Technical University of Crete
School of Electrical and Computer Engineering
Course: **Advanced Topics in Convex Optimization**
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Part A 1 Computation of the projection onto the unit simplex.

The unit Simplex: $\Delta_n = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{x} \geq \mathbf{0}, \mathbf{e}^T\mathbf{x} = 1\}$.

There for the problem can be written as:

$$\begin{aligned} \min\ & f(\mathbf{x}) = \frac{1}{2}||\mathbf{x_0} - \mathbf{x}||_2^2 \\ \text{s.t.}\quad & \mathbf{e}^T\mathbf{x} = 1 \\ & \mathbf{x} \geq \mathbf{0} \end{aligned}$$

3 Let $\mathbf{c} \in \mathbb{R}^\mathbf{n}$ and $\Delta_n = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{x} \geq \mathbf{0}, \mathbf{e}^T\mathbf{x} = 1\}$.
The problem:

$$\min\ _{\mathbf{x} \in \boldsymbol{\Delta}_\mathbf{n}} f(\mathbf{x}) = \mathbf{c}^T\mathbf{x}$$

Which can be written as:

$$\begin{aligned} \min\ _{\mathbf{x} \in \mathbb{R}^n} & f(\mathbf{x}) = \mathbf{c}^T\mathbf{x} \\ \text{s.t.}\qquad & h(\mathbf{x}) = \mathbf{e}^T\mathbf{x} - 1 = 0 \\ & f_i(\mathbf{x}) = -x_i \leq \mathbf{0},\ i = 1, ...n \end{aligned}$$

The KKT for this problem are:

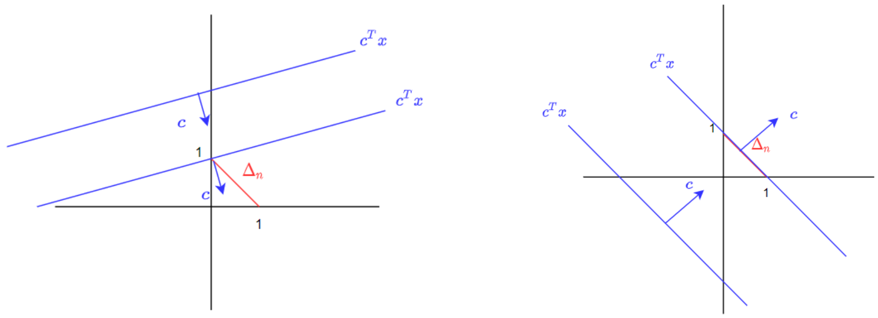
$$\nabla f(\mathbf{x}) + \sum_i^n u_i \nabla f_i(\mathbf{x}) + \mathbf{e}v = 0 \Rightarrow \mathbf{c} + \mathbf{u} + \mathbf{e}v = 0 \tag{1}$$

$$u_i \geq 0,\ i = 1, ..., n \tag{2}$$

$$u_i x_i = 0,\ i = 1, ...n \tag{3}$$

$$v \in \mathbb{R}$$

Figure 1: Examples $\mathbf{c}^T\mathbf{x}$ Level Sets and the minimum on Δ_2



As observed on the above examples we claim that a minimum $\mathbf{x}^*$ always has an element $x_k^* = 1,\ k = 1, ..., n$

Therefore $x_k^* = 1$, using (3):

$$\begin{aligned} & u_k x_k^* = 0 \overset{x_k^*=1}{\Rightarrow} u_k = 0 \\ & (1) \Rightarrow c_k + v = 0 \iff v = -c_k \end{aligned} \tag{4}$$

Replacing (4) in (1) an solving for $\mathbf{u}$ we get:

$$\mathbf{u} = \mathbf{e}c_k - \mathbf{c} \Rightarrow u_i = c_k - c_i,\ i = 1, ..., n \tag{5}$$

From (2) and (5) we get that: $c_i \geq c_k$.

Therefore we can say that the solution $\mathbf{x}^*$ is 1 at $k = min_i\mathbf{c}$ and 0 at the other indices