1. DETAILS OF ALGORITHM FOR FITTING GENERALIZED SUPERELLIPSES AND OTHER CLOSED CURVES.

1.1. *Generalized Newton method for obtaining minimum distance points.* A necessary condition for a point $\mathbf{p} = (x, y)$ on a given centered closed curve $\{(x, y) : F_c(x - x_0, y - y_0, \boldsymbol{\theta}_c) = 0\}$ to minimize the distance to a given data point $\mathbf{p}_i = (x_i, y_i)$ is that the line connecting $\mathbf{p}$ with $\mathbf{p}_i$ is parallel to the normal vector ∇F_c at $\mathbf{p}$, i.e.,

$$\nabla F_c \times (\mathbf{p}_i - \mathbf{p}) = \mathbf{0},$$

where $\nabla = (\partial/\partial x, \partial/\partial y)^T$. Imposing the additional condition that $\mathbf{p}$ must lie on the curve, the minimization problem amounts to solving the equations

$$\mathbf{f}(\boldsymbol{\theta}_c, \mathbf{p}_i, \mathbf{p}) \equiv \begin{pmatrix} F_c \\ \nabla F_c \times (\mathbf{p}_i - \mathbf{p}) \end{pmatrix} = \mathbf{0}$$

for $\mathbf{p}$. This is a system of two nonlinear equations in the two unknowns x and y, which we solve using a generalized Newton method. This amounts to solving the equation

$$\left.\frac{\partial \mathbf{f}}{\partial \mathbf{p}}\right|_{\mathbf{p}=\mathbf{p}^{(k)}} \Delta \mathbf{p} = -\mathbf{f}(\mathbf{p}^{(k)})$$

for $\Delta \mathbf{p}$ and then updating $\mathbf{p}^{(k)}$ via $\mathbf{p}^{(k+1)} = \mathbf{p}^{(k)} + \Delta \mathbf{p}$. It is easily shown that

$$\frac{\partial \mathbf{f}}{\partial \mathbf{p}} = \begin{pmatrix} 0 & 0 \\ y_i - y & -(x_i - x) \end{pmatrix} \mathbf{H} + \begin{pmatrix} \frac{\partial F_c}{\partial x} & \frac{\partial F_c}{\partial y} \\ \frac{\partial F_c}{\partial y} & -\frac{\partial F_c}{\partial x} \end{pmatrix}$$

where $\mathbf{H} = \frac{\partial}{\partial \mathbf{p}} \nabla F_c$. This is repeated for every point $\mathbf{p}_i$, $i = 1, \ldots, I$.

1.2. *Gauss-Newton method for updating parameter estimates.* Given the data $\{\mathbf{p}_i : i = 1, \ldots, I\}$ and a set of minimum distance points $\{\mathbf{p}'_i : i = 1, \ldots, I\}$, the first-order necessary condition for a minimizer of $\Phi(\boldsymbol{\theta})$ is

$$\left(\frac{\partial \Phi}{\partial \boldsymbol{\theta}}\right)^T = 2\mathbf{M}^T \mathbf{d} = \mathbf{0}$$

where $\mathbf{d} = (d_i) = (\|\mathbf{p}_i - \mathbf{p}'_i\|)$,

$$\mathbf{M} = \frac{\partial \mathbf{d}}{\partial \boldsymbol{\theta}} = \begin{pmatrix} \mathbf{M}_1 \\ \vdots \\ \mathbf{M}_n \end{pmatrix}$$

and $\mathbf{M}_i = \partial d_i / \partial \boldsymbol{\theta}$. The Gauss-Newton method applied here yields the equation

$$\left.\mathbf{M}^T \mathbf{M}\right|_{\boldsymbol{\theta}=\boldsymbol{\theta}^{(k)}} \Delta \boldsymbol{\theta} = -\left.\mathbf{M}^T \mathbf{d}\right|_{\boldsymbol{\theta}=\boldsymbol{\theta}^{(k)}},$$

which is solved for $\Delta \boldsymbol{\theta}$, whereupon $\hat{\boldsymbol{\theta}}^{(k)}$ is updated via $\hat{\boldsymbol{\theta}}^{(k+1)} = \hat{\boldsymbol{\theta}}^{(k)} + \Delta \boldsymbol{\theta}$. It is easily shown that

$$\begin{aligned} \left.\mathbf{M}_i\right|_{\boldsymbol{\theta}=\boldsymbol{\theta}^{(k)}} \quad &= \quad \left(\frac{\operatorname{sign}((\mathbf{p}_i - \mathbf{p}'_i)^T \nabla F_c)}{\|\nabla F_c\|} \left.\frac{\partial F_c}{\partial \boldsymbol{\theta}_c}\right|_{\mathbf{p}=\mathbf{p}'_i, \boldsymbol{\theta}=\boldsymbol{\theta}^{(k)}} \quad \mathbf{0}_{1\times 2} \right) \\ &\qquad - \frac{(\mathbf{p}_i - \mathbf{p}'_i)^T}{\|\mathbf{p}_i - \mathbf{p}'_i\|} (\mathbf{0}_{2\times(p-2)} \quad \mathbf{I}_{2\times 2}). \end{aligned}$$

1.3. *Initial values and stopping criteria.* Data points $\{\mathbf{p}_i : i = 1, \ldots, I\}$ were used as initial values for the generalized Newton method of obtaining minimum distance points. As initial values for the estimation of superellipse parameters, we used $x_0 = \frac{1}{I}\sum_{i=1}^{I} x_i$, $y_0 = \frac{1}{I}\sum_{i=1}^{I} y_i$, $a = b = \frac{1}{I}\sum_{i=1}^{I}\sqrt{(x_i - x_0)^2 + (y_i - y_0)^2}$, and $r = 1$. Final estimates of superellipse parameters were used as initial values for the estimation of an affine-transformed superellipse and a latitudinally asymmetric superellipse, and final estimates of the affine-transformed superellipse were used as initial values for the estimation of an affine-transformed latitudinally asymmetric superellipse.

For the generalized Newton method, we stopped updating $\mathbf{p}$ when the L_2-norm of $\Delta\mathbf{p}$ was less than 0.001 (in feet). For the Gauss-Newton method, we stopped updating $\boldsymbol{\theta}$ when the L_2-norm of $\mathbf{M}^T\mathbf{d}$ was less than 0.001.

1.4. *Expressions for ∇F_c, $\mathbf{H}$, and $\partial F_c/\partial\boldsymbol{\theta}_c$.* Expressions for ∇F_c, $\mathbf{H}$, and $\partial F_c/\partial\boldsymbol{\theta}_c$ for a superellipse, ATS, LAS, and ATLAS are as follows.

- Superellipse:

$$\begin{aligned}
\nabla F_c &= r\begin{pmatrix} J/x \\ K/y \end{pmatrix}, \\
\mathbf{H} &= r(2r-1)\begin{pmatrix} J/x^2 & 0 \\ 0 & K/y^2 \end{pmatrix}, \\
\frac{\partial F_c}{\partial\boldsymbol{\theta}_c} &= \begin{pmatrix} -rJ/a & -rK/b & [J\log J + K\log K]/(2r) \end{pmatrix}^T,
\end{aligned}$$

where $J = (x/a)^{2r}$ and $K = (y/b)^{2r}$.

- Affine-transformed superellipse (ATS):

$$\begin{aligned}
\nabla F_c &= r\begin{pmatrix} J/(x+sy) \\ sJ/(x+sy) + K/y \end{pmatrix}, \\
\mathbf{H} &= r(2r-1)\begin{pmatrix} J/(x+sy)^2 & sJ/(x+sy)^2 \\ sJ/(x+sy)^2 & [s^2J/(x+sy)^2] + K/y^2 \end{pmatrix}, \\
\frac{\partial F_c}{\partial\boldsymbol{\theta}_c} &= \begin{pmatrix} -rJ/a & -rK/b & [J\log J + K\log K]/(2r) & sJy/(s+sy) \end{pmatrix}^T,
\end{aligned}$$

where $J = [(x+sy)/a]^{2r}$ and $K = (y/b)^{2r}$.

- Latitudinally asymmetric superellipse (LAS):

$$\nabla F_c = \begin{cases} r_1\begin{pmatrix} J_1/x & K_1/y \end{pmatrix}^T & y \geq 0 \\ r_2\begin{pmatrix} J_2/x & K_2/y \end{pmatrix}^T & y < 0, \end{cases}$$

$$\mathbf{H} = \begin{cases} r_1(2r_1-1)\begin{pmatrix} J_1/x^2 & 0 \\ 0 & K_1/y^2 \end{pmatrix} & y \geq 0 \\ r_2(2r_2-1)\begin{pmatrix} J_2/x^2 & 0 \\ 0 & K_2/y^2 \end{pmatrix} & y < 0, \end{cases}$$

$$\frac{\partial F_c}{\partial\boldsymbol{\theta}_c} = \begin{pmatrix} (-r_1J_1/a)I(y \geq 0) + (-r_2J_2/a)I(y < 0) \\ (-r_1K_1/b)I(y \geq 0) + (-r_2K_2/b)I(y < 0) \\ [J_1\log J_1 + K_1\log K_1]/(2r_1)I(y \geq 0) \\ [J_2\log J_2 + K_2\log K_2]/(2r_2)I(y < 0) \end{pmatrix},$$

where $J_1 = (x/a)^{2r_1}$, $K_1 = (y/b)^{2r_1}$, $J_2 = (x/a)^{2r_2}$, $K_2 = (y/b)^{2r_2}$.

- Affine-transformed latitudinally asymmetric superellipse (ATLAS):

$$\nabla F_c = \begin{cases} r_1 \begin{pmatrix} J_1/(x+sy) & sJ_1/(x+sy) + K_1/y \end{pmatrix}^T & y \geq 0 \\ r_2 \begin{pmatrix} J_2/(x+sy) & sJ_2/(x+sy) + K_2/y \end{pmatrix}^T & y < 0, \end{cases}$$

$$\mathbf{H} = \begin{cases} r_1(2r_1 - 1) \begin{pmatrix} J_1/(x+sy)^2 & sJ_1/(x+sy)^2 \\ sJ_1/(x+sy)^2 & s^2 J_1/(x+sy)^2 + K_1/y^2 \end{pmatrix} & y \geq 0 \\ r_2(2r_2 - 1) \begin{pmatrix} J_2/(x+sy)^2 & sJ_2/(x+sy)^2 \\ sJ_2/(x+sy)^2 & s^2 J_2/(x+sy)^2 + K_2/y^2 \end{pmatrix} & y < 0, \end{cases}$$

$$\frac{\partial F_c}{\partial \boldsymbol{\theta}_c} = \begin{pmatrix} (-r_1 J_1/a) I(y \geq 0) + (-r_2 J_2/a) I(y < 0) \\ (-r_1 K_1/b) I(y \geq 0) + (-r_2 K_2/b) I(y < 0) \\ [J_1 \log J_1 + K_1 \log K_1]/(2r_1) I(y \geq 0) \\ [J_2 \log J_2 + K_2 \log K_2]/(2r_2) I(y < 0) \\ r_1 J_1 y/(s+sy) I(y \geq 0) + r_2 J_2 y/(s+sy) I(y < 0) \end{pmatrix},$$

where $J_1 = [(x+sy)/a]^{2r_1}$, $K_1 = (y/b)^{2r_1}$, $J_2 = [(x+sy)/a]^{2r_2}$, $K_2 = (y/b)^{2r_2}$.