Module: Group Theory and Number Theory
CS 203: Discrete Structures
Course Instructor : Prof. Prabuchandran K J

INSTRUCTIONS: The following are the practice problems to improve your understanding of the concepts in Group and Number Theory module. Try to solve all problems. You do not have to submit the solutions.

1. Define a group. Give different examples of group.
2. Define a subgroup of a group. Give different examples of subgroups of group.
3. Prove in a group identity is unique.
4. Prove cancellation law holds in a group, i.e., $a * b = a * c \implies b = c$.
5. If G is a group such that $(ab)^2 = a^2b^2 \ \ \forall a, b \in G$, show that G must be Abelian.
6. Define Isomorphism and Homomorphism. Give examples.
7. Give an example of homomorphism which is not an isomorphism.
8. Let G be a permutation group on a set A of 6 symbols. Then the order of proper normal subgroup of G is: a)120 b)240 c) 360 d)720.
9. Let G be the group of all nonzero complex numbers under multiplication and let $\hat{G}$ be the group of all real 2 ×2 matrices of the form $\begin{pmatrix} a & b \\ -b & a \end{pmatrix}$, where not both a and b are 0, under matrix multiplication. Show that G is isomorphic to $\hat{G}$ by exhibiting an isomorphism.
10. Let G be a finite group and $(\mathbb{Z}/10\mathbb{Z}, +)$ be a homomorphic image of G. Then
    a order of G divides 10.
    b order of G does not divide 10
    c order of G is divisible by 10
    d order of G is not divisible by 10
11. Define normal subgroup. Give example of a normal subgroup.
12. Define right cosets and left cosets of a subgroup.
13. Define a normal subgroup. Give an example of subgroup which is not a normal group.
14. Is a group Homomorphism always a group Isomorphism? If not, give a counter-example and justify.
15. Prove the kernel of a homomorphism is always a normal subgroup. Give an example.
16. Define a cyclic subgroup of a finite group. Give an example.

17. Define a cyclic subgroup of a infinite group. Give an example.
18. State and prove Lagrange's theorem.
19. Define Quotient Group. Give examples.
20. If $GL_n(\mathbb{R})$ is the group of invertible $n \times n$ real matrices, and N is the subgroup of $n \times n$ real matrices with determinant 1, then prove that N is normal subgroup. What are the cosets? What does the quotient group represent?
21. Define $(\mathbb{Z}/p\mathbb{Z})^*$. Prove that $1, p-1$ are the only elements in this group for which inverse is itself.
22. State and prove Chinese Remainder Theorem.
23. Define Euler totient function $\Phi(n)$. Give a formula to compute it. Prove that $\Phi(mn) = \Phi(m)\Phi(n)$, where $gcd(m,n) = 1$, i.e., $\Phi(\cdot)$ is multiplicative.
24. State and prove Fermat's little theorem and prove its generalization.
25. If H is a subgroup of G, then by the centralizer $C(H)$ is defined as the set $\{g \in G | gh = hg \ \ \forall h \in H\}$. Prove that C(H) is a subgroup of G.
26. What is the remainder when 1037^{10} is divided by a) 5 b) 6 c) 3 d) 1000
27. If 4 structures are given (A,*),(B,#),(C,$) and (D,@). Now you are given the following data. Comment what are each of the following structures (Group, Monoid, Semi Group, Magma, ...)

    (a) A,B,C,D are closed
    (b) C has an identity element and inverse exists but inverse is not in domain
    (c) $a * (b * c) = (a * b) * c \ \ \forall a, b, c \in A$
    (d) D does not obey associative law
    (e) C and B obeys associative law and commutative law
    (f) B does not have an identity element.
    (g) C has a unique identity element
28. Let $(G, *)$ be a group in which the square of every element is the identity. Show that G is abelian.
29. Give an example of a semi group without an identity element.
30. Give an example of an infinite semi group with an identity element e such that no element except e has an inverse.