ORIE 5530: Modeling Under Uncertainty

Lecture 8 (Long-run averages)

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Adapted from Professor Itai Gurvich's original notes

Let us stick to the example we had in the previous handout and which has three states.

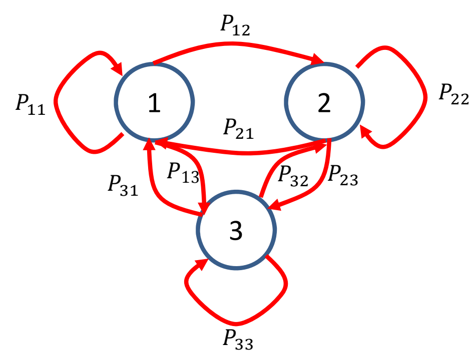


Figure 8.1: A 3-state Markov chain

In the last couple of lectures we saw how to derive multi-step transition probabilities (by taking powers of the one-step transition probability matrix) as well as how to evaluate infinite horizon discounted performance. In both cases we reduced performance evaluation to matrix algebra.

Specifically, we know now to compute the expectation of a function of the three-state Markov chain in Figure 8.1 at its k^{th} step:

$$\mathbb{E}[f(X_k)|X_0 = x] = \sum_{y=1}^{3} f(y)\mathbb{P}\{X_k = y|X_0 = x\} = \sum_{y=1}^{3} f(y)[P^k]_{xy},$$

and, in turn, we can compute cumulative finite horizon performance (over an horizon that has n periods)

$$\mathbb{E}[\sum_{k=0}^{n-1} f(X_k)|X_0 = x] = \sum_{k=0}^{n-1} \mathbb{E}[f(X_k)|X_0 = x] = \sum_{k=0}^{n-1}\sum_{y=1}^{3} f(y)[P^k]_{xy}. \qquad \text{(Finite horizon performance)}$$

This, notice, is just about taking powers of the matrix

$$P = \begin{bmatrix} P_{11} & P_{12} & P_{13} \\ P_{21} & P_{22} & P_{23} \\ P_{31} & P_{32} & P_{33} \end{bmatrix}.$$

We also know how to compute the infinite horizon discounted performance

$$V(x) = \mathbb{E}[\sum_{k=0}^{\infty} \beta^k f(X_k)|X_0 = x], \qquad \text{(Infinite-horizon discounted performance)}$$

using the matrix P. Specifically, it is the solution of the set of equation

$$\begin{bmatrix} V(1) \\ V(2) \\ V(3) \end{bmatrix} = \begin{bmatrix} R(1) \\ R(2) \\ R(3) \end{bmatrix} + \beta \begin{bmatrix} P_{11} & P_{12} & P_{13} \\ P_{21} & P_{22} & P_{23} \\ P_{31} & P_{32} & P_{33} \end{bmatrix} \begin{bmatrix} V(1) \\ V(2) \\ V(3) \end{bmatrix}.$$

See the previous handout.

One sort of performance analysis we turn to next is *long-run average performance*. Say, we are interested in the average over a very long horizon of n periods. We can reasonably approximate it by the infinite horizon average given by

$$\lim_{n\to\infty} \mathbb{E}[\frac{1}{n}\sum_{k=0}^{n-1} f(X_k)|X_0 = x]. \qquad \text{(Long-run average performance)}$$

Compare this to the finite horizon performance. Let us take one more step — using the fact that the sum of expectation equals the expectation of the sum — to write

$$\lim_{n\to\infty} \mathbb{E}[\frac{1}{n}\sum_{k=0}^{n-1} f(X_k)|X_0 = x] = \lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1} \mathbb{E}[f(X_k)|X_0 = x].$$

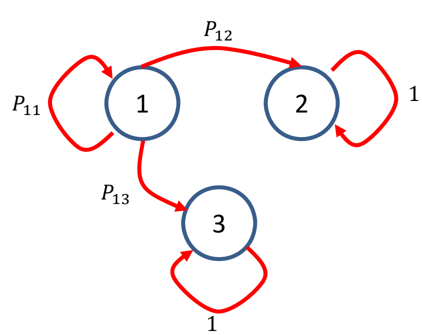
To study this criterion we need to introduce the notions of *stationary distributions* for Markov chains. First some preliminaries.

8.1 Long-run averages and stationary distributions

Suppose that in the three-state Markov chain of Figure 8.1 we have $P_{22} = P_{32} = 0$: once you reach 2 or 3 you get stuck there forever. The matrix is of the form

$$P = \begin{bmatrix} P_{11} & P_{12} & P_{13} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

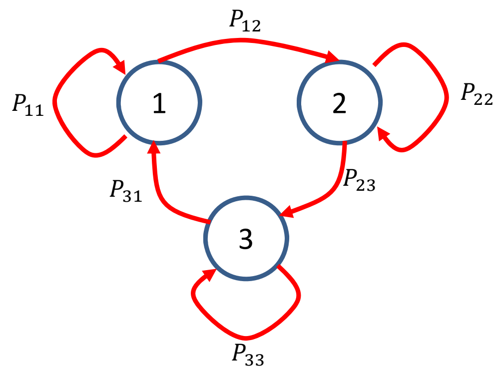
and the figure has the simple form



We could still talk about long-run average performance here. But the result will depend on where we start at time 0. If we start at 2 (that is $X_0 = 2$), we stay there forever and the long-run average is just $f(2)$. If we start at 3 we stay there forever so the long-run average is $f(3)$.

There are chains where the long-run average does not depend on where we start. These are *irreducible chains*. Simply, a chain is irreducible if from any pair of states x and y there is a path (of positive probability) from x to y in the graph.

Irreducibility does not require that you can go in one step from one state to another, rather just that there is a path. In yet another instance of the 3-state chain depicted below, you can get from any state to any other state but it will take, for example, at least two steps to get from 1 to 3.



If the chain is irreducible, there is, it turns out, an algebraic way to compute the long-run average. First, let's think just about the fraction of time the 3-state Markov chain spends in state 1. That is, you count step k if $X_k = 1$ (we write this as $\mathbb{1}\{X_k = 1\}$). You sum these up and divide by the number of steps:

$$\mathbb{E}\left[\frac{1}{n}\sum_{k=0}^{n-1}\mathbb{1}\{X_k = 1\}|X_0 = x\right].$$

We will take n to be large to get the long-run average. That is, we are looking for the long-run fraction of time, π_1, spent in state 1:

$$\pi_1 = \lim_{n\to\infty}\mathbb{E}\left[\frac{1}{n}\sum_{k=0}^{n-1}\mathbb{1}\{X_k = 1\}|X_0 = x\right].$$

We could similarly look for π_2 and π_3. Notice that we can write this "forgetting" about the fact that $X_0 = x$. In irreducible chains, the long-run average does not depend on whether we started at state 2, 3 or 1 (we can, in other words, drop the conditioning). The fundamental result here is that you can find these long-run fractions by solving a simple system of linear equations

$$\begin{aligned}
\pi_1 &= \pi_1 P_{11} + \pi_2 P_{21} + \pi_3 P_{31},\\
\pi_2 &= \pi_1 P_{12} + \pi_2 P_{22} + \pi_3 P_{32},\\
\pi_3 &= \pi_1 P_{13} + \pi_2 P_{23} + \pi_3 P_{33},
\end{aligned}$$

with the added requirements that $\pi_1, \pi_2, \pi_3 \geq 0$ and that $\pi_1 + \pi_2 + \pi_3 = 1$. These two latter conditions are obvious, you want fractions to be positive and sum up to 1. Define $\pi = (\pi_1, \pi_2, \pi_3)$ and e to be a (column) vector of all ones. Then, in matrix notation, the above is written as

$$\pi \geq 0, \quad \pi e = 1, \quad \pi = \pi P.$$

Before proceeding, let me convince you — still focusing on the 3-state chain and on the fraction of time spent in state 1 — that this makes sense. Let's first agree that we can partition the event that $X_k = 1$ into three mutually exclusive sub-events:

$$\{X_k = 1\} = \{X_k = 1, X_{k-1} = 1\} \cup \{X_k = 1, X_{k-1} = 2\} \cup \{X_k = 1, X_{k-1} = 3\}.$$

So, we can write

$$\begin{aligned}\mathbb{P}\{X_k = 1\} &= \mathbb{P}\{X_k = 1, X_{k-1} = 1\} + \mathbb{P}\{X_k = 1, X_{k-1} = 2\} + \mathbb{P}\{X_k = 1, X_{k-1} = 3\} \\ &= \mathbb{P}\{X_k = 1 | X_{k-1} = 1\}\mathbb{P}\{X_{k-1} = 1\} + \mathbb{P}\{X_k = 1 | X_{k-1} = 2\}\mathbb{P}\{X_{k-1} = 2\} \\ &\quad + \mathbb{P}\{X_k = 1 | X_{k-1} = 3\}\mathbb{P}\{X_{k-1} = 3\} \\ &= \sum_{j=1}^{3} \mathbb{P}\{X_{k-1} = j\} P_{j1}.\end{aligned}$$

This is very useful. Why? It means we can write the amount of time spent in state 1 as a weighted average of the other states, namely

$$\begin{aligned}\frac{1}{n}\sum_{k=1}^{n-1} \mathbb{E}\left[\mathbb{1}\{X_k = 1\}\right] &= \frac{1}{n}\sum_{k=1}^{n-1} \mathbb{P}\{X_k = 1\} \\ &= \frac{1}{n}\sum_{k=1}^{n-1} \mathbb{P}\{X_{k-1} = 1\} P_{11} + \frac{1}{n}\sum_{k=1}^{n-1} \mathbb{P}\{X_{k-1} = 2\} P_{21} + \frac{1}{n}\sum_{k=1}^{n-1} \mathbb{P}\{X_{k-1} = 3\} P_{31}\end{aligned}$$

But, recall that we defined $\pi_j = \lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n-1} \mathbb{P}\{X_{k-1} = j\}$ (you could complain that actually in my definition I counted from $k = 0$ to $n - 1$, but as n grows, we can ignore the first term). So what we have shown above is that (map the colors)

$$\pi_1 = \pi_1 P_{11} + \pi_2 P_{21} + \pi_3 P_{31}.$$

This explains the equations we introduced for the long-run fractions.

Example computation for finite-space Markov chain As noted above, we can compute the desired long-run fractions π by solving a simple system of linear equations and there exists numerical software to do so. To illustrate one approach, as discussed in class, we can rewrite $\pi = \pi P$ as $\pi(I - P) = 0$, where I denotes the identity matrix. We can incorporate $\pi e = 1$ in this system of linear equations by defining Z to be the matrix $(I - P)$ with any column, say the last column, replaced by all ones (the vector e) and by defining b to be the (column) vector $(0, 0, \ldots, 0, 1)$. Then we have $\pi = bZ^{-1}$, from which we obtain π by inverting the matrix Z and multiplying the vector b on the right by this inverse matrix Z^{-1}.

From time-fractions to rewards Our original objective was to figure out the long-run rewards. When we are in state i we collect a reward $f(i)$. But what we had above is only a way to compute the fraction of steps we spend in state i. Well, obviously, if we know that we spend 20 percent of the steps in state 1, we also know that on 20 percent of the steps we collect a reward of $f(1)$, etc. Mathematically, for the three-state example:

$$\mathbb{E}[f(X_k) | X_0 = x] = \sum_{j=1}^{3} f(j)\mathbb{P}\{X_k = j | X_0 = x\}.$$

Thus, the long-run average reward is

$$\frac{1}{n}\sum_{k=0}^{n-1} f(X_k) = \frac{1}{n}\sum_{k=0}^{n-1}\sum_{j=1}^{3} f(j) 1\{X_k = j\} = \sum_{j=1}^{3} f(j)\frac{1}{n}\sum_{k=0}^{n-1} 1\{X_k = j\}.$$

As n grows large we know that $\mathbb{E}[\frac{1}{n}\sum_{k=0}^{n-1} 1\{X_k = j\}]$ converges to π_j. Thus, we have

$$\lim_{n\to\infty} \mathbb{E}[\frac{1}{n}\sum_{k=0}^{n-1} f(X_k)] = \sum_{j=1}^{3} f(j) \lim_{n\to\infty} \mathbb{E}[\frac{1}{n}\sum_{k=0}^{n-1} 1\{X_k = j\}] = \sum_{j=1}^{3} f(j)\pi_j.$$

So, recalling that to solve for π_j all we need is to solve linear equations, this means that for any reward function we can build on those linear equations to evaluate the long-run average rewards.

Why is π called the stationary distribution? First notice that π is a legitimate distribution. It is non-negative ($\pi \geq 0$) and sums up to 1 ($\sum_{j=1}^{3} \pi_j = 1$). It turns out, *if you start at time $k = 0$ in π — that is $X_0 = j$ with probability π_j — then you get stuck there*, meaning that

$$\mathbb{P}\{X_k = j\} = \pi_j, \text{ for all times } k.$$

We can prove that. For example, if X_0 follows the distribution π then

$$\mathbb{P}\{X_k = 1\} = \sum_{j=1}^{3} \mathbb{P}\{X_k = 1|X_0 = j\}\mathbb{P}\{X_0 = j\} = \sum_{j=1}^{3} \pi_j P_{j1} = \pi_1.$$

The last equality is just from our system of equations for π.

In what practical contexts do we use long-run average? This is a question that we should open with but, actually, there is value in addressing this now that we have seen the analysis.

One family of settings where this is relevant concerns those where it is natural to think about long-run average and discounting does not make sense because the rewards, say, are not monetary. For example, say you run the emergency service (ambulance and fire department) for LA county. You might be interested in the fraction of ambulances that arrive to the location of the incident within 2 minutes over a year. It is not trivial to model such a network as a Markov chain. The point here is that long-run averages make sense here in contrast to discounting.

Another family is one where you are actually interested in performance over short time periods but long-run averages provide a reasonable approximation and one that is computationally simple **(remember – you only have to solve linear equations).**

8.2 Limit distributions

So π does not only capture long-run averages. It also is a stationary distribution: if you start there you stay there. It turns out, with some additional conditions, you do not even have to start there to get there. That is, no matter where you start, you will end up with π far enough in the future. In more mathematical terms: No matter what is the distribution of X_0,

$$\mathbb{P}\{X_k = j\} \approx \pi_j, \text{ for all } k \text{ large}$$

To have this, **in addition to irreducibility**, we have to require **aperiodicity**. It is useful to think here of yet another instance of our three-state Markov chain as illustrated in Figure 8.2.

Figure 8.2 is a trivial Markov chain where, with probability 1, you go from state 1 to state 2, from state 2 to state 3 and from state 3 to state 1. Here, you can return from 1 to itself in n steps only if n is divisible by 3 – 3 steps, 6 steps, 9 steps, etc. The probability of returning from 1 to itself is non-zero only if n is divisible by 3. We say that state 1 has period 3. In fact, this Markov chain is *periodic* with each state having period 3. The less trivial chain in Figure 8.3 is very different in this regard.

In Figure 8.3, we can return from state 1 to itself in 3 steps ($1- > 2- > 3- > 1$) but also in 4 steps ($1- > 2- > 2- > 3- > 1$ or $1- > 2- > 3- > 3- > 1$) and also in 5 steps, etc. Here **the period of state 1 is 1.**

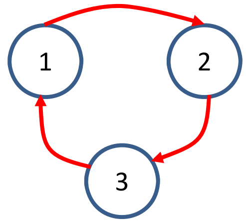


Figure 8.2: Simple periodic Markov chain

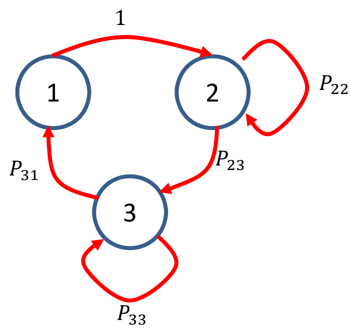


Figure 8.3: Simple aperiodic Markov chain

When all states have a period of 1, we say that the Markov chain is *aperiodic*. The Markov chain in Figure 8.3 is aperiodic. The fundamental result is then that **for irreducible aperiodic chains,** π **(computed by the linear system of equations) is a limit distribution** in the sense that

$$\mathbb{P}\{X_k = j\} \to \pi_j, \text{ as } k \to \infty.$$

Why is aperiodicity needed here? Well, let's look again at Figure 8.2.

My statement above is clearly not true here. If we start in state 1, $\mathbb{P}\{X_{3k} = 1 | X_0 = 1\} = 1$ for all k (no matter how large) but $\mathbb{P}\{X_{2k} = 1 | X_0 = 1\} = 0$ for all k. So, if we just "visit/sample" this chain at random periods, what we will see will not be really informative of the long-run average behavior of the chain.

If it is irreducible and aperiodic then we are on solid ground. Sampling the chain gives us a true understanding of its long-run average.

There are also cases where we are truly interested in the limit distribution (rather than the long-run average):

1. Macro economics: you introduce a new taxation policy that is designed to affect the distribution of income in society. It will take some time (indeed years) until this has a visible effect, but what you are making is a long-term decision and you are interested in how the distribution of income will eventually look.

2. Algorithms: Look at the case of applying the results in the next example based on page rank.

8.3 Some cool facts about moving on graphs: page rank

Consider a connected (undirected) graph that has a weight w_{ij} associated with the edge between i and j. **Notice that $w_{ij} = w_{ji}$ (there is no importance to order)**. Suppose that when in state i you move to state j with a probability that is proportional to the weight $w_{i,j}$. Specifically,

$$P_{ij} = \frac{w_{ij}}{\sum_k w_{ik}}.$$

Then, it turns out we have a beautifully simple expression for the stationary distribution π. It is given by

$$\pi_i = \frac{\sum_k w_{ik}}{\sum_{j,l} w_{jl}} = \frac{\text{total outgoing weight from node } i}{\text{total outgoing weight of all nodes}}.$$

We can verify that this is indeed correct. When we have a guess for a stationary distribution, all we have to do is verify that it satisfies the equations $\pi \geq 0$, $\sum_i \pi_i = 1$ and $\pi = \pi P$. It is trivial here that the sum is 1 and that all π_i are non-negative. Let's check $\pi_j = \sum_i \pi_i P_{ij}$:

$$\sum_i \pi_i P_{ij} = \sum_i \frac{\sum_k w_{ik}}{\sum_{kl} w_{kl}} \frac{w_{ij}}{\sum_k w_{ik}} = \frac{\sum_i w_{ij}}{\sum_{kl} w_{kl}} = \pi_j.$$

Now, let's use it on **page rank**. The basic version of page rank is that the "surfer" moves from webpage i to webpage j (linked to i) with probability $1/n_i$ where n_i is the number of outgoing links from page i. Suppose that the web we consider always has a back link. Namely that if i links to j then j links to i. So we have a movement on a graph. What are the corresponding weights though? Well, take $w_{ij} = 1$ for all edges. Then, consistent with what we want

$$P_{ij} = \frac{w_{ij}}{\sum_j w_{ij}} = \frac{1}{n_i}.$$

So, we can apply our results on graphs to argue that, in the long run, a fraction

$$\pi_i = \frac{\sum_j w_{ij}}{\sum_{kl} w_{kl}} = \frac{n_i}{\sum_k n_k} = \frac{\text{page } i \text{ outward links}}{\text{total web outward links}}$$

of a surfer's page-visits will be to page i. This is cool because it means that the frequency of page visits is proportional to the **outgoing links** of a page rather than ingoing links. Kind of surprising but not too much because you have to remember that, in our model, the more outward links you have the more inward links you have as well in our model. In the real web, you might not have *back links*.