CS310 2021
Exercise problem set 6

March 20, 2021

1. For the following languages give context-free grammars to accept them. You must write a 2-3 line justification for each. Also, for each langauge, give an informal and formal descriptions of a PDA accepting the language. Since for PDA acceptance by empty stack and final state are equivalent, you can use either definition.

   (a) $L_1 = \{w \in \{0,1\}^* \mid w \text{ contains at least four 1's } \}$

   (b) $L_2 = \{w \in \{0,1\}^* \mid w \text{ has odd length and middle letter is } 0\}$

   (c) $L_3 = \{w \in \{0,1\}^* \mid \text{number of 0's in } w \text{ is equal to number of 1's in } w\}$.

   (d) $L_4 = \{0^n 1^m 0^n \mid m, n \geq 1\}$.

2. Consider $L_5 = \{a^i b^j c^k \mid i, j, k \geq 0 \text{ and } i = j \text{ or } i = k\}$.

   (a) Construct a PDA to accept L_5.

   (b) Give a context-free grammar (CFG) that accepts L_5.

   (c) Prove or disprove: L_5 cannot be accepted by any DPDA.

3. (From many states to one state) Consider $L_6 = \{0^n 1 0^n \mid n \geq 1\}$.

   (a) Give a PDA with three states accepting L_6.

   (b) Can you give a PDA with just one state (but maybe a larger stack alphabet) which accepts L_6?

   (c) Generalize this argument to show why for ANY PDA P accepting L, we can construct another PDA P' which has only one state which accepts L.

4. (Acceptance by empty stack in PDA) Prove formally that PDA with acceptance by empty stack are expressively equivalent to PDA with acceptance by final state. Recall that expressively equivalent means that they accept the same set of languages. (The idea of the proof was given in class, this exercise is to ask you to write the details formally!).

5. Conversions:

(a) Convert the grammar $S \to S0S1S0S \mid S0S0S1S \mid S1S0S0S \mid \epsilon$ into a PDA that accepts the same language by empty stack.

(b) Consider $\{a^i b^j c^k \mid i = 2j \text{ or } j = 2k\}$. Construct a CFG for it and then convert it into a PDA.

(c) Convert the following PDA which accepts by empty stack into a CFG using the algo given in lectures. Also simplify the resultant CFG.

$$(\{q\}, \{0, 1\}, \{\perp, A, B\}, \delta, q, \perp, \emptyset)$$

where δ is given by

- $\delta(q, 0, \perp) = (q, A\perp)$, $\delta(q, 1, \perp) = (q, B\perp)$
- $\delta(q, 0, A) = (q, AA)$, $\delta(q, 1, B) = (q, BB)$
- $\delta(q, 1, A) = (q, \epsilon)$, $\delta(q, 0, B) = (q, \epsilon)$
- $\delta(q, \epsilon, \perp) = (q, \epsilon)$

6. (Closure properties) Let L and L' be two context-free languages and let R be a regular language. Recall that a language is context-free if it can be accepted by a PDA or if some CFG accepts it. Show that:

   (a) (concat) $L \circ L'$ is always a context-free language.

   (b) (intersection with regular lang) $L \cap R$ is always a context-free language.

7. (Pumping Lemma for CFLs) For each of the following languages, state if they are context-free or not. If yes, give a PDA/CFG for them, if not give a proof via pumping lemma for CFLs. You may use closure properties from question above or from class also.

   (a) $L_7 = \{0^n 1^5 2^n 1^5 \mid n \geq 0\}$

   (b) $L_8 = \{0^n 1 0^{2n} 1 0^{3n} \mid n \geq 0\}$

   (c) $L_9 = \{w \in \{a, b, c\}^* \mid w \text{ contains equal number of } a\text{'s, } b\text{'s and } c\text{'s}\}$.

8. Prove or disprove: any language over a singleton alphabet, i.e., $|\Sigma| = 1$ is context-free.

9. (Algorithms) Give an algorithm, if possible, for the following problems. What is the complexity of your algorithm?

   (a) Given a CFG G, w, to check if $w \in L(G)$.

   (b) Given a CFG G, is $L(G)$ finite?

   (c) (*)Given a CFG G, is its language universal, i.e., is $L(G) = \Sigma^*$.

10. Consider the pumping lemma for CFLs stated in the lectures. Note that it is an implication, but not an iff. Can you show why? That is, give an example of a language that is not context-free language, but which satisfies the pumping lemma conditions. (hint: take inspiration from the example for regular languages!)