Solutions First Partial for MATE-1214 Cálculo Integral con Ecuaciones Diferenciales

28/08/2018

1. Compute the following integral

$$\int x \tan^{-1} x \, dx.$$

Solution: We integrate by parts. Call:

$$u = \tan^{-1} x \qquad dv = x\,dx$$
$$du = \frac{1}{1+x^2} \qquad v = \frac{x^2}{2}.$$

Then

$$\begin{aligned}
\int x \tan^{-1} x \, dx &= \frac{x^2}{2} \tan^{-1} x - \int \frac{1}{2} \frac{x^2}{x^2+1} \, dx \\
&= \frac{x^2}{2} \tan^{-1} x - \frac{1}{2} \int \frac{x^2+1-1}{x^2+1} \, dx \\
&= \frac{x^2}{2} \tan^{-1} x - \frac{1}{2} \cdot \left(\int dx - \int \frac{1}{x^2+1} \, dx \right) \\
&= \frac{x^2}{2} \tan^{-1} x - \frac{x}{2} + \frac{1}{2} \cdot \tan^{-1} x + c \\
&= \frac{1}{2} \big((x^2+1) \tan^{-1} x - x) \big) + c
\end{aligned}$$

2. Decompose into partial fractions the following integral:

$$\int \frac{1}{x^4 - 3x^2 - 4} \, dx.$$

Solution: Since $P(x) = 1$, $\deg P(x) = 0$, which is smaller that $4 = \deg Q(x) = x^4 - 3x^2 - 4$, there is no need to divide the numerator by the denominator. So, we factor $Q(x)$.

$$x^4 - 3x^2 - 4 = (x^2 - 4)(x^2 + 1) = (x + 2)(x - 2)(x^2 + 1).$$

Hence

$$\frac{1}{x^4 - 3x^2 - 4} = \frac{A}{x+2} + \frac{B}{x-2} + \frac{Cx + D}{x^2+1}.$$

Summing the right hand side, we get:

$$\frac{A(x-2)(x^2+1)+B(x+2)(x^2+1)+(Cx+D)(x^2-4)}{x^4-3x^2-4}$$
$$=\frac{A(x^3-2x^2+x-2)+B(x^3+2x^2+x+2)+Cx^3-4Cx+Dx^2-4D}{x^4-3x^2-4}$$
$$=\frac{x^3(A+B+C)+x^2(2B-2A+D)+x(A+B-4C)-2A+2B-4D}{x^4-3x^2-4}.$$

Imposing that the previous numerator coincides with $P(x)=1$, we get

$$\begin{aligned} A+B+C&=0\\ 2B-2A+D&=0\\ A+B-4C&=0\\ 2B-2A-4D&=1. \end{aligned}$$

(The solution is $A=-1/10$, $B=1/10$, $C=0$, $D=-1/5$.)

3. Let $f\colon[0,\infty)\to\mathbb{R}$ is a continuous function such that $f(x)\geq 0$ for all $x\in[0,\infty)$. Show that if

$$\int_0^\infty f(x)\,dx$$

is convergent, then the same holds for

$$\int_0^\infty e^{-x}f(x)\,dx.$$

Solution: The function e^{-x} is continuous and $0\leq e^{-x}\leq 1$ for $x\geq 0$. This implies that $0\leq e^{-x}f(x)\leq f(x)$ and

$$\int_0^\infty e^{-x}f(x)\,dx\leq\int_0^\infty f(x)\,dx.$$

By the comparison criterion for indefinite integrals we did in class, since the rightmost integral converges, so does the middle one.

4. The figure eight curve called *Gerono lemniscate* is given by the equation $x^4=9(x^2-y^2)$.

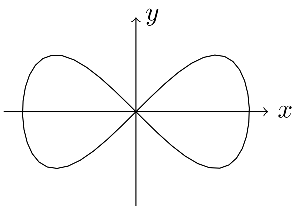

Figure 1: Gerono lemniscate.

(a) Write down the integral to compute the area bounded by it;

(b) Evaluate that integral.

Solution: (a): We start by computing the points where the curve hits the axis $y = 0$.

Since these points belong to the curve, they satisfy the equation. In other words we have $x^4 = 9x^2$. A solution of this equation is $x = 0$, as you could see from the picture.

Now we assume $x \neq 0$ and divide by x to find the other solutions. We get $x^2 = 9$, which implies that $x = \pm 3$.

By symmetry it is enough to compute the area bounded by the portion of the curve in the first quadrant and the positive x axis. This corresponds to $1/4$ of the total area by symmetry, as in the case of the ellipse.

Going back to the equation for the curve, we rewrite it as

$$9y^2 = 9x^2 - x^4 \implies y = \pm\sqrt{x^2 - \frac{x^4}{9}}.$$

Call A the total area we have to compute. Using that the part in the first quadrant has positive y coordinate, we have:

$$A = 4\,\text{area}\left\{(x, y) \in \mathbb{R}^2 : 0 \leq x \leq 3, 0 \leq y \leq \sqrt{x^2 - \frac{x^4}{9}}\right\}.$$

That is

$$A = 4\int_0^3 \sqrt{x^2 - \frac{x^4}{9}}\,dx = \frac{4}{3}\int_0^3 \sqrt{9x^2 - x^4}\,dx = \frac{4}{3}\int_0^3 x\sqrt{9 - x^2}\,dx.$$

(b): We now compute this integral. There are two possible ways to do this.

The first one is the following. Call $u = 9 - x^2$. Then $du = -2x^2$ and when $x = 0$ we have $u = 9$ and $x = 3$ implies $u = 0$. Hence

$$A = -\frac{4}{3}\int_9^0 \sqrt{u}\,\frac{du}{2} = \frac{2}{3}\int_0^9 \sqrt{u}\,du = \frac{2}{3}\cdot\left[\frac{2}{3}u^{\frac{3}{2}}\right]_{u=0}^{u=9} = \frac{4}{9}\cdot 27 = 12.$$

Another way to go was to do the trigonometric substitution $x = 3\sin\theta$. In this case, when $x = 0$ we have that $\sin\theta = 0$ and hence $\theta = 0$, and when $x = 3$, $\sin\theta = 1$ and so $\theta = \pi/2$. Moreover $dx = 3\cos\theta\,d\theta$. Hence

$$A = \frac{4}{3}\int_0^{\frac{\pi}{2}} 3\sin\theta\sqrt{9 - 9\sin^2\theta}\cdot 3\cos\theta\,d\theta = 36\int_0^{\frac{\pi}{2}} \cos^2\theta\sin\theta\,d\theta.$$

Now we do another substitution, calling $t = \cos\theta$. When $\theta = 0$, $t = 1$, and when $\theta = \pi/2$ we have $t = 0$. Finally $dt = -\sin\theta$. So

$$A = 36\int_1^0 t^2\cdot(-dt) = 36\int_0^1 t^2\,dt = 36\left[\frac{1}{3}t^3\right]_{t=0}^{t=1} = 12.$$

5. Compute the length of the curve $C : y = 1 - e^{-x}$ between the points $(0, 0)$ and $(1, 1 - e^{-1})$.

[*Hint: Use substitution.*]

The formula for the length of a curve between $(0,0)$ and a point $(1, f(1))$ is

$$\int_0^1 \sqrt{1+(f'(x))^2}\,dx.$$

In this case, $f(x) = 1-e^{-x}$, so the derative is $f'(x) = e^{-x}$ and the integral above becomes

$$\int_0^1 \sqrt{1+e^{-2x}}\,dx.$$

We begin with the subsitution $u = \sqrt{1+e^{-2x}}$. Then we have

$$du = -\frac{e^{-2x}}{\sqrt{1+e^{-2x}}}\,dx.$$

From the substitution we get that $u^2 = 1+e^{-2x}$, or $-e^{-2x} = 1-u^2$. So

$$du = -\frac{e^{-2x}}{\sqrt{1+e^{-2x}}}\,dx = \frac{1-u^2}{u}\,dx$$

When $x = 0$, we have $u = \sqrt{2}$, and when $x = 1$ we get $u = \sqrt{1+e^{-2}}$. So the integral becomes

$$\begin{aligned}
\int_0^1 \sqrt{1+e^{-2x}}\,dx &= \int_{\sqrt{2}}^{\sqrt{1+e^{-2}}} \frac{u^2}{1-u^2}\,du \\
&= \int_{\sqrt{1+e^{-2}}}^{\sqrt{2}} \frac{-u^2}{1-u^2}\,du \\
&= \int_{\sqrt{1+e^{-2}}}^{\sqrt{2}} 1 - \frac{1}{1-u^2}\,du \\
&= \int_{\sqrt{1+e^{-2}}}^{\sqrt{2}} 1 - \frac{1}{2}\left(\frac{1}{1+u} + \frac{1}{1-u}\right)\,du \\
&= \left[u - \frac{1}{2}\left(\ln|1+u| - \ln|1-u|\right)\right]_{u=\sqrt{1+e^{-2}}}^{u=\sqrt{2}} \\
&= \sqrt{2} + \frac{1}{2}\ln\frac{\sqrt{2}-1}{\sqrt{2}+1} - \sqrt{1+e^{-2}} + \frac{1}{2}\ln\frac{\sqrt{1+e^{-2}}+1}{\sqrt{1+e^{-2}}-1},
\end{aligned}$$

where we used the properties of logarithms and that $|1-\sqrt{2}| = \sqrt{2}-1$ and $|1-\sqrt{1+e^{-2}}| = \sqrt{1+e^{-2}}-1$.

One could also start with the trigonometric substitution $e^{-2x} = \tan\theta$*, but the integral becomes much more complicated and the computation of the extremes of integration becomes really tricky.*