MA-403, Introduction to Number Theory
Mid-Semester exam. (2022-23 Autumn)
IIT Dharwad.

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Time: 2 hours
Max. marks: 100
23 September 2022

Note: Your answers and proofs should be clear and complete. Rough works should be separated and marked as such. Incomplete answers will be ignored.

Answer all the questions.

1. (a) For each $0 \neq a \in \mathbb{Z}_{11}$, find $0 \neq b \in \mathbb{Z}_{11}$ such that $ab \equiv 1 (\text{mod}\, \mathbb{Z}_{11})$. (10+10)

   (b) Find all elements $a \in \mathbb{Z}_{16}$ such that there is an element $b \in \mathbb{Z}_{16}$ with $ab \equiv 1 (\text{mod}\, \mathbb{Z}_{16})$. Find b, if it exists.

2. Find all integers which are the solutions to the following linear congruences: (5+10)

   (a) $14x \equiv 5 (\text{mod}\, 21)$

   (b) $17x \equiv 3 (\text{mod}\, 29)$

3. Find the greatest common divisor $d = (a, b)$, where $a = 43$ and $b = 245$. Find integers x and y such that $d = ax + by$. Does there exists another pair of integers x' and y' such that $d = ax' + by'$. Justify. (6+8+6)

4. Find the remainder on dividing $3^{100,000}$ by 53. (Hint: use Fermat's theorem.) (10)

5. Find all integers x such that each of the following holds: (10)

$$x \equiv 1 (\text{mod}\, 3), x \equiv 2 (\text{mod}\, 4), x \equiv 3 (\text{mod}\, 5).$$

6. (a) Find positive integers x and y such that $\sqrt{x} + y = 7$ and $x + \sqrt{y} = 11$. (Hint: use Fundamental theorem of Arithmetic.) (10)

   (b) Define precisely the Euler's φ-function and the Möbius function μ on the set of natural numbers. Calculate: $\varphi(4 \times 53^{101})$ and $\mu(482)$. (3+2+4+2)

   (c) Define a primitive n^{th}-root of unity. Does a primitive 9^{th}-root of unity exists? If it exists, give an example. (3+4+2)

Note: The total marks is 105. It would be considered as out of 100.

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