Online Convex Optimization (OCO)

[1]ECE
Technical University of Crete

March 31, 2023

Introduction to Online Convex Optimization (OCO)

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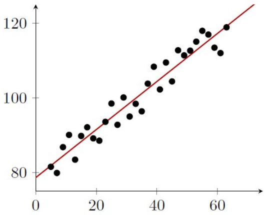
Applications: Online Regression



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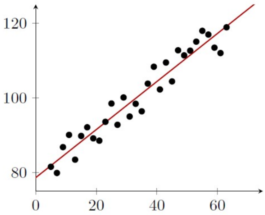

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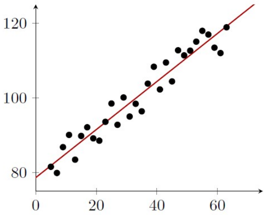

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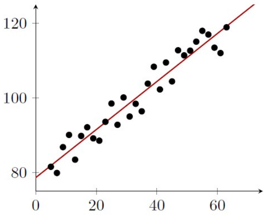

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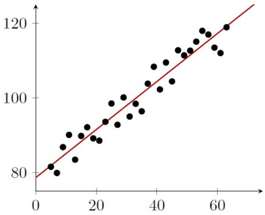

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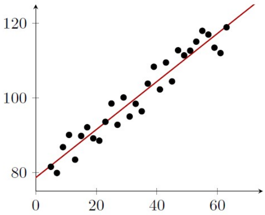

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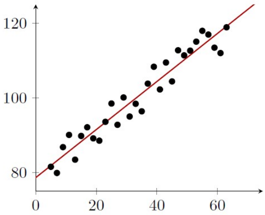

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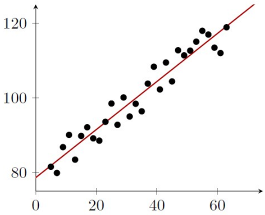

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- Observe that the "oracle/optimal" performance here is just the performance of **offline regression** (what we'd pick if we had all samples available immediately).

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- The above defines our (convex) set S (together with $w_i \in [0,1]$)

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First things first: Solving (non-online) Convex Optimization Problems

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- What do you observe?

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Theorem

If $f(x)$ convex and differentiable, and L is its maximum eigenvalue, gradient descent with $\alpha_t = 1/L$ **always converges to the global minimum** of f.

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- Most traditional ML problems (regression, SVM, etc.) are convex optimization problems
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ML Concern 1: Full Gradient is Expensive

- Assume we fit a regression model to a large dataset: $\min_w \sum_{i=1}^N f_i(x_i|w) = \sum_{i=1}^N (h(x_i|w) - y_i)^2$
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- We will see SGD and mini-batch A LOT in Reinforcement Learning

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- Forces model weights to be small (or zero - in L_1) $\Rightarrow$ improves overfitting (η is a hyperparameter)

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- Seems reasonable to try some gradient descent type of algorithm (we'll get back to this soon)

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- Turns out that this is **FTRL with Euclidean Regularizer**.

- Q: What about FTRL with Entropy Regularizer?
- A: This is a very popular/fast algorithm called **Online Mirror Descent**
- FTRL achieves order optimal $O(\sqrt{T})$ regret.