Quiz 1 Marking Scheme.

Q.1 (a) let $\omega \in \mathbb{C}$ be s.t.

$$\omega^4 = ci$$
$$= c\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)$$

let $\omega = r(\cos\alpha + i\sin\alpha)$

Then by De Moivre's formula,

$$r^4(\cos 4\alpha + i\sin 4\alpha) = c\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)$$

$\Rightarrow r = c^{1/4}$ (positive 4^{th} root of c)

and $4\alpha = \frac{\pi}{2} + 2n\pi, \; n \in \mathbb{Z}$

$\therefore \alpha = \frac{\pi}{8} + \frac{n\pi}{2}, \; n \in \mathbb{Z},$ (But out of these 4 will be distinct)

$n=0 \Rightarrow \alpha = \frac{\pi}{8} \Rightarrow \omega = c^{1/4}\left(\cos\frac{\pi}{8} + i\sin\frac{\pi}{8}\right)$ [0.5]

$n=1 \Rightarrow \alpha = \frac{5\pi}{8} \Rightarrow \omega = c^{1/4}\left(\cos\frac{5\pi}{8} + i\sin\frac{5\pi}{8}\right)$ [0.5]

$n=2 \Rightarrow \alpha = \frac{9\pi}{8} \Rightarrow \omega = c^{1/4}\left(\cos\frac{9\pi}{8} + i\sin\frac{9\pi}{8}\right)$ [0.5]

$n=3 \Rightarrow \alpha = \frac{13\pi}{8} \Rightarrow \omega = c^{1/4}\left(\cos\frac{13\pi}{8} + i\sin\frac{13\pi}{8}\right)$ [0.5]

For calculation mistake in 1 root [-0.2]

Q.1 (b) let $\omega \in \mathbb{C}$ be s.t.

$$\omega^4 = -ci$$
$$= c\left(\cos\frac{3\pi}{2} + i\sin\frac{3\pi}{2}\right)$$

let $\omega = r(\cos\alpha + i\sin\alpha)$

Then by De Moivre's formula,

$$r^4(\cos 4\alpha + i\sin 4\alpha) = c\left(\cos\frac{3\pi}{2} + i\sin\frac{3\pi}{2}\right)$$

$\Rightarrow r = c^{1/4}$ (positive 4^{th} root of c)

and $4\alpha = \frac{3\pi}{2} + 2n\pi, \; n \in \mathbb{Z}$

$\therefore \alpha = \frac{3\pi}{8} + \frac{n\pi}{2}, \; n \in \mathbb{Z},$ (But out of these 4 will be distinct)

$n=0 \Rightarrow \alpha = \frac{3\pi}{8} \Rightarrow \omega = c^{1/4}\left(\cos\frac{3\pi}{8} + i\sin\frac{3\pi}{8}\right)$ [0.5]

$n=1 \Rightarrow \alpha = \frac{7\pi}{8} \Rightarrow \omega = c^{1/4}\left(\cos\frac{7\pi}{8} + i\sin\frac{7\pi}{8}\right)$ [0.5]

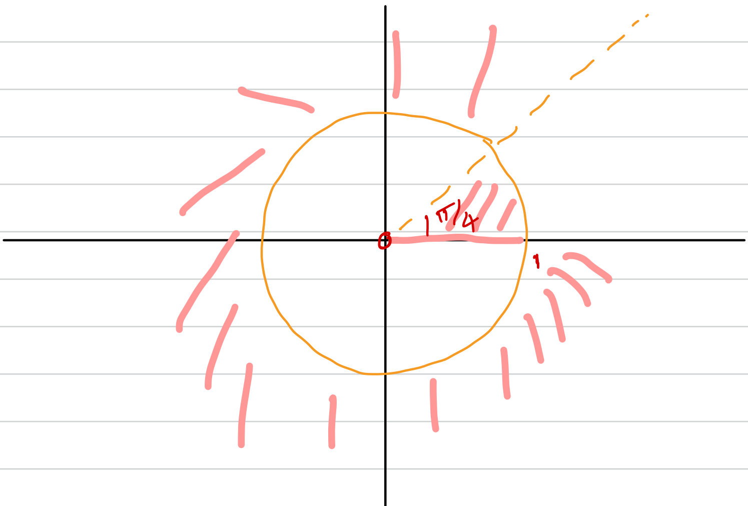
$n=2 \Rightarrow \alpha = \frac{11\pi}{8} \Rightarrow \omega = c^{1/4}\left(\cos\frac{11\pi}{8} + i\sin\frac{11\pi}{8}\right)$ [0.5]

$n=3 \Rightarrow \alpha = \frac{15\pi}{8} \Rightarrow \omega = c^{1/4}\left(\cos\frac{15\pi}{8} + i\sin\frac{15\pi}{8}\right)$ [0.5]

For calculation mistake in 1 root [-0.2]

Q.2] (a) Given

$S := \{ z \in \mathbb{C} \setminus \{0\} : |z| \leq 1 \text{ and } 0 \leq Arg(z) < \frac{\pi}{4} \}$
$\cup \{ z \in \mathbb{C} : |z| > 1 \text{ and } \frac{\pi}{4} < Arg(z) < 2\pi \}$



Limit points of S

$= \{ z \in \mathbb{C} \setminus \{0\} : |z| \leq 1 \text{ and } 0 \leq Arg(z) \leq \frac{\pi}{4} \}$
$\cup \{0\} \cup \{ z \in \mathbb{C} : |z| \geq 1 \text{ and } \frac{\pi}{4} \leq Arg(z) < 2\pi \}$
$\cup \{ z \in \mathbb{C} : |z| \geq 1 \text{ \& } Im(z) = 0 \}$ [1]

Boundary points of S

$= \{ z \in \mathbb{C} : |z| = 1 \} \cup \{ z \in \mathbb{C} : Arg(z) = \frac{\pi}{4} \}$
$\cup \{ z \in \mathbb{C} : z \text{ is real } \& \ z \geq 0 \}$ [1]

Interior points of S

$= \{ z \in \mathbb{C} \setminus \{0\} : |z| < 1 \text{ and } 0 < Arg(z) < \frac{\pi}{4} \}$
$\cup \{ z \in \mathbb{C} : |z| > 1 \text{ and } \frac{\pi}{4} < Arg(z) < 2\pi \}$ [1]

$\bar{S} := S \cup \{\text{limit points of } S\}$

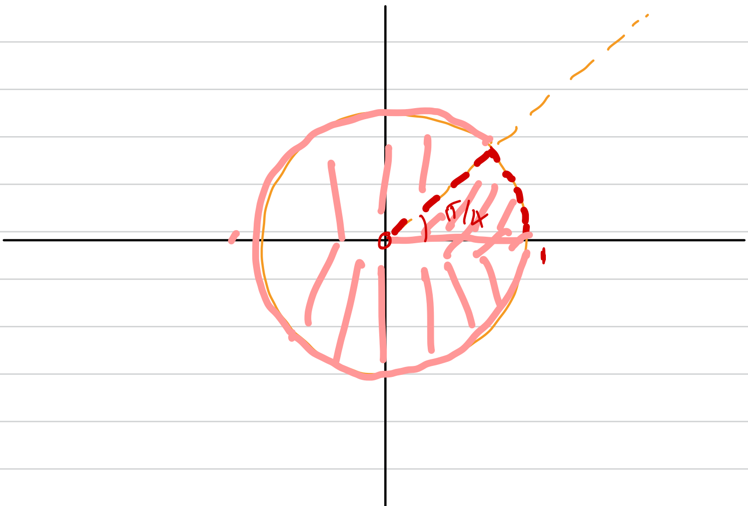
$= \{z \in \mathbb{C} \setminus \{0\} : |z| \leq 1 \text{ and } 0 \leq Arg(z) \leq \frac{\pi}{4}\}$
$\cup \{0\} \cup \{z \in \mathbb{C} : |z| \geq 1 \text{ and } \frac{\pi}{4} \leq Arg(z) < 2\pi\}$
$\cup \{z \in \mathbb{C} : |z| \geq 1 \text{ and } Im(z) = 0\}$ [1]

- For partial correct set [0.5 m]
- For drawing the set (in case nothing is done) [0.5 m]

Q.2 (b) Given

$S := \{z \in \mathbb{C} \setminus \{0\} : |z| \leq 1$ and $\frac{\pi}{4} < Arg(z) < 2\pi\}$

$\cup \{z \in \mathbb{C} : |z| < 1$ and $0 \leq Arg(z) < \frac{\pi}{4}\}$



Limit points of S

$= \{z \in \mathbb{C} : |z| \leq 1\}$ [1m]

Boundary points of S

$= \{z \in \mathbb{C} : |z| = 1\} \cup \{0\}$

$\cup \{z \in \mathbb{C} : Arg(z) = \frac{\pi}{4}$ & $|z| \leq 1\}$ [1]

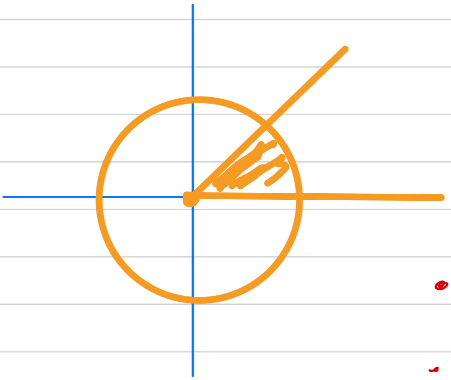
Interior points of S

$= \{z \in \mathbb{C} : |z| < 1\} \setminus \{0\}$ [1m]

$\bar{S} = \{z \in \mathbb{C} : |z| \leq 1\}$ [1m]

Q.3 (a) Given

$$S = \{z \in \mathbb{C} : |z| \leq 1 \text{ and } 0 \leq \operatorname{Arg}(z) \leq \frac{\pi}{4}\} \cup \{0\}$$



Here

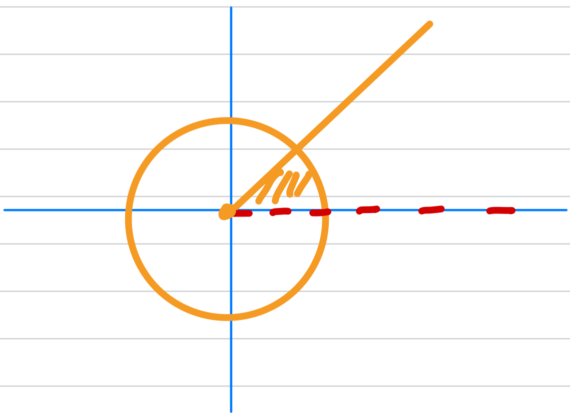
S is compact [0.5]

since it is closed and bounded [0.5]

- For saying compact [0.5]
- For justification [0.5]

(b) Here given

$$S = \{z \in \mathbb{C} : |z| \leq 1 \text{ and } 0 < \operatorname{Arg}(z) \leq \pi/4\} \cup \{0\}$$



S is not compact [0.5]

because S is not closed [0.5]

- For saying not compact [0.5]
- For justification [0.5]

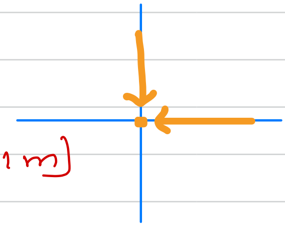
Q. 4] (a) Given

$$f(z) = \begin{cases} \dfrac{|z|}{\operatorname{Im}(z)} & \text{if } \operatorname{Im}(z) \neq 0 \\ 0 & \text{if } \operatorname{Im}(z) = 0. \end{cases}$$

$\therefore$

$$\lim_{\substack{z \to 0 \\ \operatorname{Im}(z)=0}} f(z) = \lim_{\substack{z \to 0 \\ \operatorname{Im}(z)=0}} 0 = 0$$

[1m]



and

$$\lim_{\substack{z \to 0 \\ \operatorname{Re}(z)=0}} f(z) = \lim_{\substack{y \to 0 \\ z = x+iy \\ x=0}} \frac{\sqrt{x^2+y^2}}{y}$$

$$= \lim_{y \to 0} \frac{\sqrt{y^2}}{y}$$

$$= \lim_{y \to 0} \frac{|y|}{y} \qquad (\because |z| = |y|)$$

$$= \begin{cases} 1 & \text{if } y > 0 \\ -1 & \text{if } y < 0 \end{cases}$$

[1m]

Since limit along the path $y=0$ and $x=0$ are not the same, $\lim\limits_{z \to 0} f(z)$ doesn't exist.

Hence f is not continuous at 0.

- For saying f is not continuous [1m]
- Justification [2m]

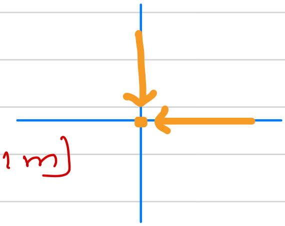
Q. 4] (b) Given

$$f(z) = \begin{cases} \dfrac{|z|}{Re(z)} & \text{if } Re(z) \neq 0 \\ 0 & \text{if } Re(z) = 0. \end{cases}$$

$\therefore$

$$\lim_{\substack{z \to 0 \\ Re(z)=0}} f(z) = \lim_{\substack{z \to 0 \\ Re(z)=0}} 0 = 0$$

[1m]



and

$$\lim_{\substack{z \to 0 \\ Im(z)=0}} f(z) = \lim_{\substack{x \to 0 \\ z = x+iy \\ y=0}} \frac{\sqrt{x^2+y^2}}{x}$$

$$= \lim_{x \to 0} \frac{\sqrt{x^2}}{x}$$

$$= \lim_{x \to 0} \frac{|x|}{x} \quad (\because |z| = |x|)$$

$$= \begin{cases} 1 & \text{if } x > 0 \\ -1 & \text{if } x < 0 \end{cases}$$ [1m]

Since limit along the path $y=0$ and $x=0$ are not the same, $\lim_{z \to 0} f(z)$ doesn't exist.

Hence f is not continuous at 0.

- For saying f is not continuous [1m]
- Justification [2m]

Q.5 (a) Given $f(z) = z\,Re(z)$

Write $z = x + iy$

$\therefore \quad f(z) = (x+iy)x = x^2 + i\,xy.$

$\therefore \quad u = Re\, f = x^2 \quad \& \quad v = Im\, f = xy$ [0.5]

$\Rightarrow \quad u_x = 2x \qquad v_x = y$

$u_y = 0 \qquad v_y = x$ [1]

Suppose f is differentiable at $z = x + iy$

$\therefore$ By Cauchy-Riemann equations

$u_x = v_y \Rightarrow 2x = x \Rightarrow x = 0$

&

$v_x = -u_y \Rightarrow y = 0$

$\therefore$ f satisfies the CR equations only at 0. [0.5]

Also,

clearly u_x, v_x, u_y, v_y are continuous on $\mathbb{C}$ (since they are polynomials)

$\Rightarrow f$ is differentiable at 0. [0.5]

Also,

$f'(0) = u_x(0,0) + v_x(0,0) = 0.$ [0.5]

$\therefore$ f is differentiable only at 0 & $f'(0) = 0$.

Alternative solution:

Suppose f is differentiable at $z = x+iy$.
Then by definition,

$$f'(z) = \lim_{h\to 0} \frac{f(z+h) - f(z)}{h} \qquad [0.5]$$

$$= \lim_{h\to 0} \frac{(z+h)\,Re(z+h) - z\,Re(z)}{h}$$

$$= \lim_{h\to 0} \frac{z\,\cancel{Re(z)} + h\,Re(z+h) + z\,Re(h) - z\,\cancel{Re(z)}}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{h}\,Re(z+h)}{\cancel{h}} + z\lim_{h\to 0}\frac{Re(h)}{h}$$

$$= Re(z) + z\lim_{h\to 0}\frac{Re(h)}{h} \qquad [0.5]$$

If $z=0$, then $z\lim_{h\to 0}\frac{Re(h)}{h} = 0$ & hence

f is differentiable at 0. [0.5]

Also, $f'(0) = Re(z) + 0\cdot\lim_{h\to 0}\frac{Re(h)}{h}$

$$= 0 \qquad [0.5]$$

If $z \neq 0$, then $z\lim_{h\to 0}\frac{Re(h)}{h}$ doesn't

exist because $\lim_{h\to 0}\frac{Re(h)}{h} = \begin{cases} \lim\limits_{\substack{h\to 0\\ h \text{ real}}} \frac{h}{h} = 1 \\ \lim\limits_{\substack{h\to 0\\ h \text{ purely imaginary}}} 0 = 0. \end{cases}$ [0.5]

$\therefore f$ is not differentiable at $z \neq 0$. [0.5]

Q.5 (b) Given $f(z) = z\, Im(z)$

Write $z = x + iy$

$\therefore f(z) = (x+iy)y = xy + iy^2$

$\therefore u = Re\, f = xy$ & $v = Im\, f = y^2$ [0.5]

$\Rightarrow u_x = y$ $\quad v_x = 0$

$u_y = x$ $\quad v_y = 2y$ [1]

Suppose f is differentiable at $z = x + iy$

$\therefore$ By Cauchy-Riemann equations

$u_x = v_y \Rightarrow y = 2y \Rightarrow y = 0$

&

$v_x = -u_y \Rightarrow x = 0$

$\therefore f$ satisfies the CR equations only at 0. [0.5]

Also,

clearly u_x, v_x, u_y, v_y are continuous on $\mathbb{C}$ (since they are polynomials)

$\Rightarrow f$ is differentiable at 0. [0.5]

Also,

$f'(0) = u_x(0,0) + v_x(0,0) = 0.$ [0.5]

$\therefore f$ is differentiable only at 0 & $f'(0) = 0.$

Alternative solution:

Suppose f is differentiable at $z = x+iy$.
Then by definition,

$$f'(z) = \lim_{h\to 0} \frac{f(z+h)-f(z)}{h} \qquad [0.5]$$

$$= \lim_{h\to 0} \frac{(z+h)\,\mathrm{Im}(z+h) - z\,\mathrm{Im}(z)}{h}$$

$$= \lim_{h\to 0} \frac{z\,\cancel{\mathrm{Im}(z)} + h\,\mathrm{Im}(z+h) + z\,\mathrm{Im}(h) - \cancel{z\,\mathrm{Im}(z)}}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{h}\,\mathrm{Im}(z+h)}{\cancel{h}} + z\lim_{h\to 0}\frac{\mathrm{Im}(h)}{h}$$

$$= \mathrm{Im}(z) + z\lim_{h\to 0}\frac{\mathrm{Im}(h)}{h} \qquad [0.5]$$

If $z=0$, then $z\lim_{h\to 0}\frac{\mathrm{Im}(h)}{h} = 0$ & hence

f is differentiable at 0. [0.5]

Also, $f'(0) = \mathrm{Im}(z) + 0\cdot\lim_{h\to 0}\frac{\mathrm{Im}(z)}{h}$

$$= 0 \qquad [0.5]$$

If $z \neq 0$, then $z\lim_{h\to 0}\frac{\mathrm{Im}(h)}{h}$ doesn't

exist because $\lim_{h\to 0}\frac{\mathrm{Im}(h)}{h} = \begin{cases} \lim\limits_{\substack{h\to 0\\ h \text{ real}}} \frac{0}{h} = 0 \\ \lim\limits_{\substack{h\to 0\\ h \text{ purely imaginary}}} \frac{h}{h} = 1 \end{cases}$ [0.5]

$\therefore$ f is not differentiable at $z \neq 0$. [0.5]

Q. 6 Given

$$x_n = c + i\frac{(-1)^n}{n}$$

Since $Re(x_n) = c \to c$ as $n \to \infty$ [0.5]

& $Im\, x_n = \frac{(-1)^n}{n} \to 0$ as $n \to \infty$ [0.5]

($\because$ Given $\varepsilon > 0$ choose N s.t. $\frac{1}{N} < \varepsilon$

then

$$\left|\frac{(-1)^n}{n}\right| = \frac{1}{n} \le \frac{1}{N} \quad \forall\, n \ge N$$

$$< \varepsilon$$)

$\therefore$ the sequence (x_n) converges

and $\lim_{n \to \infty} x_n = c$ [1m]

Q.6 Given

$$x_n = \frac{(-1)^n}{n} + ic$$

Since $Re(x_n) = \frac{(-1)^n}{n} \longrightarrow 0$ as $n \to \infty$ [0.5]

& $Im\, x_n = c \longrightarrow c$ as $n \to \infty$ [0.5]

$\left(\because \text{Given } \varepsilon > 0 \text{ choose } N \text{ s.t } \frac{1}{N} < \varepsilon \right.$
then
$\left. \left| \frac{(-1)^n}{n} \right| = \frac{1}{n} \leq \frac{1}{N} < \varepsilon \quad \forall\, n \geq N \right)$

$\therefore$ the sequence (x_n) converges
and $\lim_{n \to \infty} x_n = ic$ [1m]