8. Fourier Transform

1. Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ such that

   (a) f is infinitely many time differentiable,
   (b) $f^n(x) \to 0$ as $|x| \to \infty$,
   (c) $\int_{\mathbb{R}} |f^n(x)|dx < \infty, \;\; f^n(x) = \frac{d^n f}{dx^n}$.

   Then Prove that

   $\mathcal{F}\big(f^n(x)\big) = (i\xi)^n \mathcal{F}\big(f(x)\big)$, where $\mathcal{F}$ is the notation for fourier transform.

2. Find the Fourier transform of the following functions defined on $\mathbb{R}$:

   (i) $f(x) = x^2$, if $0 < x < 1$ and 0 otherwise.
   (ii) $f(x) = |x|$, if $x \in (-1, 1)$ and zero otherwise.
   (iii) $f(x) = e^{-4x^2}, \; x \in \mathbb{R}$ and $a > 0$.
   (iv) $f(x) = xe^{-x}, \; x > 0$ and zero otherwise.
   (v) $f(x) = e^{-|x|}, \; x \in \mathbb{R}$.
   (vi) $f(x) = 1, \;$ if $\; x \in (-1, 1)$ and zero otherwise.

3. Let $g(x) = e^{-\frac{x^2}{b^2}}$ and $f(x) = e^{-a^2x^2}, \; a, \; b \in \mathbb{R}$.

   Then find the explicit expression of the Fourier transform of $h(x) = (g * f)(x)$ (convolution of f and g).

4. Solve the following initial value problems using Fourier transform techniques

   (i) $u_t = k^2 u_{xx}, \; x \in \mathbb{R}, \; t > 0, \; k \in \mathbb{R}$
   with initial condition : $u(x, 0) = 1 + x^2$.

   (ii) $u_t = k^2 u_{xx}, \; x \in \mathbb{R}, \; t > 0, \; k \in \mathbb{R}$
   with initial condition : $u(x, 0) = 1$, if $x \in (-1, 1)$ and zero otherwise.

   (iii) $u_t = k^2 u_{xx}, \; x \in \mathbb{R}, \; t > 0, \; k \in \mathbb{R}$
   with initial condition : $u(x, 0) = e^{-a|x|}, \; a > 0$.

(iv) $u_t = k^2 u_{xx},\ x \in \mathbb{R},\ t > 0,\ k \in \mathbb{R}$
with initial condition : $u(x,0) = e^{-ax^2},\ a > 0.$

5. Use Plancherels identity to find a relation between the solution and the initial data (in terms of L^2-norm) for the following Problems:

   (i) $u_t + u_x + bu = 0$ with given $u(x,0) = u_0(x)$.
   (ii) $u_t = u_{xxx}$ with given $u(x,0) = u_0(x)$.
   (iii) $u_t = bu_{xx} + au_x + cu$ with given $u(x,0) = u_0(x)$.
   (iv) $u_{tt} = a^2 u_{xx}$ with $u(x,0) = u_0(x),\quad u_t(x,0) = u_1(x)$.