Modern Algebra (Yiu) November 6, 2015

Irreducible polynomials of small degrees in $\mathbb{F}_3[x]$

We enumerate irreducible polynomials of degrees ≤ 4 in $\mathbb{F}_3[x]$.

Note: $\mathbb{F}_3 = \{0, 1, -1\}$.

We need only consider **monic** polynomials, those with leading coefficients equal to $1 \in \mathbb{F}_3$.

(1) Linear: all 3 linear (monic) polynomials are irreducible.

(2) Quadratic: There are altogether $3^2 = 9$ monic quadratic polynomials. 6 of these are product of linear polynomials. Therefore, there are 3 irreducible quadratics: $x^2 + 1$, $x^2 + x - 1$, $x^2 - x - 1$.

(3) Cubic: There are $3^3 = 27$ monic cubic polynomials.

$3 + 3 \cdot 2 + 1 = 10$ are products of three polynomials.

$3 \times 3 = 9$ are product of a linear and a quadratic polynomials.

There are $27 - 10 - 9 = 8$ irreducible cubic polynomials:

$$\begin{array}{llll} x^3+x^2-x+1 & x^3-x^2+x+1 & x^3-x^2+1 & x^3-x+1 \\ x^3+x^2+x-1 & x^3-x^2-x-1 & x^3+x^2-1 & x^3-x-1 \end{array}$$

(4) Quartic: There are $3^4 = 64$ monic polynomials of degree 4.

Cubic $\times$ linear: $8 \times 3 = 24$,

Quadratic $\times$ quadratic: $3 + 3 = 6$,

Quadratic $\times$ linear $\times$ linear: $3 \times (3 + 3) = 18$,

Four linear: $3 + 3 \times 2 + 3 + 3 = 15$.

These account for 63 monic quartic polynomials.

Therefore, there are $81 - 63 = 18$ irreducible monic quartic polynomials in $\mathbb{F}_3[x]$.

$$\begin{array}{lll} x^4+x^3+x^2+x+1 & x^4+x^3+x^2+1 & x^4+x^3-x+1 \\ x^4-x^3+x^2-x+1 & x^4-x^3+x^2+1 & x^4-x^3+x+1 \\ x^4+x^2+x+1 & x^4+x^2-x+1 & x^4+x^3+x^2-x-1 \\ x^4+x^3-x^2-x-1 & x^4+x^3-1 & x^4-x^3+x^2+x-1 \\ x^4-x^3-x^2+x-1 & x^4-x^3-1 & x^4+x^2-1 \\ x^4-x^2-1 & x^4+x-1 & x^4-x-1 \end{array}$$

Irreducible cubic polynomials in $\mathbb{F}_5[x]$

An irreducible monic cubic polynomial is of the form

$$f(x) = x^3 + ax^2 + bx + c,$$

for a, b, $c \in \mathbb{F}_5$ and $c \neq 0$. None of the following values should be 0:

$$\begin{aligned} f(1) =& 1 + a + b + c, \\ f(2) =& 3 - a + 2b + c, \\ f(3) =& 2 - a - 2b + c, \\ f(4) =& -1 + a - b + c. \end{aligned}$$

In each of the following four tables, corresponding to $c = 1, 2, 3, 4$, an entry under in the spot (a, b) means $f(k) = 0$ for $f(x) = x^3 + ax^2 + bx + c$ (so that $f(x)$ has $x - c$ as a linear factor). The blank entries therefore correspond to irreducible polynomials which we list at the end of each row. There are altogether 40 irreducible monic cubic polynomials.

$c = 1$:

$a \setminus b$	0	1	2	3	4			
0	4			$1, 2$	3	$x^3 + x + 1$	$x^3 + 2x + 1$	
1		$2, 3, 4$	1			$x^3 + x^2 + 1$	$x^3 + x^2 + 3x + 1$	$x^3 + x^2 + 4x + 1$
2		1	4	3	2	$x^3 + 2x^2 + 1$		
3	$1, 3$		2	4		$x^3 + 3x^2 + x + 1$	$x^3 + 3x^2 + 4x + 1$	
4	2		3		$1, 4$	$x^3 + 4x^2 + x + 1$	$x^3 + 4x^2 + 3x + 1$	

$c = 2$:

$a \setminus b$	0	1	2	3	4			
0	2	4	$1, 3$			$x^3 + 2x + 2$	$x^3 + 3x + 2$	
1		1	4	2	3	$x^3 + x^2 + 2$		
2	1	$2, 3$		4		$x^3 + 2x^2 + 2x + 2$	$x^3 + 2x^2 + 4x + 2$	
3				3	$1, 2, 4$	$x^3 + 3x^2 + 2$	$x^3 + 3x^2 + x + 2$	$x^3 + 3x^2 + 2x + 2$
4	$3, 4$		2	1		$x^3 + 4x^2 + x + 2$	$x^3 + 4x^2 + 4x + 2$	

$c = 3$:

$a \setminus b$	0	1	2	3	4			
0	3	1	$2, 4$			$x^3 + 3x + 3$	$x^3 + 4x + 3$	
1	2		3	4		$x^3 + x^2 + x + 3$	$x^3 + x^2 + 2x + 3$	
2				2	$1, 3, 4$	$x^3 + 2x^2 + 3$	$x^3 + 2x^2 + x + 3$	$x^3 + 2x^2 + 2x + 3$
3	4	$2, 3$		1		$x^3 + 3x^2 + 2x + 3$	$x^3 + 3x^2 + 4x + 3$	
4		4	1	3	2	$x^3 + 4x^2 + 3$		

$c = 4$:

$a \setminus b$	0	1	2	3	4			
0	1			$3, 4$	2	$x^3 + x + 4$	$x^3 + 2x + 4$	
1	3		2		$1, 4$	$x^3 + x^2 + x + 4$	$x^3 + x^2 + 3x + 4$	
2	$2, 4$		3	1		$x^3 + 2x^2 + x + 4$	$x^3 + 2x^2 + 2x + 4$	
3		4	1	2	3	$x^3 + 3x^2 + 4$		
4		$1, 2, 3$	4			$x^3 + 4x^2 + 4$	$x^3 + 4x^2 + 3x + 4$	$x^3 + 4x^2 + 4x + 4$