Chapter 1

AN ENTROPY-BASED MEASURE OF MASS POLITICAL POLARIZATION

The past decade witnessed growing scholarly interest in mass polarization and concerns that divided electorates are pulling political systems apart. While early debates were centered more on *whether* electorates have become more polarized (Abramowitz and Saunders, 2008; Abramowitz, 2010; Baldassarri and Gelman, 2008; Fiorina, Abrams and Pope, 2010; Hill and Tausanovitch, 2015; Lelkes, 2016), other works have shifted to recognizing the affective (between party animosity), identity, and cognitive *roots* of polarization (e.g. Abramowitz and Webster, 2016; DellaPosta, 2020; Hetherington, Long and Rudolph, 2016; Iyengar, Sood and Lelkes, 2012; Kalmoe and Mason, 2022; Levendusky, 2013; Mason, 2015; Wagner, 2021). Yet the question still remains about how value and issue polarization in the mass public connect with the growing identity-based antagonism. With a few exceptions (Bischof and Wagner, 2019; Bosancianu, 2017; Lauka, McCoy and Firat, 2018; Lupu, 2015; Pontusson and Rueda, 2008), research on opinion and ideology polarization tends to focus on single-country contexts (mostly the U.S.) with pre-defined context and groups. An additional need is to understand the variation and sources of mass political polarization, from a single case and comparative perspective, with an empirical measure that works effectively. Limitations in

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our knowledge of mass political polarization in these contexts stems mainly from a lack of appropriate measurement.

This chapter serves as the starting point of the whole dissertation as it tackles the issue of how we can measure polarization at the first place. In the chapter, we propose a new *nonparametric*, *entropy*-based measure of mass political polarization. It follows the fundamental conceptualization of polarization in the literature but relaxes the unnecessary distinction between dispersion (variability of the measured categories) and modality. The proposed measure exploits the specific structure of ordinal variables in public opinion survey data, makes no prior assumptions about the distribution and spacing of the data, and is able to capture both the ordering and the distribution of the data simultaneously. We demonstrate that the proposed measure is theoretically and conceptually more relevant to the standard intuition of polarization, and thus is always able draw reliable and clear measures of aggregated polarization in a spatial context.

Many of the existing statistical measures of mass polarization (means, variances, kurtosis) are not suitable for ordinal variables common in public opinion survey data because they assume interval measure or equal spacing between categories. It is well known in political science, and survey research in general, that the far left and right choices are typically more spaced out on the latent dimension than the middle categories creating unequal spacing. In this setting the use of the mean, and statistics that incorporate the mean, are therefore mathematically incorrect, even though this practice is common (Gill, 2005; Homola, Jackson and Gill, 2016). In addition, nearly all published measures of polarization are only intended to capture single aspects of the phenomenon, or they make strong assumptions about the specific forms of the distribution assigned. Some existing measures focus only on distinguishing between general tail properties versus bimodality, which is a very rough distinction that may miss more subtle forms of polarization. In addition, influential early work found that both the variance and kurtosis can be biased for different choices of distributions (Downey and Huffman, 2001)

The entropy-based measure of polarization developed here emphasizes both the *ordering* and the *distribution* of ordinal data. This measure has two main rationales. First, the starting point for the measurement is a binary entropy term that measures to what degree two categories are evenly distributed. This is also substantively meaningful as it reflects the comparison between two sides in the context of measurement. Second, a cumulative approach is used to reflect the overall distribution of ordinal variables. Another challenge in developing such a measure is how to verify its validity as there is no such thing as *absolute* polarization in real political science data. We utilize hypothetical distributions, simulated data, and crowd-sourcing validation to demonstrate the properties and advantages of our proposed measure. Then, we apply this entropy-based measure of polarization to questions about mass polarization in the U.S., the relationship between radical party and polarization in Europe, and cross-country trends in affective and ideological polarization.

The chapter contributes to the research of polarization by providing the first measure that is designed specifically to reflect the dynamics of preference and opinion polarization. The measure will be of interest to scholars of comparative political behavior that want to investigate the variations of polarization. It can also be used for comparing polarization for a single country across time. The measure does not limit to ideological polarization. Rather, it can be used for a variety of survey objects, including issue opinions, policy preferences, social values, political cultures, and so on. Building on this chapter, the rest of the dissertation discuss the related inferential issue of aggregate political behavior and employs the proposed measure to the context of how economic inequality affects mass polarization across countries.

1.1 Existing Measurements of Polarization

The concept of polarization can be both simple and complex. The basic definition of polarization is not very controversial as it emphasizes simply to what extent preferences or opinions are opposed (DiMaggio, Evans and Bryson, 1996; Fiorina and Abrams, 2008). Polarization is often identified by statements along the lines of "I know it when I see it" whereby most people can distinguish what is more polarized when they are presented with two alternative distributions with numeric summaries or graphics (Fiorina and Abrams, 2008). To conceptualize polarization in a more principled way, many of the theoretical developments focus on different aspects of the distribution, such as what sub-components constitute a contribution to diffusion towards poles. Scholars generally distinguish between dispersion and bimodality, and additionally, other principles of polarization such as divergence, spread, regionalization, fragmentation, distinctness, and so on (Bramson et al., 2016; Lelkes, 2016; Bramson et al., 2017)[1]. Many of these principles significantly overlap with each other, both conceptually and

[1] Admittedly, some of the principles have particular theoretical significance and thus became a major part of the polarization research, such as constraint (i.e. issue alignment) and consolidation (issue partisanship) (DiMaggio, Evans and Bryson, 1996; Abramowitz and Saunders, 2008; Baldassarri and Gelman, 2008; Munzert and Bauer, 2013; Hetherington, Long and Rudolph, 2016; Lelkes, 2016; DellaPosta, 2020). These principles capture the fact that the degree to which people consistently align themselves with an ideological or partisan camp contributes to polarization. This is essentially a different concept than what this chapter focuses on as the former emphasizes the dimensionality of preferences and opinions rather than

empirically, and yet each single concept does not alone necessarily signify real polarization[2].

For instance, as a distribution becomes increasingly bimodal it will naturally in some way go through the process of dispersion as few values land in between the poles and existing measures often show confused pictures of polarization motivating some scholars to use multiple limited views in an effort to compensate (DiMaggio, Evans and Bryson, 1996; Bramson et al., 2016). These distinctions between different principles of polarization are likely due to the lack of a direct metric that provides a full coherent summary measure.

To actually measure polarization, studies have relied mainly on existing metrics, using the alternative rubrics of *dispersion* and *bimodality*[3]. The former refers to the breadth of preferences: to what extent preferences are diverse and "far apart"[4]. The latter captures the fact that, when being polarized, people with different positions cluster into separate camps (DiMaggio, Evans and Bryson, 1996). Scholars often think of polarization as the increasing degree of bimodality versus unimodality–polarization increases as two camps are increasingly isolated from each other (Fiorina, Abrams and Pope, 2010; Lelkes, 2016).

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the magnitude or strength of polarization.

[2]For example, Fiorina and Abrams (2008) shows a bimodal distribution can be a necessary condition for polarization but not a sufficient one.

[3]DiMaggio, Evans and Bryson also conceptualize two other aspects of polarization: constraints and consolidation, but these are tangential to the main definition.

[4]See also Bramson et al. (2017) for a more comprehensive review on the conceptualization issues for polarization.

1.1.1 Variance, Kurtosis, Ordinal Dispersion Measuring Polarization

Dispersion is most commonly measured using the *variance*. DiMaggio, Evans and Bryson (1996) originally used the variance to represent the extent to which respondents are likely to differ in their opinions, which then become the most popular measure of mass polarization in subsequent studies (e.g. Bischof and Wagner, 2019; Bosancianu, 2017; Evans, 2003; Hill and Tausanovitch, 2015; Levendusky and Pope, 2010; Mouw and Sobel, 2001). The variance (and of course the mean that its calculation contains) assumes the presence of continuous data or equal spacing of discrete values, and is therefore not mathematically correct as a way to measure opinions and preferences from typical Likert scale questions ubiquitous in survey data (Blair and Lacy, 2000; Downey and Huffman, 2001; Homola, Jackson and Gill, 2016). Perhaps worse, the variance does not directly capture polarization in a way that political scientists want—as in the literature, scholars usually want to distinguish between polarization and sorting (Levendusky, 2009). For instance a uniform distribution of responses, across say five or seven ordered response choices, will lead to a high variance but such a pattern does not comport with the idea of polarization that most scholars of public opinion have.

For measuring bimodality, kurtosis and related statistics are choices that measure the "tailedness" (not peakedness) of a distribution by comparing the tail density of a given frequency-distribution curve to any normal probability density function based on the scaled fourth moment: $\kappa = \mu_4/\sigma^4$, where μ_4 is the fourth central moment[5] and σ^2 is the variance

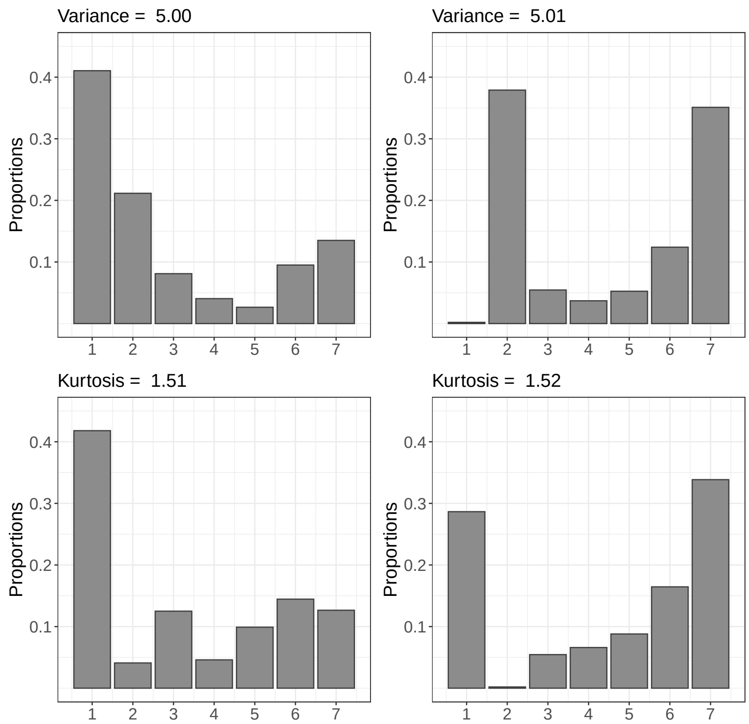
[5] The pth central moment of a given distribution is given by $\mu_p = \frac{1}{n}\sum_{i=1}^{n}\left[\frac{X_i - \bar{X}}{\sigma}\right]^p$.

of the observed data (Chissom, 1970; Pearson, 1905). Since the kurtosis of any normal distribution is 3, higher values than this indicate heavier tails on either side of the mean and the distribution is then called *leptokurtic*. Notice that no part of this technical definition addresses actual bimodality. To address this problem scholars as early as 1945 (Mosier, Myers and Price, 1945) proposed the *bimodality coefficient* to measure bimodality, which combines the kurtosis and skewness of a distribution:

$$B_c = \frac{g^2 + 1}{\kappa^* + \frac{3(n-1)^2}{(n-2)(n-3)}}, \tag{1.1}$$

where $g = \mu^3/\sigma^3$ is the skewness (the scaled third moment), κ^* is the excess kurtosis subtracting 3 from κ, and n is the size of the data. A uniform distribution gives a B_c of approximately 0.555, where lesser values are evidence of unimodality, and greater values are evidence of bimodality. The statistic was eventually adopted by political scientists to assess the degree of bimodality for mass political polarization (Abramowitz, 2010; Lelkes, 2016). The logic behind the bimodality coefficient is that a bimodal distribution will have high skewness, low kurtosis, or both. Kurtosis and the bimodality coefficient indeed can capture some ideal characteristics of the distribution regarding polarization, but *they are still based on continuous data measurement and are tied to the normal distribution as a reference point.* There have been other efforts to develop measures for ordinal dispersion-concentration similar to kurtosis (Blair and Lacy, 2000; Leik, 1966). The key objective here is summarizing the distributional information of ordinal data by its *cumulative relative frequency.* This is a compromise between imposing an assumption about the nature of the continuum underlying the categories and totally neglecting the ordering of the categories, which is the task that we take on here.

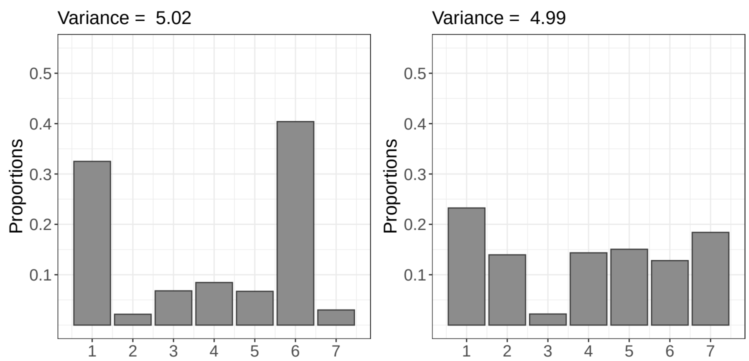
Figure 1.1: Different Patterns of Distribution but Similar Variance and Kurtosis



1.1.2 Illustrating Problems with the Standard Approaches

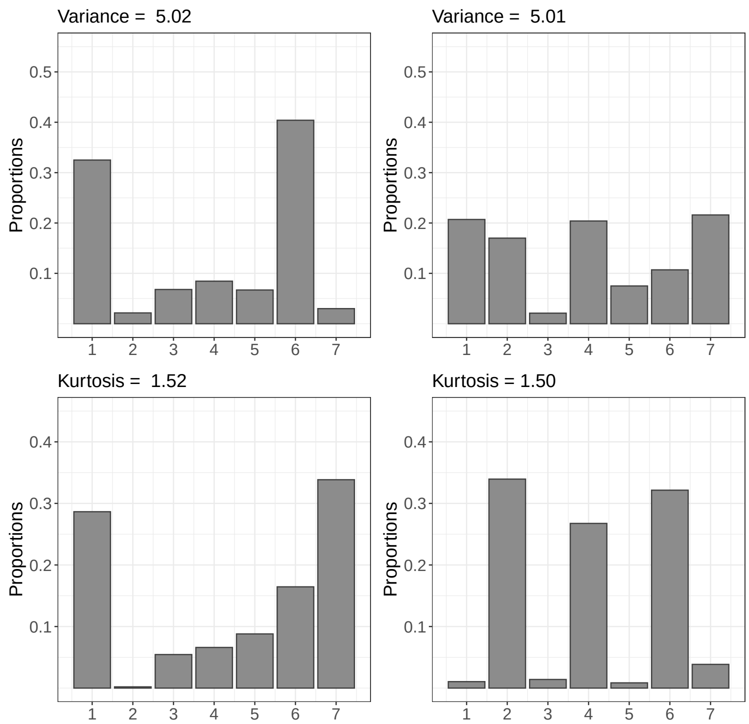
As noted, the calculation of the variance and kurtosis are based on the mean of the data, which not only violates assumptions about measurement by summing incomparable units, but it also brings along sensitivity to large outlying values (Abascal and Rada, 2014; Blair and Lacy, 2000; Downey and Huffman, 2001). Figure 1.1 shows that very different patterns of distributions can essentially result in the same variance and kurtosis values. Specifically, in this particular example, because of their reliance on mean (which does not make a lot of sense here in the ordinal context), variance and kurtosis cannot distinguish the very skewed distributions (left) and bimodal distributions (right), i.e. polarization. Note that the equal spacing between categories in this and later figures are done merely by convenience and tradition; we are *not* implying or assuming equal spacing on the true latent scale.

Figure 1.2: Polarization vs. Equal Dispersion



In terms of construct validity the existing measures are each also only capable of capturing one of the aspects of polarization. The variance and related measures are designed to describe dispersion in the classic sense regardless of modal distributional features, and the bimodality coefficient indicates the strengths of modes regardless of dispersion. Yet from a substantive political point of view, we want a measure of polarization that includes both perspectives. For example, consider the use of the variance in Figure 1.2, where the left panel shows a strongly bimodal structure and the right panel shows something much closer to a uniform distribution. Yet, the variance values are nearly identical whereas substantively all observers would consider the left panel to be a more polarized group of respondents. In addition, both the variance and the bimodality coefficient provide misleading results in the presence of more than two modes. This is illustrated in Figure 1.3 where bimodal and trimodal structures return nearly identical numerical values for both measures.

Figure 1.3: Bimodality vs. Trimodality



1.1.3 What Should an Effective Measure of Polarization Look Like?

Following the original definition of polarization as "the extent to which preferences or opinions are opposed in relation to some theoretical maximum" (DiMaggio, Evans and Bryson, 1996), there are two fundamental features that an effective measurement of polarization should incorporate. *First*, since mass polarization is a distributional concept, an efficacious measurement of polarization should distinguish the complete dynamics between minimum polarization (when all respondents lie on the same discrete location) and maximum polarization (when respondents are split evenly at the two extreme discrete locations). Also, an ideal measure should describe the distribution of opinions or preferences as a macro-level concept, which is different from individual-oriented measures of polarization such as ideological constraint/affective polarization (Abramowitz and Saunders, 2008; Baldassarri and Gelman, 2008; DellaPosta, 2020), or perceptions and attitudes towards out/in-group member (Druckman and Levendusky, 2019; Iyengar, Sood and Lelkes, 2012). *Second*, an effective measurement of polarization should reflect the inherent nature of ordinal data from standard survey instruments. For a 7-point item, 1 and 7 are obviously more extreme than 2 and 6. Therefore, the measure of polarization should capture the aggregate distribution of ordinal data via such differential spacing. Third, the measure of polarization also needs to be comparable across time, measurement level, and circumstances[6].

1.2 A New Entropy-Based Measure of Mass Polarization

Here we propose an entropy-based measure of mass polarization. Contrary to the previous literature that makes strict distinctionbetween dispersion and bimodality, this measure instead focuses on the aggregate, ordinal distribution as a whole. As it accounts for both distribution and ordering, and it can naturally capture the dynamics of dispersion and multimodality simultaneously.

[6]Variance, skewness, and kurtosis only have theoretical minimum values. And they are not comparable if the original scales are different. Also, for comparability without loss of generality we focus on 7-point variables throughout.

1.2.1 Entropy Background

It is often the case that social and political data are not continuous, especially data generated from survey research. Shannon (general) entropy (Shannon, 1948) is the classic means of describing information in discrete streams of possible outcomes $(x_1, \ldots, x_n)$ occurring with probability $p(x_1), \ldots, p(x_n)$, giving $E_S = -\sum_{i=1}^{n} p(x_i) \log p(x_i)$. This simple formula belies its power to explain natural and human generated data. General entropy and its modified forms, such as Simpson's Index (Simpson, 1949), have been one of the most popular metrics to describe categorical data structure since it is a direct measure of uncertainty for discrete random variables (Gill, 2005). It increases as every category of the responses becomes more equally likely and decreases as values concentrate in fewer categories (Bramson et al., 2017; Homola, Jackson and Gill, 2016). Measures based on entropy have already been used as fractionalization indices used for measuring the dispersion of groups in political science (e.g. Laakso and Taagepera, 1979; Golosov, 2010; Montalvo and Reynal-Querol, 2005; Esteban and Ray, 2008; Matakos, Troumpounis and Xefteris, 2016). Among all related measures, entropy and ordinal concentration measures are the most promising tools to capture polarization in ordinal data. Entropy also has a naturally intuitive range: the measure is minimized when all values fall into a single category and it is maximized when the values are uniformly distributed across all categories. However, because entropy is generally a metric for categorical data, there are two obvious limitations of directly using entropy and related indices for measuring polarization: (1) the maximum value is at uniform distribution since entropy measures the dispersion of the categorical variable; (2) it it cannot detect the *direction* the dispersion.

These hinders its use as a direct measure of polarization. We show here that this is corrected by using an ordinal modification of entropy starting with (1) a binary entropy and (2) a cumulative statement.

1.2.2 Cumulative Entropy

Following the literature on ordinal dispersion and concentration in sociology (Blair and Lacy, 2000; Leik, 1966), we develop the cumulative entropy as a measure of opinion polarization. A generalized version is called Tsallis Entropy (Tsallis, 2011). Cumulative entropy measures are also used in some natural science fields in this context and appear to have been originally developed in chemistry (Giauque et al., 1970; Pace, Dennis and Berg, 1955; Wynblatt, 1969). There are three required methodological steps to produce a *cumulative entropy measure* to account for polarization in the way discussed in the last section.

First, define the *binary entropy measure*, which shows to what degree the two categories of a dichotomous outcome are similar in magnitude:

$$H(p, 1-p) = 2^{-[p \cdot \log_2(p) + (1-p) \cdot \log_2(1-p)]} - 1 \tag{1.2}$$

where p and $1-p$ correspond to the proportions of the two categories from some survey or other data source. The exponential component, $-[p \cdot \log_2(p) + (1-p) \cdot \log_2(1-p)]$, is a basic binary entropy function, which is a special case of Shannon entropy for a Bernoulli process with p and $1-p$ as the probabilities of an event landing in either category. Mathematically

$H(p, 1-p)$ has a maximum value when $p = 1-p = 0.5$, and a minimum value if either p or $1-p$ is zero. Here we require the added assumption that $0 \times \log_2(0) = 0$. This is a common assumption in the literature, and it is not material in real survey data settings since there are no zero response categories except for trivially small number of subjects a study. Using 2 as a base for the exponent means that the first term of $H(p, 1-p)$ is scaled between 2^0 and 2^1 (a *typical set* in information theory, see: Cover and Thomas (1991)). The form of $H(p, 1-p)$ makes no parametric assumptions about the underlying scale of uncertainty (Shannon, 1948; Jaynes, 1968, 1982). Also, both base 2 in the logarithms and base 2 in the exponent reflect the diverging notion of pushing mass to the two extremes. The choice of logarithm base here is arbitrary, but using 2 leads to a more intuitive measure for our purposes.

Second, generalize the binary case to an ordinal measured variable with k discrete categories. The corresponding observed proportions for each category are denoted $p_1, p_2, \ldots, p_k$. For the j^{th} category, $j \in \{1, \ldots, k\}$, define two complementary cumulative response proportions from an ordinal variable in the data as:

$$S_j = \sum_{i=1}^{j} p_i \qquad\qquad S_{\neg j} = \sum_{i=j+1}^{k} p_i = 1 - S_j, \tag{1.3}$$

where j and $\neg j$ indicate the two summed lower and upper mass regions $(1 \leq j < k; 2 \leq \neg j \leq k)$ for which we will apply the binary entropy in equation (1.2) $k-1$ times. Note that any selection of $j = 1, \ldots, k$ produces a pair of cumulative values. Also observe that the full set of these $S_j, S_{\neg j}$ pairs is $k-1$ in length since $S_{\neg j}$ does not exist for the k^{th} category, and are analogous to the thresholds on the latent dimension in ordered logit/probit regression models. Note that for any j the sum of these two terms is always equal to 1: $S_{\neg j} = 1 - S_j$. As a simple illustration, consider category 3 from a total of 7 categories ($j = 3, k = 7$), so that $S_3 = p_1 + p_2 + p_3$ and $S_{\neg 3} = p_4 + p_5 + p_6 + p_7$.

Third, using the $k - 1$ pairs defined by equation (1.3) the *cumulative entropy* is defined by feeding each of the $(S_j, S_{\neg j})$ pairs into equation (1.2) and summing the results:

$$E_c = \frac{\sum_{j=1}^{k-1} H(S_j, S_{\neg j})}{k-1} = \frac{H(S_1, S_{\neg 1}) + H(S_2, S_{\neg 2}) + \ldots + H(S_{k-1}, S_k)}{k-1}, \tag{1.4}$$

where dividing by $k - 1$ is just a scaling factor. Hence the $(S_j, S_{\neg j})$ contrast is compared for each of the $k - 1$ pairs providing a nonparametric ordinal description of the distribution moving from left to right in the sum[7]. So E_c is a summary measure of both modal features and dispersion across the range of the item.

Mathematically, at each of the three steps the calculated values have convenient limits by design:

$$H(p, 1-p) \in [0:1] \quad \text{by log rules and subtracting by 1} \tag{1.5}$$

$$S_j, S_{\neg j} \in [0:1] \quad \text{since } S_j + S_{\neg j} = 1 \; \forall (k-1) \text{ pairs} \tag{1.6}$$

$$E_c \in (0:1] \quad \text{from standardization of the sum by } k-1. \tag{1.7}$$

In addition, the magnitude of j does not alter these properties. Thus E_c is easily interpretable for any value of j, and can be used for cases from low-category Likert scales ($k = 3, 5, 7, \ldots$)

[7]It is nonparametric in a sense we do not assume any particular family of probability mass functions

to standard feeling thermometers ($k = 101$) and even higher numbers of categories. In the latter type of instruments it is likely that the sum in E_c will have a lengthy sum of small probabilities.

Intuitively, we can look at extreme data cases with a 7-point ordinal variable ($k = 7$) as a way to understand the properties of E_c. In the most polarized case we have equal proportions at the poles, $[0.5, 0, 0, 0, 0, 0, 0.5]$, so that:

$$\begin{aligned} E_c &= \frac{H(0.5,0.5) + H(0.5,0.5) + H(0.5,0.5) + H(0.5,0.5) + H(0.5,0.5) + H(0.5,0.5)}{7-1} \\ &= \frac{1+1+1+1+1+1}{6} = 1, \end{aligned}$$

and a slightly less polarized case is given by $[0, 0.5, 0, 0, 0, 0.5, 0]$, providing:

$$\begin{aligned} E_c &= \frac{H(0,1) + H(0.5,0.5) + H(0.5,0.5) + H(0.5,0.5) + H(0.5,0.5) + H(1,0)}{7-1} \\ &= \frac{0+1+1+1+1+0}{6} = 0.667 \end{aligned}$$

Conversely, if all of the responses are concentrated at *any* single value, such as the second place, then:

$$\begin{aligned} E_c &= \frac{H(0,1) + H(1,0) + H(1,0) + H(1,0) + H(1,0) + H(1,0)}{7-1} \\ &= \frac{0+0+0+0+0+0}{6} = 0. \end{aligned}$$

Importantly, the relative space between modes is critical, rather than absolute positions, meaning that $[0.5, 0, 0, 0.5, 0, 0, 0]$ and $[0, 0, 0, 0.5, 0, 0, 0.5]$ reflect the same level of polarization giving the same value of E_c. This feature is important to compare polarization across different issues and contexts.

Substantively, the comparison of cumulative proportions describes a distributional difference for a given category along the ordinal scale. Then the cumulative entropy measures the variability of this distribution: if each side has half of the distribution, then it will result in the maximum of cumulative entropy. Also, the sum of entropy for cumulative proportions will reflect the ordering of the categories. When the distribution is highly dispersed and concentrated on two poles, it will start with a large value in the cumulated process and carry the large value until the other end. When it is concentrated in one category, the entropies for all the cumulative proportions will be very small and it will result in a small value of the measure. As a result, the cumulation of binary entropy captures both the ordering and distribution of the data since it considers both the concentration of particular categories (modes) and to what degree the modes stick together or fall apart. At the same time, this entropy-based measure of polarization imposes no assumption about the central tendency, spacing between categories, and modality of the distribution and is therefore fully nonparametric. In the following sections, we illustrate the properties of the E_c measure with contrived cases, Monte Carlo simulations, a crowd-sourcing test, and three real data applications.

1.3 Illustration, Simulation, and Validation

One challenge in developing such a measure is how to verify its performance since there is no such phenomenon as *absolute* polarization with real data. In this section we use hypothetical distributions to illustrate the properties of the proposed measurement. We then use simulations to generate ordinal data with pre-defined Gaussian mixtures, in which we can manipulate the means and standard deviations and compare the actual performances of the measures with the definitions. We also show additional evidence through crowd-sourcing validation that the proposed measure is consistent with how people evaluate and compare polarization.

1.3.1 Hypothetical Distributions Applying E_c

Here, we set up some hypothetical distributions to illustrate the properties of the proposed measure and compare it with conventional measures of variance and the bimodality coefficient. Table **??** and **??** show two sets of contrived proportions (left panel) and generated 7-choice ordinal data. We also show the barplots for the ordinal data and calculated measures of polarization. In Table **??**, Row 1 to 4 illustrates the definition of polarization–movement toward the poles of a distribution.

Table 1.1: The Definition of Polarization

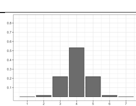
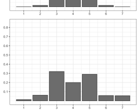
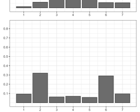
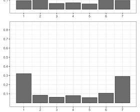
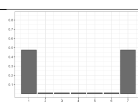
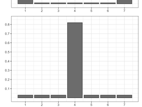
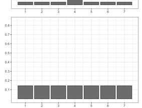
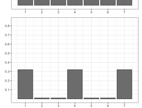
	Proportions								**Measures**		
	1	**2**	**3**	**4**	**5**	**6**	**7**		E_c	Var	$Bimod$
1	0.000	0.015	0.219	0.532	0.219	0.015	0.000		0.210	0.572	0.323
2	0.014	0.062	0.320	0.200	0.290	0.058	0.056		0.401	1.773	0.397
3	0.094	0.320	0.063	0.070	0.058	0.290	0.096		0.715	4.304	0.725
4	0.320	0.084	0.062	0.080	0.058	0.106	0.290		0.907	6.351	0.772

Table ?? describes four typical types of distributions. As all the responses concentrate on the two extreme categories, the entropy-based measure of polarization is nearly the maximum value of 1 in Row 5 (computer rounding in the intermediate calculations). Conversely, if all the responses concentrate on one category with very little elsewhere (no matter which category), the polarization approximates the minimum value of 0 (Row 6). Row 7 and 8 show a equally spread distribution and a trimodal distribution, respectively. Notice here that both the variance and the bimodality coefficient do a poor job relative to the E_c cumulative entropy measure in that their relative magnitudes do not monotonically reflect the intuition of the polarization changes.

Table 1.2: Different Types of Distributions

	Proportions								**Measures**		
	1	**2**	**3**	**4**	**5**	**6**	**7**		E_c	Var	$Bimod$
5	0.475	0.010	0.010	0.010	0.010	0.010	0.475		0.998	8.690	0.969
6	0.030	0.030	0.030	0.820	0.030	0.030	0.030		0.112	0.935	0.129
7	0.143	0.143	0.143	0.143	0.143	0.143	0.143		0.689	4.076	0.582
8	0.320	0.010	0.010	0.320	0.010	0.010	0.320		0.837	5.935	0.667

Next, in Table ?? and ?? show how E_c is able to reveal both more complicated and nuanced dynamics of polarization compared to traditional variance and the bimodality coefficient. Table ?? focus on the comparison between entropy and variance. While our measure was able to reflect the similarity between Row 9 and 10 and the difference between 11 and 12, the variance cannot distinguish the spread and polarization. For example, Row 11 shows the majority of the responses concentrate on one category, which does not indicate high-level polarization. On the contrary, Row 12 shows that two nearly equally sized camps are relatively far apart. Similarly, in Table **??** Row 13 through 16 demonstrate that the bimodality coefficient fails to capture the nuances in the changes of distributions when it's not perfectly bimodal.

Table 1.3: Comparing E_c and Variance with Hypothetical Distributions

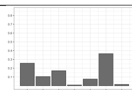
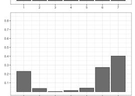
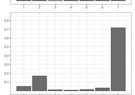
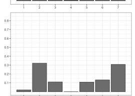
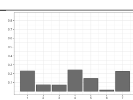
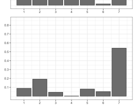
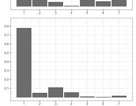
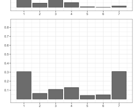
	Proportions								Measures		
	1	**2**	**3**	**4**	**5**	**6**	**7**		E_c	Var	$Bimod$
9	0.257	0.105	0.172	0.009	0.076	0.365	0.015		**0.747**	**4.489**	0.752
10	0.230	0.037	0.003	0.014	0.041	0.274	0.402		**0.753**	**5.787**	0.891
11	0.050	0.169	0.011	0.007	0.015	0.032	0.717		**0.547**	**4.893**	0.948
12	0.020	0.321	0.110	0.001	0.108	0.133	0.307		**0.770**	**4.732**	0.786

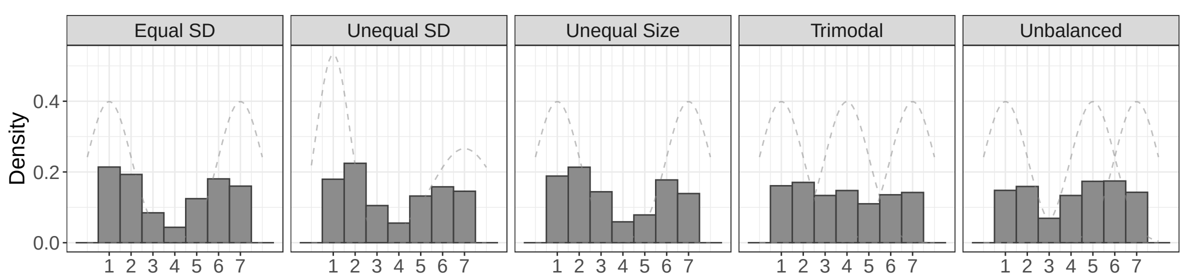
Table 1.4: Comparing E_c and Bimodality Coefficient with Hypothetical Distributions

	Proportions								Measures		
	1	**2**	**3**	**4**	**5**	**6**	**7**		E_c	Var	$Bimod$
13	0.231	0.073	0.071	0.242	0.144	0.014	0.224		**0.740**	4.753	**0.573**
14	0.088	0.192	0.044	0.002	0.081	0.052	0.541		**0.753**	5.500	**0.860**
15	0.776	0.045	0.107	0.052	0.004	0.001	0.013		**0.203**	1.206	**0.733**
16	0.305	0.064	0.108	0.129	0.041	0.047	0.307		**0.882**	6.144	**0.726**

1.3.2 Simulations of E_c

Simulations allow us to compare how metrics of polarization can recover the "ground truth." But since there is no absolute polarization that can be used as a benchmark from real data, we employ a two-step procedure creating quasi-true values. First, we use normal mixture distributions to simulate continuous data with predefined clusters and the "true polarization" is defined as the distance between the means (modes) of two normal distributions. This mimics the distinct groups identified in the literature on polarization. The next task is to cluster the continuous data into multiple categories, focusing on the 7 category case as it dominates ordinal designs in survey research. This represents the process from an underlying, continuous utility to the fixed categories of ordinal responses. We also use a modified optimal k-means algorithm with dynamic programming (Wang and Song, 2011) for the clustering to overcome the challenge of clustering one-dimensional data as it generally conveys less information, and leads to an *NP-hard* problem in a Euclidean space (Aloise et al., 2009) as discussed further in Online Appendix A.1.

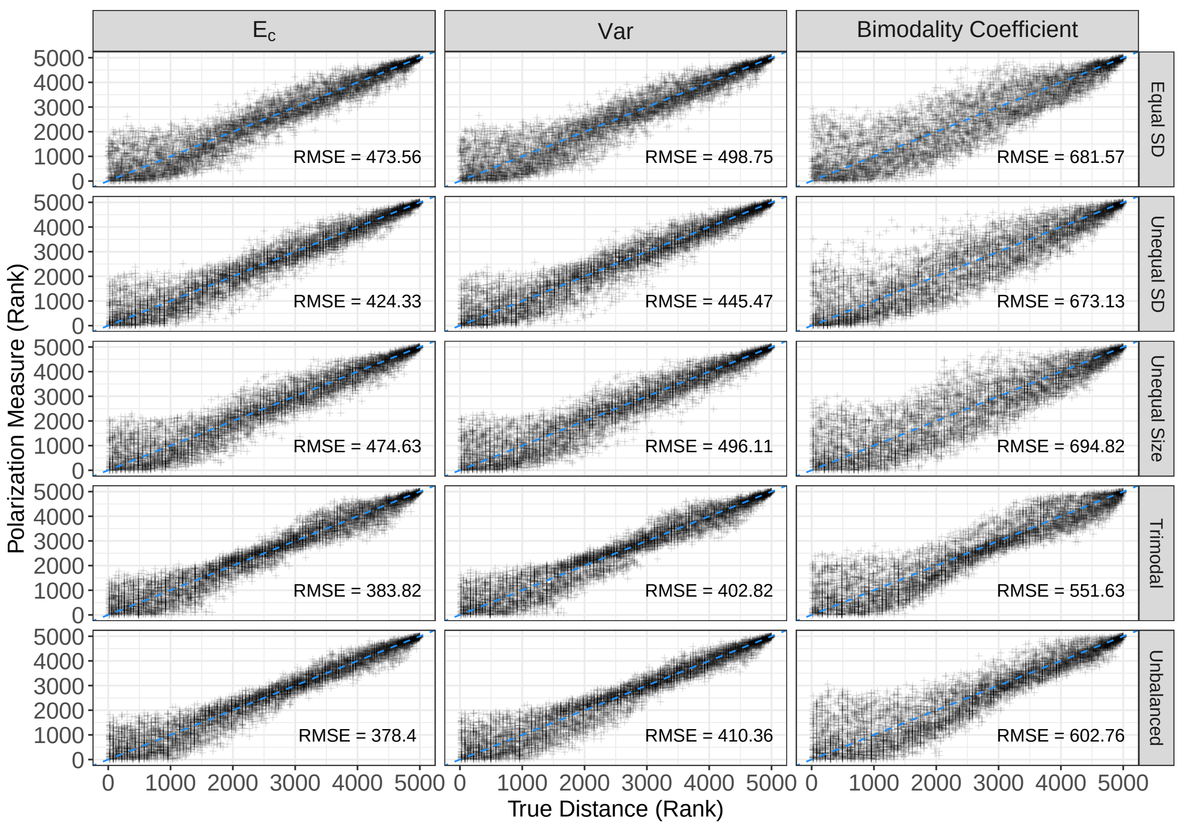
Figure 1.4: Simulation Setups



Note: The dashed curves represent the Gaussian mixture distributions. The bar plots are the ordinal data after clustering.

It is also important to note that k-means cluster estimation comes with strong assumptions that are often ignored in common practice but fit here because we control the structure of the simulation. We use five basic configurations to simulate normal mixture distributions: (1) equal standard deviation, (2) unequal standard deviation, (3) unequal size, (4) trimodality, and (5) unbalanced (middle point) as illustrated in Figure 1.4. Although (1) reflects perfectly two polarized camps, it is more common to find the other four scenarios in the real data and they are more likely to provide measurement challenges. Figure 1.4 visualizes two-step procedures and five configurations.

Figure 1.5: Simulations of Ordinal Data and Polarization Measures



Note: Each configuration is repeated 5000 times with a sample of 2000 data points each time and ranked by both the true and calculated polarization, with higher ranks indicating higher level of polarization. The blue dashed lines represent when measured and defined polarization is perfectly aligned ($y = x$). RMSEs are calculated based on errors of the rank, which scale both the polarization measures and true distances to the same unit.

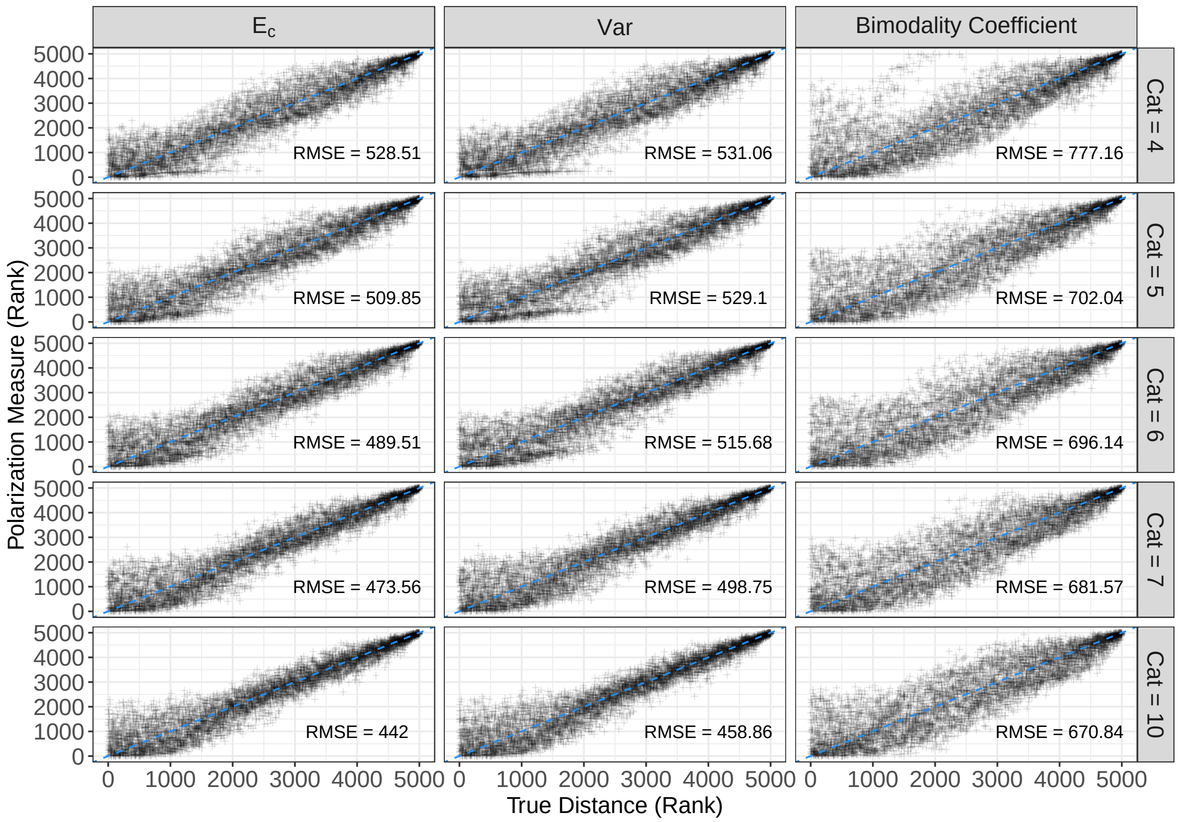
Figure 1.5 shows the results of these simulations. Our entropy-based measure of polarization clearly performs the best in all five settings where the E_c estimates are closer to the $y = x$ lines, indicated by the smallest root mean squared errors of ranks. Variance is second, in quality even though we know it violates the underlying assumption of ordinal data. It performs nearly as well in the equal standard deviation setting but becomes worse when the distributions are less perfect[8]. The bimodality coefficient performs much worse in all five configurations. Note that the simulation procedure would generally favor the measures of variance and the bimodality coefficient because we generate two spread-out clusters and equally spaced categories in order to use the difference in means as the benchmark. Yet even in this setting, E_c performs consistently better, and we would also expect so in more complex real-world data with more nuances. It is also worth mentioning that one generally cares more about the right halves of each panel in terms of measuring polarization. The right halves are more about the dynamics of polarization while the left halves contain more uniform distributions. It is generally less informative to compare two near uniform distributions in the context of polarization.

Additionally, we also test whether different numbers of categories would affect the results. Figure 1.6 compares 4, 5, 6, 7, and 10 categories, which are all very common in surveys. As we can see, the entropy-based measure of polarization clearly outperforms both of the other measures in all five settings. Generally speaking, we would suspect that, as the number of categories increases, the distribution of the ordinal data should be closer to the pattern of "continuous" data (i.e. more equally spaced, having a meaningful average point, etc.)

[8] It is expected variance will perform well in some scenarios as it measures the data variability and should be able to pick up some of the dynamics of polarization while still violating the assumption of ordinal data.

However, even with 10 categories, we still see that the entropy-based measure performs better than the other two statistics that are based on continuous data[9].

Figure 1.6: Simulations of Ordinal Data and Polarization Measures



Note: Each row of plots represent one configuration. Each configuration is repeated 5000 times with a sample of 2000 data points each time and ranked by both the true and calculated polarization, with higher ranks indicating higher level of polarization. The blue dashed lines represent when measured and defined polarization is perfectly aligned ($y = x$). RMSEs are calculated based on errors of the rank, which scale both the polarization measures and true distances to the same unit.

[9]We can see variance performs better when number of category equals four. This is largely due to the setup of simulations and the classification algorithm we used: it tends to generate more evenly distributed four categories, and thus, favors variance in this case.

1.3.3 Crowd-Sourcing Validation of E_c

As it is impossible to define a perfect "ground truth" of polarization, even in just a simulated setting. We additionally turn to a "wisdom of the crowd" approach to benchmark the metrics of polarization with more intuitive judgments by humans. We designed an online MTurk validation experiment in which respondents compare polarization with different scenarios. The goal of the crowd-sourcing approach is to solicit human judgments of different scenarios of polarization, aggregate this information based on collective opinions from multiple individuals and multiple rounds of evaluations, and finally, benchmark different metrics of polarization against the crowd assessment (Lyon and Pacuit, 2013; see Barberá et al., 2021; Sumner, Farris and Holman, 2020, for some recent crowd-sourcing approaches in political science.

Table 1.5: Agreement Rates between Measures of Polarization and Crowd-Sourcing Evaluations

Metrics	**Agreement Rates**	
	Full sample	**Baseline-satisfied**
E_c	0.650 (0.195)	0.674 (0.229)
Variance	0.421 (0.238)	0.410 (0.279)
Bimod Coef	0.403 (0.230)	0.384 (0.267)

Note: Each agreement rate is calculated as the average proportion of tasks where the metrics of polarization and human respondents select the same scenarios as more polarized (Online Appendix A.2) The first column shows the results based on the full sample; the second column is based on the sample filter out the respondents who didn't pass the baseline task (Figure A.2).

For the validation task, each respondent was presented with a pair of barplots given in Appendix A.2, Figure A.1 with a contrived context of either ideology or issue opinions and asked to choose a more polarized scenario according to the barplots of ordinal distributions.

To analyze the crowd-sourcing data, we calculate the agreement rates between the metrics of polarization (variance, bimodality coefficient, entropy-based measure) and crowd-sourcing evaluations using the standard methodology shown in Section A.2. The results, as reported in Table 1.3, show that E_c performs significantly better than the other two measures, and in addition it has a smaller standard error than the variance and the bimodality coefficient. So about two-thirds of the time, E_c and the testers agree on which scenario is more polarized, but for the other two measures it is notably worse than flipping a coin.

1.4 Empirical Applications

In this section, we apply the proposed measure to three contexts of mass polarization and compare it with the conventional measurements of polarization. First, we revisit the question of polarization in the U.S. electorate. We describe evidence of ideological, partisan, and issue polarization and compare it with conventional measurements. Second, we employ our measurement to an analysis of the relationship between radical parties and polarization in the Europe. Last, we measure the cross-country trends in ideological polarization and explore the relationship between ideological and affective polarization. The objective here is to show both similarities and differences with the previously used measures with real data as users of our new E_c would encounter.

1.4.1 Ideological and Issue Polarization in the U.S.

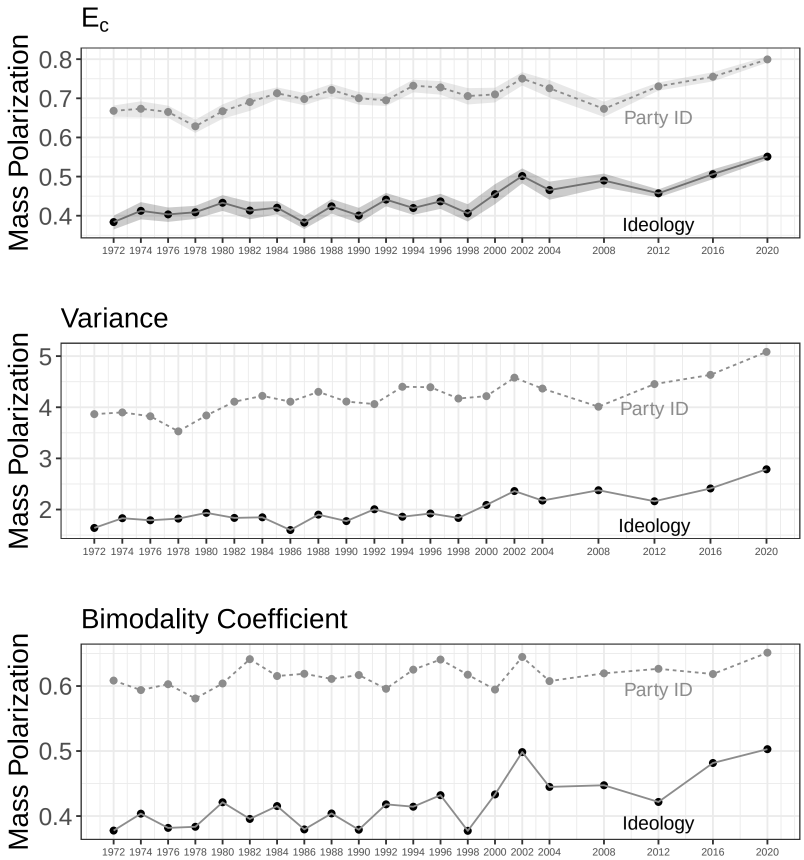
Mass polarization has been an extremely salient and important topic in American politics. While the early debates centered on whether and to what degree the mass public is polarized, recent studies have paid more attention to the different partial dynamics of partisan, ideological, and issue polarization. These scholars suggest that sorting has brought partisanship into alignment, and consequentially give rise to increasing identity-based partisan bias and affective polarization (Abramowitz and Saunders, 2008; Iyengar, Sood and Lelkes, 2012; Levendusky, 2013; Abramowitz and Webster, 2016; Hetherington, Long and Rudolph, 2016; Mason, 2018*b*; Kalmoe and Mason, 2022). In comparison, issue positions in the mass public have experienced relatively milder increases and underwent different dynamics (Fiorina, Abrams and Pope, 2010; Levendusky, 2009; Baldassarri and Gelman, 2008; Mason, 2013; Enders, 2021). To depict mass public polarization in those different dimensions, we apply both the entropy-based measure and conventional metrics of polarization to several issue opinions (as well as party identification and ideology) in longitudinal ANES surveys from 1972 to 2020.

For ideology and party identification self-placement, three metrics present overall similar trends while displaying some nuanced differences (detailed in Online Appendix A.3). Both ideology and party ID show a moderate increasing trend starting in 1998 after a long relatively stable period. This speaks to the construct validity of the proposed measure. It is not surprising that the mathematically incorrect variance and the bimodality coefficient share some features with E_c since they are designed to capture two different effects that are incorporated simultaneously in E_c (dispersion and modality). Also, data in these two dimensions are relatively balanced and has a middle point, which makes it resemble some features of continuous data. This is in comparison to other contexts where the majority of the country can lean left or right and the middle category is not perfectly aligned with the mean of the distribution due to the categorical nature of the data.

The upper panel in Figure 1.7 shows our measure of ideological and partisan polarization for the American public. From the figure, we can see both the static and dynamic trends of polarization in ideology and party identification. Both ideology and party ID show an increasing trend starting in 1998 after a long relatively stable period, which provides construct validity of the measure.

Comparing E_c to the variance and bimodality measurement, we can also see both the similarity and nuanced differences between the entropy-based measure and conventional statistics. While the entropy-based measure and variance show a very similar overall trend, bimodality coefficient tells a pretty different story. It shows, except for periodic fluctuations, partisan polarization did not really increase since 1970s while ideological polarization has experienced more dramatic but not consistent changes. Although entropy-based measure and variance depict a similar pattern, the former also provides more dynamics, especially for ideological polarization. In addition to the scaling differences, we can see E_c is able to capture issue dynamics that differ from the variance and bimodality coefficient. This is in line with our theoretical expectation since variance and bimodality are meant to measure different things–one for dispersion, the other for bimodality (DiMaggio, Evans and Bryson, 1996; Lelkes, 2016). The reason for the dramatic fluctuations between adjacent years for

Figure 1.7: Ideological and Partisan Polarization in the U.S.

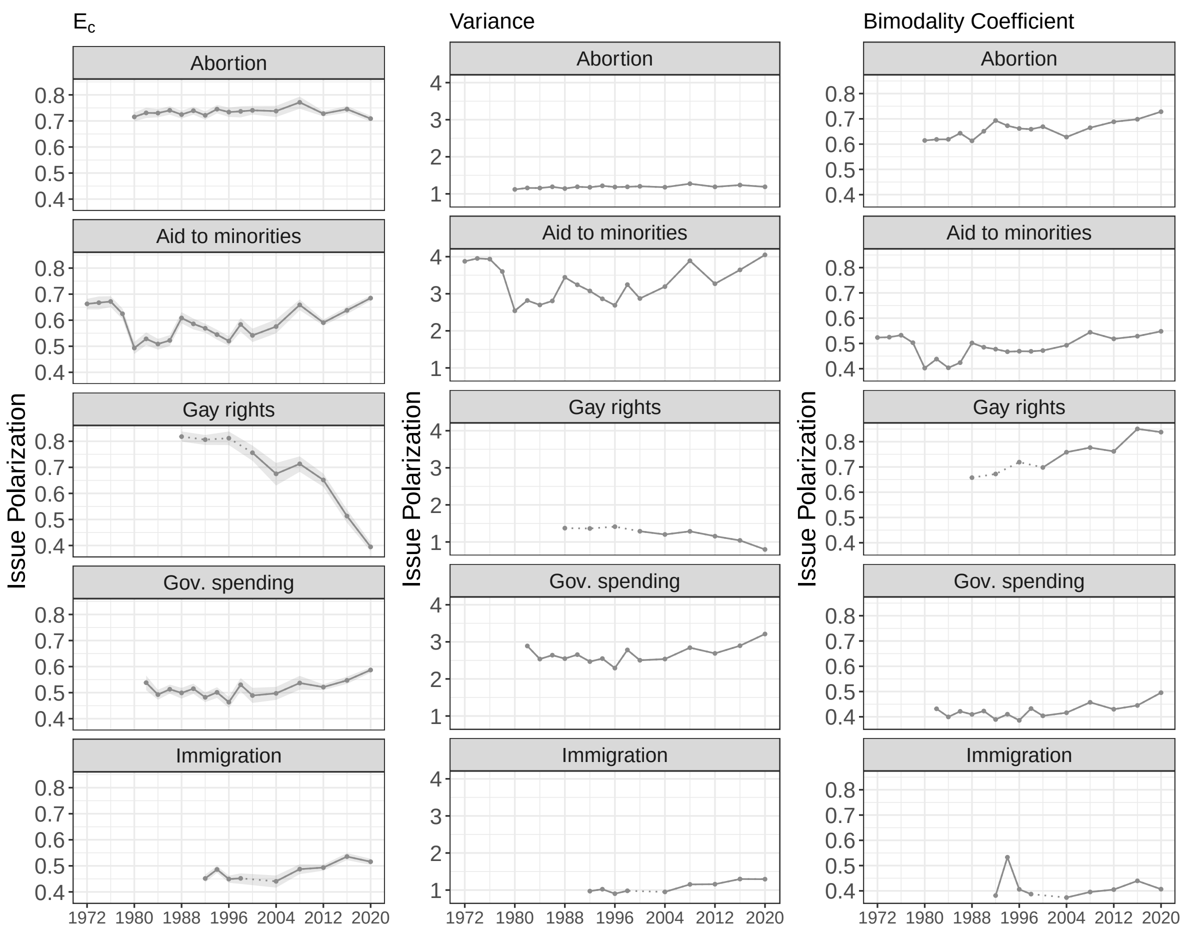


Note: Data source: ANES (1972-2020). Black solid line is ideology. Grey dotted line is party identification. The 95% non-parametric bootstrapping confidence intervals are shown for E_c.

the bimodality coefficient are likely due to the fact that this kurtosis-based statistic is very sensitive to mode changes in this ordinal setting even though the changes are extremely small and do not greatly affect the overall distribution. At the same time, variance was able to capture the dispersion but is not sensitive to other distributional dynamics of ordinal data. In addition, the variance (and the even the bimodality coefficient) is performing relatively well on the issues of party identification and ideology, which is not beyond our expectation since in this case there is a theoretical middle point, and two relatively balanced sides.

The issue polarization depicted in Figure 1.8 is where metrics start to differ substantively. These salient issues can become rather complex in terms of measuring polarization. They are measured in different scales from 4 points to 7 points and some issues do not have a theoretical middle or some do not have a balanced distribution between two sides. The spacing between categories in such settings are also generally more complex than ideological and party ID items. Comparing the metrics, we see both similarities and differences in terms of both the levels and trends of polarization.

Figure 1.8: Issue Polarization in the Mass Public



Note: Data: ANES (1972-2020). The 95% nonparametric bootstrapping confidence intervals are shown for E_c. The dotted lines represent there are disconnections between the years that ask the questions and are fitted using linear interpolation.

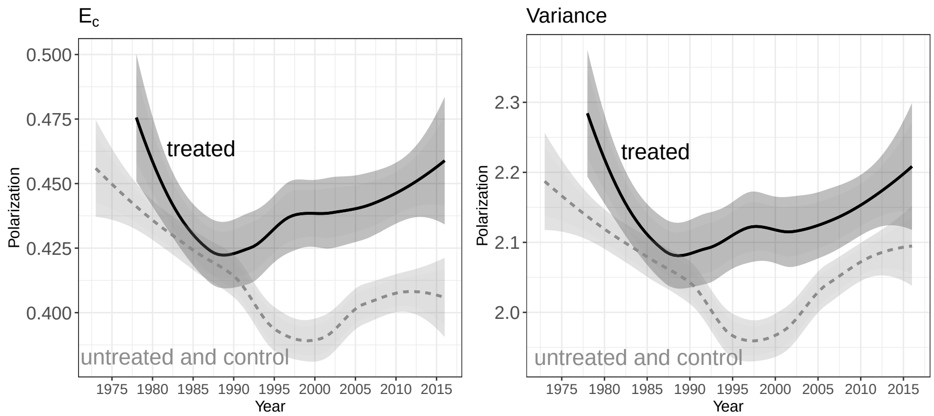
However, for some issues, there are both big and small discrepancies. For example, the abortion issue is measured as a four-point item. There is no middle category nor does it has equal spacing and balancing. The Entropy-based measure and variance suggest a relatively stable trend since the 1980s, but E_c is able to capture more subtle dynamics for the uneven decline since 2008. While the entropy-based measure finds a gradual polarization decrease in recent waves of the survey, the bimodality coefficient shows an increasing trend since 1998. The E_c measure is clearly more in line with what the substantive research suggests: the abortion issue, albeit divisive, certainly has not become more polarized (Mouw and Sobel, 2001; Fiorina, Abrams and Pope, 2010; Carsey and Layman, 2006), and there is some evidence that the proportion of citizens who favor more abortion rights has increased during this period (as shown in the detailed barplots in Figure A.3). We also know from a vast literature that support for abortion rights is not as stable over time as the variance measure implies here.

For aid to minorities the E_c measure and the variance are very similar and pick up the same fluctuations, whereas the bimodality coefficient suggest a much more stable picture of polarization as a policy issue, at least since 1992. This implies that the polarization story here is more about dispersion than multimodality. For the issue of gay rights, the three measures show very different patterns. The entropy-based measure depicts a long-term, sharp decrease in the level of polarization, which is consistent with academic (Bishin, Freebourn and Teten, 2021) and journalistic accounts (shown in Figure A.3 with detailed distributions). The variance shows a slow decline which is not consistent with such accounts. More incorrectly the bimodality coefficient gives an overall increase in polarization around gay rights since 1988 because it is overly sensitive to an immobile but shrinking mode of opposition. Interestingly, the patterns for polarization over this period are essentially the same across time for views on government spending. This implies stability in both dispersion and in bimodality. Online Appendix A.3 provides barplots to describe the detailed distributions for all the issues as well as ideology and party identification, which further demonstrate that E_c is able to provide more reasonable accounts for the dynamics and overall trends of the ordinal distributions.

1.4.2 Radical Parties and Mass Polarization in Europe

In this section, we focus on a more generalized and analytic example that compares the polarization across countries. There has indisputably been rise of extreme parties and radical political elites and increasing polarization and across continents. Theories suggest that party and elite polarization is conducive to fueling the polarization in the mass public as it publicizes extreme views, increases mobilization of discontent voters, and accelerates realignment in the electorate (Hetherington, 2001; Layman and Carsey, 2002; Fiorina and Abrams, 2008; Adams, Ezrow and Somer-Topcu, 2014; Lupu, 2015; Silva, 2018). Recently Bischof and Wagner (2019) employ a time-series cross-sectional analysis on European countries and demonstrates that the mass ideological polarization will increase after a radical right party gains power in the legislature. For the key outcome, they use the standard deviation of left–right self-placements in each country-year unit to measure public polarization. We replicate their descriptive, inferential, and causal results using the proposed measure and compare them with the original variance measure in their paper.

Figure 1.9: Descriptives of Mass Polarization in Europe



Note: The figures show lines fitted with local polynomials and corresponding 95% and 83.7% confidence intervals. Solid curves represent the treated cases and dashed curves represent the control and untreated cases.

Figure 1.9 describes the polarization time trends between treated cases (those encountered entries of radical parties, solid curve) and controls (those never experienced or had not yet experienced, dashed curve). The pattern is similar in general trends between E_c and the original authors' use of the variance, indicating that dispersion not multimodality is dominant in these data. The entropy measure finds a greater divergence between the two groups from 1985 onward, meaning that radical parties have had a even greater impact on polarization than Bischof and Wagner found. They also estimate linear regression models for the relationship between the entrance of radical right party and mass ideological polarization. Table 1.4 reproduces the original results (right half) and replicates the models using the entropy-based measure for polarization as the outcome variable (left half)[10]. The overall results using E_c are consistent with the original findings, suggesting the entrance of radical-right parties has a reliable effect on the increase of polarization. However, the E_c measure finds noticeably greater separation in later years where we know that the effect is greater.

[10] Online Appendix A.4 reproduces the additional results for countries with electoral threshold as in the original study.

Also, the entropy-based measure results in relatively smaller standard errors (comparing to the scale of coefficient estimates), which can mean an improvement in the efficiency of the estimation. Plus, the different measurement of the outcome also changes the relationship with other explanatory variables such as *unemployment*.

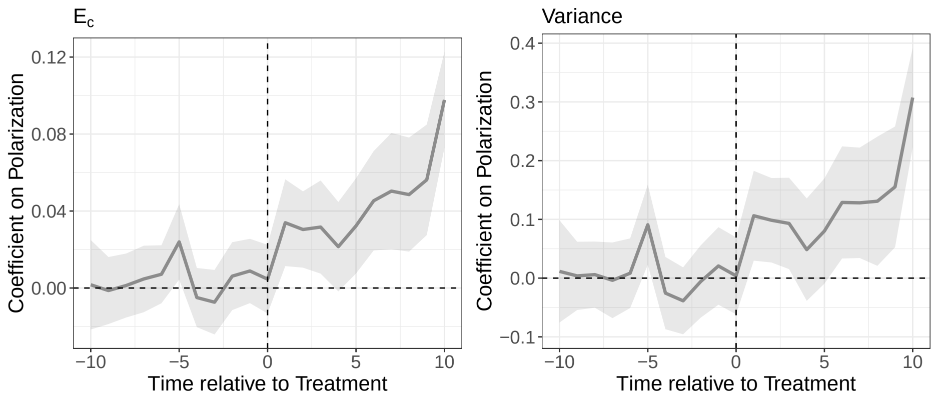
Table 1.6: OLS Estimates: Increasing Polarization after Entrance of Radical Right Party

		E_c			Variance	
	(1)	**(2)**	**(3)**	**(1)**	**(2)**	**(3)**
Radical-right enter	**0.031**	**0.043**	**0.046**	**0.090**	**0.116**	**0.131**
	(0.009)	**(0.009)**	**(0.009)**	**(0.034)**	**(0.032)**	**(0.033)**
GDP growth			−0.002			−0.006
			(0.001)			(0.003)
Unemployment (t-1)			0.0004			0.003
			(0.001)			(0.003)
Party system polarization (t-1)			0.0001			−0.001
			(0.001)			(0.002)
Party system fragmentation (t-1)			−0.004			−0.018
			(0.003)			(0.013)
Constant	0.412	0.423	0.444	2.055	2.103	2.203
	(0.006)	(0.014)	(0.029)	(0.021)	(0.051)	(0.111)
R-squared	0.056	0.661	0.683	0.035	0.674	0.690
N (elections)	164	164	145	164	164	145
N	534	534	503	534	534	503
Country fixed effects		✓	✓		✓	✓
Decade fixed effects		✓	✓		✓	✓

Note: Standard errors are clustered by country/election.

Finally, we reproduce the authors' analysis using Generalized Synthetic Control Methods (Xu, 2017), which can provide causal inference with interactive fixed-effect models and exploit synthesized counterfactuals for treated units based on information from untreated groups. Figure 1.10 reports the GSCM estimates using both entropy-based measure as the outcome (left panel) and the original standard deviations (right panel). It again shows very similar patterns between two measures with the entropy-based measure providing some more nuanced dynamics. The increased efficiency from the measure results in a more steady trend in both the pre and post treatment periods with E_c farther away from zero in the post treatment period.

Figure 1.10: GSCM Estimates: Effects of Radical-Right Parties on Polarization



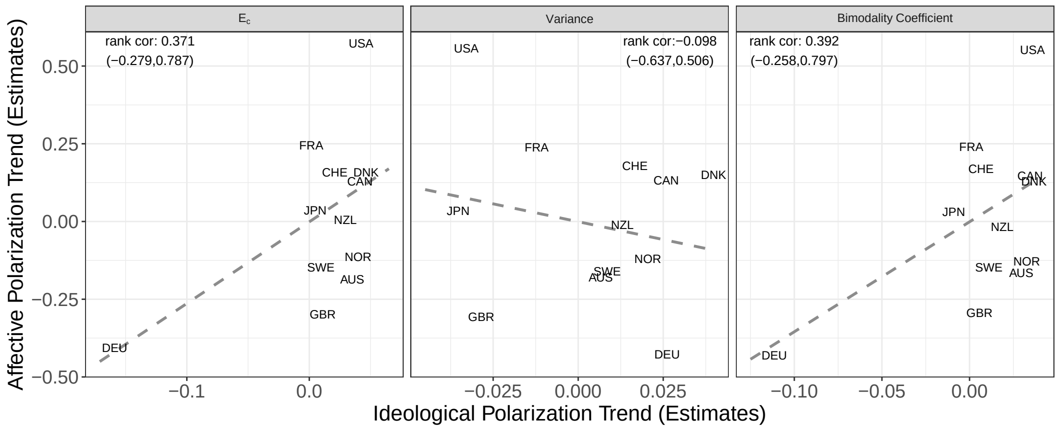
1.4.3 Cross-Country Trends in Ideological and Affective Polarization

Recent work in the polarization literature has observed that mass polarization is not only about where people stand on the issues, but also about how people emotionally dislike those from rival parties (Iyengar, Sood and Lelkes, 2012). The increasing trend of affective polarization is well observed in the U.S (Iyengar, Sood and Lelkes, 2012; Mason, 2015) and across other parts of the world (Gidron, Adams and Horne, 2020; Reiljan, 2020; Harteveld, 2021; Hobolt, Leeper and Tilley, 2021; Wagner, 2021; Boxell, Gentzkow and Shapiro, forthcoming). However, the underlying causes and consequences of this growing partisan antagonism around the world still require further explanation. A possibly important explanation of affective polarization is mass ideological polarization: the divergent values and policy attitudes in the mass public can be either the cause or the consequences of partisan animosity (Levendusky, 2009; Rogowski and Sutherland, 2016; Mason, 2018*a*). While the relationship between ideological and affective polarization has generated ongoing debates in American political studies (see: Rogowski and Sutherland, 2016; Mason, 2018*a*), cross-country comparisons are more elusive, possibly due to the lack of comparable measures of mass ideological polarization. We apply the metrics of polarization to data assembled from multiple survey projects and compare its trend with the recent finding of affective polarization in twelve OECD countries for the past three decades.

For measuring affective polarization across countries, we use the most up-to-date and comprehensive dataset possible (Boxell, Gentzkow and Shapiro, forthcoming). The affective polarization is defined as the weighted average of respondents' partisan affect and individual partisan affect is measured by the extent to which an individual expresses a more favorable attitude toward their own party than toward other parties (see also: Iyengar et al., 2019; Gidron, Adams and Horne, 2020; Reiljan, 2020; Harteveld, 2021). For ideological polarization, we assemble data from multiple surveys and match them with the affective polarization data set. We again apply three polarization metrics to survey items about left-right position and measure the polarization trends of ideology across years for each country. The survey items have similar wordings that ask respondents' about self-identified left-right positions but different numbers of categories across surveys. Additional details about the data sources and question wordings can be found in Online Appendix A.5.

Figure 1.11 provides a scatter plot and the Spearman rank correlation estimates to compare the trends between ideological polarization (X-axis) and affective polarization (Y-axis) for each measure of polarization. The correlation coefficients between the variables are not statistically reliable in all three case as the confidence intervals all contain zero. The estimates based on E_c measure and the bimodality coefficient suggest a positive relationship

Figure 1.11: Trends in Affective and Ideological Polarization



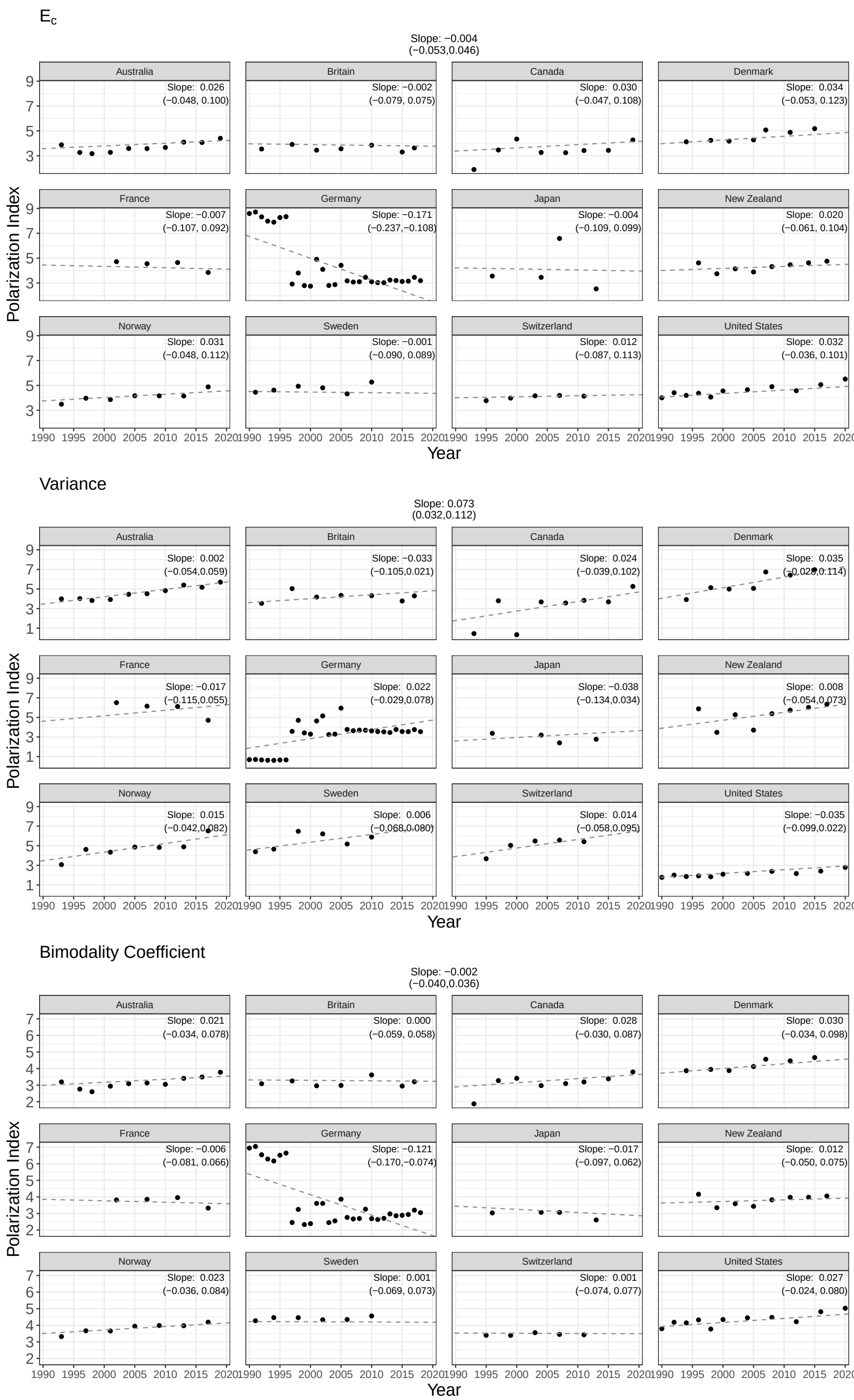
Note: Both axis variables are the estimates of linear trend based on Bayesian hierarchical partial-pooling models. The dashed line displays a fitted bivariate linear regression. The correlation estimates are from the Spearman rank correlation and 95% confidence intervals are reported in the parenthesis.

between ideological polarization and affective polarization while the variance suggests in a negative relationship. Since the y-axis is the same metric of affective polarization, this essentially reflects the differences in how three metrics capture different trends in ideological polarization[11]. Specifically, this shows variability as measured by variance can be verydifferent from polarization that is better captured by the tendency of bimodality here, which explains the similarity between E_c and bimodality coefficient. Nevertheless, there are still some small differences between E_c and bimodality coefficient as the latter also only captures the bimodality part, let alone it also assumes interval measured data.

Looking further into evidence of ideological polarization, Figure 1.12 shows the ideological polarization for each of the twelves OECD countries from 1990 to 2020. For the time patterns of ideological polarization, the three measures depict both similarity and differences. Note

[11]As it seems Germany (DEU) is a outlier in the E_c and bimodality measures, it may disproportionally influence the overall pattern. In Appendix **??**, we additionally test the robustness of this relationship dropping the case of Germany.

Figure 1.12: Trends in Ideological Polarization by Country



Note: The slope estimates are based on Bayesian hierarchical partial-pooling models and also depicted by the dashed lines. The 95% credible intervals are reported in the parentheses.

that E_c produces a more consistent linear pattern of polarization for each country, reflecting in that dots in the figure closely center around the line. This further demonstrates the construct validity of E_c as a measure of aggregate polarization since a country's polarization at the aggregate level should show a certain level of consistency and pattern over time rather than random fluctuation. Each plot also includes an estimated linear time trend and reports the associated average slope coefficient and individual slope for each country, where the variance measure of polarization indicates a substantially greater increasing trend than the other two (see Online Appendix A.4 and A.5). The other two measures are aligned with detailed single in-depth single-country studies using alternative country-specific measures (see: Lelkes, 2016; Merkley, 2021), which suggest there is little evidence that indicates the mass public is in these types of countries is significantly more ideologically polarized than 30 years ago.

This example further demonstrates the validity and consistency of the entropy-based measure: The E_c provides a more reasonable depiction here of ideological polarization and its connection to affective polarization than the variance, according to both the original data distribution and previous findings. The reason why the bimodality coefficient is similar to the E_c in this example is that cross-country comparisons provide more distributional variation with regards to modality. The alternative proxy statistics of polarization only capture some components of the dispersion and distribution in the ordinal data, which partially describes polarization. Thus the use of E_c is even more important for comparative studies as countries can present an more heterogeneous set of distributional patterns of preferences and opinions.

1.5 Conclusion

Despite the increasing prevalence of political polarization around the world, studies of this phenomenon have been limited by the lack of a consistent and tested measure of it. In this chapter, we introduce a nonparametric, entropy-based method for measuring mass political polarization that is completely new to the literature. We demonstrate here that the proposed measure is theoretically and conceptually more appropriate for the intuition and structure of polarization, and further, it measures this phenomenon in a way that does not rely on the confusing distinction between dispersion and bimodality typically used in this literature. Unlike these previous methods, our measure exploits the structure of ordinal variables in public opinion surveys such that polarization is revealed in a novel way where it captures the ordering and distribution of the data at the same time. The new measure makes no a priori assumptions about the central tendency, spacing between categories, specific forms of distributions, and is therefore fully nonparametric. The hypothetical illustrations, the simulation analysis, and the crowd-sourcing validation exercise all demonstrate that our measure is able to reliably reveal the nuanced and complicated dynamics of polarization with different types of empirical distributions. We also apply the measure to three different examples to demonstrate the utility of the entropy approach with real data.

The proposed measure also has theoretical and empirical implications. Current studies of polarization mostly focus on single cases, which rely on predefined political and social contexts usually by nation. This does not answer the big questions: *why are some countries increasingly polarized, and how are political systems being stressed by polarization.* Chapter 3 applies the proposed measure in the context of studying the relationship between inequality, information, and mass polarization. Moreover, empirically, there is another layer of its connections to affective and group-based polarization. To investigate such topics requires a reliable measure of mass polarization that can be applied to cross spatial contexts. We anticipate our measure can be used to promote more research in this area and help answer important political questions about polarization.

For further avenues of research, there are a few limitations or extensions of the current study. First, although we have demonstrated the advantages of the proposed measure in a variety of settings, since there is no ground truth to define polarization, additional analyses with read-world data can be helpful to further demonstrate the validity and reliability of the measure, especially to identify the situations where the proposed measure is superior or similar to other existing metrics. Second, scholars in public opinion are often interested in using multiple issues or survey items to construct measures of important concepts or measure multidimensional opinions. The current measure focuses on unidimensional polarization. Future studies may explore how the proposed measure can be applied to a multidimensional setting. Moreover, as noted above, we consider mass political polarization as an *aggregate* phenomenon. Therefore, how to define a multidimensional space at the aggregate level also requires further theoretical and methodological exploration. Finally, this chapter focuses primarily on the measurement issue and the nonparametric features of the measure with regard to a broader context of issue-based polarization without pre-defining specific settings. But eventually, we want to apply this measure in the broader analytic context. Therefore, how this entropy-based measure can be tightly combined with analytic methods such as different families of regression models and how we may include parametric or semiparametric choices to further make the measure suitable for particular settings can also be interesting future avenues to extend the current study.

In a much broader sense, the proposed measure should also be particularly useful for studies of aggregate political behavior and public opinion, as mass polarization itself is an aggregate political behavior. As suggested in theories, polarization can be the source, consequence, or intermediate mechanism of many macro-level political phenomenon, which poses another methodological challenge—how can we make inferences of macro-level political behavior. Next, we turn to a more analytic context of studying aggregate political behavior before applying the entropy-based measure to study the relationship between economic inequality, information, and mass polarization around the world.

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APPENDIX

A APPENDIX TO CHAPTER 1

A.1 Additional Details about Simulation

Simulation Setup

The simulation setup involves two major steps. We first use Gaussian mixture distributions to simulate continuous data with predefined number of clusters. This way we can define the "truth" of polarization as the distance between the means of two Gaussian distributions. The process can be described as follows:

$$
\begin{aligned}
x_i \sim & F(x) \\
F(x) = & \sum_{j=1}^{K} \lambda_j \mathcal{N}(x|\mu_j, \sigma_j)
\end{aligned}
\tag{1}
$$

where $x_1, x_2, ..., x_n$ is an array of n values drawing from a Gaussian mixture distribution. The Gaussian mixture has a total cluster of K ($K = 2$ for most of the time except for the tri-modal distribution) and different means and variances for subcomponents. The polarization is thus computed as the difference between the means of the two Gaussians. We start with a basic configuration with two clusters with equal variance followed by other four different settings (see Figure 1.4) to investigate how different shapes of distributions, such as trimodal and unequal standard deviation and size, affect the performance of the measures.

The second step is to cluster the continuous data into multiple categories. It is more challenging to cluster one-dimensional data as it generally conveys less information. This is also well-known that this leads to an *NP-hard* problem in a Euclidean space, even when the number of clusters k is 2 (Aloise et al., 2009). Therefore, the conventional heuristic k-means clustering tends to heavily rely on the initial cluster centers, producing neither optimal nor repeatable results. In this case, we use a modified optimal k-means algorithm with dynamic programming to classifying the continuous data into ordinal data (Wang and Song, 2011). In other words, the goal here is to assign elements of $x_1, x_2, ..., x_n$ into k clusters (k here corresponds to the eventual clusters of ordinal responses, and thus, $k \neq K$). This is done by first sorting the data vector, $x_1, x_2, ..., x_n$, by non-descending order. Then, the optimized algorithm will calculate the minimum within-cluster sum of square iteratively for the recurrent substructure $\mathbf{D}[i, m]$ from the original problem $\mathbf{D}[n, k]$:

$$\mathbf{D}[i, m] = \min_{m \leq c \leq i} \{\mathbf{D}[c-1, m-1] + d(x_c, ..., x_i)\} \quad (1 \leq i \leq n, 1 \leq m \leq k) \tag{2}$$

where c is the index of the smallest value in cluster m that guarantees the optimal within-cluster sum of square of $\mathbf{D}[i, m]$ and $d(x_c, ..., x_i)$ is the sum of squared distances from $x_c, ..., x_i$ to their mean. By this recurrence and the final minimum within-cluster sum of square $\mathbf{D}[n,k]$, we can obtain the new clustered data, $x'_1, x'_2, ..., x'_n$ ($x' \in 1, 2, ..., k$). Then, we can compute the proportions of each category and the eventual polarization metrics. It is important to note that k-means cluster estimation comes with strong assumptions that are often ignored but fit here because we control the structure of the simulation (the number of clusters is known in advance, the clusters are spatially grouped, the clusters are of similar size, and the within-cluster variance is the same).

Additional Simulation Results with Different Numbers of Categories

A.2 Crowd-Sourcing Validation

We use a "wisdom-of-the-crowd" validation approach (Barberá et al., 2021; Lyon and Pacuit, 2013; Sumner, Farris and Holman, 2020) to further compare the metrics of polarization with more intuitive human judgments. The goal of this crowd-sourcing approach is to solicit human judgments of different scenarios of polarization, aggregate this information based on collective opinions from multiple individuals and multiple rounds of evaluations, and finally, benchmark different metrics of polarization against the crowd assessment.

Validation Tasks and Online Experiment

We designed an online validation experiment in which respondents compare polarization of different scenarios. The design can be described as follows:

First, we generate ordinal data based on Dirichlet distributions:

$$(p_1, p_2, ...p_k) \sim Dir(\alpha) \; (k = 7, \; \alpha > 0) \tag{3}$$

where, $p_1, p_2, ...p_k$ are proportions for a ordinal variable with seven categories ($k = 7$). Then, using these proportions, we can formulate validation tasks as an evaluation of which scenario is more polarized. Study participants were recruited through Amazon Mechanical Turk ($N = 250$). For each task, each respondent was presented with a pair of barplots as in Figure A.1 with a contrived context of ideology or issue opinion and asked to choose a more polarized scenario according to the barplots of ordinal distribution. Each respondent was asked to evaluate eight tasks in addition to one baseline task (Figure A.2) that simply tests whether the respondents understand the most basic meaning of polarization.

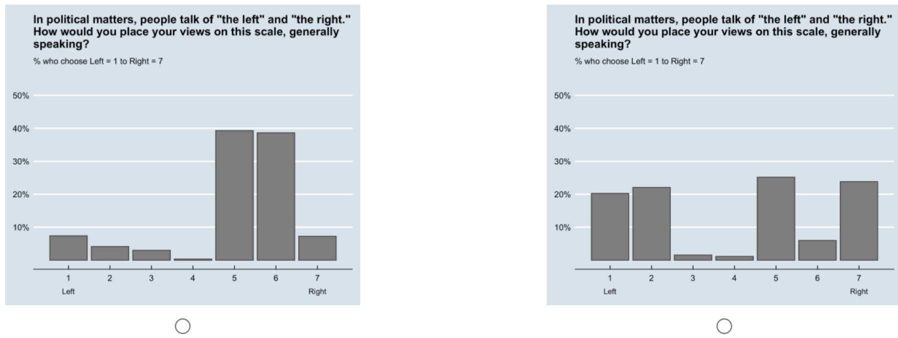
The Calculation of Agreement Rates

To analyze the crowd-sourcing data, we calculate the agreement rates between the metrics of polarization (variance, bimodality coefficient, entropy-based measure) and crowd-sourcing evaluations. Specifically, for respondents $i = 1, ..., n$, they evaluated $j = 1, ..., m$ tasks, we can determine the agreement rates as:

$$\begin{aligned}
X_{ij}^{E_c} &= \mathbb{1}(Y_{ij} = Y_{ij}^{E_c}) \\
X_{ij}^{Var} &= \mathbb{1}(Y_{ij} = Y_{ij}^{Var}) \\
X_{ij}^{Bimod} &= \mathbb{1}(Y_{ij} = Y_{ij}^{Bimod})
\end{aligned} \tag{4}$$

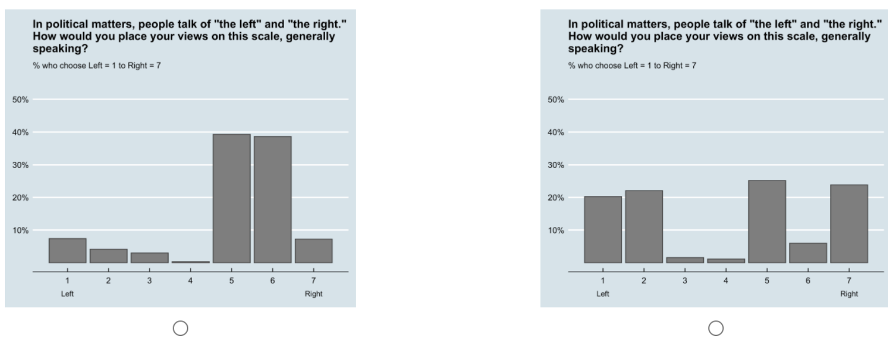
Figure A.1: Sample Crowd-Sourcing Validation Tasks

Polarization refers to the extent to which people's preferences and opinions are opposed. Please see the two figures below that are based on survey data about *people's self-placed ideology* and **choose the one that you think is more polarized**.



(a) Ideology Context

Polarization refers to the extent to which people's preferences and opinions are opposed. Please see the two figures below that are based on survey data about *people's preferences on an important government policy* and **choose the one that you think is more polarized**.

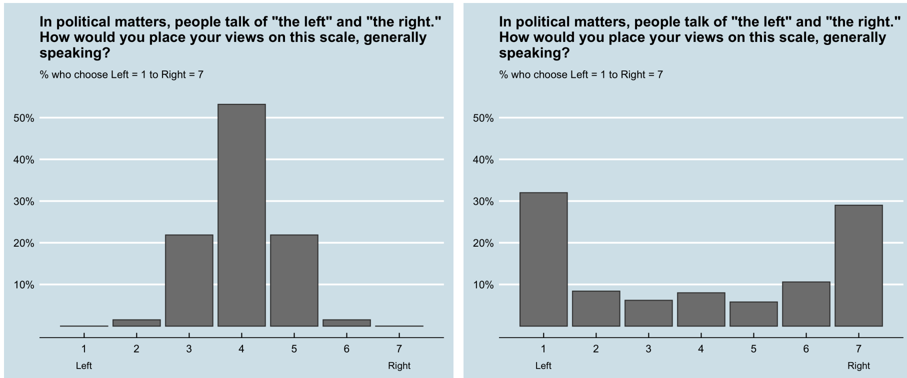


(b) Issue Opinion Context

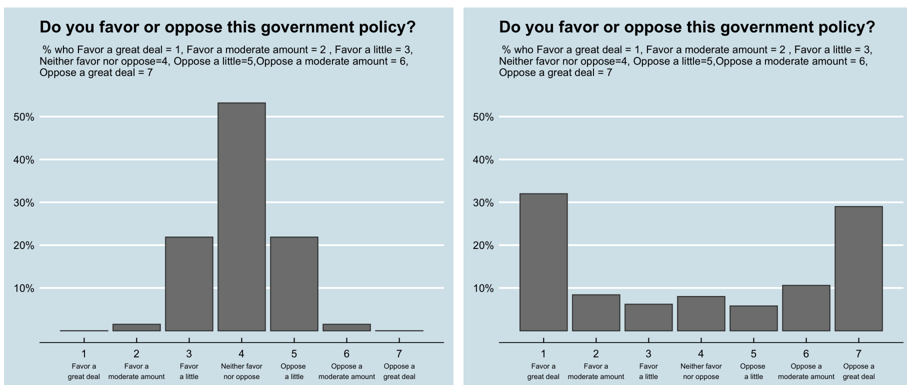
Note: The plots are the screen shots taken from the online experiment. They are intentionally created to look like charts from some popular magazine.

where X is an indicator of agreement for respondent i and task j. When the metric and the response agrees on which one is more polarized, $X_{ij} = 1$, otherwise, 0. Then, we compute

Figure A.2: Baseline Crowd-Sourcing Tasks



(a) Ideology Context



(b) Issue Opinion Context

and compare the overall agreement rates by the average agreement rate for each metric:

$$\widehat{E_c} = \frac{1}{nm}\sum_{i=1}^{n}\sum_{j=1}^{m} X_{ij}^{E_c}$$

$$\widehat{\text{Var}} = \frac{1}{nm}\sum_{i=1}^{n}\sum_{j=1}^{m} X_{ij}^{\text{Var}}$$

$$\widehat{\text{Bimod}} = \frac{1}{nm}\sum_{i=1}^{n}\sum_{j=1}^{m} X_{ij}^{\text{Bimod}} \tag{5}$$

A.3 Mass Polarization in the U.S.

Data Source and Survey Instruments

Data for the example of mass polarization in the U.S. comes from the American National Election Studies (ANES). We combined the 1948-2016 cumulative data with the newly released 2020 Time Series Study. The data is based on work supported by the National Science Foundation under grant numbers SES 1444721, 2014-2017, the University of Michigan, and Stanford University.

We use seven survey instruments in this study: party identification (7-point), liberal-conservative scale (7-point), government services-spending (7-point), opinions on abortion (4-point), aid to blacks (7-point), protect homosexuals against discrimination (4-point), opinions on immigration (6-point).

Detailed Description of Each Item

To show the dynamics of responses behind the polarization measures, in Figure A.3 we show the detailed barplots for all the ANES items that we used in Section 1.4.1. In a more nuanced way, they show how our entropy-based measure of polarization can pick up the subtle dynamics that are neglected and wrongly captured by other measures.

For the partisanship and ideology items, the barplots of distribution show that the entropy-based measure of polarization is able to capture both the long-term trend and shortterm nuances of polarization. The barplots of abortion opinions also demonstrate that E_c is able to reflect the relatively stable trend of the distributions of responses since 1980s and the recent slight decreasing polarization as more people move to the pro-choice side. For the aid to minority item, the barplots show the decreasing trend of polarization from 1970s to early 2000s, and the increasing polarization in the most recent decade, which are also reflected in the metrics of polarization. The distributions of gay rights in the barplots clearly show the once relatively polarized issue has been forming unprecedented agreement as the majority of the people becomes on the same page of supporting for protecting gay rights. As demonstrated in the barplots, all the metrics of polarization are able to capture the dynamics of government spending. For the immigration issue, the barplots show an increasing trend of polarization from early 2000s to 2016 as more people became neutral or pro-immigration and a decrease in 2020 as it forms a uniform distribution, which is only picked up by the entropy-based measure.

References for the analysis of mass polarization in the U.S.

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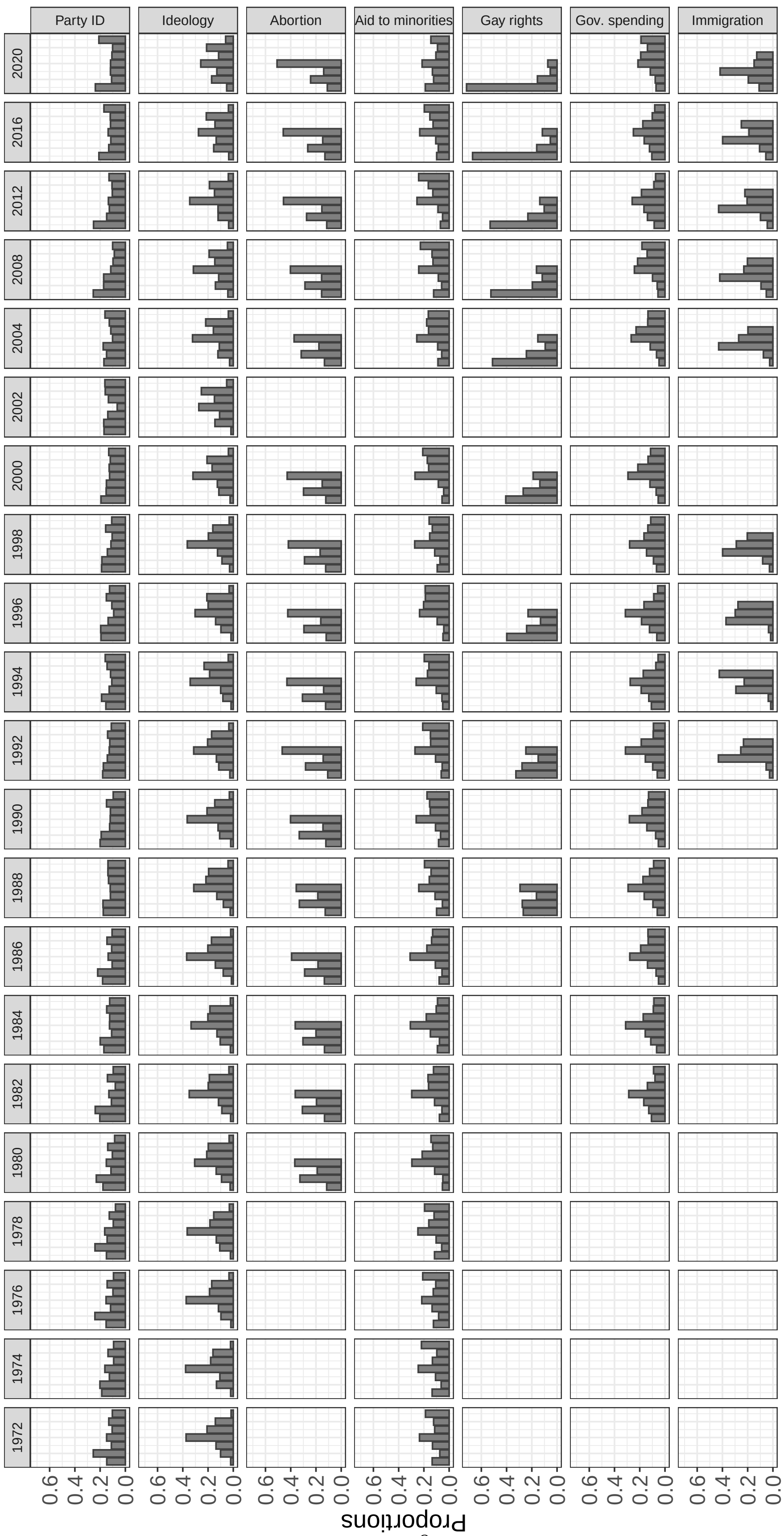
American National Election Studies, University of Michigan, & Stanford University. 2021. 2020 Time Series Study [data file and codebook]. https://electionstudies.org/data-center/2020-time-series-study/

Figure A.3: Mass Polarization in the U.S. (Bar Plots)



The barplots of the responses across years (1972-2020). The years in which the questions weren't asked are omitted. The codings all start with one but different questions have different numbers of categories.

A.4 Radical Parties and Mass Polarization in Europe

Data Source

The data for the analysis of radical parties and mass polarization in Europe comes from the replication materials of the original study by Bischof and Wagner (2019). All variables and coding are the same as the original study except for the additional measure on the ideology item.

Additional Analysis

In Bischof and Wagner (2019), the authors also identified that electoral thresholds may make entrances of new parties more difficult, and thus, give the rise of radical parties additional impact on polarization. They report the results for both the entire sample and countries with electoral threshold, and identify that the entrance of radial-right parties have even larger effects on mass polarization for countries with threshol. In the main text, due to limited space, we only report the results for the whole sample. Table A.1 shows the comparison of E_c and variance estimates for the country sample with electoral thresholds. In line with the original study, we also find the slightly larger estimates for the models with E_c as the outcome. We again confirm the entropy-based measure results in smaller standard errors, suggesting an improvement in the efficiency of the estimation.

	E_c			Variance		
	(1)	**(2)**	**(3)**	**(1)**	**(2)**	**(3)**
Radical-right enter	0.036	0.054	0.057	0.126	0.161	0.174
	(0.013)	(0.013)	(0.015)	(0.053)	(0.045)	(0.048)
GDP growth			−0.001			−0.005
			(0.001)			(0.005)
Unemployment (t-1)			−0.001			−0.003
			(0.001)			(0.005)
Party system polarization (t-1)			−0.0002			−0.001
			(0.001)			(0.004)
Party system fragmentationt (t-1)			−0.006			−0.022
			(0.005)			(0.017)
Constant	0.429	0.522	0.552	2.088	2.505	2.610
	(0.009)	(0.028)	(0.038)	(0.036)	(0.095)	(0.139)
N	253	253	243	253	253	243
R-squared	0.078	0.582	0.608	0.061	0.646	0.670

Table A.1: OLS estimates for the effects of radical party entrance on polarization. The table reports the results for only countries with electoral threshold as defined in the original study. Standard errors are clustered by country/election.

References for the analysis of radical parties and mass polarization in Europe

Bischof, Daniel and Markus Wagner. 2019. ”Replication Data for: Do Voters Polarize When Radical Parties Enter Parliament?”. https://doi.org/10.7910/DVN/DZ1NFG. *Harvard Dataverse*. V3.

Bischof, Daniel and Markus Wagner. 2019. “Do Voters Polarize When Radical Parties Enter Parliament?” *American Journal of Political Science* 63(4) :888–904

A.5 Cross-Country Trends in Ideological and Affective Polarization

Data Sources and Survey Instruments

The data for cross-country trends of affective polarization shared by Boxell, Gentzkow and Shapiro (forthcoming)[12]. The data for cross-country trends of ideological polarization is complied by the authors. The question is generally worded as "in politics people sometimes talk of left and right. Where would you place yourself on this scale?" with some small variations between surveys. The data sources and scales is described in Table A.2:

Trend Estimation

We follow a similar strategy as in Boxell, Gentzkow and Shapiro (forthcoming) to estimate the linear trend of both affective and ideological polarization. They fitted a bivariate linaer regression with affective polarization as the outcome variable and year as the explanatory variable. Because the sample size is in fact relatively small (N = 149) here[13], we take a further step and fit a Bayesian hierarchical partial-pooling model for both affective and ideological polarization, respectively as follows:

[12] Available at: https://scholar.harvard.edu/files/shapiro/files/data-for-cross-polar.zip

[13] There are 12 countries in total. Germany has the most data available for a total of 27 years from 1990 to 2020. Japan and France both only have four years of data.

Table A.2: Data Sources for Cross-Country Ideological Polarization

Country	**Sources**
Australia	CSES (1996, 2004, 2007, 2013); Australian Election Study (1993, 1998, 2001, 2010, 2016, 2019)
Britain	CSES (1997, 2005, 2015); The British Election Study (1992, 2001, 2010, 2017)
Canada	CSES (1997, 2004, 2008, 2011, 2015); Canadian Election Study (1993, 2000, 2019)
Denmark	CSES (1998, 2001, 2007); Denmark Election Project (1994, 2005, 2011, 2015)
France	CSES (2002, 2007, 2012, 2017)
Germany	CSES (1998, 2002, 2005, 2009, 2013, 2017); Polibarometers (1990, 1991, 1992, 1994, 1995, 1996, 1997, 1999, 2000, 2001, 2003, 2004, 2006, 2007, 2008, 2010, 2011, 2012, 2014, 2015, 2018)
Japan	CSES (2007, 2013); World Value Survey (1996 (1995), 2004 (2005));
New Zealand	CSES (1996, 2002, 2008, 2011, 2014, 2017); World Value Survey (1999 (1998), 2005 (2004))
Norway	CSES (1997, 2001, 2005, 2009, 2013, 2017); European Value Survey (1993 (1996))
Sweden	CSES (1998, 2002, 2006); European Value Survey (1991 (1990)), 1994 (1996), 2010 (2010)
Switzerland	CSES (1999, 2003, 2007, 2011); European Value Survey (1995 (1996))
United States	CSES (1996, 2004, 2008, 2012, 2016, 2020); ANES (1990, 1992, 1994, 1998, 2000)

$$y_i = a_j[i] + \beta_j t_i + \epsilon_i$$

$$\alpha_j \sim \mathcal{N}(\mu_\alpha, \sigma_\alpha)$$

$$\beta_j \sim \mathcal{N}(\mu_\beta, \sigma_\beta)$$

where i refers to the individual country-year case, j refers to the country as grouping unit, and the outcome variables (y_i) are either affective or ideological polarization and predicted by the survey years (t_i). Both intercepts (α_j) and slopes (α_j) are allow to vary in the models, which substantively means countries can have different baselines of polarization as well as different rates of change. Because of the features of the partial-pooling model, this allows us to draw strength from the overall trends for those cases with scarce data—or in other words, for those cases, their estimates of trend will shrink towards the means. Overall, the hierarchical model allows us to draw inferences using both between- and within-country variations while still accounting for country-specific patterns and trends of polarization.

Descriptive Trends of Ideological Polarization

Figure A.4: Distributions of Ideology Item for Each Country

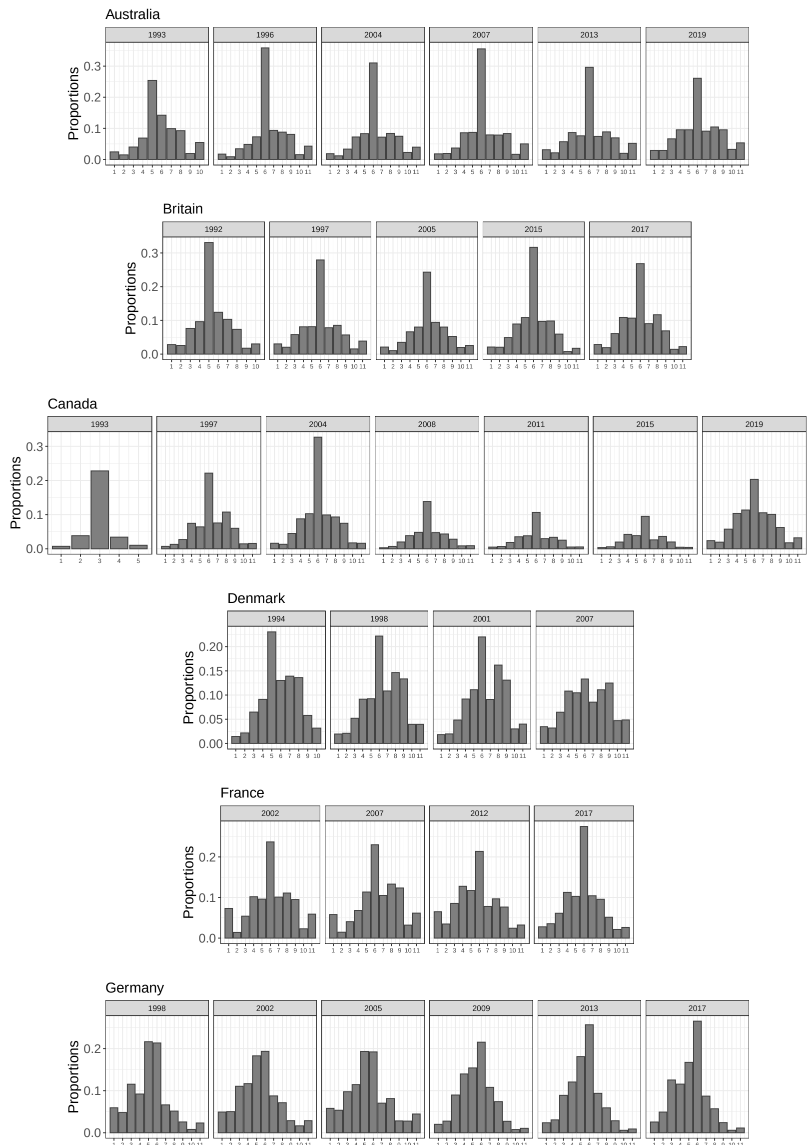
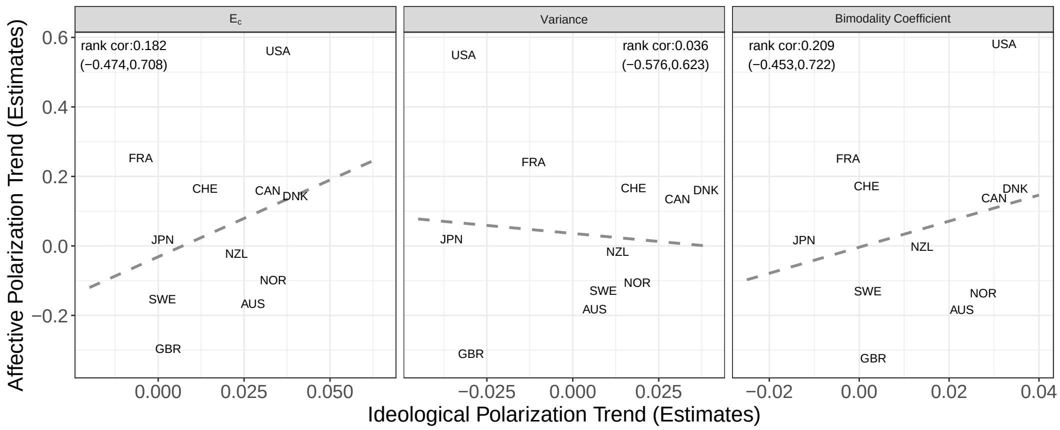


Figure A.5: Distributions of Ideology Item for Each Country (cont'd)

Robustness Check for Trends between Affective and Ideological Polarization

Figure A.6: Trends in Affective and Ideological Polarization (without Germany)



Note: Both axis variables are the estimates of linear trend based on Bayesian hierarchical partial-pooling models. The dashed line displays a fitted bivariate linear regression. The correlation estimates are from the Spearman rank correlation and 95% confidence intervals are reported in the parenthesis. Germany is not included in this analysis.

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