TP SY09

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1 Justification

$$d_M^2(x,y) = (x_y)^T M(x-y) \tag{1}$$

$$J(\{v_k, M_k\}_{k=1,\dots,K}) = \sum_{k=1}^{K}\sum_{i=1}^{n} z_{ik} d_{ik}^2 \tag{2}$$

$$v_k = \bar{x_k} = \frac{1}{n_k}\sum_{i=1}^{n} z_{ik} x_i \tag{3}$$

$$M_k^{-1} = (\rho_k det V_k)^{\frac{-1}{p}} V_k, avec V_k = \frac{1}{n_k}\sum_{i=1}^{n} z_{ik}(x_i - v_k)(x_i - v_k)^T \tag{4}$$

1.1 Question 10

$$\begin{aligned}
\frac{\partial \mathcal{L}(v_k, M_k, \lambda_k)}{\partial v_k} &= \frac{\partial J(v_k, M_k)}{\partial v_k} - \frac{\partial(\sum_{k=1}^{K}\lambda_k(det(M_k) - \rho_k))}{\partial v_k} \\
&= \frac{\partial(\sum_{k=1}^{K}\sum_{i=1}^{n} z_{ik} d_{ik}^2)}{\partial v_k} - \sum_{k=1}^{K}\frac{\partial(\lambda_k(det(M_k) - \rho_k))}{\partial v_k} \\
&= \sum_{k=1}^{K}\sum_{i=1}^{n}\frac{\partial(z_{ik} d_{M_k}^2(x_i, v_k))}{\partial v_k} - 0 \\
&= \sum_{k=1}^{K}\sum_{i=1}^{n} z_{ik}\frac{\partial((x_i - v_k)^T M_k (x_i - v_k))}{\partial v_k}
\end{aligned}$$

En utilisant (86) du matrix cook book :

$$= -2\sum_{k=1}^{K}\sum_{i=1}^{n} z_{ik} M_k (x_i - v_k)$$

On cherche ensuite à résoudre :

$$\frac{\partial \mathcal{L}(v_k, M_k, \lambda_k)}{\partial v_k} = 0$$

$$\iff -2\sum_{k=1}^{K}\sum_{i=1}^{n} z_{ik} M_k (x_i - v_k) = 0$$

$$\iff \sum_{i=1}^{n} z_{ik} M_k (x_i - v_k) = 0 \qquad \forall k$$

$$\iff \sum_{i=1}^{n} z_{ik} (x_i - v_k) = 0 \qquad \forall k$$

$$\iff \sum_{i=1}^{n} z_{ik}.x_i = v_k.\sum_{i=1}^{n} z_{ik} \qquad \forall k$$

$$\iff \sum_{i=1}^{n} z_{ik}.x_i = v_k.n_k \qquad \forall k$$

$$\iff v_k = \frac{1}{n_k}\sum_{i=1}^{n} z_{ik}.x_i \qquad \forall k$$

On retombe bien sur l'équation (3).

1.2 Question 11

$$\frac{\partial \mathcal{L}(v_k, M_k, \lambda_k)}{\partial M_k} = \sum_{k=1}^{K}\sum_{i=1}^{n} z_{ik}\frac{\partial((x_i - v_k)^T M_k (x_i - v_k))}{\partial M_k} - \sum_{k=1}^{K}\lambda_k \frac{\partial(det(M_k) - \rho_k)}{\partial M_k}$$

En utilisant (72) et (49) du matrix cook book:

$$= \sum_{k=1}^{K}\sum_{i=1}^{n}(z_{ik}(x_i - v_k)(x_i - v_k)^T) - \sum_{k=1}^{K}\lambda_k(det(M_k)(M_k^{-1})^T - \frac{\partial \rho_k}{\partial M_k})$$

En posant :

$$V_k = \frac{1}{n_k}\sum_{i=1}^{n} z_{ik}(x_i - v_k)(x_i - v_k)^T$$

On obtient :

$$\frac{\partial \mathcal{L}(v_k, M_k, \lambda_k)}{\partial M_k} = \sum_{k=1}^{K} n_k.V_k - \sum_{k=1}^{K}\lambda_k.det(M_k)(M_k^{-1})^T$$

On cherche à résoudre :

$$\frac{\partial \mathcal{L}(v_k, M_k, \lambda_k)}{\partial M_k} = 0$$

$$\iff n_k.V_k = \lambda_k.det(M_k)(M_k^{-1})^T \qquad \forall k$$

$$\iff (M_k^{-1})^T = \frac{n_k}{\lambda_k.det(M_k)}.V_k \qquad \forall k$$

$$\iff (M_k^{-1})^T = \frac{n_k}{\lambda_k.det(M_k)}.V_k \qquad \forall k$$

$$\frac{\partial \mathcal{L}(v_k, M_k, \lambda_k)}{\partial \lambda_k} = \frac{\partial J(v_k, M_k)}{\partial \lambda_k} - \sum_{k=1}^{K}\frac{\partial(\lambda_k(det(M_k) - \rho_k))}{\partial \lambda_k}$$

$$= 0 - \sum_{k=1}^{K}(det(M_k) - \rho_k)$$

On cherche à résoudre :

$$\frac{\partial \mathcal{L}(v_k, M_k, \lambda_k)}{\partial \lambda_k} = 0$$

$$\iff \sum_{k=1}^{K}(det(M_k) - \rho_k) = 0$$

$$\iff det(M_k) = \rho_k \qquad \forall k$$

En reprenant :

$$\frac{\partial \mathcal{L}(v_k, M_k, \lambda_k)}{\partial M_k} = 0 \iff (M_k^{-1})^T = \frac{n_k}{\lambda_k.det(M_k)}.V_k \qquad \forall k$$

$$\iff (M_k^{-1})^T = \frac{n_k}{\lambda_k.\rho_k}.V_k \qquad \forall k$$

En passant au déterminant :

$$\implies det(M_k^{-1}) = det(\frac{n_k}{\lambda_k.\rho_k}.V_k) \qquad \forall k$$

En utilisant (19) et (22) du matrix cook book :

$$\implies \frac{1}{det(M_k)} = (\frac{n_k}{\lambda_k.\rho_k})^p.det(V_k) \qquad \forall k$$

$$\implies \frac{n_k}{\lambda_k.\rho_k} = (det(M_k).det(V_k))^{-1/p} \qquad \forall k$$

Au final :

$$\frac{\partial \mathcal{L}(v_k, M_k, \lambda_k)}{\partial M_k} = 0 \iff (M_k^{-1})^T = \frac{n_k}{\lambda_k.det(M_k)}.V_k \qquad \forall k$$

$$\iff (M_k^{-1})^T = (\rho_k.det(V_k))^{-1/p}.V_k \qquad \forall k$$

Etant donné que les matrices sont symétriques :

$$(M_k^{-1})^T = M_k^{-1}$$

Donc on retrouve bien l'équation (4).