CS310 2020 (Automata Theory)
Exercise problem set 8: Turing machines, decidability and undecidability

1. Design a Turing machine to recognize the language of palindromes.

2. Prove or disprove: the following are *decidable* languages.

   (a) $L_1 = \{\langle R, w\rangle \mid R \text{ is a regular expression that generates } w\}$

   (b) $L_2 = \{\langle G, w\rangle \mid G \text{ is a CFG that generates } w\}$ (be careful! is it decidable or just Turing recognizable?)

   (c) $L_3 = \{\langle G\rangle \mid G \text{ is a CFG and } L(G) = \emptyset\}$

   (d) $L_4 = \{\langle A\rangle \mid A \text{ is a DFA and } L(A) = \Sigma^*\}$

3. Prove that the set of real numbers is uncountable, i.e., there is no bijection between the set of real numbers and the set of natural numbers.

4. In lecture slides, we have a proof of the undecidability of halting problem for Turing machines (i.e., given a Turing machine, M and input word w, does M halt on w).

   (a) Give another proof of the above undecidability by directly using the diagonalizability technique.

   (b) Suppose we define another problem, called ALL-HALT, which given a Turing machines, asks whether it halts on *all* inputs.

      i. Formulate this new problem as a language.

      ii. Prove or disprove: ALL-HALT is decidable.

      iii. Prove or disprove: ALL-HALT is Turing-recognizable.

5. Consider the following languages:

   (a) $\{\langle M\rangle \mid M \text{ is a Turing machine and } 1011 \in L(M)\}$

   (b) $\{\langle M\rangle \mid M \text{ is a Turing machine and } L(M) = \Sigma^*\}$

   (c) $\{\langle M\rangle \mid M \text{ is a Turing machines and } L(M) \text{ is Turing recognizable}\}$

   (d) $\{\langle M\rangle \mid M$ is a Turing machine that never moves more than 2 steps to left or right of its initial position$\}$

(e) $\{\langle M\rangle \mid M$ is a Turing machine and $L(M)$ is the set of all palindromes.$\}$

(f) $\{\langle M_1, M_2\rangle \mid M_1, M_2$ are Turing machines and $L(M_1) = L(M_2)\}$

Answer the following questions for *each* of the above languages.

(i) Can Rice's theorem be used to show undecidability? why or why not?

(ii) Are they decidable or undecidable?

(iii) For languages that you think are undecidable, give a proof by reduction (i.e., without using Rice's theorem).

6. Let M be a TM that has 2-tapes. Consider the problem of determining whether this TM ever writes a nonblank symbol on its second tape, on input w.

   (a) Formulate this problem as a language.

   (b) Prove or disprove: the above language is decidable.