Lecture 1: Propositions and logical connectives

One of the stated objectives of the course is to teach students how to understand and fashion mathematical arguments. Essential to and characteristic of these arguments is a precise logical structure. This first preliminary lecture hopes to make these logical notions clear and to illustrate how to use them when building arguments.

1 Propositions

In mathematics we are in the business of proving or disproving certain types of sentences. As such we are concerned with sentences that are either true or false. These are called propositions.

Definition 1.1. A **proposition** is a sentence which is either true or false, but not both.

Example (Propositions)**.** • '2 is an even number.'

- 'The sun revolves around the earth.'

Example (Non-propositions)**.** • 'What time is it?'

- 'Go to bed!'

2 Compound propositions

We can build up more complicated, *compound propositions* using the logical operations of *conjunction*, *disjunction* and *implication*, associated most commonly in English with the constructions 'and', 'or', and 'if...then', respectively.

2.1 Conjunction and disjunction

Let P and Q be two propositions.

Definition 2.1. The **conjunction** of P and Q is the proposition 'P and Q'. This new proposition is true exactly when both P and Q are true. In other words, the conjunction is false when either one of P and Q is false.

Comment 2.1. Note for our purposes, to understand 'P and Q' is simply to understand when it is true. "What is truth?", you may ask. As the poet Keats would have it: beauty is truth, truth beauty–that is all ye know on earth, and all ye need to know.

Example. Let P be the proposition '2 is even', and let Q be the proposition 'The sun revolves around the earth'. Then the sentence 'P and Q' is false since one of the component sentences, Q, is false.

Let P and Q be two propositions.

Definition 2.2. The **disjunction** of P and Q is the proposition 'P or Q'. This new proposition is true when P is true, or Q is true, or both. In other words, it is false only when both P and Q are false.

This notion of disjunction is sometimes described as the *inclusive or*. The *exclusive or* construction, which is true when exactly one of P and Q is true, is usually rendered in English as 'P or Q, but not both'.

Example. Let P be the proposition '2 is even', and let Q be the proposition 'The earth revolves around the sun'. Then the sentence 'P or Q' is true since at least one of the sentences P and Q is true. In fact both sentences are true in this case. This means that the *exclusive or* statement, 'P or Q, but not both', is false.

Here are some other ways in which these logical connectives are sometimes rendered in English.

- **Conjunction**. 'P and Q', 'P while Q, 'P, but Q', 'P, yet Q'.
- **Disjunction**. 'P or Q', 'Either P or Q'.
- **Exclusive disjunction**. 'P or Q, but not both', 'Either P or Q, but not both'.

The truth value of a compound proposition is defined in terms of the truth value of its component propositions. We can express this in a succinct way using *truth tables*.

- Conjunction.

P	Q	P and Q
T	T	T
T	F	F
F	T	F
F	F	F

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- Disjunction.

P	Q	P or Q
T	T	T
T	F	T
F	T	T
F	F	F

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2.2 Implication

Let P and Q be two propositions.

Definition 2.4. The proposition 'If P, then Q' is called an **implication**. We will often write it symbolically as $P \Rightarrow Q$.

When is this new proposition true?

P	Q	$P \Rightarrow Q$
T	T	T
T	F	F
F	T	T
F	F	T

. The truth table for implication is not exactly intuitive. The two first rows where P is true are straightforward enough. For the rows where P is false consider the following justification: first, $P \to Q$ is supposed to be a proposition, so it needs to be true or false; next, if P is false, then no matter what the truth value of Q, we wouldn't say that the implication is false, because intuitively the implication only asserts something when P is the case; but if the truth value of $P \Rightarrow Q$ is not F for the last two rows, it must be T.

Example. Let P be the proposition 'Aaron is human', and let Q be 'Aaron is mortal'. Consider $P \Rightarrow Q$. The only situation in which $P \Rightarrow Q$ is not true is when Aaron is human, yet Aaron is not mortal. Suppose Aaron is not human. Then the implication $P \Rightarrow Q$ is true, no matter whether this nonhuman Aaron is mortal or not!

Example (Contract interpretation)**.** Footballer Lionel enters into the following contract with manager Pep: if Lionel scores 50 goals during the regular season, then Lionel will be given a new Xbox. We can represent this contract as $P \Rightarrow Q$ in the obvious way. Suppose Lionel does not score 50 goals during the season (i.e., that P is false). Whether or not Pep gives Lionel a new Xbox, will the contract have been violated?

- English variants of '$P \Rightarrow Q$': 'If P, then Q', 'P implies Q','P only if Q', 'Q if P', 'Q when P'.
- In the implication $P \Rightarrow Q$, P is referred to variously as the **antecedent**, the **hypothesis**, or as the **sufficient condition**; and Q is referred to as the **consequent**, the **conclusion**, or as the **necessary condition**.

2.3 Equivalence

- Given implication $P \Rightarrow Q$, the implication $Q \Rightarrow P$ is called the **converse** of $P \Rightarrow Q$.
- Note that an implication $P \Rightarrow Q$ being true does not necessarily mean that the converse $Q \Rightarrow P$ is true.
P	Q	$P \Rightarrow Q$	$Q \Rightarrow P$
T	T	T	T
T	F	F	T
F	T	T	F
F	F	T	T

Definition 2.5. The proposition 'P if and only if Q', written symbolically as $P \Leftrightarrow Q$, is called an **equivalence**.

The equivalence $P \Leftrightarrow Q$ is true if the implications $P \Rightarrow Q$ and $Q \Rightarrow P$ are *both* true. Looking at the truth tables for $P \Rightarrow Q$ and its converse, we see that $P \Leftrightarrow Q$ is true when P and Q have the *same truth* values.

P	Q	$P \Rightarrow Q$	$Q \Rightarrow P$
T	T	T	T
T	F	F	T
F	T	T	F
F	F	T	T
P	Q	$P \Leftrightarrow Q$
T	T	T
T	F	F
F	T	F
F	F	T
- Variant renderings of 'P if and only if Q': $P \Leftrightarrow Q$, 'P iff Q', 'P is equivalent to Q', 'P is necessary and sufficient for Q'.
- Recall: $P \Leftrightarrow Q$ is true when $P \Rightarrow Q$ is true and $Q \Rightarrow P$ is true. The first implication asserts P is sufficient for Q, the second asserts P is necessary for Q. Thus our 'necessary and sufficient'.

2.4 Negation

Definition 2.6. The **negation** of a proposition P is the proposition 'Not P', written symbolically as $\neg P$. The negation 'Not P' is true exactly when P is false.

P	Not P
T	F
F	T

2.5 Truth table examples

Compute the truth table for the proposition 'Not (P or Q)', or perhaps more naturally in English, 'It is not the case that P or Q'.

P	Q	P or Q	Not (P or Q)
T	T	T	F
T	F	T	F
F	T	T	F
F	F	F	T

3 Exercises

1.1 Compute the truth table of '(Not P) or (Not Q)'. Your table should have 5 columns: namely, P, Q, Not P, Not Q, Not P or Not Q.

1.2 Compute the truth table of '(Not P) and (Not Q)'. Again, your table should have 5 columns. Compare with the truth table of 'Not (P or Q)'.

1.3 Show that '$P\Rightarrow Q$' and '(Not Q)$\Rightarrow$ (Not P)' have the same truth table. More specifically, make a truth table whose columns are P, Q $P \Rightarrow Q$, and (Not Q)$\Rightarrow$ (Not P).