Third Partial for MATE-1214 Cálculo Integral con Ecuaciones Diferenciales

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Name:
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1. Evaluate
$$\lim_{n\to\infty} \frac{\cos(n)}{n}.$$

   Solution: For all n we have $-1 \leq \cos(n) \leq 1$. Hence
$$-\frac{1}{n} \leq \frac{\cos(n)}{n} \leq \frac{1}{n}.$$
   Since both $-1/n$ and $1/n$ tend to 0 as n tends to infinity, by the sandwich theorem also $\frac{\cos(n)}{n}$ tends to 0.

2. Determine whether the following series converges or diverges. If it converges, evaluate the sum.
$$\sum_{n=1}^{\infty} \frac{2}{n(n+1)}.$$

   Solution: One has
$$c_n = \frac{2}{n(n+1)} = 2\left(\frac{1}{n} - \frac{1}{n+1}\right) = 2(a_n - a_{n+1}),$$
   so this is a telescopic series. Since
$$a_{n+1} = \frac{1}{n+1} \to 0,$$
   as $n \to \infty$, the series converges. The sum is equal to $2a_1 = 2$.
   [The series
$$\sum_{n=1}^{\infty} \frac{2}{n^2(n^2+1)},$$
   is a series with positive terms. Since $n^2+1 > n^2$ we have that
$$\sum_{n=1}^{\infty} \frac{2}{n^2(n^2+1)} \leq \sum_{n=1}^{\infty} \frac{2}{n^2 \cdot n^2}.$$

The right hand side is twice the p-series with $p = 4$. Since $4 > 1$ the right hand side converges. By comparison, also the left hand side does it. However, the sum of the original series is hard to compute.]

3. Determine whether the following series converges or diverges. If it converges, evaluate the sum.

$$\sum_{n=0}^{\infty} \frac{n^2 + 2n + 2}{\sqrt{n^6 + 3n^2 + 1}}.$$

Solution: This is a series with positive terms. We compare it with the diverging series

$$\sum_{n=1}^{\infty} \frac{1}{n}.$$

We have

$$\lim_{n\to\infty} \frac{\frac{n^2+2n+2}{\sqrt{n^6+3n^2+1}}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n^3 + 2n^2 + 2n}{\sqrt{n^6 + 3n^2 + 1}} = \lim_{n\to\infty} \frac{n^3(1 + \frac{2}{n} + \frac{2}{n^2})}{n^3\sqrt{1 + \frac{3}{n^4} + \frac{1}{n^6}}} = 1.$$

Since the limit is neither 0 nor ∞, the two series have the same behavior. As the harmonic series diverges, so does the other one.

4. Find the interval of convergence of the following power series.

$$\sum_{n=2}^{\infty} \frac{(-1)^n (x-1)^n}{\ln n \cdot 2^n}.$$

Solution: The ratio test give us the radius of convergence R. We have

$$\lim_{n\to\infty} \left| \frac{\frac{(-1)^{n+1}(x-1)^{n+1}}{\ln (n+1) \cdot 2^{n+1}}}{\frac{(-1)^n (x-1)^n}{\ln n \cdot 2^n}} \right| = \lim_{n\to\infty} |x-1| \cdot \frac{1}{2} \cdot \frac{\ln n}{\ln (n+1)}.$$

The limit can be evaluated using L'Hopital rule

$$\lim_{n\to\infty} \frac{\ln n}{\ln (n+1)} = \lim_{n\to\infty} \frac{\frac{1}{n}}{\frac{1}{(n+1)}} = \lim_{n\to\infty} \frac{n+1}{n} = \lim_{n\to\infty} \frac{n(1 + \frac{1}{n})}{n} = 1.$$

Hence, we want

$$\lim_{n\to\infty} |x-1| \cdot \frac{1}{2} \cdot \frac{\ln (n+1)}{\ln n} = \frac{1}{2}|x-1|$$

to be smaller than 1. This means that $R = 2$. So, the series converges for $|x-1| < 2$.

We now look what happens on the extremes of the interval. When $x - 1 = 2$ the original series becomes

$$\sum_{n=2}^{\infty} \frac{(-1)^n 2^n}{\ln n \cdot 2^n} = \sum_{n=2}^{\infty} \frac{(-1)^n}{\ln n}.$$

This is an alternating series. Since $\ln n$ is increasing, $\frac{1}{\ln n}$ is decreasing. Moreover, as $\ln n \to \infty$ when $n \to \infty$, we have $\frac{1}{\ln n} \to 0$. Hence, by the alternating series theorem, the series converges.

Finally, when $x - 1 = -2$ the original series becomes

$$\sum_{n=2}^{\infty} \frac{(-1)^n(-2)^n}{\ln n \cdot 2^n} = \sum_{n=2}^{\infty} \frac{(-1)^n(-1)^n(2)^n}{\ln n \cdot 2^n} = \sum_{n=2}^{\infty} \frac{1}{\ln n}.$$

This is a series with positive terms. Since $\ln n \leq n$ for all $n \geq 2$ we have that

$$\sum_{n=2}^{\infty} \frac{1}{n} \leq \sum_{n=2}^{\infty} \frac{1}{\ln n}.$$

As the left hand side diverges, by the comparison criterion also the right hand side does.

Summing up, the interval of convergence of the series is $(-1, 3]$.

5. Expand in power series around $x = 1$ the function

$$\frac{1}{x^2}.$$

Solution: Remark that

$$\int \frac{1}{x^2}\, dx = -\frac{1}{x} + c.$$

We can take $c = 0$, as later we are going to derive it again. Let's expand the right hand side into a power series around $x = 1$ using the geometric series. We have

$$\begin{aligned} -\frac{1}{x} &= -\frac{1}{1 + x - 1} = -\frac{1}{1 - (-(x-1))} = -\sum_{n=0}^{\infty} (-1)^n (x-1)^n \\ &= \sum_{n=0}^{\infty} (-1)^{n+1} (x-1)^n. \end{aligned}$$

Since this is a power series, to derive the series we simply derive every term. We obtain:

$$\begin{aligned} \frac{1}{x^2} &= \frac{d}{dx}\left(-\frac{1}{x}\right) = \sum_{n=0}^{\infty} \frac{d}{dx}((-1)^{n+1}(x-1)^n) \\ &= \sum_{n=1}^{\infty} (-1)^{n+1} n (x-1)^{n-1} = \sum_{n=1}^{\infty} (-1)^{n-1} n (x-1)^{n-1}. \end{aligned}$$