6. Canonical form

1. Classify the following PDEs. Find the characteristic curve and transform into canonical form.

   (a) $4u_{xx} + 8u_{xy} + 5u_{yy} + 2u_x + 7u_y + u = f(y), \quad u := u(x, y)$

   (b) $9u_{xx} + 12u_{xy} + 4u_{yy} + u_x + 3u_y + 10u - f(4y - 2x) = 0$

   (c) $u_{xx} - 8u_{xy} + 15u_{yy} + 7u_x + 11u_y + 20u = f(2y + 8x)$.

7. Uniqueness Result

1. Show that each of the following problem has a unique solution:

   (a)
   $$-\Delta u = f(x), \quad u := u(x), \quad x \in \Omega \subset \mathbb{R}^n$$
   with Robinson boundary condition
   $$\alpha u + \frac{\partial u}{\partial \nu} = g, \ x \in \partial\Omega,$$
   here $\partial\Omega$ is the piecewise smooth boundary, ν is the outward normal and α is a positive constant.

   (b)
   $$-\Delta u + u = f(x), \quad x \in \Omega \subset \mathbb{R}^n,$$
   with Neumann boundary condition
   $$\frac{\partial u}{\partial \nu} = g(x), \quad x \in \partial\Omega.$$

   (c)
   $$u_t - \Delta u = f(x, t), \quad u := u(x, t), \quad x \in \Omega \subset \mathbb{R}^n, \quad t \in [0, T],$$
   with Robinson boundary condition
   $$\alpha u + \frac{\partial u}{\partial \nu} = g, \quad \alpha > 0, \quad x \in \partial\Omega, \quad t \geq 0,$$

and initial condition: $u(x,0) = u_0(x), x \in \Omega$.

(d)

$$u_{tt} - \nabla \cdot \big(a(x)\nabla u\big) = f(x,t), \quad a(x) > 0, \quad x \in \Omega \subset \mathbb{R}^n, \ t \in [0,T],$$

with Robins boundary condition

$$\alpha u + a(x)\frac{\partial u}{\partial \nu} = 0, \quad x \in \partial\Omega,$$

and Initial conditions

$$u(x,0) = 0, \ u_t(x,0) = u_1 \ \forall \ x \in \Omega.$$

Here $\partial\Omega$ is the piecewise smooth boundary, ν is the outward normal, $a(x) \geq \beta > 0$ and α is a positive constant.

2. For the heat equation

$$u_t - \nabla \cdot \big(a(x)\nabla u\big) = 0, \quad x \in \Omega \subset \mathbb{R}^n, \quad t > 0,$$

with homogeneous boundary condition: $u(x,t) = 0, \ x \in \partial\Omega, \ t \geq 0$ and with initial condition $u(x,0) = u_0(x), \ x \in \Omega$, show that the energy $E(t) := \frac{1}{2}\int_\Omega |u(x,t)|^2 dx$ is monotonically decreasing and satisfies $E(t) \leq E(0), \ t > 0$.