MA 201 Complex Analysis
End Semester Examination
IIT Dharwad (Autumn 2021)

Total Marks: 30 **Date & Time:** 23 September 2021, 03:00 pm to 04:35 p.m.

Part 1

(1) **[3M]** Let c be the last two digits of your roll number (for instance, if your roll number is 200030022, then c=22). Determine the type of singularities of the following functions at the given points. In case of poles, state the order.
(i) $\frac{\cos(z)}{z^c}$ at $z = 0$; (ii) $\frac{z^2-(c+1)z+c}{z^2-cz}$ at $z = c$; (iii) $z^c \sin(1/z)$ at $z = 0$.

(2) **[3M]** Let c be the last two digits of your roll number (for instance, if your roll number is 200030022, then c=22) and $\text{Log}(z)$ be the principal branch of the logarithm function defined using the principal argument that lies in $(0, 2\pi)$. Find the Taylor's series expansion of $\text{Log}(z)$ about the point $a = c + i$. What is the radius of convergence of this series ?

(3) **[2M]** Evaluate $\int_{|z|=c}(z-1)^2 \sin(1/z)dz$ where c is the last two digits of your roll number (for instance, if your roll number is 200030022, then c=22). Justify your answer.

(4) **[2M]** Let $D = \{z \in \mathbb{C} : |z| < 1\}$. Suppose u is a real valued harmonic function defined on D. Show that u is infinitely many times differentiable. (Hint: Use the fact that u has a harmonic conjugate on D).

(5) **[4M]** (a) **[3M]** Let f be an entire function such that for every $n \in \mathbb{N} \cup \{0\}$, $f^{(n)}(0) = \begin{cases} 0 & \text{if } n \text{ is even} \\ (-1)^k & k \text{ if } n = 2k+1. \end{cases}$ Show that $f(z) = \sin(z)$ for all $z \in \mathbb{C}$. (Hint: use the uniqueness theorem).
Bonus question [1M]: Can we conclude the same if f is a real valued differentiable function on $\mathbb{R}$? Justify your answer.

(b) **[3M]** Let f be an entire function such that for every $n \in \mathbb{N} \cup \{0\}$, $f^{(n)}(0) = \begin{cases} 0 & \text{if } n \text{ is odd} \\ (-1)^k & k \text{ if } n = 2k. \end{cases}$
Show that $f(z) = \cos(z)$ for all $z \in \mathbb{C}$. (Hint: use the uniqueness theorem).
Bonus question [1M]: Can we conclude the same if f is a real valued differentiable function on $\mathbb{R}$? Justify your answer.

(6) **[2M]** (a) Determine whether the function $\sin(z)$ is bounded on $\mathbb{C}$. Justify your answer.
(b) Determine whether the function $\cos(z)$ is bounded on $\mathbb{C}$. Justify your answer.

Part 2

(1) **[4M]** Evaluate the improper integral $\int_0^\infty \frac{dx}{(x^2+c)^2}$ where c is the is the last two digits of your roll number (for instance, if your roll number is 200030022, then c=22).

(2) **[2M]** Find the radius of the convergence of the Maclaurin series $\sum_{n=0}^{\infty}(c+(-i)^n)^n z^n$ where c is the is the last two digits of your roll number (for instance, if your roll number is 200030022, then c=22). Justify your answer.

(3) **[2M]** (a) Let $\text{Log}(z)$ be the principle branch of the logarithm function defined using the principal argument that lies in $(-\pi, \pi)$. Determine whether $\text{Log}(z)$ has an antiderivative on $\{z \in \mathbb{C} : Im(z) > 0\}$. Justify your answer.
(b) Let $\text{Log}(z)$ be the principle branch of the logarithm function defined using the principal argument that lies in $(-\pi, \pi)$. Determine whether $\text{Log}(z)$ has an antiderivative on $\{z \in \mathbb{C} : Im(z) < 0\}$. Justify your answer.

(4) **[2M]** Determine whether there exists a polynomial $p(z)$ such that

$$\frac{(1-\cos(z))e^{z^2}}{e^{z^2}+1} = p(z)$$

for all complex numbers z. Justify your answer.

(5) **[3M]** (a) Let $f(x+iy) = x^2 + iy^2$. Determine the points at which f is analytic. Justify your answer.
(b) Let $f(x+iy) = y^2 + ix^2$. Determine the points at which f is analytic. Justify your answer.

(6) **[2M]** Let c be the last two digits of your roll number (for instance, if your roll number is 200030022, then c=22). Set $a = c+1$. Evaluate the integral $\int_{|z|=1} \frac{1}{z^2+2az+1} dz$.