Technical University of Crete

School of Electrical and Computer Engineering

Course: **Convex Optimization**

Exercise 3 (50/1000)

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Instructor: Athanasios P. Liavas

Student: Alevrakis Dimitrios 2017030001

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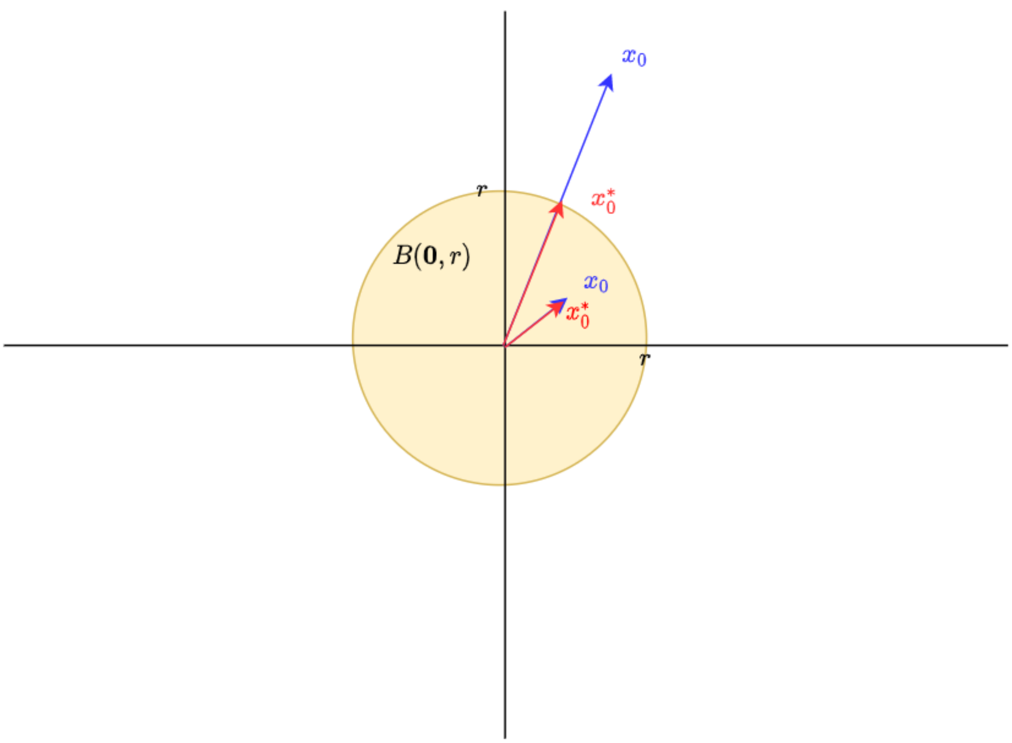
1. Compute the projection of $\mathbf{x_0} \in \mathbb{R}^n$ onto the set $B(\mathbf{0}, r) \coloneqq \{\mathbf{x} \in \mathbb{R}^n | ||\mathbf{x}||_\mathbf{2} \leq r\}$.



(a)

Figure 1: Visualization of the problem in two dimensions. We discern the two possible cases for x_0 and its projections.

(b) We can discern two different cases for our problem.

Case 1 : The vector $\mathbf{x_0}$ does not belong to the set $B(\mathbf{0}, r)$ in which case we can see that the projection is the vector $\mathbf{x}^* \in B(\mathbf{0}, r)$ for which is true that $||\mathbf{x}^*||_2 = r$ and has the same angle as $\mathbf{x_0}$. The vector $\mathbf{x}^*$ can also be described as the vector $\mathbf{x}^* \in B(\mathbf{0}, r)$ closer to $\mathbf{x_0}$.

Case 2 : The vector $\mathbf{x_0}$ belongs to the set $B(\mathbf{0}, r)$ in which case we can see that the projection of $\mathbf{x_0}$ is $\mathbf{x}^* = \mathbf{x_0}$.

We can therefore say for both cases that the computation of the projection can be written as:

$$\begin{aligned} &\text{minimize } f_0(\mathbf{x}) \coloneqq \frac{1}{2}||\mathbf{x_0} - \mathbf{x}||_2^2 \\ &\text{subject to } f_1(\mathbf{x}) = ||\mathbf{x}||_2^2 - r^2 \leq 0 \end{aligned}$$

Instead of using $f_1(x) = ||\mathbf{x}||_2 - r \leq 0$ we can use $f_1(x) = ||\mathbf{x}||_2^2 - r^2 \leq 0$ for convenience since both $||x||_2$ and r are non-negative.

(c) The KKT equations:

$$\begin{aligned} &\nabla f_0(\mathbf{x}^*) + \lambda_* \nabla f_1(\mathbf{x}^*) = 0 \qquad (1) \\ &\lambda_* \geq 0 \\ &f_1(\mathbf{x}^*) \leq 0 \\ &\lambda_* f_1(\mathbf{x}^*) = 0 \end{aligned}$$

Expanding (1):

$$\begin{aligned} &\nabla f_0(\mathbf{x}^*) + \lambda_* \nabla f_1(\mathbf{x}^*) = 0 \iff \\ &\nabla \left[\frac{1}{2}||\mathbf{x_0} - \mathbf{x}^*||_2^2\right] + \lambda_* \nabla \left[||\mathbf{x}^*||_2^2 - r^2\right] = 0 \iff \\ &\mathbf{x}^* - \mathbf{x_0} + 2\lambda_* \mathbf{x}^* = 0 \iff \\ &(1 + 2\lambda_*)\mathbf{x}^* - \mathbf{x_0} = 0 \qquad (2) \end{aligned}$$

(d) For $\lambda_* > 0$:

From the KKT equations:

$$\lambda_* f_1(\mathbf{x}^*) = 0 \overset{\lambda_* > 0}{\iff} f_1(\mathbf{x}^*) = 0 \iff ||\mathbf{x}^*||_2^2 - r^2 = 0 \iff ||\mathbf{x}^*||_2^2 = r^2 \iff ||\mathbf{x}^*||_2 = r \qquad (3)$$

Using (2):

$$
\begin{aligned}
&(1+2\lambda_*)\mathbf{x}^* - \mathbf{x_0} = 0 \iff \mathbf{x}^* = \frac{1}{1+2\lambda_*}\mathbf{x_0} \iff \\
&||\mathbf{x}^*||_2 = ||\frac{1}{1+2\lambda_*}\mathbf{x_0}||_2 \overset{(3)}{\iff} r = \frac{1}{(1+2\lambda_*)}||\mathbf{x_0}||_2 \iff \\
&r + 2r\lambda_* = ||\mathbf{x_0}||_2 \iff \lambda_* = \frac{||\mathbf{x_0}||_2 - r}{2r} > 0
\end{aligned}
$$

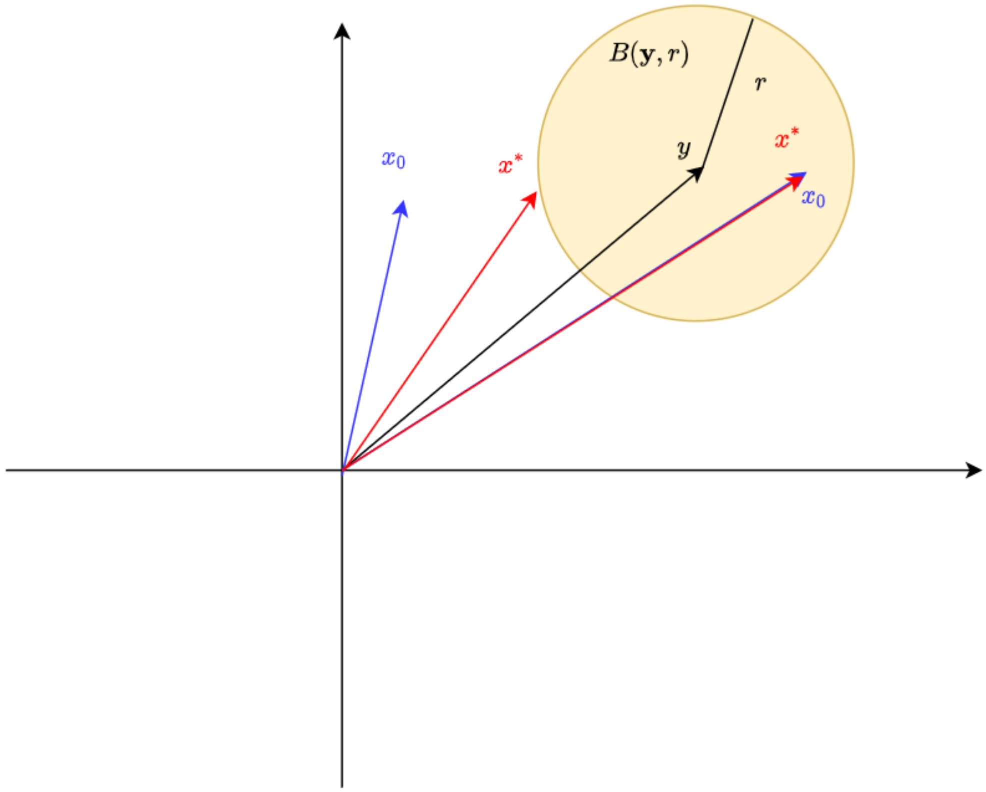
Therefore $\mathbf{x}^* = r\frac{x_0}{||x_0||}$. Since for $\lambda_* > 0$, $||\mathbf{x_0}|| > r$ we understand that $\mathbf{x_0} \notin B(\mathbf{0}, r)$ and its projection is the vector normalized and scaled by a factor of r.

(e) For $\lambda_* = 0$: Using the equation (2): $\mathbf{x}^* = \mathbf{x_0}$. From that we can conclude that $\mathbf{x_0} \in B(\mathbf{0}, r)$ and therefore its projection onto $B(\mathbf{0}, r)$ is itself.

Conclusion : If $\mathbf{x_0} \in B(\mathbf{0}, r)$ then $\mathbf{x}^* = \mathbf{x_0}$ meaning the projection is $\mathbf{x_0}$ scaled by a factor of 1. In this case the fraction $\frac{r}{||x_0||_2} > 1$. If $\mathbf{x_0} \notin B(\mathbf{0}, r)$ then $\mathbf{x}^* = r\frac{x_0}{||x_0||}$ and $\frac{r}{||x_0||_2} < 1$.

Therefore for every case we can say that $\mathbf{x}^* = min\{1, \frac{r}{||x_0||_2}\} \cdot \mathbf{x_0}$

2. Compute the projection of $\mathbf{x_0} \in \mathbb{R}^n$ onto the set $B(\mathbf{y}, r) := \{\mathbf{x} \in \mathbb{R}^n | ||\mathbf{x} - \mathbf{y}||_\mathbf{2} \leq \mathbf{r}\}$ for a given $\mathbf{y} \in \mathbb{R}^n$.



(a)

Figure 2: Visualization of the problem in two dimensions. We discern the two possible cases for x_0 and its projections.

(b) Again the computation of the projection can be written as the problem:

$$\text{minimize } f_0(\mathbf{x}) := \frac{1}{2}||\mathbf{x_0} - \mathbf{x}||_2^2$$
$$\text{subject to } f_1(\mathbf{x}) = ||\mathbf{x} - \mathbf{y}||_2^2 - r^2 \leq 0$$

(c) The KKT equations:

$$\nabla f_0(\mathbf{x}^*) + \lambda_* \nabla f_1(\mathbf{x}^*) = 0 \tag{4}$$
$$\lambda_* \geq 0$$
$$f_1(\mathbf{x}^*) \leq 0$$
$$\lambda_* f_1(\mathbf{x}^*) = 0$$

Expanding (4):

$$\begin{aligned}
&\nabla f_0(\mathbf{x}^*) + \lambda_* \nabla f_1(\mathbf{x}^*) = 0 \iff \\
&\nabla\left[\frac{1}{2}||\mathbf{x_0} - \mathbf{x}^*||_2^2\right] + \lambda_* \nabla\left[||\mathbf{x}^* - \mathbf{y}||_2^2 - r^2\right] = 0 \iff \\
&\mathbf{x}^* - \mathbf{x_0} + 2\lambda_*(\mathbf{x}^* - \mathbf{y}) = 0 \iff \\
&\mathbf{x}^* - \mathbf{y} + \mathbf{y} - \mathbf{x_0} + 2\lambda_*(\mathbf{x}^* - \mathbf{y}) = 0 \iff \\
&(1 + 2\lambda_*)(\mathbf{x}^* - \mathbf{y}) = \mathbf{x_0} - \mathbf{y} \qquad (5)
\end{aligned}$$

(d) For $\lambda_* > 0$:

From the KKT equations:

$$\begin{aligned}
&\lambda_* f_1(\mathbf{x}^*) = 0 \overset{\lambda_*>0}{\iff} f_1(\mathbf{x}^*) = 0 \iff ||\mathbf{x}^* - \mathbf{y}||_2^2 - r^2 = 0 \iff \\
&||\mathbf{x}^* - \mathbf{y}||_2^2 = r^2 \iff ||\mathbf{x}^* - \mathbf{y}||_2 = r \qquad (6)
\end{aligned}$$

Using (5):

$$\begin{aligned}
&(1 + 2\lambda_*)(\mathbf{x}^* - \mathbf{y}) = \mathbf{x_0} - \mathbf{y} \iff \\
&(1 + 2\lambda_*)||\mathbf{x}^* - \mathbf{y}||_2 = ||\mathbf{x_0} - \mathbf{y}||_2 \overset{(6)}{\iff} \\
&(1 + 2\lambda_*)r = ||\mathbf{x_0} - \mathbf{y}||_2 \iff \\
&\lambda_* = \frac{||\mathbf{x_0} - \mathbf{y}||_2 - r}{2r} > 0
\end{aligned}$$

Therefore since it must be true that $\lambda_* > 0 \iff ||\mathbf{x_0} - \mathbf{y}||_2 - r > 0$ we understand that $\mathbf{x_0} \notin B(\mathbf{y}, r)$ and the projection:

$$(5) \Rightarrow \mathbf{x}^* = \frac{\mathbf{x_0} - \mathbf{y}}{1 + 2\lambda_*} + \mathbf{y} = \frac{r}{||\mathbf{x_0} - \mathbf{y}||_2}(\mathbf{x_0} - \mathbf{y}) + \mathbf{y}$$

(e) For $\lambda_* = 0$: From (5) we get that $\mathbf{x}^* = \mathbf{x_0}$ meaning that $x_0 \in B(\mathbf{y}, r)$ and the projection of $\mathbf{x_0}$ on the set is itself.

3. Let $\mathbf{a} \in \mathbb{R}^{\mathbf{n}}$. Compute the projection of $\mathbf{x_0}$ onto set $\mathbb{S} := \{\mathbf{x} \in \mathbb{R}^n | \mathbf{a} \leq \mathbf{x}\}$

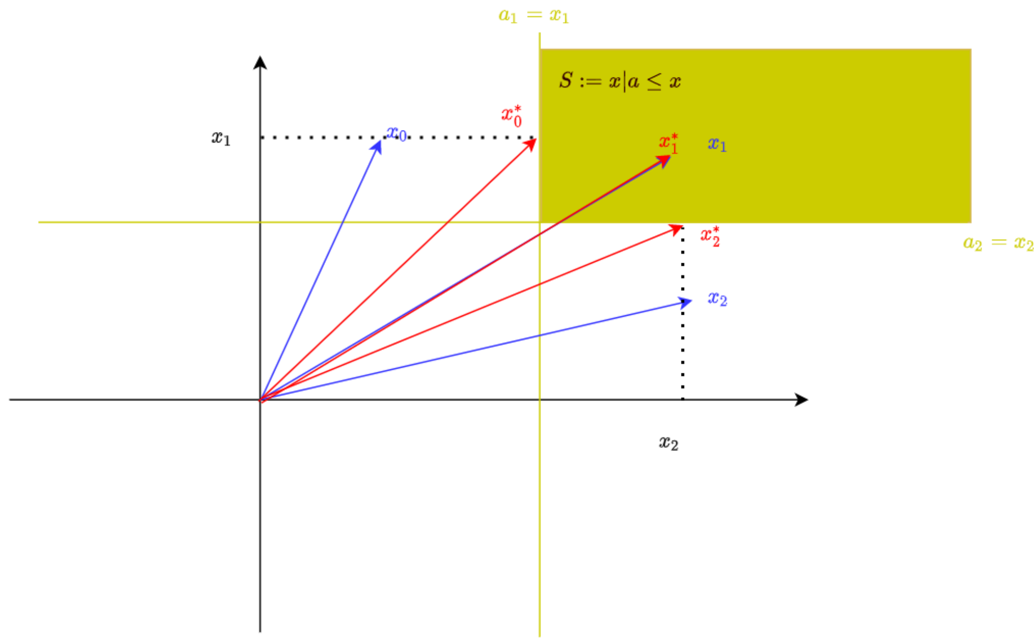


Figure 3: Visualization of the problem in two dimensions. We discern the three possible cases for x_0 and its projections.

The problem can be written as the minimization problem:

$$\text{minimize } f_0(\mathbf{x}) := \frac{1}{2}||\mathbf{x_0} - \mathbf{x}||_2^2$$
$$\text{subject to } f_i(\mathbf{x}) = a_i - x_i \leq 0,\ \ i = 0, .., 1$$

The KKT equations:

$$\nabla f_0(\mathbf{x}^*) + \sum_{i=1}^{n} \lambda_i \nabla f_i(\mathbf{x}) = 0 \tag{7}$$
$$\lambda_i \geq 0\, i = 1, .., n \tag{8}$$
$$f_i(\mathbf{x}^*) \leq 0,\ \ i = 1, .., n \tag{9}$$
$$\lambda_i f_i(\mathbf{x}^*) = 0,\ \ i = 1, .., n \tag{10}$$

Defining $\lambda \coloneqq (\lambda_1, .., \lambda_n)$ we can write equation (7) as:

$$\mathbf{x}^* - \mathbf{x_0} - \lambda = 0 \Rightarrow \mathbf{x}^* = \mathbf{x_0} + \lambda \Rightarrow x_i^* = x_{0,i} + \lambda_i, \ i = 1, .., n \tag{11}$$

We discern two cases:

- if $x_{0,1} < a_i$:

  $$x_{0,1} < a_i \Rightarrow x_{0,1} + \lambda_i < a_i + \lambda_i \overset{(11)}{\Rightarrow} x_i^* < a_i + \lambda_i \overset{(9)}{\Rightarrow} a_i \leq x_i^* < a_i + \lambda_i$$

  $$\Rightarrow a_i < a_i + \lambda_i \Rightarrow \lambda_i > 0 \tag{12}$$

  Therefore from (10) it must be true that $f_i(\mathbf{x}^*) = 0 \Rightarrow a_i - x_i = 0 \Rightarrow x_i = a_i$.

- if $x_{0,1} \geq a_i$:

  $$x_{0,1} \geq a_i \Rightarrow x_{0,1} + \lambda_i \geq a_i + \lambda_i \overset{(11)}{\Rightarrow} x_i^* \geq a_i + \lambda_i \overset{(9)}{\Rightarrow} a_i \leq x_i^* \geq a_i + \lambda_i$$

  $$\overset{(8)}{\Rightarrow} \lambda_i = 0 \tag{13}$$

  Therefore from (11): $x_i^* = x_{0,i}$.

Thus in general we can say that every element of the projection of $\mathbf{x_0}$ is given by $x_i^* = max\{a_i, x_{0,i}\}, \ i, = 1, .., n$

4. Let $\mathbf{0} \neq \mathbf{a} \in \mathbb{R}^n,\ b \in \mathbb{R}$. Solve the problem

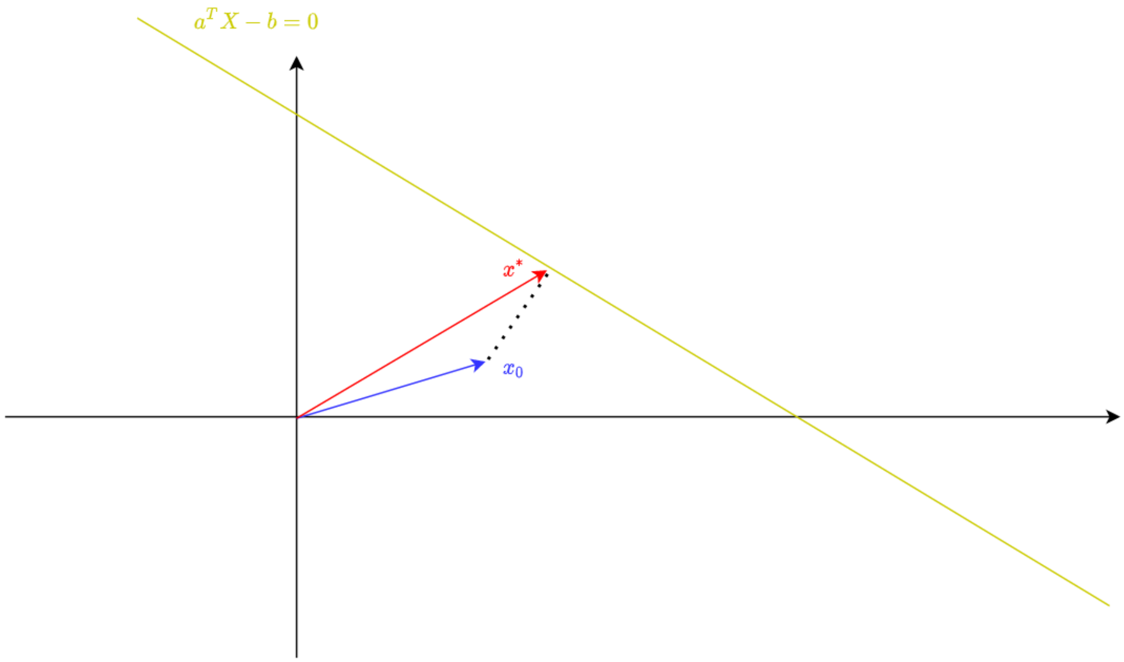


Figure 4: Visualization of the problem in two dimensions.

$$\text{minimize } \frac{1}{2}||\mathbf{x}||_2$$
$$\text{subject to } \mathbf{a}^T\mathbf{x} = b$$

This problem is the same as :

$$\text{minimize } f_0(\mathbf{x}) = \frac{1}{2}||\mathbf{x}||_2^2$$
$$\text{subject to } f_1(\mathbf{x}) = \mathbf{a}^T\mathbf{x} - b = 0$$

The KKT equations:

$$\nabla f_0(\mathbf{x}^*) + u\nabla f_1(\mathbf{x}^*) = 0, u \in \mathbb{R} \tag{14}$$
$$f_1(\mathbf{x}^*) = 0 \tag{15}$$

Expanding(14):

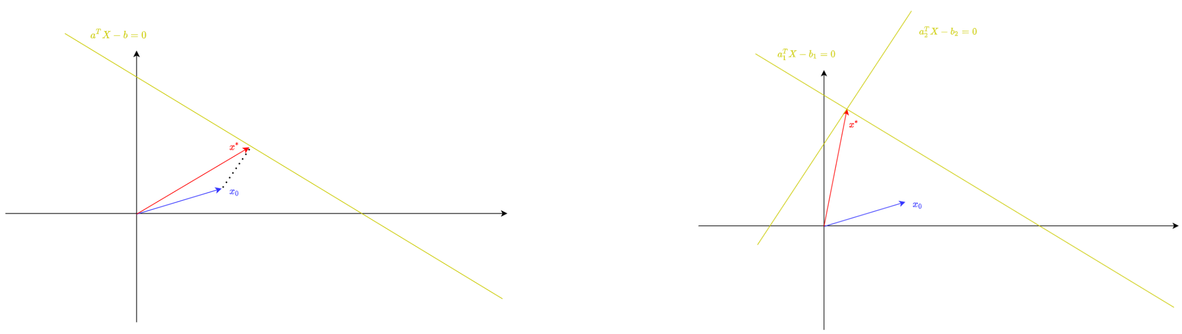
$$(14) \Rightarrow \mathbf{x}^* + u\mathbf{a} = 0 \Rightarrow \mathbf{x}^* = -u\mathbf{a} \tag{16}$$

Using (16) to (15):

$$f_1(\mathbf{x}^*) = 0 \iff \mathbf{a}^T\mathbf{x} - b = 0 \iff \mathbf{a}^T(-u\mathbf{a}) - b = 0 \overset{\mathbf{a}\neq\mathbf{0}}{\iff} u = -\frac{b}{||a||_2^2} \tag{17}$$

Therefore from (16) using (17): $\mathbf{x}^* = \dfrac{b}{||\mathbf{a}||_2^2}\mathbf{a}$

5. Let $\mathbf{A}$ $\mathbb{R}^{pxn}$, with $rank(\mathbf{A}) = p$, and $\mathbf{b} \in \mathbb{R}^p$. Find the distance from a point $\mathbf{x_0} \in \mathbb{R}^n$ from the set $\mathbb{S} \coloneqq \{\mathbf{x} \in \mathbb{R}^n | \mathbf{A}\mathbf{x} = \mathbf{b}\}$.



(a) $p = 1$ (b) $p = 2$

Figure 5: Visualization of the problem for $n = 2$. And since $n = 2$, $p \leq 2$ to ensure row linear independence.

The problem:

$$\begin{aligned} \text{minimize } & f_0(\mathbf{x}) = \frac{1}{2}||\mathbf{x} - \mathbf{x_0}||_2^2 \\ \text{subject to } & f_1(\mathbf{x}) = \mathbf{A}\mathbf{x} = \mathbf{b} \end{aligned}$$

The KKT equations:

$$\nabla f_0(\mathbf{x}^*) + \mathbf{A}^T\mathbf{v} = 0, u \in \mathbb{R} \tag{18}$$

$$\mathbf{A}\mathbf{x}^* = \mathbf{b} \tag{19}$$

Expanding (18):

$$\mathbf{x}^* - \mathbf{x_0} + \mathbf{A}^T\mathbf{v} = \mathbf{0} \iff \mathbf{x}^* = \mathbf{x_0} - \mathbf{A}^T\mathbf{v} \tag{20}$$

Since $rank(\mathbf{A}) = p$ then $\mathbf{A}\mathbf{A}^T \in \mathbb{R}^{pxp}$ with $rank(\mathbf{A}\mathbf{A}^T) = p$ therefore $\mathbf{A}\mathbf{A}^T$ is invertible. Using (20) on (19):

$$\mathbf{A}(\mathbf{x_0} - \mathbf{A}^T\mathbf{v}) = \mathbf{b} \iff \mathbf{v} = (\mathbf{A}\mathbf{A}^T)^{-1}(\mathbf{A}\mathbf{x_0} - \mathbf{b}) \tag{21}$$

Thus: Using (21) on (20) : $\mathbf{x}^* = \mathbf{x_0} - \mathbf{A}^T(\mathbf{A}\mathbf{A}^T)^{-1}(\mathbf{A}\mathbf{x_0} - \mathbf{b})$.

So the distance of $\mathbf{x_0}$ from $\mathbb{S}$ is:

$$||\mathbf{x}^* - \mathbf{x_0}||_2 = ||\mathbf{x_0} - \mathbf{A}^T(\mathbf{A}\mathbf{A}^T)^{-1}(\mathbf{A}\mathbf{x_0} - \mathbf{b}) - \mathbf{x_0}||_2 = ||\mathbf{A}^T(\mathbf{A}\mathbf{A}^T)^{-1}(\mathbf{A}\mathbf{x_0} - \mathbf{b})||_2$$

Figure 6

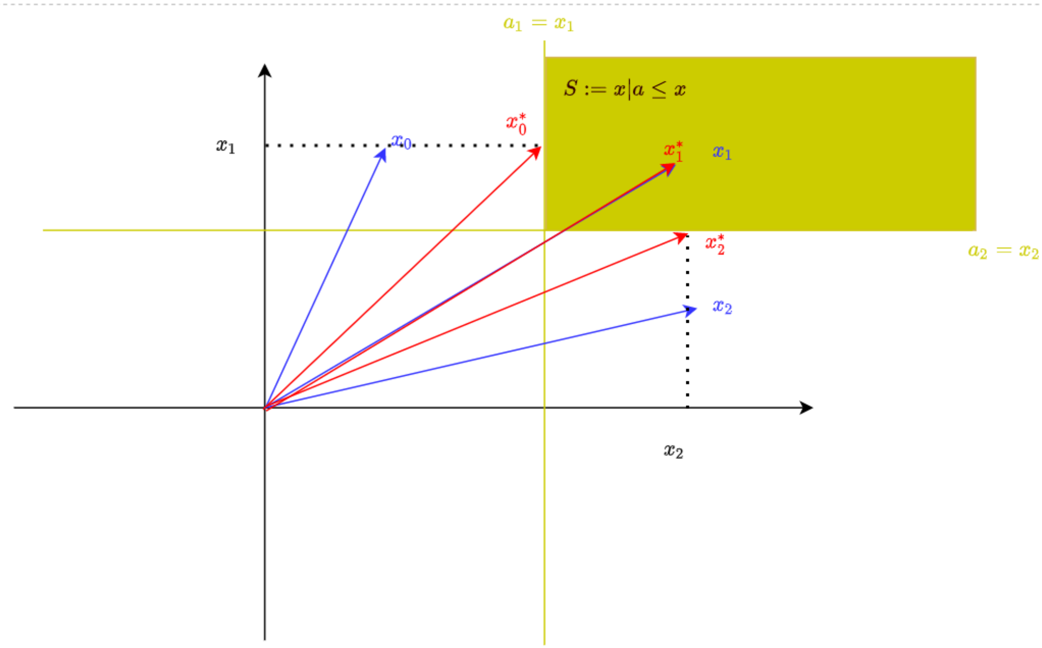


Figure 7