Mid Semester Examination

Marking Scheme

Part 1

Q.1 (a) Given $u(x,y) = \sinh x \sin y$.

$\Rightarrow u_x = \cosh x \sin y, \quad u_y = \sinh x \cos y$

$\Rightarrow u_{xx} = \sinh x \sin y, \quad u_{yy} = -\sinh x \sin y$

$\therefore \quad u_{xx} + u_{yy} = 0$

$\Rightarrow u$ is harmonic [1m]

Let v be a harmonic conjugate of u.

$\Rightarrow v_y = u_x$ and $v_x = -u_y$

$\Rightarrow v_y = \cosh x \sin y$

$\Rightarrow v = -\cosh x \cos y + \psi(x)$ [1m]

$\Rightarrow v_x = -\sinh x \cos y + \psi'(x)$

$= -u_y = -\sinh x \cos y$

$\Rightarrow \psi'(x) = 0$

$\Rightarrow \psi(x) = c, \quad c \in \mathbb{R}.$

$$\boxed{\therefore \quad v = -\cosh x \cos y + c}$$ [1m]

Q.1 (b) Given $u(x,y) = \cosh x \cos y$.

$\Rightarrow u_x = \sinh x \cos y, \quad u_y = -\cosh x \sin y$

$\Rightarrow u_{xx} = \cosh x \cos y, \quad u_{yy} = -\cosh x \cos y$

$\therefore u_{xx} + u_{yy} = 0$

$\Rightarrow u$ is harmonic [1m]

Let v be a harmonic conjugate of u.

$\Rightarrow v_y = u_x$ and $v_x = -u_y$

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$\Rightarrow v_x = \cosh x \sin y + \psi'(x)$

$= -u_y = \cosh x \sin y$

$\Rightarrow \psi'(x) = 0$

$\Rightarrow \psi(x) = c, \quad c \in \mathbb{R}.$

$$\boxed{\therefore v = \sinh x \sin y + c}$$ [1m]

Q.2 Given

$$f(z) = z^2 + z\bar{z} + c(\bar{z})^2$$

Suppose $f(z)$ is differentiable at $z = x+iy$. Here

$$f(x+iy) = (x+iy)^2 + (x+iy)(x-iy) + c(x-iy)^2$$

$$= [(x^2-y^2) + (x^2+y^2) + c(x^2-y^2)]$$

$$+ i(2xy - 2cxy)$$

$$= [(c+1)x^2 - cy^2] + i(1-c)2xy$$

$\therefore \quad u = (c+2)x^2 - cy^2 \qquad v = (1-c)2xy$

$\Rightarrow \quad u_x = 2(c+2)x \qquad v_x = 2(1-c)y$

$\quad u_y = -2cy \qquad v_y = 2(1-c)x$

f is differentiable at $z = x+iy$

$\Rightarrow f$ satisfies CR equations at z

$\therefore \quad u_x = v_y$ and $v_x = -u_y$ at (x,y) [0.5]

$\Leftrightarrow 2(c+2)x = 2(1-c)x \;\;\&\;\; 2(1-c)y = 2cy$

$\Leftrightarrow \quad (2c+1)x = 0 \;\;\&\;\; (2c-1)y = 0$

As $c \in \mathbb{N}$, $\Rightarrow x = y = 0.$ [0.5]

$\therefore$ f is not differentiable at $z \neq 0$. [0.5]

At $z=0$, f is differentiable because f satisfies CR equations at 0 and here u_x, u_y, v_x, v_y are continuous on $\mathbb{C}$. [0.5]

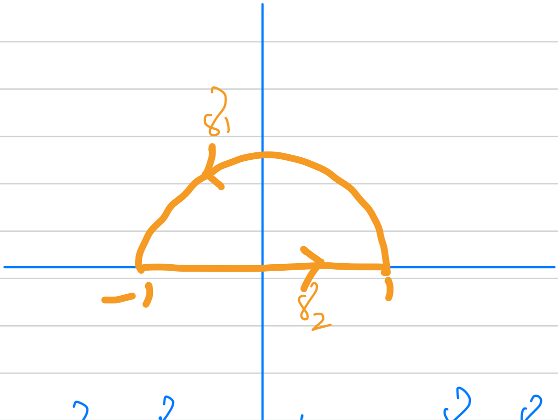
But f is <u>not</u> analytic at 0 [0.5]

since any open ball around 0 contains non zero points where f is not differentiable.

$\therefore$ f is nowhere analytic. [0.5]

- For not mentioning f is not analytic [-1]

Q.3 (a)



Let $C = \gamma_1 + \gamma_2$ where γ_1, γ_2 are given by

$$\gamma_1 : [0, \pi] \to \mathbb{C}, \quad \gamma_1(t) = e^{it}$$

$$\gamma_2 : [0, 1] \to \mathbb{C}, \quad \gamma_2(t) = (1-t)(-1) + t(1) = 2t - 1$$

[0.5]

$$\therefore \int_C \bar{z}\, dz = \int_{\gamma_1} \bar{z}\, dz + \int_{\gamma_2} \bar{z}\, dz$$

Let $f(z) = \bar{z}$. Then

$$\int_{\gamma_1} f(z)\, dz := \int_0^\pi f(\gamma_1(t))\, \gamma_1'(t)\, dt$$

$$= \int_0^\pi e^{-it} (i e^{it})\, dt$$

$$= i \int_0^\pi dt = i(\pi - 0) = \pi i$$

[1]

Now

$$\int_{\gamma_2} f(z)\,dz := \int_0^1 f(\gamma_2(t))\,\gamma_2'(t)\,dt$$

$$= \int_0^1 (2t-1)(2)\,dt$$

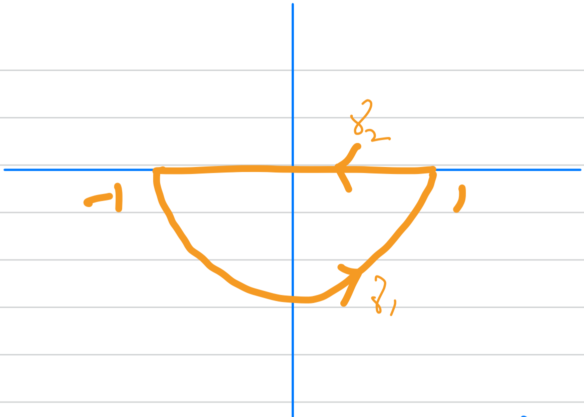
$$= 2\left[t^2\Big|_0^1 - t\Big|_0^1 \right]$$

$$= 2[1-0-(1-0)]$$

$$= 0 \qquad [1]$$

$$\therefore \int_C f(z)\,dz = \pi i + 0 = \pi i \qquad [0.5]$$

Q.3 (b)



Let $C = \gamma_1 + \gamma_2$ where γ_1, γ_2 are given by

$\gamma_1 : [\pi, 2\pi] \to \mathbb{C}, \quad \gamma_1(t) = e^{it}$,

$\gamma_2 : [0, 1] \to \mathbb{C}, \quad \gamma_2(t) = (1-t)(1) + t(-1)$
$= -2t + 1$ [0.5]

$$\therefore \int_C \bar{z}\, dz = \int_{\gamma_1} \bar{z}\, dz + \int_{\gamma_2} \bar{z}\, dz$$

Let $f(z) = \bar{z}$. Then

$$\int_{\gamma_1} f(z)\, dz := \int_{\pi}^{2\pi} f(\gamma_1(t))\, \gamma_1'(t)\, dt$$

$$= \int_{\pi}^{2\pi} e^{-it} \left(i e^{it}\right) dt$$

$$= i \int_{\pi}^{2\pi} dt = i(2\pi - \pi) = \pi i$$

[1]

Now

$$\int_{\gamma_2} f(z)\,dz := \int_0^1 f(\gamma_2(t))\,\gamma_2'(t)\,dt$$

$$= \int_0^1 (-2t+1)(2)\,dt$$

$$= 2\left[-t^2\Big|_0^1 + t\Big|_0^1\right]$$

$$= 2\left[-(1-0) + (1-0)\right]$$

$$= 0 \qquad [1]$$

$$\therefore \int_C f(z)\,dz = \pi i + 0 = \pi i \qquad [0.5]$$

Q. 4 (a) Let $z = x+iy$ be such that

$$\overline{e^{iz}} = -e^{i\bar{z}}$$

$$\Rightarrow \overline{e^{ix-y}} = -e^{i(x-iy)} = -e^{ix+y}$$

$$\Rightarrow e^{-y}(\cos x - i\sin x) = -\left[e^{y}(\cos x + i\sin x)\right]$$ [0.5]

$$\Rightarrow e^{-y}\cos x = -e^{y}\cos x \;\&\; -e^{-y}\sin x = -e^{y}\sin x$$

$$\Rightarrow (e^{2y}+1)\cos x = 0 \quad \& \quad (e^{2y}-1)\sin x = 0$$ — (1)

[0.5]

$\Rightarrow$ As $e^{2y}+1 \neq 0$

$\Rightarrow \cos x = 0.$

$\Rightarrow x = \frac{\pi}{2} + n\pi, \; n \in \mathbb{Z}$ [0.5]

$\therefore \sin x \neq 0 \overset{(1)}{\Rightarrow} e^{2y} = 1 \Rightarrow y = 0.$

[0.5]

$\therefore \overline{e^{iz}} = -e^{i\bar{z}} \Rightarrow z = \frac{\pi}{2} + n\pi, \; n \in \mathbb{Z}.$

- For writing definition of e^{z} [0.5]

Q. 4 (b) Let $z = x + iy$ be such that

$$\overline{e^{iz}} = e^{i\bar{z}}$$

$$\Rightarrow \overline{e^{ix-y}} = e^{i(x-iy)} = e^{ix+y}$$

$$\Rightarrow e^{-y}(\cos x - i\sin x) = \left[e^{y}(\cos x + i\sin x)\right]$$ [0.5]

$$\Rightarrow e^{-y}\cos x = e^{y}\cos x \ \& \ -e^{-y}\sin x = e^{y}\sin x$$

$$\Rightarrow (e^{2y}-1)\cos x = 0 \quad (1) \qquad \& \qquad (e^{2y}+1)\sin x = 0$$

$\Rightarrow$ As $e^{2y}+1 \neq 0$ [0.5]

$\Rightarrow \sin x = 0.$

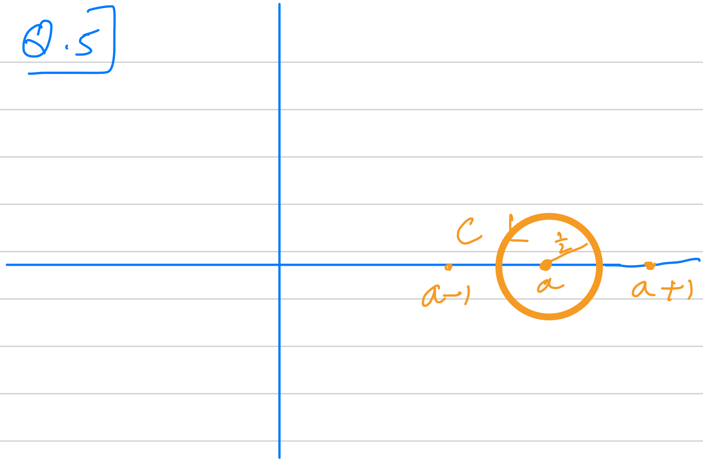
$\Rightarrow x = n\pi, \quad n \in \mathbb{Z}$ [0.5]

$\therefore \cos x \neq 0 \overset{(1)}{\Rightarrow} e^{2y} = 1 \Rightarrow y = 0.$ [0.5]

$\therefore \overline{e^{iz}} = e^{i\bar{z}} \Rightarrow z = n\pi, \quad n \in \mathbb{Z}.$

- For writing definition of e^{z} [0.5]

Q.5



Since $\frac{1}{z}$ is analytic on and inside of C $(\because 0 \notin C \cup I(C))$ [0.5]

$\therefore$ By Cauchy's theorem $\int_C \frac{1}{z}\, dz = 0$. [0.5]

Here $c(t) = a + \frac{1}{2} e^{it}$, $0 \le t \le 2\pi$ parametrizes C.

$\therefore$ By defn,

$$\int_C \frac{1}{z-a}\, dz = \int_0^{2\pi} \frac{1}{\frac{1}{2} e^{it}} \cdot \left(\frac{i}{2} e^{it}\right) dt$$

$$= i \int_0^{2\pi} dt = 2\pi i$$

[1]

Q. 6] (a) We prove that f is continuous at 0. [0.5]

Given $\varepsilon > 0$, choose $\delta = \varepsilon$.

Then $|z| < \delta$

$$\Rightarrow |f(z) - f(0)|$$

$$= \left| \frac{Re(z)}{1+|z|} \right| \leq |Re(z)| \qquad \because |z| \geq 0, \quad 1+|z| \geq 1$$

$$\leq |z|$$

$$< \delta = \varepsilon$$

[1.5]

(b) We prove that f is continuous at 0. [0.5]

Given $\varepsilon > 0$, choose $\delta = \varepsilon$.

Then $|z| < \delta$

$$\Rightarrow |f(z) - f(0)|$$

$$= \left| \frac{Im(z)}{1+|z|} \right| \leq |Im(z)| \qquad \because |z| \geq 0, \quad 1+|z| \geq 1$$

$$\leq |z|$$

$$< \delta = \varepsilon$$

[1.5]

Q. 7] By def^n,

$$\log((-1)^2) = \log(1) = \log 1 + 0 = 0$$

[0.5]

Whereas

$$2\log(-1) = 2(\log 1 + i\pi)$$

$$= 2(i\pi) = 2\pi i \neq 0.$$

$$\therefore \log((-1)^2) \neq 2\log(-1)$$

[0.5]

Part 2

Q.1] Given $f(z) = \sqrt{|x||y|}$.

Here $u = \sqrt{|x||y|}$ $\quad v = 0$.

$$\therefore u_x(0,0) = \lim_{h \to 0} \frac{u(h,0) - u(0,0)}{h}$$

$$= \lim_{h \to 0} \frac{0}{h} = 0$$

[0.5]

Also,

$$u_y(0,0) = \lim_{k \to 0} \frac{u(0,k) - u(0,0)}{k}$$

$$= \lim_{k \to 0} \frac{0}{k} = 0$$

[0.5]

$$v_x(0,0) = \lim_{h \to 0} \frac{v(h,0) - v(0,0)}{h}$$

$$= \lim_{h \to 0} \frac{0}{h} = 0$$

$$v_y(0,0) = \lim_{k \to 0} \frac{v(0,k) - v(0,0)}{k}$$

$$= \lim_{k \to 0} \frac{0-0}{k} = 0$$

[0.5]

$$\therefore u_x(0,0) = v_y(0,0) = 0 \quad \&$$

$$u_y(0,0) = -v_x(0,0) = 0.$$

$\therefore f$ satisfies CR equations at $(0,0)$.

We show f is not differentiable at $(0,0)$.

Here

$$\lim_{h\to 0}\frac{f(h)-f(0)}{h}$$

$$= \lim_{\substack{h\to 0\\ h=h_1+ih_2}} \frac{\sqrt{|h_1||h_2|}-0}{h}$$

$$= \begin{cases} \displaystyle\lim_{\substack{h\to 0\\ h_1=h_2\\ h_1>0}} \frac{\sqrt{(h_1)^2}}{(1+i)h_1} = \lim_{\substack{h_1\to 0\\ h_1>0}} \frac{|h_1|}{(1+i)h_1} = \frac{1}{1+i} \\[2ex] \displaystyle\lim_{\substack{h_1\to 0\\ h_2=0}} \frac{0}{h} = 0 \end{cases}$$

[1]

$\therefore$ limit along $y=0$ & $x=y$ are different. $\therefore$ f is not differentiable at 0.

[0.5]

Q.2] By defn, for $z \neq 0$

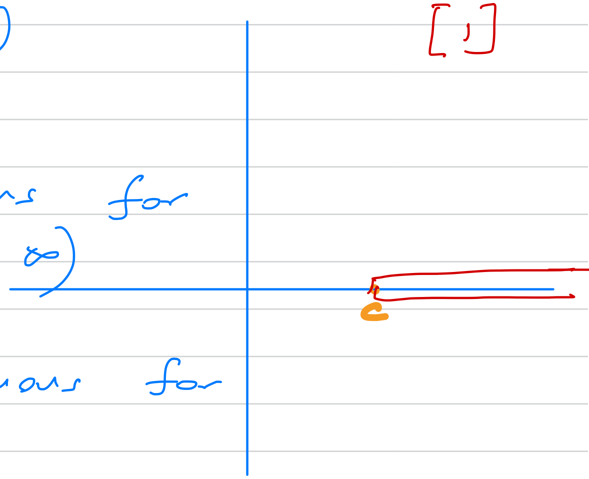
$\text{Log}(z) := \log|z| + i\theta$ where $\theta \in \arg(z)$ & $\theta \in [0, 2\pi)$.

Given $f(z) = \text{Log}(z-c)$

(1) We know $\text{Log}(z)$ is not continuous for $z \in [0, \infty)$

$\therefore$ $\text{Log}(z-c)$ is not continuous for $z \in [c, \infty)$ [1]

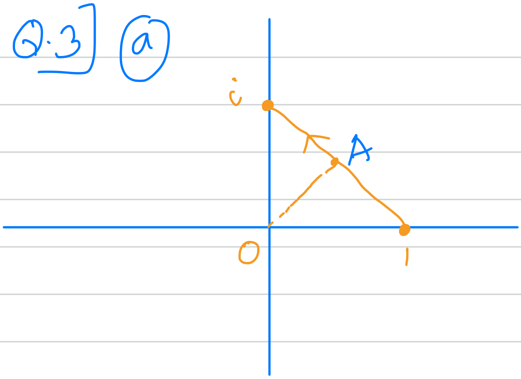
(2) Also, We know $\text{Log}(z)$ is continuous for $z \in \mathbb{C} \setminus [0, \infty)$



$\Rightarrow$ $\text{Log}(z-c)$ is continuous for $z \in \mathbb{C} \setminus [c, \infty)$

As $\text{Log}(z-c)$ is not continuous on $[c, \infty)$, the largest open subset of $\mathbb{C}$ on which f is continuous is $\mathbb{C} \setminus [c, \infty)$ [1]

Q.3] (a)



Let $f(z) = \frac{1}{z^4}$

Here the length of the curve C is $L(C) = \sqrt{2}$ [0.5]

On C, $|z| \geq l(OA)$ where A is the midpoint of C & O is the origin.

$$\Rightarrow |z| \geq \frac{\sqrt{2}}{2} = \frac{1}{\sqrt{2}}$$

$$\Rightarrow |z|^4 \geq \frac{1}{(\sqrt{2})^4} = \frac{1}{4}$$

$$\Rightarrow \frac{1}{|z|^4} \leq 4 \quad \text{on } C.$$ [1]

$\therefore$ By ML-inequality

$$\left| \int_C \frac{1}{z^4} dz \right| \leq 4\sqrt{2}.$$ [0.5]

Let $F(z) = -\frac{1}{3z^3}$

Then $F'(z) = -\frac{1}{3}(-3)z^{-4} = \frac{1}{z^4}$.

$\therefore$ By fundamental theorem of calculus,

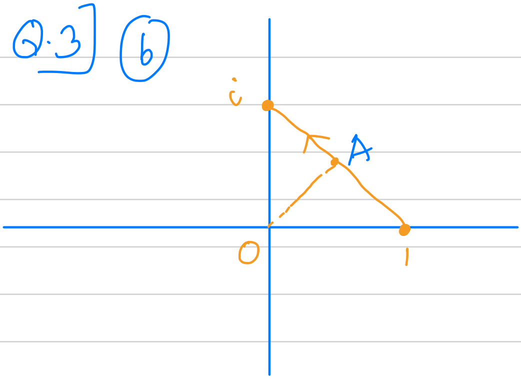
$$\int_C \frac{1}{z^4}\,dz = F(1) - F(i)$$

$$= -\frac{1}{3} + \frac{1}{3(-i)}$$

$$= -\frac{1}{3} + \frac{1}{3}i$$

[1]

Q.3] (b)



Let $f(z) = \frac{1}{z^2}$

Here the length of the curve C

$= L(C) = \sqrt{2}$ [0.5]

On C, $|z| \geq l(OA)$ where A is the midpoint of C & O is the origin.

$\implies |z| \geq \frac{\sqrt{2}}{2} = \frac{1}{\sqrt{2}}$

$\implies |z|^2 \geq \frac{1}{(\sqrt{2})^2} = \frac{1}{2}$

$\implies \frac{1}{|z|^2} \leq 2$ on C. [1]

$\therefore$ By ML-inequality

$$\left| \int_C \frac{1}{z^2}\, dz \right| \leq 2\sqrt{2}.$$

[0.5]

Let $F(z) = -\frac{1}{z}$

Then $F'(z) = (-1)(-z^{-2}) = \frac{1}{z^2}$

$\therefore$ By fundamental theorem of calculus,

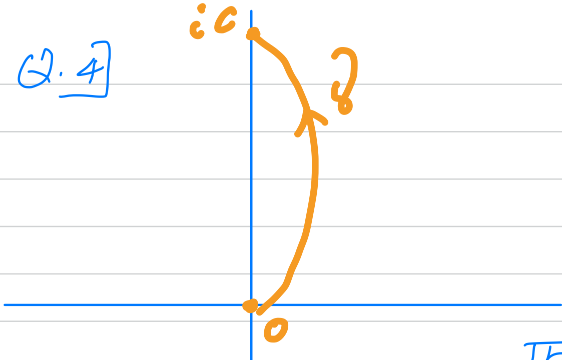
$$\int_C \frac{1}{z^2}\,dz = F(1) - F(i)$$

$$= -1 + \frac{1}{i}$$

$$= -1 - i$$

[1]

Q.4



Let $f(z) = e^{\pi z}$

Consider
$$F(z) = \frac{1}{\pi} e^{\pi z}$$

Then F is analytic and
$$F'(z) = f(z)$$

$\therefore$ By fundamental theorem of calculus,

$$\int_\gamma f(z)\,dz = \int_\gamma F'(z)\,dz$$

$$= F(ic) - F(0)$$

$$= \frac{1}{\pi}\left[e^{i\pi c} - 1\right]$$

$$= \frac{1}{\pi}\left[(-1)^c - 1\right]$$

$$= \begin{cases} -\frac{2}{\pi} & \text{if } c \text{ is odd} \\ 0 & \text{if } c \text{ is even} \end{cases}$$

Q. 5] Let $u, v : D \to \mathbb{R}$ s.t

u is a harmonic conjugate of v

& v is a " " " u

WTS: u & v are constants.

Since v is a harmonic conjugate of u,

$$u_x = v_y \quad \& \quad u_y = -v_x. \quad -① \quad [1]$$

Also, u is a harmonic conjugate of v

$$\Rightarrow v_x = u_y \quad \& \quad v_y = -u_x \quad -② \quad [1]$$

① & ② $\Rightarrow u_x = v_y = -u_x$ &

$$u_y = -v_x = v_x$$

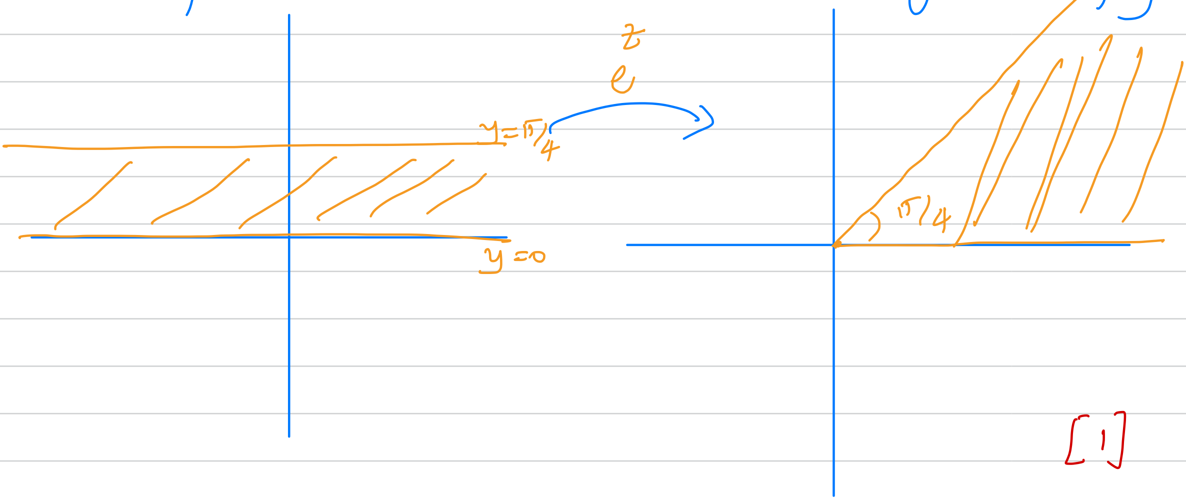
$$\Rightarrow u_x \equiv 0 \equiv v_x \equiv u_y \equiv v_y$$

D is connected $\Rightarrow$ u and v are constants. [1]

Q.6 (a) (1) Let

$$S = \{ z = x + iy : 0 \le y \le \pi/4 \}.$$

$$\exp(S) = \{ z \in \mathbb{C} : 0 \le \operatorname{Arg} z \le \pi/4 \}$$



[1]

(2) By def;

$$(-1)^{i} = e^{i \log(-1)}$$

$$= e^{i \left(i(2n+1)\pi \right)}$$

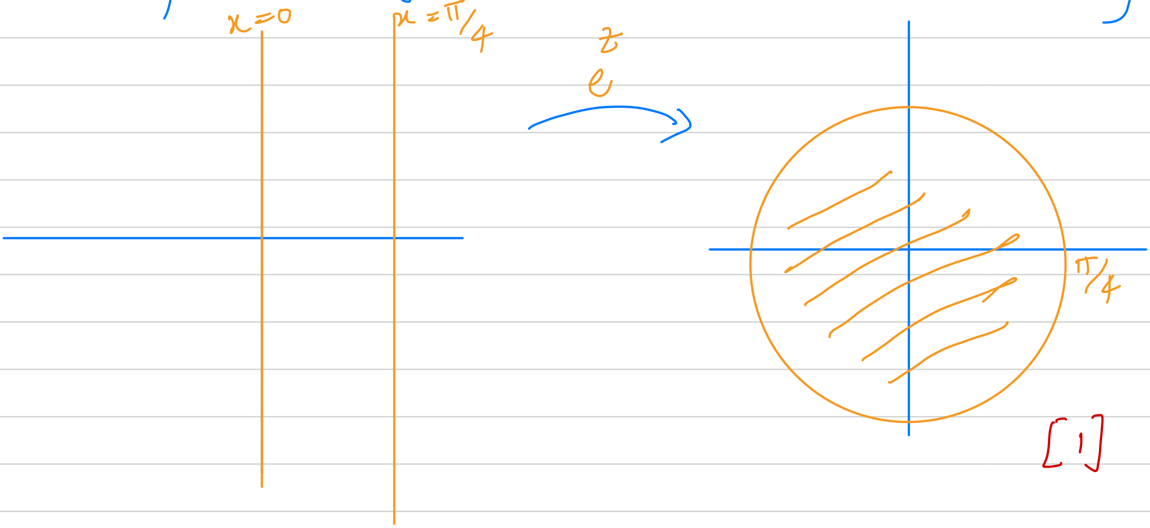
$$= e^{-(2n+1)\pi} \qquad n \in \mathbb{Z}.$$

[1]

Q. 6] (b) Let

(1) $S = \{ z = x + iy : 0 \leq x \leq \pi/4 \}$.

$\exp(S) = \{ z \in \mathbb{C} : 0 \leq |z| \leq \pi/4 \}$



[1]

(2) By def;

$$(-1)^{-i} = e^{-i \log(-1)}$$

$$= e^{-i \left(i(2n+1)\pi \right)}$$

$$= e^{+(2n+1)\pi} \qquad n \in \mathbb{Z}.$$

[1]