Module: Counting Part 2
CS 203: Discrete Structures
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INSTRUCTIONS: The following are the practice problems to improve your understanding of the concepts in Counting module. Try to solve all problems. You do not have to submit the solutions.

1. In how many ways can a given positive integer $n \geq 1$ be expressed as the sum of 2 positive integers (which are not necessarily distinct). For example, for n=3, the number of ways is 2, i.e.2+1,1+2.

2. We have a cat in a Cartesian coordinate 2D(X,Y) plane. Now let us suppose the cat is at origin and has to reach the point (4,3). It has restricted movements where it is allowed to move from coordinates (i,j) to (i+1,j) or (i,j) to (i,j+1). In how many ways is this possible?

3. The minimum number of people in a room to be present such that 3 distinct people are sure to have the same birth month.

4. Show that if seven integers are selected from the set $\{1, 2, 3, 4, \cdots, 10\}$ there must be at least two pairs of these integers with the sum 11.

5. How many number of ways can 10 balls be chosen from an urn containing 10 identical green balls, 5 identical yellow balls and 3 identical blue balls ?

6. A box contains 6 red, 8 green, 10 blue, 12 yellow and 15 white balls. What is the minimum no. of balls we have to choose randomly from the box to ensure that we get 9 balls of same color?

7. Which one the following is a closed form expression for the generating function of the sequence $\{a_n\}$, where $a_n = 2n + 3$ for all $n = 0, 1, 2, \ldots$?

   a $3/(1-x)^2$

   b $3x/(1-x)^2$

   c $(2-x)/(1-x)^2$

   d $(3-x)/(1-x)^2$

8. If the ordinary generating function of a sequence $\{a_n\}_{n=0}^{\infty}$ is $(1+z)/(1-z)^3$, then what will be the value of $a3$–$a0$?

9. Find a recurrence relation for the number of ways to climb n stairs if the person climbing the stairs can take one stair or two stairs at a time. What are the initial conditions? In how many ways can this person climb a flight of eight stairs?

10. Find a recurrence relation for the number of bit strings of length n that contain a pair of consecutive 0s. What are the initial conditions? How many bit strings of length seven contain two consecutive 0s?

11. A square has unit length of 4. Seventen points are arbitrarily chosen in the square. Prove that there exist 2 of them having a distance not more than $\sqrt{2}$.

12. What is the least value of k such that there must be a pair of numbers from $\{1, 2, 3, ..., k\}$, with a sum equal to 9?

13. Prove that $2^n \geq 1 + n\sqrt{2^{n-1}}$; $n =$ 2,3,...

14. Is it true that $n^2 + n + 41$ is prime for any natural number n ?

15. Find a recurrence relation for the number of ternary strings of length n that do not contain two consecutive 0s. What are the initial conditions? How many ternary strings of length six do not contain two consecutive 0s?

16. Find a recurrence relation for the number of bit strings of length n that do not contain three consecutive 0s. What are the initial conditions? How many bit strings of length seven do not contain three consecutive 0s?

17. Solve the recurrence relation for $a_n = 5a_{n-1} - 6a_{n-2}$ where $n \geq 2, a_0 = 1, a_1 = 0$.

18. A bag contains beads of two colors : black and white. What is the smallest number of beads which must be drawn from the bag, without looking, so that among these beads there are ten of the same color ?

19. Given 8 integers, show that two of them can be chosen whose difference is divisible by 7.

20. Using inclusion and exclusion find how many solutions does

$$x_1 + x_2 + x_3 = 11$$

have, where x_1, x_2 and x_3 are non negative integers with $x_1 \leq 3, x_2 \leq 4$ and $x_3 \leq 6$?

21. A new employee checks the hats of n people at a restaurant, forgetting to put claim check numbers on the hats. When customers return for their hats, the checker gives them back hats chosen at random from the remaining hats. What is the probability that no one receives the correct hat?

22. What is the closed form expression for the generating function of the sequence $\{a_n\}$, where $a_n = 2n + 3$ for all $n = 0, 1, 2, ...$?

23. Consider the recurrence relation $a_1 = 8$, $a_n = 6n^2 + 2n + a_{n-1}$. Let $a_{99} = k * 10^4$. Then what is the value of k?

24. Prove by mathematical induction :

$1^3 + 2^3 + ... + n^3 = \frac{n^2(n+1)^2}{4}$

25. Prove that a number written with 3^n ones is divisible by 3^n.

26. Prove that for any natural number n, $2^{3^n} + 1$ is divisible by $3^n + 1$.

27. Prove that of any 52 integers, two can always be found such that the difference of their squares is divisible by 100.

28. Prove that there exists an integer whose decimal representation consists entirely of 1's, and which is divisible by 1987.

29. Fifteen boys gathered 100 nuts. Prove that some pair of boys gathered an identical number of nuts.

30. Prove that we can choose a subset of a set of ten given integers, such that their sum is divisible by 10.

31. Given 11 different natural numbers, none greater than 20. Prove that two of these can be chosen, one of which divides the other.