1. Power Series

Determine the radius and interval of convergence of the given power series.

(1) $\displaystyle\sum_{n=0}^{\infty}(x-2)^n$, (2) $\displaystyle\sum_{n=0}^{\infty}\frac{n}{2^n}x^n$, (3) $\displaystyle\sum_{n=0}^{\infty}\frac{x^{2n}}{n!}$, (4) $\displaystyle\sum_{n=1}^{\infty}\left(x-\frac{1}{3}\right)^n$, (5) $\displaystyle\sum_{n=1}^{\infty}\frac{(2x+1)^n}{n^2}$,
(6) $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n n^2(x+1)^n}{4^n}$, (7) $\displaystyle\sum_{n=1}^{\infty}\frac{n!x^n}{n^n}$.

2. Series Solutions (around ordinary point)

Solve the following ODEs by power series (around x_0) method.

1. $y'' + 4y = 0$, around $x_0 = 0$.
2. $y'' - 9y = 0$, around $x_0 = 0$.
3. $y'' - xy' - y = 0$, around $x_0 = 1$.
4. $(1-x)y'' + y = 0$, around $x_0 = 0$.
5. $(2+x^2)y'' - xy' + 4y = 0$, around $x_0 = 0$.
6. $xy'' + y' + xy = 0$, around $x_0 = 1$.

3. The Legendre Equation

1. Show that for $n = 0, 1, 2, \cdots$ the Legendre polynomial (Rodrigues formula) is given by

$$P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n.$$

   (Hint: Show that $P_n(x)$ satisfy the Legendre equation and $P_n(1) = 1$)

2. Prove that

   (a) $P_n(-x) = (-1)^n P_n(x)$, (c) $P_n'(1) = \frac{n(n+1)}{2}$.

3. Show that

$$\int_{-1}^{1} P_n(x)P_m(x)dx = \begin{cases} 0 & \text{if } n \neq m \\ \frac{2}{2n+1} & \text{if } n = m. \end{cases}$$

4. Show that

$$P_n(x) = \frac{1}{2^n}\sum_{n=0}^{M_n}(-1)^k \frac{(2n-2k)!}{k!(n-k)!(n-2k)!}x^{n-2k},$$

where

$$M_n = \begin{cases} \frac{n}{2} & \text{if } n \textit{ is even} \\ \frac{n-1}{2} & \text{if } n \textit{ is odd.} \end{cases}$$

(Hint: Use Rodrigues formula and expand $(x^2-1)^n$)

5. Show that the generating function for Legendre polynomial is given by

$$\frac{1}{\sqrt{1-2xt+t^2}} = \sum_{n=0}^{\infty} P_n(x)t^n.$$

6. Prove the following relations

   (a) $(n+1)P_{n+1}(x) = (2n+1)xP_n(x) - nP_{n-1}(x)$

   (b) $nP_n(x) = xP_n'(x) - P_{n-1}'(x)$

   (c) $(2n+1)P_{n+1}(x) = P_{n+1}'(x) - P_{n-1}'(x)$

   (d) $P_{n+1}'(x) = xP_{n-1}'(x) - nP_{n-1}(x)$

   (e) $(1-x^2)P_n'(x) = n[P_{n-1}(x) - xP_n(x)]$

7.
$$\int_{-1}^{1} x^m P_n'(x)dx = \begin{cases} 0 & \text{if } n \leq m \textit{ and } n-m \textit{ is even} \\ 2 & \text{if } m \leq n \textit{ and } n-m \textit{ is odd.} \end{cases}$$

8. Prove that $x^n = \sum_{k=0}^{n} a_k P_k(x)$ where $a_n = \frac{2^n(n!)^2}{(2n)!}$.

9. Prove that $\int_{-1}^{1}(1-x^2)[P_n'(x)]^2dx = \frac{2n(n+1)}{2n+1}$.

3. The Bessel's Equation

1. Prove that
$$J_{-p}(x) = (-1)^p J_p(x),\ \ p \in \mathbb{N}.$$

2. Prove the following relation

   (a) $(x^p J_p(x))' = x^p J_{p-1}(x)$

   (b) $(x^{-p} J_p(x))' = -x^{-p} J_{p+1}(x)$

   (c) $J_{p+1}(x) + J_{p-1}(x) = \frac{2p}{x} J_p(x)$

   (d) $J_{p+1}(x) - J_{p-1}(x) = 2J_p(x)'$

3. When n is an integer show that (i) $J_n(x)$ is an even function if n is even, (ii) $J_n(x)$ is an odd function if n is odd.

4. Show that between any two consecutive positive zeros of $J_n(x)$ there is precisely one zero of $J_{n+1}(x)$ and one zero of $J_{n-1}(x)$.

5. Show that the generating function for Bessel functions is given by
$$e^{\frac{x}{2}(t-t^{-1})} = \sum_{n=-\infty}^{\infty} J_n(x)t^n, \quad t \neq 0, \quad n \in \mathbb{Z}.$$
   Use above formula to show that
$$J_0^2 + 2\sum_{n=1}^{\infty} J_n^2 = 1.$$
   Deduce that $|J_0(x)| \leq 1$ and $|J_n(x)| \leq \frac{1}{\sqrt{2}}$.