Technical University of Crete

School of Electrical and Computer Engineering

Course: **Advanced Topics in Convex Optimization**

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1 Computation of the projection onto the unit simplex.
The unit Simplex: $\Delta_n = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{x} \geq \mathbf{0}, \mathbf{e}^T\mathbf{x} = 1\}$.

In order to find a solution with CVX the problem can be written as:

$$\begin{aligned} \min\ & f(\mathbf{x}) = \frac{1}{2}||\mathbf{x_0} - \mathbf{x}||_2^2 \\ \text{s.t.}\quad & \mathbf{e}^T\mathbf{x} = 1 \\ & \mathbf{x} \geq \mathbf{0} \end{aligned}$$

It can be proven that for any $\mathbf{x} \in \mathbb{R}^n$ the projection onto the unit simplex,

$$P_{\Delta_n}(\mathbf{x}) = [\mathbf{x} - \mu^*\mathbf{e}]_+$$

Where μ^* is a root of the equation

$$f(\mu^*) = \mathbf{e}^T[\mathbf{x} - \mu^*\mathbf{e}]_+ - 1 = 0 \tag{1}$$

In order to find a root μ^* for (1) we will be using the bisection method.

The Bisection Method

Initialization: pick ub,lb such that $f(lb) < 0$ and $f(ub) > 0$, tolerance TOL, error offset err and maximum iterations max_iter

End Criterion: $\frac{ub-lb}{2} < TOL$ or $f(\frac{ub+lb}{2}) < err$

General Step: for any $k = 0, 1, 2, ...$ execute:

- $$mid = \frac{ub + lb}{2}$$
  $$if\ f(mid) < 0\ lb = f(mid)\ else\ ub = f(mid)$$

First we need to pick a good upper bound(ub) and lower bound(lb).

- The highest lower bound is given if every element of $x_i > lb$

  $$f(lb) < 0 \Rightarrow \mathbf{e}^T[\mathbf{x} - lb\mathbf{e}]_+ - 1 < 0 \Rightarrow sum_i^n\{x_i\} - nlb - 1 < 0 \Rightarrow lb > \frac{1}{n}sum_i^n(x_i) - \frac{1}{n}$$

- The lowest upper bound is given if only one element of $x_i > ub \Rightarrow max_i^n\{\mathbf{x}\} > ub$

  $$f(ub) > 0 \Rightarrow \mathbf{e}^T[\mathbf{x} - lb\mathbf{e}]_+ - 1 > 0 \Rightarrow max_i^n\{\mathbf{x}\} - ub - 1 > 0 \Rightarrow ub < max_i^n\{\mathbf{x}\} - 1$$

We plot the Mean Square Error of the projection found by the Bisection Algorithm against the projection found by CVX, as it progresses.

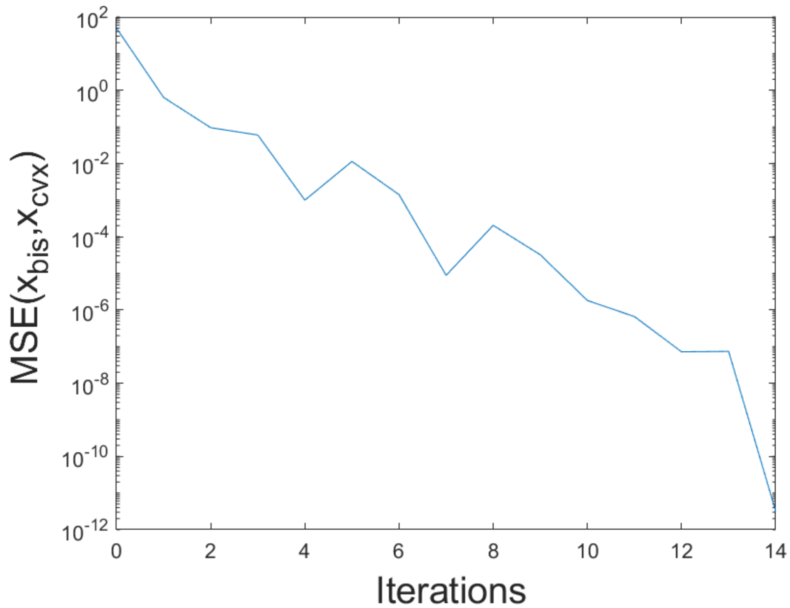


Figure 1: Mean Square Error Bisection Method-CVX, $err = 10^{-4}$, $TOL = 10^{-5}$

2 Computation of a point in the set $S = S_1 \cap S_2$ where,

$$S_1 = \{\mathbf{x} \in \mathbb{R}^n | \mathbf{A}\mathbf{x} = \mathbf{b}\},\ S_2 = \mathbb{R}^n_+$$

Since the projection cant be easily calculated in closed form we will be using the alternating projection algorithm.

Alternating Projection Algorithm
Initialization: pick $\mathbf{x_0} \in \mathbb{R}^n$

General Step: for any $k = 0, 1, 2, ...$ execute: $\mathbf{x}_{k+1} = P_{S_2}(P_{S_1(\mathbf{x}_k)})$

The projections onto S_1 and S_2 respectively

$$P_{S_1} = \mathbf{x} - \mathbf{A}^T(\mathbf{A}\mathbf{A}^T)^{-1}(\mathbf{A}\mathbf{x} - \mathbf{b})$$
$$P_{S_2} = [\mathbf{x}]_+$$

We observe that for for small problem size the algorithm solves the problem in little iterations irrelevant to the $n - m$.

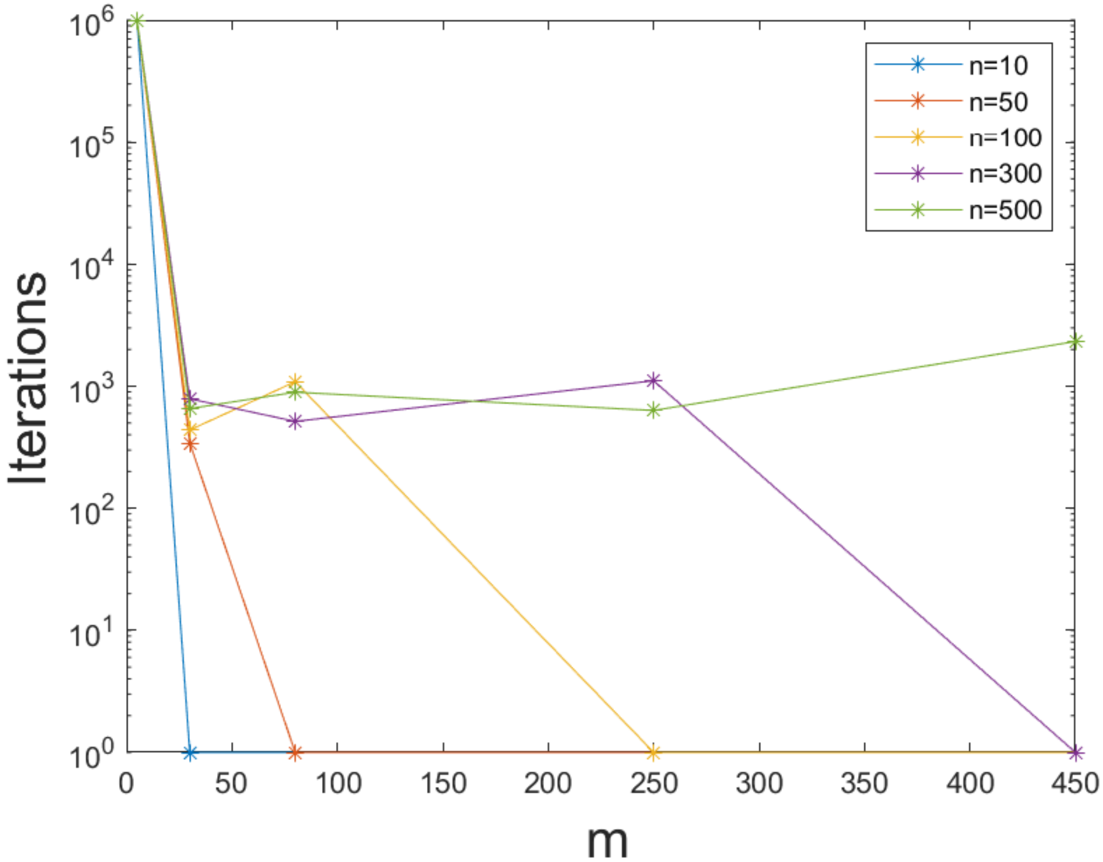


Figure 2: Alternating Projection Algorithm m-Iterations for different n

As the problem size gets bigger the iterations increase. In this case we see that the algorithm runs faster when $n << m$

3 Let $\mathbf{c} \in \mathbb{R}^{\mathbf{n}}$ and $\Delta_n = \{\mathbf{x} \in \mathbb{R}^n : \mathbf{x} \geq \mathbf{0}, \mathbf{e}^T\mathbf{x} = 1\}$.

The problem:

$$\min_{\ \mathbf{x} \in \boldsymbol{\Delta}_\mathbf{n}} f(\mathbf{x}) = \mathbf{c}^T\mathbf{x}$$

Which can be written as:

$$\begin{aligned} &\min_{\ \mathbf{x} \in \mathbb{R}^n} f(\mathbf{x}) = \mathbf{c}^T\mathbf{x} \\ &\text{s.t.} \qquad h(\mathbf{x}) = \mathbf{e}^T\mathbf{x} - 1 = 0 \\ &\qquad\qquad f_i(\mathbf{x}) = -x_i \leq \mathbf{0},\ i = 1, \ldots n \end{aligned}$$

The KKT for this problem are:

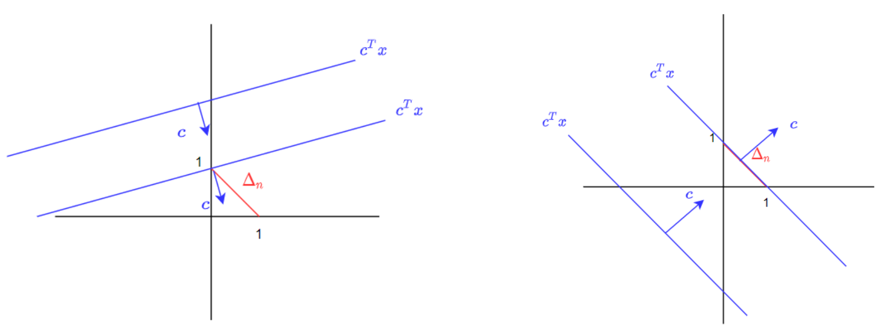
$$\nabla f(\mathbf{x}) + \sum_i^n u_i \nabla f_i(\mathbf{x}) + \mathbf{e}v = 0 \Rightarrow \mathbf{c} + \mathbf{u} + \mathbf{e}v = 0 \tag{2}$$

$$u_i \geq 0,\ i = 1, \ldots, n \tag{3}$$

$$u_i x_i = 0,\ i = 1, \ldots n \tag{4}$$

$$v \in \mathbb{R}$$

Figure 3: Examples $\mathbf{c}^T\mathbf{x}$ Level Sets and the minimum on Δ_2



As observed on the above examples we claim that a minimum $\mathbf{x}^*$ always has an element $x_k^* = 1,\ k = 1, \ldots, n$

Therefore $x_k^* = 1$, using (3):

$$u_k x_k^* = 0 \overset{x_k^*=1}{\Rightarrow} u_k = 0$$

$$(1) \Rightarrow c_k + v = 0 \iff v = -c_k \tag{5}$$

Replacing (4) in (1) an solving for $\mathbf{u}$ we get:

$$\mathbf{u} = \mathbf{e}c_k - \mathbf{c} \Rightarrow u_i = c_k - c_i, \ i = 1, ..., n \tag{6}$$

From (2) and (5) we get that: $c_i \geq c_k$.

Therefore we can say that the solution $\mathbf{x}^*$ is 1 at $k = min_i\{\mathbf{c}\}$ and 0 at the other indices

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