


End Semester Exam (Marking Scheme)
23 Sep 2021



Part 1

Q.1] (i)

we know

$$\cos(z) = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n}}{(2n)!} \quad \forall z \in \mathbb{C}$$

$$\Rightarrow \frac{\cos(z)}{z^c} = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n-c}}{(2n)!} \quad \forall z \neq 0$$

$$= \frac{1}{z^c} - \frac{1}{2!} \frac{1}{z^{c-2}} + \frac{1}{4!} \frac{1}{z^{c-4}} - \dots \quad [0.5]$$

$\forall z \in \mathbb{C} \setminus \{0\}$

Since $f(z) = \frac{\cos(z)}{z^c}$ has finitely many negative powers of z in its Laurent series around 0, f has a pole of order c at 0. [0.5]

(ii) $f(z) = \frac{z^2 - (c+1)z + c}{z^2 - cz}$ at $z=c$

$$= \frac{(z-c)(z-1)}{z(z-c)}$$

Here $\lim_{z \rightarrow c} (z-c)f(z)$

$$= \lim_{z \rightarrow c} \frac{(z-c)(z-1)}{z} = 0 \quad [0.5]$$

$\therefore f$ has a removable singularity at $z=c$. [0.5]

Q.2] Let $a = c + i$ where $c =$ last two digits of your roll number.
 Note that $\log(z)$ is analytic at a .

Let $\log(z) = \sum_{n=0}^{\infty} a_n (z-a)^n$ be a power series expansion of $\log(z)$ around a .

Then

$$\frac{d}{dz} (\log(z)) = \sum_{n=1}^{\infty} n a_n (z-a)^{n-1}$$

(\because power series can be differentiated term by term)

$$\Rightarrow \frac{1}{z} = \sum_{n=1}^{\infty} n a_n (z-a)^{n-1} \quad [1]$$

But $\frac{1}{z} = \frac{1}{a + z - a} = \frac{1}{a \left(1 + \frac{z-a}{a}\right)}$

$$= \frac{1}{a} \sum_{n=0}^{\infty} (-1)^n \left(\frac{z-a}{a}\right)^n \quad \text{for } |z-a| < |a|$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{a^{n+1}} (z-a)^n \quad \text{--- (2)}$$

[1]

Comparing (1) and (2) gives,

$$n a_n = \frac{(-1)^{n-1}}{a^n} \quad n \geq 1$$

$$\Rightarrow a_n = \frac{(-1)^{n-1}}{a^{n+1}}$$

$$\text{and } a_0 = \log(a)$$

[0.5]

$$\therefore \log(z) = \sum_{n=0}^{\infty} a_n (z-a)^n \quad \text{where } c$$

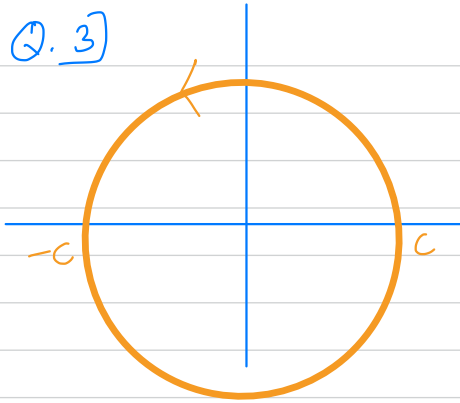
$$a_n = \begin{cases} \frac{(-1)^{n+1}}{a^{n+1}} & \forall n \geq 1 \\ \log(a) & \text{for } n=0 \end{cases}$$

$$\forall |z-a| < |a| = \sqrt{c^2+1}$$

\therefore Radius of convergence of $\sum_{n=0}^{\infty} a_n (z-a)^n$ is $\sqrt{c^2+1}$.

[0.5]

Q. 3]



Let

$$f(z) = (z-1)^2 \sin(1/z)$$

Then f is analytic everywhere on $\{z \in \mathbb{C} : |z| \leq 1\}$ except at $z=0$.

\therefore By Cauchy's Residue theorem

$$\int_{|z|=1} (z-1)^2 \sin(1/z) = 2\pi i \operatorname{Res}(f; 0) \quad [0.5]$$

The Laurent series expansion of $\sin(1/z)$ around 0 is,

$$\sin(1/z) = \sum_{n=0}^{\infty} (-1)^n \frac{1}{(2n+1)!} \frac{1}{z^{2n+1}}$$

$$\Rightarrow (z-1)^2 \sin(1/z) = (z^2 - 2z + 1) \left(\frac{1}{z} - \frac{1}{3!} z^{-3} + \frac{1}{5!} z^{-5} - \dots \right)$$

$$= \text{lower order terms} + \left(-\frac{1}{3!} + 1 \right) \frac{1}{z} + \text{higher order terms}$$

$$\therefore \operatorname{Res}(f; 0) = -\frac{1}{6} + 1 = 5/6 \quad [1]$$

$$\therefore \int_{|z|=1} (z-1)^2 \sin(1/z) = (2\pi i) (5/6)$$

$$= \frac{5\pi i}{3} \quad [0.5]$$

Q. 4] Let u be a real valued harmonic function defined on $D = \{z \in \mathbb{C} : |z| < 1\}$

Let v be a harmonic conjugate of u on D .

$\therefore f = u + iv$ is analytic on D . [0.5]

Hence f is infinitely many times differentiable [0.5]

\Rightarrow all partials of u and v exist and are continuous on D [0.5]

$\Rightarrow u$ is infinitely many times differentiable. [0.5]

Q.5] (a) Since f is entire, f is analytic at 0.

\therefore By Taylor's theorem

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \quad \text{in } B(0; r) \text{ for some } r > 0 \quad [0.5]$$

$$\text{where } a_n = \frac{f^{(n)}(0)}{n!} = \begin{cases} 0 & \text{if } n \text{ even} \\ (-1)^k & \text{if } n = 2k+1 \end{cases}$$

$$\therefore f(z) = \sum_{n=0}^{\infty} (-1)^k \frac{z^{2k+1}}{(2k+1)!} \quad \text{in } B(0; r) \quad [1]$$

$$\Rightarrow f(z) = \sin(z) \quad \text{in } B(0; r) \quad [0.5]$$

Clearly, $B(0; r)$ has a limit point in \mathbb{C} .

\therefore by the uniqueness theorem,

$$f(z) = \sin(z) \quad \forall z \in \mathbb{C}. \quad [1]$$

Bonus question: we can not conclude the same if $f: \mathbb{R} \rightarrow \mathbb{R}$ is differentiable. [0.5]

For example,

$$\text{Let } f(x) = \begin{cases} \sin(x) + e^{-1/x^2} & \text{if } x > 0 \\ \sin(x) & \text{if } x \leq 0 \end{cases}$$

Then clearly f is differentiable at $x \neq 0$.

$$\text{At } x=0, \quad g(x) = \begin{cases} e^{-1/x^2} & \text{if } x > 0 \\ 0 & \text{if } x \leq 0 \end{cases} \quad \text{is}$$

differentiable infinitely many times and $g^{(n)}(0) = 0$.

$$\Rightarrow f^{(n)}(0) = \sin^{(n)}(0) + g^{(n)}(0) = \sin^{(n)}(0)$$

$$= \begin{cases} 0 & \text{if } n \text{ is even} \\ (-1)^k & \text{if } n = 2k+1 \end{cases}$$

But ^{clearly} $f(x) \neq \sin(x) \quad \forall x \in \mathbb{R}$. [0.5]

(b) Since f is entire, f is analytic at 0.

\therefore By Taylor's theorem

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \quad \text{in } B(0; r) \text{ for some } r > 0 \quad [0.5]$$

$$\text{where } a_n = \frac{f^{(n)}(0)}{n!} = \begin{cases} 0 & \text{if } n \text{ odd} \\ (-1)^k & \text{if } n = 2k \end{cases}$$

$$\therefore f(z) = \sum_{n=0}^{\infty} (-1)^k \frac{z^{2k}}{(2k)!} \quad \text{in } B(0; r) \quad [1]$$

$$\Rightarrow f(z) = \cos(z) \quad \text{in } B(0; r) \quad [0.5]$$

Clearly, $B(0; r)$ has a limit point in \mathbb{C} .

\therefore by the uniqueness theorem,

$$f(z) = \cos(z) \quad \forall z \in \mathbb{C}. \quad [1]$$

Bonus question: we can not conclude the same if $f: \mathbb{R} \rightarrow \mathbb{R}$ is differentiable. [0.5]

For example,

$$\text{Let } f(x) = \begin{cases} \sin(x) + e^{-1/x^2} & \text{if } x \geq 0 \\ \sin(x) & \text{if } x \leq 0 \end{cases}$$

Then clearly f is differentiable at $x \neq 0$.

$$\text{At } x=0, \quad g(x) = \begin{cases} e^{-1/x^2} & \text{if } x \geq 0 \\ 0 & \text{if } x \leq 0 \end{cases}$$

is differentiable infinitely many times and $g^{(n)}(0) = 0$.

$$\begin{aligned} \Rightarrow f^{(n)}(0) &= \sin^{(n)}(0) + g^{(n)}(0) = \sin^{(n)}(0) \\ &= \begin{cases} 0 & \text{if } n \text{ is even} \\ (-1)^k & \text{if } n = 2k+1 \end{cases} \end{aligned}$$

But ^{clearly} $f(x) \neq \sin(x) \quad \forall x \in \mathbb{R}$. [0.5]

Q. 6] (a) ^{We prove that} the function $\sin z$ is not bounded on \mathbb{C} .

[0.5]

Suppose $\sin z$ is bounded on \mathbb{C} .

Since $\sin(z)$ is entire, by Liouville's theorem $\sin(z)$ is a constant.

But we know ^{that} $\sin(z)$ is not constant

For example, $\sin(0) = 0$ &

$$\sin\left(\frac{\pi}{2}\right) = 1$$

[1.5]

Q. 6] (b) ^{We prove that} the function $\cos(z)$ is not bounded on \mathbb{C} .

[0.5]

Suppose $\cos(z)$ is bounded on \mathbb{C} .

Since $\cos(z)$ is entire, by Liouville's theorem $\cos(z)$ is [^]constant.

But we know ^{that} $\cos(z)$ is not constant

For example, $\cos(0) = 1$ &
 $\cos\left(\frac{\pi}{2}\right) = 0$

[1.5]

Part 2

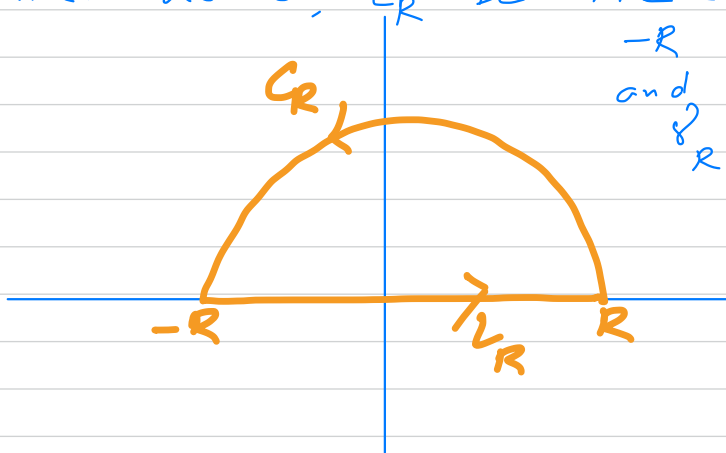
Q.1] Let $f(z) = \frac{1}{(z^2 + c)^2}$

Since $f(z) = f(-z)$, f is an even function.

$$\therefore \int_0^{\infty} \frac{dx}{(x^2 + c)^2} = \frac{1}{2} \text{P.V.} \int_{-\infty}^{\infty} \frac{dx}{(x^2 + c)^2} \quad [0.5]$$

Let C_R be the circle of radius R centered at O , L_R be the line joining $-R$ to R

and $\gamma_R = C_R + L_R$.



Then

$$\int_{-R}^R f(x) dx = \int_{\gamma_R} f(z) dz - \int_{C_R} f(z) dz \quad (1)$$

Here f is analytic on $\mathbb{C} \setminus \{\pm\sqrt{c}i\}$ and f has a pole of order 2 at $\pm\sqrt{c}i$. Choose $R > \sqrt{c}$ so that $\pm\sqrt{c}i$ lies in γ_R . [0.5]

\therefore By Cauchy's residue theorem

$$\oint_R f(z) dz = 2\pi i \operatorname{Res}(f; \alpha) \quad (2)$$

We claim that $\lim_{R \rightarrow \infty} \int_{C_R} f(z) dz = 0$.

Proof of the claim:

On C_R , $|z^2 + c|^2 \geq (|z|^2 - c)^2 = (R^2 - c)^2$

$$\Rightarrow |f(z)| = \left| \frac{1}{z^2 + c} \right| \leq \frac{1}{(R^2 - c)^2} \quad \text{on } C_R$$

\therefore By ML-inequality

$$\left| \int_{C_R} f(z) dz \right| \leq \frac{1}{(R^2 - c)^2} \times \pi R$$

$$\longrightarrow 0 \quad \text{as } R \rightarrow \infty$$

$$\therefore \lim_{R \rightarrow \infty} \int_{C_R} f(z) dz = 0 \quad [1]$$

\therefore By (1) and (2)

$$\begin{aligned} \text{D.V. } \int_{-\infty}^{\infty} f(x) dx &= \lim_{R \rightarrow \infty} \int_{-R}^R f(x) dx \\ &= 2\pi i \operatorname{Res}(f; \alpha) \end{aligned}$$

$$= g(z)$$

Here $f(z) = \frac{1}{(z-\alpha)^2} (z+\alpha)^2$

Since f has a pole of order 2 at α ,

$$\text{Res}(f; \alpha) = \lim_{z \rightarrow \alpha} g'(z) \quad \text{where } g(z) = \frac{1}{(z+\alpha)^2}$$

$$\therefore g(z) = \frac{-2}{(z+\alpha)^3}$$

$$\therefore g'(\alpha) = \frac{-2}{8\alpha^3} = -\frac{1}{4\alpha^3}$$

[1]

$$\therefore \text{P.V.} \int_{-\infty}^{\infty} f(x) dx = \frac{-\pi i}{2\alpha^3}$$

$$\text{P.V.} \int_{-\infty}^{\infty} f(x) dx = \frac{-\pi i}{2c^{3/2}} (-i)$$

[0.5]

$$\therefore \int_0^{\infty} f(x) dx = \frac{\pi}{4c^{3/2}}$$

[0.5]

Q.2] Given $\sum_{n=0}^{\infty} (c + (-i)^n)^n z^n$

$$\therefore a_n = (c + (-i)^n)^n$$

By Cauchy-Hadamard formula,

$$\frac{1}{R} = \limsup \{ \sqrt[n]{|a_n|} : n \in \mathbb{N} \} \quad [0.5]$$

Here

$$\sqrt[n]{|a_n|} = |c + (-i)^n|$$

$$= \begin{cases} c+1 \\ \sqrt{c^2+1} \\ c-1 \end{cases}$$

$$n=4k$$

$$n=4k+1 \text{ or } 4k+3$$

$$n=4k+2$$

$$\therefore \limsup \{ \sqrt[n]{|a_n|} : n \in \mathbb{N} \} = c+1 \quad [1]$$

$$\therefore \frac{1}{R} = c+1$$

$$\Rightarrow \boxed{R = \frac{1}{c+1}}$$

$$[0.5]$$

Q.3] (a) Let $\mathcal{D} = \{z \in \mathbb{C} : \operatorname{Im}(z) > 0\}$

Given $\operatorname{Log}(z) = \ln |z| + i \theta(z)$ where $\theta(z) \in \arg(z) \cap (-\pi, \pi)$

$\therefore \operatorname{Log}(z)$ is analytic on $\mathbb{C} \setminus (-\infty, 0]$ [0.5]

We show that $\operatorname{Log}(z)$ has antiderivative on \mathcal{D} . [0.5]



Since $\operatorname{Log}(z)$ is analytic on \mathcal{D} and \mathcal{D} is simply connected, by Cauchy's theorem

$$\oint \operatorname{Log}(z) dz = 0 \quad \text{for any}$$

closed contour γ .

[0.5]

\therefore By Morera's theorem, $\operatorname{Log}(z)$ has an antiderivative on \mathcal{D} . [0.5]

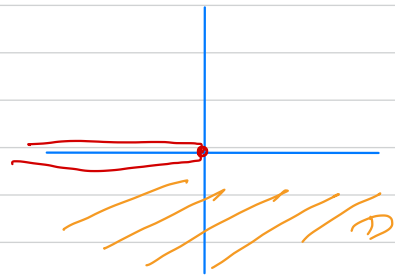
- For saying that $\operatorname{Log}(z)$ has an antiderivative [0.5]
- For justification of antiderivative [1.5]
(they may use existence theorem stated in the class)

Q.3] (b) Let $\mathcal{D} = \{z \in \mathbb{C} : \operatorname{Im}(z) < 0\}$

Given $\operatorname{Log}(z) = \ln |z| + i \theta(z)$ where $\theta(z) \in \arg(z) \cap (-\pi, \pi)$

$\therefore \operatorname{Log}(z)$ is analytic on $\mathbb{C} \setminus (-\infty, 0]$ [0.5]

We show that $\operatorname{Log}(z)$ has antiderivative on \mathcal{D} . [0.5]



Since $\operatorname{Log}(z)$ is analytic on \mathcal{D} and \mathcal{D} is simply connected, by Cauchy's theorem

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closed contour γ .

[0.5]

\therefore By Morera's theorem, $\operatorname{Log}(z)$ has an antiderivative on \mathcal{D} . [0.5]

- For saying that $\operatorname{Log}(z)$ has an antiderivative [0.5]
- For justification of antiderivative [1.5]
(they may use existence theorem stated in the class)

Q. 4] No.

Suppose there exists a polynomial $p(z)$ s.t. [0.5]

$$\frac{(1 - \cos(z)) e^{z^2}}{(e^{z^2} + 1)} = p(z).$$

We know

$$\cos(z) = 1 \text{ for } z = n\pi \text{ for } n \in \mathbb{Z} \text{ for } n \text{ even} \quad [0.5]$$

$$\Rightarrow p(n\pi) = 0 \text{ for } n \in \mathbb{Z} \text{ and } n \text{ even}$$

$\Rightarrow p(z)$ has infinitely many zeroes. This contradicts the fundamental theorem of algebra unless $p(z)$ is a constant [0.5]

But $p(z)$ is constant & $p(n\pi) = 0$
 $p(z) \equiv 0 \quad \forall z \in \mathbb{C}.$

$$\Rightarrow 1 - \cos(z) = 0 \quad (\because e^{z^2} \neq 0)$$

$$\Rightarrow \cos(z) = 1 \quad \forall z \in \mathbb{C}, \quad [0.5]$$

a contradiction. (as $\cos(\pi) = -1$ for example)

- For saying $\nexists p(z)$ [0.5]
- Justification [1.5]

Q.5] (a) Given $f(z) = x^2 + iy^2$

$$\Rightarrow u = x^2$$

$$v = y^2$$

$$\therefore u_x = 2x$$

$$v_x = 0$$

$$u_y = 0$$

$$v_y = 2y$$

f is differentiable at $z = x + iy$

$\Rightarrow f$ satisfies the CR equations at $z = x + iy$

$$\Rightarrow u_x = v_y \quad \text{and} \quad u_y = -v_x \quad \text{at} \quad z = x + iy$$

Here $u_x = v_y \Leftrightarrow 2x = 2y$ &

$$\Leftrightarrow x = y.$$

$$v_x = -u_y \text{ always}$$

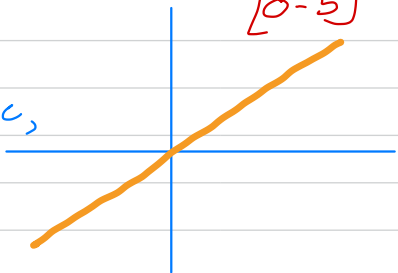
[1]

$\therefore f$ is not differentiable at $z = x + iy$ if $y \neq x$. [0.5]

If $z = x + ix$, then f satisfies the CR equations at z . Since u & v are polynomials, their partials are continuous on \mathbb{C} . $\therefore f$ is differentiable at $x + ix$. [0.5]

Since for any $z = x + ix$,
 \nexists any ball around z
 $B(z; r)$ s.t f is
differentiable on $B(z; r)$,

f is **not** analytic at any $z \in \mathbb{C}$. [1]



Q.5] (b) Given $f(x+iy) = y^2 + ix^2$

$$\Rightarrow u = y^2$$

$$\therefore u_x = 0$$

$$u_y = 2y$$

$$v = x^2$$

$$v_x = 2x$$

$$v_y = 0$$

f is differentiable at $z = x+iy$
 $\Rightarrow f$ satisfies the CR equations
at $z = x+iy$

$$\Rightarrow u_x = v_y \quad \text{and} \quad u_y = -v_x \quad \text{at} \quad z = x+iy$$

Here $u_x = v_y$ always
 $v_x = -u_y \Leftrightarrow 2x = -2y$
 $\Leftrightarrow x = -y$

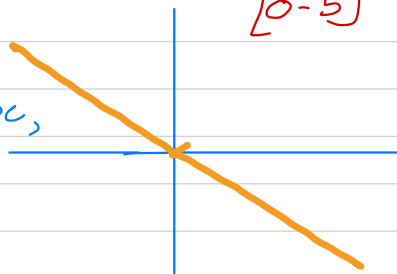
[1]

$\therefore f$ is not differentiable at $z = x+iy$
if $y \neq -x$. [0.5]

If $z = x - ix$, then f satisfies the CR equations at z . Since u & v are polynomials, their partials are continuous on \mathbb{C} . $\therefore f$ is differentiable at $x - ix$. [0.5]

Since for any $z = x - ix$,
 \nexists any ball around z
 $B(z; r)$ s.t f is
differentiable on $B(z; r)$,

f is **not** analytic at any $z \in \mathbb{C}$. [1]



Q.6] Given $a > 1$
we've

$$z^2 + 2az + 1 = 0$$

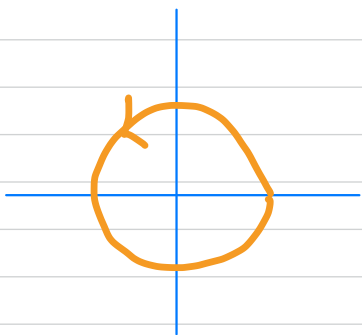
$$\Leftrightarrow z = \frac{-2a \pm \sqrt{4a^2 - 4}}{2} \\ = -a \pm \sqrt{a^2 - 1}$$

$$\text{Let } \alpha = -a + \sqrt{a^2 - 1} \quad \& \quad \beta = -a - \sqrt{a^2 - 1}$$

$$\therefore \frac{1}{z^2 + 2az + 1} = \frac{1}{\alpha - \beta} \left[\frac{1}{z - \alpha} - \frac{1}{z - \beta} \right]$$

$$\therefore \int_{|z|=1} \frac{1}{z^2 + 2az + 1} dz = \frac{1}{\alpha - \beta} \left[\int_{|z|=1} \frac{1}{z - \alpha} dz - \int_{|z|=1} \frac{1}{z - \beta} dz \right]$$

[0.5] \rightarrow (1)



$$\text{Since } a > 1, \\ \Rightarrow 2a > 2$$

$$\Rightarrow (a-1)^2 = a^2 - 2a + 1 \\ < a^2 - 2 + 1 \\ = a^2 - 1$$

$$\Rightarrow a-1 < \sqrt{a^2 - 1}$$

$$\Rightarrow \alpha = -a + \sqrt{a^2 - 1} > -1$$

Clearly $\alpha \leq 0$

$$\therefore \alpha \in B(0; 1)$$

But $\beta = -a - \sqrt{a^2 - 1}$ is clearly
doesn't lie in $B(0; 1)$.

∴ By Cauchy's theorem,

$$\int_{|z|=1} \frac{1}{z-\beta} dz = 0.$$

[0.5]

& by Cauchy's integral formula

$$\int_{|z|=1} \frac{1}{z-\alpha} dz = 2\pi i (1) = 2\pi i$$

[0.5]

∴ By (1)

$$\int_{|z|=1} \frac{1}{z^2+2az+1} dz = \frac{1}{\alpha-\beta} (2\pi i)$$

$$= \frac{1}{-2\sqrt{a^2-1}} \times 2\pi i$$

$$\boxed{\int_{|z|=1} \frac{1}{z^2+2az+1} dz = \frac{-\pi i}{\sqrt{a^2-1}}}$$

[0.5]