R Tools for Understanding Credit Risk Modelling

QRM: Concepts, Techniques & Tools

Alexander J. McNeil
Heriot-Watt University, Edinburgh

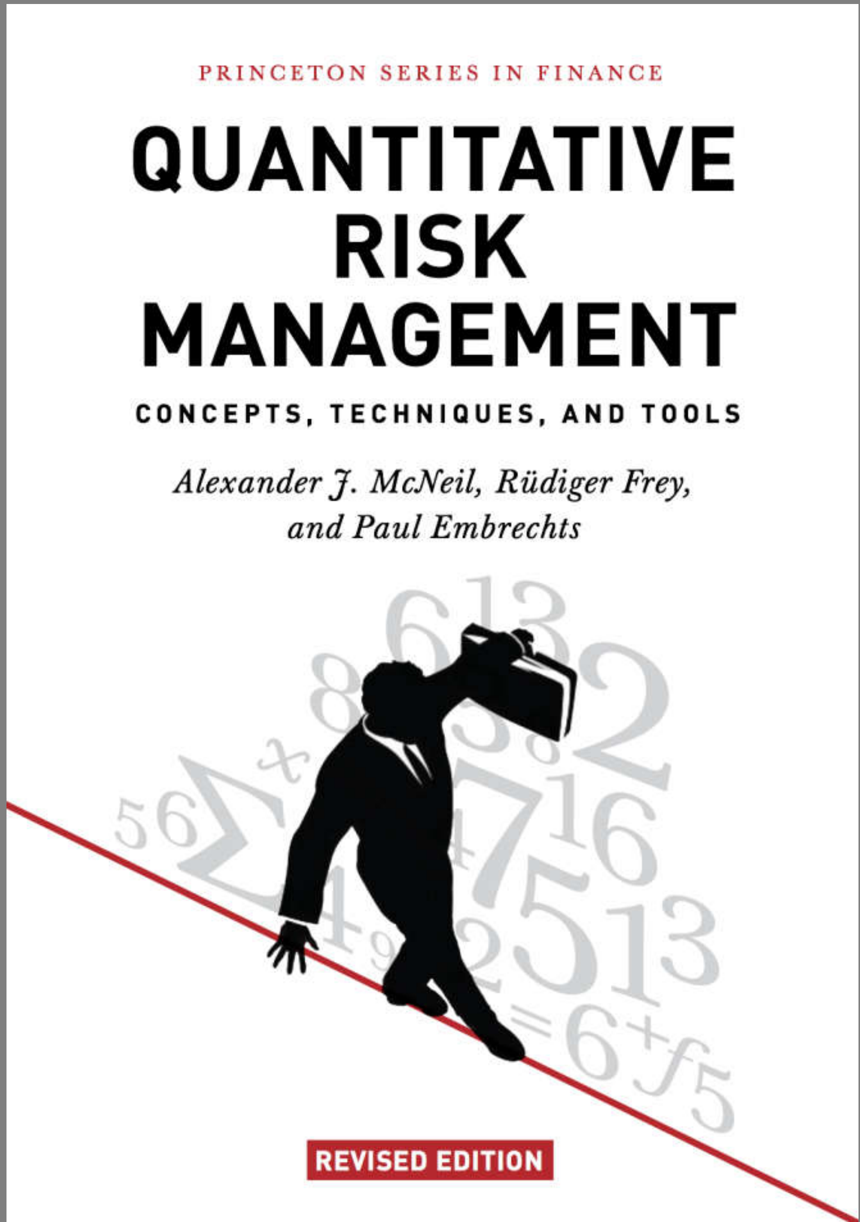
Chicago, 30 May 2015

Overview

1. R Credentials
2. Markov Chains for Rating Migrations
3. Merton’s Model
4. Distance-to-Default Calculations
5. Portfolio Loss Distributions with FFT
6. Estimation of Credit Risk Models from Default Data

Overview

QRM 2nd edition, May 2015



R Credentials

- User of the S language since circa 1990.
- EVIS (1997). Extreme Values in S-Plus. A library that was later integrated into S+FinMetrics.
- QRMlib (2005). S-Plus library to accompany 1st edition of QRM book (McNeil, Frey, and Embrechts, 2005). Originally ported to R by Scott Ulmann; additional input by Jonah Beckford.
- QRM. R package maintained by Bernhard Pfaff based on QRMlib. Additional listed authors: Marius Hofert, AJM and Scott Ulmann.
- www.qrmtutorial.org. Learning support site for the 2nd edition of QRM book (McNeil, Frey, and Embrechts, 2015). R scripts illustrating analyses in book.

Overview

Scores, Ratings & Measures Inferred from Prices

- Broadly speaking, there are two philosophies for quantifying the credit quality or default probability of an obligor.

1. Credit quality can be described by a credit rating or score that is based on empirical data describing the borrowing and repayment history and other characteristics of the obligor, or similar obligors.
2. For obligors whose equity is traded on financial markets, prices can be used to infer the market's view of the credit quality of the obligor.

Rating Migration as a Markov Chain

- Let (R_t) denote a continuous-time stochastic process taking values in the set $S = \{D, C, B3, B2, B1, A3, A2, A1\}$.
- We assume that (R_t) is a Markov chain so that migration probabilities depend only on the current rating.
- However, there is evidence that empirical rating histories show momentum and stickiness (Lando and Skodeberg, 2002).
- Transition probabilities are summarized by a generator matrix $\Lambda = (\lambda_{jk})$. Over any small time step of duration δt the probability of a transition from rating j to k is given approximately by $\lambda_{jk}\delta t$. The probability of staying at rating j is given by $1 - \sum_{k \neq j} \lambda_{jk}\delta t$.
- Let $\boldsymbol{P}(t)$ be the matrix of transition probabilities for the period $[0, t]$.
- Using the so-called matrix exponential we have

$$\boldsymbol{P}(t) = \exp(\Lambda t)\,.$$

Estimating the Generator Matrix

- A Markov chain with generator Λ can be constructed in the following way. An obligor remains in rating state j for an exponentially distributed amount of time with parameter $\lambda = \sum_{k \neq j} \lambda_{jk}$. When a transition takes place the probability that it is from j to state k is given by λ_{jk}/λ.
- This construction leads to natural estimators for the matrix Λ.
- Since λ_{jk} is the instantaneous rate of migrating from j to k we can estimate it by

$$\hat{\lambda}_{jk} = \frac{N_{jk}(T)}{\int_0^T Y_j(t)dt},$$

  where $N_{jk}(T)$ is the total number of observed transitions from j to k over the time period $[0, T]$ and $Y_j(t)$ is the number of obligors with rating j at time t.
- The denominator represents the total time spent in state j by all the companies in the dataset.
- This is in fact the maximum likelihood estimator.

Illustration of Data

We start with 20 years of rating migration data:

```
head(RatingEvents,n=6)

##   id starttime  endtime startrating endrating       time
## 1  1   0.00000  6.16813          B2        B3  6.1681297
## 2  1   6.16813 16.32281          B3         C 10.1546802
## 3  1  16.32281 17.28772           C         D  0.9649059
## 4  2   0.00000 10.10462          A1        A2 10.1046193
## 5  2  10.10462 12.67101          A2        B2  2.5663897
## 6  2  12.67101 15.49927          B2        B3  2.8282614

(Njktable = table(RatingEvents$startrating,RatingEvents$endrating))

##
##        A1   A2   A3   B1   B2   B3    C    D
##   A1  141  405   30   12    2    0    0    0
##   A2   86  504  996   41    5    4    0    0
##   A3   17  402  954 1167   49   22    0    0
##   B1    3   41  781  843 1198  137   13   19
##   B2    2   10   48  744  475 1411   96   39
##   B3    0    9   25   30  555  376 1055  305
##   C     2    0   12    6   32  268   68 1276
```

Implementation of Estimator

```r
# Load library for matrix exponential
require(Matrix)

# compute total time spent in each rating state
RiskSet <- by(RatingEvents$time,RatingEvents$startrating,sum)
Jlevels <- levels(RatingEvents$startrating)
Klevels <- levels(RatingEvents$endrating)
Njmatrix <- matrix(nrow=length(Jlevels),ncol=length(Klevels),
                   as.numeric(RiskSet),byrow=FALSE)

# basic form of estimator
Lambda.hat <- Njktable/Njmatrix

# complete matrix by adding default row and fixing diagonal
D <- rep(0,dim(Lambda.hat)[2])
Lambda.hat <- rbind(Lambda.hat,D)
diag(Lambda.hat) <- D
rowsums <- apply(Lambda.hat,1,sum)
diag(Lambda.hat) <- -rowsums

# compute estimated transition probabilities
P.hat <- expm(Lambda.hat)
```

Estimated Annual Transition Probabilities

Results:

```
P.hat

## 8 x 8 Matrix of class "dgeMatrix"
##              A1          A2          A3          B1           B2
## A1 9.228262e-01 0.066737442 0.007697783 0.002284359 0.0004060488
## A2 6.477260e-03 0.912775249 0.074355393 0.005382232 0.0006087827
## A3 9.292008e-04 0.020157475 0.915501238 0.057602427 0.0042015008
## B1 1.786315e-04 0.002460217 0.038446159 0.890662745 0.0563027502
## B2 1.495989e-04 0.000828126 0.004514951 0.050088183 0.8373087370
## B3 4.643126e-05 0.000826357 0.002585119 0.004107536 0.0463984391
## C  6.825022e-04 0.000131263 0.004260374 0.002677500 0.0129685207
## D  0.000000e+00 0.000000000 0.000000000 0.000000000 0.0000000000
##              B3            C            D
## A1 4.242895e-05 2.956203e-06 2.786187e-06
## A2 3.711982e-04 1.729870e-05 1.258679e-05
## A3 1.449703e-03 8.251465e-05 7.594058e-05
## B1 9.476403e-03 9.786031e-04 1.494491e-03
## B2 9.159274e-02 8.486709e-03 7.030957e-03
## B3 8.292587e-01 6.395807e-02 5.281934e-02
## C  8.600773e-02 4.385426e-01 4.547295e-01
## D  0.000000e+00 0.000000e+00 1.000000e+00
```

Overview

Assumptions in Merton's Model (1974)

- Consider firm with stochastic asset-value (V_t), financing itself by equity (i.e. by issuing shares) and debt.
- Assume that debt consists of single zero coupon bond with face or nominal value B and maturity T.
- Denote by S_t and B_t the value at time $t \leq T$ of equity and debt so that

$$V_t = S_t + B_t, \quad 0 \leq t \leq T.$$

- Assume that default occurs if the firm misses a payment to its debt holders and hence only at T.
- At T we have two possible cases:
  1. $V_T > B$. In that case the debtholders receive B; shareholders receive residual value $S_T = V_T - B$, and there is no default.
  2. $V_T \leq B$. In that case the firm cannot meet its financial obligations, and shareholders hand over control to the bondholders, who liquidate the firm; hence we have $B_T = V_T$, $S_T = 0$.

Equity and Debt as Contingent Claims on Assets

In summary we obtain

$$\begin{aligned} S_T &= (V_T - B)^+ \\ B_T &= \min(V_T, B) = B - (B - V_T)^+ . \end{aligned}$$

- The value of equity at T equals the pay-off of a European call option on V_T with exercise price equal to B.
- The value of the debt at T equals the nominal value of debt minus the pay-off of a European put option on V_T.

Generating Merton Model

It is assumed that asset value (V_t) follows a diffusion of the form

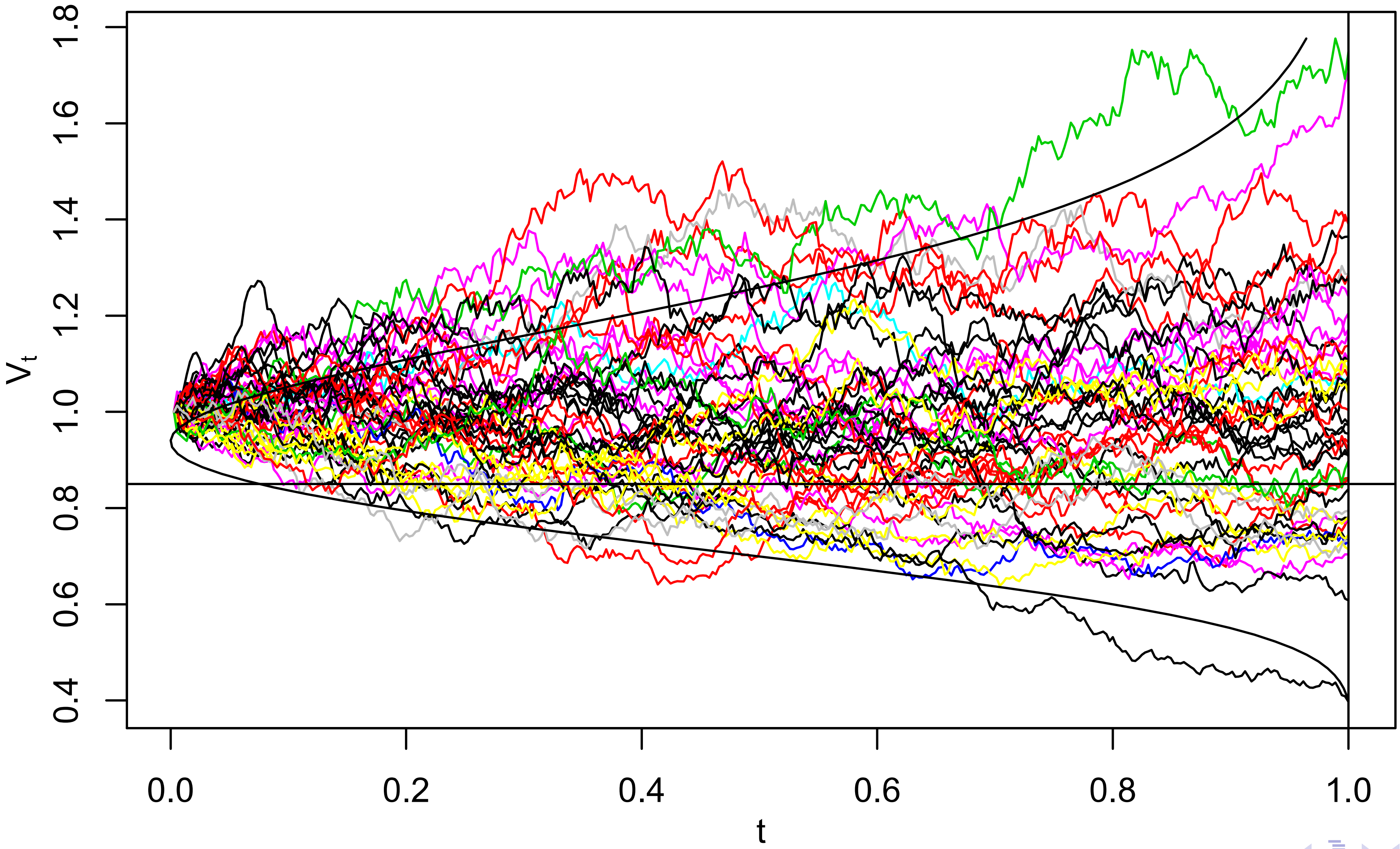
$$dV_t = \mu_V V_t dt + \sigma_V V_t dW_t$$

for constants $\mu_V \in \mathbb{R}$, $\sigma_V > 0$, and a Brownian motion $(W_t)_{t \geq 0}$, so that

$$V_T = V_0 \exp\Big((\mu_V - \frac{1}{2}\sigma_V^2)T + \sigma_V W_T\Big).$$

```
require(sde)
## Parameters for Merton model
V0 <- 1; muV <- 0.03; sigmaV <- 0.25
r <- 0.02; B <- 0.85; T <- 1
N <- 364

## Simulated asset value trajectories for Merton model
npaths <- 50
paths <- matrix(NA,nrow=N+1,ncol=npaths)
for (i in 1:npaths)
{paths[,i] <- GBM(x=V0,r=muV,sigma=sigmaV,T=T,N=N)}
```

Generating Merton Model (cont.)



Pricing of Equity and Debt

- Since equity is a call option on the asset value (V_t) theBlack–Scholes formula yields

$$S_t = C^{\mathrm{BS}}(t, V_t; \sigma_V, r, T, B) := V_t \Phi(d_{t,1}) - B e^{-r(T-t)} \Phi(d_{t,2}),$$

where the arguments are given by

$$d_{t,1} = \frac{\ln \frac{V_t}{B} + (r + \frac{1}{2}\sigma_V^2)(T-t)}{\sigma_V \sqrt{T-t}}, \quad d_{t,2} = d_{t,1} - \sigma_V \sqrt{T-t}.$$

- The price at $t \leq T$ of a default-free zero-coupon bond with maturity T and a face value of one equals

$$p_0(t, T) = \exp(-r(T-t)).$$

- The value of the firm's debt equals the difference between the value of default-free debt and a put option on (V_t) with strike B, i.e.

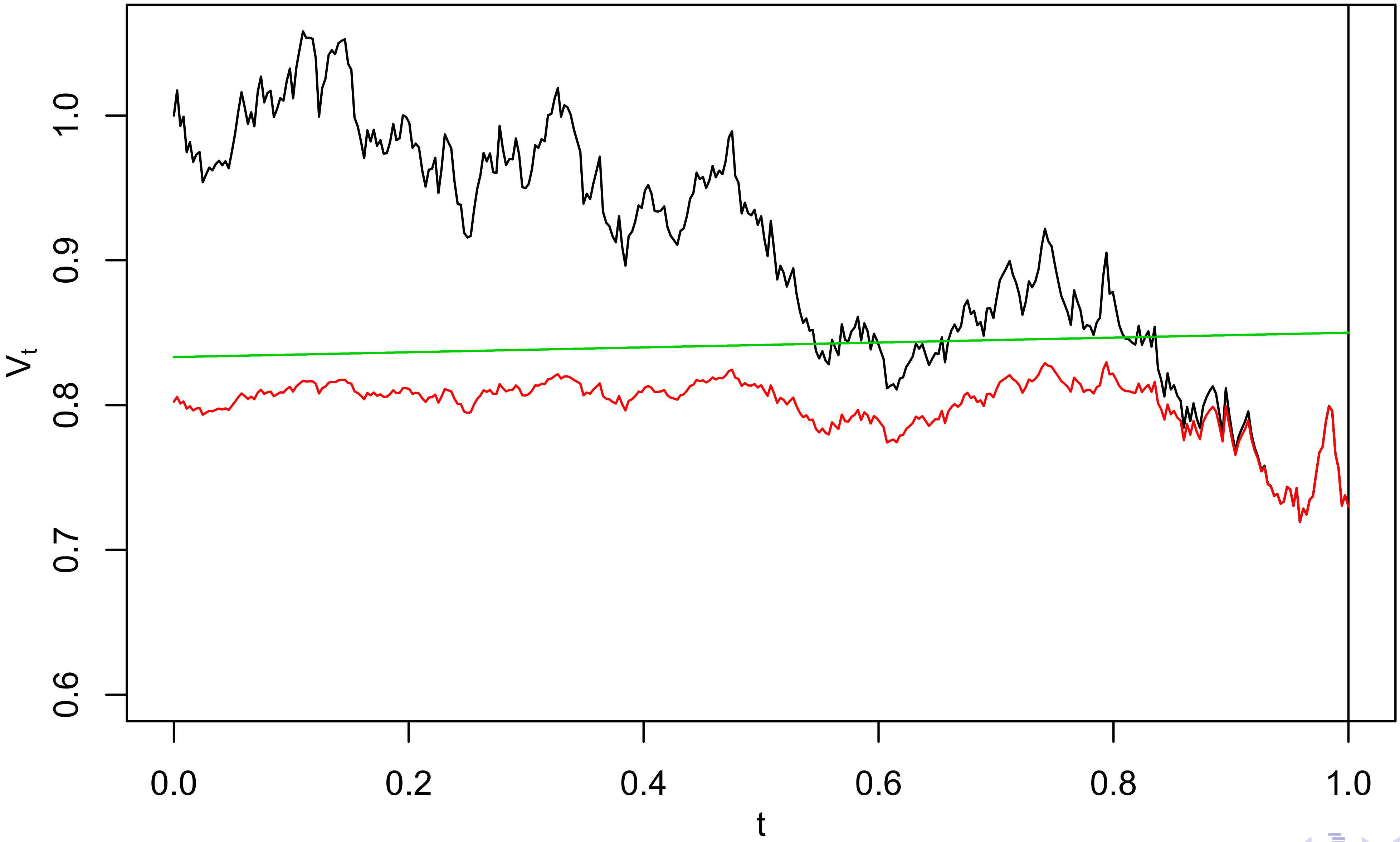
$$\begin{aligned} B_t &= B p_0(t, T) - P^{\mathrm{BS}}(t, V_t; r, \sigma_V, B, T) \\ &= p_0(t, T) B \Phi(d_{t,2}) + V_t \Phi(-d_{t,1}) \end{aligned}$$

Default Path

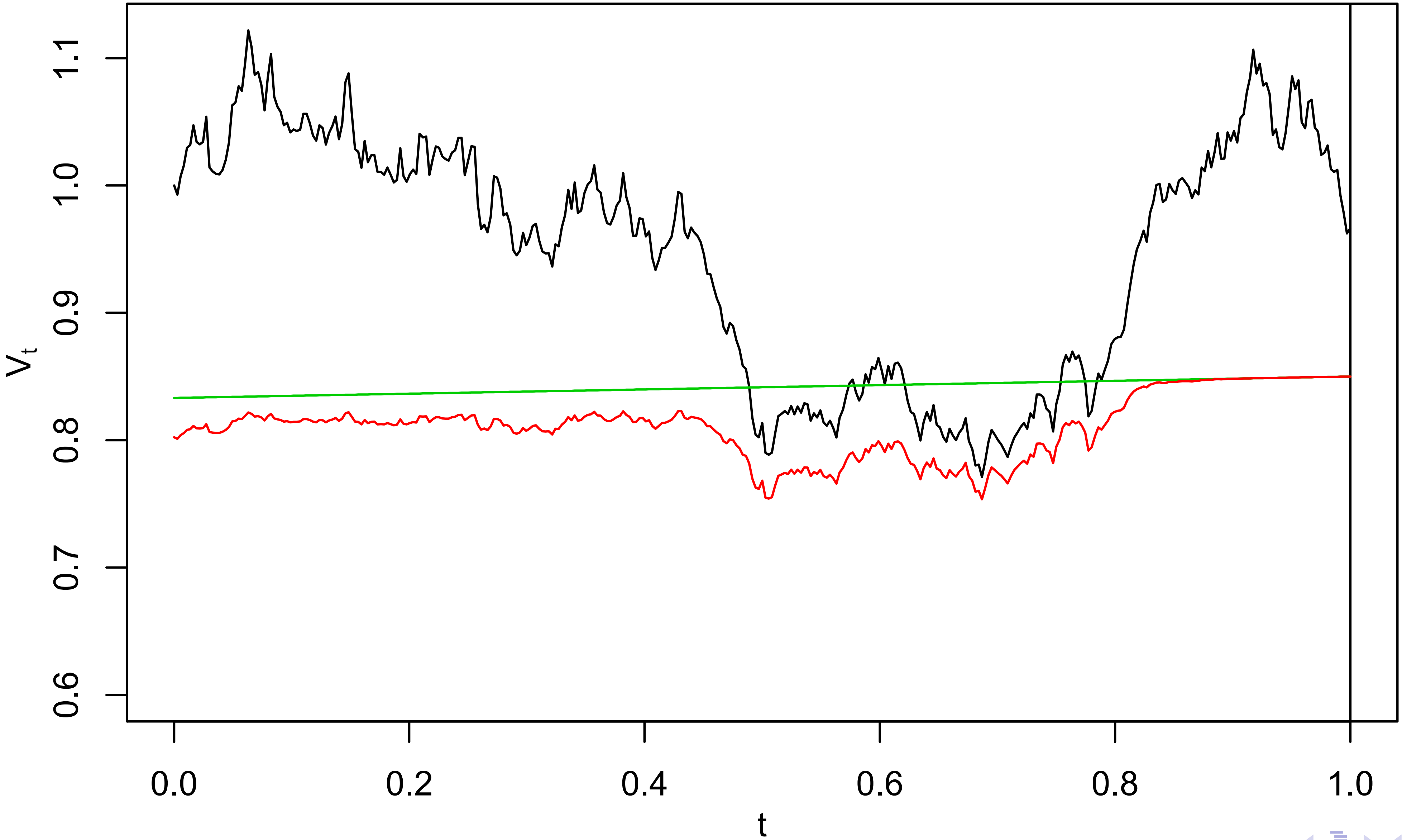
Draw a path of Merton GBM leading to default and superimpose value of default-free and defaultable debt.

```
source("Black_Scholes.R")
set.seed(63)
Vt <- GBM(x=V0,r=muV,sigma=sigmaV,T=T,N=N)

times <- seq(from=0,to=1,length=N+1)
par(mar=c(3,3,2,1),mgp=c(2,1,0))
plot(times,Vt,type="l",ylim=range(0.6,max(Vt)),
      xlab="t",ylab=expression(V[t]))
abline(v=1)

# Add default free debt value as green line
lines(times,B*exp(-r*(T-times)),col=3)

# Add defaultable debt as red line
Bt <- B*exp(-r*(T-times)) -
  BlackScholes(times,Vt,r,sigmaV,B,T,type="put")
lines(times,Bt,col=2)
```

Default Path (cont.)



Non-Default Path



Overview

Industry Adaptations of Merton

- There are a number of industry models that descend from the Merton model.
- An important example is the so-called public-firm EDF model that is maintained by Moody's Analytics.
- The methodology builds on earlier work by KMV (a private company named after its founders Kealhofer, McQuown and Vasicek) in the 1990s, and is also known as the KMV approach.
- Literature: Crosbie and Bohn, 2002 and Sun, Munves, and Hamilton, 2012.
- Expected Default Frequency. The EDF is an estimate of the default probability of a given firm over a one-year horizon.

Moody's Public-Firm EDF Model

- Suppose we use Merton's model for a company issuing debt with face value B maturing at time $T = 1$. The EDF would be

$$\mathrm{EDF}_{Merton} = 1 - \Phi\left(\frac{\ln V_0 - \ln B + (\mu_V - \frac{1}{2}\sigma_V^2)}{\sigma_V}\right).$$

- The public-firm EDF model uses a formula of the form

$$\mathrm{EDF} = g(\mathrm{DD}) = g\left(\frac{\ln V_0 - \ln \tilde{B}}{\sigma_V}\right)$$

where
  - g is an empirically estimated function;
  - the default point $\tilde{B}$ represents the structure of the firm's liabilities more closely;
  - the current asset value V_0 and the asset volatility σ_V are inferred (or 'backed out') from information about the firm's equity value.

Illustration of Public-Firm EDF Approach

Variable	J&J	RadioShack	Notes
Market value of assets V_0	236 bn	1834 m	Option pricing approach.
Asset volatility σ_V	11%	24%	Option pricing approach
Default threshold $\tilde{B}$	39 bn	1042 m	Short-term liabilities and half of long-term liabilities.
DD	16.4	2.3	Given by $(\log V_0 - \log \tilde{B})/\sigma_V$.
EDF (one year)	0.01%	3.58%	Empirical mapping g.

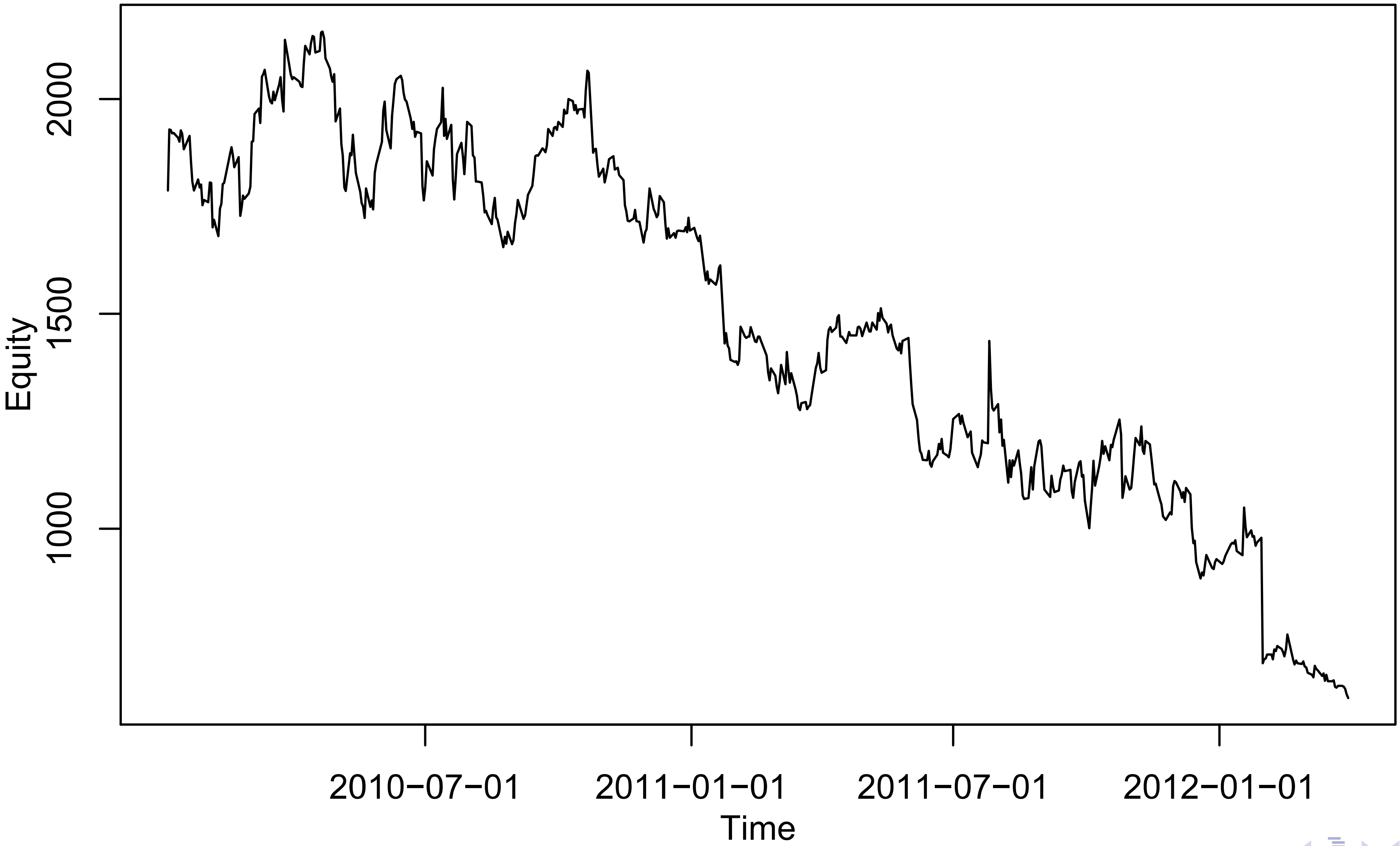
This example is taken from Sun, Munves, and Hamilton, 2012 and concerns the situations of Johnson and Johnson (J&J) and RadioShack as of April 2012. All quantities are in USD.

$$S_t = C^{\mathrm{BS}}(t, V_t; \sigma_V, r, T, \tilde{B})$$

RadioShack Example

```
require(timeSeries)
load("RadioShack.RData")
source("Black_Scholes.R")

# Use 2010 to March 2012 (as in Moody's example)
RadioShack = window(RadioShack, start="2010-01-01", end="2012-03-31")
# Number of shares in millions (approximately)
N.shares = 100
# Value of equity in millions of dollars (approx)
Svalues = RadioShack*N.shares

# Equity vol
sigmaS <- as.numeric(sd(returns(Svalues)))*sqrt(250)

# Value of one-year debt in millions approximately
B = 1042

Vvalues = timeSeries(rep(NA,length(Svalues)),time(Svalues))

par(mar=c(3,3,2,1),mgp=c(2,1,0))
plot(Svalues,ylab="Equity")
```

RadioShack Example (cont.)



Iterative Solution for Asset Process

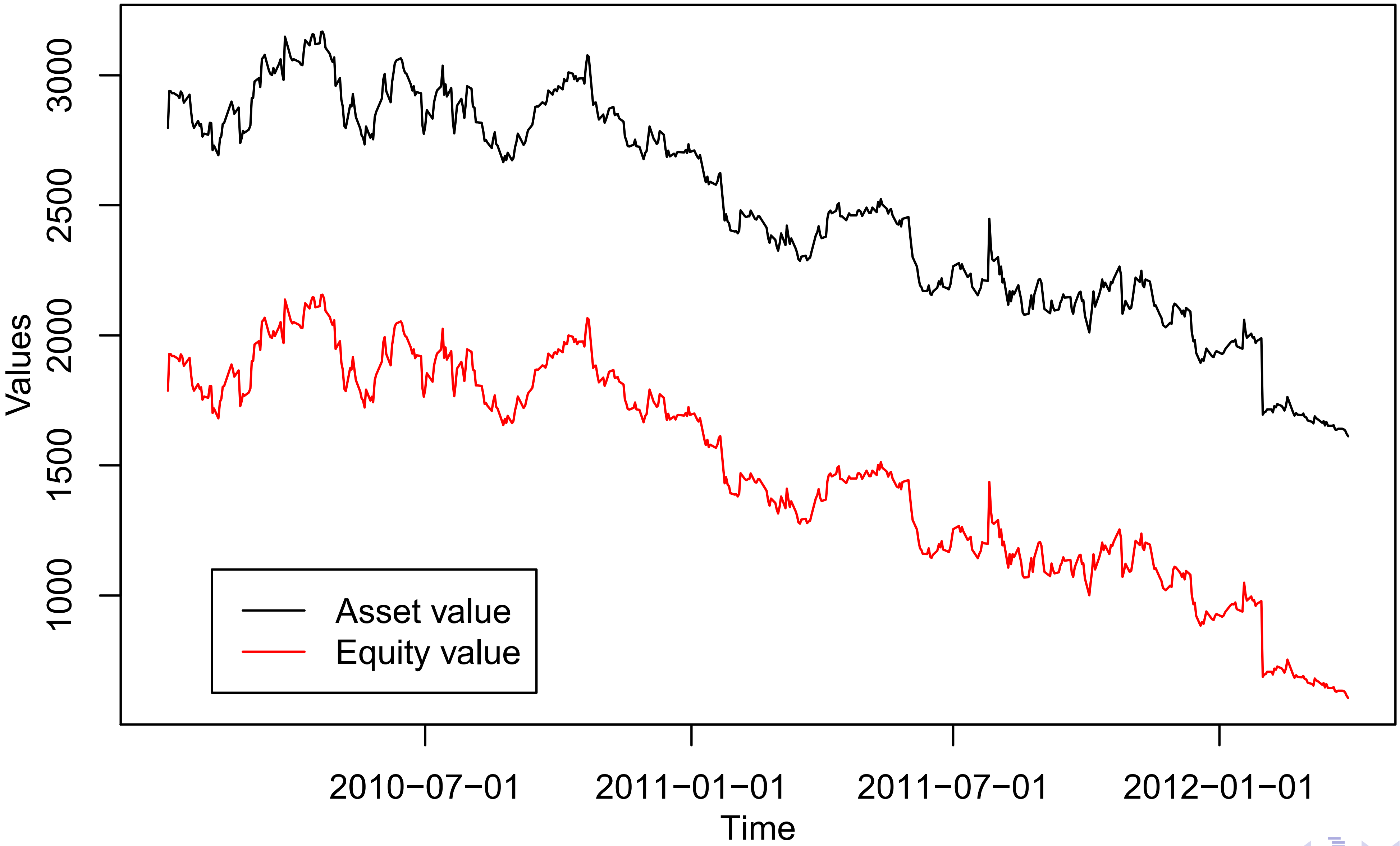
```
rootegn <- function(V,S,t,r,sigmaV,B,T)
{
  S - BlackScholes(t,V,r,sigmaV,B,T,"call")
}

# Initial estimate of volatility
sigmaV <- sigmaS
sigmaV.old <- 0
it = 1

# iterative solution for asset values
while (abs(sigmaV-sigmaV.old)/sigmaV.old > 0.000001)
{
        it = it + 1
for (i in 1:length(Svalues)){
        tmp = uniroot(rooteqn, interval =c(Svalues[i],10*Svalues[i]),
  S=Svalues[i],t=0,r=0.03,sigmaV=sigmaV,B=B,T=1)
        Vvalues[i] = tmp$root
}
sigmaV.old = sigmaV
sigmaV = as.numeric(sd(returns(Vvalues)))*sqrt(250)
}
```

Results

$$\text{EDF} = g(\text{DD}) = g\left(\frac{\ln V_0 - \ln \tilde{B}}{\sigma_V}\right)$$

```
# results
DD <- (log(Vvalues[length(Vvalues)])-log(B))/sigmaV
results<-c(it=it,sigmaS=sigmaS,V0=Vvalues[length(Vvalues)],sigmaV=sigmaV,D
results

##            it        sigmaS            V0        sigmaV            DD
##     6.0000000     0.4764729  1612.2850854     0.2613945     1.6699299

# graphs
par(mar=c(3,3,2,1),mgp=c(2,1,0))
plot(Vvalues,ylim=range(Vvalues,Svalues),ylab="Values")
lines(Svalues,col=2)
legend(x=1265200000,y=1100,legend=c("Asset value","Equity value"),
       lty=c(1,1), col=c(1,2))
```

Results (cont.)



Overview

Gaussian Firm-Value Models for Portfolio

Most industry portfolio models assume default occurs if a standard normal critical variable X_i (with an asset value interpretation) is less than a critical threshold d_i. The dependence between defaults is modelled with a factor model:

$$\begin{aligned} X_i &= \sqrt{\beta_i}\tilde{F}_i + \sqrt{1-\beta_i}\varepsilon_i \\ &= \sqrt{\beta_i}\mathbf{a}_i'\boldsymbol{F} + \sqrt{1-\beta_i}\varepsilon_i \end{aligned}$$

- $\boldsymbol{F} \sim N_p(\mathbf{0}, \Omega)$ is a random vector of normally distributed common economic factors with country-industry interpretations;
- ε_i is a standard normally distributed error, which is independent of $\boldsymbol{F}$ and of ε_j for $j \neq i$;
- $0 < \beta_i < 1$; $\text{var}(\tilde{F}_i) = 1$;
- $\boldsymbol{a}_i$ are factor weights.

Representation as Mixture Model

In this model defaults are conditionally independent Bernoulli events given $\boldsymbol{F}$ (or equivalently $\boldsymbol{\Psi} = -\boldsymbol{F}$) the systematic factors.

The conditional independence of defaults given $\boldsymbol{F}$ follows immediately from the independence of the idiosyncratic terms $\varepsilon_1, \ldots, \varepsilon_m$. The conditional default probabilities are given by

$$p_i(\psi) = \Phi\left(\frac{\Phi^{-1}(p_i) + \sqrt{\beta_i}\boldsymbol{a}_i'\psi}{\sqrt{1-\beta_i}}\right),$$

where p_i is the default probability, $\boldsymbol{a}_i$ are the factor weights and β_i is the systematic risk component for obligor i.

Computational Advantages

- Let e_i denote the exposure, δ_i the loss given default and Y_i the default indicator of obligor i.
- It is difficult to compute the df F_L of the portfolio loss $L = \sum_{i=1}^{m} e_i \delta_i Y_i$.
- However, it is easy to use the conditional independence of the defaults to show that the Laplace–Stieltjes transform of F_L is given by

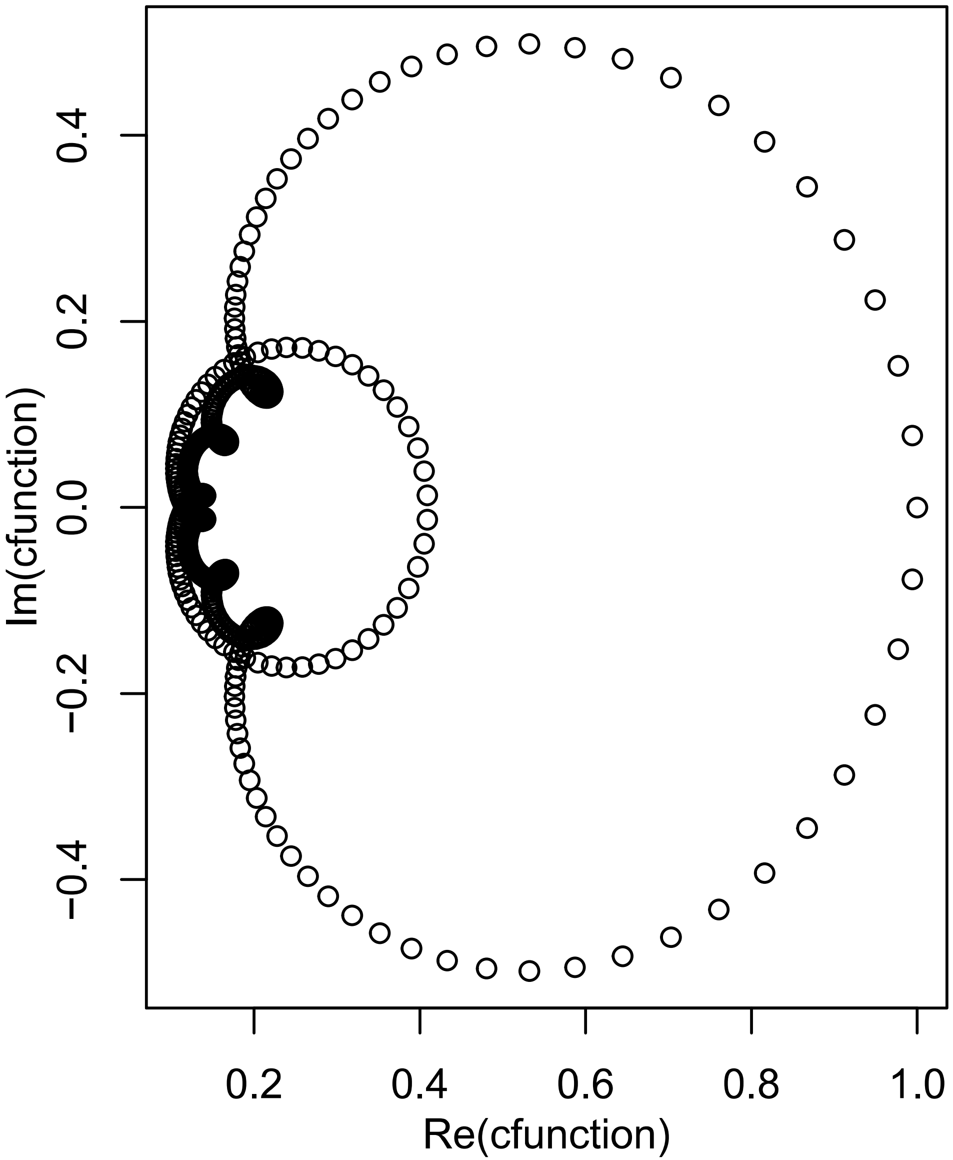
$$\begin{aligned} \hat{F}_L(t) = E(e^{-tL}) &= E\left(E(e^{-t\sum_{i=1}^{m} e_i \delta_i Y_i} \mid \boldsymbol{\Psi})\right) \\ &= E\left(\prod_{i=1}^{m} \left(p_i(\boldsymbol{\Psi}) e^{-te_i\delta_i} + 1 - p_i(\boldsymbol{\Psi})\right)\right) \end{aligned}$$

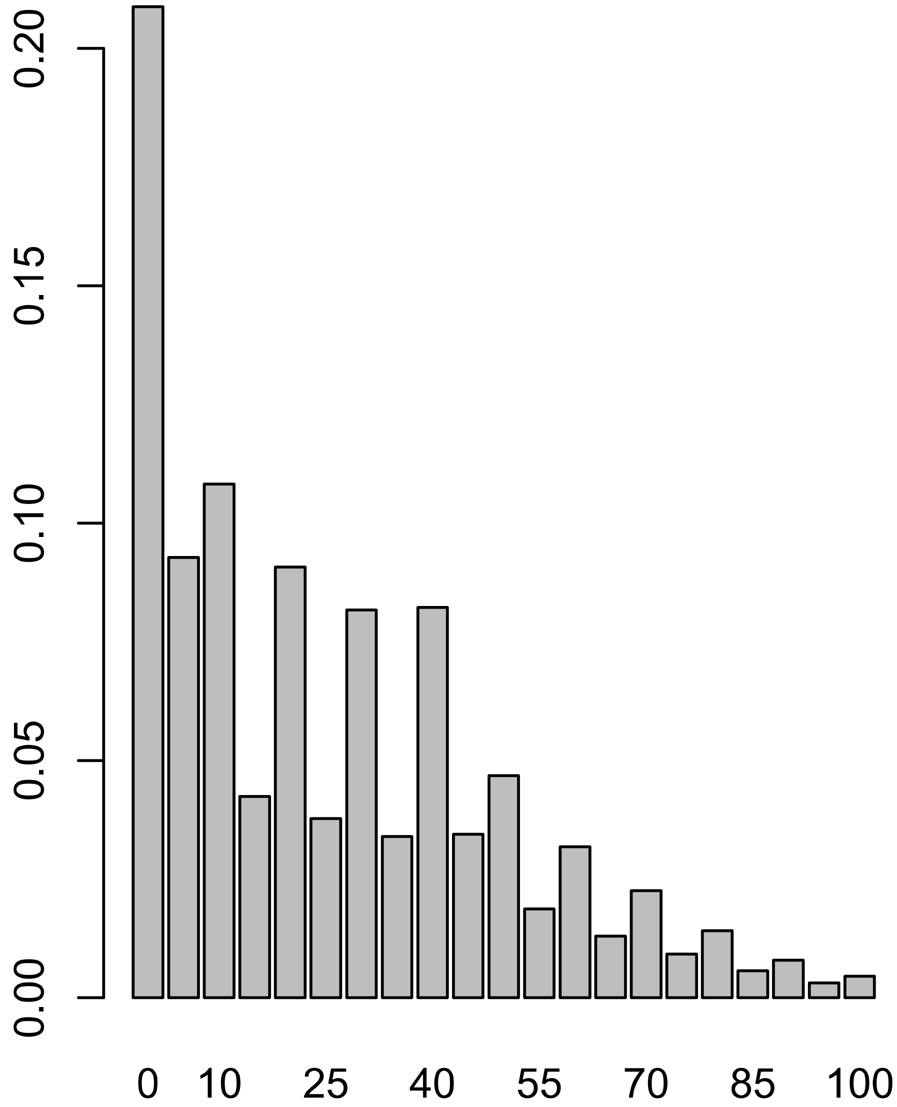
  which can be obtained by integrating over distribution of factors $\boldsymbol{\Psi}$.
- The Laplace–Stieltjes transform is useful for: sampling losses from model with importance sampling; approximating probability mass function using Fourier transform.

Loss Distribution

```
laplace.transform <- function(t,pd,exposure,lgd=rep(1,length(exposure)))
{
  output <- rep(NA,length(t))
        for (i in 1:length(t))
        output[i] <- exp(sum(log(1-pd*(1- exp(-exposure*lgd*t[i])))))
        output
}
# no common factor for simplicity
m <- 20
exposure <- c(5,5,5,5,10,10,10,10,20,20,20,20,30,30,30,30,40,40,40,40)
pd <- c(rep(0.1,10),rep(0.05,10))

N <- sum(exposure)+1
t <- 2*pi*(0:(N-1))/N
cfunction <- laplace.transform(-t*(1i),pd,exposure)
par(mar=c(3,3,2,1),mgp=c(2,1,0))
plot(cfunction)

fft.out <- round(Re(fft(cfunction)),digits=20)
probs <- fft.out/N
barplot(probs[(0:20)*5+1],names.arg=paste((0:20)*5))
```

Loss Distribution (cont.)





Overview

Estimation from Default Data

- Since defaults are sparse, industry models generally calibrate the factor model to equity return data (or asset return data).
- Where actual historical default data are available these can also be used.
- Recall that conditional default probabilities are given by

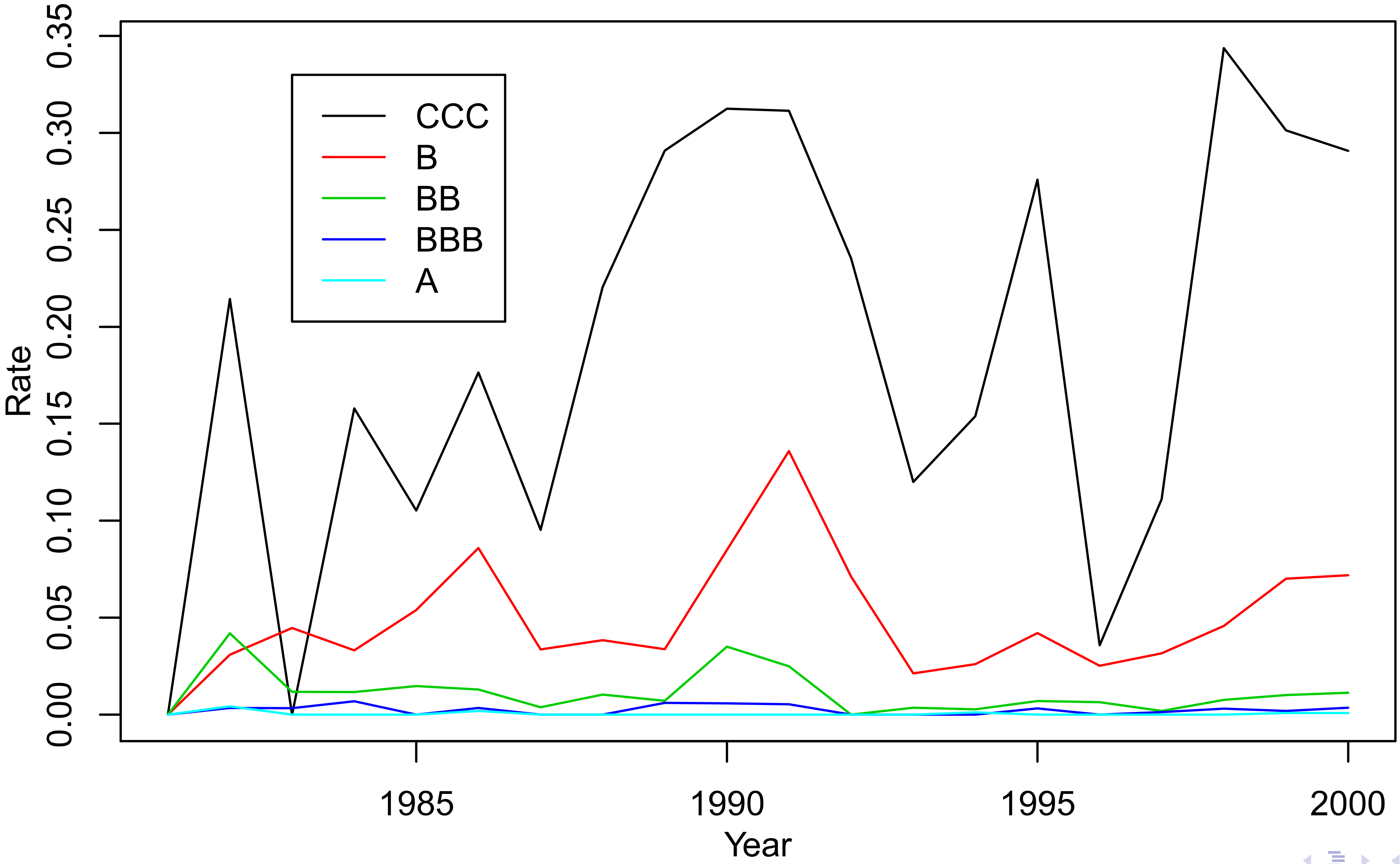
$$p_i(\psi) = \Phi\left(\frac{\Phi^{-1}(p_i) + \sqrt{\beta_i}\boldsymbol{a}_i'\psi}{\sqrt{1-\beta_i}}\right).$$

- In a one-factor version of this model we have a model of the form:

$$p_i(\psi) = \Phi\left(\mu_i + \sigma_i\psi\right).$$

- We often assume that the μ_i and σ_i are identical for all obligors in a homogeneous group (for example a rating group).
- This kind of model can be fitted to historical default data as a so-called GLMM (generalized linear mixed model).

Default Data (from S&P)



Example

```
tail(data.frame(defaults,firms,year,rating),n=10)

##     defaults firms year rating
## 91         1  1208 1999      A
## 92         2  1085 1999    BBB
## 93         8   793 1999     BB
## 94        63   899 1999      B
## 95        22    73 1999    CCC
## 96         1  1215 2000      A
## 97         4  1157 2000    BBB
## 98        10   887 2000     BB
## 99        69   961 2000      B
## 100       25    86 2000    CCC

# Fit glmm
mod <- glmer(cbind(defaults,firms-defaults) ~
               -1 + rating + (1|year), family=binomial(probit))
```

Example (cont.)

```
mod

## Generalized linear mixed model fit by maximum likelihood (Laplace
##   Approximation) [glmerMod]
##  Family: binomial  ( probit )
## Formula: cbind(defaults, firms - defaults) ~ -1 + rating + (1 | year)
##       AIC       BIC    logLik deviance df.resid
##  404.338  419.969 -196.169  392.338       94
## Random effects:
##  Groups Name        Std.Dev.
##  year   (Intercept) 0.2415
## Number of obs: 100, groups:  year, 20
## Fixed Effects:
##   ratingA  ratingBBB   ratingBB    ratingB  ratingCCC
##   -3.4318    -2.9185    -2.4039    -1.6895    -0.8378

sigma <- mod@theta; mu <- mod@beta
(beta <- sigma^2/(1+sigma^2)); (PD <- pnorm(mu*sqrt(1-beta)))

## [1] 0.05510481
## [1] 0.0004251567 0.0022776810 0.0097268556 0.0502693782 0.2077200911
```

For Further Reading

- P. J. Crosbie and J. R. Bohn. Modeling default risk. Technical document, Moody's/KMV, New York. 2002.
- D. Lando and T. Skodeberg. "Analyzing Rating Transitions and Rating Drift with Continuous Observations". In: Journal of Banking and Finance 26 (2002), pp. 423–444.
- A. J. McNeil, R. Frey, and P. Embrechts. Quantitative Risk Management: Concepts, Techniques and Tools. Princeton: Princeton University Press, 2005.
- A. J. McNeil, R. Frey, and P. Embrechts. Quantitative Risk Management: Concepts, Techniques and Tools. 2nd. Princeton: Princeton University Press, 2015.
- Z. Sun, D. Munves, and D. Hamilton. Public Firm Expected Default Frequency (EDF) Credit Measures: Methodology, Performance and Model Extensions. Technical document, Moody's Analytics. 2012.