PLH 415

Exercise Set 1

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Problem 1 The idea behind the range query on binary search tree is traversing the tree inorder while the node keys remain in range and returning every node within the range.

Therefore the algorithm follows the steps below:

- Recursively call the left sub-tree if the root key is greater or equal to the lower limit.
- If the root key is within range keep the key
- Recursively call the right sub-tree.

The pseudo-code:

```
range_query(root, lower_limit, upper_limit, keys_found){
  if(root->key >= lower_limit && root->left != NULL)
    range_query(root->left, lower_limit, upper_limit, keys_found)

  if(root->key >= lower_limit && root->key <= upper_limit)
    keys_found.add(root->key)

  if(root->right != NULL)
    range_query(root->right, lower_limit, upper_limit, keys_found)

  return
}
```

We can see that in general we have a complexity of $O(h)$ for the recursions and complexity of $O(k)$ for the keys found.

Therefore the total complexity is $O(h + k)$.

Problem 2 We know that the convex combination of two points describe all the points in the line segment between those two points. Whereas the affine combination of two points describes all the point on the line through those two points.

Convex Combination of two points $\mathbf{x_1}, \mathbf{x_2}$:

$\mathbf{x} = \lambda \mathbf{x_1} + (1-\lambda)\mathbf{x_2}, \lambda \in [0,1]$

Affine Combination of two points $\mathbf{x_1}, \mathbf{x_2}$:

$\mathbf{x} = \lambda \mathbf{x_1} + (1-\lambda)\mathbf{x_2}, \lambda \in \mathcal{R}$

Therefore for the pairs of endpoints $(\mathbf{p_1}, \mathbf{p_2})$ and $(\mathbf{q_1}, \mathbf{q_2})$, we need to calculate the λ_1 and λ_2 for which the affine combinations of $(\mathbf{p_1}, \mathbf{p_2})$ and $(\mathbf{q_1}, \mathbf{q_2})$ respectively, give the same point(the intersection). If $\lambda_1, \lambda_2 \in [0,1]$, the intersection belongs in the line segments and thus they intersect.

Calculation of λ_1, λ_2:

Let $\mathbf{p_1} = \begin{bmatrix} p_1(1) \\ p_1(2) \end{bmatrix}, \mathbf{p_2} = \begin{bmatrix} p_2(1) \\ p_2(2) \end{bmatrix}, \mathbf{q_1} = \begin{bmatrix} q_1(1) \\ q_1(2) \end{bmatrix}, \mathbf{q_2} = \begin{bmatrix} q_2(1) \\ q_2(2) \end{bmatrix}$ assuming that $\mathbf{p_1} \neq \mathbf{p_2}, \mathbf{q_1} \neq \mathbf{q_2}$

The points given by the affine combination of the pair $(\mathbf{p_1}, \mathbf{p_2})$:

$\mathbf{p} = \lambda_1 \mathbf{p_1} + (1-\lambda_1)\mathbf{p_2} = \lambda_1(\mathbf{p_1} - \mathbf{p_2}) + \mathbf{p_2}$

The points given by the affine combination of the pair $(\mathbf{q_1}, \mathbf{q_2})$:

$\mathbf{q} = \lambda_2 \mathbf{q_1} + (1-\lambda_2)\mathbf{q_2} = \lambda_2(\mathbf{q_1} - \mathbf{q_2}) + \mathbf{q_2}$

First we check if the vectors defined by the endpoints are parallel, using the cross product:

$$|(\mathbf{q_2} - \mathbf{q_1}) \times (\mathbf{q_2} - \mathbf{q_1})| = (q_2(1) - q_1(1))(p_2(2) - q_1(2)) - (q_2(2) - q_1(2))(p_2(1) - q_1(1))$$

- If $(q_2(1) - q_1(1))(p_2(2) - q_1(2)) - (q_2(2) - q_1(2))(p_2(1) - q_1(1)) = 0$ then the line segments are parallel. Then the two line segments intersect only if one or both of the endpoints of one line segment belong in the other.

Which means that one of the following equalities must be true for some $\lambda \in [0,1]$:

$$\mathbf{p_1} = \lambda_1(\mathbf{q_1} - \mathbf{q_2}) + \mathbf{q_2} \iff \begin{cases} p_1(1) = \lambda_{1,x}(q_1(1) - q_2(1)) + q_2(1) \\ p_1(2) = \lambda_{1,y}(q_1(2) - q_2(2)) + q_2(2) \end{cases}$$

$$\mathbf{p_2} = \lambda_2(\mathbf{q_1} - \mathbf{q_2}) + \mathbf{q_2} \iff \begin{cases} p_2(1) = \lambda_{2,x}(q_1(1) - q_2(1)) + q_2(1) \\ p_2(2) = \lambda_{2,y}(q_1(2) - q_2(2)) + q_2(2) \end{cases}$$

$$\mathbf{q_1} = \lambda_3(\mathbf{p_1} - \mathbf{p_2}) + \mathbf{p_2} \iff \begin{cases} q_1(1) = \lambda_{3,x}(p_1(1) - p_2(1)) + p_2(1) \\ q_1(2) = \lambda_{3,y}(p_1(2) - p_2(2)) + p_2(2) \end{cases}$$

$$\mathbf{q_2} = \lambda_4(\mathbf{p_1} - \mathbf{p_2}) + \mathbf{p_2} \iff \begin{cases} q_2(1) = \lambda_{4,x}(p_1(1) - p_2(1)) + p_2(1) \\ q_2(2) = \lambda_{4,y}(p_1(2) - p_2(2)) + p_2(2) \end{cases}$$

We can see that if $q_1(1) = q_2(1)$ or $q_1(2) = q_2(2)$ then it must be true that $q_1(1) = q_2(1) = p_1(1) = p_2(1)$ and $q_1(2) = q_2(2) = p_1(2) = p_2(2)$ respectively for the line segments to possibly intersect.

Also it must be true that $\lambda_x = \lambda_y$ for the line segments to be co-linear and to possibly intersect. Then if any $\lambda \in [0,1]$, there is an intersection and the algorithm returns TRUE. In the other cases it returns FALSE.

- If $(q_2(1) - q_1(1))(p_2(2) - q_1(2)) - (q_2(2) - q_1(2))(p_2(1) - q_1(1)) \neq 0$.

  The intersection is at the point $\mathbf{p} = \mathbf{q}$, so

$$\begin{aligned} &\mathbf{p} = \mathbf{q} \iff \\ &\lambda_1(\mathbf{p_1} - \mathbf{p_2}) + \mathbf{p_2} = \lambda_2(\mathbf{q_1} - \mathbf{q_2}) + \mathbf{q_2} \iff \\ &\lambda_1\left(\begin{bmatrix} p_1(1) \\ p_1(2) \end{bmatrix} - \begin{bmatrix} p_2(1) \\ p_2(2) \end{bmatrix} \right) + \begin{bmatrix} p_2(1) \\ p_2(2) \end{bmatrix} = \lambda_2\left(\begin{bmatrix} q_1(1) \\ q_1(2) \end{bmatrix} - \begin{bmatrix} q_2(1) \\ q_2(2) \end{bmatrix} \right) + \begin{bmatrix} q_2(1) \\ q_2(2) \end{bmatrix} \iff \\ &\begin{bmatrix} \lambda_1(p_1(1) - p_2(1)) + p_2(1) \\ \lambda_1(p_1(2) - p_2(2)) + p_2(2) \end{bmatrix} = \begin{bmatrix} \lambda_2(q_1(1) - q_2(1)) + q_2(1) \\ \lambda_2(q_1(2) - q_2(2)) + q_2(2) \end{bmatrix} \end{aligned} \tag{1}$$

  If $p_1(1) - p_2(1) = 0$ then $p_1(2) - p_2(1) \neq 0$ because $\mathbf{p_1} \neq \mathbf{p_2}$ and $q_(1) - q_2(1) \neq 0$ because $(q_2(1) - q_1(1))(p_2(2) - q_1(2)) - (q_2(2) - q_1(2))(p_2(1) - q_1(1)) \neq 0$

$$(1) \iff \begin{bmatrix} p_2(1) \\ \lambda_1(p_1(2)-p_2(2))+p_2(2) \end{bmatrix} = \begin{bmatrix} \lambda_2(q_1(1)-q_2(1))+q_2(1) \\ \lambda_2(q_1(2)-q_2(2))+q_2(2) \end{bmatrix} \iff$$

$$\begin{cases} \lambda_2 = \frac{p_2(1)-q_2(1)}{q_1(1)-q_2(1)} \\ \lambda_1(p_1(2)-p_2(2))+p_2(2) = \lambda_2(q_1(2)-q_2(2))+q_2(2) \end{cases} \iff$$

$$\begin{cases} \lambda_2 = \frac{p_2(1)-q_2(1)}{q_1(1)-q_2(1)} \\ \lambda_1(p_1(2)-p_2(2))+p_2(2)-q_2(2) = \frac{p_2(1)-q_2(1)}{q_1(1)-q_2(1)}(q_1(2)-q_2(2)) \end{cases} \iff$$

$$\begin{cases} \lambda_2 = \frac{p_2(1)-q_2(1)}{q_1(1)-q_2(1)} \\ \lambda_1 = \frac{p_2(1)-q_2(1)(q_1(2)-q_2(2))-(p_2(2)-q_2(2))(q_1(1)-q_2(1))}{(p_1(2)-p_2(2))(q_1(1)-q_2(1))} \end{cases}$$

Else if $p_1(1)-p_2(1) \neq 0$

$$(1) \iff \begin{cases} \lambda_1(p_1(1)-p_2(1))+p_2(1) = \lambda_2(q_1(1)-q_2(1))+q_2(1) \\ \lambda_1(p_1(2)-p_2(2))+p_2(2) = \lambda_2(q_1(2)-q_2(2))+q_2(2) \end{cases} \iff$$

$$\begin{cases} \lambda_1 = \frac{\lambda_2(q_1(1)-q_2(1))+q_2(1)-p_2(1)}{p_1(1)-p_2(1)} \\ \frac{\lambda_2(q_1(1)-q_2(1))+q_2(1)-p_2(1)}{p_1(1)-p_2(1)}(p_1(2)-p_2(2))+p_2(2) = \lambda_2(q_1(2)-q_2(2))+q_2(2) \end{cases} \iff$$

$$\begin{cases} \lambda_1 = \frac{\lambda_2(q_1(1)-q_2(1))+q_2(1)-p_2(1)}{p_1(1)-p_2(1)} \\ \frac{\lambda_2(q_1(1)-q_2(1))}{p_1(1)-p_2(1)}(p_1(2)-p_2(2))+\frac{q_2(1)-p_2(1)}{p_1(1)-p_2(1)}(p_1(2)-p_2(2))+p_2(2) = \lambda_2(q_1(2)-q_2(2))+q_2(2) \end{cases} \iff$$

$$\begin{cases} \lambda_1 = \frac{\lambda_2(q_1(1)-q_2(1))+q_2(1)-p_2(1)}{p_1(1)-p_2(1)} \\ \frac{q_2(1)-p_2(1)}{p_1(1)-p_2(1)}(p_1(2)-p_2(2))+p_2(2) = \lambda_2[(q_1(2)-q_2(2))-\frac{(q_1(1)-q_2(1))}{p_1(1)-p_2(1)}(p_1(2)-p_2(2))]+q_2(2) \end{cases} \iff$$

$$\begin{cases} \lambda_1 = \frac{\lambda_2(q_1(1)-q_2(1))+q_2(1)-p_2(1)}{p_1(1)-p_2(1)} \\ \lambda_2 = \frac{(q_2(1)-p_2(1))(p_1(2)-p_2(2))+(p_1(1)-p_2(1))(p_2(2)-q_2(2))}{(p_1(1)-p_2(1))(q_1(2)-q_2(2))-(q_1(1)-q_2(1))(p_1(2)-p_2(2))} \end{cases} \iff$$

$$\begin{cases} \lambda_1 = \frac{(q_2(1)-p_2(1))(p_1(2)-p_2(2))+(p_1(1)-p_2(1))(p_2(2)-q_2(2))}{(p_1(1)-p_2(1))(q_1(2)-q_2(2))-(q_1(1)-q_2(1))(p_1(2)-p_2(2))} \frac{(q_1(1)-q_2(1))+q_2(1)-p_2(1)}{p_1(1)-p_2(1)} \\ \lambda_2 = \frac{(q_2(1)-p_2(1))(p_1(2)-p_2(2))+(p_1(1)-p_2(1))(p_2(2)-q_2(2))}{(p_1(1)-p_2(1))(q_1(2)-q_2(2))-(q_1(1)-q_2(1))(p_1(2)-p_2(2))} \end{cases}$$

In both these cases if $\lambda_1, \lambda_2 \in [0,1]$, the line segments intersect and we return TRUE, else we return FALSE.

Problem 5 Let $\mathbf{p_i} = \begin{bmatrix} p_i(1) \\ p_i(2), p_i(3) \end{bmatrix}, i = 1, 2, 3,\ \mathbf{a} = \begin{bmatrix} a(1) \\ a(2) \\ a(3) \end{bmatrix},\ \mathbf{D} = \begin{bmatrix} D(1) \\ D(2) \\ D(3) \end{bmatrix}$.

In order to determine if the half-line intersects the triangle:

- We need to find the point at which th half-line intersects the supporting plane of the triangle.
- Determine if the point lies in the triangle.

First we will calculate the plane defined by the vertices of the 3-d triangle.

The normal vector of the plane: $\mathbf{n} = \dfrac{(\mathbf{p_2} - \mathbf{p_1}) \times (\mathbf{p_3} - \mathbf{p_1})}{||\mathbf{p_2} - \mathbf{p_1}||||\mathbf{p_3} - \mathbf{p_1}||}$

Therefore the plane defined by the points $\mathbf{p_i}, i = 1, 2, 3$

$n(1)x + n(2)y + n(3)z + k = 0.$

Where k is calculated by replacing one of the vertices on the equation:

$n(1)p_1(1) + n(2)p_1(2) + n(3)p_1(3) + k = 0 \iff k = -n(1)p_1(1) - n(2)p_1(2) - n(3)p_1(3)$

So we know that:

$n(1)x + n(2)y + n(3)z - n(1)p_1(1) - n(2)p_1(2) - n(3)p_1(3) = 0$

We will calculate the point at which the half-line intersects the plane.

We replace $\mathbf{a} + \lambda\mathbf{D}$ at the equation of the plane and solve for λ:

$$n(1)(a(1) + \lambda D(1)) + n(2)(a(2) + \lambda D(2)) + n(3)(a(3) + \lambda D(3)) - n(1)p_1(1) - n(2)p_1(2) - n(3)p_1(3) = 0 \iff$$
$$\lambda(n(1)D(1) + n(2)D(2) + n(3)D(3)) + n(1)(a(1) - p_1(1)) + n(2)(a(2) - p_1(2)) + n(3)(a(3) - p_1(3)) = 0 \iff$$
$$\lambda = \frac{n(1)(a(1) - p_1(1)) + n(1)(a(1) - p_1(1)n(2)(a(2) - p_1(2)) + n(3)(a(3) - p_1(3))}{n(1)D(1) + n(2)D(2) + n(3)D(3)} = \frac{\mathbf{n} \cdot (\mathbf{a} - \mathbf{p_1})}{\mathbf{n} \cdot \mathbf{D}} \tag{2}$$

If the denominator $n(1)D(1)+n(2)D(2)+n(3)D(3) = \mathbf{n} \cdot \mathbf{D} = 0$, then the line is parallel to the support plane of the triangle. In this case the line could belong in the plane and the intersection would be infinite or the line does not intersect the plane. In both cases we say there is not a (unique) intersection.

If $\lambda < 0$ then the half-line does not intersect the plane and therefore not the triangle.

If the half-line does intersect the plane then we need to determine if the intersection belongs to the inside of the triangle. This is true if the intersection is at the left of each side of the triangle.

Let $\mathbf{p}$ be the intersection point of $\mathbf{a} + \lambda\mathbf{D}$ with the support plane of the triangle defined by the vertices $\mathbf{p_1}, \mathbf{p_2}, \mathbf{p_3}$ The following expression must be true for the intersection to belong inside the triangle:

- $(\mathbf{p_1} - \mathbf{p_2}) \times (\mathbf{p_1} - \mathbf{p}) > 0$
- $(\mathbf{p_2} - \mathbf{p_3}) \times (\mathbf{p_2} - \mathbf{p}) > 0$
- $(\mathbf{p_3} - \mathbf{p_1}) \times (\mathbf{p_3} - \mathbf{p}) > 0$

In conclusion the predicate calculates the λ using the formula (2). If the lambda does not exist or negative it returns FALSE. If the λ is positive, it checks if the above condition are met. If they are, the predicate returns TRUE and the intersection, else it returns FALSE.