5. Fourier Series

1. Find the Fourier series of the function f with period 4,

$$f(x) = \begin{cases} 0 & \text{if } -2 \le x < 0 \\ x & \text{if } 0 \le x \le 2. \end{cases}$$

Deduce that

$$1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \cdots = \frac{\pi^2}{8}.$$

2. Show that $\frac{1}{8}\pi x(\pi - x) = \sum_{n=0}^{\infty} \frac{\sin(2n+1)x}{(2n+1)^3}, \quad 0 \le x \le \pi.$

(Hint: Use half Range sine series formula)

3. Show that $\frac{\pi}{2}(\pi - 2x) = 4\sum_{n=0}^{\infty} \frac{\cos(2n+1)x}{(2n+1)^2}, \quad 0 < x < \pi.$

(Hint: Use half range cosine series)

4. (a) Find the half range sine series of

$$f(x) = \begin{cases} \frac{x}{2} & \text{if } 0 \le x < 2 \\ \frac{1}{2}(4-x) & \text{if } 2 \le x \le 4. \end{cases}$$

(b) Also, find the half range cosine series for the same $f(x)$.

5. Find the Fourier series for $f(x) = x, \quad -\pi < x < \pi$ and deduce that $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$ (Hint: Use Parseval Identity)

6. Find the Fourier series of $f(x) = x^2, \quad -\pi < x < \pi$ and deduce that

$$\frac{\pi^2}{12} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2}.$$

Use Parseval Identity to show that

$$\frac{\pi^4}{90} = \sum_{n=1}^{\infty} \frac{1}{n^4}.$$

7. Find the Fourier series of $f(x) = |x|, \quad -\pi < x < \pi.$