ORIE 5530: Modeling Under Uncertainty

Lecture 7 (Performance evaluation of Markov chains)

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Adapted from Professor Itai Gurvich's original notes

It will be useful to have a single example to be used throughout.

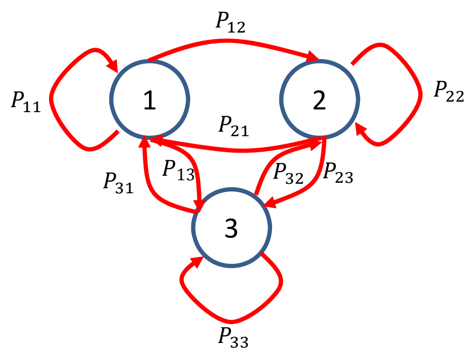

Figure 7.1: A 3-state Markov chain

7.1 Multi-step transition probability

In Figure 7.1 the probability of going in one step from state i to state j is P_{ij} – entry i, j of the one-step transition probability matrix. What, however, if we ask for the probability of doing so in n steps:

$$p_{ij}^n = \mathbb{P}\{X_n = j | X_0 = i\}.$$

Well, that can be more difficult. If $n = 2$ we can figure this out easily: you can get, for example, from state 1 to itself in two steps, by either staying there for two steps—with the corresponding probability $(P_{1,1})^2$—or by first going to state 2 and then back to state 1 which would have the probability $P_{12}P_{21}$.

But if I ask what happens in n steps it becomes a bit insane to count all possible trajectories. So let's build a simple formula that can be generalized to other Markov Chains. Starting with two steps, we just have to count all the options of where we could be in one step:

$$p_{12}^2 = \sum_{k=1}^{3} \mathbb{P}\{X_2 = 2, X_1 = k, X_0 = 1\} = \sum_{k=1}^{3} \mathbb{P}\{X_1 = k | X_0 = 1\}\mathbb{P}\{X_2 = 2 | X_1 = k, X_0 = 1\} = \sum_{k=1}^{3} P_{1k}P_{k2}.$$

Now notice that what we have obtained on the right hand side is the product of row 1 of the transition matrix P with the first column of P:

$$\begin{bmatrix} P_{11} & P_{12} & P_{13} \\ P_{21} & P_{22} & P_{23} \\ P_{31} & P_{32} & P_{33} \end{bmatrix} \begin{bmatrix} P_{11} & P_{12} & P_{13} \\ P_{21} & P_{22} & P_{23} \\ P_{31} & P_{32} & P_{33} \end{bmatrix}.$$

Thus, we have found that

$$p_{12}^2 = [P \times P]_{12} = [P^2]_{12}.$$

How about the three step probability $p_{12}^3 = \mathbb{P}\{X_3 = 2, X_0 = 1\}$?

$$p_{12}^3 = \sum_{k=1}^{3} \mathbb{P}\{X_3 = 2, X_1 = k, X_0 = 1\} = \sum_{k=1}^{3} \mathbb{P}\{X_1 = k|X_0 = 1\}\mathbb{P}\{X_3 = 2|X_1 = k, X_0 = 1\}.$$

Notice that $\mathbb{P}\{X_3 = 2|X_1 = k, X_0 = 1\} = \mathbb{P}\{X_3 = 2|X_1 = k\} = \mathbb{P}\{X_2 = 2|X_0 = k\} = p_{k2}^2$ is just the probability of getting from state k to state 2 in two steps. We already know that $p_{k2}^2 = [P^2]_{k2}$. Thus, we have that

$$p_{12}^3 = \sum_{k=1}^{3} \mathbb{P}\{X_1 = k|X_0 = 1\}\mathbb{P}\{X_3 = 2|X_1 = k, X_0 = 1\} = \sum_{k=1}^{3} P_{1k}[P^2]_{k2} = [P^3]_{12}.$$

So we see that there is a pattern here that indicates

$$\mathbb{P}\{X_n = j|X_0 = i\} = [P^n]_{ij}. \qquad (n\text{-steps transitions})$$

This is rather beautiful because it says that all we need is matrix algebra.

Let's see a numerical example. Suppose that

$$P = \begin{bmatrix} 0.3 & 0.3 & 0.4 \\ 0.2 & 0.35 & 0.45 \\ 0.18 & 0.52 & 0.3 \end{bmatrix}.$$

Then,

$$P^2 = \begin{bmatrix} 0.198 & 0.403 & 0.375 \\ 0.159 & 0.3925 & 0.3405 \\ 0.1704 & 0.392 & 0.396 \end{bmatrix},$$

which, because the Markov chain has only three states, you can verify by yourself that it make sense. For example $[P^2]_{12}$ is the probability of getting in two steps from state 1 to state 2. We can do that in three ways: (i) first step back to 1 and second step to 2, which has probability $P_{11}P_{12} = 0.3 * 0.3$; (ii) first step to 2 and then from 2 to itself, which has probability $P_{12}P_{22} = 0.3 * 0.35$; (iii) from 1 to 3 and 3 to 2, which has probability $P_{13}P_{32} = 0.4 * 0.52$. The total is $0.3 * 0.3 + 0.3 * 0.35 + 0.4 * 0.52 = 0.403$.

So, even in two steps, the fact that we can just raise the matrix to the power of 2, saves us some work. Of course, it saves us a LOT of work if we want $P\{X_{10} = 3|X_0 = 1\}$. But we can easily compute

$$P^{10} = \begin{bmatrix} 0.2137 & 0.4043 & 0.3820 \\ 0.2137 & 0.4043 & 0.3820 \\ 0.2137 & 0.4043 & 0.3820 \end{bmatrix},$$

and conclude that $P\{X_{10} = 3|X_0 = 1\} = [P^{10}]_{13} = 0.382$. Interestingly, we see that (while the rows of P^2 were different than each other) those of P^{10} are equal to each other. We will be able to explain that soon.

7.2 Performance analysis

Suppose that, in the same Markov chain of Figure 7.1, we collect a reward $R(1)$ every time we visit state 1, a reward $R(2)$ every time we visit state 2, and a reward $R(3)$ when we visit state 3. You can think of the three Markov chain as capturing weather and $R(i)$ is the amount of sales I am making when the weather is in state i.

Consider discounting as a standard way of thinking about present value. If putting \$1 in the bank gives you an interest of r percent per day. Then, a dollar you will get tomorrow is worth $\beta = 1/(1 + r) < 1$ in today's terms. A dollar received on day k is worth $1 \times \beta^k$ in todays terms. So if you seek to understand whether you will sell this product, you want to compare the investment you are making in today's dollars to the expected discounted reward

$$\sum_{k=0}^{\infty} \beta^k R(X_k).$$

This is random so you have to decided how you want to deal with this. It is standard just to look at expected present value. Suppose today (i.e., at time 0) $X_0 = 2$ so we are seeking to compute

$$\mathbb{E}[\sum_{k=0}^{\infty} \beta^k R(X_k)|X_0 = 2].$$

This seems difficult so let's go back to Figure 7.1. If you start at state 2, you collect today a reward $R(2)$ – that, for one, is certain.

Now, what happens from tomorrow and on. With probability P_{21} you will be in state 1 tomorrow and your expected reward from then on will be $\mathbb{E}[\sum_{k=0}^{\infty} \beta^k R(X_k)|X_0 = 1]$. With probability P_{22} you will be in state 2 tomorrow and your reward from then on will be $\mathbb{E}[\sum_{k=0}^{\infty} \beta^k R(X_k)|X_0 = 2]$. Finally, with probability P_{23} you will be in state 3 tomorrow and your reward from then and on will be $\mathbb{E}[\sum_{k=0}^{\infty} \beta^k R(X_k)|X_0 = 3]$.

But you also have to take into account that every dollar made from tomorrow is worth only β today. This logic is basically the proof that

$$\begin{aligned}\mathbb{E}[\sum_{k=0}^{\infty} \beta^k R(X_k)|X_0 = 2] = R(2) + \beta \Bigg(& P_{21}\mathbb{E}[\sum_{k=0}^{\infty} \beta^k R(X_k)|X_0 = 1] + P_{22}\mathbb{E}[\sum_{k=0}^{\infty} \beta^k R(X_k)|X_0 = 2] \\ & + P_{23}\mathbb{E}[\sum_{k=0}^{\infty} \beta^k R(X_k)|X_0 = 3]\Bigg).\end{aligned}$$

To have a cleaner equation, let's define

$$V(i) = \mathbb{E}[\sum_{k=0}^{\infty} \beta^k R(X_k)|X_0 = i].$$

Then, what we proved is that

$$\begin{aligned}V(2) &= R(2) + \beta\left(P_{21}V(1) + P_{22}V(2) + P_{23}V(3)\right) \\ &= R(2) + \beta \sum_{i=1}^{3} P_{2i}V(i).\end{aligned}$$

From this we can in the same way create the equations for the cases that today we are in state 1 or 3 (instead of 2), i.e., $X_0 = 1$ or $X_0 = 3$. We overall have then the set of equations

$$\begin{aligned}V(1) &= R(1) + \beta \sum_{i=1}^{3} P_{1i}V(i) \\ V(2) &= R(2) + \beta \sum_{i=1}^{3} P_{2i}V(i) \\ V(3) &= R(3) + \beta \sum_{i=1}^{3} P_{3i}V(i).\end{aligned}$$

This is just a linear system of equations with the unknowns being $V(1), V(2), V(3)$. We can solve this system using linear programming that you are covering in the optimization class. Fortunately, in this case we can also directly compute the solution.

Re-writing this system in matrix form we have

$$\begin{bmatrix} V(1) \\ V(2) \\ V(3) \end{bmatrix} = \begin{bmatrix} R(1) \\ R(2) \\ R(3) \end{bmatrix} + \beta \begin{bmatrix} P_{11} & P_{12} & P_{13} \\ P_{21} & P_{22} & P_{23} \\ P_{31} & P_{32} & P_{33} \end{bmatrix} \begin{bmatrix} V(1) \\ V(2) \\ V(3) \end{bmatrix}.$$

Notice that this is the same as

$$\left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \beta \begin{bmatrix} P_{11} & P_{12} & P_{13} \\ P_{21} & P_{22} & P_{23} \\ P_{31} & P_{32} & P_{33} \end{bmatrix} \right) \begin{bmatrix} V(1) \\ V(2) \\ V(3) \end{bmatrix} = \begin{bmatrix} R(1) \\ R(2) \\ R(3) \end{bmatrix},$$

or inverting

$$\begin{bmatrix} V(1) \\ V(2) \\ V(3) \end{bmatrix} = \left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \beta \begin{bmatrix} P_{11} & P_{12} & P_{13} \\ P_{21} & P_{22} & P_{23} \\ P_{31} & P_{32} & P_{33} \end{bmatrix} \right)^{-1} \begin{bmatrix} R(1) \\ R(2) \\ R(3) \end{bmatrix}.$$

Inverting matrices is something that computers do beautifully. Here is a numerical example (you will be doing such a computation in the homework). Suppose that, $\beta = 0.95$, $R(i) = i^2$ and that, as before,

$$P = \begin{bmatrix} 0.3 & 0.3 & 0.4 \\ 0.2 & 0.35 & 0.45 \\ 0.18 & 0.52 & 0.3 \end{bmatrix}.$$

Then, the equation becomes

$$\begin{aligned} \begin{bmatrix} V(1) \\ V(2) \\ V(3) \end{bmatrix} &= \left(\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - 0.95 \begin{bmatrix} 0.3 & 0.3 & 0.4 \\ 0.2 & 0.35 & 0.45 \\ 0.18 & 0.52 & 0.3 \end{bmatrix} \right)^{-1} \begin{bmatrix} 1 \\ 4 \\ 9 \end{bmatrix} \\ &= \begin{pmatrix} 0.715 & -0.285 & -0.38 \\ -0.19 & 0.6675 & -0.4275 \\ -0.171 & -0.494 & 0.715 \end{pmatrix}^{-1} \begin{bmatrix} 1 \\ 4 \\ 9 \end{bmatrix} \\ &= \begin{pmatrix} 5.151249685 & 7.579327435 & 7.26942288 \\ 4.045311986 & 8.63928523 & 7.315402784 \\ 4.026920024 & 7.781639014 & 8.191440962 \end{pmatrix} \begin{bmatrix} 1 \\ 4 \\ 9 \end{bmatrix} = \begin{matrix} 100.8933653 \\ 104.441078 \\ 108.8764447 \end{matrix} \end{aligned}$$

In this case, then, our total discounted reward is of the order of a 100 with small differences depending on where we start.