Question #1 (A) $f(x) = e^{-\frac{x^2}{b}}$ $b > 0$

$$F(f) = \hat{f}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-i\xi x} \, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{x^2}{b}} e^{-i\xi x} \, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\left[\left(\frac{x}{\sqrt{b}}\right)^2 + 2\frac{x}{\sqrt{b}} \frac{\sqrt{b}\, i\xi}{2} + \left(\frac{i\xi\sqrt{b}}{2}\right)^2 - \left(\frac{i\xi\sqrt{b}}{2}\right)^2\right]} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\left(\frac{x}{\sqrt{b}} + \frac{i\xi\sqrt{b}}{2}\right)^2} e^{-\frac{\xi^2 b}{4}} \, dx$$

$$= e^{-\frac{\xi^2 b}{4}} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\left(\frac{x}{\sqrt{b}} + \frac{i\xi\sqrt{b}}{2}\right)^2} dx$$

put $\frac{x}{\sqrt{b}} + i\xi\frac{\sqrt{b}}{2} = t$

$$= e^{-\frac{b}{4}\xi^2} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-t^2} \sqrt{b} \, dt$$

$$= \frac{e^{-\frac{b}{4}\xi^2}}{\sqrt{2\pi}} \sqrt{b} \int_{-\infty}^{\infty} e^{-t^2} dt$$

$\because \int_{-\infty}^{\infty} e^{-t^2} dt = \sqrt{\pi}$

$$= \sqrt{\frac{b}{2}} \, e^{-\frac{b}{4}\xi^2}$$

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(B) $u_t = \frac{1}{5} u_{xx}$, $u(x,0) = e^{-\sqrt{3}x^2}$ $\quad x \in \mathbb{R},\ t > 0$

$u := u(x,t)$

$u_t = \frac{1}{5} u_{xx}$

Take Fourier transform w.r.to x

1

$$\hat{u}_t(\xi,t) = \frac{1}{5}\hat{u}_{xx}(\xi,t)$$

$$= \frac{1}{5}(i\xi)^2\,\hat{u}(\xi,t) \qquad \Big|\ \mathcal{F}(f''(x)) = (i\xi)^2\mathcal{F}(f)$$

$$= -\frac{\xi^2}{5}\hat{u}(\xi,t)$$

$$\Rightarrow \frac{d\hat{u}}{dt} = -\frac{\xi^2}{5}\hat{u} \qquad \left(\hat{u} := \hat{u}(\xi,t)\right)$$

1

$$\Rightarrow \log \hat{u} = -\frac{\xi^2}{5}t + \log c$$

$$\Rightarrow \hat{u} = c\,e^{-\frac{\xi^2 t}{5}} \qquad c \text{ is a constant}$$

$$\Rightarrow \hat{u}(\xi,t) = c\,e^{-\frac{\xi^2 t}{5}} \quad \text{—} \ (*)$$

1

Now we have

$$u(x,0) = e^{-\sqrt{3}x^2}$$

$$\Rightarrow \hat{u}(\xi,0) = \frac{1}{\sqrt{2\sqrt{3}}}e^{-\frac{\xi^2}{4\sqrt{3}}}$$

From $(*)$ we have $\hat{u}(\xi,0) = c$

$$\Rightarrow c = \frac{1}{\sqrt{2\sqrt{3}}}e^{-\frac{\xi^2}{4\sqrt{3}}}$$

From (*) we get

$$(t>0)$$

$$\hat{u}(\xi,t) = \frac{1}{\sqrt{2\sqrt{3}}} e^{-\left[\frac{t}{5}+\frac{1}{4\sqrt{3}}\right]\xi^2}$$

$$= \frac{1}{\sqrt{2\sqrt{3}}} e^{-\left(\frac{4\sqrt{3}t+5}{20\sqrt{3}}\right)\xi^2} \quad (**)$$

(1)

we know from part (A) that

$$\mathcal{F}\left(e^{-\frac{x^2}{b}}\right) = \sqrt{\frac{b}{2}}\, e^{-\frac{b}{4}\xi^2}$$

$$\boxed{\mathcal{F}(f) = \hat{f}(\xi) \to \text{Fourier transform}}$$

$$e^{-\frac{b}{4}\xi^2} = \sqrt{\frac{2}{b}}\, \mathcal{F}\left(e^{-\frac{x^2}{b}}\right)$$

put $b = \frac{4\sqrt{3}t+5}{5\sqrt{3}}$

$$\Rightarrow e^{-\left(\frac{4\sqrt{3}t+5}{20\sqrt{3}}\right)\xi^2} = \sqrt{\frac{10\sqrt{3}}{4\sqrt{3}t+5}}\, \mathcal{F}\left(e^{-\left(\frac{5\sqrt{3}}{4\sqrt{3}t+5}\right)x^2}\right)$$

From (**) we write

(2)

$$\hat{u}(\xi,t) = \sqrt{\frac{5}{4\sqrt{3}t+5}}\, \mathcal{F}\left(e^{-\left(\frac{5\sqrt{3}}{4\sqrt{3}t+5}\right)x^2}\right)$$

$$\Rightarrow u(x,t) = \sqrt{\frac{5}{4\sqrt{3}t+5}}\, e^{-\left(\frac{5\sqrt{3}}{4\sqrt{3}t+5}\right)x^2}$$

(3)

Question # 2

$$u_{xx} + u_{yy} = 0 \text{ in } D = \{(x,y) \in \mathbb{R}^2 : x^2+y^2 < 1\}$$

$u := u(x,y)$

$$\frac{\partial u}{\partial \vec{n}} = (y^6 + 3x^4y^2 + 3x^2y^4 + x^6)^{\sqrt{3}} \left(\tan^{-1}\frac{y}{x}\right)^2 \text{ on } \partial D$$

put $x = r\cos\theta \quad y = r\sin\theta$

$r = \sqrt{x^2+y^2}, \quad \theta = \tan^{-1}\frac{y}{x}$

$r_x = \frac{x}{\sqrt{x^2+y^2}} = \frac{x}{r}, \quad r_y = \frac{y}{\sqrt{x^2+y^2}} = \frac{y}{r}$

$r_{xx} = \frac{1}{r} - \frac{x}{r^2} r_x$
$= \frac{1}{r} - \frac{x}{r^2}\frac{x}{r}$
$= \frac{1}{r} - \frac{x^2}{r^3}$
$= \frac{y^2}{r^3}$

$r_{yy} = \frac{1}{r} - \frac{y}{r^2} r_y$
$= \frac{1}{r} - \frac{y}{r^2}\frac{y}{r}$
$= \frac{1}{r} - \frac{y^2}{r^3}$
$= \frac{x^2}{r^3}$

1

$\theta_x = \frac{1}{1+\left(\frac{y}{x}\right)^2}\left(-\frac{y}{x^2}\right)$
$= -\frac{y}{x^2+y^2}$
$= -\frac{y}{r^2}$

$\theta_y = \frac{1}{1+\left(\frac{y}{x}\right)^2}\frac{1}{x}$
$= \frac{x}{x^2+y^2} = \frac{x}{r^2}$

$\theta_{yy} = -\frac{2x}{r^3} r_y$
$= -\frac{2xy}{r^4}$

$\theta_{xx} = \frac{2y}{r^3} r_x$
$= \frac{2xy}{r^4}$

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put the value of r_x, θ_x

$$u_x = u_r r_x + u_\theta \theta_x$$

$$u_{xx} = \frac{\partial}{\partial x}(u_r r_x) + \frac{\partial}{\partial x}(u_\theta \theta_x)$$

$$= u_r r_{xx} + r_x(u_{rr} r_x + u_{r\theta}\theta_x) + \theta_{xx} u_\theta + \theta_x(u_{\theta\theta}\theta_x + u_{r\theta} r_x)$$

$$= \frac{y^2}{r^3} u_r + \frac{x}{r}\left[u_{rr}\frac{x}{r} - u_{r\theta}\frac{y}{r^2}\right] + \frac{2xy}{r^4} u_\theta - \left(\frac{y}{r^2}\right)\left(-\frac{y}{r^2} u_{\theta\theta} + \frac{x}{r} u_{r\theta}\right)$$

$$u_{xx} = \frac{x^2}{r^2} u_{rr} - \frac{2xy}{r^3} u_{r\theta} + \frac{y^2}{r^4} u_{\theta\theta} + \frac{y^2}{r^3} u_r + \frac{2xy}{r^4} u_\theta$$

(1)

$$u_y = u_r r_y + u_\theta \theta_y$$

$$u_{yy} = \frac{\partial}{\partial y}(u_r r_y) + \frac{\partial}{\partial y}(u_\theta \theta_y)$$

$$= u_r r_{yy} + r_y(u_{rr} r_y + u_{r\theta}\theta_y) + \theta_{yy} u_\theta + \theta_y(u_{\theta\theta}\theta_y + u_{r\theta} r_y)$$

$$= \frac{x^2}{r^3} u_r + \frac{y}{r}\left(u_{rr}\frac{y}{r} + \frac{x}{r^2} u_{r\theta}\right) - \frac{2xy}{r^4} u_\theta + \frac{x}{r^2}\left(\frac{x}{r^2} u_{\theta\theta} + \frac{y}{r} u_{r\theta}\right)$$

$$= \frac{y^2}{r^2} u_{rr} + \frac{2xy}{r^3} u_{r\theta} + \frac{x^2}{r^4} u_{\theta\theta} + \frac{x^2}{r^3} u_r - \frac{2xy}{r^4} u_\theta$$

(1)

Now $u_{xx} + u_{yy} = 0$

$$\Rightarrow \frac{x^2+y^2}{r^2} u_{rr} + \frac{x^2+y^2}{r^4} u_{\theta\theta} + \frac{x^2+y^2}{r^3} u_r = 0$$

$$\Rightarrow u_{rr} + \frac{1}{r^2} u_{\theta\theta} + \frac{1}{r} u_r = 0$$

now $D = \{(x,y) \in \mathbb{R}^2 : x^2+y^2 < 1\}$

in polar co-ordinate it becomes

$D = \{(r,\theta) : 0 \le r < 1, \; -\pi \le \theta \le \pi\}$

$$\frac{\partial u}{\partial \vec{n}} = (y^6 + 3x^4y^2 + 3x^2y^4 + x^6)^{\sqrt{3}} (\tan^{-1}\tfrac{y}{x})^2$$

$$= (y^2+x^2)^{3\sqrt{3}} (\tan^{-1}\tfrac{y}{x})^2 \quad \text{on } \partial D$$

in polar-co-ordinate it becomes

$$\frac{\partial u}{\partial r} = (r^2)^{3\sqrt{3}} \theta^2 \quad \text{on } \partial D$$

$$= \theta^2 \quad \text{(as on } \partial D \; r = 1)$$

$r = 1$ on ∂D

in polar co-ordinate we have

$$u_{rr} + \frac{1}{r^2} u_{\theta\theta} + \frac{1}{r} u_r = 0$$

and $\frac{\partial u}{\partial r} = \theta^2$ at $r = 1$

ie $\left.\frac{\partial u}{\partial r}\right|_{r=1} = \theta^2$

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Solution by separation of variable method

$$u_{rr} + \frac{1}{r^2} u_{\theta\theta} + \frac{1}{r} u_r = 0 \; , \; \frac{\partial u}{\partial r}\Big|_{r=1} = \theta^2$$

Let $u(r,\theta) = R(r)\, F(\theta)$

then we have

$$R''(r)\, F(\theta) + \frac{1}{r^2} R(r)\, F''(\theta) + \frac{1}{r} R'(r)\, F(\theta) = 0$$

$$\Rightarrow \; r^2 \frac{R''(r)}{R(r)} + r \frac{R'(r)}{R(r)} + \frac{F''(\theta)}{F(\theta)} = 0$$

$$\Rightarrow \; \frac{r^2 R''(r) + r R'(r)}{R(r)} = -\frac{F''(\theta)}{F(\theta)} = \lambda \text{ (Constant)}$$

~~Case~~ $\lambda \geq 0$

Case I

$$F''(\theta) + \lambda F(\theta) = 0$$

$$\Rightarrow \; F(\theta) = A_1 \cos(\sqrt{\lambda}\,\theta) + B_1 \sin(\sqrt{\lambda}\,\theta)$$

but in (r,θ) plane

$$F(\theta + 2\pi) = F(\theta)$$

$$\Rightarrow \; \sqrt{\lambda} = n \quad n = 0, 1, 2 \ldots$$

$$\lambda = n^2 \quad n = 0, 1, 2 \ldots$$

$$\Rightarrow \; F_n(\theta) = A_{1n} \cos(n\theta) + B_{1n} \sin(n\theta) \qquad n = 0, 1, 2 \ldots$$

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Case II $\lambda < 0$

$F''(\theta) + \lambda F(\theta) = 0$

$\Rightarrow F(\theta) = C_1 e^{(\sqrt{-\lambda})\theta} + C_2 e^{-(\sqrt{-\lambda})\theta}$

but $F(\theta+2\pi) = F(\theta)$ only when $C_1 = 0 = C_2$

so we get zero soln for $\lambda < 0$

Now for $\lambda \geq 0$

$r^2 R''(r) + r R'(r) - \lambda R(r) = 0$

is a Cauchy Euler Eqn

Put $R(r) = r^m$ we get

$m(m-1) + m - \lambda = 0$

$\Rightarrow m = \pm\sqrt{\lambda}$

$R(r) = A r^{\sqrt{\lambda}} + B r^{-\sqrt{\lambda}}$

as $r \to 0$, $R(r)$ should be bounded so we choose $B = 0$

$R(r) = A r^{\sqrt{\lambda}}$

for each $\lambda_n = n^2$ $n = 0, 1, 2 - -$

$R_n(r) = A_n r^n$

so $U(r,\theta) = \sum_{n=0}^{\infty} R_n(r) F_n(\theta)$

$= C_0 + \sum_{n=1}^{\infty} r^n (C_n \cos n\theta + D_n \sin n\theta)$

Here $C_0 = A_{10} A_0$

$C_n = A_{1n} A_n$

$D_n = B_{1n} A_n$

so we get

$U(r,\theta) = C_0 + \sum_{n=1}^{\infty} r^n (C_n \cos n\theta + D_n \sin n\theta)$

$\frac{\partial u}{\partial r} = \sum_{n=1}^{\infty} n r^{n-1} (C_n \cos n\theta + D_n \sin n\theta)$

$\left.\frac{\partial u}{\partial r}\right|_{r=1} = \theta^2$

$\Rightarrow \theta^2 = \sum_{n=1}^{\infty} (n C_n \cos n\theta + n D_n \sin n\theta)$

which is Fourier series of θ^2

$n C_n = \frac{1}{\pi} \int_{-\pi}^{\pi} \theta^2 \cos n\theta \, d\theta$

$= \frac{2}{\pi} \int_0^{\pi} \theta^2 \cos n\theta \, d\theta$

$= \frac{2}{\pi} \left[\theta^2 \frac{\sin n\theta}{n} \right]_0^{\pi} - \frac{4}{\pi} \int_0^{\pi} \theta \frac{\sin n\theta}{n} \, d\theta$

$= -\frac{4}{\pi n} \int_0^{\pi} \theta \sin n\theta \, d\theta$

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$$n C_n = -\frac{4}{\pi n}\int_0^\pi \theta \sin n\theta \, d\theta$$

$$= -\frac{4}{\pi n}\left[\theta\left(-\frac{\cos n\theta}{n}\right)\right]_0^\pi + \frac{4}{\pi n}\int_0^\pi \left(-\frac{\cos n\theta}{n}\right) d\theta$$

$$= \frac{4}{\pi n^2}\left[\theta \cos n\theta\right]_0^\pi$$

$$= \frac{4}{\pi n^2}\, \pi \cos n\pi =$$

$$= \frac{4}{n^2}\cos n\pi$$

$$n C_n = (-1)^n \frac{4}{n^2}$$

$$\Rightarrow C_n = (-1)^n \frac{4}{n^3}$$

$$n D_n = \frac{1}{\pi}\int_{-\pi}^{\pi} \theta^2 \sin n\theta \, d\theta = 0 \Rightarrow D_n = 0$$

$\frac{1}{2}$

So the soln is

$$u(r,\theta) = C_0 + \sum_{n=1}^{\infty} (-1)^n \frac{4}{n^3} r^n \cos n\theta$$

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Question #3

$$2u_{xx} + 8u_{xy} + 5u_{yy} + 2u_x - 3u_y + u = \sin(xy) \quad \text{—— (1)}$$

Here $a = 2,\ 2b = 8 \Rightarrow b = 4,\ c = 5$

$$b^2 - ac = (4)^2 - 2\cdot 5 = 16 - 10 = 6 > 0$$

$$w^{\pm} = \frac{-b \pm \sqrt{b^2 - ac}}{a}$$

$$w^{\pm} = \frac{-4 \pm \sqrt{16-10}}{2}$$

$$= \frac{-4 \pm \sqrt{6}}{2}$$

$$= -2 \pm \frac{\sqrt{6}}{2}$$

1.5

$$w^{+} = -2 + \frac{\sqrt{6}}{2}$$

$$w^{-} = -2 - \frac{\sqrt{6}}{2}$$

The characteristic are

$$\xi(x,y) = y + \left(-2 + \frac{\sqrt{6}}{2}\right)x, \quad \eta(x,y) = y - \left(2 + \frac{\sqrt{6}}{2}\right)x$$

$$\xi_y = 1 = \eta_y \quad \xi_x = -2 + \frac{\sqrt{6}}{2}, \quad \eta_x = -\left(2 + \frac{\sqrt{6}}{2}\right)$$

$$x = \frac{\xi - \eta}{\sqrt{6}}, \quad y = \left(\frac{1}{2} + \frac{2}{\sqrt{6}}\right)\xi + \left(-\frac{2}{\sqrt{6}} + \frac{1}{2}\right)\eta$$

$$\xi_y = 1 \quad \xi_x = -2 + \frac{\sqrt{6}}{2}$$

$$\eta_y = 1 \quad -2 - \frac{\sqrt{6}}{2} = \eta_x$$

$$\xi_x = -2 + \frac{\sqrt{6}}{2}$$
$$\eta_x = -2 - \frac{\sqrt{6}}{2}$$
$$\xi_y = 1 = \eta_y$$

$$u_x = u_\xi \xi_x + u_\eta \eta_x$$
$$= \left(-2 + \frac{\sqrt{6}}{2}\right) u_\xi + \left(-2 - \frac{\sqrt{6}}{2}\right) u_\eta$$

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$$u_y = u_\xi \xi_y + u_\eta \eta_y$$
$$= u_\xi + u_\eta$$

$$u_{xx} = \left(-2 + \frac{\sqrt{6}}{2}\right)\left(u_{\xi\xi}\xi_x + u_{\xi\eta}\eta_x\right) + \left(-2 - \frac{\sqrt{6}}{2}\right)\left(u_{\eta\eta}\eta_x + u_{\xi\eta}\xi_x\right)$$
$$= \left(-2 + \frac{\sqrt{6}}{2}\right)^2 u_{\xi\xi} + \left(4 - \frac{6}{4}\right) u_{\xi\eta} + \left(-2 - \frac{\sqrt{6}}{2}\right)^2 u_{\eta\eta} + \left(4 - \frac{6}{4}\right) u_{\xi\eta}$$
$$= \left(-2 + \frac{\sqrt{6}}{2}\right)^2 u_{\xi\xi} + 5 u_{\xi\eta} + \left(-2 - \frac{\sqrt{6}}{2}\right)^2 u_{\eta\eta}$$

1

$$u_{yy} = u_{\xi\xi}\xi_y + u_{\xi\eta}\eta_y + u_{\eta\eta}\eta_y + u_{\xi\eta}\xi_y$$
$$= u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta}$$

$$u_{xy} = \frac{\partial}{\partial y}(u_x) = \frac{\partial}{\partial y}\left\{\left(-2 + \frac{\sqrt{6}}{2}\right) u_\xi + \left(-2 - \frac{\sqrt{6}}{2}\right) u_\eta\right\}$$
$$= \left(-2 + \frac{\sqrt{6}}{2}\right)\left(u_{\xi\xi}\xi_y + u_{\xi\eta}\eta_y\right) + \left(-2 - \frac{\sqrt{6}}{2}\right)\left(u_{\eta\eta}\eta_y + u_{\xi\eta}\xi_y\right)$$

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① $u_{xy} = \left(-2+\frac{\sqrt6}{2}\right)u_{\xi\xi} - 4u_{\xi\eta} + \left(-2-\frac{\sqrt6}{2}\right)u_{\eta\eta}$

Put the values of u_{xx}, u_{yy}, u_{xy}, u_x, u_y, x, y in terms of ξ, η in

$2u_{xx} + 8u_{xy} + 5u_{yy} + 2u_x - 3u_y + u = \sin(xy)$

② $\Rightarrow 2\left[\left(-2+\frac{\sqrt6}{2}\right)^2 u_{\xi\xi} + 5u_{\xi\eta} + \left(-2-\frac{\sqrt6}{2}\right)^2 u_{\eta\eta}\right]$

$+8\left[\left(-2+\frac{\sqrt6}{2}\right)u_{\xi\xi} - 4u_{\xi\eta} + \left(-2-\frac{\sqrt6}{2}\right)u_{\eta\eta}\right]$

$+5\left[u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta}\right]$

$+2\left[\left(-2+\frac{\sqrt6}{2}\right)u_\xi + \left(-2-\frac{\sqrt6}{2}\right)u_\eta\right]$

$-3(u_\xi + u_\eta) + u = \sin\left[\left(\frac{\xi-\eta}{\sqrt6}\right)\left\{\left(1+\frac{2}{\sqrt6}\right)\xi + \left(-\frac{2}{\sqrt6}+\frac12\right)\eta\right\}\right]$

$\Rightarrow -12u_{\xi\eta} + \left(-7+\sqrt6\right)u_\xi + \left(-7-\sqrt6\right)u_\eta + u$

$= \sin\left[\left(\frac{\xi-\eta}{\sqrt6}\right)\left\{\left(1+\frac{2}{\sqrt6}\right)\xi + \left(-\frac{2}{\sqrt6}+\frac12\right)\eta\right\}\right]$

The required canonical form

Question #3 (B)

$$u_{tt} - \Delta u = x+y+t, \quad \frac{\partial u}{\partial \vec{n}} = g \text{ on } \partial\Omega$$

$$\Omega = \{x^2+y^2<1\} \quad (x,y,t) \in \Omega \times (0,T)$$

$$u_t(x,y,0) = e^{x+y}$$

$$u(x,y,0) = x+2y$$

u_1 and u_2 are two sol^n of (E)

Let

$$\text{s.t. } v = u_1 - u_2$$

$$\Rightarrow v_{tt} - \Delta v = 0$$

$$\text{and } \frac{\partial v}{\partial \vec{n}} = 0 \text{ on } \partial\Omega, \quad v_t(x,y,0) = 0, \quad v(x,y,0) = 0$$

$$v_{tt} - \Delta v = 0$$

$$\Rightarrow (v_{tt} - \Delta v)\, v_t = 0$$

Integration over Ω

$$\Rightarrow \int_\Omega (v_{tt} v_t - v_t \Delta v)\, dx\,dy = 0$$

$$\Rightarrow \int_\Omega v_t v_{tt}\, dx\,dy - \int_{\partial\Omega} v_t \frac{\partial v}{\partial \vec{n}}\, ds + \int_\Omega \nabla v_t \cdot \nabla v\, dx\,dy = 0$$

$$\Rightarrow \int_\Omega v_t v_{tt}\, dx\,dy + \int_\Omega \nabla v_t \cdot \nabla v\, dx\,dy = 0 \quad \boxed{\frac{\partial v}{\partial \vec{n}} = 0 \text{ on } \partial\Omega}$$

$$\Rightarrow \frac{1}{2} \frac{d}{dt} \left(\int_\Omega v_t^2(x,y,t)\, dx\,dy + \int_\Omega |\nabla v|^2\, dx\,dy \right) = 0$$

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Let $E(t) = \int_\Omega v_t^2(x,y,t)\,dx\,dy + \int_\Omega |\nabla v|^2\,dx\,dy$

So we have $E'(t) = 0$ $\quad (\because E'(t) = 0)$

$\Rightarrow E(t)$ is constant

$\Rightarrow E(t) = E(0) = \int_\Omega v_t^2(x,y,0)\,dx\,dy + \int_\Omega |\nabla v(x,y,0)|^2\,dx\,dy$

$= 0 \qquad \because v_t(x,y,0) = 0$

$v(x,y,0) = 0$

$\Rightarrow E(t) = 0$

$\Rightarrow \int_\Omega v_t^2(x,y,t)\,dx\,dy + \int_\Omega |\nabla v(x,y,t)|^2\,dx\,dy = 0$

$\Rightarrow v_t^2(x,y,t) = 0$

$\Rightarrow v_t(x,y,t) = 0$

$\Rightarrow v(x,y,t) = f(x,y)$ $\qquad$ for some f which is independent of t

$\Rightarrow$ but $v(x,y,0) = 0$

$\Rightarrow f(x,y) = 0$

$\Rightarrow v(x,y,t) = 0$

$\Rightarrow u_1 = u_2$

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Question 4 (A)

$u - 2u_{xx} + u_t = 0$ $\quad u(x,0) = e^{\sqrt{7}x^2}$, $\quad u := u(x,t)\ x \in \mathbb{R}$, $t > 0$

$u - 2u_{xx} + u_t = 0$

take Fourier transform w.r.t to x

$\hat{u}(\xi,t) - 2\hat{u}_{xx}(\xi,t) + \hat{u}_t(\xi,t) = 0$

$\hat{u}(\xi,t) - 2(i\xi)^2\hat{u}(\xi,t) + \hat{u}_t(\xi,t) = 0$

1 $\Rightarrow \hat{u}(\xi,t) = -(1+2\xi^2)\hat{u}(\xi,t)$

$\Rightarrow \dfrac{d\hat{u}(\xi,t)}{dt} = -(1+2\xi^2)\hat{u}(\xi,t)$

$\boxed{\begin{aligned} &\mathcal{F}(f''(x)) \\ &= \hat{f''}(\xi) \\ &= (i\xi)^2\hat{F}(t)\end{aligned}}$

$\Rightarrow \log\hat{u}(\xi,t) = -(1+2\xi^2)t + \log C$

$\Rightarrow \hat{u}(\xi,t) = C e^{-(1+2\xi^2)t}$ ——— (*)

Now we have $u(x,0) = e^{\sqrt{7}x^2}$

$\Rightarrow \hat{u}(\xi,0) = \dfrac{1}{\sqrt{2\sqrt{7}}} e^{-\frac{\xi^2}{4\sqrt{7}}}$

$\mathcal{F}(e^{-ax^2}) = \dfrac{1}{\sqrt{2a}} e^{-\frac{\xi^2}{4a}}$

Now from (*) we have

$C = \hat{u}(\xi,0) = \dfrac{1}{\sqrt{2\sqrt{7}}} e^{-\frac{\xi^2}{4\sqrt{7}}}$

1 so we have

$\hat{u}(\xi,t) = \dfrac{1}{\sqrt{2\sqrt{7}}} e^{-\frac{\xi^2}{4\sqrt{7}}} e^{-(1+2\xi^2)t}$

$= \dfrac{1}{\sqrt{2\sqrt{7}}} e^{-\left(\frac{1}{4\sqrt{7}}+2t\right)\xi^2} e^{-t}$

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$\Rightarrow \hat{U}(\xi,t) = \frac{e^{-t}}{\sqrt{2\sqrt{7}}} e^{-\left(\frac{1}{4\sqrt{7}}+2\gamma t\right)\xi^2}$

$= \frac{e^{-t}}{\sqrt{2\sqrt{7}}} e^{-\left(\frac{\xi^2}{4\sqrt{7}/(1+8\sqrt{7}t)}\right)}$

$= \frac{e^{-t}}{\sqrt{2\sqrt{7}}} \frac{\sqrt{2\cdot\frac{\sqrt{7}}{1+\gamma\sqrt{7}t}}}{\sqrt{2\cdot\frac{\sqrt{7}}{1+\gamma\sqrt{7}t}}} e^{-\left(\frac{\xi^2}{4\sqrt{7}/(1+\gamma\sqrt{7}t)}\right)}$

$\hat{U}(\xi,t) = \frac{e^{-t}}{\sqrt{1+\gamma\sqrt{7}t}} \frac{1}{\sqrt{2a}} e^{-\frac{\xi^2}{4a}}$

$a = \frac{\sqrt{7}}{1+\gamma\sqrt{7}t}$

$\Rightarrow$ take the Inverse F.T

$\boxed{F\left(e^{-ax^2}\right) = \frac{1}{\sqrt{2a}} e^{-\frac{\xi^2}{4a}}}$

$U(x,t) = \frac{e^{-t}}{\sqrt{1+\gamma\sqrt{7}t}} e^{-ax^2}$

$U(x,t) = \frac{e^{-t}}{\sqrt{1+\gamma\sqrt{7}t}} e^{-\left(\frac{\sqrt{7}}{1+\gamma\sqrt{7}t}x^2\right)}$

(B) $x^2y'' + 4xy' + \lambda y = 0$, $y(1) = y(3) = 0$

It is a Euler Cauchy eqn ~~Put y = x^n~~

so its auxiliary eqn is

$r(r-1) + 4r + \lambda = 0$

$\Rightarrow r^2 + 3r + \lambda = 0$

$\Rightarrow r = \dfrac{-3 \pm \sqrt{9-4\lambda}}{2}$

$= -\frac{3}{2} \pm \frac{1}{2}\sqrt{9-4\lambda}$

Case-I when $9-4\lambda > 0$ i.e. $\lambda < \frac{9}{4}$

then the soln is

$y = A x^{\left(-\frac{3}{2} + \frac{1}{2}\sqrt{9-4\lambda}\right)} + B \cdot x^{\left(-\frac{3}{2} - \frac{1}{2}\sqrt{9-4\lambda}\right)}$

$= x^{-\frac{3}{2}}\left[A x^{\frac{\sqrt{9-4\lambda}}{2}} + B x^{-\frac{\sqrt{9-4\lambda}}{2}}\right]$

$y(1) = 0 \Rightarrow A + B = 0$

$y(3) = 0 \Rightarrow A\, 3^{\frac{\sqrt{9-4\lambda}}{2}} + B\, 3^{-\frac{\sqrt{9-4\lambda}}{2}} = 0$

$\Rightarrow A = B = 0$

so no non zero soln

Case II $\lambda = \frac{9}{4}$ $r = -\frac{3}{2}, -\frac{3}{2}$

$y = A x^{-\frac{3}{2}} + B (\ln x) x^{-\frac{3}{2}}$

$= x^{-3/2}(A + B \ln x)$

(1)

T.O

$y = x^{-\frac{3}{2}}(A + B\ln x)$

$y(1) = 0 \Rightarrow A = 0$

$y(x) = Bx^{-3/2}\ln x$

$y(3) = 0 \Rightarrow B = 0$

so again we get only zero solution.

<u>Case-III</u> When $9 - 4\lambda < 0$ i.e. $\lambda > \frac{9}{4}$

$r = -\frac{3}{2} \pm i\cdot\frac{\sqrt{4\lambda-9}}{2}$

$y(x) = x^{-\frac{3}{2}}\left[A\cos\left(\frac{\sqrt{4\lambda-9}}{2}\ln x\right) + B\sin\left(\frac{\sqrt{4\lambda-9}}{2}\ln x\right)\right]$

$y(1) = 0 \Rightarrow A = 0$

$y(x) = x^{-\frac{3}{2}}B\sin\left(\frac{\sqrt{4\lambda-9}}{2}\ln x\right)$

$y(3) = 0 \Rightarrow B\sin\left(\frac{\sqrt{4\lambda-9}}{2}\ln 3\right) = 0$

$\Rightarrow \frac{\sqrt{4\lambda-9}}{2}\ln 3 = n\pi \quad n \in \mathbb{Z}$ ∵ $B \neq 0$ we want

$\Rightarrow \frac{\sqrt{4\lambda-9}}{2} = \frac{n\pi}{\ln 3}$

$\Rightarrow 4\lambda - 9 = \left(\frac{2n\pi}{\ln 3}\right)^2$

$\Rightarrow \lambda = \frac{9}{4} + \frac{1}{4}\left(\frac{2n\pi}{\ln 3}\right)^2 \quad n \in \mathbb{N}$

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so the eigen values are

$$\lambda_n = \frac{9}{4} + \frac{1}{4}\left(\frac{2n\pi}{\ln 3}\right)^2$$

~~eigenfunctions are~~

$$\lambda_n = \frac{9}{4} + \left(\frac{n\pi}{\ln 3}\right)^2$$

1

eigenfunction are

$$y_n(x) = B_n x^{3/2} \sin\left(\frac{n\pi}{\ln 3} \ln x\right), \quad n \in \mathbb{N}$$

$0 \le x \le 3$

Question 5 / A

$$(3x^2-9)y' - (9x - x^3)y'' + \lambda x^2 (\cos x) y + e^{-x^2}\sin(x^2) y = 0 \quad (*)$$

$$y'(0) = y(3) = 0$$

now the above ODE can be written as

$$(x^3 - 9x)y'' + (3x^2-9)y' + e^{-x^2}\sin(x^2) y + \lambda x^2 (\cos x) y = 0$$

1.5

$$\Rightarrow \frac{d}{dx}\left[(x^3-9x)y'\right] + e^{-x^2}\sin(x^2) y + \lambda x^2 (\cos x) y = 0$$

Let $Ly = \frac{d}{dx}\left[(x^3-9x)y'\right] + e^{-x^2}\sin(x^2) y$

then $(*)$ can be written as

$$Ly + \lambda x^2 (\cos x) y = 0$$

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Let λ_1, λ_2 are two distinct eigenvalues and φ_1, φ_2 are corresponding eigenfunction.

then $L\varphi_1 + \lambda_1 x^2 (\cos x)\varphi_1 = 0$

1.5

$\Rightarrow L\varphi_1 = -\lambda_1 x^2 (\cos x)\varphi_1$

similarly $L\varphi_2 = -\lambda_2 x^2 (\cos x)\varphi_2$

$x + \ast$

Now $\int_0^3 (\varphi_2 L\varphi_1 \, dx - \varphi_1 L\varphi_2 \, dx)$

$= \langle L\varphi_1, \varphi_2 \rangle - \langle \varphi_1, L\varphi_2 \rangle$

$= \left[(x^3 - 9x)(\varphi_1' \varphi_2 - \varphi_2' \varphi_1)\right]_0^3$

$= 0$

1.5

So $\int_0^3 (\varphi_2 L\varphi_1 - \varphi_1 L\varphi_2) \, dx = 0$

use $\ast\ast$

$\Rightarrow \int_0^3 (\lambda_2 - \lambda_1) x^2 (\cos x) \varphi_1 \varphi_2 \, dx = 0$

$\Rightarrow \int_0^3 \varphi_2(x) (\cos x) \varphi_1(x) x^2 \, dx = 0$

$\lambda_2 \neq \lambda_1$

(B) we have

$$xy' + (\cos x)\ln(x)\, y + y''(1+x) = 0$$

we write

$$(1+x)y'' + xy' + (\cos x)\ln(x)\, y = 0 \quad \text{— } (X)$$

2

Here $a(x) = 1+x$, $b(x) = x$

$$\mu(x) = \frac{1}{a(x)} e^{\int \frac{b(x)}{a(x)}dx}$$

$$= \frac{1}{1+x} e^{\int \frac{x}{1+x}dx}$$

1.5

$$= \frac{1}{1+x} e^{\int \left(1-\frac{1}{1+x}\right)dx}$$

$$= \frac{1}{1+x} e^{x} e^{-\ln(1+x)}$$

$$\mu(x) = \frac{e^x}{(1+x)^2}$$

multiply (X) by $\mu(x) = \frac{e^x}{(1+x)^2}$

1.5

$$\frac{e^x}{1+x}y'' + \frac{xe^x}{(1+x)^2}y' + \frac{(\cos x)\ln(x)\,e^x}{(1+x)^2}y = 0$$

0

$$\Rightarrow \frac{d}{dx}\left(\frac{e^x}{1+x}y'\right) + \frac{(\cos x)\ln(x)\,e^x}{(1+x)^2}y = 0$$

$$u(x) = \frac{e^x}{1+x}$$

$$v(x) = \frac{(\cos x)\ln(x)\,e^x}{(1+x)^2}$$

(c) The given series is $\sum\limits_{\substack{P \text{ is prime} \\ P > 1}} 3^{-P} x^{P}$

The above series can be written as

$$\sum_{n=1}^{\infty} a_n x^n .$$

① where $a_n = \begin{cases} 0 & \text{if } n \text{ is not prime} \\ 3^{-n} & \text{if } n \text{ is prime} \end{cases}$

① so $\{|a_n|^{1/n}\}_{n \in \mathbb{N}} = \{0, 3^{-1}\}$.

$\Rightarrow \overline{\lim} |a_n|^{1/n} = 3^{-1}$

so radius of convergence is $\rho = \dfrac{1}{\overline{\lim} |a_n|^{1/n}} = 3$