MA 201 Complex Analysis
Mid Semester Examination
IIT Dharwad (Autumn 2021)

Total Marks: 30 **Date & Time:** 04 September 2021, 03:00 pm to 04:25 p.m.

Part 1

(1) **[3M]** (a) Show that the function $u(x,y) = \sinh x \sin y$ is harmonic. Find its harmonic conjugate.
(b) Show that the function $u(x,y) = \cosh x \cos y$ is harmonic. Find its harmonic conjugate.

(2) **[3M]** Discuss at which points on the complex plane the function given by $f(z) = z^2 + z\bar{z} + c(\bar{z})^2$ is analytic where c is the last two digits of your roll number (for example, if your roll number is 190010023, then c=23).

(3) **[3M]** (a) Using the definition of contour integration compute $\int_C \bar{z}dz$ where C is the positively oriented boundary of the half disc $0 \le r \le 1$, $0 \le \theta \le \pi$.
(b) Using the definition of contour integration compute $\int_C \bar{z}dz$ where C is the positively oriented boundary of the half disc $0 \le r \le 1$, $\pi \le \theta \le 2\pi$.

(4) **[2M]** (a) Find all the complex numbers z such that $\overline{e^{iz}} = -e^{i\bar{z}}$.
(b) Find all the complex numbers z such that $\overline{e^{iz}} = e^{i\bar{z}}$.

(5) **[2M]** Let C be the circle of radius $1/2$ centered at a where a is the last two digits of your roll number (for example, if your roll number is 190010023, then a=23). Use Cauchy's theorem to show that $\int_C \frac{1}{z}dz = 0$. What is $\int_C \frac{1}{z-a}dz$? Justify your answer.

(6) **[2M]** (a) Determine whether the function $f(z) = \frac{\mathrm{Re}(z)}{1+|z|}$ is continuous at 0. Justify your answer (in case f is continuous, prove using $\epsilon - \delta$ definition).
(b) Determine whether the function $f(z) = \frac{\mathrm{Im}(z)}{1+|z|}$ is continuous at 0. Justify your answer (in case f is continuous, prove using $\epsilon - \delta$ definition).

(7) **[1M]** Let $\mathrm{Log}(z)$ denotes the principal branch of the logarithm multifunction defined using the principal argument that lies in $[0, 2\pi)$. Show that $\mathrm{Log}((-1)^2) \neq 2\,\mathrm{Log}(-1)$.

Part 2

(1) **[3M]** Consider the complex valued function f defined from $\mathbb{C}$ to $\mathbb{C}$ by $f(z) = \sqrt{|x||y|}$. Show that the real and imaginary parts of f satisfies the CR equations at 0 but f is not differentiable at 0.

(2) **[2M]** Let c be the last two digits of your roll number (for example, if your roll number is 190010023, then c=23). For $z \neq c$ consider the function $f(z) = \mathrm{Log}(z)$ where Log is the principal branch of the logarithm multifunction defined using the principal argument that lies in $[0, 2\pi)$.

(1) Describe the set of points where the function f is not continuous.
(2) Find the largest open subset of $\mathbb{C}$ where the function f is continuous.

(3) **[2M]** (a) Let C denote the line segment from $z = i$ to $z = 1$. By using ML-inequality prove that $\left|\int_C \frac{1}{z^4} dz\right| \leq 4\sqrt{2}$. (Hint: observe that of all the points on that line segment, the midpoint is the closest to the origin).
Bonus quetsion [1M] Compute $\int_C \frac{1}{z^4} dz$.
(b) Let C denote the line segment from $z = i$ to $z = 1$. By using ML-inequality prove that $\left|\int_C \frac{1}{z^2} dz\right| \leq 2\sqrt{2}$. (Hint: observe that of all the points on that line segment, the midpoint is the closest to the origin).
Bonus quetsion [1M] Compute $\int_C \frac{1}{z^2} dz$.

(4) **[3M]** By finding the antiderivative, evaluate the integral $\int_\gamma e^{\pi z} dz$ where γ is any contour joining 0 and ic. Here c is the last two digits of your roll number (for example, if your roll number is 190010023, then c=23).

(5) **[3M]** Suppose that u and v are real valued functions defined in a domain D such that u is a harmomic conjugate of v, and v is a harmonic conjugate of u. Then show that u and v are both constants.

(6) **[2M]** (a) (1) Determine and sketch the image of the horizontal strip $0 \leq y \leq \pi/4$ under the exponential map.
(2) Find all the complex numbers z such that $z = (-1)^i$.
(b) (1) Determine and sketch the image of the vertical strip $0 \leq x \leq \pi/4$ under the exponential map.
(2) Find all the complex numbers z such that $z = (-1)^{-i}$.