9. Separation of variables method

1. Solve the Heat equation by separable of variables method

$$u_t = u_{xx},\ u := u(x,t),\quad 0 < x < \ell,\quad t > 0$$

$$u(0,t) = 0,\ u(\ell,t) = 0,\quad u(x,0) = f(x).$$

When (i) $f(x) = 6\sin\frac{\pi x}{\ell}$, (ii) $f(x) = 12\sin\frac{9\pi x}{\ell} - 7\sin\frac{4\pi x}{\ell}$.

2. Using the method of separation of variables, solve:

$$u_t = c^2 u_{xx},\ u := u(x,t),\quad x \in (0,1),\ t > 0,\ c \in \mathbb{R},$$

with data
(i) $u(0,t) = u(1,t) = 0$ and $u(x,0) = x(1-x)$.

(ii) $u_x(0,t) = 0 = u_x(1,t)$ and $u(x,0) = x(1-x)$.

3. Using separation of variables, compute the solution of :

$$u_t - c^2 u_{xx} + a^2 u = 0,\ 0 < x < \ell,\ t > 0$$

with initial condition $u(x,0) = f(x),\ 0 < x < \ell$ and Dirichlet boundary conditions $u(0,t) = u(l,t) = 0$. Find $\lim\limits_{t\to\infty} u(x,t)$.

4. Use separation of variables to obtain the solution of the following wave equation: $u_{tt} - c^2 u_{xx} = 0,\ 0 < x < l,\ t > 0,\ u := u(x,t)$, with initial and boundary conditions:

(A) Boundary condition (Dirichlet): $u(0,t) = 0,\ u(\ell,t) = 0,\ t > 0$
Initial condition: $u(x,0) = x(x-\ell),\ u_t(x,0) = 0,\ x \in (0,\ell)$.

(B) Boundary condition (Dirichlet): $u(0,t) = 0,\ u(\ell,t) = 0,\ t > 0$
Initial condition: $u(x,0) = 0,\ u_t(x,0) = \sin\left(\frac{7\pi x}{\ell}\right)$.

(C) Boundary condition (Neumann): $u_x(0,t) = 0,\ u_x(\ell,t) = 0,\ t > 0$.
Initial condition: $u(x,0) = x,\ u_t(x,0) = 0\ x \in (0,\ell)$.

(D) Boundary condition (Neumann): $u_x(0,t) = 0,\ u_x(\ell,t) = 0,\ t > 0$.
Initial condition: $u(x,0) = 0,\ u_t(x,0) = 1,\ x \in (0,\ell)$.

5. Using separation of variables, solve the Laplace equation:

$$\Delta u = \frac{d^2u}{dx^2} + \frac{d^2u}{dy^2} = 0 \ on \ [0,a] \times [0,b], \ u := u(x,y).$$

with the boundary conditions:

(A) $u(0,y) = y(y-b), \ u(a,y) = 0 \ \& \ u(x,0) = 0, \ u(x,b) = 0.$

(B) $u(x,0) = x, \ u(x,b) = 0 \ \& \ u(0,y) = 0, \ u(a,y) = 0.$

(C) $u(0,y) = u(a,y) = 0, \ u_y(x,0) = x(x-a), u_y(x,b) = g(x).$

6. Consider the Neumann problem

$$\Delta u = u_{xx} + u_{yy} = 0 \ in \ \mathbb{D}, \ \ and \ \ \frac{\partial u}{\partial \overrightarrow{n}} = g \ on \ \partial\mathbb{D}.$$

Here $u := u(x,y), \ \mathbb{D} = \{(x,y) \in R^2 : x^2 + y^2 < 1\}, \ \overrightarrow{n}$ is the outward normal unit vector to $\partial\mathbb{D}$, the boundary of $\mathbb{D}$.

Solve the above problem after reducing it to Polar co-ordinate, when (A) $g(x,y) = \sin^3\left(\tan^{-1}\frac{y}{x}\right)$, (B) $g(x,y) = (x^2+y^2)\left(\tan^{-1}\frac{y}{x}\right)$.