ORIE 5530 HW2 Konstantinos Ntalis (kn442), Maxwell Wulff (mcw232)

In [2]:
```
import numpy as np
```

Question 1 ¶

Part (a)

In [3]:
```
p = 12
c = 3
income = []

# the driver can work for integers hours <= 10
for i in range(1,11):
    in_per_hour = []
    mean = np.log(1+i) # mean of the poisson process
    cost = c*i
    # simulating poisson random variables
    for j in range(1000000):
        D = np.random.poisson(mean)
        in_per_hour.append(p*D-cost)
    income.append(in_per_hour)
```

We can now calculate the mean obtained from the simulations for each value of t as

In [18]:
```
np.mean(income, axis = 1)
```

Out[18]:
```
array([ 5.310888,  7.1919  ,  7.615512,  7.29918 ,  6.505692,  5.330196,
        3.956184,  2.354328,  0.631944, -1.226892])
```

and the standard deviation

In [19]:
```
np.std(income, axis = 1)
```

Out[19]:
```
array([ 9.99000724, 12.59499069, 14.10858551, 15.21371287, 16.05924468,
       16.7448959 , 17.31702192, 17.78146506, 18.2111761 , 18.58403229])
```

Part (b)

$$P(t) = p\mu(t) - ct$$

where P is the total profit, p the average dollar amount per ride and c the cost. We can now differentiate and set the derivative to zero. We obtain

$$\frac{\partial P}{\partial t} = 0 \implies \frac{p}{1+t} = c$$

Thus for the given values of p and c we get $t = 3$.

Question 2

Part (a)

We have

$$\mathbb{P}[W = 2, B = 3, R = 1] = \frac{\binom{3}{2}\binom{5}{3}\binom{6}{1}}{\binom{14}{6}} = \frac{3 \cdot 10 \cdot 6}{3003} = 0.06$$

Part (b)

We have that 3 black balls have already been chosen. Therefore we have 2 black balls remaining.

$$\mathbb{P}[W = t | B = 3] = \frac{\binom{3}{t}\binom{6}{3-t}}{\binom{9}{3}}$$

Part (c)

$$\mathbb{P}[W = t | B = 3] = \binom{3}{t}\left(\frac{3}{14}\right)^t\left(\frac{11}{14}\right)^{3-t}$$

Question 3

Part (a)

We can show that for independent Poisson processes, the following holds

$$Poisson(\lambda_1) + Poisson(\lambda_2) = Poisson(\lambda_1 + \lambda_2)$$

Suppose that X, Y are Poisson random variables and $Z = X + Y$ is also a random variable. Then

$$\mathbb{P}(Z = z) = \mathbb{P}(X + Y = z) = \sum_{x+y=z} \mathbb{P}(X = x, Y = y)$$

Then from the independence assumption we have

$$\begin{aligned}\mathbb{P}(Z=z) &= \sum_{x+y=z} \mathbb{P}(X=x)P(Y=y) \\ &= \sum_{x+y=z} \frac{e^{-\lambda_1}\lambda_1^x}{x!}\frac{e^{-\lambda_2}\lambda_2^y}{y!} \\ &= e^{-(\lambda_1+\lambda_2)} \sum_{x+y=z} \frac{\lambda_1^x}{x!}\frac{\lambda_2^y}{y!}\end{aligned}$$

Now using the definition of the binomial coefficient $\binom{z}{x}$ we can write the above as

$$\frac{e^{-(\lambda_1+\lambda_2)}}{z!}\sum_{x+y=z}\binom{z}{x}\lambda_1^x\lambda_2^y = \frac{e^{-(\lambda_1+\lambda_2)}(\lambda_1+\lambda_2)^z}{z!}$$

and this is a poisson distribution with mean and variance $\lambda_1 + \lambda_2$.

Part (b)

Well the probability for a customer to go to store 1 is $\frac{\lambda_1}{\lambda_1+\lambda_2}$ and $\frac{\lambda_2}{\lambda_1+\lambda_2}$ to go to store 2. Then we have:

$$\mathbb{P}(X_1 = i|Y=n) = \binom{n}{i}\Big(\frac{\lambda_1}{\lambda_1+\lambda_2}\Big)^i\Big(\frac{\lambda_2}{\lambda_1+\lambda_2}\Big)^{n-i}$$

Similarly,

$$\mathbb{P}(X_2 = i|Y=n) = \binom{n}{i}\Big(\frac{\lambda_2}{\lambda_1+\lambda_2}\Big)^i\Big(\frac{\lambda_1}{\lambda_1+\lambda_2}\Big)^{n-i}$$

Given the above distributions and the fact that Y is a poisson with rate $\lambda_1 + \lambda_2$, we can consider

$$\mathbb{P}(X_1 = i, X_2 = j) = \mathbb{P}(X_1 = i, X_2 = j|Y = i+j)\mathbb{P}(Y = i+j)$$

$$= \frac{(i+j)!}{i!j!}\Big(\frac{\lambda_1}{\lambda_1+\lambda_2}\Big)^i\Big(\frac{\lambda_2}{\lambda_1+\lambda_2}\Big)^j\frac{e^{-(\lambda_1+\lambda_2)}(\lambda_1+\lambda_2)^{i+j}}{(i+j)!}$$

$$= \frac{e^{-\lambda_1}\lambda_1^i}{i!}\frac{e^{-\lambda_2}\lambda_2^j}{j!}$$

Now to get the marginal distribution of X_1 and X_2, we sum out i and j respectively to obtain

$$\begin{aligned}\mathbb{P}(X_1 = i) &= \frac{e^{-\lambda_1}\lambda_1^i}{i!} \\ \mathbb{P}(X_1 = j) &= \frac{e^{-\lambda_2}\lambda_2^j}{j!}\end{aligned}$$

Another interesting approach is to consider the limiting case of a Binomial distribution. As shown in lectures, to understand $\mathbb{P}(X_1 = i)$ and $\mathbb{P}(X_2 = i)$ we can consider the Poisson approximation to the binomial. If we consider a sequence of these random variables such that n is increasing along the sequence but p is decreasing so as to keep $np = \lambda$ constant. In our case $p = \frac{\lambda_1}{\lambda_1 + \lambda_2}$ or $p = \frac{\lambda_2}{\lambda_1 + \lambda_2}$ according to the store. Then,

$$\mathbb{P}(X_{n,p} = i) \rightarrow \frac{e^{-\lambda}\lambda^i}{i!}$$

Part (c)

The mean of two random variables is additive. If $X_1 \sim Poisson(\lambda_1)$ and $X_2 \sim Poisson(\lambda_2)$. Then

$$\mathbb{E}[X_1 + X_2] = \mathbb{E}[X_1] + \mathbb{E}[X_2] = \lambda_1 + \lambda_2$$

As for the variance,

$$\begin{aligned} \mathrm{Var}[X_1 + X_2] &= \mathrm{Var}[X_1] + \mathrm{Var}[X_2] + 2\mathrm{Cov}[X_1, X_2] \\ &= \lambda_1 + \lambda_2 + 1.6\sqrt{\lambda_1 \lambda_2} \end{aligned}$$