Experts and Adversarial Bandits

Algorithms and Analysis

[1]ECE
Technical University of Crete

March 31, 2023

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- A: Yes! This is also part of our framework so far (and is what is asked in your 1st assignment).

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  Is *this* a "stochastic" MAB?
- A: Not really...(food for thought: where does the proof break?)

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- A: This is an "adversarial" bandit $\rightarrow$ seems like a more difficult (hopeless?) environment

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Theorem

Any deterministic algorithm fails: it achieves $\mathbf{\Omega(T)}$ (i.e. linear) regret!

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- In other words *even UCB would not give sublinear regret in this setup*

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- True label $y(t)$ is revealed after $\hat{y}(t)$ is chosen, and a **loss** is incurred (e.g., -1) for every misclassification

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- Solution: Need to keep track of "error-prone" experts, without eliminating!

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Performance of Algorithm 2 (WMR) - cnt'd

Proof(cnt'd):

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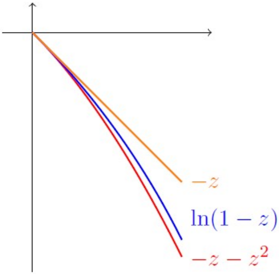

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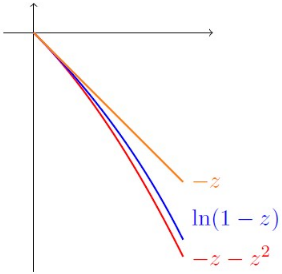

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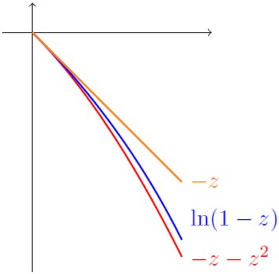

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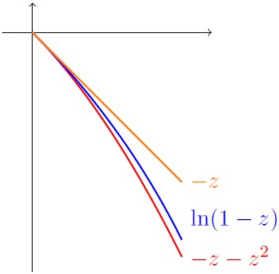

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8. Combine lower and upper bound:
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(use again) $-z - z^2 \leq ln(1-z) \leq -z$

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Performance of Multiplicative Weights

Proof (cnt'd):

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- **Note:** An alternative but **equivalent** formulation requires $w_i^{t+1} = w_i^t \cdot e^{-\eta \cdot l_i^t}$. This is known as **Hedge**.

Performance of Multiplicative Weights or Hedge

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2. Then the same algorithm with $l_i^t \in [0,\rho] \;\; (\rho > 1)$ achieves $L_{ALG} \leq \alpha \cdot \min_i \sum_{t=1}^T l_i^t + \rho \cdot \beta$

Multiplicative Weights to Bandits: Attempt 2

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- $\rho = \frac{k}{\epsilon}$ in this case $\Rightarrow$ we can apply the previous "black box" method (and proof).

Performance of MW for adversarial Bandits

Theorem

- The total loss of the Multiplicative Weights algorithm for the adversarial bandits setting is $\mathbf{L_{ALG}} \leq (\mathbf{1}+\eta) \cdot \mathbf{min_i} \sum_{\mathbf{t=1}}^{\mathbf{T}} \mathbf{l_i^t} + \frac{\mathbf{k \ln k}}{\epsilon\eta} + \epsilon\mathbf{T}$

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