Q1 $y'' - xy = 0$ —(*) | $x_0 = 0$ is a ordinary point

let $y = \sum_{n=0}^{\infty} a_n x^n$

$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$

$y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$

$\frac{1}{2}$

substitute in (*) we get

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} - x \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\Rightarrow \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} - \sum_{n=0}^{\infty} a_n x^{n+1}$$

put $n-2 = m$ in 1st series

$n+1 = k$ in 2nd series

so we get

$$\sum_{m=0}^{\infty} (m+2)(m+1) a_{m+2} x^m - \sum_{k=1}^{\infty} a_{k-1} x^k = 0$$

m, k are dummy variables

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n - \sum_{n=1}^{\infty} a_{n-1} x^n = 0$$

$$\Rightarrow 2a_2 + \sum_{n=1}^{\infty} (n+2)(n+1) a_{n+2} x^n - \sum_{n=1}^{\infty} a_{n-1} x^n = 0$$

$$\Rightarrow 2a_2 + \sum_{n=1}^{\infty} \{ (n+2)(n+1) a_{n+2} - a_{n-1} \} x^n = 0$$

$$\Rightarrow 2a_2 + \sum_{n=1}^{\infty} \{ (n+2)(n+1) a_{n+2} - a_{n-1} \} x^n = 0 \quad (1)$$

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Now we get

$a_2 = 0$

and $(n+2)(n+1)a_{n+2} - a_{n-1} = 0$

$\Rightarrow \quad a_{n+2} = \dfrac{a_{n-1}}{(n+2)(n+1)} \quad \forall\, n \geq 1$

put $n=1$, $\quad a_3 = \dfrac{a_0}{3.2} = \dfrac{a_0}{2.3}$

put $n=2$ $\quad a_4 = \dfrac{a_1}{4.3} = \dfrac{a_1}{3.4}$

put $n=3$, $\quad a_5 = \dfrac{a_2}{5.4} = 0 \quad$ $\boxed{\text{as } a_2 = 0}$

put $n=4$, $\quad a_6 = \dfrac{a_3}{6.5} = \dfrac{a_0}{2.3.5.6}$

put $n=5$, $\quad a_7 = \dfrac{a_4}{7.6} = \dfrac{a_1}{3.4.6.7}$

put $n=6$, $\quad a_8 = \dfrac{a_5}{8.7} = 0 \rightarrow \boxed{\text{as } a_5 = 0}$

$1\frac{1}{2}$

we get

$a_{3k} = \dfrac{a_0}{2.3.5.6 \cdots (3k-1)\,3k} \quad k \geq 1$

$a_{3k+1} = \dfrac{a_1}{3.4.6.7 \cdots 3k(3k+1)}, \quad k \geq 1$

$a_{3k+2} = 0 \quad \forall\, k \geq 0$

1

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$$y(x) = \sum_{n=0}^{\infty} a_n x^n$$

$$= a_0 + \sum_{k=1}^{\infty} a_{3k} x^{3k} + a_1 x + \sum_{k=1}^{\infty} a_{3k+1} x^{3k+1} + a_2 x^2 + \sum_{k=1}^{\infty} a_{3k+2} x^{3k+2}$$

$$= a_0 + \sum_{k=1}^{\infty} \frac{a_0}{2.3.5.6\cdots(3k-1)3k} x^{3k} + a_1 x + \sum_{k=1}^{\infty} \frac{a_1}{3.4.6.7\cdots3k(3k+1)} x^{3k+1}$$

1

$$= a_0 \left[1 + \sum_{k=1}^{\infty} \frac{1}{2.3.5.6\cdots(3k-1)3k} x^{3k}\right] + a_1 \left[x + \sum_{k=1}^{\infty} \frac{1}{3.4.6.7\cdots3k(3k+1)} x^{3k+1}\right]$$

Here we used $a_2 = 0$, $a_{3k+2} = 0$ $\forall\, k \geq 1$

$$y(x) = a_0 y_1(x) + a_1 y_2(x) \qquad a_0, a_1 \in \mathbb{R}$$

$$y_1(x) = 1 + \sum_{k=1}^{\infty} \frac{x^{3k}}{2.3.5.6\cdots(3k-1)3k}$$

$$y_2(x) = x + \sum_{k=1}^{\infty} \frac{x^{3k+1}}{3.4.6.7\cdots3k(3k+1)}$$

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(Q2) $y'' + \lambda y = 0, \quad 0 \le x \le 1, \quad y'(0) = 0, \quad y(1) = 0$ $0 < 1$

For $\lambda = 0$ (Case I)

$y'' = 0$

$\Rightarrow y = Ax + B$

$\Rightarrow y' = A$

$y'(0) = 0 \Rightarrow A = 0$

$y = B$

$y(1) = 0 \Rightarrow B = 0$

so $y \equiv 0$ not non zero solⁿ

so $\lambda = 0$ is not eigenvalue

For $\lambda < 0$ Case-II

$y'' + \lambda y = 0$

auxiliary eqⁿ $m^2 + \lambda = 0$

$\Rightarrow m = \pm\sqrt{-\lambda}$

$y = A e^{(\sqrt{-\lambda})x} + B e^{-(\sqrt{-\lambda})x}$

$y(1) = 0$

$\Rightarrow A e^{(\sqrt{-\lambda})1} + B e^{-(\sqrt{-\lambda})1} = 0$ — (i)

$y'(x) = \sqrt{-\lambda}\left[A e^{(\sqrt{-\lambda})x} - B e^{-(\sqrt{-\lambda})x}\right]$

$y'(0) = 0 \Rightarrow A = B$. Now From (i)

$A\left[e^{(\sqrt{-\lambda})1} + e^{-(\sqrt{-\lambda})1}\right] = 0$

$\Rightarrow A = 0$

$\Rightarrow B = 0$

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so $y=0$, therefore $\lambda<0$ is not an eigenvalue.

For $\lambda>0$, Case - III

$\frac{1}{2}$

$y''+\lambda y=0$

auxiliary eq^n $m^2+\lambda=0$

$m=\pm i\sqrt{\lambda}$

$y(x)=A\cos(\sqrt{\lambda}x)+B\sin(\sqrt{\lambda}x)$

$y'(x)=\sqrt{\lambda}\left(-A\sin(\sqrt{\lambda}x)+B\cos(\sqrt{\lambda}x)\right)$

$y'(0)=0 \Rightarrow B=0$

$y(x)=A\cos(\sqrt{\lambda}x)$

$y(1)=0 \Rightarrow$ ~~$A\cos(\sqrt{\lambda}x)=0$~~

$\Rightarrow$ ~~$\cos\sqrt{\lambda}x$~~

$1\frac{1}{2}$

$\Rightarrow A\cos(\sqrt{\lambda}1)=0$ [$\because$ we want $A\neq 0$]

$\Rightarrow \cos(\sqrt{\lambda}1)=0$

$\Rightarrow \sqrt{\lambda}1=(2n+1)\frac{\pi}{2}$ ~~$n=0,\pm1,\pm2$~~ $n=0,1,2\ldots$

$\sqrt{\lambda}=(2n+1)\frac{\pi}{2}$

$\Rightarrow \lambda=\left(\frac{(2n+1)\pi}{2}\right)^2$, ~~$n=0,\pm1,\pm2,$~~ $n=0,1,2\ldots$

So the eigen values are $\lambda_n = \left(\frac{(2n+1)\pi}{2l}\right)^2$ $n = 0, 1, 2 \ldots$

and eigenfunctions are

$$y_n(x) = A_n \cos(\sqrt{\lambda_n}\, x)$$

$$= A_n \cos\left(\frac{(2n+1)\pi x}{2l}\right) \quad n = 0, 1, 2 \ldots$$

to get normalised eigen function we need

$$\int_0^l y_n^2\, dx = 1$$

$$\Rightarrow A_n^2 \int_0^l \cos^2\left(\frac{(2n+1)\pi x}{2l}\right) dx = 1$$

$$\Rightarrow \frac{A_n^2}{2} \int_0^l 2\cos^2\left(\frac{(2n+1)\pi x}{2l}\right) dx = 1$$

$$\Rightarrow \frac{A_n^2}{2} \int_0^l \left[1 + \cos\frac{(2n+1)\pi x}{l}\right] dx = 1$$

$$\Rightarrow \frac{A_n^2}{2}\, l = 1$$

$$\Rightarrow A_n = \sqrt{\frac{2}{l}} \quad \left(\text{or } A_n = -\sqrt{\frac{2}{l}}\right)$$

OR both

So the normalised eigen functions are

$$y_n = \sqrt{\frac{2}{l}} \cos\left(\frac{(2n+1)\pi x}{2l}\right) \quad n = 0, 1, 2 \ldots$$

$\left(\text{OR } y_n = -\sqrt{\frac{2}{l}} \cos\left(\frac{(2n+1)\pi x}{2l}\right)\right)$

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Q3 (A) $P_n(x) = \frac{1}{2^n \lfloor n} \frac{d^n}{dx^n}(x^2-1)^n$

Now

$$P_{2n}(x) = \frac{1}{2^{2n} \lfloor 2n} \frac{d^{2n}}{dx^{2n}}(x^2-1)^{2n}$$

$$= \frac{1}{2^{2n}\lfloor 2n} \frac{d^{2n}}{dx^{2n}}\left[\sum_{k=0}^{2n} \binom{2n}{k} (x^2)^{2n-k} (-1)^k\right]$$

Binomial expansion of $(x^2-1)^{2n}$

$$= \frac{1}{2^{2n}\lfloor 2n} \sum_{k=0}^{2n} (-1)^k \binom{2n}{k} \frac{d^{2n}}{dx^{2n}}(x^2)^{2n-k}$$

$$= \frac{1}{2^{2n}\lfloor 2n} \sum_{k=0}^{2n} (-1)^k \binom{2n}{k} \frac{d^{2n}}{dx^{2n}}(x^{4n-2k}) \quad \text{—— (*)}$$

(1)

Now $\left.\frac{d^{2n}}{dx^{2n}}(x^{4n-2k})\right|_{x=0} = 0$ if $n \neq k$

(2)

So from (*) we get

$$P_{2n}(0) = \frac{1}{2^{2n}\lfloor 2n} (-1)^n \binom{2n}{n} \left.\frac{d^{2n}}{dx^{2n}}(x^{2n})\right|_{x=0}$$

$$= \frac{1}{2^{2n}\lfloor 2n} (-1)^n \binom{2n}{n} \lfloor 2n$$

$$= \frac{1}{2^{2n}} (-1)^n \frac{\lfloor 2n}{\lfloor n \lfloor n} = (-1)^n \frac{\lfloor 2n}{2^{2n} (\lfloor n)^2}$$

(1)

(7)

Q3 (B) $P_n(x) = \dfrac{1}{2^n \lfloor n} \dfrac{d^n}{dx^n}(x^2-1)^n$

1

$$P_n'(x) = \frac{1}{2^n \lfloor n} \frac{d^{n+1}}{dx^{n+1}} (x^2-1)^n$$

$$= \frac{1}{2^n \lfloor n} \frac{d^{n+1}}{dx^{n+1}} \{(x+1)^n (x-1)^n\}$$

(*) $$= \frac{1}{2^n \lfloor n} \sum_{k=0}^{n+1} \binom{n+1}{k} \frac{d^k}{dx^k}(x+1)^n \frac{d^{n-k+1}}{dx^{n-k+1}}(x-1)^n$$ | Leibnitz rule

1

Now $\left.\dfrac{d^k}{dx^k}(x+1)^n\right|_{x=-1} = 0$ if $k \neq n$

so From (*) we get

$$P_n'(-1) = \frac{1}{2^n \lfloor n} \binom{n+1}{n} \left.\frac{d^n}{dx^n}(x+1)^n\right|_{x=-1} \left.\frac{d}{dx}(x-1)^n\right|_{x=-1}$$

1

$$= \frac{1}{2^n \lfloor n} \frac{\lfloor n+1}{\lfloor 1 \lfloor n} (\lfloor n) n (-2)^{n-1}$$

$$= (-1)^{n-1} \frac{n(n+1)}{2}$$

=

//

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Q4 (i)

$$J_n(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{\Gamma(1+k)\,\Gamma(k+n+1)} \left(\frac{x}{2}\right)^{2k+n}$$

~~put~~ Replace n by $-n$ in above we set

$$J_{-n}(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{\Gamma(1+k)\,\Gamma(k-n+1)} \left(\frac{x}{2}\right)^{2k-n}$$

1

put $k-n=m$ in above $(k=m+n)$

$$J_{-n}(x) = \sum_{m=-n}^{\infty} \frac{(-1)^{m+n}}{\Gamma(1+m+n)\,\Gamma(m+1)} \left(\frac{x}{2}\right)^{2m+n}$$

2

$$\boxed{\text{Now use } \Gamma(m+1) = \infty \text{ for } 1+m \le 0 \text{ i.e. } m \le -1 \quad \text{i.e. } \frac{1}{\Gamma(m+1)} = 0}$$

So we set

$$J_{-n}(x) = (-1)^n \sum_{m=0}^{\infty} \frac{(-1)^m}{\Gamma(1+m+n)\,\Gamma(1+m)} \left(\frac{x}{2}\right)^{2m+n}$$

1

$$= (-1)^n \sum_{k=0}^{\infty} \frac{(-1)^k}{\Gamma(1+k)\,\Gamma(k+n+1)} \left(\frac{x}{2}\right)^{2k+n}$$

$\boxed{m \text{ is dummy variable}}$

$$\Rightarrow J_{-n}(x) = (-1)^n J_n(x)$$

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(Q4) (B) we have

$$e^{\frac{x}{2}(z-z^{-1})} = \sum_{n=-\infty}^{\infty} J_n(x) z^n \quad z \in \mathbb{C} \; z \neq 0$$

put $z = e^{i\phi}$ in above

$$e^{\frac{x}{2}(e^{i\phi}-e^{-i\phi})} = \sum_{n=-\infty}^{\infty} J_n(x) e^{in\phi}$$

use $e^{i\phi} = \cos\phi + i\sin\phi$

$$\Rightarrow e^{ix\sin\phi} = \sum_{n=-\infty}^{\infty} J_n(x)(\cos n\phi + i\sin n\phi)$$

$$\Rightarrow \cos(x\sin\phi) + i\sin(x\sin\phi) = \sum_{n=-\infty}^{\infty} J_n(x)\cos(n\phi) + i\sum_{n=-\infty}^{\infty} J_n(x)\sin(n\phi)$$

Equating Real and Imaginary part of in above

$$\cos(x\sin\phi) = \sum_{n=-\infty}^{\infty} J_n(x)\cos(n\phi)$$

$$\sin(x\sin\phi) = \sum_{n=-\infty}^{\infty} J_n(x)\sin(n\phi)$$

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(8.5) (A) $\sum_{n=1}^{\infty} \frac{\lfloor 2n}{2^{2n} (\lfloor n)^2} x^n$

here $a_n = \frac{\lfloor 2n}{2^{2n} (\lfloor n)^2}$ $\frac{1}{2}$

radius of convergence $\rho = \lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right|$

$= \lim_{n\to\infty} \frac{\lfloor 2n}{2^{2n}(\lfloor n)^2} \cdot \frac{2^{2(n+1)} (\lfloor n+1)^2}{\lfloor 2(n+1)}$

$= \lim_{n\to\infty} \frac{\lfloor 2n \cdot 2^{2n} \cdot 2^2 \, (n+1)^2 (\lfloor n)^2}{2^{2n} (\lfloor n)^2 (2n+2)(2n+1) \cdot \lfloor 2n}$

$= \lim_{n\to\infty} \frac{2^2 (n+1)^2}{(2n+2)(2n+1)}$

$= \lim_{n\to\infty} 2^2 \frac{(1+\frac{1}{n})^2}{(2+\frac{2}{n})(2+\frac{1}{n})}$

$= \lim_{n\to\infty} 2^2 \frac{1}{4} = 1$

$\frac{1}{2}$

Q5 (B) $\sum_{n=1}^{\infty} \frac{(x-1)^n}{n2^n}$

here $a_n = \frac{1}{n2^n}$ $\quad \frac{1}{2}$

radius of convergence

$\rho = \lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right|$

$= \lim_{n\to\infty} \frac{1}{n2^n} (n+1)2^{n+1}$

$= \lim_{n\to\infty} 2\left(1+\frac{1}{n}\right) = 2$

$1\frac{1}{2}$

$[-1,1] = [-L, L]$

$L=1$

Q6 $f(x) = 1-x^2,\ x \in [-1,1]$

here $L=1$

$f(x) \simeq a_0 + \sum_{n=1}^{\infty} \left[a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)\right]$

$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx = \frac{1}{2}\int_{-1}^{1} (1-x^2)\,dx$

$= \frac{1}{2}\left[x - \frac{x^3}{3}\right]_{-1}^{1}$

$= \frac{2}{3}$

1

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$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx$$

$$= \int_{-1}^{1}(1-x^2)\cos(n\pi x)\,dx \qquad \text{Let } L=1$$

$$= 2\int_0^1 (1-x^2)\cos(n\pi x)\,dx$$

$$= 2\left[(1-x^2)\left(\frac{\sin n\pi x}{n\pi}\right)\right]_0^1 - 2\int_0^1 (-2x)\frac{\sin(n\pi x)}{n\pi}dx$$

$$= \frac{4}{n\pi}\int_0^1 x\sin(n\pi x)\,dx$$

$$= \frac{4}{n\pi}\left\{\left[x\left(-\frac{\cos(n\pi x)}{n\pi}\right)\right]_0^1 - \int_0^1\left(-\frac{\cos(n\pi x)}{n\pi}\right)dx\right\}$$

$$= \frac{4}{n^2\pi^2}\left[-x\cos(n\pi x)\right]_0^1 + \frac{4}{n^2\pi^2}\int_0^1 \cos(n\pi x)\,dx$$

$$= \frac{4}{n^2\pi^2}(-\cos n\pi)$$

$$= -\frac{4}{n^2\pi^2}(-1)^n \qquad \cos n\pi = (-1)^n$$

$$= (-1)^{n+1}\frac{4}{n^2\pi^2}$$

2

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$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

$$= \int_{-1}^{1}(1-x^2)\sin(n\pi x)\,dx$$

$$= 0$$

$\frac{1}{2}$

$L=1$

$(1-x^2)\sin n\pi x$ is odd

So the Fourier series for $f(x)=1-x^2$ on $(-1,1)$ is

$$f(x) \simeq a_0 + \sum_{n=1}^{\infty} a_n \cos(n\pi x) + \sum_{n=1}^{\infty} b_n \sin(n\pi x)$$

$$= \frac{2}{3} + \frac{4}{\pi^2}\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\cos(n\pi x)}{n^2}$$

$\frac{1}{2}$

~~But~~ at $x=0$ $f(x)$ is continuous so the value of the Fourier series at $x=0$ is $f(0)=1$ $\boxed{f(x)=1-x^2}$

1

$$1 = \frac{2}{3} + \frac{4}{\pi^2}\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n^2}$$

$$\Rightarrow \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2} = \frac{1}{3}$$

$$\Rightarrow \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2} = \frac{\pi^2}{12}$$

$$\Rightarrow 1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} + \frac{1}{25} + \cdots = \frac{\pi^2}{12}$$

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