# End Semester Exam (Marking Scheme)

23 Sep 2021

## Part 1

**Q.1** (i)

we know

$$\cos(z) = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!} \quad \forall z \in \mathbb{C}$$

$$\Rightarrow \frac{\cos(z)}{z^c} = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n-c}}{(2n)!} \quad \forall z \neq 0$$

$$= \frac{1}{z^c} - \frac{1}{2}\frac{1}{z^{c-2}} + \frac{1}{2!}\frac{1}{z^{c-4}} + \cdots \quad [0.5]$$

$$\forall z \in \mathbb{C}\setminus\{0\}$$

Since $f(z) = \frac{\cos(z)}{z^c}$ has finitely many negative powers of $z$ in its Laurent series around $0$, f has a pole of order $c$ at $0$. [0.5]

(ii) $$f(z) = \frac{z^2 - (c+1)z + c}{z^2 - cz} \quad \text{at } z = c$$

$$= \frac{(z-c)(z-1)}{z(z-c)}$$

Here $$\lim_{z \to c} (z-c) f(z)$$

$$= \lim_{z \to c} \frac{(z-c)(z-1)}{z} = 0 \quad [0.5]$$

$\therefore$ f has a removable singularity at $z = c$. [0.5]

(iii) $f(z) = z^c \sin\left(\frac{1}{z}\right)$

$$= z^c \left[\sum_{n=0}^{\infty} (-1)^n \frac{1}{(2n+1)!} \cdot \frac{1}{z^{2n+1}}\right]$$

$\forall \quad z \in \mathbb{C} \setminus \{0\}$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \frac{1}{z^{2n+1-c}} \quad \forall\, z \neq 0$$

[0.5]

$\therefore$ f has infinitely many negative powers of z in its Laurent expansion around 0.

$\Rightarrow$ f has an essential singularity at 0.

[0.5]

- For mentioning type of singularity [0.5]
- For justification [0.5]

Q.2] Let $a = c + i$ where $c$ = last two digits of your roll number.
Note that $\operatorname{Log}(z)$ is analytic at $a$.

Let $\operatorname{Log}(z) = \sum_{n=0}^{\infty} a_n (z-a)^n$ be a power series expansion of $\operatorname{Log}(z)$ around $a$.

Then

$$\frac{d}{dz}\left(\operatorname{Log}(z)\right) = \sum_{n=1}^{\infty} n\, a_n (z-a)^{n-1}$$

( $\because$ power series can be differentiated term by term by term)

$$\Rightarrow \quad \frac{1}{z} = \sum_{n=1}^{\infty} n\, a_n (z-a)^{n-1} \quad [1]$$

$$= \sum_{n=0}^{\infty} (n+1)\, a_{n+1} (z-a)^n \quad \text{—— (1)}$$

But

$$\frac{1}{z} = \frac{1}{a + z - a}$$

$$= \frac{1}{a\left(1 + \frac{z-a}{a}\right)}$$

$$= \frac{1}{a} \sum_{n=0}^{\infty} (-1)^n \left(\frac{z-a}{a}\right)^n \quad \text{for } |z-a| < |a|$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{a^{n+1}} (z-a)^n \quad \text{—— (2)}$$

[1]

Comparing (1) and (2) gives,

$$n a_n = \frac{(-1)^{n-1}}{a^n} \qquad n \geq 1$$

$$\Rightarrow \quad a_n = \frac{(-1)^{n-1}}{n a^n}$$

and $\quad a_0 = \text{Log}(a)$ [0.5]

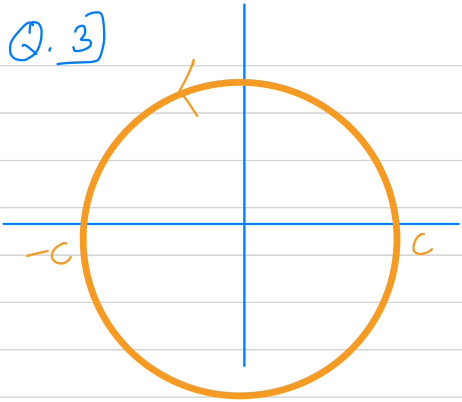
$$\therefore \quad \text{Log}(z) = \sum_{n=0}^{\infty} a_n (z-a)^n \quad \text{where}$$

$$a_n = \begin{cases} \dfrac{(-1)^{n+1}}{n a^n} & \forall \; n \geq 1 \\ \text{Log}(a) & \text{for } n = 0 \end{cases}$$

$$\forall \quad |z-a| < |a| = \sqrt{c^2+1}$$

$\therefore$ Radius of convergence of $\sum_{n=0}^{\infty} a_n (z-a)^n$ is $\sqrt{c^2+1}$. [0.5]

Q. 3




Let

$f(z) = (z-1)^2 \sin(1/z)$

Then $f$ is analytic everywhere on $\{z \in \mathbb{C} : |z| \le 1\}$ except at $z = 0$.

$\therefore$ By Cauchy's Residue theorem

$$\int_{|z|=1} (z-1)^2 \sin(1/z) = 2\pi i \, \mathrm{Res}(f; 0) \qquad [0.5]$$

The Laurent series expansion of $\sin(1/z)$ around 0 is,

$$\sin(1/z) = \sum_{n=0}^{\infty} (-1)^n \frac{1}{(2n+1)!} \frac{1}{z^{2n+1}}$$

$$\Rightarrow (z-1)^2 \sin(1/z) = (z^2 - 2z + 1)\left(\frac{1}{z} - \frac{1}{3!\, z^3} + \frac{1}{5!\, z^5} + \cdots\right)$$

$$= \text{lower order terms} + \left(\frac{-1}{3!} + 1\right)\frac{1}{z} + \text{higher order terms}$$

$$\therefore \mathrm{Res}(f; 0) = -\frac{1}{6} + 1 = 5/6 \qquad [1]$$

$$\therefore \int_{|z|=1} (z-1)^2 \sin(1/z) = (2\pi i)(5/6)$$

$$= \frac{5\pi i}{3} \qquad [0.5]$$

Q. 4] Let $u$ be a real valued harmonic function defined on $D = \{ z \in \mathbb{C} : |z| < 1 \}$

Let $v$ be a harmonic conjugate of $u$ on $D$.

$\therefore$ $f = u + iv$ is analytic on $D$. [0.5]

Hence $f$ is infinitely many times differentiable [0.5]

$\Rightarrow$ all partials of $u$ and $v$ exist and are continuous on $D$ [0.5]

$\Rightarrow$ $u$ is infinitely many times differentiable. [0.5]

Q.5 (a) Since $f$ is entire, $f$ is analytic at $0$.

$\therefore$ By Taylor's theorem

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \quad \text{in } B(0;r) \text{ for some } r>0 \quad [0.5]$$

where $a_n = \frac{f^{(n)}(0)}{n!} = \begin{cases} 0 & \text{if } n \text{ even} \\ (-1)^k & \text{if } n=2k+1 \end{cases}$

$$\therefore f(z) = \sum_{n=0}^{\infty} (-1)^k \frac{z^{2k+1}}{(2k+1)!} \quad \text{in } B(0;r) \quad [1]$$

$$\Rightarrow f(z) = \sin(z) \quad \text{in } B(0;r) \quad [0.5]$$

Clearly, $B(0;r)$ has a limit point in $\mathbb{C}$.

$\therefore$ by the uniqueness theorem,

$$f(z) = \sin(z) \quad \forall z \in \mathbb{C}. \quad [1]$$

Bonus question: We can not conclude the same if $f : \mathbb{R} \to \mathbb{R}$ is differentiable. [0.5]

For example,

Let $f(x) = \begin{cases} \sin(x) + e^{-1/x^2} & \text{if } x > 0 \\ \sin(x) & \text{if } x \le 0 \end{cases}$

Then clearly $f$ is differentiable at $x \neq 0$.

At $x = 0$, $g(x) = \begin{cases} e^{-1/x^2} & \text{if } x > 0 \\ 0 & \text{if } x \le 0 \end{cases}$ is differentiable infinitely many times and $g^{(n)}(0) = 0$.

$\Rightarrow f^{(n)}(0) = \sin^{(n)}(0) + g^{(n)}(0) = \sin^{(n)}(0)$

$= \begin{cases} 0 & \text{if } n \text{ is even} \\ (-1)^k & \text{if } n = 2k+1 \end{cases}$

But clearly $f(x) \neq \sin(x) \ \forall\, x \in \mathbb{R}$. [0.5]

(b) Since $f$ is entire, $f$ is analytic at $0$.

$\therefore$ By Taylor's theorem

$f(z) = \sum_{n=0}^{\infty} a_n z^n$ in $B(0;r)$ for some $r > 0$ [0.5]

where $a_n = \frac{f^{(n)}(0)}{n!} = \begin{cases} 0 & \text{if } n \text{ odd} \\ (-1)^k & \text{if } n = 2k \end{cases}$

$\therefore f(z) = \sum_{n=0}^{\infty} (-1)^k \frac{z^{2k}}{(2k)!}$ in $B(0;r)$ [1]

$\Rightarrow f(z) = \cos(z)$ in $B(0;r)$ [0.5]

Clearly, $B(0;r)$ has a limit point in $\mathbb{C}$.

$\therefore$ by the uniqueness theorem,

$f(z) = \cos(z) \ \forall\, z \in \mathbb{C}$. [1]

**Bonus question:** We can not conclude the same if $f: \mathbb{R} \longrightarrow \mathbb{R}$ is differentiable. [0.5]

For example,

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$$\Rightarrow f^{(n)}(0) = \sin^{(n)}(0) + g^{(n)}(0) = \sin^{(n)}(0)$$

$$= \begin{cases} 0 & \text{if } n \text{ is even} \\ (-1)^k & \text{if } n = 2k+1 \end{cases}$$

But clearly $f(x) \neq \sin(x) \ \forall x \in \mathbb{R}$. [0.5]

Q. 6] (a) We prove that the function $\sin z$ is not bounded on $\mathbb{C}$. [0.5]

Suppose $\sin z$ is bounded on $\mathbb{C}$.

Since $\sin(z)$ is entire, by Liouville's theorem $\sin(z)$ is a constant.

But we know that $\sin(z)$ is not constant

For example, $\sin(0) = 0$ &

$\sin(\pi/2) = 1$ [1.5]

Q. $\boxed{6}$ (b) We prove that the function $\cos(z)$ is not bounded on $\mathbb{C}$. [0.5]

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But we know that $\cos(z)$ is not constant

For example, $\cos(0) = 1$ &

$\cos(\pi/2) = 0$ [1.5]

## Part 2

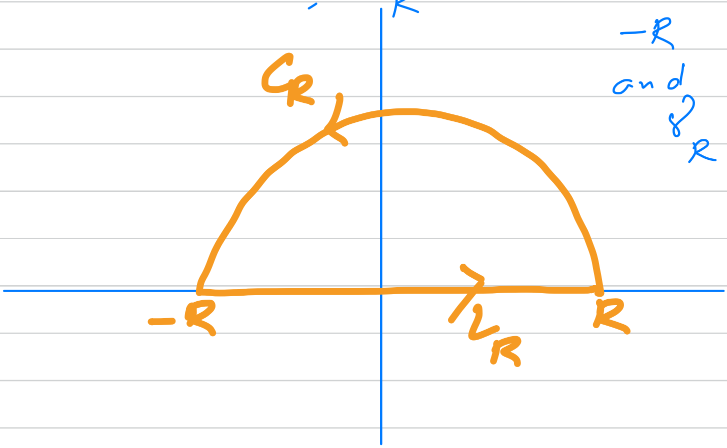
**Q.1** Let $f(z) = \dfrac{1}{(z^2+c)^2}$

Since $f(z) = f(-z)$, $f$ is an even function.

$$\therefore \int_0^\infty \frac{dx}{(x^2+c)^2} = \frac{1}{2}\,\text{P.V.}\int_{-\infty}^{\infty} \frac{dx}{(x^2+c)^2}$$

[0.5]

Let $C_R$ be the circle of radius $R$ centered at $O$, $L_R$ be the line joining $-R$ to $R$ and $\gamma_R = C_R + L_R$.




Then

$$\int_{-R}^{R} f(x)\,dx = \int_{\gamma_R} f(z)\,dz - \int_{C_R} f(z)\,dz \qquad -(1)$$

Here $f$ is analytic on $\mathbb{C}\setminus\{\pm\sqrt{c}\,i\}$ and $f$ has a pole of order 2 at $\pm\sqrt{c}\,i$. Choose $R > \sqrt{c}$ so that $\alpha = \sqrt{c}\,i$ lies in $\gamma_R$.

[0.5]

$\therefore$ By Cauchy's residue theorem

$$\oint_{\gamma_R} f(z)\,dz = 2\pi i \operatorname{Res}(f;\alpha) \quad \text{—(2)}$$

We claim that $\displaystyle\lim_{R\to\infty} \int_{C_R} f(z)\,dz = 0.$

Proof of the claim:

On $C_R$, $\quad |z^2+c|^2 \geq (|z|^2-c)^2 = (R^2-c)^2$

$$\Rightarrow |f(z)| = \left|\frac{1}{(z^2+c)^2}\right|$$

$$\leq \frac{1}{(R^2-c)^2} \quad \text{on } C_R$$

$\therefore$ By ML-inequality

$$\left|\int_{C_R} f(z)\,dz\right| \leq \frac{1}{(R^2-c)^2} \times \pi R$$

$$\longrightarrow 0 \quad \text{as } R\to\infty$$

$$\therefore \lim_{R\to\infty} \int_{C_R} f(z)\,dz = 0 \qquad [1]$$

$\therefore$ By (1) and (2)

$$\text{P.V.} \int_{-\infty}^{\infty} f(x)\,dx = \lim_{R\to\infty} \int_{-R}^{R} f(x)\,dx$$

$$= 2\pi i \operatorname{Res}(f;\alpha)$$

$= g(z)$

Here $f(z) = \dfrac{1}{(z-\alpha)^2 (z+\alpha)^2}$

Since $f$ has a pole of order 2 at $\alpha$,

$\text{Res}(f; \alpha) = \dfrac{g'(\alpha)}{1!}$ where $g(z) = \dfrac{1}{(z+\alpha)^2}$

$\therefore g'(z) = \dfrac{-2}{(z+\alpha)^3}$

$\therefore g'(\alpha) = \dfrac{-2}{8\alpha^3} = -\dfrac{1}{4\alpha^3}$ [1]

$\therefore \text{P.V.} \displaystyle\int_{-\infty}^{\infty} f(x)\,dx = \dfrac{-\pi i}{2\alpha^3}$

$= \dfrac{-\pi i}{2c^{3/2}(-i)}$ [0.5]

$\text{P.V.} \displaystyle\int_{-\infty}^{\infty} f(x)\,dx = \dfrac{\pi}{2c^{3/2}}$

$$\boxed{\therefore \int_{0}^{\infty} f(x)\,dx = \frac{\pi}{4c^{3/2}}}$$ [0.5]

Q.2] Given $\sum_{n=0}^{\infty} \left(c + (-i)^n\right)^n z^n$

$\therefore a_n = \left(c + (-i)^n\right)^n$

By Cauchy-Hadamard formula,

$$\frac{1}{R} = \limsup \left\{ \sqrt[n]{|a_n|} : n \in \mathbb{N} \right\}$$ [0.5]

Here

$$\sqrt[n]{|a_n|} = \left| c + (-i)^n \right|$$

$$= \begin{cases} c+1 & n = 4k \\ \sqrt{c^2+1} & n = 4k+1 \text{ or } 4k+3 \\ c-1 & n = 4k+2 \end{cases}$$

$$\therefore \limsup \left\{ \sqrt[n]{|a_n|} : n \in \mathbb{N} \right\} = c+1$$ [1]

$$\therefore \frac{1}{R} = c+1$$

$$\Rightarrow \boxed{R = \frac{1}{c+1}.}$$ [0.5]

Q.3] (a) Let $D = \{ z \in \mathbb{C} : \operatorname{Im}(z) > 0 \}$

Given $\operatorname{Log}(z) = \ln|z| + i\,\theta(z)$ where $\theta(z) \in \arg(z) \cap (-\pi, \pi)$

$\therefore$ $\operatorname{Log}(z)$ in analytic on $\mathbb{C} \setminus (-\infty, 0]$ [0.5]

We show that $\operatorname{Log}(z)$ has antiderivative on $D$. [0.5]


Since $\operatorname{Log}(z)$ is analytic on $D$ and $D$ is simply connected, by Cauchy's theorem

$$\int_\gamma \operatorname{Log}(z)\,dz = 0$$ for any closed contour $\gamma$. [0.5]

$\therefore$ By Morera's theorem, $\operatorname{Log}(z)$ has an antiderivative on $D$. [0.5]

- For saying that $\operatorname{Log}(z)$ has an antiderivative [0.5]
- For justification [1.5]

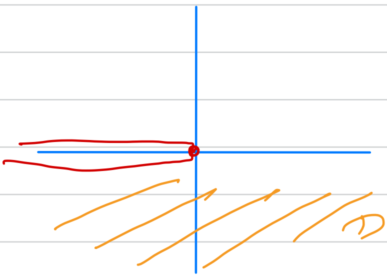
(they may use existence of antiderivative theorem stated in the class)

Q.3 (b) Let $D = \{ z \in \mathbb{C} : \operatorname{Im}(z) < 0 \}$

Given $\operatorname{Log}(z) = \ln|z| + i\,\theta(z)$ where $\theta(z) \in \arg(z) \cap (-\pi, \pi)$

$\therefore$ $\operatorname{Log}(z)$ is analytic on $\mathbb{C} \setminus (-\infty, 0]$ [0.5]

We show that $\operatorname{Log}(z)$ has antiderivative on $D$. [0.5]

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- For saying that $\operatorname{Log}(z)$ has an antiderivative [0.5]
- For justification of antiderivative [1.5]

(they may use existence theorem of antiderivative stated in the class)

Q. 4] No. [0.5]

Suppose there exists a polynomial $p(z)$ s.t.

$$\frac{(1-\cos(z))e^{z^2}}{(e^{z^2}+1)} = p(z).$$

we know $n \in \mathbb{Z}$

$\cos(z) = 1$ for $z = n\pi$ for $n$ even [0.5]

$\Rightarrow p(n\pi) = 0$ for $n \in \mathbb{Z}$ and $n$ even

$\Rightarrow p(z)$ has infinitely many zeroes. This contradicts the fundamental theorem of algebra unless $p(z)$ is a constant. [0.5]

But $p(z)$ is constant & $p(n\pi) = 0$

$p(z) \equiv 0 \quad \forall \; z \in \mathbb{C}.$

$\Rightarrow 1 - \cos(z) = 0 \quad (\because e^{z^2} \neq 0)$

$\Rightarrow \cos(z) = 1 \quad \forall \; z \in \mathbb{C},$ [0.5]

a contradiction. (as $\cos(\pi) = -1$ for example)

- For saying $\nexists \; p(z)$. [0.5]
- Justification [1.5]

Q.5] (a) Given $f(x+iy) = x^2 + iy^2$

$\Rightarrow \quad u = x^2 \qquad\qquad v = y^2$

$\therefore \quad u_x = 2x \qquad\qquad v_x = 0$

$u_y = 0 \qquad\qquad v_y = 2y$

$f$ is differentiable at $z = x+iy$

$\Rightarrow$ $f$ satisfies the CR equations at $z = x+iy$

$\Rightarrow \quad u_x = v_y$ and $u_y = -v_x$ at $z = x+iy$

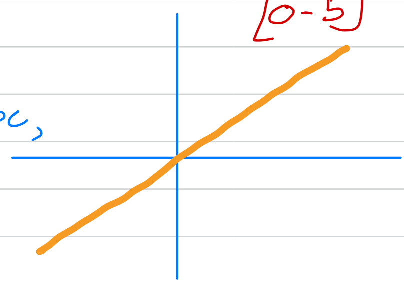
Here $u_x = v_y \iff 2x = 2y$ &

$\iff x = y.$

$v_x = -u_y$ always [1]

$\therefore$ $f$ is not differentiable at $z = x+iy$ if $y \neq x$. [0.5]

If $z = x+ix$, then $f$ satisfies the CR equations at $z$. Since $u$ & $v$ are polynomials, their partials are continuous on $\mathbb{C}$. $\therefore$ $f$ is differentiable at $x+ix$. [0.5]



Since for any $z = x+ix$, $\nexists$ any ball around $z$ $B(z;r)$ s.t $f$ is differentiable on $B(z;r)$,

$f$ is not analytic at any $z \in \mathbb{C}$. [1]

Q.5 (b) Given $f(x+iy) = y^2 + ix^2$

$\Rightarrow \quad u = y^2 \qquad\qquad v = x^2$

$\therefore \quad u_x = 0 \qquad\qquad v_x = 2x$

$u_y = 2y \qquad\qquad v_y = 0$

$f$ is differentiable at $z = x+iy$

$\Rightarrow$ $f$ satisfies the CR equations at $z = x+iy$

$\Rightarrow$ $u_x = v_y$ and $u_y = -v_x$ at $z = x+iy$

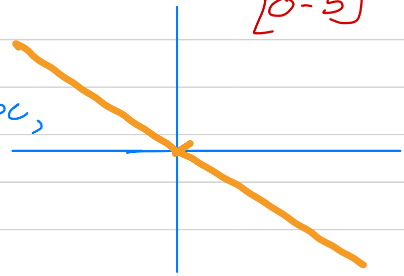
Here $u_x = v_y$ always &

$v_x = -u_y \iff 2x = -2y$

$\iff x = -y$ [1]

$\therefore$ $f$ is not differentiable at $z = x+iy$ if $y \neq -x$. [0.5]

If $z = x - ix$, then $f$ satisfies the CR equations at $z$. Since $u$ & $v$ are polynomials, their partials are continuous on $\mathbb{C}$. $\therefore$ $f$ is differentiable at $x - ix$. [0.5]



Since for any $z = x - ix$, $\nexists$ any ball around $z$ $B(z;r)$ s.t $f$ is differentiable on $B(z;r)$,

$f$ is not analytic at any $z \in \mathbb{C}$. [1]

Q.6] Given $a = c+1$

we're

$$z^2 + 2az + 1 = 0$$

$$\Longleftrightarrow \quad z = \frac{-2a \pm \sqrt{4a^2 - 4}}{2}$$

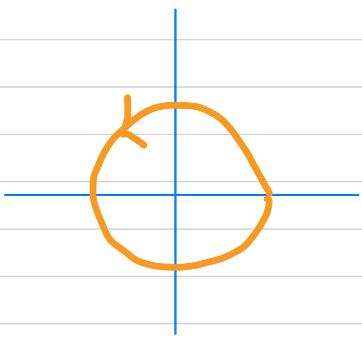
$$= -a \pm \sqrt{a^2 - 1}$$

Let $\alpha = -a + \sqrt{a^2-1}$ & $\beta = -a - \sqrt{a^2-1}$

$$\therefore \frac{1}{z^2+2az+1} = \frac{1}{\alpha - \beta}\left[\frac{1}{z-\alpha} - \frac{1}{z-\beta}\right]$$

$$\therefore \int_{|z|=1} \frac{1}{z^2+2az+1}\,dz = \frac{1}{\alpha-\beta}\left[\int_{|z|=1} \frac{1}{z-\alpha}\,dz - \int_{|z|=1} \frac{1}{z-\beta}\,dz\right] \longrightarrow (1)$$

[0.5]



Since $a > 1$,

$$\Rightarrow 2a > 2$$

$$\Rightarrow (a-1)^2 = a^2 - 2a + 1$$

$$< a^2 - 2 + 1$$

$$= a^2 - 1$$

$$\Rightarrow a - 1 < \sqrt{a^2-1}$$

$$\Rightarrow \alpha = -a + \sqrt{a^2-1} > -1$$

Clearly $\alpha < 0$

$$\therefore \alpha \in B(0;1)$$

But $\beta = -a - \sqrt{a^2-1}$ is clearly doesn't lie in $B(0;1)$.

$\therefore$ By Cauchy's theorem,

$$\int_{|z|=1} \frac{1}{z-\beta}\, dz = 0. \qquad [0.5]$$

& by Cauchy's integral formula

$$\int_{|z|=1} \frac{1}{z-\alpha}\, dz = 2\pi i\,(1) = 2\pi i$$

[0.5]

$\therefore$ By (1)

$$\int_{|z|=1} \frac{1}{z^2+2az+1}\, dz = \frac{1}{\alpha-\beta}\,(2\pi i)$$

$$= \frac{1}{2\sqrt{a^2-1}} \times 2\pi i$$

$$\boxed{\int_{|z|=1} \frac{1}{z^2+2az+1}\, dz = \frac{\pi i}{\sqrt{a^2-1}}} \qquad [0.5]$$