**ORIE 5530: Modeling Under Uncertainty**

# Lecture 11 (The Poisson Process)

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*Adapted from Professor Itai Gurvich's original notes*

This is a short introduction to the Poisson process. The Poisson process is a very useful way to model inputs (arrivals to services, people browsing the internet, etc.). It is flexible enough to accommodate temporal variation, forecasting uncertainty and other realistic features. Here, we focus on some basics that will allow you to use and simulate a Poisson process. More discussion of modeling, forecast uncertainty, etc. are postponed to a service-operations class.

Chapter 5 of the Ross book covers the Poisson process with a bit more detail than we need but you can get an alternative view there.

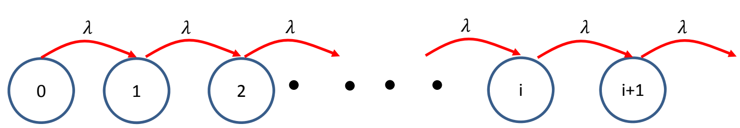
## 11.1 A Poisson process as a special case of a CTMC

The Poisson process models the following:

- No customers arrive before time 0
- Customers arrive one at a time
- The time between consecutive arrivals is $exp(\lambda)$. With $T_i$ being the time of the $i^{th}$ arrival, this means $T_{i+1} - T_i \sim exp(\lambda)$. Consecutive interarrival times are independent of each other.

  $\lambda$ is then referred to as *the rate of the Poisson process.*

It is a process that evolves over time. It is, specifically, a continuous time Markov chain that jumps up by 1 every exponential amount of time. In a transition diagram this is




Let's call this Markov Chain $N(t)$. Then, the number of arrivals by time $t$ is the state of this Markov chain at $t$. There is, of course, a reason that this is called a Poisson process.

**Property 1.** The number of arrivals by time $t$, $N(t)$, has a Poisson distribution with mean $\lambda t$. That is,

$$\mathbb{P}\{N(t) = k\} = \frac{e^{-\lambda t}(\lambda t)^k}{k!}.$$

In particular, $\mathbb{E}[N(t)] = Var(N(t)) = \lambda t$.

**Remark.** You might ask yourself how to make this consistent with what we learned about finite horizon performance? What is written here is that $P_{0,k}(t) = e^{-\lambda t}(\lambda t)^k/k!$. We can check that it solves the differential equation $P'(t) = QP(t)$. Let us take the first item of this set of equations. On the left we have $P'_{0,k}(t)$ (the derivative of $P_{0,k}(t)$ at time $t$) and on the right we have the product of the first row of $Q$ with the $k^{th}$ column of $P(t)$. This (check) gives

$$P'_{0,k}(t) = -\lambda P_{0,k}(t) + \lambda P_{1,k}(t).$$

Let us check that our "guess" in property 1 actually satisfies this. With $P_{0,k}(t) = e^{-\lambda t}(\lambda t)^k/k!$ we have that

$$\begin{aligned} P'_{0,k}(t) &= -\lambda \frac{e^{-\lambda t}(\lambda t)^k}{k!} + \lambda \frac{e^{-\lambda t}k(\lambda t)^{k-1}}{k!} \\ &= -\lambda \frac{e^{-\lambda t}(\lambda t)^k}{k!} + \lambda \frac{e^{-\lambda t}(\lambda t)^{k-1}}{(k-1)!} \\ &= -\lambda P_{0,k}(t) + \lambda P_{1,k}(t). \end{aligned}$$

The last line follows because it is clear that in this Markov chain $P_{0,k-1}(t)$ is the same as $P_{1,k}(t)$ (its the probability of $k-1$ arrivals in $t$ time units) and equals $e^{-\lambda t}(\lambda t)^{k-1}/(k-1)!$. Thus, our "guess" solves the necessary differential equation. ∎

Because of the homogeneous nature of this chain (for all states $\lambda_i = \lambda$ and $P_{i,i+1} = 1$) it is clear that

$$\mathbb{P}\{N(t+s) - N(s) = k | N(s) = \ell\} = \mathbb{P}\{N(t) = k\},$$

and does not depend on $N(s)$. In other words, for any interval of length $t$,

$$\mathbb{P}\{N(t+s) - N(s) = k\} = \frac{e^{-\lambda t}(\lambda t)^k}{k!}.$$

**Exercise:** People are hired into a workplace according to a Poisson process with rate $\lambda/week$. There is a learning curve. Let $X(t,z)$ be the number of employees at time $t$ that have that have been with the firm for $z$ weeks or more. What is the distribution of $X(t,z)$? ∎

Now, take $t_1 < t_2 \leq t_3 < t_4$ and consider arrivals on the intervals $(t_1, t_2]$ and $(t_3, t_4]$.

**Property 2.** (**independent increments**) $N(t_4) - N(t_3)$ is independent from $N(t_2) - N(t_1)$ meaning that for every $k, \ell$,

$$\begin{aligned} \mathbb{P}\{N(t_4) - N(t_3) = k, N(t_2) - N(t_1) = \ell\} &= \mathbb{P}\{N(t_4) - N(t_3) = k\}\mathbb{P}\{N(t_2) - N(t_1) = \ell\} \\ &= \frac{e^{-\lambda(t_4-t_3)}(\lambda(t_4 - t_3))^k}{k!} \frac{e^{-\lambda(t_2-t_1)}(\lambda(t_2 - t_1))^\ell}{\ell!}. \end{aligned}$$

This should be clear again from homogeneity. $N(t_4) - N(t_3)$ is Poisson$(\lambda(t_4 - t_3))$ independently of anything that happens before $t_3$. In particular, it should not depend on $N(t_2) - N(t_1)$.

**Remark 1 (simulation)** Simulating a Poisson process is rather straightforward given the simulation you had already constructed in your homework for a more complicated continuous time Markov chains. Here is the basic pseudo code for simulating a Poisson process on $[0, T]$

- Set the time to $T_0 = 0$. Set the counter $i$ to 0
- As long as $T_i \leq T$:
  - Generate an exponential random variable $Y_i$ with parameter $\lambda$. (we do this, recall, by generating a uniform U and then computing $ln(U)/\lambda$.)

- Advance $i \leftarrow i+1$ Advance the time to $T_i \leftarrow T_{i-1} + Y_i$.
- Set $N(T_i) = i$.

Now, computers are not very good at keeping functions, such as $N(t)$. They are better at discrete things. So one way to keep this information is to generate a table where one records the times of events and where the Poisson process was (the same applies to any CTMC). Then, if you are asked what is the value of $N(t)$, you can search the table for the interval in which $t$ falls.

Notice that if you are interested in just one specific $t$ (rather than the sample path of the process – how the process evolves) you don't need this. Since $N(t) \sim Poisson(\lambda t)$, you can just simulate one random variable with this distribution.

Below, for example, is data from a simulation run for a Poisson process with rate $\lambda = 1$.

| Arrival $i$ | $T_i$ |
|---|---|
| 1 | 1.637013794 |
| 2 | 3.617584853 |
| 3 | 5.607789129 |
| 4 | 6.274811479 |
| 5 | 8.921637384 |
| 6 | 10.01218765 |
| 7 | 10.64001714 |
| 8 | 11.01940243 |
| 9 | 11.0229644 |
| 10 | 11.32889342 |
| 11 | 11.4233002 |
| 12 | 12.32754224 |
| 13 | 14.11642354 |
| 14 | 14.27324588 |
| 15 | 16.74586619 |
| 16 | 16.75468146 |
| 17 | 16.76373638 |
| 18 | 17.74894105 |
| 19 | 18.6708546 |
| 20 | 19.11551817 |

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## 11.2 Splitting/Thinning a Poisson processes

Think about the following (see Figure 11.1): There is a total input process that follows a Poisson process with rate $\lambda$. Each of the customers is of type $A$ with probability $p$ and of type $B$ with the remaining probability $1-p$. Customers are independent of each other for their types. What is the input process to queues $A$ and $B$?

In total then there are $\lambda$ customers arriving per hour and a fraction $p$ of them are type $A$. So, on average, you expect (and you'd be correct) that the number of type $A$ customers per hour is $\lambda p$. But it turns out we can say more than that. The input to $A$ and the input to $B$ maintain the Poisson structure of the input process and are independent of each other. Let

$$N_A(t) = \#\text{of customers of type } A \text{ that arrived by time } t,$$
$$N_B(t) = \#\text{of customers of type } B \text{ that arrived by time } t.$$

Then, we have

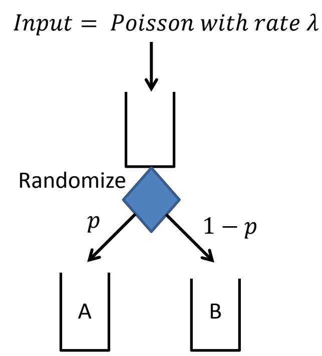


Figure 11.1: Splitting of Poisson process

**Property 3.** $N_A(t)$ is a Poisson process with rate $\lambda p$ and $N_B(t)$ is a Poisson process with rate $\lambda(1-p)$. In particular,

- $\mathbb{P}\{N_A(t)=k\} = e^{-\lambda pt}(\lambda pt)^k/k!$ and $\mathbb{P}\{N_B(t)=k\} = e^{-\lambda(1-p)t}(\lambda(1-p))^k/k!$
- The time between type $A$ arrivals is exponential with parameter $\lambda p$ (and mean $\frac{1}{\lambda p}$).
- The time between type $B$ arrivals is exponential with parameter $\lambda(1-p)$ (and mean $\frac{1}{\lambda(1-p)}$).

Moreover, the two processes are independent of each other. That is, for each $t, k, \ell$

$$\mathbb{P}\{N_A(t)=k, N_B(t)=\ell\} = \mathbb{P}\{N_A(t)=k\}\mathbb{P}\{N_B(t)=\ell\}.$$

This is very useful because it tells us that the probabilistic structure is maintained. This makes it easier to simulate the system to which $A$ and $B$ are inputs. Moreover, we can treat whatever comes after the randomization node as independent processes.

The fact that the time between $A$-arrivals is exponential with parameters $\lambda p$ and hence mean $1/(\lambda p)$ makes intuitive sense: If we just had a type $A$ customer, the number of customers until we get another one is Geometric with parameter $p$ (because each customer is of type $A$ with probability $p$), right? On average we have to wait then $1/p$ customers. The mean time between consecutive customers is $1/\lambda$ (the expectation of the exponential) and hence *the expected time between $A$-arrivals is* $\frac{1}{p}\frac{1}{\lambda} = \frac{1}{\lambda p}$.

But the result goes beyond average time between arrival. It says that this time — a *Geom*($p$) sum of *exp*($\lambda$) random variables — is exponential with parameters $\lambda p$:

**Implication.** Let $M \sim Geom(p)$. Let $Y_1, Y_2, \ldots$ be independent $exp(\lambda)$ random variables (that are also independent of $M$). Then,

$$\sum_{i=1}^{M} Y_i \sim exp(\lambda p).$$

In particular,

$$\mathbb{P}\left\{\sum_{i=1}^{M} Y_i \leq x\right\} = 1 - e^{-(\lambda p)x}.$$

**Example 1 (The infinite server queue)** This idea of Poisson splitting is kind of powerful. As an example consider the following setting which is called "the infinite server queue". We have the following:

- Customers arrive according to a Poisson process with rate $\lambda$
- Customer $i$ stays in the system for a service time $S_i$ that follows a distribution $G$. That is, $\mathbb{P}\{S_i \leq s\} = G(s)$ and $\mathbb{P}\{S_i > s\} = 1 - G(s)$. We use the notation $\bar{G}(s) = 1 - G(s)$.

What can we say about the number of people in the system at time $t$. That is, how many people arrived by time $t$ but did not leave the system yet.

Poisson thinning gives us an answer. Let's divide the time $[0, t]$ into buckets of a size $h$.

Since the arrival process is Poisson, the number of arrivals on $[0, h]$ is Poisson with mean $\lambda h$. Out of those arriving on this interval, each customer will be still be around at time $t$ if her service time is longer (roughly) than $t$. The fraction of those is $\mathbb{P}\{S_i > t\} = \bar{G}(t)$. Let's think of those customers as type $A$ in our thinning example. The probability of being $A$ is here $p = \bar{G}(t)$. Then, the number of type $A$ customer arriving over $[0, h]$ is by thinning $N_0 = Poisson(\lambda h \bar{G}(t))$.

We can apply this also to the second time bucket and we will have that the number that arrived over $[h, 2h]$ that are still around at time $t$ is $N_1 = Poisson(\lambda h \bar{G}(t - h))$, for the third $N_2 = Poisson(\lambda h \bar{G}(t - 2h))$, and so on.

Now, we are going to add all these Poisson random variables. By *addition* of independent Poisson random variables we have that

$$\sum_{i=1}^{t/h} N_i \sim Poisson(\sum_{k=1}^{t/h} \lambda h \bar{G}(t - kh)).$$

As we shrink $h$ to zero we get

$$\sum_{k=1}^{t/h} \lambda h \bar{G}(t - kh) \to \int_0^t \lambda \bar{G}(s) ds.$$

Thus, **we found that the number of people in the system at time $t$ is a Poisson random variable with parameters (and hence mean)** $\lambda \int_0^t \bar{G}(s) ds$.

**Exercise:** The doors to a show open at 8:00pm. People start arriving at 6:00pm and lining up. Arrivals follow a Poisson process with rate $50/hour$ until doors open.

Those that arrive at 6:01 will wait 1:59 hours while those that arrive at 7:59 will wait just one minute. What is the expected total waiting time (across all customers). How much will the average customer wait?