Quiz 2 Marking Scheme

15 september 2021

Q.1] Given $f(z) = \frac{1}{z^2}$

Here

$$\frac{1}{z} = \frac{1}{z-c+c}$$

$$= \frac{1}{c\left(1 + \frac{z-c}{c}\right)}$$

$$= \frac{1}{c} \sum_{n=0}^{\infty} (-1)^n \left(\frac{z-c}{c}\right)^n \quad \text{for } \left|\frac{z-c}{c}\right| < 1$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{c^{n+1}} (z-c)^n \quad \text{for } |z-c| < c$$ [1]

Since power series can be differentiated term by term for $|z-c| < c$, we've

$$\frac{d}{dz}\left(\frac{1}{z}\right) = \sum_{n=1}^{\infty} \frac{(-1)^n}{c^{n+1}} . n (z-c)^{n-1}$$ [1]

$$\Rightarrow -\frac{1}{z^2} = \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{c^{n+2}} (n+1) (z-c)^n$$

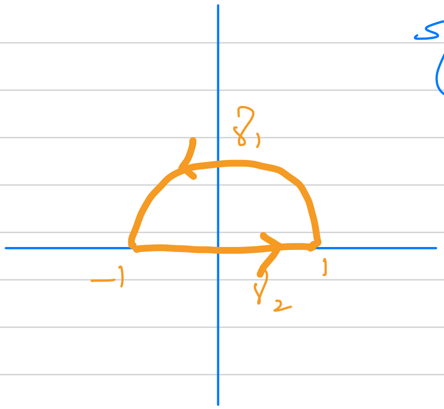
$$\Rightarrow \frac{1}{z^2} = \sum_{n=0}^{\infty} \frac{(-1)^n}{c^{n+2}} (n+1)(z-c)^n \quad \text{for } |z-c| < c$$ [2]

• For not mentioning $|z-c| < c$ — [0.5]

Q.2] Given $f(z) = |z|$.

We claim that f has no antideri-
-vative in $\mathbb{C}$. [0.5]

Consider the simple closed contour γ shown in the diagram (boundary of the half disc $0 \le r \le 1$ & $0 \le \theta \le \pi$) oriented positively



$\therefore \; \gamma = \gamma_1 + \gamma_2$

$$\therefore \int_{\gamma} f(z)\,dz = \int_{\gamma_1} f(z)\,dz + \int_{\gamma_2} f(z)\,dz$$

Here $\gamma_1(t) = e^{it}$, $0 \le t \le \pi$

$$\therefore \int_{\gamma_1} f(z)\,dz = \int_0^{\pi} f(\gamma_1(t))\,\gamma_1'(t)\,dt$$

$$= \int_0^{\pi} (1)\; i(e^{it})\,dt$$

$$= i\left[\int_0^{\pi} \cos t\,dt + i\int_0^{\pi} \sin t\,dt\right]$$

$$= i\left[\sin t\Big|_0^{\pi} + (-i)\cos t\Big|_0^{\pi}\right]$$

$$= i\left[0 - i\,(-1-1)\right]$$

$$= -2$$

Now, $\gamma_2(t) = (1-t)(-1) + t(1)$, $0 \le t \le 1$

$= 2t - 1$

$\therefore \int_{\gamma_2} f(z)\,dz = \int_0^1 |2t-1|\,(2)\,dt$

$= 2\left[\int_0^{1/2} (1-2t)\,dt + \int_{1/2}^{1} (2t-1)\,dt\right]$

$= 2\left[t\Big|_0^{\frac{1}{2}} - t^2\Big|_0^{\frac{1}{2}} + t^2\Big|_{\frac{1}{2}}^{1} - t\Big|_{\frac{1}{2}}^{1}\right]$

$= 2\left[\frac{1}{2} - \frac{1}{4} + \left(1 - \frac{1}{4}\right) - \left(\frac{1}{2}\right)\right]$

$= 2\left(\frac{1}{2}\right) = 1$

$\therefore \int_{\gamma} f(z)\,dz = -2 + 1 = -1 \neq 0$

Suppose f has an antiderivative on $\mathbb{C}$. Then by fundamental theorem of calculus, $\int_{\gamma} f(z)\,dz = 0$, a contradiction.

$\therefore$ f has no antiderivative on $\mathbb{C}$. [1.5]

- For saying no [0.5]
- For justification [1.5]

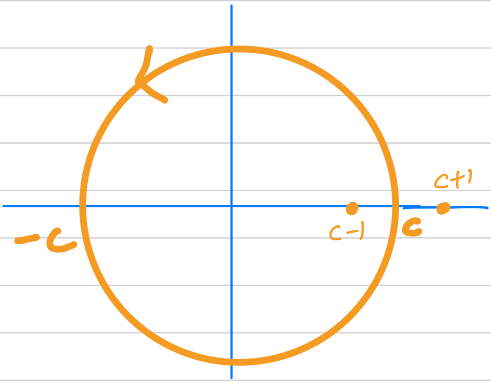
Alternative argument

Suppose $\exists$ an entire function F s.t. $F'(z) = f(z)$ $\forall$ $z \in \mathbb{C}$.

F is entire $\Rightarrow$ F' is entire
$\Rightarrow$ $f(z) = |z|$ is entire,
a contradiction.

Q.3] Given

$$g(\omega) = \int_{|z|=c} \frac{z^3 + 2z^2 - z}{(z-\omega)^3}\, dz$$



Let $\gamma = \{ z \in \mathbb{C} : |z| = c \}$

Let $f(z) = z^3 + 2z^2 - z$.

Since f is entire & $c-1 \in$ inside of γ, by Cauchy's integral formula,

$$\int_{|z|=c} \frac{f(z)\,dz}{(z-(c-1))^3} = \frac{(2\pi i)}{2!} f''(c-1).$$ [1]

Here

$$f(z) = z^3 + 2z^2 - z$$

$$\Rightarrow f'(z) = 3z^2 + 4z - 1$$

$$\Rightarrow f''(z) = 6z + 4$$

$$\therefore f''(c-1) = 6(c-1) + 4 = 6c - 2$$ [0.5]

$$\therefore g(c-1) = \int_{|z|=c} \frac{f(z)}{(z-(c-1))^3} dz$$

$$= \frac{2\pi i}{2!} \times (6c-2)$$

$$\boxed{g(c-1) = (6c-2)\pi i}$$ [0.5]

Note that $c+1$ lies outside of γ.

$\therefore \dfrac{f(z)}{(z-(c+1))^3}$ is analytic on and inside of γ. Hence by Cauchy's theorem, [0.5]

$$g(c+1) = \int_{|z|=c} \frac{f(z)}{z-(c+1)^3}\, dz = 0$$

$$\boxed{\therefore\ g(c+1) = 0}$$ [0.5]

- For finding $g(c-1)$ [2]
- For finding $g(c+1)$ [1]

Q. 4) Given that f is entire and $|f'(z)| > M \ \forall z \in \mathbb{C}$.

As f is entire, f' is entire. [0.5]

we're $|f'(z)| > M \ \forall z \in \mathbb{C}$

$\Rightarrow f'$ is non-zero &

$$\left|\frac{1}{f'(z)}\right| < M \quad \forall z \in \mathbb{C}.$$

$\Rightarrow \frac{1}{f'(z)}$ is bounded.

As $f'(z) \neq 0$ and $f'(z)$ is entire

$\Rightarrow \frac{1}{f'(z)}$ is also entire [0.5]

$\therefore$ By Liouville's theorem, $\frac{1}{f'(z)}$ is constant.

$\Rightarrow f'(z)$ is constant. [1]

Let $f'(z) = c$, $c \in \mathbb{C}$.

$= c_1 + i c_2$

Suppose $f(z) = u + iv$.

Then $f' = u_x + i v_x = v_y - i u_y$

$= c_1 + i c_2$

$\Rightarrow u_x = v_y = c_1$ & $v_x = -u_y = c_2$

$\Rightarrow u = x c_1 + \phi(y)$

$\therefore u_y = \phi'(y) = -c_2$

$\Rightarrow \phi(y) = -c_2 y + k_1, \ k_1 \in \mathbb{R}.$

$\therefore\ u = x c_1 - c_2 y + k_1$

Now $\quad v_y = c_1 \Rightarrow v = c_1 y + \psi(x)$

$\Rightarrow v_x = \psi'(x) = c_2$

$\Rightarrow \psi(x) = c_2 x + k_2, \quad k_2 \in \mathbb{R}$

$\therefore\ v = c_1 y + c_2 x + k_2$

$$\begin{aligned} \therefore\ f &= u + iv \\ &= (c_1 x - c_2 y) + i(c_1 y + c_2 x) + (k_1 + ik_2) \\ &= (c_1 + ic_2)(x + iy) + (k_1 + ik_2) \\ &= cz + k \quad \text{where } k = k_1 + ik_2 \end{aligned}$$

$\therefore\ f$ is linear. $[1]$

Q.5] Given $\sum_{n=0}^{\infty} n(z+ic)^n$

Here $a_n = n$.

By Cauchy-Hadamard formula,

$$\frac{1}{R} = \limsup \{ \sqrt[n]{|a_n|} : n \in \mathbb{N} \}$$ [0.5]

$$= \limsup \{ n^{1/n} : n \in \mathbb{N} \}$$

$$= 1$$

$\Rightarrow R = 1$ [1.5]

Q.6 We prove that $f \equiv c$ is the only entire function s.t. $f(1/n) = c \quad \forall n \in \mathbb{N}$. [1]

Let $S = \{ \frac{1}{n} : n \in \mathbb{N} \}$.
Then S has a limit point in $\mathbb{C}$.
Let $g(z) \equiv c \quad \forall z \in \mathbb{C}$. Then g is entire and

$$g\left(\frac{1}{n}\right) = c = f(1/n) \quad \forall n \in \mathbb{N}$$
(by assumption)

i.e
$$g(z) = f(z) \quad \forall z \in S.$$

$\therefore$ by uniqueness theorem, $f \equiv g$ on $\mathbb{C}$. [1]

- For saying $f \equiv c$ has this property [0.5]
- For saying $f \equiv c$ is the <u>only</u> function [0.5]
- Justification [1]