \text{in } \Omega \end{cases} \tag{6}$$

and

$$\begin{cases} \nabla \times (\kappa\,(\nabla \times \mathbf{A}_1)) + \nabla \times (\chi\,(\nabla \times \mathbf{A}_2)) + j\omega\,(\nabla \times \mathbf{A}_2) = -\mathbf{J}_m & \text{in } \Omega \\ \nabla \times (\gamma\,(\nabla \times \mathbf{A}_1)) + \nabla \times (\nu\,(\nabla \times \mathbf{A}_2)) - j\omega\,(\nabla \times \mathbf{A}_1) = \mathbf{J}_e & \text{in } \Omega. \end{cases} \tag{7}$$

We can state these facts as the following proposition

Proposition 1. *Let Ω be an open ball and assume HS1 and HM1. For any $(\mathbf{E}, \mathbf{B}, \mathbf{H}, \mathbf{D}) \in H(\mathrm{curl}, \Omega) \times H(\mathrm{div}^0, \Omega) \times H(\mathrm{curl}, \Omega) \times H(\mathrm{div}^0, \Omega)$ satisfying (2)-(3), there is a not uniquely determined pair of vector potentials $(\mathbf{A}_1, \mathbf{A}_2) \in (H^1(\Omega))^3 \times (H^1(\Omega))^3$ such that $\nabla \cdot \mathbf{A}_1 = 0$, $\nabla \cdot \mathbf{A}_2 = 0$ and that satisfies (5) and the system of second order partial differential equations (7).*

In that follows, we will look for assumptions on κ, χ, γ, ν, $\mathbf{J}_e$ and $\mathbf{J}_m$ that ensure an improved regularity of $(\mathbf{A}_1, \mathbf{A}_2)$, as solution of (7). Then, additional regularity for $(\mathbf{E}, \mathbf{B}, \mathbf{H}, \mathbf{D})$ will follow from (5) and (6).

3 An alternative formulation more suited to assess regularity

In order to prove a result of regularity for the electromagnetic fields, one usually tries to prove that such fields are solutions of elliptic systems of partial differential equations because for such systems several results of regularity are known [1]. Following the same approach, in this section we firstly show that system (7) is not suited to our purpose, being not elliptic. Then, we will introduce an alternative system, still satisfied by the pair of divergence-free vector potentials $(\mathbf{A}_1, \mathbf{A}_2)$, which is elliptic, at least in standard passive anisotropic materials. This alternative system will be heavily used in Sections 4, 6 and 7 to assess the results of regularity of interest in most of anisotropic or bianisotropic media and metamaterials (in the following we will use the adjectives isotropic, biisotropic, anisotropic or bianisotropic in the strictest sense. In particular, we will say that a medium is isotropic when it is uni-isotropic, that is when κ and ν are scalar fields and χ and γ are trivial; biisotropic when κ, χ, γ and ν are scalar fields and at least one of χ or γ is not identically zero; anisotropic when it is uni-anisotropic, that is when at least one of κ or ν is a 3-by-3 matrix-valued complex function which is not everywhere multiple of the identity matrix and χ and γ are trivial; bianisotropic when the constitutive relations (3) apply but it is not isotropic, biisotropic or anisotropic).

With this aim, we firstly introduce some definitions and some notations, which will be heavily exploited later on. Without loss of generality for studies related to interior regularity, we consider a Cartesian system of coordinates (x_1, x_2, x_3) and use the notation ∂_j for the partial derivative $\frac{\partial}{\partial x_j}$ in the sense of distributions.

According to [1], a system of m second-order partial differential equations in m scalar unknowns, formulated by using derivatives with respect to the spatial coordinates x_i, $i = 1, \ldots, n$, is said to be elliptic if and only if the characteristic determinant is not zero for any real algebraic vector of n coefficients $\mathbf{l}_{1,n}^T = [\lambda_1, \ldots, \lambda_n]$, of which at least one different from zero. The characteristic determinant is the determinant of a square $m \times m$ matrix C, obtained from the system of interest. The entry C_{ij}, $i, j = 1, \ldots, m$, is given by the sum of the terms of the i-th equation that involve the second order derivatives $\partial_l \partial_m A_j$ of the j-th scalar unknown A_j, with $\partial_l \partial_m A_j$ substituted by $\lambda_l \, \lambda_m$.

In order to develop some partial but also simpler considerations, we firstly study in some detail the well known case of anisotropic media (i. e., $\chi = \gamma = 0$). In these cases, the principal part [1] (p. 210) of system (7), given by the terms involving second order differential operators, reduces to two terms without any common unknown. Hence, the characteristic determinant of system (7) is the product of the characteristic determinants of each of them. Since these terms have the same form, namely

$$\nabla \times (R \, (\nabla \times \mathbf{A})) \, , \tag{8}$$

with R given by κ or ν and $\mathbf{A}$ given by $\mathbf{A}_1$ or $\mathbf{A}_2$, let us consider it. The different Cartesian components are

$$\begin{aligned}
&\nabla \times (R \, (\nabla \times \mathbf{A})) \\
&\quad = \hat{x}_1 \Big(R_{31} \, \partial_2 \partial_2 A_3 - R_{31} \, \partial_2 \partial_3 A_2 + R_{32} \, \partial_2 \partial_3 A_1 - R_{32} \, \partial_2 \partial_1 A_3 + R_{33} \, \partial_2 \partial_1 A_2 - R_{33} \, \partial_2 \partial_2 A_1 \\
&\qquad - R_{21} \, \partial_3 \partial_2 A_3 + R_{21} \, \partial_3 \partial_3 A_2 - R_{22} \, \partial_3 \partial_3 A_1 + R_{22} \, \partial_3 \partial_1 A_3 - R_{23} \, \partial_3 \partial_1 A_2 + R_{23} \, \partial_3 \partial_2 A_1 \Big) \\
&\quad + \hat{x}_1 \Big((\partial_2 R_{31}) \, \partial_2 A_3 - (\partial_2 R_{31}) \, \partial_3 A_2 + (\partial_2 R_{32}) \, \partial_3 A_1 - (\partial_2 R_{32}) \, \partial_1 A_3 \\
&\qquad + (\partial_2 R_{33}) \, \partial_1 A_2 - (\partial_2 R_{33}) \, \partial_2 A_1 - (\partial_3 R_{21}) \, \partial_2 A_3 + (\partial_3 R_{21}) \, \partial_3 A_2 \\
&\qquad - (\partial_3 R_{22}) \, \partial_3 A_1 + (\partial_3 R_{22}) \, \partial_1 A_3 - (\partial_3 R_{23}) \, \partial_1 A_2 + (\partial_3 R_{23}) \, \partial_2 A_1 \Big) \\
&\quad + \hat{x}_2 \Big(R_{11} \, \partial_3 \partial_2 A_3 - R_{11} \, \partial_3 \partial_3 A_2 + R_{12} \, \partial_3 \partial_3 A_1 - R_{12} \, \partial_3 \partial_1 A_3 + R_{13} \, \partial_3 \partial_1 A_2 - R_{13} \, \partial_3 \partial_2 A_1 \\
&\qquad - R_{31} \, \partial_1 \partial_2 A_3 + R_{31} \, \partial_1 \partial_3 A_2 - R_{32} \, \partial_1 \partial_3 A_1 + R_{32} \, \partial_1 \partial_1 A_3 - R_{33} \, \partial_1 \partial_1 A_2 + R_{33} \, \partial_1 \partial_2 A_1 \Big) \\
&\quad + \hat{x}_2 \Big((\partial_3 R_{11}) \, \partial_2 A_3 - (\partial_3 R_{11}) \, \partial_3 A_2 + (\partial_3 R_{12}) \, \partial_3 A_1 - (\partial_3 R_{12}) \, \partial_1 A_3 \\
&\qquad + (\partial_3 R_{13}) \, \partial_1 A_2 - (\partial_3 R_{13}) \, \partial_2 A_1 - (\partial_1 R_{31}) \, \partial_2 A_3 + (\partial_1 R_{31}) \, \partial_3 A_2 \\
&\qquad - (\partial_1 R_{32}) \, \partial_3 A_1 + (\partial_1 R_{32}) \, \partial_1 A_3 - (\partial_1 R_{33}) \, \partial_1 A_2 + (\partial_1 R_{33}) \, \partial_2 A_1 \Big) \\
&\quad + \hat{x}_3 \Big(R_{21} \, \partial_1 \partial_2 A_3 - R_{21} \, \partial_1 \partial_3 A_2 + R_{22} \, \partial_1 \partial_3 A_1 - R_{22} \, \partial_1 \partial_1 A_3 + R_{23} \, \partial_1 \partial_1 A_2 - R_{23} \, \partial_1 \partial_2 A_1 \\
&\qquad - R_{11} \, \partial_2 \partial_2 A_3 + R_{11} \, \partial_2 \partial_3 A_2 - R_{12} \, \partial_2 \partial_3 A_1 + R_{12} \, \partial_2 \partial_1 A_3 - R_{13} \, \partial_2 \partial_1 A_2 + R_{13} \, \partial_2 \partial_2 A_1 \Big) \\
&\quad + \hat{x}_3 \Big((\partial_1 R_{21}) \, \partial_2 A_3 - (\partial_1 R_{21}) \, \partial_3 A_2 + (\partial_1 R_{22}) \, \partial_3 A_1 - (\partial_1 R_{22}) \, \partial_1 A_3 \\
&\qquad + (\partial_1 R_{23}) \, \partial_1 A_2 - (\partial_1 R_{23}) \, \partial_2 A_1 - (\partial_2 R_{11}) \, \partial_2 A_3 + (\partial_2 R_{11}) \, \partial_3 A_2 \\
&\qquad - (\partial_2 R_{12}) \, \partial_3 A_1 + (\partial_2 R_{12}) \, \partial_1 A_3 - (\partial_2 R_{13}) \, \partial_1 A_2 + (\partial_2 R_{13}) \, \partial_2 A_1 \Big)
\end{aligned} \tag{9}$$

By using the substitution described in the definition of characteristic determinant, from (9) we obtain the following matrix determining the characteristic determinant corresponding to (8)

$$S(R,\mathbf{l}_{1,3}) = \begin{bmatrix} \begin{array}{l} R_{32}\,\lambda_2\lambda_3 - R_{33}\,\lambda_2\lambda_2 \\ -R_{22}\,\lambda_3\lambda_3 + R_{23}\,\lambda_3\lambda_2 \end{array} & \begin{array}{l} R_{33}\,\lambda_2\lambda_1 - R_{31}\,\lambda_2\lambda_3 \\ +R_{21}\,\lambda_3\lambda_3 - R_{23}\,\lambda_3\lambda_1 \end{array} & \begin{array}{l} R_{31}\,\lambda_2\lambda_2 - R_{32}\,\lambda_2\lambda_1 \\ -R_{21}\,\lambda_3\lambda_2 + R_{22}\,\lambda_3\lambda_1 \end{array} \\ \\ \begin{array}{l} R_{12}\,\lambda_3\lambda_3 - R_{13}\,\lambda_3\lambda_2 \\ -R_{32}\,\lambda_1\lambda_3 + R_{33}\,\lambda_1\lambda_2 \end{array} & \begin{array}{l} R_{13}\,\lambda_3\lambda_1 - R_{33}\,\lambda_1\lambda_1 \\ -R_{11}\,\lambda_3\lambda_3 + R_{31}\,\lambda_1\lambda_3 \end{array} & \begin{array}{l} R_{11}\,\lambda_3\lambda_2 - R_{12}\,\lambda_3\lambda_1 \\ +R_{32}\,\lambda_1\lambda_1 - R_{31}\,\lambda_1\lambda_2 \end{array} \\ \\ \begin{array}{l} R_{22}\,\lambda_1\lambda_3 - R_{23}\,\lambda_1\lambda_2 \\ -R_{12}\,\lambda_2\lambda_3 + R_{13}\,\lambda_2\lambda_2 \end{array} & \begin{array}{l} R_{23}\,\lambda_1\lambda_1 - R_{21}\,\lambda_1\lambda_3 \\ +R_{11}\,\lambda_2\lambda_3 - R_{13}\,\lambda_2\lambda_1 \end{array} & \begin{array}{l} R_{21}\,\lambda_1\lambda_2 - R_{22}\,\lambda_1\lambda_1 \\ -R_{11}\,\lambda_2\lambda_2 + R_{12}\,\lambda_2\lambda_1, \end{array} \end{bmatrix}, \tag{10}$$

where its dependence on R and $\mathbf{l}_{1,3}$ is explicitly reported. It is easy to check that, by adding the first column multiplied by λ_1, the second column multiplied by λ_2 and the third column multiplied by λ_3, one obtains a trivial column, so that, as it is well known,

$$\mathrm{determinant}(S(R,\mathbf{l}_{1,3})) = 0 \tag{11}$$

for any $\mathbf{l}_{1,3}$, independently of R. Thus, the characteristic determinant is everywhere zero in Ω for $\mathbf{l}_{1,3} \neq 0$, independently of R, and then any system having the operator in (8) as its principal part is nowhere elliptic in Ω, independently of R (the operation described between equations (10) and (11) corresponds to the calculation of $S(R,\mathbf{l}_{1,3})\,\mathbf{l}_{1,3}$. The result must necessarily be zero since, as the reader can easily check, it corresponds to the principal part of $\nabla \times (R\,(\nabla \times \nabla u))$, for any scalar field u).

Now, it is easily seen that the 6-by-6 block matrix determining the characteristic determinant of the system (7) is

$$W\big(\kappa,\chi,\gamma,\nu,\mathbf{l}_{1,3}\big) = \begin{bmatrix} S(\kappa,\mathbf{l}_{1,3}) & S(\chi,\mathbf{l}_{1,3}) \\ S(\gamma,\mathbf{l}_{1,3}) & S(\nu,\mathbf{l}_{1,3}) \end{bmatrix}. \tag{12}$$

Then, by proceeding as before, it can be shown that the first (or the last) three rows are linearly dependent for any $\mathbf{l}_{1,3}$, independently of κ and χ (respectively, γ and ν). Hence, system (7), like any system having the operator in (8) as its principal part, is nowhere elliptic in Ω, independently of κ, χ, γ and ν.

As a consequence, for our study we need to define an alternative formulation of the system satisfied by $(\mathbf{A}_1,\mathbf{A}_2)$. Fortunately, it is well known that the divergence-free character of a solution can be exploited to obtain an elliptic system having the same solution. For example, any solenoidal solution of the system

$$\nabla \times (R\,(\nabla \times \mathbf{A})) = \mathbf{F} \tag{13}$$

is also a solution of

$$\nabla \times (R\,(\nabla \times \mathbf{A})) - \nabla\,(\nabla \cdot \mathbf{A}) = \mathbf{F}. \tag{14}$$

When $R = I$ it is very well known that the left-hand side of (14) reduces to $-\nabla^2\mathbf{A}$ and that the resulting system of equations is elliptic.

By using the same approach, we exploit the solenoidality of the vector potentials $\mathbf{A}_1$ and $\mathbf{A}_2$ of Proposition 1 and modify system (7) as follows:

$$\begin{cases} \nabla \times (\kappa\,(\nabla \times \mathbf{A}_1)) - \nabla\,(u_1\,\nabla \cdot \mathbf{A}_1) + \nabla \times (\chi\,(\nabla \times \mathbf{A}_2)) + j\omega\,(\nabla \times \mathbf{A}_2) = -\mathbf{J}_m & \text{in } \Omega \\ \nabla \times (\gamma\,(\nabla \times \mathbf{A}_1)) + \nabla \times (\nu\,(\nabla \times \mathbf{A}_2)) - \nabla\,(u_2\,\nabla \cdot \mathbf{A}_2) - j\omega\,(\nabla \times \mathbf{A}_1) = \mathbf{J}_e & \text{in } \Omega, \end{cases} \tag{15}$$

where u_1 and u_2 are arbitrary auxiliary complex-valued scalar fields which, consistently with HM1, we assume to belong at least to $C^0(\Omega) \cap L^\infty(\Omega)$. Notice that, among the terms like (8), only those that do not vanish for anisotropic materials have been supplemented by an additional term, like from (13) to (14).

The above deduction allows us to state the following proposition.

Proposition 2. *The pair* $(\mathbf{A}_1,\mathbf{A}_2)$ *of Proposition 1 satisfies also (15) with arbitrary* $u_1, u_2 \in C^0(\Omega) \cap L^\infty(\Omega)$.

We observe that by dropping the subscript, any term like $\nabla\left(u_i \nabla \cdot \mathbf{A}_i\right)$ has the following Cartesian components:

$$\begin{aligned}
&\nabla\left(u \nabla \cdot \mathbf{A}\right) \\
&\quad = u\Big(\hat{x}_1\left(\partial_1\partial_1 A_1 + \partial_1\partial_2 A_2 + \partial_1\partial_3 A_3\right) + \hat{x}_2\left(\partial_2\partial_1 A_1 + \partial_2\partial_2 A_2 + \partial_2\partial_3 A_3\right) \\
&\qquad + \hat{x}_3\left(\partial_3\partial_1 A_1 + \partial_3\partial_2 A_2 + \partial_3\partial_3 A_3\right)\Big) + \left(\partial_1 A_1 + \partial_2 A_2 + \partial_3 A_3\right)\left(\hat{x}_1\partial_1 u + \hat{x}_2\partial_2 u + \hat{x}_3\partial_3 u\right)
\end{aligned} \tag{16}$$

and the corresponding matrix $T(u_i, \mathbf{l}_{1,3})$ determining the characteristic determinant of a system having this term as principal part is

$$T(u_i, \mathbf{l}_{1,3}) = u_i \cdot \begin{bmatrix} \lambda_1^2 & \lambda_1\lambda_2 & \lambda_1\lambda_3 \\ \lambda_2\lambda_1 & \lambda_2^2 & \lambda_2\lambda_3 \\ \lambda_3\lambda_1 & \lambda_3\lambda_2 & \lambda_3^2 \end{bmatrix}. \tag{17}$$

Thus, from (15), by using (9), (10), (16) and (17), we deduce that the 6-by-6 block matrix determining the characteristic determinant of the modified system is:

$$U\left(\kappa, \chi, \gamma, \nu, u_1, u_2, \mathbf{l}_{1,3}\right) = \begin{bmatrix} S(\kappa, \mathbf{l}_{1,3}) - T(u_1, \mathbf{l}_{1,3}) & S(\chi, \mathbf{l}_{1,3}) \\ S(\gamma, \mathbf{l}_{1,3}) & S(\nu, \mathbf{l}_{1,3}) - T(u_2, \mathbf{l}_{1,3}) \end{bmatrix}. \tag{18}$$

Hence, by definition [1] (p. 211), system (15) is said to be elliptic in Ω for a specific pair of auxiliary functions u_1 and u_2 and for the constitutive matrix-valued complex functions κ, χ, γ and ν, characterizing the material filling Ω, if and only if

$$\text{determinant}\Big(U\left(\kappa, \chi, \gamma, \nu, u_1, u_2, \mathbf{l}_{1,3}\right)\Big) \neq 0 \quad \forall \mathbf{l}_{1,3} \in \mathbb{R}^3, \mathbf{l}_{1,3} \neq 0, \forall \mathbf{x} \in \Omega. \tag{19}$$

For the sake of conciseness, in the following, under the indicated circumstances, we will often say that system (15) is elliptic, without any additional specification.

Remark 2. *In order to deduce some additional regularity results for the solenoidal vector potentials $\mathbf{A}_1$ and $\mathbf{A}_2$ satisfying system (15) for arbitrary $u_1, u_2 \in C^0(\Omega) \cap L^\infty(\Omega)$ (by Proposition 2), any specific pair (u_1, u_2) of auxiliary functions can be exploited to prove the ellipticity of system (15).*

On the contrary, for the constitutive matrix-valued complex functions κ, χ, γ and ν characterizing the material filling Ω, it is not possible to deduce any result of additional regularity for $\mathbf{A}_1$ and $\mathbf{A}_2$, with the indicated approach, only when no pair of functions $u_1, u_2 \in C^0(\Omega) \cap L^\infty(\Omega)$ that guarantees that condition (19) is satisfied does exist. In this case we will just say that system (15) is not elliptic.

Remark 3. *Suppose that $\exists \mathbf{x} \in \Omega$ such that $u_1(\mathbf{x}) = 0$ ($u_2(\mathbf{x}) = 0$). From equations (12), (17) and (18) we deduce that the first (last) three rows of $W\left(\kappa, \chi, \gamma, \nu, \mathbf{l}_{1,3}\right)$ and $U\left(\kappa, \chi, \gamma, \nu, u_1, u_2, \mathbf{l}_{1,3}\right)$ are the same, for that $\mathbf{x}$. But below equation (12) we show that the first (last) three rows of $W\left(\kappa, \chi, \gamma, \nu, \mathbf{l}_{1,3}\right)$ are linearly dependent for any $\mathbf{l}_{1,3}$, independently of κ and χ (γ and ν) and this implies that condition (19) cannot be satisfied. For this reason, in any analysis aiming at proving that system (15) is either elliptic or not elliptic, the considered set of auxiliary functions $u_1, u_2 \in C^0(\Omega) \cap L^\infty(\Omega)$ can be restricted with the additional conditions $u_1 \neq 0$ in Ω and $u_2 \neq 0$ in Ω.*

4 Local regularity of the strong solutions of the system consisting of Maxwell equations and constitutive relations

In this section we will show that, under the hypotheses HS1 and HM1, if condition (19) holds true for a sufficiently regular pair (u_1, u_2), additional regularity of the constitutive parameters and the sources entails a corresponding additional regularity of the pair $(\mathbf{A}_1, \mathbf{A}_2)$ of Proposition 1 and, then, of any strong solution $(\mathbf{E}, \mathbf{B}, \mathbf{H}, \mathbf{D})$ of (2) and (3).

In particular, we will show that, under these circumstances, we can exploit Theorem 6.4.3, p. 246 of [1] to determine under which additional regularity assumptions concerning the constitutive parameters and the sources, the vector potentials $\mathbf{A}_1$ and $\mathbf{A}_2$ belong to $(W^{n+2,p}_{\mathrm{loc}}(\Omega))^3$ (where [43] (p. 2) $W^{m,p}_{\mathrm{loc}}(\Omega)$ is the set of functions which belong to $W^{m,p}(\mathcal{O})$ for every open $\mathcal{O}$ such that $\overline{\mathcal{O}} \subset \Omega$) or to $(C^{n+2,\lambda}(\Omega))^3$, with n a non-negative integer, $p \in \mathbb{R}, p \geq 2$ and $\lambda \in (0,1]$, and, thus, $\mathbf{E}$, $\mathbf{B}$, $\mathbf{H}$ and $\mathbf{D}$ belong to $(W^{n+1,p}_{\mathrm{loc}}(\Omega))^3$ or to $(C^{n+1,\lambda}(\Omega))^3$. Conditions under which both the potentials and the fields belong to $(C^\infty(\Omega))^3$ or have analytic components will be also given.

Since, by Proposition 2, the pair $(\mathbf{A}_1, \mathbf{A}_2)$ of Proposition 1 satisfies (15), it satisfies also the corresponding weak form, which reads

$$\begin{aligned}
&(\kappa\,(\nabla \times \mathbf{A}_1), \nabla \times \mathbf{v}_1)_{0,\Omega} + (u_1\, \nabla \cdot \mathbf{A}_1, \nabla \cdot \mathbf{v}_1)_{0,\Omega} + (\chi\,(\nabla \times \mathbf{A}_2), \nabla \times \mathbf{v}_1)_{0,\Omega} \\
&\qquad + (\gamma\,(\nabla \times \mathbf{A}_1), \nabla \times \mathbf{v}_2)_{0,\Omega} + (\nu\,(\nabla \times \mathbf{A}_2), \nabla \times \mathbf{v}_2)_{0,\Omega} + (u_2\, \nabla \cdot \mathbf{A}_2, \nabla \cdot \mathbf{v}_2)_{0,\Omega} \\
&\qquad + j\omega\,(\nabla \times \mathbf{A}_2, \mathbf{v}_1)_{0,\Omega} - j\omega\,(\nabla \times \mathbf{A}_1, \mathbf{v}_2)_{0,\Omega} \\
&\qquad = -(\mathbf{J}_m, \mathbf{v}_1)_{0,\Omega} + (\mathbf{J}_e, \mathbf{v}_2)_{0,\Omega} \qquad \forall \mathbf{v}_1, \mathbf{v}_2 \in (\mathcal{D}(\Omega))^3
\end{aligned} \tag{20}$$

where $(\cdot,\cdot)_{0,\Omega}$ denotes the scalar product in $(L^2(\Omega))^3$ and $\mathcal{D}(\Omega)$ the space of infinitely differentiable functions with compact support in Ω and values in $\mathbb{C}$ [47].

The above formulation can be recast in the form (6.4.1), p. 242 of [1], with $s_j = 0$, $m_j = 1$, $t_k = 2$ and $\rho_{jk} = 1$ for $j, k = 1, \ldots, 6$.

Notice that (6.4.2) of [1] are satisfied and we may have $a^{\alpha\,\beta}_{jk} \neq 0$ only for $|\beta| = 1$ and $f^\alpha_j \neq 0$ only for $|\alpha| = 0$. Moreover, when $|\alpha| = 0$ and $|\beta| = 1$, then either $a^{\alpha\,\beta}_{jk} = 0$ or $a^{\alpha\,\beta}_{jk} = \pm j\omega$, while when $|\alpha| = |\beta| = 1$ the coefficients $a^{\alpha\,\beta}_{jk}$ can be equal only to 0, u_1, u_2 or to elements of κ, χ, γ, or ν, possibly with a change of sign. Finally, all the non-vanishing f^α_j are components of $\mathbf{J}_e$ or $-\mathbf{J}_m$.

Now, we set $h_0 = -1$ and $h = n \geq 0$, n integer, so that conditions (6.4.3) of [1] are satisfied. Then the coefficients $a^{\alpha\,\beta}_{jk}$ fulfill the $h-$conditions in Ω (Definition 6.4.1 of [1]) if we assume

HM2(n). $\kappa, \chi, \gamma, \nu \in (C^{n,1}(\Omega))^{3\times 3}$

and

HA1(n). $u_1, u_2 \in C^{n,1}(\Omega)$.

Then, since $(\mathbf{A}_1, \mathbf{A}_2) \in (H^1(\Omega))^3 \times (H^1(\Omega))^3$ and satisfies the weak form (20) of the elliptic system (15), we have just to assume

HS2(n, p). $\mathbf{J}_e, \mathbf{J}_m \in (W^{n,p}_{\mathrm{loc}}(\Omega))^3$, $p \in \mathbb{R}$, $p \geq 2$

to conclude, according to Theorem 6.4.3 of [1], that $(\mathbf{A}_1, \mathbf{A}_2) \in (W^{n+2,p}_{\mathrm{loc}}(\Omega))^3 \times (W^{n+2,p}_{\mathrm{loc}}(\Omega))^3$.

Hence, taking into account that HM2(n) implies HM1 $\forall n \geq 0$, we have proved the following lemma:

Lemma 1. *For any $n \geq 0$, n integer, and any $p \in \mathbb{R}$, $p \geq 2$, under the hypotheses HS1 and HS2(n, p) concerning the sources, and HM2(n) concerning the constitutive parameters, if two auxiliary functions u_1 and u_2 satisfying HA1(n) and such that condition (19) is fulfilled do exist, then the solenoidal vector potentials $\mathbf{A}_1$ and $\mathbf{A}_2$ of Proposition 1 belong to $(W^{n+2,p}_{\mathrm{loc}}(\Omega))^3$.*

In order to achieve the aimed result, we need the following lemma.

Lemma 2. *For any $n \geq 0$, n integer, and any $p \in \mathbb{R}$, $p \geq 2$, if $\mathbf{A}_1, \mathbf{A}_2 \in (W^{n+2,p}_{\mathrm{loc}}(\Omega))^3$ and HM2(n) holds true, then $\mathbf{E}, \mathbf{B}, \mathbf{H}, \mathbf{D}$ obtained from (5) and (6) belong to $(W^{n+1,p}_{\mathrm{loc}}(\Omega))^3$.*

Proof. Just a first order differential operator appears in equations (5) and (6). □

By combining Lemma 1 and Lemma 2, we obtain the following theorem, which gives the Sobolev regularity of any strong solution of (2) and (3).

Theorem 1. *For any $l \geq 0$ and $m \geq 0$, l and m integers, and any $p \in \mathbb{R}$, $p \geq 2$, under the hypotheses HS1, HS2(l,p) concerning the sources, and HM2(m) concerning the constitutive parameters, if two auxiliary functions u_1 and u_2 satisfying HA1(n), where $n = \min\{l, m\}$, and such that condition (19) is fulfilled do exist, then any strong solution $(\mathbf{E}, \mathbf{B}, \mathbf{H}, \mathbf{D}) \in H(\mathrm{curl}, \Omega) \times H(\mathrm{div}^0, \Omega) \times H(\mathrm{curl}, \Omega) \times H(\mathrm{div}^0, \Omega)$ of (2) and (3) actually belongs to $(W^{n+1,p}_{\mathrm{loc}}(\Omega))^3 \times (W^{n+1,p}_{\mathrm{loc}}(\Omega))^3 \times (W^{n+1,p}_{\mathrm{loc}}(\Omega))^3 \times (W^{n+1,p}_{\mathrm{loc}}(\Omega))^3$.*

By substituting in the same line of reasoning exploited to prove Lemma 1 the assumptions HM2(n), HA1(n) and HS2(n,p) by

HM3(n,λ). $\kappa, \chi, \gamma, \nu \in (C^{n+1,\lambda}(\overline{\Omega}))^{3\times 3}$,

HA2(n,λ). $u_1, u_2 \in C^{n+1,\lambda}(\overline{\Omega})$

and

HS3(n,λ). $\mathbf{J}_e, \mathbf{J}_m \in (C^{n,\lambda}(\Omega))^3$

we easily prove the following lemma

Lemma 3. *For any $n \geq 0$, n integer, and $\lambda \in (0,1]$, under the hypotheses HS1 and HS3(n,λ) concerning the sources, and HM3(n,λ) concerning the constitutive parameters, if two auxiliary functions u_1 and u_2 satisfying HA2(n,λ) and such that condition (19) is fulfilled do exist, then the solenoidal vector potentials $\mathbf{A}_1$ and $\mathbf{A}_2$ of Proposition 1 belong to $(C^{n+2,\lambda}(\Omega))^3$.*

Now, again from (5) and (6), we get the following lemma

Lemma 4. *For any $n \geq 0$, n integer, and $\lambda \in (0,1]$, if $\mathbf{A}_1, \mathbf{A}_2 \in (C^{n+2,\lambda}(\Omega))^3$ and HM3(n,λ) holds true, then $\mathbf{E}, \mathbf{B}, \mathbf{H}, \mathbf{D}$ obtained from (5) and (6) belong to $(C^{n+1,\lambda}(\Omega))^3$.*

Proof. Just a first order differential operator appears in equations (5) and (6). □

Finally, from Lemma 3 and Lemma 4, we obtain the following theorem, which gives the Hölder regularity of any strong solution of (2) and (3).

Theorem 2. *For any $p \geq 0$ and $q \geq 0$, p and q integers, and any $\rho, \sigma \in (0,1]$, under the hypotheses HS1, HS3(p,ρ) concerning the sources, and HM3(q,σ) concerning the constitutive parameters, if two auxiliary functions u_1 and u_2 satisfying HA2(n,λ), where $n = \min\{p, q\}$ and $\lambda = \min\{\rho, \sigma\}$, and such that condition (19) is fulfilled do exist, then any strong solution $(\mathbf{E}, \mathbf{B}, \mathbf{H}, \mathbf{D}) \in H(\mathrm{curl}, \Omega) \times H(\mathrm{div}^0, \Omega) \times H(\mathrm{curl}, \Omega) \times H(\mathrm{div}^0, \Omega)$ of (2) and (3) actually belongs to $(C^{n+1,\lambda}(\Omega))^3 \times (C^{n+1,\lambda}(\Omega))^3 \times (C^{n+1,\lambda}(\Omega))^3 \times (C^{n+1,\lambda}(\Omega))^3$.*

Remark 4. *Let us stress that all the hypotheses assumed in deducing the results previously obtained in this section are such that if they hold in Ω, then they hold in any open subset of Ω. Moreover, all the aforementioned results concern local regularity and their proofs hold when Ω is an open ball. Hence, by standard arguments on coverings, they hold for any open domain $\Omega \subset \mathbb{R}^3$.*

Finally, it is useful to state the following corollaries.

Corollary 1. *Under the hypotheses HS1, if $\kappa, \chi, \gamma, \nu \in (C^\infty(\overline{\Omega}))^{3\times 3}$, $\mathbf{J}_e, \mathbf{J}_m \in (C^\infty(\Omega))^3$ and two auxiliary functions $u_1, u_2 \in C^\infty(\overline{\Omega})$ and such that condition (19) is fulfilled do exist, then any strong solution $(\mathbf{E}, \mathbf{B}, \mathbf{H}, \mathbf{D}) \in H(\mathrm{curl}, \Omega) \times H(\mathrm{div}^0, \Omega) \times H(\mathrm{curl}, \Omega) \times H(\mathrm{div}^0, \Omega)$ of (2) and (3) actually belongs to $(C^\infty(\Omega))^3 \times (C^\infty(\Omega))^3 \times (C^\infty(\Omega))^3 \times (C^\infty(\Omega))^3$.*

Proof. Theorem 1 applies for any $n \in \mathbb{N}$ and any $p \in \mathbb{R}$, $p \geq 2$. Then, by the Sobolev embedding the result follows. □

Corollary 2. *Under the hypotheses HS1, if all entries of $\kappa, \chi, \gamma, \nu$ belong to $(C^\infty(\overline{\Omega}))$ and are analytic in Ω, all components of $\mathbf{J}_e, \mathbf{J}_m$ are analytic in Ω and two auxiliary functions u_1 and u_2 do exist that belong to $(C^\infty(\overline{\Omega}))$, are analytic in Ω and are such that condition (19) is fulfilled, then any strong solution $(\mathbf{E}, \mathbf{B}, \mathbf{H}, \mathbf{D}) \in H(\mathrm{curl}, \Omega) \times H(\mathrm{div}^0, \Omega) \times H(\mathrm{curl}, \Omega) \times H(\mathrm{div}^0, \Omega)$ of (2) and (3) is composed by fields having analytic components in Ω.*

Proof. Lemma 1 applies. Then, by the Sobolev embedding we have $\mathbf{A}_1, \mathbf{A}_2 \in (C^\infty(\Omega))^3$. By Theorem 6.6.1, p. 262 of [1] the vector potentials $\mathbf{A}_1$ and $\mathbf{A}_2$ have analytic components in Ω and the thesis follows from (5) and (6). $\square$

Since in this paper we are interested in the limitations possibly set by the very general structure of the constitutive relations of innovative materials to the interior regularity of the electromagnetic field, in our typical configuration the domain Ω is filled by a single material and the electromagnetic field is generated by sources external to the material itself. Therefore, in this situation, no hypothesis about the source currents is required and, in order to apply the previous theorems and corollaries, besides assuming some regularity of the constitutive parameters of the material, we are left just with the crucial task of verifying condition (19).

However, since condition (19) is not easily managed in terms of constitutive parameters, as an intermediate step, in Sections 6 and 7 we will look for sufficient conditions more suited to typical practical situations.

5 Other forms of practical interest of the constitutive relations

Even though the form of the constitutive relations indicated in (3) is the most convenient one for our developments we observe that two other forms are commonly used in studies and applications [19] (pp. 4-9).

One of these forms is the inverse relation of (3) [19] (pp. 4-9) and is considered for example in [48], [35], [28]. It is moreover the universal form adopted for anisotropic media (when $\xi = \zeta = 0$) and reads as follows

$$\begin{cases} \mathbf{D} = \varepsilon\, \mathbf{E} + \xi\, \mathbf{H} & \text{in } \Omega \\ \mathbf{B} = \zeta\, \mathbf{E} + \mu\, \mathbf{H} & \text{in } \Omega. \end{cases} \tag{21}$$

The other form of the constitutive relations having a great importance for applications is the following [19] (pp. 4-9)

$$\begin{cases} \mathbf{D} = (1/c)\, P\, \mathbf{E} + L\, \mathbf{B} & \text{in } \Omega \\ \mathbf{H} = M\, \mathbf{E} + \mathrm{c}\, Q\, \mathbf{B} & \text{in } \Omega, \end{cases} \tag{22}$$

where c is the velocity of light in vacuum. This form is the one used in relativity, in which almost any moving media is bianisotropic, because is Lorentz-covariant [19] (pp. 7-8). In these cases, too, ε, ξ, ζ and μ or P, L, M and Q are four 3-by-3 matrix-valued complex functions of the space point only.

It is easy to deduce that the constitutive parameters appearing in (21) and (22) are linked together by the following relations, if the matrix Q can be inverted

$$\begin{cases} \varepsilon = \frac{1}{c}\,(P - L\, Q^{-1}\, M) \\ \xi = \frac{1}{c}\, L\, Q^{-1} \\ \zeta = -\frac{1}{c}\, Q^{-1}\, M \\ \mu = \frac{1}{c}\, Q^{-1}. \end{cases} \tag{23}$$

Vice versa, if μ can be inverted (the reader can note that the invertibility of Q and the invertibility of μ are actually the same hypothesis) we obtain

$$\begin{cases} P = c\,(\varepsilon - \xi\, \mu^{-1}\, \zeta) \\ L = \xi\, \mu^{-1} \\ M = -\mu^{-1}\, \zeta \\ Q = \frac{1}{c}\, \mu^{-1}. \end{cases} \tag{24}$$

In an analogous way, the constitutive parameters appearing in (3) and (22) are linked together by the following relations, if the matrix P can be inverted

$$\begin{cases} \kappa = c\,P^{-1} \\ \chi = -c\,P^{-1}\,L \\ \gamma = c\,M\,P^{-1} \\ \nu = c\,(Q - M\,P^{-1}\,L). \end{cases} \tag{25}$$

Vice versa, if κ can be inverted (in this case, too, the invertibility of P and the invertibility of κ are actually the same hypothesis) we obtain

$$\begin{cases} P = c\,\kappa^{-1} \\ L = -\kappa^{-1}\,\chi \\ M = \gamma\,\kappa^{-1} \\ Q = \frac{1}{c}\,(\nu - \gamma\,\kappa^{-1}\,\chi). \end{cases} \tag{26}$$

Finally, as already pointed out, the 6-by-6 block matrix made up of ε, ξ, ζ and μ is the inverse of the 6-by-6 block matrix made up of κ, χ, γ and ν, when this inverse exists, and vice versa. If, in particular, κ can be inverted together with its Schur complement [49] (p. 123) $\nu - \gamma\,\kappa^{-1}\,\chi$, from exercise 3.7.11 of [49] (p. 123) we get

$$\begin{cases} \varepsilon = \kappa^{-1} + \kappa^{-1}\,\chi\,(\nu - \gamma\,\kappa^{-1}\,\chi)^{-1}\,\gamma\,\kappa^{-1} \\ \xi = -\kappa^{-1}\,\chi\,(\nu - \gamma\,\kappa^{-1}\,\chi)^{-1} \\ \zeta = -(\nu - \gamma\,\kappa^{-1}\,\chi)^{-1}\,\gamma\,\kappa^{-1} \\ \mu = (\nu - \gamma\,\kappa^{-1}\,\chi)^{-1}. \end{cases} \tag{27}$$

For completeness, under the corresponding hypotheses of invertibility of μ and $\varepsilon - \xi\,\mu^{-1}\,\zeta$, one obtains

$$\begin{cases} \kappa = (\varepsilon - \xi\,\mu^{-1}\,\zeta)^{-1} \\ \chi = -(\varepsilon - \xi\,\mu^{-1}\,\zeta)^{-1}\,\xi\,\mu^{-1} \\ \gamma = -\mu^{-1}\,\zeta\,(\varepsilon - \xi\,\mu^{-1}\,\zeta)^{-1} \\ \nu = \mu^{-1} + \mu^{-1}\,\zeta\,(\varepsilon - \xi\,\mu^{-1}\,\zeta)^{-1}\xi\,\mu^{-1}. \end{cases} \tag{28}$$

Two other similar formulas directly linking (3) and (21) can be obtained under the invertibility of both ν and $\kappa - \chi\,\nu^{-1}\,\gamma$ and both ε and $\mu - \zeta\,\varepsilon^{-1}\,\xi$, respectively.

For the sake of completeness, we also mention the other possible form of the constitutive relations [19] (p. 9). In this form the fields $\mathbf{E}$ and $\mathbf{B}$ are expressed in terms of the fields $\mathbf{D}$ and $\mathbf{H}$ by

$$\begin{cases} \mathbf{E} = \mathcal{P}\,\mathbf{D} + \mathcal{L}\,\mathbf{H} & \text{in } \Omega \\ \mathbf{B} = \mathcal{M}\,\mathbf{D} + \mathcal{Q}\,\mathbf{H} & \text{in } \Omega, \end{cases} \tag{29}$$

where, in this case too, $\mathcal{P}$, $\mathcal{L}$, $\mathcal{M}$ and $\mathcal{Q}$ are four 3-by-3 matrix-valued complex functions of the space point only.

The links between the constitutive matrices $\mathcal{P}$, $\mathcal{L}$, $\mathcal{M}$ and $\mathcal{Q}$ appearing in (29) and those appearing in (21) and (3) are

$$\begin{cases} \varepsilon = \mathcal{P}^{-1} \\ \xi = -\mathcal{P}^{-1}\,\mathcal{L} \\ \zeta = \mathcal{M}\,\mathcal{P}^{-1} \\ \mu = \mathcal{Q} - \mathcal{M}\,\mathcal{P}^{-1}\,\mathcal{L}, \end{cases} \tag{30}$$

if the matrix $\mathcal{P}$ can be inverted,

$$\begin{cases} \mathcal{P} = \varepsilon^{-1} \\ \mathcal{L} = -\varepsilon^{-1}\,\xi \\ \mathcal{M} = \zeta\,\varepsilon^{-1} \\ \mathcal{Q} = \mu - \zeta\,\varepsilon^{-1}\,\xi, \end{cases} \tag{31}$$

if the matrix ε can be inverted,

$$\begin{cases} \kappa = \mathcal{P} - \mathcal{L}\,\mathcal{Q}^{-1}\,\mathcal{M} \\ \chi = \mathcal{L}\,\mathcal{Q}^{-1} \\ \gamma = -\mathcal{Q}^{-1}\,\mathcal{M} \\ \nu = \mathcal{Q}^{-1}, \end{cases} \tag{32}$$

if the matrix $\mathcal{Q}$ can be inverted,

$$\begin{cases} \mathcal{P} = \kappa - \chi\,\nu^{-1}\,\gamma \\ \mathcal{L} = \chi\,\nu^{-1} \\ \mathcal{M} = -\nu^{-1}\,\gamma \\ \mathcal{Q} = \nu^{-1}, \end{cases} \tag{33}$$

if the matrix ν can be inverted.

Finally, the 6-by-6 block matrix made up of $\mathcal{P}$, $\mathcal{L}$, $\mathcal{M}$ and $\mathcal{Q}$ is the inverse of the 6-by-6 block matrix made up of $(1/c)\,P$, L, M and $c\,Q$, when this inverse exists. If, in particular, the matrix Q can be inverted together with its Schur complement $P - L\,Q^{-1}\,M$ we get, in analogy with (28)

$$\begin{cases} \mathcal{P} = c\,(P - L\,Q^{-1}\,M)^{-1} \\ \mathcal{L} = -(P - L\,Q^{-1}\,M)^{-1}\,L\,Q^{-1} \\ \mathcal{M} = -Q^{-1}\,M\,(P - L\,Q^{-1}\,M)^{-1} \\ \mathcal{Q} = \frac{1}{c}\,Q^{-1} + \frac{1}{c}\,Q^{-1}\,M\,(P - L\,Q^{-1}\,M)^{-1}\,L\,Q^{-1}, \end{cases} \tag{34}$$

and, if the matrix $\mathcal{P}$ can be inverted together with its Schur complement $\mathcal{Q} - \mathcal{M}\,\mathcal{P}^{-1}\,\mathcal{L}$, we get, as in (27)

$$\begin{cases} P = c\,\mathcal{P}^{-1} + c\,\mathcal{P}^{-1}\,\mathcal{L}\,(\mathcal{Q} - \mathcal{M}\,\mathcal{P}^{-1}\,\mathcal{L})^{-1}\,\mathcal{M}\,\mathcal{P}^{-1} \\ L = -\mathcal{P}^{-1}\,\mathcal{L}\,(\mathcal{Q} - \mathcal{M}\,\mathcal{P}^{-1}\,\mathcal{L})^{-1} \\ M = -(\mathcal{Q} - \mathcal{M}\,\mathcal{P}^{-1}\,\mathcal{L})^{-1}\,\mathcal{M}\,\mathcal{P}^{-1} \\ Q = \frac{1}{c}\,(\mathcal{Q} - \mathcal{M}\,\mathcal{P}^{-1}\,\mathcal{L})^{-1}. \end{cases} \tag{35}$$

These links will be heavily exploited in Section 7, where several bianisotropic materials of interest in practical applications are considered.

6 The case of anisotropic media or metamaterials: results of ellipticity and regularity and their applications

When just an anisotropic medium is present, the matrix in (18) becomes block diagonal and its determinant is obviously given by the product of the determinants of the submatrices on the main diagonal. To find the conditions which guarantee that for a pair of auxiliary functions they do not vanish or that at least one of them does vanish for any choice of the corresponding auxiliary function, we firstly deduce the explicit form of the submatrices of $U\big(\kappa, \chi, \gamma, \nu, u_1, u_2, \mathbf{l}_{1,3}\big)$ of interest in this case. By using (10) and (17) and by substituting κ or ν with R and u_1 or u_2 with u we obtain:

$$S(R, \mathbf{l}_{1,3}) - T(u, \mathbf{l}_{1,3}) = \tag{36}$$
$$\begin{bmatrix} \begin{matrix}(R_{32} + R_{23})\,\lambda_2\lambda_3 - u\,\lambda_1\lambda_1 \\ -R_{22}\,\lambda_3\lambda_3 - R_{33}\,\lambda_2\lambda_2\end{matrix} & \begin{matrix}(R_{33} - u)\,\lambda_2\lambda_1 - R_{31}\,\lambda_2\lambda_3 \\ +R_{21}\,\lambda_3\lambda_3 - R_{23}\,\lambda_3\lambda_1\end{matrix} & \begin{matrix}(R_{22} - u)\,\lambda_3\lambda_1 - R_{32}\,\lambda_2\lambda_1 \\ R_{31}\,\lambda_2\lambda_2 - R_{21}\,\lambda_3\lambda_2\end{matrix} \\ \\ \begin{matrix}(R_{33} - u)\,\lambda_1\lambda_2 - R_{13}\,\lambda_3\lambda_2 \\ R_{12}\,\lambda_3\lambda_3 - R_{32}\,\lambda_1\lambda_3\end{matrix} & \begin{matrix}(R_{31} + R_{13})\,\lambda_3\lambda_1 - u\,\lambda_2\lambda_2 \\ -R_{11}\,\lambda_3\lambda_3 - R_{33}\,\lambda_1\lambda_1\end{matrix} & \begin{matrix}(R_{11} - u)\,\lambda_3\lambda_2 - R_{12}\,\lambda_3\lambda_1 \\ -R_{31}\,\lambda_1\lambda_2 + R_{32}\,\lambda_1\lambda_1\end{matrix} \\ \\ \begin{matrix}(R_{22} - u)\,\lambda_1\lambda_3 - R_{23}\,\lambda_1\lambda_2 \\ -R_{12}\,\lambda_2\lambda_3 + R_{13}\,\lambda_2\lambda_2\end{matrix} & \begin{matrix}+(R_{11} - u)\,\lambda_2\lambda_3 + R_{23}\,\lambda_1\lambda_1 \\ -R_{21}\,\lambda_1\lambda_3 - R_{13}\,\lambda_2\lambda_1\end{matrix} & \begin{matrix}(R_{21} + R_{12})\,\lambda_2\lambda_1 - u\,\lambda_3\lambda_3 \\ -R_{22}\,\lambda_1\lambda_1 - R_{11}\,\lambda_2\lambda_2\end{matrix} \end{bmatrix}.$$

The calculation of the determinant of this matrix can be simplified by substituting the first row of the matrix with the sum of the first row itself multiplied by λ_1, the second row multiplied by

λ_2 and the third row multiplied by λ_3. In this way one obtains the following new matrix

$$V(R,u,\mathbf{l}_{1,3}) = \tag{37}$$
$$\begin{bmatrix} -u\,\lambda_1(\lambda_1^2+\lambda_2^2+\lambda_3^2) & -u\,\lambda_2(\lambda_1^2+\lambda_2^2+\lambda_3^2) & -u\,\lambda_3(\lambda_1^2+\lambda_2^2+\lambda_3^2) \\ \begin{matrix}(R_{33}-u)\,\lambda_1\lambda_2 - R_{13}\,\lambda_3\lambda_2 \\ R_{12}\,\lambda_3\lambda_3 - R_{32}\,\lambda_1\lambda_3\end{matrix} & \begin{matrix}(R_{31}+R_{13})\,\lambda_3\lambda_1 - u\,\lambda_2\lambda_2 \\ -R_{11}\,\lambda_3\lambda_3 - R_{33}\,\lambda_1\lambda_1\end{matrix} & \begin{matrix}(R_{11}-u)\,\lambda_3\lambda_2 - R_{12}\,\lambda_3\lambda_1 \\ -R_{31}\,\lambda_1\lambda_2 + R_{32}\,\lambda_1\lambda_1\end{matrix} \\ \begin{matrix}(R_{22}-u)\,\lambda_1\lambda_3 - R_{23}\,\lambda_1\lambda_2 \\ -R_{12}\,\lambda_2\lambda_3 + R_{13}\,\lambda_2\lambda_2\end{matrix} & \begin{matrix}+(R_{11}-u)\,\lambda_2\lambda_3 + R_{23}\,\lambda_1\lambda_1 \\ -R_{21}\,\lambda_1\lambda_3 - R_{13}\,\lambda_2\lambda_1\end{matrix} & \begin{matrix}(R_{21}+R_{12})\,\lambda_2\lambda_1 - u\,\lambda_3\lambda_3 \\ -R_{22}\,\lambda_1\lambda_1 - R_{11}\,\lambda_2\lambda_2\end{matrix} \end{bmatrix}.$$

whose determinant is a-priori related to the determinant of the matrix $S-T$ by [49] (p. 463)

$$\text{determinant}(V(R,u,\mathbf{l}_{1,3})) = \lambda_1 \cdot \text{determinant}(S(R,\mathbf{l}_{1,3}) - T(u,\mathbf{l}_{1,3})). \tag{38}$$

The calculation of the determinant of $V(R,u,\mathbf{l}_{1,3})$ can be performed directly and after some long but trivial calculations one deduces that

$$\text{determinant}(V(R,u,\mathbf{l}_{1,3})) = -u\,\lambda_1\left(\lambda_1^2+\lambda_2^2+\lambda_3^2\right)^2(\mathbf{l}_{1,3}^T\,Z(R)\,\mathbf{l}_{1,3}), \tag{39}$$

where $Z(R)$ is a 3-by-3 matrix whose entry $Z(R)_{ij}$ is the cofactor [49] (p. 477) of R_{ji} (or, equivalently, R_{ij}).

Thus,

$$\text{determinant}(S(R,\mathbf{l}_{1,3}) - T(u,\mathbf{l}_{1,3})) = -u\left(\lambda_1^2+\lambda_2^2+\lambda_3^2\right)^2(\mathbf{l}_{1,3}^T\,Z(R)\,\mathbf{l}_{1,3}), \tag{40}$$

if $\lambda_1 \neq 0$. The conclusion is valid in general, however, since, by definition of ellipticity, at least one of the λ_i, $i=1,\ldots,3$, is different from zero.

Hence, for anisotropic media or metamaterials, condition (19), which defines the ellipticity of system (15), can be substituted by the following equivalent condition on the auxiliary functions u_1 and u_2 and on the matrices $Z(\kappa)$ and $Z(\nu)$:

$$u_1 \neq 0 \text{ and } u_2 \neq 0 \text{ and } \mathbf{l}_{1,3}^T\,Z(\kappa)\,\mathbf{l}_{1,3} \neq 0 \text{ and } \mathbf{l}_{1,3}^T\,Z(\nu)\,\mathbf{l}_{1,3} \neq 0 \quad \forall \mathbf{l}_{1,3} \in \mathbb{R}^3, \mathbf{l}_{1,3} \neq 0, \forall \mathbf{x} \in \Omega. \tag{41}$$

The conditions on the arbitrary auxiliary functions u_1 and u_2 can be easily fulfilled owing to their arbitrariness. Since these requirements are exactly the ones discussed in Remark 3, they do not actually enforce any new condition on u_1 and u_2. The last two requirements appearing in (41) involve the matrices $Z(\kappa)$ and $Z(\nu)$, which are complex in general, and the vector $\mathbf{l}_{1,3}$, which, on the contrary, is real by definition. For this reason a deduction of equivalent and simpler conditions could be not easy. Fortunately, a part of these difficulties can be easily removed, since at least the invertibility of κ and ν should be taken for granted everywhere in Ω. As a matter of fact, from Section 5, under the condition

HM4. $\chi = \gamma = 0 \quad \forall \mathbf{x} \in \Omega$,

when the inverse matrices involved have a meaning in Ω, we deduce

$$\begin{cases} \varepsilon = \kappa^{-1} \\ \mu = \nu^{-1}, \end{cases} \quad \begin{cases} \kappa = \varepsilon^{-1} \\ \nu = \mu^{-1}, \end{cases} \tag{42}$$

or

$$\begin{cases} P = c\,\kappa^{-1} \\ Q = \frac{1}{c}\,\nu, \end{cases} \quad \begin{cases} \kappa = c\,P^{-1} \\ \nu = c\,Q, \end{cases} \tag{43}$$

or

$$\begin{cases} \mathcal{P} = \kappa \\ \mathcal{Q} = \nu^{-1}, \end{cases} \quad \begin{cases} \kappa = \mathcal{P} \\ \nu = \mathcal{Q}^{-1}. \end{cases} \tag{44}$$

Thus, if κ or ν cannot be inverted everywhere in Ω, the other constitutive parameters and, in particular, ε and μ, which are universally adopted in these cases, are not everywhere defined in Ω and the problem would seem physically meaningless.

For the indicated reason, in the following we define

HM5(R). $|\text{determinant}(R)| > 0 \; \forall \mathbf{x} \in \Omega$

and assume HM5(κ) and HM5(ν).

Now we observe that for a generic invertible matrix R it is well known [49] (p. 479) that $Z(R) = \text{determinant}(R)\, R^{-1}$. Hence, under the assumptions HM5(κ) and HM5(ν), the necessary and sufficient condition (41) for the ellipticity of system (15) for any $u_1 \neq 0$ and any $u_2 \neq 0$ in Ω becomes equivalent to HM6(κ) and HM6(ν), having defined

HM6(R). $\mathbf{l}_{1,3}^T\, R^{-1}\, \mathbf{l}_{1,3} \neq 0 \; \forall \mathbf{l}_{1,3} \in \mathbb{R}^3, \mathbf{l}_{1,3} \neq 0, \; \forall \mathbf{x} \in \Omega,$

(the reader can notice a strong analogy of HM5(κ), HM5(ν), HM6(κ) and HM6(ν) with the even stronger hypotheses assumed in Section 3 of [50]).

Hence, we conclude that the following theorem holds true

Theorem 3. *Under the hypotheses HM1, HM4, HM5(κ) and HM5(ν), conditions HM6(κ) and HM6(ν) hold true if and only if condition (19) holds true for any $u_1 \neq 0$ and any $u_2 \neq 0$ in Ω.*

In order to ease the exploitation of this theorem, more explicit and simpler conditions implying HM5(κ), HM5(ν), HM6(κ) and HM6(ν) are deduced in the following.

Before doing that, let us introduce some notations and some well known results concerning matrices or matrix functions.

For any matrix R we denote by R_s and R_{ss} the Hermitian symmetric matrices $R_s = \frac{R+R^*}{2}$ and $R_{ss} = \frac{R^*-R}{2j}$, so that $R = R_s - jR_{ss}$. We will say that a 3×3 Hermitian symmetric matrix N is positive (negative) semidefinite if $\mathbf{v}^* N \mathbf{v} \geq 0$ ($\mathbf{v}^* N \mathbf{v} \leq 0$), $\forall \mathbf{v} \in \mathbb{C}^3$ [49] (p. 558). The same matrix is said to be positive (negative) definite [49] (p. 559) if $\mathbf{v}^* N \mathbf{v} > 0$ ($\mathbf{v}^* N \mathbf{v} < 0$), $\forall \mathbf{v} \in \mathbb{C}^3$, $\mathbf{v} \neq 0$ and indefinite if it is neither positive nor negative semidefinite. A 3×3 Hermitian symmetric matrix-valued function $N(\mathbf{x})$ with domain D, is said to be uniformly positive (negative) definite in an open set $D_o \subset D$ if there exists $C > 0$ such that $\mathbf{v}^* N(\mathbf{x}) \mathbf{v} \geq C|\mathbf{v}|^2$ ($\mathbf{v}^* N(\mathbf{x}) \mathbf{v} \leq -C|\mathbf{v}|^2$) $\forall \mathbf{x} \in D_o$, $\forall \mathbf{v} \in \mathbb{C}^3$.

It is known that an invertible Hermitian symmetric matrix has an inverse which is Hermitian symmetric, too, [49] (p. 120) and that (see Theorem 1 of [37]) the matrix-valued function N has an inverse $\mathcal{N} = N^{-1}$ in D_o whenever at least one of the Hermitian symmetric matrix-valued functions $N_s = \frac{N+N^*}{2}$ or $N_{ss} = \frac{N^*-N}{2j}$ is either uniformly positive or uniformly negative definite in D_o. If, moreover, $N|_{D_o} \in (C^0(D_o) \cap L^\infty(D_o))^{3\times 3}$ then $\mathcal{N}|_{D_o} \in (C^0(D_o) \cap L^\infty(D_o))^{3\times 3}$.

Finally, (see again Theorem 1 of [37]) the Hermitian symmetric matrix-valued function $\mathcal{N}_s = \frac{\mathcal{N}+\mathcal{N}^*}{2}$ is uniformly positive (negative) definite in D_o if N_s is uniformly positive (negative) definite in D_o and the entries of N are bounded in D_o. On the contrary, the Hermitian symmetric matrix-valued function $\mathcal{N}_{ss} = \frac{\mathcal{N}^*-\mathcal{N}}{2j}$ is uniformly negative (positive) definite in D_o if N_{ss} is uniformly positive (negative) definite in D_o and the entries of N are bounded in D_o. Furthermore, $\mathcal{N}_{ss}$ is negative (positive) semidefinite in D_o if $\mathcal{N}$ exists and N_{ss} is positive (negative) semidefinite in D_o.

With these preliminary considerations it is easy to prove the following theorem.

Theorem 4. *Under hypotheses HM1 and HM4, whenever ε_s and μ_s or ε_{ss} and μ_{ss} or ε_s and μ_{ss} or μ_s and ε_{ss} are uniformly definite in Ω, hypotheses HM5(κ), HM5(ν), HM6(κ) and HM6(ν) are satisfied. We can replace ε_s, μ_s, ε_{ss} and μ_{ss} respectively with κ_s, ν_s, κ_{ss} and ν_{ss} or P_s, Q_s, P_{ss} and Q_{ss} or $\mathcal{P}_s$, $\mathcal{Q}_s$, $\mathcal{P}_{ss}$ and $\mathcal{Q}_{ss}$ without changing the conclusion.*

Proof. From the indicated preliminary considerations we deduce that, in any of the indicated cases, ε and μ can be inverted in Ω, so that κ and ν have a meaning in Ω together with their inverse. Then HM5(κ) and HM5(ν) are satisfied. Moreover, in any case, κ_s and ν_s or κ_{ss} and ν_{ss} or κ_s and ν_{ss} or κ_{ss} and ν_s are uniformly definite in Ω. Then $|Re(\mathbf{v}^*\kappa^{-1}\mathbf{v})| = |\mathbf{v}^*(\kappa^{-1})_s\mathbf{v}| \geq C|\mathbf{v}|^2 > 0$ or $|Im(\mathbf{v}^*\kappa^{-1}\mathbf{v})| = |\mathbf{v}^*(\kappa^{-1})_{ss}\mathbf{v}| \geq C|\mathbf{v}|^2 > 0$ and $|Re(\mathbf{v}^*\nu^{-1}\mathbf{v})| = |\mathbf{v}^*(\nu^{-1})_s\mathbf{v}| \geq C|\mathbf{v}|^2 > 0$ or $|Im(\mathbf{v}^*\nu^{-1}\mathbf{v})| = |\mathbf{v}^*(\nu^{-1})_{ss}\mathbf{v}| \geq C|\mathbf{v}|^2 > 0$. Hence $\mathbf{v}^*\kappa^{-1}\mathbf{v} \neq 0$ and $\mathbf{v}^*\nu^{-1}\mathbf{v} \neq 0$, $\forall \mathbf{x} \in \Omega, \forall \mathbf{v} \in \mathbb{C}^3, \mathbf{v} \neq 0$, and these conditions imply HM6(κ) and HM6(ν). □

Since, under conditions HM1, HM4, HM5(κ) and HM5(ν), hypotheses HM6(κ) and HM6(ν) are necessary for the ellipticity of system (15), it could be useful to know in advance under which explicit conditions hypotheses HM6(κ) or HM6(ν) cannot hold true. Several combinations are possible, but among these the one which appears to be more interesting refers to the modelling of anisotropic media having negligible losses. For these media one can then assume that $\varepsilon_{ss} = 0$ and $\mu_{ss} = 0$ and it is trivial to check that

Theorem 5. *Under hypotheses HM1 and HM4, whenever $\varepsilon_{ss} = \mu_{ss} = 0\ \forall \mathbf{x} \in \Omega$ and $\exists \mathbf{x} \in \Omega$ such that $\varepsilon_s(\mathbf{x})$ or $\mu_s(\mathbf{x})$ is indefinite, hypotheses HM6(κ) and HM6(ν) are not both satisfied. Provided that κ_s and ν_s or Q_s or $\mathcal{P}_s$ are invertible in Ω, in the previous statement we can replace ε_s, μ_s, ε_{ss} and μ_{ss} respectively with κ_s, ν_s, κ_{ss} and ν_{ss} or P_s, Q_s, P_{ss} and Q_{ss} or $\mathcal{P}_s$, $\mathcal{Q}_s$, $\mathcal{P}_{ss}$ and $\mathcal{Q}_{ss}$ without changing the conclusion.*

It could be important to notice that, when neither Theorem 4 nor Theorem 5 can be applied, nonetheless system (15) may be elliptic. We can assess whether this is the case by checking directly the validity of the hypotheses of Theorem 3, as it will be shown in the next subsection.

In Figure 1 we summarize our results on the ellipticity of system (15) for anisotropic media or metamaterials.

We are now in a position to give a result about the regularity of the electromagnetic fields inside anisotropic materials. In fact, by combining Theorems 1 and 2 and Corollaries 1 and 2 with Theorems 3 and 4 and noticing that we can set u_1 and u_2 to any constant different from zero in order to satisfy any conceivable regularity requirement on them, we obtain the following theorem

Theorem 6. *Suppose that HS1, HM1 and HM4 are satisfied. Suppose, moreover, that one of the following alternatives holds true:*

- *HM5(κ), HM5(ν), HM6(κ) and HM6(ν) are satisfied*
- *either κ_s and ν_s or κ_{ss} and ν_{ss} or κ_s and ν_{ss} or κ_{ss} and ν_s are uniformly definite in Ω.*

In any of the above cases, for any $n \geq 0$, n integer, $p \in \mathbb{R}, p \geq 2$ ($\lambda \in (0,1]$), if HS2(n,p) and HM2(n) (respectively, HS3(n,λ) and HM3(n,λ)) are satisfied, then all strong solutions $(\mathbf{E}, \mathbf{B}, \mathbf{H}, \mathbf{D}) \in H(\mathrm{curl}, \Omega) \times H(\mathrm{div}^0, \Omega) \times H(\mathrm{curl}, \Omega) \times H(\mathrm{div}^0, \Omega)$ of (2) and (3) actually belong to $(W^{n+1,p}_{\mathrm{loc}}(\Omega))^3 \times (W^{n+1,p}_{\mathrm{loc}}(\Omega))^3 \times (W^{n+1,p}_{\mathrm{loc}}(\Omega))^3 \times (W^{n+1,p}_{\mathrm{loc}}(\Omega))^3$ (respectively, $(C^{n+1,\lambda}(\Omega))^3 \times (C^{n+1,\lambda}(\Omega))^3 \times (C^{n+1,\lambda}(\Omega))^3 \times (C^{n+1,\lambda}(\Omega))^3$). Moreover, if $\kappa, \chi, \gamma, \nu \in (C^\infty(\overline{\Omega}))^{3\times 3}$ and $\mathbf{J}_e, \mathbf{J}_m \in (C^\infty(\Omega))^3$ (respectively, all entries of $\kappa, \chi, \gamma, \nu \in C^\infty(\overline{\Omega})$ and are analytic in Ω and all components of $\mathbf{J}_e, \mathbf{J}_m$ are analytic in Ω) then the aforementioned strong solution $(\mathbf{E}, \mathbf{B}, \mathbf{H}, \mathbf{D})$ actually belong to $(C^\infty(\Omega))^3 \times (C^\infty(\Omega))^3 \times (C^\infty(\Omega))^3 \times (C^\infty(\Omega))^3$ (respectively, is composed by fields having analytic components in Ω).

Remark 5. *From Theorem 4 one can observe that the alternatives of Theorem 6 could be completed with*

- *either ε_s and μ_s or ε_{ss} and μ_{ss} or ε_s and μ_{ss} or μ_s and ε_{ss} are uniformly definite in Ω*
- *either P_s and Q_s or P_{ss} and Q_{ss} or P_s and Q_{ss} or P_{ss} and Q_s are uniformly definite in Ω*
- *either $\mathcal{P}_s$ and $\mathcal{Q}_s$ or $\mathcal{P}_{ss}$ and $\mathcal{Q}_{ss}$ or $\mathcal{P}_s$ and $\mathcal{Q}_{ss}$ or $\mathcal{P}_{ss}$ and $\mathcal{Q}_s$ are uniformly definite in Ω*

which are written in terms of the other types of constitutive parameters. These additional alternatives are provided to ease the exploitation of the result but do not extend the coverage of Theorem 6.

Finally, it could be interesting to point out that in some cases involving hypothetical anisotropic media and solenoidal sources, not only a result of ellipticity of system (15) cannot be achieved, but also a result of H^1_{loc} regularity of the fields, the lowest level of additional regularity we are interested in, is not possible. This is shown in the following example, where Ω is considered as an open ball centered at the origin and the constitutive parameters are assumed to satisfy

$$\kappa = \nu = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{45}$$

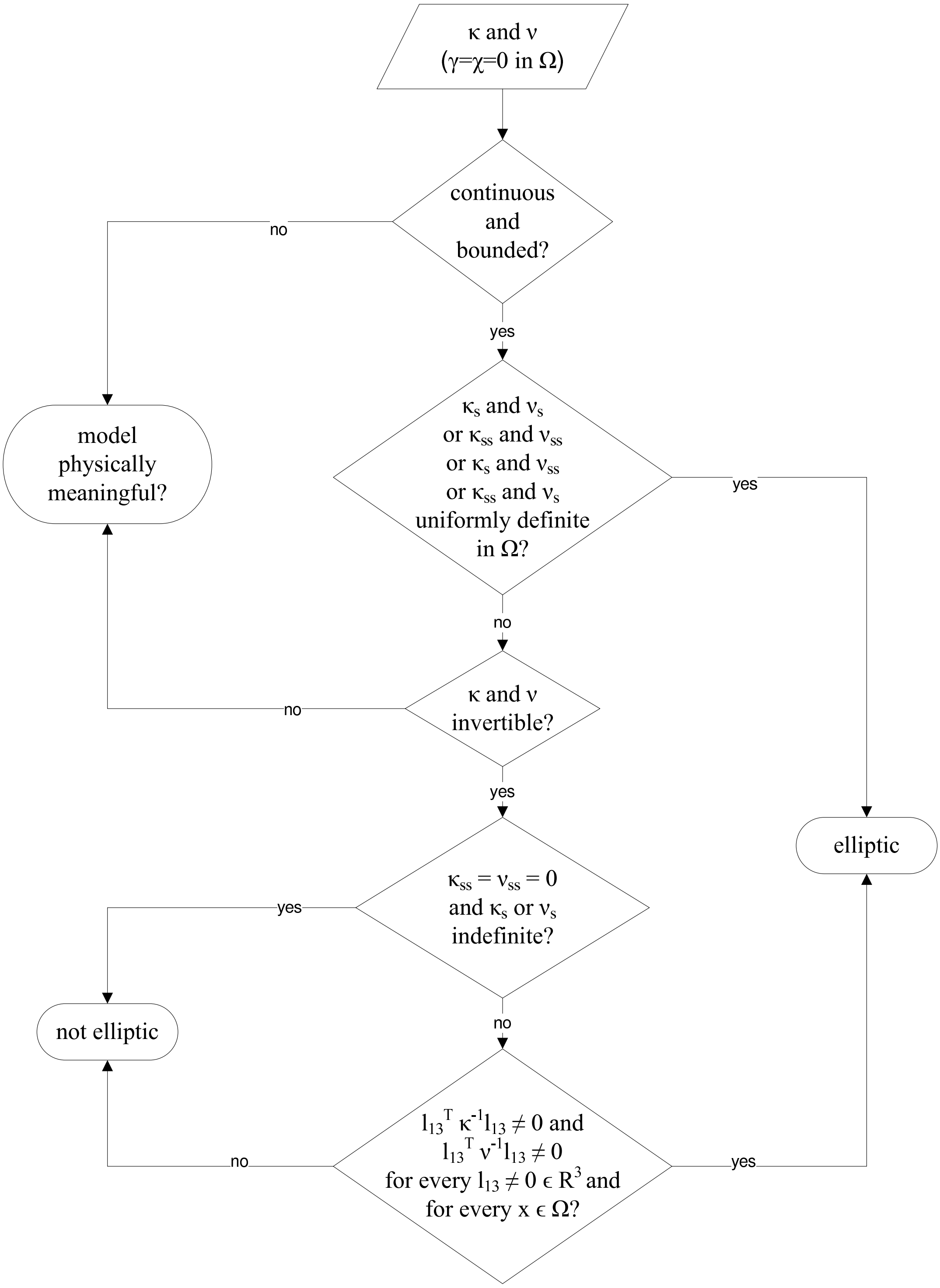

Figure 1: Flow chart giving indications on the ellipticity of system (15) and, then, on the regularity of the electromagnetic field in the interior of a single anisotropic substance. Analogous flow charts could be obtained in terms of ε and μ or P and Q or $\mathcal{P}$ and $\mathcal{Q}$.

In this case one can consider

$$\mathbf{E} = \mathbf{H} = \begin{cases} 0 & \forall (x_1, x_2, x_3) \in \Omega : x_1 - x_2 < 0 \\ \hat{x}_1 - \hat{x}_2 & \forall (x_1, x_2, x_3) \in \Omega : x_1 - x_2 > 0 \end{cases} \tag{46}$$

and

$$\mathbf{D} = \mathbf{B} = \frac{j}{\omega}\mathbf{J}_e = \frac{j}{\omega}\mathbf{J}_m = \begin{cases} 0 & \forall (x_1, x_2, x_3) \in \Omega : x_1 - x_2 < 0 \\ \hat{x}_1 + \hat{x}_2 & \forall (x_1, x_2, x_3) \in \Omega : x_1 - x_2 > 0 \end{cases} \tag{47}$$

so that $\kappa, \nu, \kappa^{-1}, \nu^{-1} \in (W^{n,\infty}(\Omega))^{3\times 3}$, for any $n \in \mathbb{N}$, $\mathbf{E}, \mathbf{H} \in H(\mathrm{curl}^0, \Omega)$ (i. e., $\mathbf{E} \in H(\mathrm{curl}, \Omega)$ and $\nabla \times \mathbf{E} = 0$), $\mathbf{D}, \mathbf{B}, \mathbf{J}_e, \mathbf{J}_m \in H(\mathrm{div}^0, \Omega)$, Maxwell equations (2) and the constitutive relations (3) are satisfied in Ω but $\mathbf{E}, \mathbf{H} \notin (H^1_{loc}(\Omega))^3$, as it is easy to verify.

6.1 Practical applications

It is well known that the constitutive parameters of standard passive anisotropic substances have ε_s and μ_s uniformly positive definite. As a matter of fact, it is precisely the lack of this property that is pointed out when one refers to innovative anisotropic media like double-negative or single-negative materials [16]. In particular, many recent artificially made materials that do not exist in nature have been claimed to have effective constitutive parameters ε and μ characterized by uniformly negative definite ε_s and μ_s [16], [51], or by a uniformly negative definite ε_s and a uniformly positive definite μ_s or, finally, by a uniformly positive definite ε_s and a uniformly negative definite μ_s [16] (see also [37] for the corresponding definitions of double-negative, epsilon-negative or mu-negative materials).

In all these cases Theorem 4 applies and we directly conclude that system (15) is elliptic for any $u_1 \neq 0$ and $u_2 \neq 0$ in Ω and, in particular, when u_1 and u_2 are constants different from zero. Hence, the regularity of the fields depends on the regularity of the source currents and the constitutive parameters, as stated in Theorem 6 (see also Remark 5).

The reader can note that losses cannot affect this conclusion so that for most of the (known models of) anisotropic media [16] the ellipticity can be taken for granted. The same conclusion could even be obtained for active media, as it is trivial to check.

However, it is important to notice that some anisotropic metamaterials or effective models of metamaterials are believed to have or are modelled as having ε_s or μ_s indefinite and ε_{ss} or μ_{ss} at most positive semidefinite [34], [52] [53] [54].

Some of these metamaterials, in particular, seems to be modelled as lossless media by using a Hermitian symmetric ε and μ, with ε_s and μ_s being invertible and indefinite [34], [54]. In all these cases, Theorem 5 applies so that system (15) is not elliptic in the interior of these media (for any choice of u_1 and u_2). In this case, Theorem 6 does not apply and, with our approach, we can conclude nothing about field regularity. Other approaches should be considered if one is interested in the regularity of the time-harmonic electromagnetic fields which can be generated in these (models of) materials (see the beginning of Section 3).

Fortunately, in many other cases considered in practice with ε_s and μ_s being invertible and indefinite, the ellipticity of system (15) can still be deduced. As an example representative of wide classes of media of interest, in [52] and [53] the following effective dielectric permittivity and magnetic permeability are considered (a Cartesian coordinate system is assumed)

$$\varepsilon = \varepsilon_0 \begin{bmatrix} \varepsilon_{11} & 0 & 0 \\ 0 & \varepsilon_{22} & 0 \\ 0 & 0 & \varepsilon_{33} \end{bmatrix}, \tag{48}$$

$$\mu = \mu_0 \begin{bmatrix} \mu_{11} & 0 & 0 \\ 0 & \mu_{22} & 0 \\ 0 & 0 & \mu_{33} \end{bmatrix} \tag{49}$$

where two (one) entries (entry) of ε and μ are (is) real and positive while the remaining one (two) is (are) complex with negative real and imaginary parts. In these cases, too, one can easily verify that whenever $(\mathbf{l}_{1,3}^T\,\varepsilon_s\,\mathbf{l}_{1,3}) = 0$, or $(\mathbf{l}_{1,3}^T\,\mu_s\,\mathbf{l}_{1,3}) = 0$, we have $(\mathbf{l}_{1,3}^T\,\varepsilon_{ss}\,\mathbf{l}_{1,3}) \neq 0$, or $(\mathbf{l}_{1,3}^T\,\mu_{ss}\,\mathbf{l}_{1,3}) \neq 0$, so that HM5($\kappa$), HM5($\nu$), HM6($\kappa$) and HM6($\nu$) are satisfied, system (15) is elliptic in these media, too, for any $u_1 \neq 0$ and $u_2 \neq 0$ in Ω and Theorem 6 directly gives the aimed relations between the regularity of the fields and those of the source currents and the constitutive parameters.

Remark 6. *From a physical point of view it seems reasonable that the diagonal entries having a negative real part have also a negative imaginary part, since the mechanisms giving negative effective parameters cannot take place without losses. For this reason, as a reviewer pointed out, the constitutive parameters considered above, for which, with our approach, we can conclude nothing about field regulatity, could be regarded as simplified effective constitutive parameters.*

Remark 7. *One can note that in the open literature one can find examples of anisotropic media of all the types considered in the flow chart of Figure 1, except for the models with non-bounded or non-continuous entries or having non-invertible constitutive matrix functions, which, indeed, we consider just to discard them as physically questionable.*

Remark 8. *The considerations at the beginning of this subsection apply, in particular, to isotropic substances. In this case, for the ellipticity of system (15), it is sufficient that either the real or the imaginary part of both ε and μ (or P and Q or κ and ν or $\mathcal{P}$ and $\mathcal{Q}$) is uniformly definite in Ω. Let us also mention that, however, in order to prove just the H^1_{loc} regularity of the fields in isotropic media, the alternative procedure exploited in [6] can be considered, too. As a matter of fact, by taking the divergence of the simplified version of the constitutive relation $(3)_1$ ($(3)_2$) given by $\mathbf{E} = \kappa\,\mathbf{D}$ ($\mathbf{H} = \nu\,\mathbf{B}$), where κ (respectively ν) is a scalar field, we obtain (in the sense of distributions)*

$$\nabla \cdot \mathbf{E} = \nabla\kappa \cdot \mathbf{D} + \kappa\nabla \cdot \mathbf{D} \quad (\nabla \cdot \mathbf{H} = \nabla\nu \cdot \mathbf{B} + \nu\nabla \cdot \mathbf{B}). \tag{50}$$

Then, if we assume $\kappa \in W^{1,\infty}(\Omega)$ ($\nu \in W^{1,\infty}(\Omega)$) and, possibly, weaken the hypothesis $\mathbf{D} \in H(\mathrm{div}^0,\Omega)$ ($\mathbf{B} \in H(\mathrm{div}^0,\Omega)$) to $\mathbf{D} \in H(\mathrm{div},\Omega)$ ($\mathbf{B} \in H(\mathrm{div},\Omega)$), it follows that $\nabla \cdot \mathbf{E} \in L^2(\Omega)$ ($\nabla \cdot \mathbf{H} \in L^2(\Omega)$).

Now, since $\mathbf{E} \in H(\mathrm{curl},\Omega)$ ($\mathbf{H} \in H(\mathrm{curl},\Omega)$), by Corollary 2.10 of [43] (p. 36) one can deduce $\mathbf{E} \in (H^1_{loc}(\Omega))^3$ ($\mathbf{H} \in (H^1_{loc}(\Omega))^3$).

If, moreover, $\kappa^{-1} \in L^\infty(\Omega)$ ($\nu^{-1} \in L^\infty(\Omega)$) then

$$\nabla \times \mathbf{D} = \frac{1}{\kappa}\nabla \times \mathbf{E} - \frac{1}{\kappa}\nabla\kappa \times \mathbf{D} \quad (\nabla \times \mathbf{B} = \frac{1}{\nu}\nabla \times \mathbf{H} - \frac{1}{\nu}\nabla\nu \times \mathbf{B}), \tag{51}$$

so that $\mathbf{D} \in H(\mathrm{curl},\Omega)$ ($\mathbf{B} \in H(\mathrm{curl},\Omega)$) and again by Corollary 2.10 of [43] (p. 36) one deduces $\mathbf{D} \in (H^1_{loc}(\Omega))^3$ ($\mathbf{B} \in (H^1_{loc}(\Omega))^3$).

The reader can observe, however, that the hypotheses assumed on the constitutive parameters for this alternative procedure aimed at proving the H^1_{loc} regularity of the fields in isotropic media are almost the same (just a bit stronger, indeed) which allow to obtain the same regularity result by using our analysis.

7 The case of bianisotropic media or metamaterials: results of ellipticity and regularity and their applications

As we have already pointed out at the end of Section 4, condition (19) is not easily managed in terms of constitutive parameters, especially when the constitutive relations (3) are complex as in the case of bianisotropic media. Thus, in this section we look for simpler and more explicit conditions on the constitutive parameters defining the electromagnetic behaviour of bianisotropic materials which imply the ellipticity of system (15).

In order to obtain this result we firstly consider a simple and reasonable hypothesis on the constitutive parameters which allows to deduce a condition simpler than the ellipticity condition (19) and equivalent to it, under some simple additional hypotheses. This condition will then be exploited to obtain the main results of this section. Technically speaking, these results apply every time the magnetoelectric effects due to χ and γ are much smaller than the effects due to κ and ν and some abstract theoretical considerations suggest that this should always be the case, in practice [18]. Several practical applications of these results confirm their importance.

According to the above plan, we observe that it seems physically meaningless not to assume that

HM7. All forms of the constitutive relations have a meaning everywhere in Ω.

This hypothesis guarantees that κ, ν, ε, μ, P, Q, $\mathcal{P}$ and $\mathcal{Q}$ can be inverted everywhere in Ω. As a matter of fact, by multiplying equation $(3)_1$ by $\frac{1}{c}\,P$ we obtain

$$\frac{1}{c}\,P\,\mathbf{E} = \frac{1}{c}\,P\,\kappa\,\mathbf{D} + \frac{1}{c}\,P\,\chi\,\mathbf{B} \tag{52}$$

and, by comparing the result with equation $(22)_1$, here reported to ease the reading

$$\frac{1}{c}\,P\,\mathbf{E} = \mathbf{D} - L\,\mathbf{B}, \tag{53}$$

for any $\mathbf{D},\mathbf{B} \in \mathbb{C}^3$ (we assume that $(\mathbf{D},\mathbf{B})$ can span $\mathbb{C}^3 \times \mathbb{C}^3$), we deduce that κ and P can be inverted everywhere in Ω. Analogously, by multiplying equation $(29)_2$ by ν (respectively, equation $(29)_1$ by ε or equation $(21)_2$ by $c\,Q$) and comparing the result with equation $(3)_2$ (respectively, equation $(21)_1$ or equation $(22)_2$) for any $\mathbf{H},\mathbf{D} \in \mathbb{C}^3$ (respectively, for any $\mathbf{D},\mathbf{H} \in \mathbb{C}^3$ or for any $\mathbf{E},\mathbf{H} \in \mathbb{C}^3$) we deduce that ν and $\mathcal{Q}$ (respectively, ε and $\mathcal{P}$ or Q and μ) can be inverted everywhere in Ω.

On the basis of the above considerations we do not worry about the invertibility, everywhere in Ω, of κ and ν and, thus, conditions HM5(κ) or HM5(ν), reported in Section 6, will be considered, whenever useful, as hypotheses in all statements of this section.

HM5(κ) and HM5(ν) are of great help but without further hypotheses the analysis of interest in this section remains too complex without covering additional significant applications. For this reason, we strenghten the above hypotheses by assuming HM6(κ) or HM6(ν). It should be noted that, however, with this assumption some abstract models of media like those considered in [34], [54] (see Section 6) are excluded. By using all the indicated hypotheses we can now look for conditions equivalent to (19) which are simpler to manage. In particular, under the hypotheses HM1, HM5(κ) and HM6(κ) (respectively, HM5(ν) and HM6(ν)), (40) and $Z(R) = \text{determinant}(R)\,R^{-1}$ guarantee that the matrix function $S(\kappa,\mathbf{l}_{1,3}) - T(u_1,\mathbf{l}_{1,3})$ (respectively $S(\nu,\mathbf{l}_{1,3}) - T(u_2,\mathbf{l}_{1,3})$) on the main diagonal of the matrix function (18) is invertible everywhere in Ω, provided that $u_1 \neq 0$ (respectively $u_2 \neq 0$) in Ω and $\mathbf{l}_{1,3} \neq 0$. The invertibility everywhere in Ω of the matrices on the main diagonal allows us to use a well known [49] (p. 475) and simple manipulation of a square matrix function made up of four square submatrix functions, like the one of interest in (18). For example, if M_1 and M_4 are invertible,

$$\begin{pmatrix} M_1 & M_2 \\ M_3 & M_4 \end{pmatrix} = \begin{pmatrix} M_1 & 0 \\ 0 & M_4 \end{pmatrix} \begin{pmatrix} I & M_1^{-1}\,M_2 \\ M_4^{-1}\,M_3 & I \end{pmatrix} \tag{54}$$

and, consequently,

$$\begin{aligned} &\text{determinant}\begin{pmatrix} M_1 & M_2 \\ M_3 & M_4 \end{pmatrix} = \\ &\qquad \text{determinant}(M_1) \,\cdot\, \text{determinant}(M_4) \,\cdot\, \text{determinant}(I - M_4^{-1}\,M_3\,M_1^{-1}\,M_2). \end{aligned} \tag{55}$$

Then, by observing that the matrix in (18) has the same structure as the matrix in equation (54) and by defining

$$\begin{aligned} &A\big(\kappa,\chi,\gamma,\nu,u_1,u_2,\mathbf{l}_{1,3}\big) = \\ &\qquad \big(S(\nu,\mathbf{l}_{1,3}) - T(u_2,\mathbf{l}_{1,3})\big)^{-1} \cdot S(\gamma,\mathbf{l}_{1,3}) \cdot \big(S(\kappa,\mathbf{l}_{1,3}) - T(u_1,\mathbf{l}_{1,3})\big)^{-1} \cdot S(\chi,\mathbf{l}_{1,3}), \end{aligned} \tag{56}$$

the ellipticity condition (19) becomes equivalent to

$$\text{determinant}\Big(I - A\big(\kappa, \chi, \gamma, \nu, u_1, u_2, \mathbf{l}_{1,3}\big)\Big) \neq 0 \quad \forall \mathbf{l}_{1,3} \in \mathbb{R}^3, \mathbf{l}_{1,3} \neq 0, \forall \mathbf{x} \in \Omega, \tag{57}$$

under HM1, HM5(κ), HM5(ν), HM6(κ), HM6(ν), $u_1 \neq 0$ in Ω and $u_2 \neq 0$ in Ω.

The invertibility of $I - A\big(\kappa, \chi, \gamma, \nu, u_1, u_2, \mathbf{l}_{1,3}\big)$ can be easily studied by the Neumann series [49] (p. 126), when the entries in $A\big(\kappa, \chi, \gamma, \nu, u_1, u_2, \mathbf{l}_{1,3}\big)$ are sufficiently small. In particular, (57) is satisfied provided that, $\forall \mathbf{l}_{1,3} \in \mathbb{R}^3$, $\mathbf{l}_{1,3} \neq 0$ and $\forall \mathbf{x} \in \Omega$,

$$\lim_{n \to \infty} \Big(A\big(\kappa, \chi, \gamma, \nu, u_1, u_2, \mathbf{l}_{1,3}\big)\Big)^n = 0 \tag{58}$$

and this is implied by

$$\big\|A\big(\kappa, \chi, \gamma, \nu, u_1, u_2, \mathbf{l}_{1,3}\big)\big\| < 1 \tag{59}$$

and by the even stronger condition

$$\big\|\big(S(\nu, \mathbf{l}_{1,3}) - T(u_2, \mathbf{l}_{1,3})\big)^{-1}\big\| \, \big\|S(\gamma, \mathbf{l}_{1,3})\big\| \, \big\|\big(S(\kappa, \mathbf{l}_{1,3}) - T(u_1, \mathbf{l}_{1,3})\big)^{-1}\big\| \, \big\|S(\chi, \mathbf{l}_{1,3})\big\| < 1 \tag{60}$$

independently of the matrix norm.

As a matter of fact, for any pair of conformable matrices A and B [49] (p. 96) we have [49] (p. 280) $\|A\,B\| \leq \|A\|\,\|B\|$ and, then, (60) implies (59). Moreover, for a square matrix A, $\|A^n\| \leq \|A\|^n$, independently of the matrix norm. Therefore (59) implies

$$0 \leq \lim_{n \to \infty} \Big\|\Big(A\big(\kappa, \chi, \gamma, \nu, u_1, u_2, \mathbf{l}_{1,3}\big)\Big)^n\Big\| \leq \lim_{n \to \infty} \big\|A\big(\kappa, \chi, \gamma, \nu, u_1, u_2, \mathbf{l}_{1,3}\big)\big\|^n = 0 \tag{61}$$

so that (58) is satisfied.

It is a remarkable fact that condition (60) holds true, to the best of authors' knowledge, for all realistic models of bianisotropic materials considered in the open literature, where the effects due to κ and ν (or ε and μ or P and Q or $\mathcal{P}$ and $\mathcal{Q}$ in the other forms of the constitutive relations) are dominant with respect to the electrically induced or magnetically induced magnetoelectric effect [18]. In summary we can say that all results of this section can be interpreted as a proof of this statement. In particular, we now state two theorems. The first makes no assumptions on the structure of the constitutive matrix functions while the second provides sharper results for bianisotropic materials characterized by constitutive matrix functions which are real, diagonal and definite in Ω. Useful intermediate results dealing with materials having only some of the constitutive matrix functions proportional to real, diagonal and uniformly definite in Ω matrix functions can be easily deduced and are not reported. One of these results is considered and exploited in the subsection devoted to the practical applications.

To state the first theorem, let us define the versor (in the euclidean norm) $\mathbf{l}_{1,3,n} = \frac{\mathbf{l}_{1,3}}{\sqrt{\lambda_1^2 + \lambda_2^2 + \lambda_3^2}}$ when $\mathbf{l}_{1,3} \neq 0$, and the hypotheses

HM8. $\exists\exists C_{\kappa,d} > 0,\ C_{\nu,d} > 0 :\ |\text{determinant}(\kappa)| \geq C_{\kappa,d},\ |\text{determinant}(\nu)| \geq C_{\nu,d}\ \forall \mathbf{x} \in \Omega$

and

HM9. $\exists\exists C_{\kappa,r} > 0,\ C_{\nu,r} > 0 :\ |\mathbf{l}_{1,3,n}^T\,\kappa^{-1}\,\mathbf{l}_{1,3,n}| \geq C_{\kappa,r},\ |\mathbf{l}_{1,3,n}^T\,\nu^{-1}\,\mathbf{l}_{1,3,n}| \geq C_{\nu,r}\ \forall \mathbf{l}_{1,3,n} \in \mathbb{R}^3 : \|\mathbf{l}_{1,3,n}\|_2 = 1,\ \forall \mathbf{x} \in \Omega$.

We have

Theorem 7. *Under the hypotheses HM1, HM8 and HM9, condition (19) holds true for two specific constant functions $u_1 \neq 0$ and $u_2 \neq 0$ if κ, χ, γ and ν are such that*

$$\frac{4\Big(\big(\sum_{i,j=1}^3 |\gamma_{ij}|\big) - \min_{i=1,2,3}|\gamma_{ii}|\Big)\Big(\big(\sum_{i,j=1}^3 |\chi_{ij}|\big) - \min_{i=1,2,3}|\chi_{ii}|\Big)}{\big(-C_{\kappa,s} + \sqrt{C_{\kappa,s}^2 + 4\,C_{\kappa,d}\,C_{\kappa,r}}\big)\big(-C_{\nu,s} + \sqrt{C_{\nu,s}^2 + 4\,C_{\nu,d}\,C_{\nu,r}}\big)} < 1 \quad \forall \mathbf{x} \in \Omega, \tag{62}$$

where $C_{\kappa,s} > 0$ and $C_{\nu,s} > 0$ are such that

$$\Big(\sum_{i,j=1}^{3} |\kappa_{ij}|\Big) - \min_{i=1,2,3} |\kappa_{ii}| \leq C_{\kappa,s} \quad \forall \mathbf{x} \in \Omega, \tag{63}$$

$$\Big(\sum_{i,j=1}^{3} |\nu_{ij}|\Big) - \min_{i=1,2,3} |\nu_{ii}| \leq C_{\nu,s} \quad \forall \mathbf{x} \in \Omega. \tag{64}$$

Proof. First of all, let us notice that HM1 implies that the constant values $C_{\kappa,s} > 0$ and $C_{\nu,s} > 0$ exist. Moreover, HM8 implies both HM5(κ) and HM5(ν) and HM9 implies both HM6(κ) and HM6(ν). Furthermore, we have already noticed that, on the one hand, under these hypotheses, condition (19) and condition (57) are equivalent, provided that $u_1 \neq 0$ and $u_2 \neq 0$ in Ω, and, on the other hand, that condition (60) is a sufficient condition for condition (57) to hold true. Hence, we can conclude the proof by noticing that Lemma 8 and Lemma 18 of Appendix A allow to deduce that, for two specific constant functions $u_1 \neq 0$ and $u_2 \neq 0$, the sufficient condition (60) is satisfied for the $\| \, \|_2$ norm when inequality (62) holds true. □

Theorem 8. *Under the hypotheses HM1, HM5(κ), HM5(ν), HM6(κ) and HM6(ν), condition (19) holds true for some constant functions $u_1 \neq 0$ and $u_2 \neq 0$ if κ, χ, γ and ν are real, diagonal matrix functions uniformly definite in Ω and are such that*

$$\frac{\max_{i=1,2,3} |\gamma_{ii}| \; \max_{i=1,2,3} |\chi_{ii}|}{\min_{i=1,2,3} |\nu_{ii}| \; \min_{i=1,2,3} |\kappa_{ii}|} < 1 \quad \forall \mathbf{x} \in \Omega. \tag{65}$$

Proof. From Lemmas 19 and 20 we deduce that, for some constant functions $u_1 \neq 0$ and $u_2 \neq 0$,

$$\begin{aligned} \big\| \big(S(\nu, \mathbf{l}_{1,3}) - T(u_2, \mathbf{l}_{1,3})\big)^{-1} \big\|_2 \, \big\|S(\gamma, \mathbf{l}_{1,3})\big\|_2 \, \big\| \big(S(\kappa, \mathbf{l}_{1,3}) - T(u_1, \mathbf{l}_{1,3})\big)^{-1} \big\|_2 \, \big\|S(\chi, \mathbf{l}_{1,3})\big\|_2 \\ \leq \|\nu^{-1}\|_2 \, \|\gamma\|_2 \, \|\kappa^{-1}\|_2 \, \|\chi\|_2. \end{aligned} \tag{66}$$

Since the right-hand side of (66) is equal to the left-hand side of (65), condition (60) is satisfied. Finally, (60) implies (57) and the latter is equivalent to (19) under the assumptions HM1, HM5(κ), HM5(ν), HM6(κ), HM6(ν), $u_1 \neq 0$ in Ω and $u_2 \neq 0$ in Ω. □

We can now summarize the deductions of Section 7 with the following result about the regularity of the electromagnetic fields inside a bianisotropic material. In particular, by combining Theorems 1 and 2 and Corollaries 1 and 2 with Theorems 7 and 8 and noticing that u_1 and u_2 are (or can be set to) constants, we obtain the following theorem

Theorem 9. *Suppose that HS1 and HM1 are satisfied. Suppose, moreover, that one of the following alternatives holds true:*

- *HM8, HM9 and condition (62) are satisfied;*
- *κ, χ, γ and ν are real and diagonal matrix functions uniformly definite in Ω and HM5(κ), HM5(ν), HM6(κ), HM6(ν) and condition (65) are satisfied.*

In any of the above cases, for any $n \geq 0$, n integer, $p \in \mathbb{R}, p \geq 2$ ($\lambda \in (0, 1]$), if HS2(n,p) and HM2(n) (respectively, HS3(n,λ) and HM3(n,λ)) are satisfied, then all strong solutions $(\mathbf{E}, \mathbf{B}, \mathbf{H}, \mathbf{D}) \in H(\mathrm{curl}, \Omega) \times H(\mathrm{div}^0, \Omega) \times H(\mathrm{curl}, \Omega) \times H(\mathrm{div}^0, \Omega)$ of (2) and (3) actually belong to $(W^{n+1,p}_{\mathrm{loc}}(\Omega))^3 \times (W^{n+1,p}_{\mathrm{loc}}(\Omega))^3 \times (W^{n+1,p}_{\mathrm{loc}}(\Omega))^3 \times (W^{n+1,p}_{\mathrm{loc}}(\Omega))^3$ (respectively, $(C^{n+1,\lambda}(\Omega))^3 \times (C^{n+1,\lambda}(\Omega))^3 \times (C^{n+1,\lambda}(\Omega))^3 \times (C^{n+1,\lambda}(\Omega))^3$). Moreover, if $\kappa, \chi, \gamma, \nu \in (C^\infty(\overline{\Omega}))^{3\times 3}$ and $\mathbf{J}_e, \mathbf{J}_m \in (C^\infty(\Omega))^3$ (respectively, all entries of $\kappa, \chi, \gamma, \nu \in C^\infty(\overline{\Omega})$ and are analytic in Ω and all components of $\mathbf{J}_e, \mathbf{J}_m$ are analytic in Ω) then the aforementioned strong solution $(\mathbf{E}, \mathbf{B}, \mathbf{H}, \mathbf{D})$ actually belong to $(C^\infty(\Omega))^3 \times (C^\infty(\Omega))^3 \times (C^\infty(\Omega))^3 \times (C^\infty(\Omega))^3$ (respectively, have analytic components in Ω).

Figure 2 provide a graphical overview of the results obtained in this section.

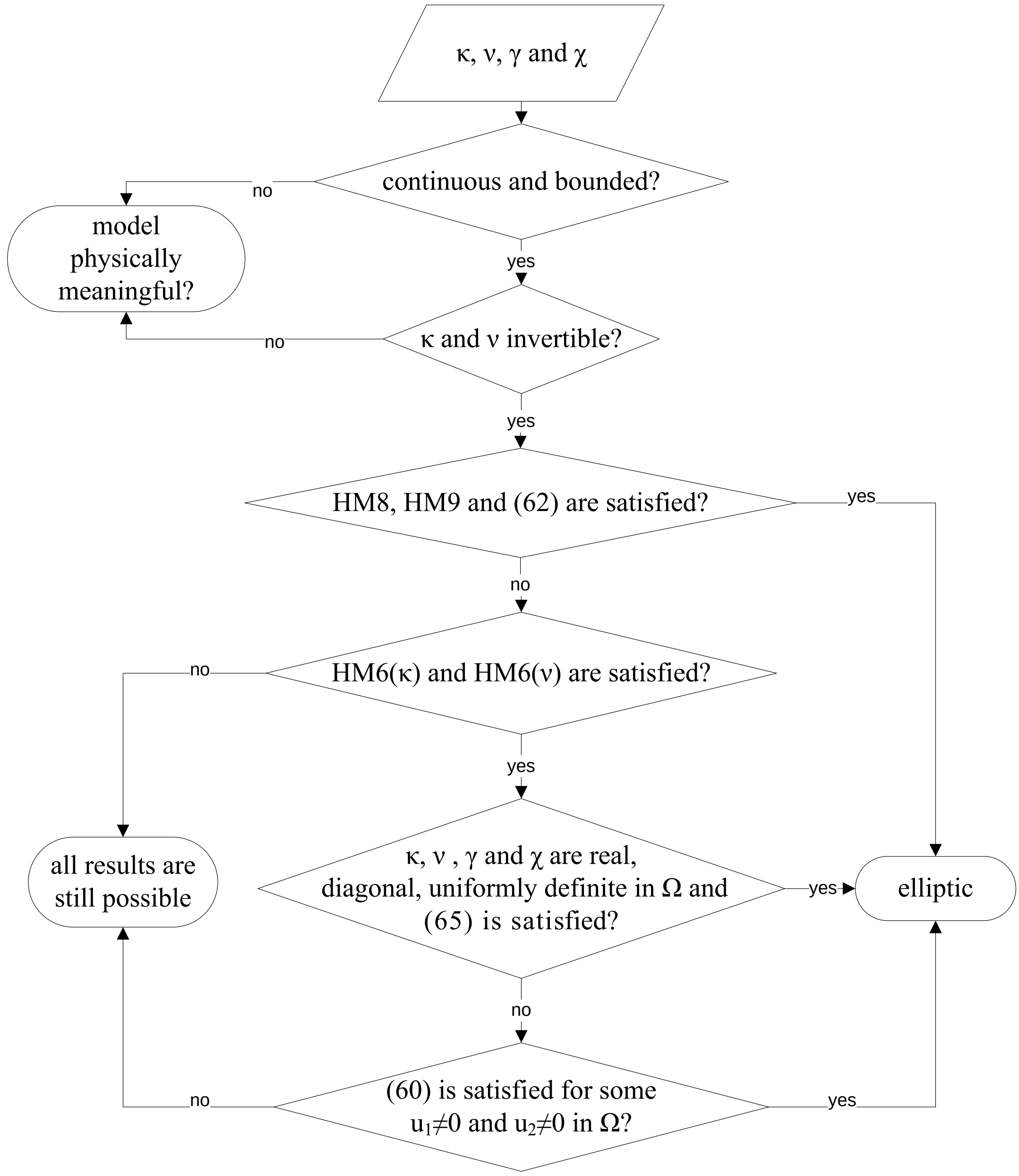

Figure 2: Flow chart giving indications on the ellipticity of system (15) and, then, on the regularity of the electromagnetic field in the interior of a single bianisotropic substance. Analogous flow charts could be obtained in terms of ε, ξ, ζ and μ or P, L, M and Q or $\mathcal{P}$, $\mathcal{L}$, $\mathcal{M}$ and $\mathcal{Q}$.

Remark 9. *It could be useful to point out that the theoretical results deduced and the practical applications considered in this work are affected just by the properties of the constitutive parameters and of the impressed current densities. In particular, these results and applications are not affected by any particular value of the angular frequency ω, $\omega > 0$, at which the considered constitutive parameters and sources show up. The only role of ω in this analysis is related to the fact that in time dispersive materials the constitutive matrix-valued complex functions κ, χ, γ and ν depend also on ω, so that, for a given material, different results could be obtained for different values of ω.*

7.1 Practical applications

In the book by Landau and Lifshitz [18] (see pp. 130, 131, 176 and 177) it is clearly pointed out that the effects due to κ and ν (or ε and μ or P and Q or $\mathcal{P}$ and $\mathcal{Q}$ in the other forms of the constitutive relations) are dominant with respect to the electrically induced or magnetically induced magnetoelectric effect. This consideration suggests that with Theorem 9 we could be able to obtain the result of interest in most cases considered in practical applications.

In order to verify if this is the case, let us firstly consider the kind of homogeneous bianisotropic materials considered in [21], [22], [23], [24], [25], [26], [27].

The constitutive relations for the materials considered in these works are in the form (21), with diagonal tensors ε, μ, ζ and ξ. In particular, in all these contributions we have

$$\varepsilon = \begin{bmatrix} \varepsilon_{11} & 0 & 0 \\ 0 & \varepsilon_{11} & 0 \\ 0 & 0 & \varepsilon_{33} \end{bmatrix}, \tag{67}$$

$$\mu = \begin{bmatrix} \mu_{11} & 0 & 0 \\ 0 & \mu_{11} & 0 \\ 0 & 0 & \mu_{33} \end{bmatrix}, \tag{68}$$

$$\xi = \zeta = \begin{bmatrix} \zeta_{11} & 0 & 0 \\ 0 & \zeta_{11} & 0 \\ 0 & 0 & \zeta_{33} \end{bmatrix}, \tag{69}$$

and in [26] it is stated that this form of the constitutive tensors "can be used to represent a wide range of magnetoelectric materials, including tetragonal and hexagonal crystals, such as Cr_2O_3, and transversely isotropic polycrystals and composites."

In this sense we use some data on a single crystal of Cr_2O_3 from [26]. From Table 1 of [26] we deduce $\varepsilon_{11} = 1.18\ 10^{-10}\ [\frac{\text{Cb}}{\text{V m}}]$, $\varepsilon_{33} = 1.05\ 10^{-10}\ [\frac{\text{Cb}}{\text{V m}}]$, $\mu_{11} = 1.6\ 10^{-3}\ [\frac{\text{V s}}{\text{A m}}]$ and $\mu_{33} = 0.787\ 10^{-3}\ [\frac{\text{V s}}{\text{A m}}]$ and, from Figure 3 of [26] we deduce, when the orientation distribution coefficient approaches zero (giving the single-crystalline values), $\zeta_{11} \simeq 2.5\ 10^{-14}\ [\frac{\text{Cb}}{\text{A m}} = \frac{\text{s}}{\text{m}}]$, $\zeta_{33} \simeq 5.2\ 10^{-13}\ [\frac{\text{Cb}}{\text{A m}} = \frac{\text{s}}{\text{m}}]$. We assume that this data are measured at $\omega > 0$ as indicated in [21] or in [25] since [26] refers to these works for the experimental data considered as terms of comparison.

One can easily verify that μ can be inverted. Moreover, with these values one deduces that $\xi\,\mu^{-1}\,\zeta$ has entries whose biggest amplitude is of the order of $10^{-22}\ [\frac{\text{Cb}}{\text{V m}}]$. Thus, $\varepsilon - \xi\,\mu^{-1}\,\zeta \simeq \varepsilon$, with an approximation on the twelfth significant digit. Since it admits an inverse, from (28) we obtain $\kappa \simeq \varepsilon^{-1}$, $\chi \simeq -\varepsilon^{-1}\,\xi\,\mu^{-1}$, $\gamma \simeq -\mu^{-1}\,\zeta\,\varepsilon^{-1}$ and $\nu \simeq \mu^{-1} + \mu^{-1}\,\zeta\,\varepsilon^{-1}\,\xi\,\mu^{-1} \simeq \mu^{-1}$, again with an approximation on the twelfth significant digit. In particular, we obtain $\kappa_{11} \simeq \frac{1}{\varepsilon_{11}} = 8.4746\ 10^{9}\ [\frac{\text{V m}}{\text{Cb}}]$, $\kappa_{33} \simeq \frac{1}{\varepsilon_{33}} = 9.5238\ 10^{9}\ [\frac{\text{V m}}{\text{Cb}}]$, $\chi_{11} = \gamma_{11} \simeq -\frac{\zeta_{11}}{\varepsilon_{11}\,\mu_{11}} = -1.3242\ 10^{-1}\ [\frac{\text{m}}{\text{s}}]$, $\chi_{33} = \gamma_{33} \simeq -\frac{\zeta_{33}}{\varepsilon_{33}\,\mu_{33}} = -6.293\ [\frac{\text{m}}{\text{s}}]$, $\nu_{11} \simeq \frac{1}{\mu_{11}} = 6.25\ 10^{2}\ [\frac{\text{A m}}{\text{V s}}]$ $\nu_{33} \simeq \frac{1}{\mu_{33}} = 1.2706\ 10^{3}\ [\frac{\text{A m}}{\text{V s}}]$

With these values we deduce that HM8 and HM9 are satisfied provided that

$$C_{\kappa,d} = 6.8399\ 10^{29}\ \Big[\frac{\text{V}^3\ \text{m}^3}{\text{Cb}^3}\Big], \tag{70}$$

$$C_{\nu,d} = 4.9635\ 10^{8} \left[\frac{\mathrm{A}^3\ \mathrm{m}^3}{\mathrm{V}^3\ \mathrm{s}^3}\right], \tag{71}$$

$$C_{\kappa,r} = 1.0500\ 10^{-10} \left[\frac{\mathrm{Cb}}{\mathrm{V\ m}}\right], \tag{72}$$

$$C_{\nu,r} = 7.8700\ 10^{-4} \left[\frac{\mathrm{V\ s}}{\mathrm{A\ m}}\right]. \tag{73}$$

Moreover, we set

$$C_{\kappa,s} = 1.7998\ 10^{10} \left[\frac{\mathrm{V\ m}}{\mathrm{Cb}}\right], \tag{74}$$

$$C_{\nu,s} = 1.8956\ 10^{3} \left[\frac{\mathrm{A\ m}}{\mathrm{V\ s}}\right], \tag{75}$$

and have

$$\Big(\sum_{i,j=1}^{3} |\gamma_{ij}|\Big) - \min_{i=1,2,3} |\gamma_{ii}| = \Big(\sum_{i,j=1}^{3} |\chi_{ij}|\Big) - \min_{i=1,2,3} |\chi_{ii}| = 6.4254 \left[\frac{\mathrm{m}}{\mathrm{s}}\right] \tag{76}$$

so that the first member of (62), a dimensionless quantity, becomes

$$\frac{4\ \ 6.4254^2}{6.7245\ 10^9\ 3.7504\ 10^2} = 6.5482\ 10^{-11}, \tag{77}$$

which is significantly lower than 1 as required to apply the first alternative in Theorem 9 (HM1 is trivially verified being the medium homogeneous). Obviously, the approximations considered cannot affect our conclusion.

One can observe that, with the indicated diagonal structure of the constitutive matrix functions, the entries of γ and χ (or of ξ and ζ) could be incremented, with respect to the values considered above, by a factor equal to 10^5 without changing the conclusion. The bianisotropic substance with the largest magnetoelectric effect discovered so far [55], [27] has some entries in ζ and ξ having an amplitude equal to $730\ 10^{-12}\ [\frac{\mathrm{s}}{\mathrm{m}}]$. This amplitude is almost 1500 times larger than the largest amplitude and almost 30000 times larger than the smallest one considered above. Thus, we should be able to apply our result even in the presence of the largest magnetoelectric effect known. We do not write this consideration as a statement since the structure of the four constitutive matrix functions for this material and all their entries are not reported in the quoted papers [55], [27] since their emphasis is not on the traditional dielectric or magnetic effects but rather on the magnetoelectric effect and, in particular, on the largest values of the corresponding matrix entries.

As another application of Theorem 9 we consider the biisotropic media studied in [30]. The medium is homogeneous and is characterized by $\frac{1}{c} P = \varepsilon_0\, \varepsilon_r\ I$, $L = M = -j\, \xi_c\ I$, $c\,Q = \frac{1}{\mu_0}\ I$, $\varepsilon_r, \xi_c \in \mathbb{R}$, $\varepsilon_r \in [1, 4]$, $\xi_c \in [5\ 10^{-4}, 3\ 10^{-3}]$. P is then clearly invertible and, from (25), we deduce $\kappa = \frac{1}{\varepsilon_0\, \varepsilon_r}\ I$, $\chi = -\gamma = \frac{j\, \xi_c}{\varepsilon_0\, \varepsilon_r}\ I$ and $\nu = \big(\frac{1}{\mu_0} + \frac{\xi_c^2}{\varepsilon_0\, \varepsilon_r}\big)\ I$. Thus, HM1 is satisfied. It is easy to verify that conditions HM8 and HM9 are satisfied, as well. However, after some trivial calculations, one can verify that the dimensionless first member of (62) is strictly lower than 1 just for a subset of materials considered in [30]. Thus the first alternative in Theorem 9, the one obtained from Theorem 7, does not allow to obtain the result of interest. Fortunately, for these materials the sharper condition of Theorem 8 can be used. In particular, we have

$$\max_{i=1,2,3} |\gamma_{ii}| = \max_{i=1,2,3} |\chi_{ii}| = \frac{|\xi_c|}{|\varepsilon_0|\, |\varepsilon_r|} \left[\frac{\mathrm{m}}{\mathrm{s}}\right], \tag{78}$$

$$\min_{i=1,2,3} |\kappa_{ii}| = \frac{1}{|\varepsilon_0|\, |\varepsilon_r|} \left[\frac{\mathrm{V\ m}}{\mathrm{Cb}}\right], \tag{79}$$

and

$$\min_{i=1,2,3} |\nu_{ii}| = \Big|\frac{1}{\mu_0} + \frac{\xi_c^2}{\varepsilon_0\, \varepsilon_r}\Big| \left[\frac{\mathrm{A\ m}}{\mathrm{V\ s}}\right]. \tag{80}$$

One can easily verify that the left-hand side in equation (65) is equal to

$$\frac{1}{1+\frac{\varepsilon_0\,\varepsilon_r}{\mu_0\,\xi_c^2}}. \tag{81}$$

This quantity is lower than 1 for any $\varepsilon_r, \xi_c \in \mathbb{R}$, $\varepsilon_r > 0$, $\xi_c > 0$, and thus (65) is always verified for the same sets of values of ε_r and ξ_c. We need not verify that HM5(κ), HM5(ν), HM6(κ) and HM6(ν) are satisfied since they are implied by HM8 and HM9. Thus, we obtain the result of interest for all materials considered in [30] by using the second alternative in Theorem 9, the one obtained from Theorem 8.

Remark 10. *In this application χ and γ have purely imaginary entries. The imaginary unit j, however, does not affect the estimate of $\|S(\gamma,\mathbf{l}_{1,3})\|_2$ and $\|S(\chi,\mathbf{l}_{1,3})\|_2$ (see, for example, (83) and (84)). Thus, Theorems 8 and 9 can be trivially generalized to cover these cases, too.*

As a final application of Theorem 9 we consider another possible development of our procedure, even if it is not stated as a formal result. However, it can be easily obtained from our previous deductions. It is important to notice that this result should be considered as representative of the many similar conclusions which can be devised in an analogous way in “intermediate” cases between the two alternatives in Theorem 9.

In particular we refer to the set of bianisotropic media considered in [31]. In this case, too, the medium is homogeneous and is characterized by $\frac{1}{c}\,P = \varepsilon_0\,\varepsilon_r\,I$, $L = M = -j\,\xi_c\,I$, $c\,Q = \mu^{-1}$, with

$$\mu = \mu_0 \begin{bmatrix} 1 & j\mu_a & 0 \\ -j\mu_a & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \tag{82}$$

where $\varepsilon_r \in \mathbb{C}$, $\xi_c \in \mathbb{R}$, $\mu_a \in \mathbb{R}$, $\varepsilon_r \neq 0$, $\xi_c > 0$.

P is again clearly invertible and, from (25), we deduce $\kappa = \frac{1}{\varepsilon_0\,\varepsilon_r}\,I$ and $\chi = -\gamma = \frac{j\,\xi_c}{\varepsilon_0\,\varepsilon_r}\,I$. From the same equation $\nu = \left(\mu^{-1} + \frac{\xi_c^2}{\varepsilon_0\,\varepsilon_r}\,I\right)$ whenever $\mu_a \neq \pm 1$. μ is not invertible if $\mu_a = \pm 1$ but we do not consider this case as already discussed at the beginning of this section.

Since for $\mu_a \neq 0$ the constitutive matrix function ν is not diagonal, we cannot apply the alternative of Theorem 9 corresponding to Theorem 8 in all these cases of interest [31]. It is immediately seen that HM5(κ) and HM6(κ) are satisfied. HM5(ν) and HM6(ν) hold true, too, but some calculations are needed to show that. Unfortunately, one can verify, again after some calculations, that inequality (62) does not hold true for most of the cases of interest and, thus, even the alternative of Theorem 9 corresponding to Theorem 7 cannot be applied.

However, one can notice that the first member of inequality (60) is the product of four norms of four different matrices and that there is no need to use for all of them the estimate deduced for general matrices or the one obtained for real and diagonal matrices. In particular, in all the cases of interest in this example we can use the sharper estimate for the three norms involving κ, χ and γ and the general one for the matrix function involving ν. Other combinations of general and particular matrices are possible and the reader can easily deduce the corresponding alternatives generalizing Theorem 9.

As for $S(\chi,\mathbf{l}_{1,3})$ and $S(\gamma,\mathbf{l}_{1,3})$, from Lemma 5 we deduce

$$S(\gamma,\mathbf{l}_{1,3}) = -S(\chi,\mathbf{l}_{1,3}) = \frac{j\,\xi_c}{\varepsilon_0\,\varepsilon_r}\,S(I,\mathbf{l}_{1,3}). \tag{83}$$

Since I satisfies the hypotheses of Lemma 19 we can use it and deduce

$$\|S(\gamma,\mathbf{l}_{1,3})\|_2 = \|S(\chi,\mathbf{l}_{1,3})\|_2 \leq \frac{|\xi_c|}{\varepsilon_0\,|\varepsilon_r|}\,\|\mathbf{l}_{1,3}\|_2^2. \tag{84}$$

Let us now consider the term involving $S(\kappa,\mathbf{l}_{1,3}) - T(u_1,\mathbf{l}_{1,3})$ in equation (60). By setting $u_1 = \frac{1}{\varepsilon_0\,\varepsilon_r}$ we obtain $S(\kappa,\mathbf{l}_{1,3}) - T(u_1,\mathbf{l}_{1,3}) = \frac{1}{\varepsilon_0\,\varepsilon_r}\left(S(I,\mathbf{l}_{1,3}) - T(1,\mathbf{l}_{1,3})\right)$. Thus $\left(S(\kappa,\mathbf{l}_{1,3}) - T(u_1,\mathbf{l}_{1,3})\right)^{-1}$

is equal to $\varepsilon_0\,\varepsilon_r\,\big(S(I,\mathbf{l}_{1,3})-T(1,\mathbf{l}_{1,3})\big)^{-1}$ and, since I and u_1 satisfy the hypotheses of Lemma 20, we deduce

$$\|\big(S(\kappa,\mathbf{l}_{1,3})-T(u_1,\mathbf{l}_{1,3})\big)^{-1}\|_2 \le \frac{\varepsilon_0\,|\varepsilon_r|}{\|\mathbf{l}_{1,3}\|_2^2} \qquad \forall \mathbf{l}_{1,3}\in\mathbb{R}^3, \mathbf{l}_{1,3}\neq 0 \tag{85}$$

for the indicated choice of u_1.

Finally, in order to evaluate $\|\big(S(\nu,\mathbf{l}_{1,3})-T(u_2,\mathbf{l}_{1,3})\big)^{-1}\|_2$ we firstly observe that the authors of the quoted paper consider $\mu_a=\{-0.25,0,0.25\}$ and that for this reason and, as already pointed out, to avoid problems with the very definition of ν we limit the range of μ_a, the unique variable in this set of materials, to $(-1,1)$.

By definition one can easily check that we can set

$$C_{\nu,d}=\Big|\frac{1}{\mu_0}+\frac{\xi_c^2}{\varepsilon_0\,\varepsilon_r}\Big|\Big|\frac{1}{\mu_0^2\,(1-\mu_a^2)}+\frac{\xi_c^4}{\varepsilon_0^2\,\varepsilon_r^2}+\frac{2\,\xi_c^2}{\mu_0\,\varepsilon_0\,\varepsilon_r\,(1-\mu_a^2)}\Big| \tag{86}$$

and, $\forall \mu_a\in(-1,1)$

$$C_{\nu,s}=2\Big|\frac{1}{\mu_0\,(1-\mu_a^2)}+\frac{\xi_c^2}{\varepsilon_0\,\varepsilon_r}\Big|+2\Big|\frac{\mu_a}{\mu_0\,(1-\mu_a^2)}\Big|. \tag{87}$$

A sharp estimate of the largest possible value of $C_{\nu,r}$ could require some work. However, we can simplify this task by using an underestimate of it. This non-optimal value can be obtained from the calculation of the minimum eigenvalue of the positive definite Hermitian symmetric matrix $\frac{\nu^{-1}+(\nu^{-1})^*}{2}$, which is equal to the real part of $\frac{\mu_0\,\varepsilon_0\,\varepsilon_r}{\varepsilon_0\,\varepsilon_r+\xi_c^2\,\mu_0}$, $\forall \mu_a\in(-1,1)$.

In this way, by applying Lemma 18 with the calculated constants $C_{\nu,d}$ and $C_{\nu,s}$ and with the indicated non-optimal value of $C_{\nu,r}$ we deduce an overestimate of $\|\big(S(\nu,\mathbf{l}_{1,3})-T(u_2,\mathbf{l}_{1,3})\big)^{-1}\|_2$ for a constant nonvanishing u_2.

Even with this overestimate the first member of inequality (60) remains well below 0.5, $\forall \mu_a\in(-1,1)$, and, taking into account that HM1, HM5(κ), HM5(ν), HM6(κ) and HM6(ν) hold true, we obtain the result of interest for the considered superset of the set of materials analyzed in [31].

Remark 11. *It could be interesting to point out that, the same simple procedure used in Remark 8 to prove the H^1_{loc} regularity of the fields in isotropic media can be applied also to biisotropic media, even though, so far, this has not been pointed out in the literature, to the best of authors' knowledge. Hence, also in biisotropic media the H^1_{loc} regularity of the fields can be proved without resorting to our analysis. In fact, again under the assumptions $\mathbf{D}\in H(\mathrm{div},\Omega)$, $\mathbf{B}\in H(\mathrm{div},\Omega)$ and $\kappa,\chi,\gamma,\nu\in W^{1,\infty}(\Omega)$, from the constitutive relations (3) we obtain*

$$\begin{cases} \nabla\cdot\mathbf{E}=\nabla\kappa\cdot\mathbf{D}+\kappa\nabla\cdot\mathbf{D}+\nabla\chi\cdot\mathbf{B}+\chi\nabla\cdot\mathbf{B} \\ \nabla\cdot\mathbf{H}=\nabla\gamma\cdot\mathbf{D}+\gamma\nabla\cdot\mathbf{D}+\nabla\nu\cdot\mathbf{B}+\nu\nabla\cdot\mathbf{B}. \end{cases} \tag{88}$$

Hence, $\nabla\cdot\mathbf{E}\in L^2(\Omega)$ and $\nabla\cdot\mathbf{H}\in L^2(\Omega)$. Then, again $\mathbf{E}\in(H^1_{loc}(\Omega))^3$ and $\mathbf{H}\in(H^1_{loc}(\Omega))^3$.

As for the regularity of $\mathbf{D}$ and $\mathbf{B}$ we observe that

$$\begin{cases} \nabla\times\mathbf{E}=\nabla\kappa\times\mathbf{D}+\kappa\nabla\times\mathbf{D}+\nabla\chi\times\mathbf{B}+\chi\nabla\times\mathbf{B} \\ \nabla\times\mathbf{H}=\nabla\gamma\times\mathbf{D}+\gamma\nabla\times\mathbf{D}+\nabla\nu\times\mathbf{B}+\nu\nabla\times\mathbf{B} \end{cases} \tag{89}$$

from which we deduce that

$$\begin{cases} \kappa\nabla\times\mathbf{D}+\chi\nabla\times\mathbf{B}=\nabla\times\mathbf{E}-\nabla\kappa\times\mathbf{D}-\nabla\chi\times\mathbf{B} \\ \gamma\nabla\times\mathbf{D}+\nu\nabla\times\mathbf{B}=\nabla\times\mathbf{H}-\nabla\gamma\times\mathbf{D}-\nabla\nu\times\mathbf{B} \end{cases} \tag{90}$$

Thus, under the above assumptions, the right-hand sides have entries in $(L^2(\Omega))^3$ and, to obtain $\mathbf{D},\mathbf{B}\in H(\mathrm{curl},\Omega)$, from which their $(H^1_{loc}(\Omega))^3$ regularity follows, it is sufficient that $(\kappa\nu-\chi\gamma)^{-1}\in L^\infty(\Omega)$.

In this case, too, the reader can observe that the hypotheses assumed on the constitutive parameters for this alternative procedure aimed at proving the H^1_{loc} regularity of the fields in biisotropic media are just a bit stronger than those which allow to obtain the same regularity result by using our analysis.

Remark 12. *If the single material filling Ω is bianisotropic in the Cartesian system of coordinates (x_1, x_2, x_3) and if one can find an inertial frame (x_1', x_2', x_3'), in uniform translation with respect to (x_1, x_2, x_3), in which the medium is isotropic or anisotropic, then the regularity of interest of the fields* $\mathbf{E}$, $\mathbf{B}$, $\mathbf{H}$ *and* $\mathbf{D}$ *in (x_1, x_2, x_3) can be deduced in a much simpler way by firstly deducing the regularity of the fields* $\mathbf{E}'$, $\mathbf{B}'$, $\mathbf{H}'$ *and* $\mathbf{D}'$ *in (x_1', x_2', x_3') (for example by applying our results of Section 6) and, then, by observing that the Lorentz transformations between the field quantities in (x_1', x_2', x_3') and (x_1, x_2, x_3) [56] (equations (9) and (10) or their inverse relations; see also the comments above equation (8)) do not affect the regularity being simple linear combinations.*

8 Conclusions

In this paper we investigate, for the first time to the best of the authors' knowledge, the regularity of the four time-harmonic vector fields composing any strong solution of the system obtained from Maxwell's curl equations and the constitutive relations in the interior of a single inhomogeneous bianisotropic material.

In particular, sufficient conditions for either the interior Sobolev or Hölder regularity of the electric field, magnetic field, electric displacement and magnetic induction are deduced. Sufficient conditions for the interior C^∞ regularity or for the interior analyticity of the indicated vector fields are deduced, too.

The results of interest are obtained under specific conditions on the constitutive parameters of the bianisotropic material considered and on the impressed current densities but it is shown that such conditions do not significantly limit the coverage of this investigation in terms of applications.

A Appendix: some properties of the matrix functions $S(R, \mathbf{l}_{1,3})$ and $S(R, \mathbf{l}_{1,3}) - T(u, \mathbf{l}_{1,3})$

The following lemmas establish the properties we are most interested in. In this Appendix we use the notation

$$R = \sum_{i,j=1}^{3} r_{ij}\, \hat{x}_i \hat{x}_j^T, \tag{91}$$

where r_{ij} are complex functions of the point position only, $\hat{x}_i$ is the Cartesian unit vector along the x_i coordinate axis and $\hat{x}_i \hat{x}_j^T$ denotes the matrix obtained from the row by column multiplication of $\hat{x}_i$ (an algebraic column vector) and $\hat{x}_j^T$ (an algebraic row vector).

Lemma 5. *$S(R, \mathbf{l}_{1,3})$ is linear with respect to its first arguments, that is*

$$S(c_1\, R_1 + c_2\, R_2, \mathbf{l}_{1,3}) = c_1\, S(R_1, \mathbf{l}_{1,3}) + c_2\, S(R_2, \mathbf{l}_{1,3}), \tag{92}$$

for all $c_1, c_2 \in \mathbb{C}$, for all 3×3 matrix-valued complex functions R_1 and R_2 and for all $\mathbf{l}_{1,3} \in \mathbb{R}^3$.

Proof. Directly deduced from equation (10). □

Lemma 6. $\|S(\hat{x}_i \hat{x}_i^T, \mathbf{l}_{1,3})\|_2 = \lambda_j^2 + \lambda_k^2$, *$\forall i, j, k = 1, 2, 3$, $i \neq j \neq k$, $\forall \mathbf{l}_{1,3} \in \mathbb{R}^3$.*

Proof. The proof is carried out for $i = 1$ but in all other cases the considerations are exactly the same.

For $i = 1$ we have

$$S(\hat{x}_1 \hat{x}_1^T, \mathbf{l}_{1,3}) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -\lambda_3^2 & \lambda_2\lambda_3 \\ 0 & \lambda_2\lambda_3 & -\lambda_2^2 \end{bmatrix}, \tag{93}$$

so that

$$\left(S(\hat{x}_1 \hat{x}_1^T, \mathbf{l}_{1,3})\right)^* \cdot S(\hat{x}_1 \hat{x}_1^T, \mathbf{l}_{1,3}) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \lambda_3^4 + \lambda_2^2\lambda_3^2 & -\lambda_2\lambda_3^3 - \lambda_2^3\lambda_3 \\ 0 & -\lambda_2\lambda_3^3 - \lambda_2^3\lambda_3 & \lambda_2^4 + \lambda_2^2\lambda_3^2 \end{bmatrix}. \tag{94}$$

As it is easy to verify this matrix has an eigenvalue equal to zero (with multiplicity 2) and an eigenvalue equal to $(\lambda_2^2 + \lambda_3^2)^2$. Therefore, [49] (p. 281),

$$\|S(\hat{x}_1 \hat{x}_1^T, \mathbf{l}_{1,3})\|_2 = \lambda_2^2 + \lambda_3^2 \quad \forall \mathbf{l}_{1,3} \in \mathbb{R}^3. \tag{95}$$

$\square$

Lemma 7. $\|S(\hat{x}_i \hat{x}_j^T, \mathbf{l}_{1,3})\|_2 \le \|\mathbf{l}_{1,3}\|_2^2 = \lambda_1^2 + \lambda_2^2 + \lambda_3^2$, $\forall i, j = 1, 2, 3$, $i \neq j$, $\forall \mathbf{l}_{1,3} \in \mathbb{R}^3$.

Proof. The proof is carried out for $i = 1$ and $j = 2$ but in all other cases the considerations are exactly the same.

For $i = 1$ and $j = 2$ we have

$$S(\hat{x}_1 \hat{x}_2^T, \mathbf{l}_{1,3}) = \begin{bmatrix} 0 & 0 & 0 \\ \lambda_3^2 & 0 & -\lambda_1\lambda_3 \\ -\lambda_2\lambda_3 & 0 & \lambda_1\lambda_2 \end{bmatrix}, \tag{96}$$

so that

$$\left(S(\hat{x}_1 \hat{x}_2^T, \mathbf{l}_{1,3})\right)^* \cdot S(\hat{x}_1 \hat{x}_2^T, \mathbf{l}_{1,3}) = \begin{bmatrix} \lambda_3^4 + \lambda_2^2\lambda_3^2 & 0 & -\lambda_1\lambda_3^3 - \lambda_1\lambda_2^2\lambda_3 \\ 0 & 0 & 0 \\ -\lambda_1\lambda_3^3 - \lambda_1\lambda_2^2\lambda_3 & 0 & \lambda_1^2\lambda_3^2 + \lambda_1^2\lambda_2^2 \end{bmatrix}. \tag{97}$$

As it is easy to verify this matrix has an eigenvalue equal to zero (with multiplicity 2) and an eigenvalue equal to $(\lambda_1^2 + \lambda_3^2) \cdot (\lambda_2^2 + \lambda_3^2)$. Therefore, [49] (p. 281),

$$\|S(\hat{x}_1 \hat{x}_2^T, \mathbf{l}_{1,3})\|_2 = \sqrt{(\lambda_1^2 + \lambda_3^2) \cdot (\lambda_2^2 + \lambda_3^2)} \le \lambda_1^2 + \lambda_2^2 + \lambda_3^2 \quad \forall \mathbf{l}_{1,3} \in \mathbb{R}^3. \tag{98}$$

$\square$

Lemma 8. $\|S(R, \mathbf{l}_{1,3})\|_2 \le \|\mathbf{l}_{1,3}\|_2^2 \left(\left(\sum_{i,j=1}^3 |r_{ij}| \right) - \min_{i=1,2,3} |r_{ii}| \right)$ *for all* 3×3 *matrix-valued complex functions* R *and* $\forall \mathbf{l}_{1,3} \in \mathbb{R}^3$.

Proof. From Lemma 5 we deduce that

$$S(R, \mathbf{l}_{1,3}) = S(\sum_{i,j=1}^3 r_{ij}\, \hat{x}_i \hat{x}_j^T, \mathbf{l}_{1,3}) = \sum_{i,j=1}^3 r_{ij}\, S(\hat{x}_i \hat{x}_j^T, \mathbf{l}_{1,3}) \tag{99}$$

and by the triangle inequality

$$\|S(R, \mathbf{l}_{1,3})\|_2 \le \sum_{i,j=1}^3 |r_{ij}|\, \|S(\hat{x}_i \hat{x}_j^T, \mathbf{l}_{1,3})\|_2. \tag{100}$$

By using Lemmas 6 and 7 one then obtains

$$\begin{aligned} \|S(R, \mathbf{l}_{1,3})\|_2 \le & \Big(\sum_{i,j=1}^3 |r_{ij}|\, \|\mathbf{l}_{1,3}\|_2^2 \Big) \qquad (101) \\ & -|r_{11}|\, \lambda_1^2 - |r_{22}|\, \lambda_2^2 - |r_{33}|\, \lambda_3^2 \le \|\mathbf{l}_{1,3}\|_2^2 \Big(\big(\sum_{i,j=1}^3 |r_{ij}| \big) - \min_{i=1,2,3} |r_{ii}| \Big) \end{aligned}$$

$\square$

Lemma 9. $\mathbf{l}_{1,3}$ *is an eigenvector for* $T(u, \mathbf{l}_{1,3})$ *and the corresponding eigenvalue is* $u\,(\lambda_1^2 + \lambda_2^2 + \lambda_3^2)$. *Moreover,* $\mathbf{l}_{1,3}$ *is an eigenvector for* $S(R, \mathbf{l}_{1,3})$ *and the corresponding eigenvalue is zero, independently of* R. *Finally,* $\mathbf{l}_{1,3}$ *is an eigenvector for* $S(R, \mathbf{l}_{1,3}) - T(u, \mathbf{l}_{1,3})$ *and the corresponding eigenvalue is* $-u\,(\lambda_1^2 + \lambda_2^2 + \lambda_3^2)$, *independently of* R.

Proof.

$$T(u,\mathbf{l}_{1,3})\,\mathbf{l}_{1,3} = u\begin{bmatrix} \lambda_1^2 & \lambda_1\lambda_2 & \lambda_1\lambda_3 \\ \lambda_2\lambda_1 & \lambda_2^2 & \lambda_2\lambda_3 \\ \lambda_3\lambda_1 & \lambda_3\lambda_2 & \lambda_3^2 \end{bmatrix}\begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{bmatrix} \tag{102}$$

$$= u\begin{bmatrix} \lambda_1^3+\lambda_1\lambda_2^2+\lambda_1\lambda_3^2 \\ \lambda_1^2\lambda_2+\lambda_2^3+\lambda_2\lambda_3^2 \\ \lambda_1^2\lambda_3+\lambda_2^2\lambda_3+\lambda_3^3 \end{bmatrix} = u\,(\lambda_1^2+\lambda_2^2+\lambda_3^2)\,\mathbf{l}_{1,3}.$$

Moreover, as already pointed out in Section 3, it is easy to check that by adding the first column multiplied by λ_1, the second column multiplied by λ_2 and the third column multiplied by λ_3 one obtains a trivial column.

Finally, the above results on $T(u,\mathbf{l}_{1,3})$ and $S(R,\mathbf{l}_{1,3})$ imply the last one on $S(R,\mathbf{l}_{1,3})-T(u,\mathbf{l}_{1,3})$. □

Lemma 10. *The only other eigenvalue of $T(u,\mathbf{l}_{1,3})$ is zero and its multiplicity is 2. $\mathbf{G}-(\mathbf{G}\cdot\mathbf{l}_{1,3,n})\mathbf{l}_{1,3,n}$ is in the kernel of $T(u,\mathbf{l}_{1,3})$ for any vector $\mathbf{G}=(G_1,G_2,G_3)^T\in\mathbb{C}^3$.*

Proof.

$$\text{determinant}\Big(T(u,\mathbf{l}_{1,3})-\alpha\,I\Big) = \begin{vmatrix} u\lambda_1^2-\alpha & u\lambda_1\lambda_2 & u\lambda_1\lambda_3 \\ u\lambda_2\lambda_1 & u\lambda_2^2-\alpha & u\lambda_2\lambda_3 \\ u\lambda_3\lambda_1 & u\lambda_3\lambda_2 & u\lambda_3^2-\alpha \end{vmatrix} \tag{103}$$

$$= (u\lambda_1^2-\alpha)\Big[(u\lambda_2^2-\alpha)(u\lambda_3^2-\alpha)-u^2\lambda_2^2\lambda_3^2\Big] - u\lambda_1\lambda_2\Big[u\lambda_1\lambda_2(u\lambda_3^2-\alpha)-u^2\lambda_1\lambda_2\lambda_3^2\Big]$$

$$+u\lambda_1\lambda_3\Big[u^2\lambda_1\lambda_2^2\lambda_3\Big] - u\lambda_1\lambda_3(u\lambda_2^2-\alpha)$$

$$= (u\lambda_1^2-\alpha)\Big[-\alpha u\lambda_2^2-\alpha u\lambda_3^2+\alpha^2\Big] - u\lambda_1\lambda_2\Big[-\alpha u\lambda_1\lambda_2\Big] + u\lambda_1\lambda_3\Big[\alpha u\lambda_1\lambda_3\Big]$$

$$= \alpha^2\Big(-\alpha+u(\lambda_1^2+\lambda_2^2+\lambda_3^2)\Big)$$

From equation (102) it is easy to deduce that

$$T(u,\mathbf{l}_{1,3})\,\mathbf{G} = u\,(\lambda_1G_1+\lambda_2G_2+\lambda_3G_3)\,\mathbf{l}_{1,3} = u\,(\mathbf{G}\cdot\mathbf{l}_{1,3})\,\mathbf{l}_{1,3}. \tag{104}$$

But by using again equation (102) we deduce

$$T(u,\mathbf{l}_{1,3})\,(\mathbf{G}\cdot\mathbf{l}_{1,3,n})\mathbf{l}_{1,3,n} = (\mathbf{G}\cdot\mathbf{l}_{1,3,n})\Big(T(u,\mathbf{l}_{1,3})\,\mathbf{l}_{1,3,n}\Big) = (\mathbf{G}\cdot\mathbf{l}_{1,3,n})\,u\|\mathbf{l}_{1,3}\|_2\,\mathbf{l}_{1,3} = (\mathbf{G}\cdot\mathbf{l}_{1,3})u\,\mathbf{l}_{1,3} \tag{105}$$

and the conclusion follows. The previous scalar product involve the real vector $\mathbf{l}_{1,3}$ and can be taken in $\mathbb{C}^3$ or in $\mathbb{R}^3$. □

Lemma 11. $\|T(u,\mathbf{l}_{1,3})\|_2 = |u|\,\|\mathbf{l}_{1,3}\|_2^2 = |u|\,(\lambda_1^2+\lambda_2^2+\lambda_3^2)$*, for all functions u and $\forall\mathbf{l}_{1,3}\in\mathbb{R}^3$.*

Proof. Lemmas 9 and 10 easily imply that $\big(T(u,\mathbf{l}_{1,3})\big)^*\cdot T(u,\mathbf{l}_{1,3})$ has an eigenvalue equal to zero with multiplicity 2 and an eigenvalue equal to $|u|^2\,(\lambda_1^2+\lambda_2^2+\lambda_3^2)^2$. The thesis follows by definition of $\|\cdot\|_2$ [49] (p. 281). □

Lemmas 8 and 11 together with the triangle inequality imply the following result.

Lemma 12. $\|S(R,\mathbf{l}_{1,3})-T(u,\mathbf{l}_{1,3})\|_2 \le \|\mathbf{l}_{1,3}\|_2^2\left(|u|+\Big(\big(\sum_{i,j=1}^3|r_{ij}|\big)-\min_{i=1,2,3}|r_{ii}|\Big)\right)$*, for all 3×3 matrix-valued complex functions R, for all functions u and $\forall\mathbf{l}_{1,3}\in\mathbb{R}^3$.*

This Lemma has as a direct consequence, since $\|A\,B\|\le\|A\|\|B\|$ for any matrix norm [49] (p. 280) and $\|A^*\|_2=\|A\|_2$ [49] (p. 283)

Lemma 13. *$\|\big(S(R,\mathbf{l}_{1,3})-T(u,\mathbf{l}_{1,3})\big)^* \big(S(R,\mathbf{l}_{1,3})-T(u,\mathbf{l}_{1,3})\big)\|_2 \le \|\mathbf{l}_{1,3}\|_2^4 \Bigg(|u|+\Big(\big(\sum_{i,j=1}^3 |r_{ij}|\big) - \min_{i=1,2,3}|r_{ii}|\Big)\Bigg)^2$.*

Directly from equations (10) and (17) one deduces

Lemma 14. *$S(R,\mathbf{l}_{1,3}) = (S(R^*,\mathbf{l}_{1,3}))^*$. In particular, $S(R,\mathbf{l}_{1,3})$ is Hermitian symmetric if R is Hermitian symmetric. $T(u,\mathbf{l}_{1,3}) = (T(u^*,\mathbf{l}_{1,3}))^*$. In particular, $T(u,\mathbf{l}_{1,3})$ is Hermitian symmetric if $u \in \mathbb{R}$.*

Lemma 15. *Suppose R is Hermitian symmetric and suppose that $S(R,\mathbf{l}_{1,3})$ has an eigenvector $\mathbf{G}$ corresponding to a non-trivial eigenvalue α. Then $\mathbf{G}$ is an eigenvector for $S(R,\mathbf{l}_{1,3})-T(u,\mathbf{l}_{1,3})$ and the corresponding eigenvalue is α.*

Proof. By lemma 14 $S(R,\mathbf{l}_{1,3})$ is Hermitian symmetric. Then it can be diagonalized and two eigenvectors corresponding to different eigenvalues are orthogonal in $\mathbb{C}^3$. By lemma 9 $\mathbf{l}_{1,3}$ is an eigenvector of $S(R,\mathbf{l}_{1,3})$ and the corresponding eigenvalue is zero. If $\mathbf{G}$ is an eigenvector of $S(R,\mathbf{l}_{1,3})$ corresponding to an eigenvalue $\alpha \neq 0$ we have $\mathbf{G}\cdot\mathbf{l}_{1,3} = 0$. Then

$$\Big(S(R,\mathbf{l}_{1,3})-T(u,\mathbf{l}_{1,3})\Big)\,\mathbf{G} = \Big(S(R,\mathbf{l}_{1,3})\,\mathbf{G}\Big)-\Big(T(u,\mathbf{l}_{1,3})\,\mathbf{G}\Big) = \alpha\mathbf{G}-T(u,\mathbf{l}_{1,3})\Big(\mathbf{G}-(\mathbf{G}\cdot\mathbf{l}_{1,3,n})\mathbf{l}_{1,3,n}\Big) \tag{106}$$

and the conclusion follows by Lemma 10. □

Now, from Lemmas 9 and 14 we deduce

Lemma 16. *$\Big(\big(S(R,\mathbf{l}_{1,3})-T(u,\mathbf{l}_{1,3})\big)^* \big(S(R,\mathbf{l}_{1,3})-T(u,\mathbf{l}_{1,3})\big)\Big)\,\mathbf{l}_{1,3} = |u|^2\,\|\mathbf{l}_{1,3}\|_2^4\,\mathbf{l}_{1,3}$, that is $\mathbf{l}_{1,3}$ is an eigenvector and the corresponding eigenvalue is $|u|^2\,\|\mathbf{l}_{1,3}\|_2^4$.*

Proof. $\Big(\big(S(R,\mathbf{l}_{1,3})-T(u,\mathbf{l}_{1,3})\big)^* \big(S(R,\mathbf{l}_{1,3})-T(u,\mathbf{l}_{1,3})\big)\Big)\,\mathbf{l}_{1,3} = \big(S(R,\mathbf{l}_{1,3})-T(u,\mathbf{l}_{1,3})\big)^* \Big(\big(S(R,\mathbf{l}_{1,3})-T(u,\mathbf{l}_{1,3})\big)\,\mathbf{l}_{1,3}\Big) = \big(S(R^*,\mathbf{l}_{1,3})-T(u^*,\mathbf{l}_{1,3})\big)\,(-u\,\|\mathbf{l}_{1,3}\|_2^2)\;\mathbf{l}_{1,3} = (-u^*\,\|\mathbf{l}_{1,3}\|_2^2)\;\mathbf{l}_{1,3}\;(-u\,\|\mathbf{l}_{1,3}\|_2^2) = |u|^2\,\|\mathbf{l}_{1,3}\|_2^4\;\mathbf{l}_{1,3}$. □

Lemma 17. *Suppose that the matrix function R satisfies $|\text{determinant}(R)| > 0\ \forall\mathbf{x}\in\Omega$ and $\mathbf{l}_{1,3}^T\,R^{-1}\,\mathbf{l}_{1,3} \neq 0\ \forall\mathbf{l}_{1,3}\in\mathbb{R}^3, \mathbf{l}_{1,3}\neq 0,\ \forall\mathbf{x}\in\Omega$. Then, we deduce*

$$\|\big(S(R,\mathbf{l}_{1,3})-T(u(R),\mathbf{l}_{1,3})\big)^{-1}\|_2 = \frac{1}{\|\mathbf{l}_{1,3}\|_2^2\,u(R)} \quad \forall\mathbf{l}_{1,3}\in\mathbb{R}^3, \mathbf{l}_{1,3}\neq 0. \tag{107}$$

where the real function $u(R)$ is defined by

$$u(R) = \frac{1}{2}\cdot \tag{108}$$

$$\left(\min_{i=1,2,3}|r_{ii}| - \big(\sum_{i,j=1}^3 |r_{ij}|\big) + \sqrt{\Big(\big(\sum_{i,j=1}^3 |r_{ij}|\big) - \min_{i=1,2,3}|r_{ii}|\Big)^2 + 4\,|\text{determinant}(R)|\,m(R)}\right),$$

being $m(R)$ the following minimum of a continuous function on the unit sphere $\lambda_1^2+\lambda_2^2+\lambda_3^2=1$

$$m(R) = \min_{\lambda_1^2+\lambda_2^2+\lambda_3^2=1} |\mathbf{l}_{1,3}^T\,R^{-1}\,\mathbf{l}_{1,3}| = \min_{(\lambda_1,\lambda_2,\lambda_3)\neq 0} |\mathbf{l}_{1,3,n}^T\,R^{-1}\,\mathbf{l}_{1,3,n}|. \tag{109}$$

Proof. Since transposition does not alter determinant [49] (p. 463), we easily deduce that

$$\text{determinant}\big((S(R,\mathbf{l}_{1,3})-T(u,\mathbf{l}_{1,3}))^*\big) = \Big(\text{determinant}\big(S(R,\mathbf{l}_{1,3})-T(u,\mathbf{l}_{1,3})\big)\Big)^*. \tag{110}$$

Moreover, [49] (p. 467), determinant(AB) = determinant(A) determinant(B). Then, by using also equation (40), we deduce

$$\begin{aligned}&\text{determinant}\Big(\big(S(R,\mathbf{l}_{1,3})-T(u,\mathbf{l}_{1,3})\big)^*\,\big(S(R,\mathbf{l}_{1,3})-T(u,\mathbf{l}_{1,3})\big)\Big) \\ &\quad = |\text{determinant}\big(S(R,\mathbf{l}_{1,3})-T(u,\mathbf{l}_{1,3})\big)|^2 = |u|^2\,\big(\lambda_1^2+\lambda_2^2+\lambda_3^2\big)^4\,|\mathbf{l}_{1,3}^T\,Z(R)\,\mathbf{l}_{1,3}|^2.\end{aligned} \tag{111}$$

But for any matrix A the product $A^*\,A$ gives an Hermitian symmetric matrix, having real [49] (p. 549) and non-negative [49] (p. 553) eigenvalues. Hermitian matrices are normal [49] (p. 548) and their determinants are given by the product of their eigenvalues [49] (p. 547).

Thus, if we use the notation $b_i, i=1,2,3$, for the eigenvalues of $\big(S(R,\mathbf{l}_{1,3})-T(u,\mathbf{l}_{1,3})\big)^*\,\big(S(R,\mathbf{l}_{1,3})-T(u,\mathbf{l}_{1,3})\big)$, we deduce

$$|u|^2\,\big(\lambda_1^2+\lambda_2^2+\lambda_3^2\big)^4\,|\mathbf{l}_{1,3}^T\,Z(R)\,\mathbf{l}_{1,3}|^2 = b_1\,b_2\,b_3. \tag{112}$$

But by using Lemma 16 we know that one of these eigenvalues, let us say b_2, satisfies

$$b_2 = |u|^2\,\big(\lambda_1^2+\lambda_2^2+\lambda_3^2\big)^2. \tag{113}$$

Moreover, by Lemma 13 the biggest eigenvalue, let us say b_3, is such that

$$b_3 \le \big(\lambda_1^2+\lambda_2^2+\lambda_3^2\big)^2\left(|u| + \big(\sum_{i,j=1}^3 |r_{ij}|\big) - \min_{i=1,2,3}|r_{ii}|\right)^2. \tag{114}$$

Thus, for any matrix function R we deduce, from equation (112), if $u\ne 0$ in Ω, that the remaining eigenvalue b_1 is such that

$$b_1 \ge \frac{|\mathbf{l}_{1,3}^T\,Z(R)\,\mathbf{l}_{1,3}|^2}{\Big(|u| + \big(\sum_{i,j=1}^3 |r_{ij}|\big) - \min_{i=1,2,3}|r_{ii}|\Big)^2}. \tag{115}$$

But for the invertibility of R we know that $Z(R)$ = determinant$(R)\,R^{-1}$ and, from equation (115), we obtain

$$\begin{aligned} b_1 \ge & \frac{|\text{determinant}(R)|^2\,|\mathbf{l}_{1,3}^T\,R^{-1}\,\mathbf{l}_{1,3}|^2}{\Big(|u| + \big(\sum_{i,j=1}^3 |r_{ij}|\big) - \min_{i=1,2,3}|r_{ii}|\Big)^2} = \\ & \frac{|\text{determinant}(R)|^2\,|\mathbf{l}_{1,3,n}^T\,R^{-1}\,\mathbf{l}_{1,3,n}|^2}{\Big(|u| + \big(\sum_{i,j=1}^3 |r_{ij}|\big) - \min_{i=1,2,3}|r_{ii}|\Big)^2}\,\big(\lambda_1^2+\lambda_2^2+\lambda_3^2\big)^2. \end{aligned} \tag{116}$$

Since $|\mathbf{l}_{1,3,n}^T\,R^{-1}\,\mathbf{l}_{1,3,n}|$ has a minimum $m(R)$ with respect to $\mathbf{l}_{1,3,n}$ (defined in (109)), we have

$$b_1 \ge \frac{|\text{determinant}(R)|^2\,(m(R))^2}{\Big(|u| + \big(\sum_{i,j=1}^3 |r_{ij}|\big) - \min_{i=1,2,3}|r_{ii}|\Big)^2}\,\big(\lambda_1^2+\lambda_2^2+\lambda_3^2\big)^2. \tag{117}$$

The function u can now be defined in such a way that also b_2 (see (113)) is not lower than the right-hand side of (117). To this aim, u must satisfy

$$|u(R)| \ge \frac{|\text{determinant}(R)|\,m(R)}{\Big(|u(R)| + \big(\sum_{i,j=1}^3 |r_{ij}|\big) - \min_{i=1,2,3}|r_{ii}|\Big)} \tag{118}$$

from which we deduce the equivalent condition

$$\begin{aligned} & u(R) \ge \frac{1}{2}\cdot \\ & \left(\min_{i=1,2,3}|r_{ii}| - \big(\sum_{i,j=1}^3 |r_{ij}|\big) + \sqrt{\Big(\big(\sum_{i,j=1}^3 |r_{ij}|\big) - \min_{i=1,2,3}|r_{ii}|\Big)^2 + 4\,|\text{determinant}(R)|\,m(R)}\right). \end{aligned} \tag{119}$$

Noticing that (117) gives the sharpest bound when u is minimum, we define u by (108). With this definition also (118) holds as an equality and we have

$$b_1 \geq b_2 = u(R)^2 \left(\lambda_1^2 + \lambda_2^2 + \lambda_3^2\right)^2. \tag{120}$$

Hence, the conclusion follows from equation (5.2.8) of [49] (p. 281). □

Lemma 18. *Suppose that the matrix function R satisfies $|\text{determinant}(R)| \geq C_d > 0\ \forall \mathbf{x} \in \Omega$ and $|\mathbf{l}_{1,3,n}^T\, R^{-1}\, \mathbf{l}_{1,3,n}| \geq C_r > 0\ \forall \mathbf{l}_{1,3,n} \in \mathbb{R}^3 : \|\mathbf{l}_{1,3,n}\|_2 = 1,\ \forall \mathbf{x} \in \Omega$. Suppose, moreover, that $\Big(\big(\sum_{i,j=1}^3 |r_{ij}|\big) - \min_{i=1,2,3} |r_{ii}|\Big) \leq C_s\ \forall \mathbf{x} \in \Omega$.*

Then, we deduce

$$\|\big(S(R, \mathbf{l}_{1,3}) - T(u_{R,c}, \mathbf{l}_{1,3})\big)^{-1}\|_2 = \frac{1}{\|\mathbf{l}_{1,3}\|_2^2\, u_{R,c}} \quad \forall \mathbf{l}_{1,3} \in \mathbb{R}^3, \mathbf{l}_{1,3} \neq 0, \tag{121}$$

where the constant real function $u_{R,c}$ is defined by

$$u_{R,c} = \frac{1}{2}\Big(- C_s + \sqrt{C_s^2 + 4\, C_d\, C_r}\Big). \tag{122}$$

Proof. Under the assumed hypotheses, from (117) we get

$$b_1 \geq \frac{C_d^2\ C_r^2}{(|u| + C_s)^2} \left(\lambda_1^2 + \lambda_2^2 + \lambda_3^2\right)^2. \tag{123}$$

Then, by proceeding as in the last part of the proof of Lemma 17, the conclusion follows. □

Lemma 19. $\|S(R, \mathbf{l}_{1,3})\|_2 \leq \|\mathbf{l}_{1,3}\|_2^2 \max_{i=1,2,3} |R_{ii}| = \|\mathbf{l}_{1,3}\|_2^2\, \|R\|_2$*, for all* 3×3 *diagonal matrix-valued real functions R definite in Ω and $\forall \mathbf{l}_{1,3} \in \mathbb{R}^3$.*

Proof. Since R is Hermitian (for any $\mathbf{x} \in \Omega$) Lemma 14 implies that $S(R, \mathbf{l}_{1,3})^* = S(R^*, \mathbf{l}_{1,3})^* = S(R, \mathbf{l}_{1,3})$. Thus any eigenvalue of $S(R, \mathbf{l}_{1,3})^* \cdot S(R, \mathbf{l}_{1,3})$ is exactly the square of an eigenvalue of $S(R, \mathbf{l}_{1,3})$. Consequently, $\|S(R, \mathbf{l}_{1,3})\|_2 = \max_{i=1,2,3} |\eta_i|$, where η_i, $i = 1, 2, 3$, are the (real) eigenvalues of $S(R, \mathbf{l}_{1,3})$ [49] (p. 281). Exactly the same reasoning applies to R and we deduce $\|R\|_2 = \max_{i=1,2,3} |R_{ii}|$.

Under the assumed hypotheses we have

$$S(R, \mathbf{l}_{1,3}) = \begin{bmatrix} -R_{33}\lambda_2^2 - R_{22}\lambda_3^2 & R_{33}\lambda_1\lambda_2 & R_{22}\lambda_1\lambda_3 \\ R_{33}\lambda_2\lambda_1 & -R_{33}\lambda_1^2 - R_{11}\lambda_3^2 & R_{11}\lambda_2\lambda_3 \\ R_{22}\lambda_3\lambda_1 & R_{11}\lambda_3\lambda_2 & -R_{22}\lambda_1^2 - R_{11}\lambda_2^2 \end{bmatrix}. \tag{124}$$

From Lemma 9 we know that one eigenvalue of $S(R, \mathbf{l}_{1,3})$ is zero. The other two real eigenvalues can be easily shown to be the roots of

$$x^2 + bx + c \tag{125}$$

where

$$b = R_{11}(\lambda_2^2 + \lambda_3^2) + R_{22}(\lambda_1^2 + \lambda_3^2) + R_{33}(\lambda_1^2 + \lambda_2^2), \tag{126}$$

$$c = (\lambda_1^2 + \lambda_2^2 + \lambda_3^2)\left(R_{22}R_{33}\lambda_1^2 + R_{11}R_{33}\lambda_2^2 + R_{11}R_{22}\lambda_3^2\right). \tag{127}$$

If R is definite, then $c > 0$. Moreover, $b > 0$ if R is positive definite and $b < 0$ if R is negative definite.

Without loss of generality we assume that the axes have been ordered in such a way that $R_{11} \geq R_{22}$ and $R_{11} \geq R_{33}$. Moreover, we assume $R_{22} \geq R_{33}$ but the result is unchanged if $R_{33} \geq R_{22}$, as the reader can easily verify. It is then possible to deduce that

$$\begin{aligned} b^2 - 4c = {} & \lambda_1^4(R_{22} - R_{33})^2 + \lambda_2^4(R_{11} - R_{33})^2 + \lambda_3^4(R_{11} - R_{22})^2 \\ & + 2\lambda_1^2\lambda_2^2(R_{11} - R_{33})(R_{22} - R_{33}) - 2\lambda_1^2\lambda_3^2(R_{11} - R_{22})(R_{22} - R_{33}) \\ & + 2\lambda_2^2\lambda_3^2(R_{11} - R_{22})(R_{11} - R_{33}) \\ \leq {} & \left(\lambda_1^2(R_{22} - R_{33}) + \lambda_2^2(R_{11} - R_{33}) + \lambda_3^2(R_{11} - R_{22})\right)^2. \end{aligned} \tag{128}$$

For positive or negative definite matrices the eigenvalue η_i with biggest amplitude is given by

$$\begin{aligned}\max_{i=1,2,3}|\eta_i| &= \frac{|b|+\sqrt{b^2-4c}}{2}\\ &\leq \frac{|R_{11}(\lambda_2^2+\lambda_3^2)+R_{22}(\lambda_1^2+\lambda_3^2)+R_{33}(\lambda_1^2+\lambda_2^2)|}{2}\\ &+\frac{\lambda_1^2(R_{22}-R_{33})+\lambda_2^2(R_{11}-R_{33})+\lambda_3^2(R_{11}-R_{22})}{2}.\end{aligned} \tag{129}$$

The right-hand side of (129) is equal to $R_{22}\lambda_1^2+R_{11}\lambda_2^2+R_{11}\lambda_3^2$ when R is positive definite and to $-R_{33}\lambda_1^2-R_{33}\lambda_2^2-R_{22}\lambda_3^2$ when R is negative definite. Therefore the right-hand side of (129) is lower than or equal to $\max_{i=1,2,3}|R_{ii}|\,\left(\lambda_1^2+\lambda_2^2+\lambda_3^2\right)=\|\mathbf{l}_{1,3}\|_2^2\,\|R\|_2$. □

Lemma 20. *Consider a 3×3 diagonal matrix-valued real function R uniformly definite in Ω. Define u as a constant such that $u\geq\sup_{\mathbf{x}\in\Omega}\min_{i=1,2,3}|R_{ii}|$. Then, $\|\left(S(R,\mathbf{l}_{1,3})-T(u,\mathbf{l}_{1,3})\right)^{-1}\|_2\leq\frac{1}{\|\mathbf{l}_{1,3}\|_2^2\ \min_{i=1,2,3}|R_{ii}|}=\frac{\|R^{-1}\|_2}{\|\mathbf{l}_{1,3}\|_2^2}\ \forall\mathbf{l}_{1,3}\in\mathbb{R}^3,\mathbf{l}_{1,3}\neq 0$.*

Proof. From the proof of Lemma 19 we know that $S(R,\mathbf{l}_{1,3})$ has two non-trivial eigenvalues (the roots of (125) with $c>0$). From Lemma 15 these are eigenvalues also for $S(R,\mathbf{l}_{1,3})-T(u,\mathbf{l}_{1,3})$. Moreover, from Lemma 9, the other eigenvalue of $S(R,\mathbf{l}_{1,3})-T(u,\mathbf{l}_{1,3})$ is $u\,(\lambda_1^2+\lambda_2^2+\lambda_3^2)$.

The root of (125) having biggest amplitude have been considered in Lemma 19. From (125) - (128) we can deduce that the other non-trivial root β of the same polynomial is such that

$$\begin{aligned}|\beta| &= \frac{|b|-\sqrt{b^2-4c}}{2}\\ &\geq\frac{1}{2}\Big(\lambda_1^2\big(|R_{22}+R_{33}|-(R_{22}-R_{33})\big)+\lambda_2^2\big(|R_{11}+R_{33}|-(R_{11}-R_{33})\big)+\\ &\qquad\lambda_3^2\big(|R_{11}+R_{22}|-(R_{11}-R_{22})\big)\Big)\\ &\geq\Big(\lambda_1^2+\lambda_2^2+\lambda_3^2\Big)\min_{i=1,2,3}|R_{ii}|=\|\mathbf{l}_{1,3}\|_2^2\ \min_{i=1,2,3}|R_{ii}|.\end{aligned} \tag{130}$$

If we choose $u\geq\sup_{\mathbf{x}\in\Omega}\min_{i=1,2,3}|R_{ii}|$ we conclude that the eigenvalue of $S(R,\mathbf{l}_{1,3})-T(u,\mathbf{l}_{1,3})$ with the smallest amplitude is bigger or equal to $\|\mathbf{l}_{1,3}\|_2^2\ \min_{i=1,2,3}|R_{ii}|$.

Then, from equation (5.2.8) of [49] (p. 281), $\|\left(S(R,\mathbf{l}_{1,3})-T(u,\mathbf{l}_{1,3})\right)^{-1}\|_2\leq\frac{1}{\|\mathbf{l}_{1,3}\|_2^2\ \min_{i=1,2,3}|R_{ii}|}=\frac{\|R^{-1}\|_2}{\|\mathbf{l}_{1,3}\|_2^2}$. □

References

[1] C. B. Morrey. *Multiple Integrals in the Calculus of Variations.* Springer–Verlag, Berlin, 1966.

[2] O. A. Ladyzhenskaya and N. N. Ural'tseva. *Linear and Quasilinear Elliptic Equations.* Academic Press, New York, 1968.

[3] C. Miranda. *Partial Differential Equations of Elliptic Type.* Springer–Verlag, Berlin, 1970.

[4] C. Müller. *Foundations of the Mathematical Theory of Electromagnetic Waves.* Springer-Verlag, Berlin, 1969.

[5] R. Leis. *Initial boundary value problems in Mathematical Physics.* John Wiley & Sons, New York, 1986.

[6] C. Hazard and M. Lenoir. On the solution of time-harmonic scattering problems for Maxwell's equations. *SIAM Journal on Mathematical Analysis*, 27:1597–1630, 1996.

[7] K. Kang and S. Kim. On the Hölder continuity of solutions of a certain system related to Maxwell's equations. *SIAM Journal on Mathematical Analysis*, 34(1):87–100, 2002.

[8] H.-M. Yin. Optimal regularity of solution to a degenerate elliptic system arising in electromagnetic fields applications to microwave heating. *Communications on Pure and Applied Analysis*, 1(1):127–134, March 2002.

[9] M. Costabel. A remark on the regularity of solutions of Maxwell's equations on Lipschitz domains. *Mathematical Methods in the Applied Sciences*, 12:365–368, 1990.

[10] M. Costabel and M. Dauge. Singularities of electromagnetic fields in polyhedral domains. *Archive for Rational Mechanics and Analysis*, 151(3):221–276, 2000.

[11] H.-M. Yin. Regularity of weak solution to Maxwell's equations and applications to microwave heating. *Journal of Differential Equations*, 200(1):137–161, 2004.

[12] F. Jochmann. Regularity of weak solutions of Maxwell's equations with mixed boundary conditions. *Mathematical Methods in the Applied Sciences*, 22:1255–1274, 1999.

[13] M. Costabel, M. Dauge, and S. Nicaise. Singularities of Maxwell interface problems. *Mathematical Modelling and Numerical Analysis*, 33(3):627–649, 1999.

[14] V. G. Veselago. The electrodynamics of substances with simultaneously negative values of ε and μ. *Soviet Physics-Uspekhi*, 10(4):509–514, January-February 1968.

[15] J. B. Pendry. Negative refraction makes a perfect lens. *Physical Review Letters*, 85:3966–3969, 2000.

[16] R. W. Ziolkowski and N. Engheta, editors. *Special Issue on Metamaterials*, volume 51. IEEE Transactions on Antennas and Propagation, October 2003.

[17] B. A. Munk. *Metamaterials: critique and alternatives.* Wiley, New York, 2009.

[18] L. D. Landau and E. M. Lifshits. *Physique Theorique, Tome VIII, Electrodinamique des milieux continus.* MIR, Moscow, 1969.

[19] J. A. Kong. *Theory of Electromagnetic Waves.* Wiley, New York, 1975.

[20] E. J. Post. *Formal structure of electromagnetics - general covariance and electromagnetics.* Dover, Mineola, New York, 1997.

[21] V. J. Folen, G. T. Rado, and E. W. Stalder. Anisotropy of the magnetoelectric effect in $cr2o3$. *Physical Review Letters*, 6(11):607–608, Jun 1961.

[22] G. T. Rado. Mechanism of the magnetoelectric effect in an antiferromagnet. *Physical Review Letters*, 6(11):609–610, Jun 1961.

[23] G. T. Rado and V. J. Folen. Observation of the magnetically induced magnetoelectric effect and evidence for antiferromagnetic domains. *Physical Review Letters*, 7(8):310–311, Oct 1961.

[24] G. T. Rado. Statistical theory of magnetoelectric effects in antiferromagnetics. *Physical Review*, 128(6):2546–2556, Dec 1962.

[25] S. Shtrikman and D. Treves. Observation of the magnetoelectric effect in $cr2o3$ powders. *Physical Review*, 130(3):986–988, May 1963.

[26] Y. Y. Liu, S. H. Xie, and J. Y. Li. The effective medium approximation for annealed magnetoelectric polycrystals. *Journal of Applied Physics*, 103(2):023919–1–023919–6, 2008.

[27] J.-P. Rivera. A short review of the magnetoelectric effect and related experimental techniques on single phase (multi-) ferroics. *The European Physical Journal B - Condensed Matter and Complex Systems*, 71(3):299–313, 2009.

[28] F. Bilotti, L. Vegni, and A. Toscano. Radiation and scattering features of patch antennas with bianisotropic substrates. *IEEE Transactions on Antennas and Propagation*, 51(3):449–456, March 2003.

[29] L. Valor and J. Zapata. An efficient finite element formulation to analyze waveguides with lossy inhomogeneous bi-anisotropic materials. *IEEE Transactions on Microwave Theory and Techiques*, 44(2):291–296, February 1996.

[30] T. X. Wu and D. L. Jaggard. A comprehensive study of discontinuities in chirowaveguides. *IEEE Transactions on Microwave Theory and Techniques*, 50(10):2320–2330, October 2002.

[31] S. Maruyama and M. Koshiba. A vector finite element formulation for general bi-anisotropic waveguides. *IEEE Transactions on Magnetics*, 33(2 - Part 2):1528–1531, March 1997.

[32] M. K. Kärkkäinen. Numerical study of wave propagation in uniaxially anisotropic Lorentzian backward-wave slabs. *Physical Review E*, 68(2):026602–1–026602–6, 2003.

[33] R. Marqués, F. Mesa, J. Martel, and F. Medina. Comparative analysis of edge- and broadside-coupled split ring resonators for metamaterial design - theory and experiments. *IEEE Transactions on Antennas and Propagation*, 51(10):2572–2581, October 2003.

[34] D. R. Smith and D. Schuring. Electromagnetic wave propagation in media with indefinite permittivity and permeability tensors. *Physical Review Letters*, 90(7):077405–1 – 077405–4, February 2003.

[35] A. Ishimaru, S.-W. Lee, Y. Kuga, and V. Jandhyala. Generalized constitutive relations for metamaterials based on the quasi-static lorentz theory. *IEEE Transactions on Antennas and Propagation*, 51(10):2550–2557, October 2003.

[36] R. Marqués, F. Medina, and R. Rafii-El-Idrissi. Role of bianisotropy in negative permeability and left-handed metamaterials. *Physical Review B*, 65(144440), 2002.

[37] P. Fernandes and M. Raffetto. Existence, uniqueness and finite element approximation of the solution of time-harmonic electromagnetic boundary value problems involving metamaterials. *COMPEL*, 24(4):1450–1469, 2005.

[38] P. Fernandes and M. Raffetto. Well posedness and finite element approximability of time-harmonic electromagnetic boundary value problems involving bianisotropic materials and metamaterials. *Mathematical Models and Methods in Applied Sciences*, 19(12):2299–2335, December 2009. DOI No: 10.1142/S0218202509004121.

[39] A. S. Bonnet-Ben Dhia, P. Ciarlet Jr., and C. M. Zwölf. Two- and three-field formulations for wave transmission between media with opposite sign dielectric constants. *Journal of Computational and Applied Mathematics*, 204:408–417, 2007.

[40] A. S. Bonnet-Ben Dhia, P. Ciarlet Jr., and C. M. Zwölf. Time harmonic wave diffraction problems in materials with sign-shifting coefficients. *Journal of Computational and Applied Mathematics*, 234:1912–1919, 2010.

[41] C. M. Zwölf. *Méthodes variationnelles pour la modélisation des problèmes de transmission d'onde électromagnétique entre diélectrique et méta-matériau.* Phd thesis, Université de Versailles et Saint Quentin en Yvelines, December 2007.

[42] R. F. Harrington. *Time-Harmonic Electromagnetic Fields.* McGraw-Hill, New York, 1961.

[43] V. Girault and P. A. Raviart. *Finite element methods for Navier–Stokes equations.* Springer–Verlag, Berlin, 1986.

[44] R. Dautray and J. L. Lions. *Mathematical analysis and numerical methods for science and technology, vol. 1: physical origins and classical methods.* Springer–Verlag, Berlin, 1988.

[45] R. E. Collin. *Field theory of guided waves.* McGraw-Hill, New York, 1960.

[46] R. D. Richtmyer. *Principle of advanced mathematical physics*, volume 1. Springer-Verlag, New York, 1978.

[47] R. Dautray and J. L. Lions. *Mathematical analysis and numerical methods for science and technology, vol. 2: functional and variational methods.* Springer–Verlag, Berlin, 1988.

[48] A. H. Sihvola and O. P. M. Pekonen. Effective medium formulae for bi-anisotropic mixtures. *Journal of Physics D: Applied Physics*, 29(3):514–521, 1996.

[49] C. D. Meyer. *Matrix analysis and applied linear algebra.* SIAM, 2000.

[50] M. Costabel. A coercive bilinear form for Maxwell's equations. *Journal of Mathematical Analysis and Applications*, 157(2):527–541, 1991.

[51] I. V. Lindell and A. H. Sihvola. Negative-definite media, a class of bi-anisotropic metamaterials. *Microwave and Optical Technology Letters*, 48(3):602–606, March 2006.

[52] C. G. Parazzoli, R. B. Greegor, K. Li, B. E. C. Koltenbah, and M. Tanielian. Experimental verification and simulation of negative index of refraction using snell's law. *Physical Review Letters*, 90(10):107401, March 2003.

[53] D. R. Smith, P. Rye, D. C. Vier, A. F. Starr, J. J. Mock, and T. Perram. Design and measurement of anisotropic metamaterials that exhibit negative refraction. *IEICE Transactions on Electronics*, E87-C(3), March 2004.

[54] M. Y. Wang, J. Xu, J. Wu, B. Wei, H. L. Li, T. Xu, and D. B. Ge. FDTD study on wave propagation in layered structures with biaxial anisotropic metamaterials. *Progress In electromagnetics research*, PIER 81:253–265, 2008.

[55] G. T. Rado, J. M. Ferrari, and W. G. Maisch. Magnetoelectric susceptibility and magnetic symmetry of magnetoelectrically annealed $TbPO_4$. *Physical Review B*, 29(7):4041–4048, Apr 1984.

[56] D. K. Cheng and J.-A. Kong. Covariant descriptions of bianisotropic media. *Proceedings of the IEEE*, 56(3):248–251, March 1968.